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ERROR ESTIMATES FOR MODIFIED LOCAL SHEPARD'S FORMULAS IN SOBOLEV SPACES

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Abstract. Interest in meshfree methods in solving boundary-value problems has grown rapidly in recent years. A meshless method that has attracted considerable interest in the community of computational mechanics is built around the idea of modified local Shepard's partition of unity. For these kinds of applications it is fundamental to analyze the order of the approximation in the context of Sobolev spaces. In this paper, we study two different techniques for building modified local Shepard's formulas, and we provide a theoretical analysis for error estimates of the approximation in Sobolev norms. We derive Jackson-type inequalities for *h*-*p* cloud functions using the first construction. These estimates are important in the analysis of Galerkin approximations based on local Shepard's formulas or *h*-*p* cloud functions.

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Introduction

The idea of meshless methods for numerical analysis of partial differential equations (PDEs) has become quite popular over the last decade. In most computer-aided design work, the generation of an appropriate mesh constitutes the costliest portion of the process. For this reason, the development of techniques which do not rely on traditional mesh concepts is still very appealing.

In meshless method, *h*-*p* (spectral) types of approximations are built around a collection of nodes sprinkled within the domain on which a boundary-value problem has been posed. Associated with each node, there is an open set (cloud) that forms the support for the approximation basis functions built around the node. The boundary-value problem is then solved using these *h*-*p* cloud functions and a Galerkin method. For this kind of application, it is fundamental to analyze the order of the approximation in the context of Sobolev spaces.

A meshless method that has attracted significant interest in the community of computational mechanics is built around the idea of local Shepard's partition of unity (see [5]). In [13], Shepard introduced an interpolation scheme which is easily programmable. Given any arbitrarily spaced points $x_1, x_2, ..., x_N \in \mathbb{R}^n$ and values

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$u(x_1), u(x_2), ..., u(x_N)$ of a function u, the first version of Shepard's formula is given by

$$S_k^0 u(x) = \sum_{i=1}^{N} u(x_i) \cdot \mathcal{W}_i(x),$$

with basis functions

$$\mathcal{W}_i(x) = \frac{||x - x_i||^{-k}}{\sum_j ||x - x_j||^{-k}}.$$

Here $k > 1$, and $||.||$ is the Euclidean norm.

In practical applications, the global character of Shepard's interpolation formula is totally undesirable. This disadvantage is avoided by using a local version of Shepard's formula, where the basis functions $\mathcal{W}_i$ have small compact supports which may even depend on the local distribution of data points [12]. Another drawback is that the interpolating function $S_k^0 u$ has flat spots in the neighborhood of all data points. This drawback can be avoided by using Taylor polynomials of degree m at the data points. The generalized Shepard's formula is

$$ST_k^m u(x) = \sum_{i=1}^{N} T_i^m(x) \cdot \mathcal{W}_i(x), \tag{0.1}$$

where Taylor polynomials $T_i^m(x) = \sum_{0 \le |\alpha| \le m} a_\alpha (x - x_i)^\alpha$ are selected in some way. On the other hand, the space X^m, consisting of all functions which have the form (0.1), is considered in *h-p* cloud methods.

In this paper, we mainly consider two different ways of building approximations (0.1) in Sobolev spaces. In the first one, polynomials T_i^m are Verfürth's averaged polynomials [14]. Interpolation operators built in this way are of theoretical interest: error estimates obtained in this case are used to derive Jackson-type inequalities for *h-p* cloud functions. The second one deals with a widely used interpolation method: polynomials T_i^m are built by a least square fit of a function

$$T_i^m[u](x) = u_i + \sum_{1 \le |\alpha| \le m} a_\alpha (x - x_i)^\alpha, \tag{0.2}$$

to function values on a set of nearby nodes of node x_i. The set of nearby nodes of node x_i where the weighed least square approximation (0.2) is made, is called the *star* of x_i. In [15], we have defined a *condition number* of the *star,* which is practically computable. The *condition number* is a measure for the quality of the *star* and it is strongly related to the approximating power of the modified Shepard's interpolation formula in the uniform norm [15]. We investigate here the approximation power of this construction in Sobolev spaces. It is well known that error estimates and the Céa's lemma [2, 3] give *a priori* error estimates for the approximate solutions of boundary-value problems. A more detailed analysis of this and related questions will be developed in a forthcoming paper. Moreover, we do not discuss here numerical tests, but it is worthwhile mentioning that there is an extensive literature in the computational mechanic community confirming the theoretical error estimates obtained in [5, 6, 9, 10].

The paper is organized as follows. In Section 1, we present local Shepard's partition of unity and local Shepard's formulas, we state some conditions for the partition of unity which will be considered in this work. Section 2 deals with polynomial approximation of functions in Sobolev spaces in star-shaped domains. In Section 3, we analyze error estimates for the local Shepard's formula modified with Verfürth's averaged polynomials. An application to *h-p* cloud functions is discussed therein and the results of Duarte–Oden are improved. Section 4 is dedicated to the local Shepard's formula modified with least square fits of Taylor's polynomials.

1. Local Shepard's formulas

Let Ω be an open bounded domain in $\mathbb{R}^n$ and Q^N denote an arbitrarily chosen set of N points $x_i \in \overline{\Omega}$ referred to as *nodes*:

$$Q^N = \{x_1, x_2, ..., x_N\}, \quad x_i \in \overline{\Omega}.$$

Let $\mathcal{I}_N := \{\omega_i\}_{i=1}^N$ denote a finite open covering of $\overline{\Omega}$ consisting of N *clouds* with center at x_i, $i = 1, ..., N$, and let $\mathcal{S}_N := \{\phi_i\}_{i=1}^N$ be a class of functions having the following properties:

$\phi_i \in C_0^s(\mathbb{R}^n), \quad s \geq 0$ or $s = +\infty$;
spt $(\phi_i) = \overline{\omega}_i$, where we have denoted;
spt (ϕ_i) the support of ϕ_i:
$\phi_i(x) > 0, \quad x \in \omega_i$.

In particular, for every $x \in \overline{\Omega}$, there is at least one ϕ_j so that $\phi_j(x) > 0$.

For a fixed positive integer k and every $i = 1, ..., N$, we define functions ν_i, $\mathcal{W}_i$ by

$$\nu_i^k(x) = ||x - x_i||^{-k} \cdot \phi_i(x), \text{ for } x \neq x_i$$

and

$$\mathcal{W}_i^k(x) = \frac{\nu_i^k(x)}{\sum\limits_{j=1}^{N} \nu_j^k(x)}, \text{ if } x \notin Q^N.$$

The sets ω_i, $i = 1, ..., N$, are called *clouds* in meshless methods community [5].

