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EXPANSION FOR THE SUPERHEATING FIELD IN A SEMI-INFINITE FILM IN THE WEAK-κ LIMIT

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Abstract. Dorsey, Di Bartolo and Dolgert (Di Bartolo *et al.*, 1996; 1997) have constructed asymptotic matched solutions at order two for the half-space Ginzburg-Landau model, in the weak-κ limit. These authors deduced a formal expansion for the superheating field in powers of $\kappa^{\frac{1}{2}}$ up to order four, extending the formula by De Gennes (De Gennes, 1966) and the two terms in Parr's formula (Parr, 1976). In this paper, we construct asymptotic matched solutions at all orders leading to a complete expansion in powers of $\kappa^{\frac{1}{2}}$ for the superheating field.

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1. Introduction

The states of a superconducting material in an exterior magnetic field are described by the Ginzburg-Landau theory which introduces a functional ε depending in particular on a complex wave function and on the magnetic potential A. When the sample is a film and the exterior magnetic field is parallel to the surface, the Ginzburg-Landau model reduces to a one-dimensional problem where the wave function is real (and denoted by F) and where the functional is the following:

$$\varepsilon_d(F,A;h) = \int_{-\frac{d}{2}}^{\frac{d}{2}} \left[\frac{1}{2}(1-F(x)^2)^2 - \frac{1}{2} + \kappa^{-2}\,F'(x)^2 + F(x)^2\,A(x)^2 + (A'(x)-h)^2\right] \mathrm{d}x, \tag{1.1}$$

with $(F,A) \in \left(H^1\left(\left]-\frac{d}{2},\frac{d}{2}\right[\right)\right)^2$. Here, d is proportional to the thickness of the film, h is proportional to the exterior magnetic field and κ is the Ginzburg-Landau parameter characterizing the properties of the material.

In this paper, we restrict ourselves to the study of symmetric solutions (f even and A odd) and consider a new normalization of the functional where ε_d is replaced by $\left(\varepsilon_d - (h^2 - \frac{1}{2})d\right)$. We then restrict the problem to the interval $\left]-\frac{d}{2},0\right]$, and translate it to $\left]0,\frac{d}{2}\right[$. We get formally, by taking the limit $d = +\infty$, the second functional defined for $(F,A) \in E_\infty := \{(F,A)\ ;\ (1-F) \in H^1(]0,+\infty[),\ A \in H^1(]0,+\infty[)\}$ by

$$\varepsilon_\infty(F,A) = \int_0^{+\infty} \left[\kappa^{-2}\,F'(x)^2 + A'(x)^2 + \frac{1}{2}(1-F(x)^2)^2 + F(x)^2\,A(x)^2\right] \mathrm{d}x \ + 2\,h\,A(0)\,. \tag{1.2}$$

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The corresponding Ginzburg-Landau equations expressing the necessary conditions to have local minima are then

$$(GL)_\infty \begin{cases} -\kappa^{-2}F'' - F + F^3 + F\,A^2 = 0 \quad \text{on} \quad \mathbb{R}^+, \\ -A'' + AF^2 = 0 \quad \text{on} \quad \mathbb{R}^+, \end{cases} \tag{1.3}$$

with the boundary conditions

$$F'(0) = 0, \;\; A'(0) = h, \tag{1.4}$$

and

$$\lim_{x\to+\infty} F(x) = 1 \text{ and } \lim_{x\to+\infty} A(x) = 0. \tag{1.5}$$

The problem $(GL)_\infty$ is called the half-space model and was studied in [12] and [13] where computational solutions are given.

We consider the set $\mathcal{H}_\infty \subset \mathbb{R}^+$ of the h's such that there exist solutions of the (GL) system with $F > 0$. We know that $\mathcal{H}_\infty$ is a bounded interval $[0, h^+)$ (see [2], Prop. 2.1) and we then introduce the following definition:

Definition 1.1. The superheating field $h^{sh}(\kappa)$ is defined as the supremum of the interval $\mathcal{H}_\infty$.

In this paper, we first recall a construction of a formal solution of $(GL)_\infty$ obtained by Di Bartolo *et al.* in [14] when κ is small. We give a rigorous setting to this construction, introducing formal series.

We first construct a formal solution which is called the outer solution.

Let us introduce the outer variable $x' = \kappa x$. We look for an outer solution denoted by (F^e, A^e), solving the system obtained after the scaling $x' = \kappa x$ in (1.3) and the boundary conditions (1.5). We show that all the solutions have the form $\left(\tanh\left(\frac{x'+C(\kappa)}{\sqrt{2}}\right), 0\right)$, where $C(\kappa) :\sim \sum_{i=0}^{+\infty} C_i\kappa^i$, $C_i \in \mathbb{R}$.

Then, we construct solutions solving the (GL) system and the boundary conditions at zero (1.4), which is called the inner solution.

We look for an inner solution denoted by (F^i, A^i) and defined by

$$F^i \sim \sum_0^\infty F_i\kappa^i, \;\; A^i \sim \kappa^{-\frac{1}{2}}Q^i :\sim \kappa^{-\frac{1}{2}}\sum_0^\infty Q_i\kappa^i,$$

where F_i, and Q_i are C^∞ functions defined on $\overline{\mathbb{R}^+}$.

The first problem is to match the outer and inner solutions in order to get a good candidate for representing a global solution. We present a natural notion of matching and we show that it is equivalent to the van Dyke rule [18]. We prove the following theorem:

Theorem 1.2. *There exists a unique pair* $((F^e, A^e); (F^i, A^i))$ *solution of the (GL) equations, solving formally the boundary conditions*

$$F^i(0,\kappa) \sim t, \;\; (\partial_x F^i)(0,\kappa) \sim 0, \;\; \lim_{x'\to+\infty} F^e(x',\kappa) = 1$$

for $t \in]0,1[$ *fixed, and matched at all orders.*

Then, following a procedure proposed by Kaplun (see [15, 18] and [20]), we present an asymptotic matched solution of $(GL)_\infty$ at all orders.

The last part of the paper is devoted to the construction of a formal expansion for the superheating field. We indicate the dependency of the formal series A^i with respect to the parameter t in the following way: $A^i(x; t, \kappa)$. Moreover, we show that $(\partial_x A^i)(0; t, \kappa)$ is represented in the form of a formal series in powers of $\kappa^{\frac{1}{2}}$ with coefficients in $C^\infty(]0,1[)$. Then, considering, by an implicit functions theorem the zeros of $t \mapsto (\partial^2_{t\,x} A^i)(0;t,\kappa)$ on $]0,1[$, we define a notion of maximum of a formal series for which the principal term admits a unique non degenerate maximum on $]0,1[$. We deduce the following theorem:

Theorem 1.3. *The formal series $(\partial_x A^i)(0;t,\kappa)$ admits as a function of t a "formal" maximum on $]0,1[$, which is attained at a unique formal series $t(\kappa) := \sum_0^{+\infty} t_i \kappa^i$ with $t_0 \in]0,1[$.*

This formal maximum is expected to be the candidate for a formal expansion in powers of $\kappa^{\frac{1}{2}}$ for the superheating field.

The plan of this paper is the following. In Section 2, we recall the formal construction due to Di Bartolo *et al.* In Section 3, we prove Theorem 1.2. We deduce the construction of an asymptotic matched solution at all orders. In Section 4, we prove Theorem 1.3. In Section 5, we discuss a conjecture due to Fink *et al.* [19]. We note that the obtained formal expansion shows that the conjecture of these authors is false at the level of formal series.

2. Construction of a formal solution of $(GL)_\infty$

In all the following sections, for $i = (i_0, \cdots, i_n) \in \mathbb{N}^{n+1}$, we set

$$|i|_{0,n} := i_0 + i_1 + \cdots + i_n, \quad |i|_{1,n} := i_1 + i_2 + \cdots + i_n, \quad |i|_{2,n} := i_1 + 2i_2 + \cdots + ni_n. \tag{2.1}$$

2.1. Construction of an outer solution

Let $H = A'$. We get the new system

$$\begin{cases} -\kappa^{-2}F'' - F + F^3 + A^2F = 0 & \text{on } \mathbb{R}^+, \\ -A'' + AF^2 = 0 & \text{on } \mathbb{R}^+, \\ H = A' & \text{on } \mathbb{R}^+, \end{cases} \tag{2.2}$$

with the boundary conditions

$$F'(0) = 0, \quad H(0) = h, \tag{2.3}$$

and

$$\lim_{x\to+\infty} F(x) = 1, \quad \lim_{x\to+\infty} A(x) = 0. \tag{2.4}$$

We make the scaling $x' = \kappa x$ in the system (2.2) and set

$$F^e(x',\kappa) := F(x,\kappa), \quad Q^e(x',\kappa) := A(x,\kappa), \quad H^e(x',\kappa) := H(x,\kappa).$$

We get

$$\begin{cases} (F^e)'' - (Q^e)^2F^e + F^e - (F^e)^3 = 0, \\ \kappa^2\,(Q^e)'' - Q^e(F^e)^2 = 0, \\ H^e = \kappa(Q^e)', \end{cases} \tag{2.5}$$

where the differentiation is performed with respect to the variable x'.

Let us introduce the following definition:

Definition 2.1. We call formal outer solution, a triplet (F^e, Q^e, H^e) where

$$F^e \sim \sum_0^\infty f_i\kappa^i, \;\; Q^e \sim \sum_0^\infty q_i\kappa^i \;\text{ and }\; H^e \sim \sum_0^\infty h_i\kappa^i, \tag{2.6}$$

are formal solutions in powers of κ of the system (2.5), and whose coefficients verify

$$f_0 \to 1,\; f_j \to 0,\; q_i \to 0,\; h_i \to 0, \quad j \in \mathbb{N}^*, \;\; i \in \mathbb{N},$$

when x' tends to $+\infty$.

