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A NOTE ON (2K + 1)-POINT CONSERVATIVE MONOTONE SCHEMES

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Abstract. First–order accurate monotone conservative schemes have good convergence and stability properties, and thus play a very important role in designing modern high resolution shock-capturing schemes. *Do the monotone difference approximations always give a good numerical solution in sense of monotonicity preservation or suppression of oscillations?* This note will investigate this problem from a numerical point of view and show that a $(2K+1)$-point monotone scheme may give an oscillatory solution even though the approximate solution is total variation diminishing, and satisfies maximum principle as well as discrete entropy inequality.

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1. Introduction

Consider one-dimensional scalar hyperbolic conservation laws:

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \tag{1.1}$$

together with initial data $u(x,0) = u_0(x)$, $x \in \mathbb{R}$.

An explicit $(2K+1)$-point finite-difference scheme approximating (1.1) can be written as

$$u_j^{n+1} = G(u_{j-K}^n, \cdots, u_j^n, \cdots, u_{j+K}^n), \tag{1.2}$$

where K is any positive integer, $K \geq 1$. We say that the scheme (1.2) is monotone, if the function G is monotone with respect to all its arguments, *i.e.*

$$\frac{\partial}{\partial v_i} G(v_{-K}, \cdots, v_0, \cdots, v_K) \geq 0, \quad -K \leq i \leq K. \tag{1.3}$$

We call (1.2) a conservative scheme, if it can be casted in the form:

$$u_j^{n+1} = u_j^n - \lambda \left(h_{j+\frac{1}{2}}^n - h_{j-\frac{1}{2}}^n \right), \quad \lambda = \frac{\Delta t}{\Delta x}, \tag{1.4}$$

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with

$$h_{j+\frac{1}{2}} = h(u_{j-K+1}^n, \cdots, u_{j+K}^n). \tag{1.5}$$

Here Δt and Δx are step sizes in time and space, respectively, $u_j^n = u(j\Delta x, n\Delta t)$, and $h_{j+\frac{1}{2}}$ is a numerical flux function. We assume that λ is a constant and require the numerical flux $h_{j+\frac{1}{2}}$ to be consistent with the flux $f(u)$ in the following sense:

$$h(u, \cdots, u) = f(u).$$

There exist some studies on the theoretical analysis of the monotone schemes. Harten, Hyman, and Lax [4] proved that if the monotone difference approximations converge as $\Delta x, \Delta t \to 0$, they converge to the unique entropy weak solution of hyperbolic conservation laws (1.1). But monotone schemes are at most first-order accurate. Kuznetsov [6] proved that monotone schemes for conservation laws converge to the entropy solution in several space dimensions and provided suitable error estimates. Later, Crandall and Majda [1] proved a similar result without the error estimates. Sanders [9] proved convergence with error estimates for certain three-point monotone schemes with variable spatial differencing. The sharpness of the Kuznetsov's error bound was first established by Tang and Teng [11] who proved that the best L^1 convergence rate for monotone schemes to (1.1) is one half if it includes the linear flux case. This result was then extended to nonlinear fluxes by Sabac [8]. However, the half-order rate of convergence can be improved to order one for piecewise smooth solutions with convex flux [12].

Due to the good property of monotone schemes, they have played a very important role in designing modern high resolution shock-capturing schemes. However, to our knowledge, most of studies on numerical approximation and constructions of the high resolution shock-capturing schemes are conducted by using three-point monotone schemes. *Do the monotone difference approximations always give a good numerical solution in sense of monotonicity preservation or suppression of oscillation?*

The purpose of this note is to give an answer to the above problem from a numerical point of view. The results will show that a $(2K+1)$-point monotone scheme may give an oscillatory solution even though the solution is total variation diminishing (TVD), and satisfies a maximum principle as well as a discrete entropy inequality.