The *diameter* of ω_i, $d_i := \sup_{x,y \in \omega_i}\{||x - y||\}$, and $h := \max_{i=1,...,N}\{d_i\}$ will be key ingredients in error estimates.

The class of functions $\mathcal{W}_N^k := \left\{\mathcal{W}_i^k\right\}_{i=1}^N$ is called a Shepard's *partition of unity* [5, 6], subordinated to the open covering $\mathcal{I}_N$ and it has the following well known properties:

Every $\mathcal{W}_i^k$ can be defined by continuity at nodes x_j in such a way that the Kronecker-delta property is verified; that is, $\mathcal{W}_i^k(x_j) = \delta_{ij}$. From this definition and the above assumptions, it follows that

- $\mathcal{W}_i^k \in C_0^{k-1}(\mathbb{R}^n)$ in general and $\mathcal{W}_i^k \in C_0^{\infty}(\mathbb{R}^n)$ if k is an even number and $s = \infty$.
- $\mathcal{W}_i^k$ is $(k-1)$-flat at nodes x_j. In particular, $D^\nu \mathcal{W}_i^k(x_j) = 0$, for every multi-index ν, $1 \leq |\nu| < k$.
- $\mathcal{S}_N$ is a *partition of unity* on $\overline{\Omega}$: $\sum_{i=1}^N \mathcal{W}_i^k(x) = 1$, for every $x \in \overline{\Omega}$.

Assumption. In what follows, we shall not deal with question related to the differentiability of functions $\mathcal{W}_i^k$. For the sake of simplicity, from now on, we make the assumption that k is an even number, $\mathcal{W}_i^k \in C_0^{\infty}(\mathbb{R}^n)$, $i = 1, ..., N$, and we will omit any reference to the number k in our notation.

Thus, $\mathcal{W}_i = \mathcal{W}_i^k$, and so on. This assumption requires the constant s above be equal to ∞.

Let m be any integer ≥ 0. For $i = 1, ..., N$, let $\mathcal{P}_i^m$ denotes the vector space of m-Taylor's polynomials at x_i

$$\mathcal{P}_i^m := \left\{ Q : Q(x) = \sum_{0 \leq |\nu| \leq m} a_\nu (x - x_i)^\nu \right\}.$$

Let $\mathcal{F}$ be some space of functions. Given a linear operator

$$\mathcal{T}^m : \mathcal{F} \to \prod_{i=1}^{N} \mathcal{P}_i^m, \tag{1.1}$$

the associated *m-modified local Shepard's interpolation operator* is the linear operator $\mathcal{ST}^m : \mathcal{F} \to C^\infty(\overline{\Omega})$ defined by

$$\mathcal{ST}^m(u) := \sum_{i=1}^{N} T_i^m(u) \cdot \mathcal{W}_i, \quad u \in \mathcal{F} \text{ and } \mathcal{T}^m(u) = (\mathcal{T}_i^m(u))_{i=1,\dots,N} \,.$$

In this work, we are mainly interested in the case where $(\mathcal{F}, ||\cdot||_{\mathcal{F}})$ is some normed space of functions over Ω and in estimating the interpolation error:

$$||u - \mathcal{ST}^m(u)||_{\mathcal{F}} \,.$$

Sections 3 and 4 are dedicated to our main examples of *m-modified local Shepard's interpolation operators*. The first one deals with the use of local averaged Taylor's polynomials of Sobolev functions. The second one deals with continuous functions and is related to the methodology described in the introduction, and which was analyzed in [15] in the context of C^m-spaces. For this purpose, we need to state some regularity properties of the class $\{\mathcal{W}_i\}$ which will be useful later.

In establishing error estimates the following constants are cornerstones:

A1: Constants $G_{m,i} > 0$ satisfying

$$||D^\beta \mathcal{W}_j||_{L^\infty(\mathbb{R}^n)} \leq \frac{G_{m,i}}{d_j^{|\beta|}}, \qquad |\beta| \leq m, \, \forall j : \; j \in \widehat{i} \,.$$

In deriving uniform error estimates, conditions **A2** is changed to:

A1U: A constant G_m satisfying

$$||D^\beta \mathcal{W}_i||_{L^\infty(\mathbb{R}^n)} \leq \frac{G_m}{h^{|\beta|}}, \qquad |\beta| \leq m, \, i = 1, ..., N \quad .$$

In the next section we will discuss other conditions related to the geometry of Ω and the open covering $\mathcal{I}_N := \{\omega_i\}_{i=1}^N$ which will play an important role in error estimates.

2. Polynomial approximation in Sobolev spaces

Given $u \in \mathcal{D}'(\Omega)$ and $\alpha \in \mathbb{N}_0^n$ we denote, as usual,

$$D^\alpha u = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \dots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} u, \quad |\alpha| = \alpha_1 + \dots + \alpha_n, \quad \alpha! = \alpha_1! \cdots \alpha_n!.$$

For $p \geq 1$ and $m \in \mathbb{N}_0$, we call $W_p^m(\Omega)$ the Sobolev space which consists of all the functions $u \in L^p(\Omega)$ such that $D^\alpha u \in L^p(\Omega)$ for $|\alpha| \leq m$. Given $j \in \mathbb{N}_0$, we define

$$|\, u \,|_{j,p} = \left(\sum_{|\alpha|=j} ||D^\alpha u \,||^p_{L^p(\Omega)} \right)^{1/p}$$

and, therefore, the usual norm in $W_p^m(\Omega)$ is defined by

$$||\, u \,||_{m,p} = \left(\sum_{j=0}^{m} |\, u \,|^p_{j,p} \right)^{1/p} .$$

If $p = 2$, we denote, as usual, $W_2^m(\Omega) = H^m(\Omega)$. Moreover, when an explicit reference to the domain is needed, we denote instead $||\, u \,||_{m,p} = ||\, u \,||_{\Omega,m,p}$ and $|\, u \,|_{j,p} = |\, u \,|_{\Omega,\, j,p}$.

Let $U \subset \mathbb{R}^n$ be an open set with diameter d. In this section, we are interested in sharp upper bounds of the constant $c_{m,j}$ in the Jackson-type inequalities

$$\sup_{u \in W_p^{m+1}(U)} \inf_{p \in \mathcal{P}^m} \frac{|u-p|_{j,p}}{|u|_{m+1,p}} \leq c_{m,j} d^{m+1-j} \quad \forall \quad 0 \leq j \leq m, \tag{2.1}$$

when U is star-shaped w.r.t. a point in U. Here, $\mathcal{P}^m$ is the space of all polynomials in n variables of degree at most m. The best estimates of $c_{m,j}$, which are known to us, are due to Verfürth [14] and Durán [7], which is a modification of Dupont–Scott's approach.