Let us denote by $C(\kappa)$ the formal series in powers of κ defined by

$$C(\kappa) \sim \sum_{n=0}^\infty C_n\kappa^n, \;\; (C_n \in \mathbb{R}) \tag{2.7}$$

and let f_0 be the function defined on $\mathbb{R}^+$ by

$$f_0(x') := \tanh\left(\frac{x'+C_0}{\sqrt{2}}\right)\cdot \tag{2.8}$$

We denote by $f_0^{(n)}$ the derivative at order n of f_0.

We can completely describe the outer solutions (see [9] and [14] for a proof).

Proposition 2.2. *All formal outer solutions are equal to*

$$\begin{cases} F^e(x',\kappa) \sim \tanh\left(\dfrac{x'+C(\kappa)}{\sqrt{2}}\right), \\ Q^e(x',\kappa) \sim 0, \\ H^e(x',\kappa) \sim 0. \end{cases} \tag{2.9}$$

Furthermore, for all $n \in \mathbb{N}^*$, we have

$$f_n = \sum_{m=1}^n \sum_{\substack{|i|_{2,n}=n \\ |i|_{1,n}=m}} \frac{m!}{i_1! \, \cdots \, i_n!} \prod_{k=1}^n (C_k)^{i_k} \; f_0^{(m)}, \tag{2.10}$$

where f_0 is defined in (2.8).

Remark 2.3. This solution does not satisfy the condition $(F^e)'(0,\kappa) = 0$. We will use it for x' large. Let us also remark that the C_j $(\in \mathbb{R})$ are for the moment arbitrary.

2.2. Construction of an inner solution

We want to construct an expansion of F, A, H in powers of κ, such that (F, A, H) is a formal solution of (2.2) and that F verifies the condition $F'(0,\kappa) = 0$. We hope to use this solution in a neighbourhood of zero.

We know from the De Gennes' formula (see [11]) that

$$\lim_{\kappa\to 0} \kappa^{\frac{1}{2}} h^{sh}(\kappa) = 2^{-\frac{3}{4}}. \tag{2.11}$$

This equality leads one to look for A of the form

$$A(x) = \kappa^{-\frac{1}{2}} Q(x). \tag{2.12}$$

If we make the scaling (2.12) in the system (2.2), we get

$$\begin{cases} F'' - \kappa Q^2 F + \kappa^2 (F - F^3) = 0, \\ Q'' - F^2 Q = 0, \\ \kappa^{\frac{1}{2}} H = Q', \end{cases} \tag{2.13}$$

with the boundary conditions

$$F'(0) = 0, \;\; Q'(0) = \kappa^{\frac{1}{2}} h. \tag{2.14}$$

Let us introduce the following definition:

Definition 2.4. We call a formal inner solution the triplet (F^i, Q^i, H^i) of formal series in powers of κ, with C^∞ coefficients on $\overline{\mathbb{R}^+}$, such that

(i)

$$F^i \sim \sum_0^\infty F_n \kappa^n, \;\; Q^i \sim \sum_0^\infty Q_n \kappa^n \;\text{ and }\; H^i \sim \kappa^{-\frac{1}{2}} \sum_0^\infty H_n \kappa^n; \tag{2.15}$$

(ii) (F^i, Q^i, H^i) is a formal solution of (2.13);

(iii) F^i satisfies formally the Neumann condition at zero $F_n'(0) = 0$, for all $n \in \mathbb{N}$.

If we consider the formal series with real coefficients

$$A(\kappa) \sim \sum_0^\infty A_n \kappa^n, \;\; B(\kappa) \sim \sum_0^\infty B_n \kappa^n, \;\; D(\kappa) \sim \kappa^{-\frac{1}{2}} \sum_0^\infty D_n \kappa^n,$$

we say that a formal inner solution has for initial data at zero $(A(\kappa), B(\kappa), D(\kappa))$, if

$$F^i(0,\kappa) \sim A(\kappa), \;\; Q^i(0,\kappa) \sim B(\kappa), \;\; H^i(0,\kappa) \sim D(\kappa).$$

In the next sections, we consider formal inner solutions with initial data $(A(\kappa), B(\kappa), D(\kappa))$.

We introduce for any $n \in \mathbb{N}$ the following notations:

$$\bar{A}_n := (A_0, \cdots, A_n), \;\; \bar{B}_n := (B_0, \cdots, B_n), \;\text{ and }\; \bar{C}_n := (C_0, \cdots, C_n). \tag{2.16}$$

As observed in [14], it is rather easy to compute the first terms.

Let us recall that

$$F_0(x; A_0) = A_0, \tag{2.17}$$

$$Q_0(x; A_0, B_0) = B_0 \exp(-A_0 x), \tag{2.18}$$

and

$$H_0(x; A_0, B_0) = -A_0 B_0 \exp(-A_0 x), \tag{2.19}$$

with

$$0 < A_0 < 1 \ \text{ and } \ B_0 < 0. \tag{2.20}$$

Furthermore,

$$F_1(x; \bar{A}_1, B_0) = A_1 - \frac{B_0^2}{4A_0} + \frac{B_0^2}{2}x + \frac{B_0^2}{4A_0}\exp(-2A_0x), \tag{2.21}$$

$$Q_1(x; \bar{A}_1, \bar{B}_1) = \frac{B_0^3}{16A_0^2}\exp(-3A_0x) + \left(B_1 - \frac{B_0^3}{16A_0^2} - B_0A_1x - \frac{B_0^3}{4}x^2\right)\exp(-A_0x), \tag{2.22}$$

and

$$\begin{aligned} H_1(x; \bar{A}_1, \bar{B}_1) = {} & \frac{-3B_0^3}{16A_0}\exp\left(-3A_0x\right) + \left(-B_0A_1 - A_0B_1\right)\exp\left(-A_0x\right) \\ & + \left(\frac{B_0^3}{16A_0} + \left(A_0B_0A_1 + \frac{B_0^3}{2}\right)x + \frac{A_0B_0^3}{4}x^2\right)\exp\left(-A_0x\right). \end{aligned} \tag{2.23}$$

For $n \geq 2$, the expression of F_n'' is given by construction by

$$F_n'' = -F_{n-2} + I_n + J_n, \tag{2.24}$$

where the functions I_n and J_n represent the coefficient of κ^n respectively in the expansion of κ^2F^3 and of κQ^2F.

We have

$$I_n = \sum_{\substack{|i|_{0,n-2} = 3 \\ |i|_{2,n-2} = n-2}} \frac{3!}{i_0! \cdots i_{n-2}!}\prod_{\ell=0}^{n-2} F_\ell^{i_\ell} \tag{2.25}$$

and

$$J_n = \sum_{\substack{\ell + |i|_{2,n-1} = n-1, \\ |i|_{0,n-1} = 2}} \frac{2!}{i_0!..i_{n-1}!}F_\ell\prod_{k=0}^{n-1} Q_k^{i_k}. \tag{2.26}$$

The function Q_n satisfies the equation

$$Q_n'' - A_0^2Q_n = \sum_{\substack{\ell + |i|_{2,n} = n,\ \ell \leq n-1 \\ |i|_{0,n} = 2}} \frac{2!}{i_0! \cdots i_n!}\,Q_\ell\,\prod_{k=0}^{n} F_k^{i_k}. \tag{2.27}$$

More generally, in [9], we have given the following description of the inner solution in the following proposition:

Proposition 2.5. *For all $n \geq 2$, the function F_n solution of* (2.15) *is equal to a sum of products of exponential polynomials. More precisely,*

$$F_n = F_n^{\text{pol}} + \psi(.)P_n(., \psi(.)), \tag{2.28}$$

where $F_n^{\rm pol}$ is a polynomial of degree n, P_n a polynomial and $\psi(x) = \exp(-2A_0\, x)$, A_0 being defined in (2.20). *Furthermore, for $n \geq 2$, $U_n(x) := P_n(x, 0)$ is of degree $2n - 2$.*

For all $n \in \mathbb{N}$, the function Q_n, defined in (2.15) *satisfies*

$$Q_n = \phi(.)R_n(., \phi(.)), \tag{2.29}$$

where R_n is a polynomial and $\phi(x) = \exp(-A_0\, x)$.

Furthermore, the polynomial $V_n(x) := R_n(x, 0)$ is of degree $2n$.

In order to match the outer and inner solutions, we can make explicit the dependency of the functions F_n, Q_n and H_n with respect to the constants $\bar{A}_n$, $\bar{B}_n$.