In this note, we will investigate the behaviour of a special $(2K+1)$-point scheme:

$$u_j^{n+1} = u_j^n - \frac{\lambda}{2K}\big(f(u_{j+K}^n) - f(u_{j-K}^n)\big) + \frac{\alpha\lambda}{2K}(u_{j+K}^n - 2u_j^n + u_{j-K}^n), \tag{1.6}$$

where $\alpha = \max_u\{|f'(u)|\}$. The scheme (1.6) with $K = 1$ is considered as a generalized Lax-Friedrichs scheme.

The scheme (1.6) can be rewritten in the conservative form (1.4) with the numerical flux function:

$$h_{j+\frac{1}{2}} = \begin{cases} \dfrac{1}{2K}\sum\limits_{\nu=-K+1}^{K} f(u_{j+\nu}) - \dfrac{\alpha}{2K}\left(\sum\limits_{\nu=-K+1}^{K-1}(K-|\nu|)\Delta u_{j+\nu}\right), & K > 1, \\ \dfrac{1}{2}\big(f(u_{j+1}) + f(u_j)\big) - \dfrac{\alpha}{2}\Delta u_j, & K = 1, \end{cases}$$

where $\Delta u_{j+\nu} = u_{j+\nu+1} - u_{j+\nu}$. Obviously, (1.6) is consistent with the partial differential equation (1.1). Moreover, it also satisfies the following properties:

Lemma 1.1 (L^∞–stability)**.** *If the initial data u_j^0, $j \in \mathbb{Z}$, are bounded, i.e.*

$$C_1 \le u_j^0 \le C_2, \quad \forall j \in \mathbb{Z}, \tag{1.7}$$

then the solution u_j^n, $j \in \mathbb{Z}$, to the scheme (1.6) are also bounded:

$$C_1 \le u_j^n \le C_2, \quad \forall j \in \mathbb{Z}, \tag{1.8}$$

under the CFL condition

$$\lambda \max_u \{|f'(u)|\} = \lambda\alpha \le K. \tag{1.9}$$

Especially, the scheme (1.6) is monotone under the condition (1.9).

Proof. In fact, under the CFL condition (1.9), we have

$$\frac{\partial G}{\partial u_j} = 1 - \frac{\alpha\lambda}{K} \ge 0, \quad \frac{\partial G}{\partial u_{j\pm K}} = \frac{\lambda}{2K}\big(\alpha \mp f'(u_{j\pm K})\big) \ge 0. \tag{1.10}$$

Thus, the scheme (1.6) is monotone under (1.9).

Using (1.10) gives

$$\begin{aligned} u_j^{n+1} - C_1 &= G(u_{j-K}^n, \cdots, u_{j+K}^n) - G(C_1, \cdots, C_1) \\ &= G(u_{j-K}^n, \cdots, u_{j-K}^n, u_{j+K}^n) - G(u_{j-K}^n, \cdots, u_{j-K}^n, C_1) \\ &\quad + \cdots + G(u_{j-K}^n, C_1 \cdots, C_1) - G(C_1, \cdots, C_1) \ge 0, \end{aligned}$$

and

$$\begin{aligned} u_j^{n+1} - C_2 &= G(u_{j-K}^n, \cdots, u_{j+K}^n) - G(C_2, \cdots, C_2) \\ &= G(u_{j-K}^n, \cdots, u_{j-K}^n, u_{j+K}^n) - G(u_{j-K}^n, \cdots, u_{j-K}^n, C_2) \\ &\quad + \cdots + G(u_{j-K}^n, C_2 \cdots, C_2) - G(C_2, \cdots, C_2) \le 0. \end{aligned}$$

This completes the proof of the first part of Lemma 1.1. □

Lemma 1.2 (TV–stability)**.** *If the total variation of the initial data u_j^0, $j \in \mathbb{Z}$, is bounded, i.e.*

$$TV(u^0) = \sum_{j\in\mathbb{Z}} |u_{j+1}^0 - u_j^0| \le C, \tag{1.11}$$

then under the CFL restriction

$$\lambda \max_{j\in\mathbb{Z}} \left\{a_{j+\frac{1}{2}}\right\} \le \lambda\alpha \le K, \tag{1.12}$$