Verfürth's bounds do not depend on eccentricity in case U is a convex set. Moreover, for non-convex domains with a re-entrant corner, the bounds are uniform w.r.t. the exterior angle. We will need later, however, a bound in the L^∞-norm which we could not obtain with Verfürth's approach. For this reason, we shall expose here both approach.

Verfürth's projection operator

Let $B \subset U$ be a set of positive measure $|B|$. For any integer m and $p \geq 1$, a projection operator Q_B^m of $W_p^m(U)$ onto $\mathcal{P}^m$ can be built, which has the following properties:

$$D^\beta \left(Q_B^m u\right) = Q_B^{m-j}\left(D^\beta u\right), \tag{2.2}$$

$$\int_B D^\beta \left(u - Q_B^m u\right)(y)\,\mathrm{d}y = 0 \tag{2.3}$$

for all $u \in W_p^m(\Omega)$, all $0 \leq j \leq m$, and all $\beta \in \mathbb{N}^n$ with $|\beta| = j$.

We denote by

$$\pi_B(f) := \frac{1}{|B|}\int_B f(y)\,\mathrm{d}y$$

the mean value of f w.r.t. B. For any $u \in W_p^m(\Omega)$, we recursively define polynomials $q_B^m, ..., q_B^0$ in $\mathcal{P}^m$ by

$$q_B^m(u) := \sum_{|\alpha|=m} \frac{1}{\alpha!} x^\alpha \pi_B(D^\alpha u)$$

and for $k = m, m-1, ..., 1$

$$q_B^{k-1}(u) := q_B^k(u) + \sum_{|\alpha|=k-1} \frac{1}{\alpha!} x^\alpha \pi_B(D^\alpha u - q_B^k(u)).$$

Finally, we set

$$Q_B^m\, u := q_B^0(u).$$

Using (2.2) and (2.3), it is easily proved that Q_B^m is really a projection operator.

Proposition 2.1. *Let $B \subset U$ be a set of positive measure $|B|$. For any integer m and $P \in \mathcal{P}^m$, $Q_B^m P = P$.*

Definition 2.2. Given $u \in W_p^m(\Omega)$ and $P_{m,B}\, u$, the *remainder term* is $R^m u := u - P_{m,B}\, u$.

Definition 2.3. A set U is star-shaped w.r.t. a set B if, for all $x \in U$, the closed convex hull of $\{x\} \cup B$ is a subset of U.

The star-shaped condition is a key ingredient in polynomial approximation in Sobolev spaces. In order to state Verfürth's results for non-convex but star-shaped domains we need to state some more definitions. For $z \in U$ we define

$$\chi(z) := \max_{y \in \partial U} ||y - z|| \,/\, \min_{y \in \partial U} ||y - z||.$$

Now, assume that U is non-convex but star-shaped w.r.t. one point, and

$$\mathcal{S}_U := \{\, z \in U : U \text{ is star-shaped w.r.t. } z\}\cdot$$

It is clear that there exists a point $z_m \in U$ such that $\chi(z_m) = \min_{z\in\mathcal{S}}\{\chi(z)\}$. Then, the number θ is defined by

$$\theta := \chi(z_m).$$

The main Verfürth's result in [14] is:

Theorem 2.4. *Let U be a domain with diameter d_U and which is star-shaped w.r.t. one point. For $1 \le p \le \infty$ and $m \in \mathbb{N}_0$, there exist constants $c_{m,j}$, $0 \le j \le m$, such that*

$$||\, u - Q^m_{B_U}\, u\,||_{j,p} \le c_{m,j}\; d_U^{m+1-j}\,|\, u\,|_{m+1,p}\,, \quad \forall u \in W_p^{m+1}(U).$$

When U is a convex domain, $B_U = U$ and $c_{m,j} = c_{m,j}(n,m)$, i.e., the bounds $c_{m,j}$ depend only on n and m. In the non-convex case, $B_U = B(z_m,\varrho)$, $\varrho =$dist$(z_m,\partial U)$, and $c_{m,j} = c_{m,j}(n,m,\theta)$.

Dupont–Scott's representation formula

Let $B = B(x_0,\varrho)$. A function $\sigma \in C_c^\infty(\mathbb{R}^n)$ with the properties (i) *spt* $\sigma = \overline{B}$ and $\int_{\mathbb{R}^n}\sigma(x)\mathrm{d}x = 1$ will be called a *cut-off function*.

Given a function u of class C^q, its Taylor polynomial of order q at y will be denoted by $T^q_y u$.

Definition 2.5. Suppose u has weak derivatives of order q in a domain U and $B \subset\subset U$, the (σ,q)-Taylor polynomial of u averaged over B is defined as

$$Q^q u(x) := \int_B T^q_y u(x)\,\sigma(y)\,\mathrm{d}y.$$

As before, the m^{th}-order remainder term is given by

$$R^q u := u - Q^q u.$$

Let U be star-shaped w.r.t. B and $u \in C^{q+1}(U)$. Taylor's theorem gives us an error representation of $u(x) - Q^q u(x)$ which is very useful in order to obtain error estimates [2]. A key parameter in Dupont–Scott's estimates is the chunkiness of U.

Definition 2.6. Suppose U has diameter d_U and is star-shaped w.r.t. an open ball B. Let

$$\varrho_{\max} := \sup\{\varrho : U \text{ is star-shaped w.r.t. a ball }\}\cdot$$

Then, the *chunkiness parameter* of U is defined by

$$\gamma_U := \frac{d_U}{\varrho_{\max}}\cdot$$

Dupont–Scott's error estimates are proportional to γ_U^n. Their main results (see [2]) are:

Theorem 2.7. *Suppose U has diameter d_U and is star-shaped w.r.t. an open ball B. There exists a constant $C = C(n,q,\gamma_U)$ such that*

$$||\, R^q u\,||_{U,j,p} \le C\, d_U^{q+1-j}|\, u\,|_{U,q+1,p}$$

for every $u \in W_p^{q+1}(U)$ and $0 \le j \le q$.

Theorem 2.8. *Let U be star-shaped w.r.t an open ball B. If $0 < p < \infty$ and $(q+1) > n/p$ or $p = 1$ and $(q+1) \geq n$, then there exists a constant $C = C(n, q, \gamma_U)$ such that*

$$|| R^q u ||_{L^\infty(U)} \leq C\, d_U^{q+1-n/p} |\, u\, |_{U,q+1,p}$$

for every $u \in W_p^{q+1}(U)$.

The star-shaped property is, as we have seem, a key tool in approximating by polynomials. We will need some conditions in the domain and *clouds* in order to guaranty the appropriate use of this property.

A2: Ω has a Lipschitz continuous boundary $\partial\Omega$ [1].

If Ω has satisfies **A2**, it can be proved that there exists a number $\epsilon_\Omega > 0$, such that the intersection

$$\Omega \cap B(x, r), \quad x \in \overline{\Omega},$$

is star-shaped w.r.t an open ball, provides $r < \epsilon_\Omega$. If Ω is a convex set, $\epsilon_\Omega = \infty$.