Proposition 2.6. *Let us consider the formal series in powers of κ, with coefficients in $\mathbb{R}$, $A(\kappa), B(\kappa)$. Then there exists a unique formal series in powers of κ, with coefficients in $\mathbb{R}$, $D(\kappa)$, and a unique inner solution admitting as initial data $(A(\kappa), B(\kappa), D(\kappa))$.*

For $n \geq 1$, $F_n(.)$ depends only on the parameters $\bar{A}_n, \bar{B}_{n-1}$, defined in (2.16), *and $F_n \in \mathrm{C}^\infty(\mathbb{R}\times]0,1[\times\mathbb{R}^{2n})$.*

Moreover, we have the equality

$$F_n^{\rm pol}(.; \bar{A}_n, \bar{B}_{n-1}) = A_n + \tilde{P}_n(.; \bar{A}_{n-1}, \bar{B}_{n-1}), \quad \tilde{P}_n \in \mathrm{C}^\infty\left(\mathbb{R}\times]0,1[\times\mathbb{R}^{2n-1}\right). \tag{2.30}$$

For $n \in \mathbb{N}$, the functions Q_n and H_n depends only on 2n+2 parameters, $\bar{A}_n, \bar{B}_n$. Precisely, for $n \geq 1$, we have the equality

$$Q_n(.; \bar{A}_n, \bar{B}_n) = \phi(.)(B_n + \tilde{R}_n(., \phi(.); \bar{A}_n, \bar{B}_{n-1}), \quad \tilde{R}_n \in \mathrm{C}^\infty\left(\mathbb{R}^2\times]0,1[\times\mathbb{R}^{2n}\right). \tag{2.31}$$

Proof. From (2.17, 2.21, 2.18) and (2.22), we see that the data of A_0, B_0, A_1, B_1 determine completely the functions F_0, F_1, Q_0, Q_1, H_0 and H_1. Furthermore, when $n = 1$, equations (2.30) and (2.31) are satisfied with

$$F_1^{\rm pol}(x; \bar{A}_1, B_0) = A_1 + \tilde{P}_1(x; A_0, B_0), \text{ with } \tilde{P}_1(x; A_0, B_0) = -\frac{B_0^2}{4A_0} + \frac{B_0^2}{2}x,$$

and

$$\tilde{R}_1(x, y; \bar{A}_1, \bar{B}_0) = \frac{B_0^3}{16A_0^2}y^3 - \left(\frac{B_0^3}{16A_0^2} + B_0A_1x + \frac{B_0^3}{4}x^2\right).$$

Let $n \geq 2$. Let us suppose that the proposition is true for $i \in \{0, \cdots, n-1\}$.

The function F_n is completely determined by (2.24) and the boundary conditions

$$F_n'(0) = 0, \text{ and } F_n(0) = A_n. \tag{2.32}$$

From (2.25, 2.26) and by recursion hypothesis, the function $I_n(.)$ depends only on $\bar{A}_{n-2}$, $\bar{B}_{n-2}$ and $I_n \in \mathrm{C}^\infty\left(\mathbb{R}\times]0,1[\times\mathbb{R}^{2n-3}\right)$. Moreover, from (2.26) and by recursion hypothesis, the function $J_n(.)$ depends only on $\bar{A}_{n-1}$, $\bar{B}_{n-1}$ and $J_n \in \mathrm{C}^\infty(\mathbb{R}\times]0,1[\times\mathbb{R}^{2n-1})$. According (2.24, 2.25, 2.26) and (2.32), it results that F_n depends only on $\bar{A}_n$, $\bar{B}_{n-1}$ and $F_n \in \mathrm{C}^\infty(\mathbb{R}\times]0,1[\times\mathbb{R}^{2n+1})$. From (2.24) and the boundary conditions (2.32), one gets that the function $F_n^{\rm pol}$ depends only on $\bar{A}_n$, $\bar{B}_{n-1}$. Moreover, we have $F_n^{\rm pol}(., \bar{A}_n, \bar{B}_{n-1}) \equiv A_n + \tilde{P}_n(.; \bar{A}_{n-1}, \bar{B}_{n-1})$ where $\tilde{P}_n \in \mathrm{C}^\infty(\mathbb{R}\times]0,1[\times\mathbb{R}^{2n-1})$. We get (2.30) for all $n \in \mathbb{N}^*$.

From Proposition 2.5, the right-hand side of (2.27) is a sum of exponential polynomials, which tend to 0 when x tends to infinity. Furthermore, from the recursion hypothesis, it only depends on $\bar{A}_{n-1}$ and $\bar{B}_{n-1}$. As $\lim_{x\to\infty} Q_n(x) = 0$ (see (2.29)), we have the equality

$$Q_n(x) = c_1 \exp(-A_0x) + \exp(-A_0x)g(x, \exp(-A_0x)),$$

where the function $g \in \mathrm{C}^\infty(\mathbb{R}^2\times]0,1[\times\mathbb{R}^{2n})$. From the boundary condition $Q_n(0) = B_n$, the function Q_n is determined by $\bar{A}_n$ and $\bar{B}_n$. It is equal to

$$Q_n(x) = \exp(-A_0x)(B_n + \tilde{R}_n(x, \exp(-A_0x))),$$

where $\tilde{R}_n \in \mathrm{C}^\infty(\mathbb{R}^2\times]0,1[\times\mathbb{R}^{2n})$.

As $H_n = Q'_n$ and for all $n \in \mathbb{N}$, $H_n(0) = D_n$, the formal series D_n is completely determined by $A(\kappa)$ and $B(\kappa)$. $\square$

2.3. Definition of the matching of the inner and outer solution

In order to clarify a more formal matching condition proposed by van Dyke in [18], we introduce the following definitions:

Definition 2.7. Let κ_0 be a positive real and $n \in \mathbb{N}$. For all $\kappa \in]0, \kappa_0]$, let $I(\kappa)$ be an interval of $\mathbb{R}$ and $u(x,\kappa)$ be a function defined on $I(\kappa)$. We say that

$$u(x,\kappa) = \mathcal{O}(\kappa^n) \ \text{ on } \ I(\kappa) \ \text{ when } \ \kappa \to 0,$$

if there exist $C > 0$ and $\tilde{\kappa}_0 > 0$ such that

$$|u(x,\kappa)| \le C\kappa^n, \ \ \forall\, x \in \ I(\kappa) \ \text{ and } \ \forall \kappa \le \tilde{\kappa}_0.$$

Definition 2.8. The truncated inner solution at order n is the triplet $(F^{i,(n)}, Q^{i,(n)}, H^{i,(n)})$ defined by $F^{i,(n)} := \sum_0^n F_i\kappa^i$, $Q^{i,(n)} := \sum_0^n Q_i\kappa^i$, and $H^{i,(n)} := \kappa^{-\frac{1}{2}}\sum_0^n H_i\kappa^i$. We denote by $F^{\mathrm{pol},(n)}$ the polynomial part of $F^{i,(n)}$. The truncated outer solution at order n is the triplet $(F^{e,(n)}, Q^{e,(n)}, H^{e,(n)})$ defined by $F^{e,(n)}(x';\kappa) := \sum_0^n f_i(x')\kappa^i$; $Q^{e,(n)}(x';\kappa) := \sum_0^n q_i(x')\kappa^i$ and $H^{e,(n)}(x';\kappa) := \sum_0^n h_i(x')\kappa^i$.

We introduce the function $\tilde{F}^{e,(n)}$ defined by $\tilde{F}^{e,(n)}(x;\kappa) = F^{e,(n)}(\kappa x;\kappa)$.

Definition 2.9. Let $n \in \mathbb{N}$, $(F^{i,(n)}, Q^{i,(n)}, H^{i,(n)})$ and $(F^{e,(n)}, Q^{e,(n)}, H^{e,(n)})$ the triplets of functions introduced in Definition 2.8. We say that the inner and outer solutions are matched at order n on the interval $I_n(\delta_1,\delta_2,\kappa) := \left[\delta_1\kappa^{-\frac{1}{n+1}}, \delta_2\kappa^{-\frac{1}{n+1}}\right]$ if and only if

$$\begin{aligned} &F^{i,(n)}(x,\kappa) - F^{e,(n-1)}(\kappa x,\kappa) = \mathcal{O}(\kappa^n),\\ &Q^{i,(n)}(x,\kappa) - Q^{e,(n-1)}(\kappa x,\kappa) = \mathcal{O}(\kappa^n),\\ &H^{i,(n)}(x,\kappa) - H^{e,(n-1)}(\kappa x,\kappa) = \mathcal{O}(\kappa^n), \end{aligned} \tag{2.33}$$

in the sense of Definition 2.7.

Proposition 2.10. *For all $n \in \mathbb{N}$, the formal series Q^i and Q^e (also H^i and H^e) are equal modulo $\mathcal{O}(\kappa^n)$ in the sense of Definition* 2.9.

Proof. Using Proposition 2.5 (see (2.29)), for any $j \in \mathbb{N}$, we can show the existence of $m_j \in \mathbb{N}$ such that

$$Q_j = \mathcal{O}\left(\kappa^{-\frac{m_j}{n+1}}\right).\exp\left(-\delta_1.A_0\kappa^{-\frac{1}{n+1}}\right) \ \text{ and } \ H_j = \mathcal{O}\left(\kappa^{-\frac{m_j}{n+1}}\right).\exp\left(-\delta_1.A_0\kappa^{-\frac{1}{n+1}}\right) \ \text{ on } \ I_n(\delta_1,\delta_2,\kappa).$$

It results that

$$Q_j = \mathcal{O}(\kappa^n) \ \text{ and } \ H_j = \mathcal{O}(\kappa^n) \ \text{ on } \ I_n(\delta_1,\delta_2,\kappa).$$

From Proposition 2.2, $Q^e \sim 0$ and $H^e \sim 0$. Then, on $I_n(\delta_1, \delta_2, \kappa)$,

$$\begin{aligned} Q^{i,(n)}(x,\kappa) - Q^{e,(n-1)}(\kappa x, x) = Q^{i,(n)}(x,\kappa) = \mathcal{O}(\kappa^n), \\ H^{i,(n)}(x,\kappa) - H^{e,(n-1)}(\kappa x, x) = H^{i,(n)}(x,\kappa) = \mathcal{O}(\kappa^n), \end{aligned}$$

in the sense of Definition 2.7. □

In all the following sections, we introduce the set

$$\tilde{E} := \{(i,j) \in \mathbb{N}^2, \ \text{such that} \ 0 \leq i \leq j \leq n, \quad (i,j) \neq (0,n)\}\cdot$$