where $a_{j+\frac{1}{2}}$ satisfies $a_{j+\frac{1}{2}}\Delta u_j = \Delta f(u_j)$, the solution u_j^n, $j \in \mathbb{Z}$, to the scheme (1.6) is also TV-bounded:

$$TV(u^n) \le C. \tag{1.13}$$

Proof. We rewrite the scheme (1.6) in an incremental form as follows:

$$u_j^{n+1} = u_j^n + \sum_{\nu=0}^{K-1} C_{j+\nu+\frac{1}{2}}^n \Delta u_{j+\nu}^n - \sum_{\nu=-K}^{-1} D_{j+\nu+\frac{1}{2}}^n \Delta u_{j+\nu}^n, \tag{1.14}$$

where

$$C_{j+\nu+\frac{1}{2}}^n = \frac{\lambda}{2K}\left(\alpha - a_{j+\nu+\frac{1}{2}}^n\right), \quad D_{j+\nu+\frac{1}{2}}^n = \frac{\lambda}{2K}\left(\alpha + a_{j+\nu+\frac{1}{2}}^n\right), \tag{1.15}$$

and $a_{j+\nu+\frac{1}{2}}$ satisfies $a_{j+\nu+\frac{1}{2}}\Delta u_{j+\nu} = \Delta f(u_{j+\nu})$. Subtracting (1.14) at j from (1.14) at $j+1$ gives

$$\Delta u_j^{n+1} = \left(1 - C^n_{j+\frac{1}{2}} - D^n_{j+\frac{1}{2}}\right) \Delta u_j^n + C^n_{j+K+\frac{1}{2}}\Delta u^n_{j+K} + D^n_{j-K+\frac{1}{2}}\Delta u^n_{j-K}. \tag{1.16}$$

Taking the absolute value of (1.16) and using the triangle inequality and the CFL restriction (1.12), we get

$$|\Delta u_j^{n+1}| \le \left(1 - C^n_{j+\frac{1}{2}} - D^n_{j+\frac{1}{2}}\right) |\Delta u_j^n| + C^n_{j+K+\frac{1}{2}}|\Delta u^n_{j+K}| + D^n_{j-K+\frac{1}{2}}|\Delta u^n_{j-K}|. \tag{1.17}$$

Summing (1.17) from $j = -\infty$ to $+\infty$, we get by shifting indices

$$TV(u^{n+1}) \le TV(u^n).$$

It will complete the proof of Lemma 1.2. □

Remark 1.1. (1) From the proof of the Lemma 1.2, we can also conclude that the scheme (1.6) is monotonicity preserving, that is to say, if the initial data u_j^0 are monotone (either nonincreasing or nondecreasing) as a function of j, then the solution u_j^n should have the same property for all n.

(2) Following the existing results, e.g. [1], the solution of the scheme (1.6) should also satisfy discrete entropy condition and converge to the unique entropy solution of (1.1).

2. Numerical analysis

In this section we conduct some numerical experiments by using the scheme (1.6) to solve scalar conservation laws (1.1) with the flux $f(u) = cu$ or $f(u) = \frac{1}{2}u^2$, where c is a constant. In the following, unless stated otherwise, λ is generally taken to be 3, and the computational domain $[-8, 10]$ is divided by 400 grid cells.