3. Taylor averaged local Shepard's formulas

Throughout this section we assume the following:

Condition 3.1. Given the *partition of unity* $\mathcal{S}_N := \{\mathcal{W}_i\}_{i=1}^N$ over $\overline{\Omega}$, the *clouds* $\{\omega_i\}$ have been defined before with the condition $\overline{\omega}_i$ =*spt* φ_i. In what follows, it is better to have the condition that every ω_i is an open ball. So, if this is not the case, the symbol ω_i is reassigned in the following way:

$$\omega_i := B(x_i, r_{\min}),$$

where

$$r_{\min} := \min\{r : \omega_i \subset B(x_i, r)\}\cdot$$

Remark 3.2. The equality above must be understood in the context of computer science, not in a mathematical sense.

For future use, we set $\widetilde{\omega}_i := \omega_i \cap \Omega$, and $\widehat{i} := \{j : \omega_i \cap \omega_j \neq \emptyset\}$. The following condition is crucial in this section.

A3: **A2** is satisfies and $h := \max_{i=1,\ldots,N}\{d_i\} \leq \epsilon_\Omega$.

In particular, if **A3** is satisfies, every set $\widetilde{\omega}_i$ is star-shaped w.r.t. an open ball.

Therefore, for each i, $i = 1, \ldots N$, we can choose a subset $B_i \subset \widetilde{\omega}_i$ where Verfürth's projection operator applies (see Th. 2.4). We assume that $B_i = \widetilde{\omega}_i$ in case $\widetilde{\omega}_i$ is a convex set.

Given an integer $m \geq 0$ and $p \geq 1$, an *m-modified local Shepard's interpolation operator* is a linear operator $\mathcal{ST}^m : W_p^{m+1}(\Omega) \to C^\infty(\overline{\Omega})$, called from now on, a $\mathcal{TA}(m)$ *local Shepard's interpolation operator.* We will define now our first $\mathcal{TA}(m)$ *local Shepard's interpolation operator* using Verfürth's projection operators.

Let $u \in W_p^{m+1}(\Omega)$. For $i = 1, \ldots, N$, we set $Q_i^m u = Q_{B_i}^m u$. Then,

$$\mathcal{ST}^m(u) = \sum_{i=1}^N Q_i^m u\, \mathcal{W}_i\,. \tag{3.1}$$

Remark 3.3. Note that, even when $u \in W_p^{m+1}(\Omega) \cap C(\overline{\Omega})$, $\mathcal{ST}^m(u)(x_i) \neq u(x_i)$, so $\mathcal{ST}^m(u)$ is an interpolant of u in a generalized sense.

Remark 3.4. In this section we will never use the *delta-krocneker* property of the Shepard's partition of unity $\{\mathcal{W}_i\}$. As a matter of fact, all the results here are also valid assuming that $\{\mathcal{W}_i\}$ is a partition of unity.

A natural and crucial parameter in error estimates is a measure of the overlap of *clouds*:

A4: A measure of the overlap of *clouds:*

$$M = \sup_{i=1,\dots,N} \{\#\widehat{i}\},$$

where $\#S$ denote the number of elements in a finite set S.

Remark 3.5. Other authors [5,8], use the pointwise condition **A4P**:

$$M = \sup_{x\in\overline{\Omega}}\{\#(j : x \in \omega_j)\}\cdot$$

In fact, **A4** and **A4P** are different requirements. We could not obtain these results here with **A4P**.

For every i, $i = 1,...N$, we have a set of bounds $c_{m,j}(i)$ given by theorem 2.4. We define:

$$\widetilde{C}_{m,i} := \max_{0\le j\le m}\{c_{m,j}(i)\}, \tag{3.2}$$

and

$$\widetilde{C}_m := \max_{i=1,\dots N}\{\widetilde{C}_{m,i}\}\cdot \tag{3.3}$$

We are interested in estimating the error $u - \mathcal{ST}^m(u)$ in Sobolev norms. The following result will be useful in passing from local to global estimates.

Lemma 3.6. *Let $f, g \in L^1(\Omega)$ be two positive functions. Suppose that, for every $i = 1, ..., N$, $n(i)$ is a subset of indexes such that:*

$$\#n(i) \le M. \tag{3.4}$$

If

$$\int\limits_{\omega_i} f(x)\mathrm{d}x \le \sum_{j\in n(i)} \int\limits_{\omega_j} g(x)\mathrm{d}x\,, \quad \forall i : i, ..., N,$$

then

$$\int\limits_{\Omega} f(x)\mathrm{d}x \le M^2 \int\limits_{\Omega} g(x)\mathrm{d}x.$$

Proof. First, we have $\int\limits_{\Omega} f = \int\limits_{\Omega} f \ \left(\sum_{i=1}^N \mathcal{W}_i\right) \le \sum_{i=1}^N \int\limits_{\omega_i} f\ \mathcal{W}_i \le \sum_{i=1}^N \int\limits_{\omega_i} f$. By the hypotheses, we have

$$\sum_{i=1}^N \int\limits_{\omega_i} f \ \le \sum_{i=1}^N \sum_{j\in n(i)} \int\limits_{\omega_j} g\,.$$

Condition (3.4) implies that each $\int\limits_{\omega_j} g$ appears no more than M times in $\sum_{i=1}^N \sum\limits_{j\in n(i)} \int\limits_{\omega_j} g$. So,

$$\sum_{i=1}^N \sum_{j\in n(i)} \int\limits_{\omega_j} g \le M \sum_{j=1}^N \int\limits_{\omega_j} g\,.$$

Now,

$$\sum_{j=1}^{N}\int_{\omega_j} g = \sum_{j=1}^{N}\int_{\omega_j} g\left(\sum_{k_j\in\widehat{j}}\mathcal{W}_{k_j}\right)$$
$$\leq \sum_{j=1}^{N}\sum_{k_j\in n(j)}\int_{\omega_{k_j}} g\mathcal{W}_{k_j}\,.$$

Applying another time (3.4), it follows that each $\int_{\omega_k} g\mathcal{W}_k$ appears no more than M times in $\sum_{j=1}^{N}\sum_{k_j\in n(j)}\int_{\omega_{k_j}} g\mathcal{W}_{k_j}$. Therefore,

$$\sum_{j=1}^{N}\sum_{k_j\in n(j)}\int_{\omega_{k_j}} g\mathcal{W}_{k_j} \leq M\sum_{k=1}^{N}\int_{\omega_k} g\mathcal{W}_k$$
$$= M\int_{\Omega} g$$

and the lemma is proved. □

Our first global error estimate is:

Theorem 3.7. *Assume* ***A1U**, **A3**, **A4*** *and let* $p\geq 1$, $l\leq m$. *If* $u\in W_p^{m+1}(\Omega)$, *then*

$$|u-\mathcal{ST}^m(u)|_{l,p}\leq C_{m,l}\,h^{m+1-l}|\,u\,|_{m+1,p}\,,$$

where

$$C_{m,l}=(\#\{\alpha:|\alpha|=l\})^{1/p}\,M^{2/p}C(n,m)\,G_m\,\widetilde{C}_m\,.$$

The value of constant $C(n,m)$ *will be clarified along the proof of the theorem.*

Proof. Given $\alpha\in\mathbb{N}_0^n$, $|\alpha|=l$, we will estimate $||D^\alpha(u-\mathcal{ST}^m(u))||_{L^p(\Omega)}$. We have

$$\int_{\Omega}|D^\alpha(u-\mathcal{ST}^m(u))(x)|^p\mathrm{d}x\leq\sum_{i=1}^{N}\int_{\widetilde{\omega}_i}|D^\alpha(u-\mathcal{ST}^m(u))(x)|^p\mathcal{W}_i(x)\mathrm{d}x$$
$$\leq\sum_{i=1}^{N}\int_{\widetilde{\omega}_i}|D^\alpha(u-\mathcal{ST}^m(u))(x)|^p\mathrm{d}x\,.$$

For $x\in\widetilde{\omega}_i$, we can write

$$u(x)-\mathcal{ST}^m(u)(x)=\sum_{j\in\widehat{i}}\left(u(x)-Q_j^m(u)(x)\right)\mathcal{W}_j(x)$$
$$=\sum_{j\in\widehat{i}}R_j^m(u)(x)\mathcal{W}_j(x)\,.$$

Therefore,

$$\int_{\widetilde{\omega}_i}|D^\alpha(u-\mathcal{ST}^m(u))(x)|^p\mathrm{d}x\leq\sum_{j\in\widehat{i}}\int_{\widetilde{\omega}_j}|D^\alpha(R_j^m(u)\mathcal{W}_j)(x)|^p\mathrm{d}x\,.$$

We will now turn to $\int_{\widetilde{\omega}_j} |D^\alpha(R_j^m(u)\mathcal{W}_j)(x)|^p \mathrm{d}x$, for a fixed $j \in \widehat{i}$. Given $\gamma \in \mathbb{N}_0^n$, $\gamma \leq \alpha$, by (2.4) we have

$$||D^\gamma(R_j^m(u))||^p_{L^p(\widetilde{\omega}_j)} \leq \widetilde{C}_m^p \, h^{(m+1-|\gamma|)p} |\, u \,|^p_{\widetilde{\omega}_j, m+1, p} \, .$$

Taking into account that

$$||D^\delta \mathcal{W}_j||_{L^\infty} \leq \frac{G_m}{h^{|\delta|}} \, , \quad |\delta| \leq m$$

and Leibniz's rule, it follows that

$$D^\alpha(R_j^m(u)\mathcal{W}_j) = \sum_{\gamma+\delta=\alpha} C(\alpha,\gamma,\delta) D^\gamma R_j^m(u) D^\delta \mathcal{W}_j,$$

where

$$C(\alpha,\gamma,\delta) = \prod_{i=1}^n \binom{\alpha_i}{\delta_i} .$$

By choosing a constant $C(n,m)$ satisfying $\sum_{\gamma+\delta=\alpha} C(\alpha,\gamma,\delta)^p \leq C(n,m)^p$ for all $|\alpha| \leq m$, we get

$$||\, D^\alpha(R_j^m(u)\mathcal{W}_j) \,||^p_{L^p(\widetilde{\omega}_j)} \leq \Big(C(n,m) G_m \, \widetilde{C}_m\Big)^p \; h^{(m+1-l)p} |\, u \,|^p_{\widetilde{\omega}_j, m+1, p} \, .$$

Now, we can use Lemma 3.6 with

$$f = |D^\alpha(R_j^m(u)\mathcal{W}_j)|^p$$

and

$$g = \Big(C(n,m) C_{D,m} \, \widetilde{C}_m\Big)^p \; h^{(m+1-l)p} \left(\sum_{|\beta|=m+1} |D^\beta u|^p \right)$$

in order to get

$$||\, D^\alpha(R_j^m(u)\mathcal{W}_j) \,||^p_{L^p(\Omega)} \leq M^2 \, (C(n,m) G_m \, \widetilde{C}_m)^p \, h^{(m+1-l)p} \, |\, u \,|^p_{m+1,p}.$$

Hence

$$|\, D^\alpha(R_j^m(u)\mathcal{W}_j) \,|_{l,p} \leq (\#\{\alpha : |\alpha| = l\})^{1/p} \, M^{2/p} \, (C(n,m) G_m \, \widetilde{C}_m) \, h^{m+1-l} \, |\, u \,|_{m+1,p}$$

and the theorem is proved. □

The next local error estimate follows easily from the method of proof of the previous theorem.

Theorem 3.8. *Assume* ***A1****,* ***A3*** *and* ***A4*** *and let* $p \geq 1$, $l \leq m$. *If* $u \in W_p^{m+1}(\Omega)$, *then*

$$|\, u - \mathcal{ST}^m(u) \,|_{\widetilde{\omega}_i, l, p} \leq C_{m,l,i} \, h_i^{m+1-l} |\, u \,|_{\widehat{\omega}_i, m+1, p} \, ,$$

where

$$C_{i,m,l} = M^{1/p} (\, C(n,m) \, G_{m,i} \, \widehat{C}_{m,i}) \, (\#\{\alpha : |\alpha| = l)^{1/p} \, ,$$

$$h_i = \max_{j \in \widehat{i}} \{d_j\}, \text{ and } \widehat{C}_{m,i} = \max_{j \in \widehat{i}} \{\widetilde{C}_{m,j}\}.$$

3.1. **Application to *h-p* cloud functions**

In the *h-p* cloud method [5], the vectorial space $\mathcal{F}^m$ defined by

$$\mathcal{F}^m = \left\{ v : v = \sum_{i=1}^{N} P_i^m \mathcal{W}_i,\ P_i^m \in \mathcal{P}_i^m,\ i = 1, ..., N \right\},$$

is utilized to solve elliptic PDEs in a Galerkin scheme over a Sobolev space $H^m(\Omega)$.

In the notation of Duarte–Oden, $\mathcal{F}^m$ corresponds to $\mathcal{F}_N^{k=0,m}$. Numerical experiments performed in [4,5] have shown that the family of functions $\mathcal{F}_N^{k=0,m}$ are the best choice for the *h-p* cloud method. We can use the error estimates obtained here to improve the results of [4,5].