To specify the conditions of matching for the inner and outer solutions, we use the elementary lemma (see [9] for a proof):

Lemma 2.11. *Let $n \in \mathbb{N}^*$, $(\delta_1, \delta_2) \in \mathbb{R}^+ \times \mathbb{R}^+$ and $\gamma_{i,j}$ be a family of reals, where $(i,j) \in \tilde{E}$. Let S be the function defined on $[0,\kappa_0] \times [\delta_1, \delta_2]$ by*

$$S(\kappa, y) = \sum_{(i,j)\in\tilde{E}} \gamma_{i,j} \kappa^{j-\frac{i}{n+1}} y^i + \mathcal{O}(\kappa^n). \tag{2.34}$$

The equality

$$S(\kappa, y) = 0 \ \ on \ \ [0,\kappa_0] \times [\delta_1, \delta_2] \tag{2.35}$$

implies

$$\gamma_{i,j} = 0, \ \ \forall \, (i,j) \in \tilde{E}.$$

We can then give the conditions of matching modulo $\mathcal{O}(\kappa^n)$ for the outer and inner solutions. Let us write, for any $j \in \mathbb{N}$

$$F_j^{\mathrm{pol}}(x) = \sum_{i=0}^{j} \alpha_{i,j} x^i, \tag{2.36}$$

and let

$$\beta_{i,j} := \frac{f_{j-i}^{(i)}(0)}{i!}, \tag{2.37}$$

where f_i is defined in (2.6).

Proposition 2.12. *Let $n \in \mathbb{N}$. The inner and outer solutions are equal modulo $\mathcal{O}(\kappa^n)$ if and only if*

$$\alpha_{i,j} = \beta_{i,j}, \ \forall \, (i,j) \in \tilde{E}. \tag{2.38}$$

Proof. From Proposition 2.10, the formal series Q^i and Q^e are equal modulo $\mathcal{O}(\kappa^n)$.

Let $k \in \{1, \cdots, n\}$. On the interval $I_n(\delta_1, \delta_2, \kappa)$, from Proposition 2.5 (see (2.28)), for κ small, we have $\psi(x) P_k(x, \psi(x)) = \mathcal{O}(\kappa^n)$.

For $x \in I_n(\delta_1, \delta_2, \kappa)$, for κ small, we get the estimate

$$F^{i,(n)}(x,\kappa) = \sum_{(i,j)\in\tilde{E}} \alpha_{i,j} \kappa^j x^i + \mathcal{O}(\kappa^n). \tag{2.39}$$

For $k \in \{0, \cdots, n-1\}$, we can make a Taylor expansion of $x \mapsto f_k(\kappa x)$ where the function f_k is defined in (2.10) at the point $x = 0$. We can then express $F^{e,(n-1)}(\kappa x; \kappa)$ in the form of

$$F^{e,(n-1)}(\kappa x; \kappa) = \sum_{k=0}^{n-1} \kappa^k \sum_{i=0}^{\infty} \frac{f_k^{(i)}(0)}{i!} \kappa^i x^i.$$

Let us introduce $j = i + k$. We can write

$$F^{e,(n-1)}(\kappa x; \kappa) = \sum_{j-n+1 \le i \le j} \beta_{i,j} \kappa^j x^i, \tag{2.40}$$

where $\beta_{i,j}$ is defined in (2.37). As $x \in I_n(\delta_1, \delta_2, \kappa)$, for (i,j) such that $j \ge n+1$ and $i \le j$, we have the estimate

$$\kappa^j x^i = \mathcal{O}\left(\kappa^{j - \frac{i}{n+1}}\right) = \mathcal{O}(\kappa^n).$$

We deduce the estimate

$$F^{e,(n-1)}(\kappa x, \kappa) = \sum_{(i,j) \in \tilde{E}} \beta_{i,j}\ \kappa^j x^i + \mathcal{O}(\kappa^n). \tag{2.41}$$

The estimate $F^{i,(n)}(x) - F^{e,(n-1)}(\kappa x) = \mathcal{O}(\kappa^n)$ on $I_n(\delta_1, \delta_2, \kappa)$ is satisfied if and only if

$$\sum_{(i,j) \in \tilde{E}} (\alpha_{i,j} - \beta_{i,j})\, \kappa^j x^i + \mathcal{O}(\kappa^n) = 0 \ \text{ on } \ ]0, \kappa_0] \times I_n(\delta_1, \delta_2, \kappa). \tag{2.42}$$

Then, we can apply Lemma 2.11 to achieve the proof of Proposition 2.12. □

Remark 2.13. The van Dyke rule [18] consists in taking the truncated inner solution at order n and the truncated outer solution at order $n-1$, then to make an identification of the coefficients of $\kappa^j x^i$ for all pairs $(i,j) \in \tilde{E}$. We have shown that this procedure is equivalent to Definition 2.9.

3. Matching of the outer and the inner solutions at all orders

3.1. Conditions of matching modulo $\mathcal{O}(\kappa^n)$

Let us consider the formal series $F^{i,\infty}$ defined by

$$F^{i,\infty}(x, \kappa) :\sim \sum_{j=0}^{+\infty} F_j^{\mathrm{pol}}(x) \kappa^j, \tag{3.1}$$

where F_j^{pol} is defined in Proposition 2.5.

From (2.36), we can write (3.1) in the form of

$$F^{i,\infty}(x, \kappa) \sim \sum_{\ell=0}^{+\infty} \sum_{i=0}^{+\infty} \alpha_{i,i+\ell} x^i \kappa^{i+\ell}. \tag{3.2}$$

We set

$$\phi_\ell(x') :\sim \sum_{i=0}^{+\infty} \alpha_{i,i+\ell} {x'}^i. \tag{3.3}$$

So, from (3.2), we deduce that

$$F^{i,\infty}(x,\kappa) \sim \sum_{\ell=0}^{+\infty} \phi_\ell(\kappa x)\kappa^\ell.$$

Let us remark that $F^{i,\infty}$ satisfies formally the equation

$$-\kappa^{-2}(F^{i,\infty})'' - F^{i,\infty} + (F^{i,\infty})^3 = 0.$$

Let us introduce

$$G(x',\kappa) \sim \sum_{\ell=0}^{+\infty} \phi_\ell(x')\kappa^\ell, \tag{3.4}$$

where $x' = \kappa x$ is the outer variable.

The formal series G satisfies formally the equation

$$G'' = -G + G^3. \tag{3.5}$$

To show equalities (2.38), we use the following Lemma (see [9], p. 69 for a proof):

Lemma 3.1. *Let S_1 and S_2 be two formal series in powers of κ whose coefficients are formal series in powers of x' with coefficients in $\mathbb{R}$ defined by*

$$S_1(x',\kappa) \sim \sum_0^{+\infty} \phi_i(x')\kappa^i \ \ \textit{and} \ \ S_2(x',\kappa) \sim \sum_0^{+\infty} \psi_i(x')\kappa^i.$$

Let us suppose that S_1 and S_2 satisfy formally the differential equation

$$y''(x') = -y(x') + y(x')^3. \tag{3.6}$$

Furthermore, let us suppose that the equalities

$$\phi_i(0) = \psi_i(0) \ \ \textit{and} \ \ \phi_i'(0) = \psi_i'(0), \quad \forall\, i \in \mathbb{N}, \tag{3.7}$$

are satisfied.

Then, for all $i \in \mathbb{N}$, we have the equalities

$$\phi_i^{(n)}(0) = \psi_i^{(n)}(0), \ \ \forall\, n \in \mathbb{N}.$$

For all $i \in \mathbb{N}$, let us suppose that for all $\bar{A}_{i+1}$, with $A_0 \in]0,1[$, the system

$$\alpha_{0,j} = \beta_{0,j} \ \text{ and } \ \alpha_{1,j+1} = \beta_{1,j+1}, \ \text{ for } \ j \in \{0,\cdots,i+1\}, \tag{3.8}$$

admits a unique solution with $B_0 < 0$ (this point is the subject of Prop. 3.4).

Then, we can show the following proposition:

Proposition 3.2. *For all $\bar{A}_{n-1}$, with $A_0 \in]0,1[$, the formal inner and outer solutions are equal modulo $\mathcal{O}(\kappa^n)$, if and only if the system of $2n$ equations with $2n$ unknowns $\bar{B}_{n-1}, \bar{C}_{n-1}$ defined by*

$$\alpha_{0,i} = \beta_{0,i} \ \ \alpha_{1,i+1} = \beta_{1,i+1}, \ \ \textit{for} \ \ i \in \{0,\cdots,n-1\}, \tag{3.9}$$

admits a unique solution with $B_0 < 0$.