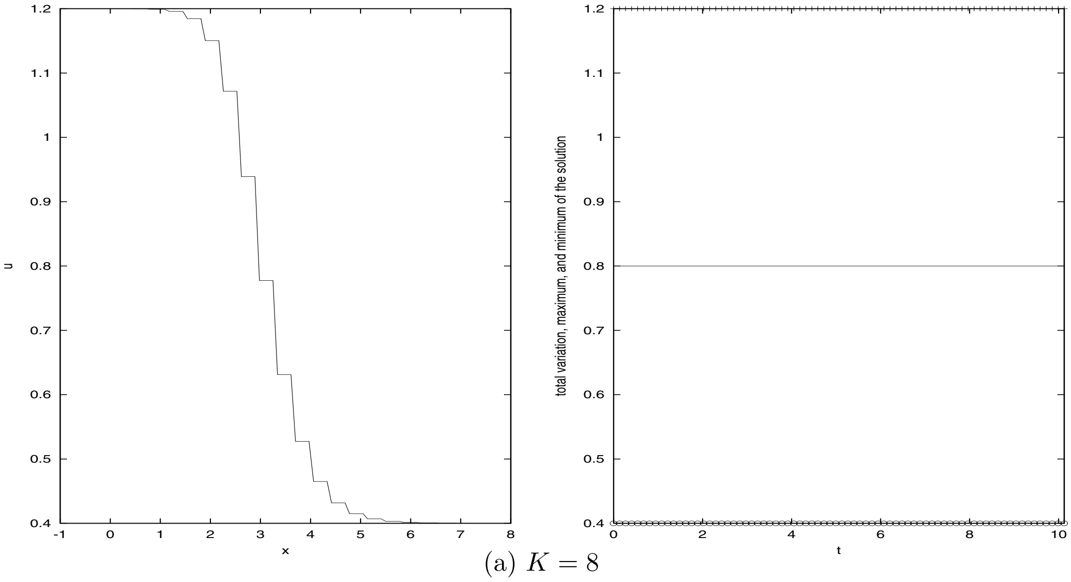
Example 1. The first case is to solve initial value problem of scalar conservation laws (1.1) with the initial data

$$u(x,0) = \begin{cases} 1.2, & -8 \le x \le -5, \\ 0.4, & -5 \le x \le 10. \end{cases} \tag{2.18}$$

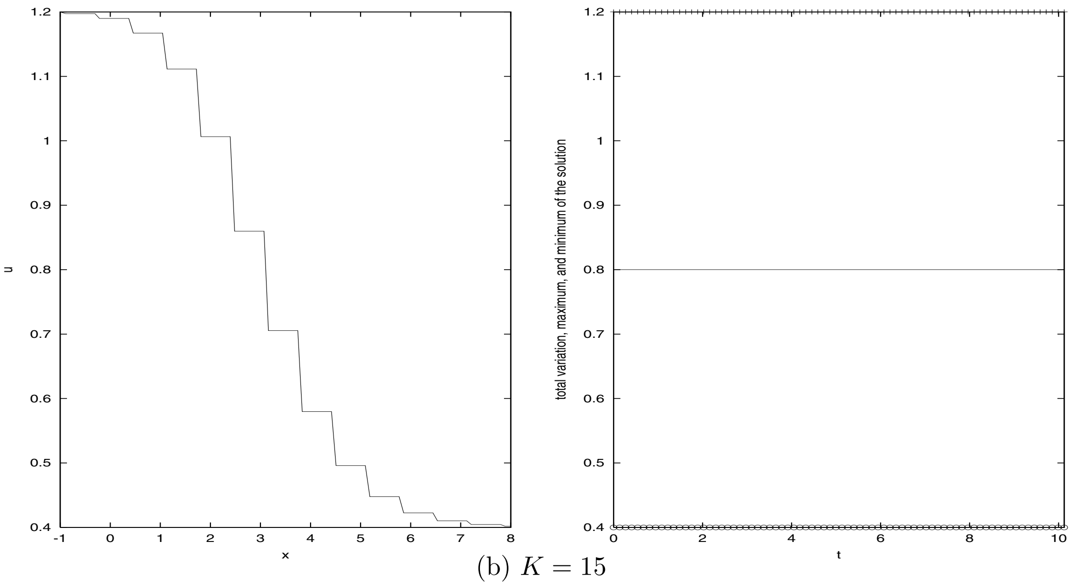
Our purpose of solving this example to check the monotonicity-preserving property of the solutions calculated by the scheme (1.6). Figures 1 and 2 show the computed solutions (left) at $t = 10$ and the recorded total variation ("solid line"), maximum ("plus"), and minimum ("circle") of the solutions (right) for $f(u) = 0.8u$ and $\frac{1}{2}u^2$, and $K = 8$ and 15, respectively. In this case, the Courant number equals to 3.6. We can see that the recorded total variation, maximum, and minimum of the solutions are kept constant, and the computed solutions are monotone even if they have become piecewise step functions.