If $u \in H^{m+1}(\Omega)$ is the exact solution of the boundary-value problem, by Céa's lemma [2,3], the error is estimated by an expression like

$$C \inf_{v \in \mathcal{F}^m} || \, u - v \, ||_{m,2}\,.$$

As $\mathcal{ST}^m(u) \in \mathcal{F}^m$, the result above can be applied in order to obtain error estimates of the boundary-value problem.

The following Jackson-type inequalities follow from Theorem 3.7 .

Corollary 3.1. *Assume* ***A1***, ***A2U*** *and* ***A4***. *Let* m *be an integer* > 0 *and* $p \geq 1$. *Then*

$$\sup_{u \in W_p^{m+1}(\Omega)} \inf_{v \in \mathcal{F}^m} \frac{|\, u - v \,|_{l,p}}{|\, u \,|_{m+1,p}} \leq C_{m+1,l}\, h^{m+1-l} \quad \forall\, 0 \leq l \leq m,$$

where

$$C_{m+1,l} = (\#\{\alpha : |\alpha| = l\})^{1/p} M^{2/p} (C(n,m) G_m\, CM_m).$$

4. Local Shepard's formulas with least square fits

Let $\mathcal{F} := \mathbb{R}^N$ be the set of possible values $\mathbf{f} = (f_i)_{i=1}^N$ of functions at the nodes x_i and let m be a positive integer. In this section we study linear operators

$$\mathcal{T}^m : \mathcal{F} \to \prod_{i=1}^{N} \mathcal{P}_i^m$$

built by least squares fits of Taylor's polynomial

$$T_i^m[\mathbf{f}](x) = f_i + \sum_{1 \leq |\alpha| \leq m} a_\alpha (x - x_i)^\alpha \tag{4.1}$$

to function values on a set of nearby nodes of node x_i. The set of nearby nodes of node x_i where the weighed least square approximation (4.1) is made, is called the *star* of x_i [5].

Assume that we have functions $T_i^m : \mathcal{F} \to \mathcal{P}_i^m$, $i = 1, ..., N$, such that:

- T_i^m is a linear transformation.
- For every $u \in \mathcal{F}$, $T_i^m(u)(x_i) = u_i$. That is, the constant term a_0 of $T_i^m(u)$ is equal to u_i.
- If $u=(1, 1, ..., 1)$ then $T_i^m(u) = 1$, for every $i = 1, ..., N$.

Given $u \in \mathcal{F}$, the *m-modified local Shepard's* interpolating function of the data values $u = (u_i)$ is

$$\mathcal{ST}^m(u) = \sum_{i=1}^{N} T_i^m(u) \cdot \mathcal{W}_i. \tag{4.2}$$

4.1. **Shape functions**

If $\{\mathbf{e}_i\}_{i=1,...,N}$ is the canonical basis of $\mathcal{F}$, the functions $\{\phi_i\}$ defined by

$$\phi_i := \mathcal{ST}^m(\mathbf{e}_i)$$

are called the *shape* functions of the modified local Shepard's formula (4.2).

We list bellow some properties of the *shape* functions which are easily proved.

- The class $\{\phi_i\}_{i=1,...,N}$ form a C^∞ partition of unity.
- For $i, j = 1, ..., N$, $\phi_i(x_j) = \delta_{ij}$.
- For every $u \in \mathcal{F}$

$$\mathcal{ST}^m(u) = \sum_{i=1}^{N} u_i \phi_i \,. \tag{4.3}$$

In practical applications, a property of localization of T_i^m is desirable. That is, every T_i^m depends only on values u_j at nodes x_j in a selected neighborhood of x_i. So, it is assumed that:

Every node x_i has a *star* of indexes of nodes

$$\mathcal{ST}(i) = \{i, j_1(i), ..., j_{k_i}(i)\}\,, \quad j_k(i) \neq i,$$

selected in same way, such that T_i^m depends only on values

$$r_i(\ f) := (f_i,\ f_{j_1(i)}, ..., f_{j_{k_i}(i)})\,.$$

However, T_i^m can be considered as defined in all $\mathcal{F}$ by the standard linear extension scheme.

A natural candidate for *star* $\mathcal{ST}(i)$ is clearly $\{j : x_j \in \omega_i\}$, but this is not necessarily the best choice. Renka obtained good results with $\omega_i = B_{R_{1(i)}}(x_i)$ and $\mathcal{ST}(i) = \left\{j : x_j \in B_{R_{2(i)}}(x_i)\right\}$ with $R_{1(i)} \neq R_{2(i)}$ [11].

In this work however, we always assume that $\mathcal{ST}(i) = \{j : x_j \in \omega_i\}$.

4.2. **Building T_i^m by least square fitting**

Let u be a function defined in $\overline{\Omega}$. We set $r_i(u) := (u_j)_{j\in\mathcal{ST}(i)}$, where we have written $u_j = u(x_j)$ in order to simplify.

Let $\mathcal{V} = \{p_t\}_{1\leq t\leq N_m}$ be a basis of the set of polynomial of degree at most m. An efficient scheme for approximating derivatives of a function consists in seeking a polynomial function

$$T_i^m[r_i(u)](x) = u_i + \sum_{t=1}^{N_m} a_t p_t(x - x_i) \tag{4.4}$$

that satisfies

$$T_i^m[r_i(u)](x_j) =_{ls} u_j \tag{4.5}$$

on the set of nodes of the reduced *star* $\mathcal{ST}'(i)$ in a weighed least square sense. Here,

$$\mathcal{ST}'(i) := \mathcal{ST}(i)\backslash\{i\}\ ,$$

and the precise meaning of symbol $=_{ls}$ is defined bellow. Note that we exclude i from our calculus because value u_i is fixed at x_i.

Let $w : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be a selected weigh function. The symbol $=_{ls}$ in (4.5) means that $T_i^m[r_i(u)]$ is chosen in order to minimize the function

$$E(a) = \frac{1}{2} \sum_{j \in \mathcal{ST}'(i)} w(||\, x_j - x_i||) \cdot (T_i^m[r_i(u)](x_j) - u_j)^2 \,, \tag{4.6}$$

where $a = (a_t)_{1 \leq t \leq N_m}$.

A common and recommended selection for w is $w(x) = 1/x^2$.

Let $B_i = B_i(\mathcal{V})$ be the Vandermonde matrix

$$(p_t\,(xj - x_i))_{1 \leq t \leq N_m,\ j \in \mathcal{ST}'(i)} \ ,$$

and W the diagonal matrix with

$$W_{kk} = ||\, x_{j_k} - x_i||^{-2} \ .$$

The normal equations for problem (4.6) are, in matrix notation

$$A\, a = C\, \widetilde{r}_i(u) \ , \tag{4.7}$$

where

$$\widetilde{r}_i(u) := \Big(u_{j_1} - u_i, u_{j_2} - u_i, ..., u_{j_{k_i}} - u_i\Big)\,,$$

$A = B_i W B_i^T$ and $C = B_i W$.