Proof. Let us consider the formal series F^e defined in (2.6) and G defined in (3.4). Let us express F^e in the form of $F^e \sim \sum_{i=0}^{\infty} \psi_i(x')\kappa^i$ where (ψ_i) is the formal series defined by

$$\psi_i(x') :\sim \sum_{j=0}^{\infty} \frac{f_i^{(j)}(0)}{j!}(x')^j \sim \sum_{j=0}^{\infty} \beta_{j,i+j}(x')^j. \tag{3.10}$$

Hypothesis (3.8) is equivalent to

$$\phi_i(0) = \psi_i(0) \text{ and } \phi_i'(0) = \psi_i'(0), \quad \forall\, i \in \mathbb{N}.$$

From Lemma 3.1 applied to the formal series G and F^e, we get, for all $i \in \mathbb{N}$

$$\phi_i^{(n)}(0) = \psi_i^{(n)}(0), \ \ \forall\, n \in \mathbb{N}.$$

Then, using (3.3) and (3.10), it results that

$$\alpha_{n,n+i} = \beta_{n,n+i}, \ \ \forall\, n \in \mathbb{N} \text{ and } \forall\, i \in \mathbb{N}.$$

We deduce that the system (2.38) admits a unique solution. □

To show that it is possible to get (3.8), we have to analyze how $\alpha_{i,j}$ and $\beta_{i,j}$ which are defined in (2.36) and (2.37), depend on A_i, B_i and C_i.

Lemma 3.3. *We have the equalities*

$$\alpha_{0,0} = A_0, \ \ \alpha_{1,1} = A_1 - \frac{B_0^2}{4A_0} \ \ \textit{and} \ \ \beta_{0,0} = f_0(0), \ \ \beta_{1,1} = f_0'(0). \tag{3.11}$$

For $i \in \mathbb{N}^*$, $\alpha_{0,i} - A_i$ and $\alpha_{1,i+1} + B_0B_i$ depend only on $\bar{A}_{i-1}$ and $\bar{B}_{i-1}$. Moreover, we have $\alpha_{0,i} - A_i \in \mathrm{C}^\infty(]0,1[\times\mathbb{R}^{2i-1})$ and $\alpha_{1,i+1} + B_0B_i \in \mathrm{C}^\infty(]0,1[\times\mathbb{R}^{2i-1})$.

For $i \in \mathbb{N}^*$, $\beta_{0,i} - C_i f_0'(0)$, and $\beta_{1,i+1} - C_i f_0''(0)$ depend only on $\bar{C}_{i-1}$ and belong to $\mathrm{C}^\infty(\mathbb{R}^i)$.

Proof. According to (2.17, 2.21, 2.36) and by definition (see (2.37)), we get equalities (3.11).

The real $\alpha_{0,i}$ is equal to the coefficient of x^0 in the expression of F_i^{pol}, and from Proposition 2.6, it is equal to A_i modulo a function depending on $\bar{A}_{i-1}, \bar{B}_{i-1}$ and $\mathrm{C}^\infty\left(]0,1[\times\mathbb{R}^{2i-1}\right)$. The real $\alpha_{1,i+1}$ is equal to the coefficient of x in the expression of F_{i+1}^{pol}. It is equal to the constant of integration obtained after an integration of the function F_{i+1}'' with the boundary condition $F_{i+1}'(0) = 0$. According to (2.24, 2.26) and (2.31), the unique function where the parameter B_i in the expression of F_{i+1}'' appears is $2F_0Q_0Q_i$. From (2.31), this term is given by $-B_0B_i$ modulo a function depending on $\bar{A}_{i-1}, \bar{B}_{i-1}$ and belongs to $\mathrm{C}^\infty\left(]0,1[\times\mathbb{R}^{2i-1}\right)$.

According to (2.10) and (2.37), we get $\beta_{0,i} = C_i f_0'(0)$ and $\beta_{1,i+1} = C_i f_0''(0)$ modulo a function depending on $\bar{C}_{i-1}$ and belongs to $\mathrm{C}^\infty(\mathbb{R}^i)$. □

From Lemma 3.3, we can deduce the following proposition:

Proposition 3.4. *Let $n \in \mathbb{N}^*$. Let us consider the system of $2n$ equations with $3n$ unknowns, $\bar{A}_{n-1}, \bar{B}_{n-1}, \bar{C}_{n-1}$*

$$\alpha_{0,i} = \beta_{0,i}, \ \ \alpha_{1,i+1} = \beta_{1,i+1} \ \ \textit{for} \ \ i \in \{0, \cdots, n-1\}\,\cdot \tag{3.12}$$

For all $\bar{A}_{n-1} \in \mathbb{R}^n$, such that $A_0 \in]0,1[$, the system (3.12) admits a unique solution in $\mathbb{R}^{2n}$ such that $B_0 < 0$.

Proof. The result is true for $n = 1$. From Lemma 3.3, the system consists in two equations,

$$A_0 = f_0(0), \quad \frac{B_0^2}{2} = f_0'(0), \tag{3.13}$$

and can be solved by choosing as principal unknowns B_0 and C_0. From the hypothesis $A_0 \in]0,1[$ and $B_0 < 0$, the solution of the system (3.13) is unique. According to (2.8), we get

$$B_0 = -2^{\frac{1}{4}}(1 - A_0^2)^{\frac{1}{2}}, \quad C_0 = \sqrt{2}\tanh^{-1}(A_0). \tag{3.14}$$

Let $n \geq 1$. Let us suppose that the result is true for $k \in \{0, \cdots, n-1\}$, then C_k and B_k can be expressed in a unique way as a function of $A_0, \cdots, A_k$ and $B_k \in \mathrm{C}^\infty\left(]0,1[\times\mathbb{R}^k\right)$ and $C_k \in \mathrm{C}^\infty\left(]0,1[\times\mathbb{R}^k\right)$.

From Lemma 3.3 and by hypothesis, the equalities $\alpha_{0,n} = \beta_{0,n}$ and $\alpha_{1,n+1} = \beta_{1,n+1}$ are equivalent to the equalities

$$A_n = f_0'(0)C_n + \mathcal{G}(\bar{A}_{n-1}), \quad \mathcal{G} \in \mathrm{C}^\infty\left(]0,1[\times\mathbb{R}^{n-2}\right) \tag{3.15}$$

and

$$-B_0B_n = f_0''(0)C_n + \mathcal{F}(\bar{A}_{n-1}), \quad \mathcal{F} \in \mathrm{C}^\infty\left(]0,1[\times\mathbb{R}^{n-2}\right). \tag{3.16}$$

From (3.13), we have the equalities

$$f_0'(0) = \frac{1}{\sqrt{2}}(1 - A_0^2) \text{ and } f_0''(0) = -\frac{1}{\sqrt{2}}A_0(1 - A_0^2). \tag{3.17}$$

We can then solve the system (3.15, 3.16) by considering B_n and C_n as principal unknowns and $A_0, \cdots, A_n$ as parameters.

From Proposition 3.4, it results that hypothesis (3.8) is verified. □

3.2. Formal solutions of the Ginzburg-Landau equation

We say that the inner and outer solutions match at all orders, if they match modulo $\mathcal{O}(\kappa^n)$, for all $n \in \mathbb{N}$. From Proposition 2.12, we observe that the matching of the outer and inner solutions modulo $\mathcal{O}(\kappa^{n+1})$ implies their matching modulo $\mathcal{O}(\kappa^n)$. Propositions 3.2 and 3.4 lead us to introduce the following definition:

Definition 3.5. We call formal solution of the Ginzburg-Landau equations (2.2) with boundary conditions (2.3, 2.4), a pair composed of an inner solution in the sense of Definition 2.4 with initial data $(A(\kappa), B(\kappa), D(\kappa))$ and an outer solution in the sense of Definition 2.2, which match at all orders.

Then, we can express the following theorem:

Theorem 3.6. *For all formal series* $A(\kappa) := \sum_{i=0}^{+\infty} A_i \kappa^i$ *with* $A_0 \in]0,1[$, *there exists a unique formal solution of the (GL) system with* $B_0 < 0$.

Proof.

Step 1. Existence

We have constructed in Proposition 2.2 a formal outer solution, and in Proposition 2.5 a formal inner solution. Then we match them at all orders using Proposition 3.2.

Step 2. Uniqueness

From Propositions 2.6 and 3.4, when $A(\kappa)$ is given with $A_0 \in]0,1[$, and if we suppose $B_0 < 0$, $B(\kappa)$ and $C(\kappa)$ are completely determined. From Propositions 2.2 and 2.6, the data of $A(\kappa)$, $B(\kappa)$ and $D(\kappa)$ definite completely the coefficients of the outer and inner solution, so, by definition of a formal solution, the uniqueness. $\square$

In the following sections, using Proposition 3.2, we suppose that the functions $F_n(x, \bar{A}_n, \bar{B}_{n-1})$, $Q_n(x, \bar{A}_n, \bar{B}_n)$ and $H_n(x, \bar{A}_n, \bar{B}_n)$ are expressed in terms of the parameters A_i, for $i \in \{0, \cdots, n\}$.