Example 2. The second case is to solve the initial value problem of scalar conservation laws (1.1) with the initial data

$$u(x,0) = \begin{cases} 0, & -8 \le x \le -2.5, \\ (x+2.5)/10, & -2.5 \le x < 0, \\ (x-2.5)/10, & 0 \le x < 2.5, \\ 0, & 2.5 \le x \le 10. \end{cases} \tag{2.19}$$



(a) $K = 8$

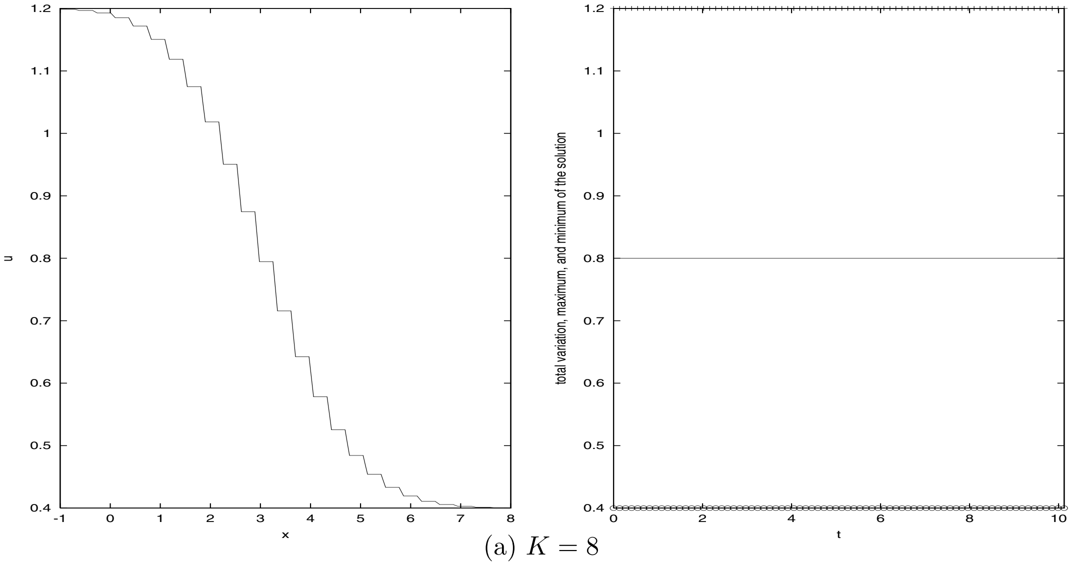


(b) $K = 15$

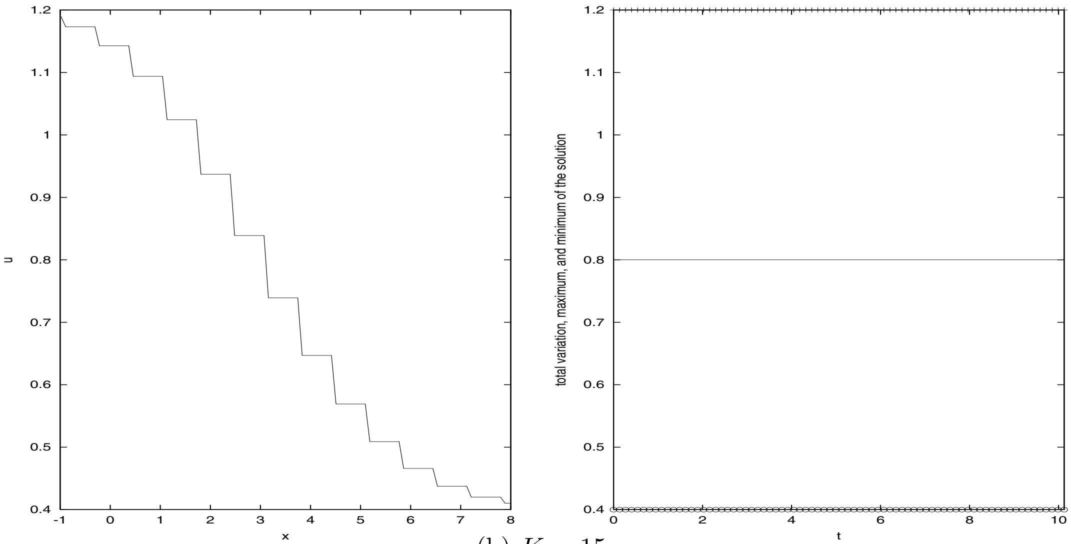
FIGURE 1. The computed solution (left) and the recorded total variation ("solid line"), maximum ("plus"), and minimum ("circle") of the solutions (right). $f(u) = \frac{1}{2}u^2$.