A5: We assume from now on that all *stars* are regular in the sense that matrix A is non singular at every node x_i.

The next result gives us a necessary and sufficient condition to satisfy the assumption above in a generic and stable way [15]. It should be remarked that a related issue was considered in [8, 16] in the context of other approximations also based on partition of unity.

Definition 4.1. The set of nodes $\{x_j\}_{j \in \mathcal{ST}'(\alpha)}$ is called $\mathcal{P}_i^m$ – unisolvent if the Vadermonde matrix satisfies $rank(B_i) = N_m$.

It is clear that this property does not depend on the basis $\mathcal{V}$.

Theorem 4.2. *A necessary and sufficient condition for the satisfaction of condition* ***A5*** *is that the set of nodes* $\{x_j\}_{j \in \mathcal{ST}'(i)}$ *is* $\mathcal{P}_i^m$ *- unisolvent* $\forall i : i = 1, ..., N$.

Corollary 4.3. *If* $m = 1$, $\{x_j\}_{j \in \mathcal{ST}'(i)}$ *is* $\mathcal{P}_i^1$ *- unisolvent if and only if the set of vectors* $\{x_j - x_i\}_{j \in \mathcal{ST}'(i)}$ *contains a subset of* n *lineally independent vectors.*

At node x_i, the Taylor's polynomial (4.4) will be now denoted by

$$T_i^m[r(u)](x) = u_i + \sum_{1 \leq |\nu| \leq m} a_\nu (x - x_i)^\nu. \tag{4.8}$$

A crucial ingredient in error estimates in [15] was a measure of the quality of the approximating Taylor's polynomial (4.8). Taking into account that we have set here $\mathcal{ST}(i) = \{j : x_j \in \omega_i\}$, we have

$$\max_{j \in \mathcal{ST}(i)} \{||x_j - x_i||\} \leq d_i.$$

The next result has been proved in [15].

Theorem 4.4. *Assume **A5** and $m \leq 2$. Then, for every $i = 1, ..., N$, there exists a number $C(\mathcal{ST}(i))$, which is algebraically computable, such that*

$$|a_\nu| \leq C(\mathcal{ST}(i))\, d_i^{-|\nu|}\, ||\widetilde{r}_i(u)||, \quad |\nu| = 1, ..., m, \tag{4.9}$$

where $(a_\nu)_{1\leq|\nu|\leq m}$ is the solution of (4.7).

The number $C(\mathcal{ST}(i))$ is called the *condition number* of the *star* $\mathcal{ST}(i)$.

Remark 4.5. Strictly speaking, we have proved Theorem 4.4 for $m \leq 2$. But, by inspecting the proof, the result is true in general. In practice however, only low order approximations are used because of the notorious polynomial snaking problem.

Assuming **A5** and $m \leq 2$, the *m-modified local Shepard's interpolation operator (4.3)* obtained by least square fits as described in this subsection will be called a $\mathcal{LS}(m)$ *local Shepard's interpolation operator.*

The next result is easily proved.

Theorem 4.6. *Assuming **A5** and $m \leq 2$, the $\mathcal{LS}(m)$ local Shepard's interpolation operator is m-reducible. That is, for every polynomial P of degree at most m, we have*

$$P(x) = \sum_{i=1}^{N} P(x_i)\, \phi_i(x), \quad \forall x \in \overline{\Omega}\,.$$

We will only prove in this section a global estimate. Then, all conditions will be stated in a global setting. We denote $\mathcal{C}_i = C(\mathcal{ST}(i))$, $i = 1, ..., N$, and let

$$\widehat{\mathcal{C}} := \max_{i=1,...N} \{\mathcal{C}\mathcal{N}_i\}\cdot \tag{4.10}$$

We will need a constant $A > 0$ such that

$$d_i^{-1} \leq A\, h^{-1}, \quad i = 1, ..., N. \tag{4.11}$$

Theorem 4.7. *Assume **A1U**, **A3**, **A4**, **A5** and $m \leq 2$. Then, there are constants $\widetilde{C} = \widetilde{C}(n, m, \widehat{\mathcal{C}}, G_m, M, A)$ such that*

$$||D^\alpha \phi_i||_{L^\infty(\mathbb{R}^n)} \leq \widetilde{C}\, h^{-|\alpha|}\,,\ i = 1, ..., N \text{ and } |\alpha| \leq m.$$

Proof. Given i and α, *spt* $(\phi_i) \subset \cup_{j\in\widehat{i}}\omega_j$ and we can write

$$\phi_i = \sum_{j\in\widehat{i}} T_j^m(\mathbf{e}_i)\mathcal{W}_j. \tag{4.12}$$

If

$$T_j^m(\mathbf{e}_i) = \sum_{0\leq|\nu|\leq m} a_\nu^{ij}\, (x - x_j)^\nu$$

then, for any β, $\beta \leq \alpha$,

$$D^\beta T_j^m(\mathbf{e}_i) = \sum_{0\leq|\nu|\leq m, \beta\leq\nu} C(n, \alpha, \nu)\, a_\nu^{ij}\, (x - x_j)^{\nu-\beta}.$$

We now choose a constant $C = C(n, m)$ such that

$$\sum_{0\leq|\nu|\leq m, \beta\leq\nu} C(n, \beta, \nu) \leq C, \quad |\beta| \leq m, |\nu| \leq m. \tag{4.13}$$

Now, given $x \in spt\ (\phi_i)$ and $s \in \widehat{i}$, we have

$$\begin{aligned}|D^\beta T_s^m(\mathbf{e}_i)(x)| &\leq \sum_{0\leq|\nu|\leq m,\beta\leq\nu} C(n,\beta,\nu)\,|a_\nu^{is}|\,||\,x-x_s||^{|\nu|-|\beta|}\\ &\leq \sum_{0\leq|\nu|\leq m,\beta\leq\nu} C(n,\beta,\nu)\,\mathcal{C}_s\,d_s^{-|\beta|}\\ &\leq C\,\widehat{\mathcal{C}}\,A\,h^{-|\beta|}\,.\end{aligned}$$

Among constants $\widehat{\mathcal{C}}$ and A, we have used here Theorem 4.4.