We introduce then the following notations:

$$\begin{aligned} &\check{F}_n(x; \bar{A}_n) := F_n(x, \bar{A}_n, \bar{B}_{n-1}(\bar{A}_{n-1})), \quad \check{Q}_n(x; \bar{A}_n) := Q_n(x, \bar{A}_n, \bar{B}_n(\bar{A}_n)), \\ &\check{H}_n(x; \bar{A}_n) := H_n(x, \bar{A}_n, \bar{B}_n(\bar{A}_n)). \end{aligned} \tag{3.18}$$

We denote by $\check{F}^i$, $\check{Q}^i$ and $\check{H}^i$ the formal series

$$\check{F}^i := \sum_{i=0}^{\infty} \check{F}_n(x; \bar{A}_n)\kappa^i, \quad \check{Q}^i := \sum_{i=0}^{\infty} \check{Q}_n(x; \bar{A}_n)\kappa^i \text{ and } \check{H}^i := \kappa^{-\frac{1}{2}} \sum_{i=0}^{\infty} \check{H}_n(x; \bar{A}_n)\kappa^i. \tag{3.19}$$

We can introduce the particular choice

$$A_0 = t, \quad t \in]0,1[\text{ and } A_i = 0, \quad \forall\, i \in \mathbb{N}^*. \tag{3.20}$$

According to (3.15, 3.16, 3.18) and (3.20), we can set

$$\hat{F}_n(x,t) := \check{F}_n(x; \bar{A}_n), \quad \hat{Q}_n(x,t) := \check{Q}_n(x; \bar{A}_n) \text{ and } \hat{H}_n(x,t) := \check{H}_n(x; \bar{A}_n). \tag{3.21}$$

From Theorem 3.6 with the choice (3.20), we get the existence and the uniqueness of a formal solution solving the boundary condition at zero

$$F^i(0,\kappa) \sim t, \quad (\partial_x F^i)(0,\kappa) \sim 0.$$

From (2.12, 2.15, 3.20) and (3.21), we have the equality

$$H^i(0; t, \kappa) := \kappa^{-\frac{1}{2}} \sum_{n=0}^{\infty} \hat{H}_n(0,t)\kappa^n. \tag{3.22}$$

3.3. Construction of a Matched Asymptotic Solutions

Generally speaking, when we have matched the inner and outer solutions modulo $\mathcal{O}(\kappa^n)$, we can hope to construct an approximate solution denoted by $(f^{vd,(n)}, A^{vd,(n)})$, proceeding in the following way. For $f^{vd,(n)}$, we consider the function obtained by adding the function $F^{i,(n+1)}$ to the function $\tilde{F}^{e,(n)}$, and subtracting from this sum the polynomial part of $F^{i,(n+1)}$, denoted $F^{\mathrm{pol},(n+1)}$. This rule was used by Di Bartolo *et al.* in [14] to construct an approximate solution of $(GL)_\infty$. The formal solution $\tilde{A}^e(x';\kappa)$ vanishes (see Prop. 2.2), and we take for $A^{vd,(n)}$, the function $A^{vd,(n)} = \kappa^{-\frac{1}{2}} \sum_0^n Q_i \kappa^i$.

Let us introduce the following definition:

Definition 3.7. Let $n \in \mathbb{N}^*$. We call asymptotic matched solution of $(GL)_\infty$ at order n, a pair $\left(f^{vd,(n)}, A^{vd,(n)}\right)$ defined by

$$f^{vd,(n)}(x;\kappa) = \tilde{F}^{e,(n)}(x;\kappa) + F^{i,(n+1)}(x;\kappa) - F^{\mathrm{pol},(n+1)}(x;\kappa), \tag{3.23}$$

where $\tilde{F}^{e,(n)}(x;\kappa)$, $F^{i,(n+1)}(x;\kappa)$, and $F^{\mathrm{pol},(n+1)}(x;\kappa)$ are introduced in Definition 2.8, and

$$A^{vd,(n)}(x,\kappa) = \kappa^{-\frac{1}{2}} \sum_0^n Q_i(x)\kappa^i, \tag{3.24}$$

where Q_i is defined in (2.29).

Let us remark that the asymptotic matched solution $f^{vd,(n)}$ satisfies the Neumann condition at zero.

Lemma 3.8. *The function $f^{vd,(n)}$ defined in* (3.23) *satisfies the Neumann condition at zero*

$$\left(f^{vd,(n)}\right)'(0) = 0.$$

Proof. For all $i \in \mathbb{N}$, we have $F_i'(0) = 0$. So, we get the equality

$$\left(F^{i,n+1}\right)'(0) = 0.$$

Furthermore $\left(F^{\mathrm{pol},n+1}\right)'(0) = \sum_{j=1}^{n+1} \alpha_{1,j}$, where $\alpha_{1,j}$ is defined in (2.36). We deduce the equality

$$\left(f^{vd,(n)}\right)'(0) = \left(F^{e,n}\right)'(0) + \left(F^{i,n+1}\right)'(0) - \left(F^{\mathrm{pol},n+1}\right)'(0) = \sum_{j=0}^{n} f_j'(0)\kappa^{j+1} - \sum_{j=1}^{n+1} \alpha_{1,j}\kappa^j. \tag{3.25}$$

Moreover, from Proposition 2.12, $\alpha_{1,j+1} = \beta_{1,j+1} = f_j'(0)$ for all $j \in \{0, \cdots, n\}$. $\square$

Remark 3.9. In [10], using the pair $\left(f^{vd,(1)}, A^{vd,(1)}\right)$ suitably modified, we have constructed a subsolution of $(GL)_\infty$, leading to two terms in the lower bound for the superheating field. Using the asymptotic matched solution at order n presented in Definition 3.7, we can hope to construct a subsolution of the (GL) equations leading to n terms in the lower bound for the superheating field. Moreover, constructing a supersolution by a similar method, we hope to get a localization of solutions of $(GL)_\infty$. We hope to come back to these points in a further publication.

4. Construction of a formal expansion in powers of $\kappa^{\frac{1}{2}}$ for the superheating field

The aim of this section is to determine a formal expansion for the superheating field introduced in Definition 1.1. We want to maximize with respect to $t \in]0,1[$ the formal series $\phi(t,\kappa)$ introduced in (3.22). This notion of "maximum" is defined in a formal sense which will be specified in the following sections. This problem of maximization of this formal series was not solved in general in [14].

4.1. Maximization of a formal series

Let us consider f a C^∞ real function. In the following section, $DT(f)$ represents the Taylor expansion of f at the point $x = 0$,

$$DT(f)(x) :\sim \sum_{i=0}^{\infty} \frac{f^{(i)}(0)}{i!} x^i. \tag{4.1}$$

We consider in the following sections formal series with coefficients in $C^\infty(]0,1[)$, defined by

$$\phi(t,\kappa) \sim \kappa^{-\frac{1}{2}} \sum_{0}^{\infty} \phi_i(t)\kappa^i. \tag{4.2}$$

We suppose that ϕ_0 admits a unique maximum achieved at the point $t_0 \in]0,1[$. Moreover, it satisfies

$$\phi_0'(t_0) = 0 \ \text{ and } \ \phi_0''(t_0) < 0. \tag{4.3}$$

In order to define the notion of maximum in the formal series (4.2), let us recall a lemma due to Borel.

Lemma 4.1. *Let ϕ be a formal series with C^∞ coefficients on $]0,1[$ defined by $\phi(t,\kappa) \sim \kappa^{-\frac{1}{2}} \sum_0^\infty \phi_i(t)\kappa^i$. Then, there exists a C^∞ function ψ defined on $]0,1[\times]-\kappa_0,\kappa_0[$, such that the expansion in Taylor series of $\kappa \mapsto \psi(.,\kappa)$ at the point $\kappa = 0$ of ψ coincides with ϕ.*

A function solving the conclusion of Lemma 4.1 is called a representative in the formal series ϕ. To define the maximum in the formal series (4.2), it results from the implicit functions theorem the following lemma:

Lemma 4.2. *Let $\bar{\phi}_1$ be a representative of ϕ. Let us suppose that $\bar{\phi}_1$ admits in $]0,1[$ a unique maximum on t_0, which is non degenerate*

$$\frac{\partial \bar{\phi}_1}{\partial t}(t_0,0) = 0 \ \text{ and } \ \frac{\partial^2 \bar{\phi}_1}{\partial t^2}(t_0,0) < 0. \tag{4.4}$$

Then, there exists a positive real, κ_0, such that, for all $|\kappa| \le \kappa_0$, the mapping $t \mapsto \bar{\phi}_1(t,\kappa)$ admits a maximum $t_1(\kappa)$ on $]0,1[$ such that $t_1(0) = t_0$. Furthermore, on $]-\kappa_0,\kappa_0[$, the mapping $\kappa \mapsto t_1(\kappa)$ is C^∞.

Let $\bar{\phi}_2$ be another representative of ϕ and let $t_2(\kappa)$ be a maximum of $t \to \bar{\phi}_2(t,.)$ such that $t_2(0) = t_0$. Then

$$DT(t_1) \sim DT(t_2), \tag{4.5}$$

where DT is defined in (4.1).

We can define the maximum in the formal series defined in (4.2) and solving hypothesis (4.3) by using Lemmas 4.1 and 4.2.

Definition 4.3. Let ϕ be a formal series in powers of κ with coefficients in $\mathrm{C}^\infty(]0,1[)$ defined by

$$\phi(t,\kappa) \sim \kappa^{-\frac{1}{2}} \sum_{0}^{\infty} \phi_i(t)\kappa^i,$$

and solving (4.3). Let $\bar{\phi}$ be a representative of ϕ.

We say that the formal series ϕ achieved its maximum on $]0,1[$ in the unique formal series defined by the Taylor expansion at the point $\kappa = 0$ of the function $t_1(\kappa)$ defined in Lemma 4.2 (see (4.5)).

Then, we can express the following proposition:

Proposition 4.4. *The formal series $H^i(0;t,\kappa)$ defined in* (3.22) *admits, as function of t, a unique formal maximum on* $]0,1[$.

Proof. From (2.19, 3.14, 3.20) and (3.21), the principal part of the series $H^i(0;t,\kappa)$ is equal to

$$\hat{H}_0(0,t) = 2^{\frac{1}{4}} t(1-t^2)^{\frac{1}{2}}.$$

This function admits a unique maximum on $]0,1[$ which is achieved at the point $t = \frac{1}{\sqrt{2}}$. Furthermore, this maximum is non degenerate. Then, the formal series ϕ admits a unique formal maximum on $]0,1[$ in the sense of Definition 4.3. □

Remark 4.5. To this maximum corresponds a formal expansion in powers of $\kappa^{\frac{1}{2}}$ for the superheating field $h^{sh}(\kappa)$. But, using this method, we cannot explicitly compute the coefficients in the formal series obtained in Proposition 4.4.