These initial values for $f(u) = \frac{1}{2}u^2$ consist of two rarefaction waves which are connected by a stationary shock at $x = 0$. The corresponding initial value problem has been used in [5] to check ability of the large time step Godunov scheme. In Figures 5–8 we give the computed solutions at $t = 10$ and the recorded total variation ("solid line"), maximum ("plus"), and minimum ("circle") of the solutions for $f(u) = 0.2u$ and $\frac{1}{2}u^2$, and $K = 1$, 3, 8 and 15, respectively. The results show that the computed solutions are TVD, and L^∞–stable. But numerical oscillations have been generated in the computed solutions when a large integer $K > 1$ is used.

Following the above numerical experiments, we can conclude that a $(2K+1)$-point monotone scheme for $K > 1$ may give an oscillatory solution even though the approximate solution is TVD, monotonicity-preserving,



(a) $K = 8$



(b) $K = 15$

FIGURE 2. The computed solution (left) and the recorded total variation ("solid line"), maximum ("plus"), and minimum ("circle") of the solutions (right). $f(u) = 0.8u$.

and satisfies maximum principle as well as discrete entropy inequality. Moreover, the oscillation cannot be suppressed or eliminated automatically at a later time.

Is a three-point monotone scheme non-oscillatory in the sense that the number of extrema of the solution $u_h(x, t_2)$ does not exceed that of the solution $u_h(x, t_1)$, where $t_2 > t_1$? To answer this question, we use the scheme (1.6) to solve the Burgers' equation with initial data

$$u_j^0 = \begin{cases} 0, & j \neq 0, \\ 100, & j = 0. \end{cases} \tag{2.20}$$

In this case, the Courant number, Δx, and K are taken as 0.8, 1, and $K = 1$, respectively. In Table 1 we list the solutions u_j^n at several different time levels. The result shows that the number of extrema of the solution u_j^2 is larger than that of the solution u_j^0, but it is reduced at the later time. That means that the three-point scheme (1.6) may also generate a new extremum, but the oscillation generated by it can be eliminated essentially by itself at later time. It is worth noting that a similar example has been applied by Tadmor in [10]. He use the Lax-Friedrichs (LxF) scheme with $\lambda = \Delta t/\Delta x$ to solve the initial value problem of (1.1) with $f(u) = u^2$ and

$$u_j^0 = \delta_{j,0} \equiv \begin{cases} 0, & j \neq 0, \\ 1, & j = 0. \end{cases} \tag{2.21}$$

TABLE 1. The computed solution of the Burgers' equation with initial data given in (2.20).

n	$TV(u^n)$	u_{-4}^n	u_{-3}^n	u_{-2}^n	u_{-1}^n	u_0^n	u_1^n	u_2^n	u_3^n	u_4^n
0	200	0	0	0	0	100	0	0	0	0
1	120	0	0	0	20	20	60	0	0	0
2	80	0	0	6.667	10.667	25.333	21.333	36	0	0
3	57.007	0	2.420	4.968	11.615	15.970	25.165	18.262	21.600	0
4	47.367	0.921	2.275	5.581	8.867	13.945	18.103	23.684	14.275	12.348
5	41.434	1.051	2.800	4.953	8.205	11.474	15.578	18.734	20.717	9.900