By Leibniz's rule it follows that

$$\begin{aligned}|D^\alpha\left(T_s^m(\mathbf{e}_i)\mathcal{W}_s\right)(x)| &\leq \sum_{\beta+\gamma=\alpha} C(n,\beta,\gamma)|D^\beta T_s^m(\mathbf{e}_i)(x)|\,|D^\gamma\mathcal{W}_s(x)|\\ &\leq \sum_{\beta+\gamma=\alpha} C(n,\beta,\gamma)\,\left(C\,\widehat{\mathcal{C}}\,A\,h^{-|\beta|}\right)\left(G_m\,A\,h^{-|\gamma|}\right).\end{aligned}$$

Setting constant C in (4.13) greater than any sum $\sum_{\beta+\gamma=\alpha} C(n,\beta,\gamma)$, the last inequality can be written

$$|D^\alpha\left(T_s^m(\mathbf{e}_i)\mathcal{W}_s\right)(x)| \leq \overline{C}\,h^{-|\alpha|},$$

where $\overline{C} = C^2\widehat{\mathcal{C}}\,G_m\,A^2$. Therefore, by (A1)

$$\begin{aligned}|D^\alpha\phi_i(x)| &\leq \sum_{s\in\widehat{i}} |D^\alpha\left(T_s^m(\mathbf{e}_i)\mathcal{W}_s\right)(x)|\\ &\leq M\,\overline{C}\,h^{-|\alpha|}.\end{aligned}$$

Setting $\widetilde{C} = M\,\overline{C}$, we get

$$||D^\alpha\phi_i||_{L^\infty} \leq \widetilde{C}\ h^{-|\alpha|}$$

and the theorem is proved. □

A local version of the theorem above can be stated with obvious modification.

4.3. Sobolev error estimates for $\mathcal{LS}(m)$ operators

Through the rest of this section we assume $m \leq 2$ without explicit mention to it. We assume also $(m+1)p > n$ if $p > 1$, or $m+1 \geq n$ if $p = 1$.

Let $u \in W_p^{m+1}(\Omega)$, then by the Sobolev imbedding theorem, $u \in C(\overline{\Omega})$, and it is meaningful to use pointwise values of u. We define the interpolation operator as in (4.3) by

$$\mathcal{ST}^m(u)=\sum_{i=1}^N u_i\phi_i.$$

In order to state our error estimations, we need here some condition similar to (3.1), but in a somewhat stronger form. For simplifying writing, we set

- $\psi_i :=spt\ \phi_i$.
- $\widetilde{\psi}_i = \psi_i \cap \Omega$.
- $\widehat{i} := \{j : \psi_i \cap \psi_j \neq \emptyset\}$.

Now, let ϱ_i be defined by

$$\varrho_i := \min\{\, \varrho : \widehat{\psi}_i \subset B(x_i, \varrho)\}.$$

A3': **A2** is satisfied and $\varrho_i \leq \epsilon_\Omega$, $i = 1, ..., N$.

The overlapping condition is also stronger. We define $n(i)$ by

$$n(i) := \{\, j : \psi_j \cap B(x_i, \varrho_i) \neq \emptyset\},$$

and $\widehat{\psi}_i := \cup_{j \in n(i)} \psi_j$.

A4': $n(i) \leq M$, $i = 1, ..., N$.

As before, if **A3'** is satisfies, every set $B(x_i, \varrho_i) \cap \Omega$ is star-shaped w.r.t. an open ball. Furthermore, there is a number $\gamma_\Omega > 0$ which bound bellow all the chunkiness parameter $\gamma_{B(x_i,\varrho_i)\cap\Omega}$. That is,

$$\gamma_\Omega \leq \gamma_{B(x_i,\varrho_i)\cap\Omega}, \quad \forall i : i = 1, ..., N. \tag{4.14}$$

Then, given $u \in W_p^{m+1}(\Omega)$, we will consider, over each set $B(x_i, \varrho_i) \cap \Omega$, the Dupont–Scott representation

$$u = Q_i^m(u) + R_i^m(u).$$

Finally, let $B > 0$ such that $\max_{i=1,...,N}\{\, 2\varrho_i\} \leq Bh$.

The next result have also been obtained in the context of moving least square and kernel reproducing particle methods [8,16].

Theorem 4.1. *Assume* ***A1U, A3, A4, A5****. Then, there exits a constant* $C = C(n, m, \gamma_\Omega, M, G_m, A, \widetilde{C})$ *such that*

$$|\, u - \mathcal{ST}^m(u)\,|_{l,p} \leq C\, h^{m+1-l} |\, u\,|_{m+1,p}\,, \quad 0 \leq l \leq m,\; u \in W_p^{m+1}(\Omega).$$

Proof. For $x \in \widetilde{\psi}_i$, we can write

$$\begin{aligned} u(x) - \mathcal{ST}^m(u)(x) \;&=\; Q_i^m(u)(x) - \sum_{j=1}^{N} Q_i^m(u)(x_j)\phi_j(x) \\ &\quad + R_i^m u(x) - \sum_{j \in \widehat{i}} R_i^m u(x_j)\phi_j(x)\,. \end{aligned}$$

By the polynomial reproducing property (4.6),

$$Q_i^m(u)(x) = \sum_{j=1}^{N} Q_i^m(u)(x_j)\phi_j(x)\,.$$

Thus,

$$u(x) - \mathcal{ST}^m(u)(x) = R_i^m u(x) - \sum_{j \in \widehat{i}} R_i^m u(x_j)\phi_j(x)\,.$$

Hence

$$||\, u(x) - \mathcal{ST}^m(u)\,||^p_{\widetilde{\psi}_i,l,p} \leq ||R_i^m u||^p_{\widetilde{\psi}_i,l,p} + ||R_i^m u||^p_{L^\infty(B(x_i,\varrho_i))} \sum_{j \in \widehat{i}} ||\phi_j||^p_{\widetilde{\psi}_j,l,p}\,.$$

By (2.7) and (2.8)

$$\begin{aligned} ||R_i^m u||^p_{\widetilde{\omega}_i,l,p} &\leq C_1(n, m, \gamma_\Omega, B)\, h^{(m+1-l)p}\, |u|_{B(x_i,\varrho_i),m+1,p}\,, \\ ||R_i^m u||^p_{L^\infty(B(x_i,\varrho_i))} &\leq C_1(n, m, \gamma_\Omega, B)\, h^{(m+1)p-n}\, |u|_{B(x_i,\varrho_i),m+1,p}\,. \end{aligned}$$

On the other hand, since $\widetilde{\psi}_j \subset B(x_i, \varrho_i)$, we have

$$||\phi_j||^p_{\widetilde{\psi}_j,l,p} \leq C(G_m, n, B, m)\, h^{-lp+n}.$$

Now, since $B(x_i, \varrho_i) \subset \cup_{j\in n(i)} \widetilde{\psi}_j$, we get

$$||\, u(x) - \mathcal{ST}^m(u)\,||^p_{\widetilde{\psi}_i,l,p} \leq \widetilde{C}\, h^{(m+1-l)p}\, |u|_{\widehat{\psi}_i,m+1,p}\,.$$

In order to finish the proof of the theorem, we pass from this local estimate from the global one, in a similar manner as in Theorem 3.7. □

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