4.2. Existence of a formal expansion for the superheating field

In [9], in order to compute the coefficients in the formal superheating field, we have presented another way for defining the maximum in the formal series given in (4.2) and (4.3). Let us explain this method. By hypothesis, the functions ϕ_i are smooth, then we can make a Taylor expansion at the point $A_0 \in]0,1[$ of the function $\phi_i(t)$, for all $i \in \mathbb{N}$. Then, we can substitute for t the formal series $A(\kappa) :\sim \sum_0^\infty A_i\kappa^i$, expand it in powers of κ, reorder the expression obtained in powers of κ. Then, the resulting formal series ϕ is expressed in the form of

$$\phi(A(\kappa),\kappa) \sim \sum_{0}^{\infty} D_n(\bar{A}_n)\kappa^n, \tag{4.6}$$

where

$$D_0(A_0) := \phi_0(A_0), \tag{4.7}$$

and, for $n \geq 1$,

$$D_n(\bar{A}_n) := \sum_{\ell+|i|_{2,n}=n} \frac{1}{i_1!\cdots i_n!}\, \phi_\ell^{(|i|_{1,n})}(A_0) \prod_{k=1}^{n} {A_k}^{i_k}. \tag{4.8}$$

The function D_n has the following property:

Lemma 4.6. *For $n \in \mathbb{N}^*$, let D_n be the function defined in* (4.7) *and* (4.8).
We have the equalities

$$D_{2n-1}(\bar{A}_{2n-1}) = \sum_{k=0}^{n-1} \frac{\partial D_{2k}}{\partial A_k}(\bar{A}_{2k}) A_{2n-1-k} + g(A_0, \cdots, A_{n-1}), \tag{4.9}$$

$$D_{2n}(\bar{A}_{2n}) = \sum_{k=0}^{n-1} \frac{\partial D_{2k}}{\partial A_k}(\bar{A}_{2k}) A_{2n-k} + L(A_0, \cdots, A_n), \tag{4.10}$$

where $g \in \mathrm{C}^\infty(]0,1[\times\mathbb{R}^{n-1})$ and $L \in \mathrm{C}^\infty(]0,1[\times\mathbb{R}^n)$.

Furthermore, L is a polynomial of degree two in A_n and the coefficient of A_n^2 is equal to $\dfrac{\phi_0''(A_0)}{2}$.

Proof. From (4.8), we have the equalities

$$D_1(\bar{A}_1) = \phi_0'(A_0)A_1 + \phi_1(A_0) \tag{4.11}$$

and

$$D_2(\bar{A}_2) = \phi_0'(A_0)A_2 + \frac{\phi_0''}{2}(A_0)A_1^2 + \phi_1'(A_0)A_1 + \phi_2(A_0). \tag{4.12}$$

From (4.7), $D_0(A_0) := \phi_0(A_0)$. Then, equalities (4.9) and (4.10) are verified for $n = 1$, with

$$g(A_0) = \phi_1(A_0)$$

and

$$L(A_0, A_1) = \frac{\phi_0''}{2}(A_0)A_1^2 + \phi_1'(A_0)A_1 + \phi_2(A_0).$$

Let us suppose that (4.9) and (4.10) are true for $k \in \{1, \cdots, n\}$, $n \geq 1$.

Let $(\ell, i) = (\ell, i_1, \cdots, i_j, \cdots, i_{2n+1}) \in \mathbb{N} \times \mathbb{N}^{2n+1}$ such that $\ell + |i|_{2,2n+1} = 2n+1$. According to (4.8), let us remark that, for $A_0 \in]0,1[$, $D_{2n+1}(\bar{A}_{2n+1})$ is polynomial in $A_1, \cdots, A_n$.

Let $j \in \{n+2, \cdots, 2n+1\}$ and let us suppose $i_j = 1$. Then, for $k \neq j$, $k \in \{n, \cdots, 2n+1\}$, $i_k = 0$. From (4.8), it results that $D_{2n+1}(\bar{A}_{2n+1})$ can be written in the form of

$$D_{2n+1}(\bar{A}_{2n+1}) = \sum_{j=n+2}^{2n+1} \rho_j A_j + g(A_0, \cdots, A_{n+1}), \tag{4.13}$$

where ρ_j depends only on $\bar{A}_{n-1}$ and the function g is defined as the function obtained replacing the condition $\ell + |i|_{2,n} = n$ in (4.8) with $\ell + |i|_{2,n} = 2n+1$.

Let us remark that, for $j \in \{n+2, \cdots, 2n+1\}$, the coefficient of A_j in $D_{2n+1}(\bar{A}_{2n+1})$ is equal to the coefficient of A_{j-1} in $D_{2n}(\bar{A}_{2n})$. Then, by the recursion argument, for $j \in \{n+2, \cdots, 2n+1\}$,

$$\rho_j = \frac{\partial D_{4n+2-2j}}{\partial A_{2n+1-j}} \cdot \tag{4.14}$$

Let us denote by ρ_{n+1} the coefficient of A_{n+1} in $D_{2n+1}(\bar{A}_{2n+1})$. Let $(\ell, i) \in \mathbb{N} \times \mathbb{N}^{2n}$ such that $\ell + |i|_{2,2n} = 2n$. If $i_n \geq 1$, $i_k = 0$ for $k > n$. From (4.8), the expression where A_n appears in $D_{2n}(\bar{A}_{2n})$ is equal to

$$\phi_0^{(2)}(A_0)\frac{A_n^2}{2} + \left(\sum_{\ell+|i|_{2,n-1}=n} \frac{1}{i_1! \cdots i_{n-1}!}\phi_\ell^{(|i|_{1,n-1}+1)}(A_0)\prod_{k=1}^{n-1} A_k^{i_k}\right) A_n. \tag{4.15}$$

According to (4.8), the coefficient of A_{n+1} in D_{2n+1} is equal to the derivate of the expression (4.15) with respect to A_n. We deduce the equality

$$\rho_{n+1} = \frac{\partial D_{2n}}{\partial A_n}(\bar{A}_{2n}). \tag{4.16}$$

From (4.13, 4.14) and (4.16) we get (4.9).

Moreover, we have the equality

$$D_{2n+2}(\bar{A}_{2n+2}) = \sum_{j=n+2}^{2n+2} \tilde{\rho}_j A_j + L(A_0, \cdots, A_{n+1}),$$

where L is obtained replacing the condition $\ell + |i|_{2,n} = n$ with $\ell + |i|_{2,n+1} = 2n + 2$ in (4.8). For $j \geq n + 2$, the coefficient of A_j in $D_{2n+2}(\bar{A}_{2n+2})$ is equal to the coefficient of A_{j-1} in $D_{2n+1}(\bar{A}_{2n+1})$.

Thus, we have

$$\tilde{\rho}_j = \frac{\partial D_{4n+4-2j}}{\partial A_{2n+2-j}}.$$

Let us observe that L is a polynomial of degree 2 in A_{n+1}, and the coefficient of A_{n+1}^2 is equal to $\dfrac{\phi_0^{(2)}(A_0)}{2}$. $\square$

Let us consider the system of equations for $\bar{A}_{2n} \in]0,1[\times\mathbb{R}^{2n}$

$$\frac{\partial D_{2k}}{\partial A_k}(\bar{A}_{2k}) = 0, \ \text{ for } \ k \in \{0, \cdots, n\} \cdot \tag{4.17}$$

We denote by S the set of the solutions of the system (4.17).

Proposition 4.7. *The set S is a manifold of dimension n.*

Let $\bar{A}_{2n} \in S$ and $\bar{A}_{2n+1} = (\bar{A}_{2n}, A_{2n+1})$, $(A_{2n+1} \in \mathbb{R})$. Then, $D_{2n+1}(\bar{A}_{2n+1})$ is independent of $(A_{n+1}, \cdots, A_{2n+1})$.

Let $\bar{A}_{2n} \in S$ and $\bar{A}_{2n+2} = (\bar{A}_{2n}, A_{2n+1}, A_{2n+2})$, $(A_{2n+1}, A_{2n+2}) \in \mathbb{R}^2$. Then, $D_{2n+2}(\bar{A}_{2n+2})$ is a polynomial of degree 2 in A_{n+1}, independent of $(A_{n+2}, \cdots, A_{2n+2})$, and the coefficient of A_{n+1}^2 is negative.

Proof. Let us suppose $n = 0$. By construction (see (4.7)), $D_0(A_0) := \phi_0(A_0)$. Then, from hypothesis (4.3), the function D_0 admits a unique maximum on $]0,1[$ achieved in $A_0 = t_0$.

For $n \geq 0$, let us suppose that the system (4.17) admits a solution, and let us consider this solution $\bar{A}_{2n}$. Let us consider the system (4.17) for $k \in \{0, \cdots, n+1\}$.

From (4.9) and by recursion, $D_{2n+1}(\bar{A}_{2n+1})$ is equal to $g(A_0, \cdots, A_n)$. Its does not depend on $(A_{n+1}, \cdots, A_{2n+1})$.

From Lemma 4.6 (see (4.10)), and the recursion argument, we observe that for all $(\bar{A}_{2n}, A_{2n+1}, A_{2n+2})$, $D_{2n+2}(\bar{A}_{2n+2})$ is equal to $L(A_0, \cdots, A_{n+1})$, where L is a polynomial of degree 2 in A_{n+1} whose the coefficient is negative. Then, for all $(\bar{A}_{2n}, A_{2n+1}, A_{2n+2}) \in S \times \mathbb{R}^2$, the equation $\frac{\partial D_{2n+2}}{\partial A_{n+1}}(\bar{A}_{2n+2}) = 0$ is equivalent to the equation $\frac{\partial L}{\partial A_{n+1}}(\bar{A}_{n+1}) = 0$ and admits a unique solution on $\mathbb{R}$, independent of $(A_{n+2}, \cdots, A_{2n+2})$.

The set S' of the solution of the system (4.17) for $k \in \{0, \cdots, n+1\}$ is a manifold of dimension $n+1$ and, for all $\bar{A}_{2n+2} = (\bar{A}_{2n}, A_{2n+1}, A_{2n+2})$, $\bar{A}_{2n} \in S$, $D_{2n+2}(\bar{A}_{2n+2})$ is a polynomial of degree 2 in A_{n+1}, independent of $(A_{n+2}, \cdots, A_{2n+2})$. □

From Proposition 4.7, we can maximize successively every coefficient of κ^n in the formal series ϕ. This procedure can be shown to be equivalent to the procedure exposed in Section 4.1.

Then, as suggested by Di Bartolo *et al.* in [14], we can obtain a formal expansion for the superheating field at all orders.

For all $i \in \mathbb{N}$, for $t \in]0,1[$, we set

$$\phi_i(t) := \hat{H}_i(0,t), \tag{4.18}$$

where $\hat{H}_i(0,t)$ is introduced in (3.21).

Proposition 4.8. *Let us consider $\check{H}_i(0, \bar{A}_i)$ introduced in* (3.19).
For all $i \in \mathbb{N}$, we have the equality

$$\check{H}_i(0, \bar{A}_i) = D_i(\bar{A}_i). \tag{4.19}$$

Proof. Let us remark from (3.15) and (3.16), that if we assume (3.20), for all $i \in \mathbb{N}$ and $t \in]0,1[$, $B_i \in \mathrm{C}^\infty(]0,1[)$. For $x \in \mathbb{R}^+$ and $t \in]0,1[$, from Proposition 2.6, the mappings defined in (3.21) are C^∞ on $\mathbb{R}^+ \times]0,1[$. We can make a Taylor expansion in series at the point $A_0 \in]0,1[$ of these mappings. Let us replace t by $\sum_0^\infty A_j \kappa^j$ and order the expression obtained in powers of κ and sum on n. We denote by $\tilde{F}^i$, $\tilde{Q}^i$ and $\tilde{H}^i$ the obtained formals series.

By constructions in the formal series $\tilde{F}^i$, $\tilde{Q}^i$ and $\tilde{H}^i$, we have the equalities

$$\tilde{F}^i(0,\kappa) \sim A(\kappa), \;\; \tilde{Q}^i(0,\kappa) \sim B(\kappa) \quad \text{and} \;\; \tilde{H}^i(0,\kappa) \sim D(\kappa).$$

Furthermore, the triplet $(\tilde{F}^i, \tilde{Q}^i, \tilde{H}^i)$ is a formal solution of the system of Ginzburg-Landau. By uniqueness of the inner solution with initial conditions $(A(\kappa), B(\kappa), D(\kappa))$, and using Proposition 2.6, we have the equalities

$$\tilde{F}^i \sim \check{F}^i, \;\; \tilde{Q}^i \sim \check{Q}^i \;\; \text{and} \;\; \tilde{H}^i \sim \check{H}^i,$$

where the formal series $\check{F}^i, \check{Q}^i, \check{H}^i$ are defined in (3.19). In particular, we get (4.19). □

To conclude, we can express the following theorem:

Theorem 4.9. *Let H^i be the formal series defined by*

$$H^i := \sum_{n=0}^{\infty} \check{H}_n(0, \bar{A}_n)\kappa^n, \tag{4.20}$$

where $\check{H}_n(0, \bar{A}_n)$ is introduced in (3.19).

The formal series H^i admits a unique "formal" maximum on $]0,1[$ *obtained maximizing successively every coefficient of κ^n in* (4.20).

Proof. The function $\check{H}_0(0; A_0) = 2^{\frac{1}{4}} A_0(1-A_0^2)^{\frac{1}{2}}$ introduced in (3.21) admits a unique maximum on $]0,1[$ at the point $A_0 = \frac{1}{\sqrt{2}}$. We set

$$h_0 := \check{H}_0\left(0; \frac{1}{\sqrt{2}}\right).$$

Moreover, from Propositions 4.7 and 4.8, the set S of the solutions of the system

$$\frac{\partial \check{H}_{2k}}{\partial A_k}(0; \bar{A}_{2k}) = 0, \quad \text{for} \ \ k \in \{0, \cdots, n\}, \tag{4.21}$$

is a manifold of $\mathbb{R}^n$.

For $\bar{A}_{2n} \in S$ and $\bar{A}_{2n+1} := (\bar{A}_{2n}, A_{2n+1})$, $(A_{2n+1} \in \mathbb{R})$, we set

$$h_{2n} := \check{H}_{2n}(0, \bar{A}_{2n}), \tag{4.22}$$

$$h_{2n+1} = \check{H}_{2n+1}(0; \bar{A}_{2n+1}). \tag{4.23}$$

□

Then, we can introduce the following definition:

Definition 4.10. We call "formal" superheating field, the formal maximum of the series defined in (3.22) and we denote it by $h^{sh,f}(\kappa)$. More precisely,

$$h^{sh,f}(\kappa) := \kappa^{-\frac{1}{2}} \sum_{i=0}^{+\infty} h_i \kappa^i,$$

where the reals h_i are defined in (4.22) and (4.23).

Remark 4.11. Practically, we have used the software Maple® to compute the coefficients in the formal expansion for the superheating field (see [9]). We proceed following the method which leads us to prove Theorem 4.9. We have recovered the results obtained by Di Bartolo *et al.* in [14] and shown that the numerical computation procedure is efficient (we mathematically prove that we never divide by 0 in the procedure).

5. On a conjecture due to H.J. Fink, D.S. McLachlan and B. Rothberg-Bibby

In [19], Fink *et al.* have conjectured that when h is equal to the superheating field, the solution (f_κ, A_κ) of $(GL)_\infty$ satisfies

$$-\frac{A_\kappa(0)}{A'_\kappa(0)} = \sqrt{2}.$$

When κ is small, let us look if the conjecture is true in a formal point of view. We have determined in the previous sections a formal expansion of $A'_\kappa(0)$ and $A_\kappa(0)$ in powers of $\kappa^{\frac{1}{2}}$. When h is equal to the superheating field, we have the equalities (see [14])

$$Q_0(0) = -2^{-\frac{1}{4}} \ \text{ and } \ Q'_0(0) = 2^{-\frac{3}{4}}. \tag{5.1}$$

Then, at the first order, we have the equality

$$-\frac{Q_0(0)}{Q'_0(0)} = \sqrt{2}.$$

But, at the second order, we have (see [14])

$$Q_1(0) = -\frac{9}{16} 2^{\frac{1}{4}} \ \text{ and } \ Q'_1(0) = \frac{15}{64} 2^{\frac{3}{4}}.$$

Then, we have the equality

$$-\frac{Q_1(0)}{Q_1'(0)} = \frac{6}{5}\sqrt{2}.$$

The conjecture is false at the second order and we get

$$-\frac{A_\kappa(0)}{A_\kappa'(0)} = \sqrt{2} + \frac{3}{16}\kappa + \mathcal{O}\left(\kappa^2\right).$$

When κ is large, from a formal construction due to Chapman [8] (see also [16]), which is analogous to the Di Bartolo *et al.* construction [14], the solution satisfies, at the "formal" superheating field

$$A_\kappa(0) = -1 + \frac{D}{\sqrt{2}}\kappa^{-\frac{2}{3}} + \mathcal{O}\left(\kappa^{-\frac{4}{3}}\right),$$
$$A_\kappa'(0) = \frac{1}{\sqrt{2}} + \mathcal{O}\left(\kappa^{-\frac{4}{3}}\right).$$

The constant D can be estimated as approximately $-0,3$. Then, when κ is large, we have

$$-\frac{A_\kappa(0)}{A_\kappa'(0)} = \sqrt{2} - D\kappa^{-\frac{2}{3}} + \mathcal{O}\left(\kappa^{-\frac{4}{3}}\right).$$

The conjecture is false in this case. It is not a mathematical proof because the expansions are formal, but this strongly suggests that the conjecture is false.

6. Open problems

Theorem 4.9 leads one to express the following conjecture:

Conjecture 6.1. Let $h^{sh}(\kappa)$ be the superheating field, introduced in Definition 1.1.

For all $n \in \mathbb{N}$, there exists κ_0 such that, for all $\kappa \leq \kappa_0$, we have the asymptotic expansion

$$\kappa^{\frac{1}{2}} h^{sh}(\kappa) := \sum_{i=0}^{n} h_i \kappa^i + o(\kappa^n), \tag{6.1}$$

where the reals h_i are defined in (4.22) and (4.23).

The expansions obtained in Section 2 lead us to conjecture the following result:

Conjecture 6.2. At the superheating field, there exists κ_0 such that, for all $\kappa \leq \kappa_0$, we have the asymptotic expansion

$$-\frac{A_\kappa(0)}{A_\kappa'(0)} = \sqrt{2} + \frac{3}{16}\kappa + \mathcal{O}\left(\kappa^2\right). \tag{6.2}$$

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