His result showed that, in this case, the LxF scheme amounted to a pure translation, lacking the dissipation to cause any decay. However, if $\lambda < 1$, then we can find that the solution constructed by the LxF scheme will not be a pure translation; the new extrema will also appear in it. Moreover, number of local extrema in the solution of the LxF scheme will not be diminished in this case. The main reason for generation of local extrema by the LxF scheme is that the LxF scheme is in a staggered form, *i.e.*, u_j^{n+1} only depends on u_{j+1}^n and u_{j-1}^n, for all $j \in \mathbb{Z}$. In other words, the solutions at $t = t_{n+1}$ with odd (or even) index j only depend on the solutions at $t = t_n$ with even (or odd) index j. This phenomenon can also be observed when we use the LxF scheme to solve the initial value problem of (1.1) with $f(u) = cu$ and the initial data given in (2.20) or (2.21), where c is a constant. It means that generation of local extrema by the LxF scheme is not due to nonlinearity of the flux $f(u)$ in (1.1). This seems to be different from the scheme (1.6). In Figure 3 we give a comparison of the solutions of the initial value problem of (1.1) with $f(u) = \frac{1}{2}u^2$ and the initial data (2.20) calculated by the LxF scheme and the scheme (1.6) with $K = 1$, respectively. We also use the LxF scheme to solve Example 2 with $\frac{1}{2}u^2$ up to $t = 10$ as before. The results are shown in Figure 4. We can see that the solution of the LxF scheme is oscillatory.

The main difference between the three-point scheme (1.6) and its $(2K+1)$-point version is that under a suitable CFL restriction, the three-point scheme satisfies the following local maximum principle:

$$\min_{p=0,\pm1} \left\{u_{j+p}^n\right\} \leq u_j^{n+1} \leq \max_{p=0,\pm1} \left\{u_{j+p}^n\right\}, \tag{2.22}$$

which is stronger than the global one shown in Lemma 1.1. It seems to be necessary to guarantee that the $(2K+1)$-point scheme (1.6) is nonoscillatory, at least for this special initial data.

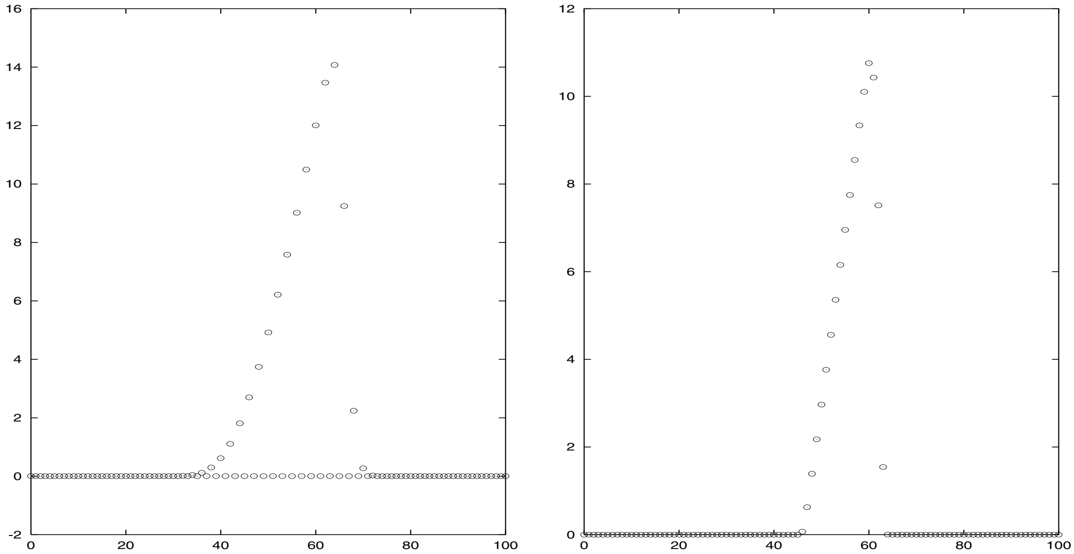


FIGURE 3. The solutions of the initial value problem of (1.1) with $f(u) = \frac{1}{2}u^2$ and the initial data (2.20). Left: the LxF scheme; right: the scheme (1.6) with $K = 1$.

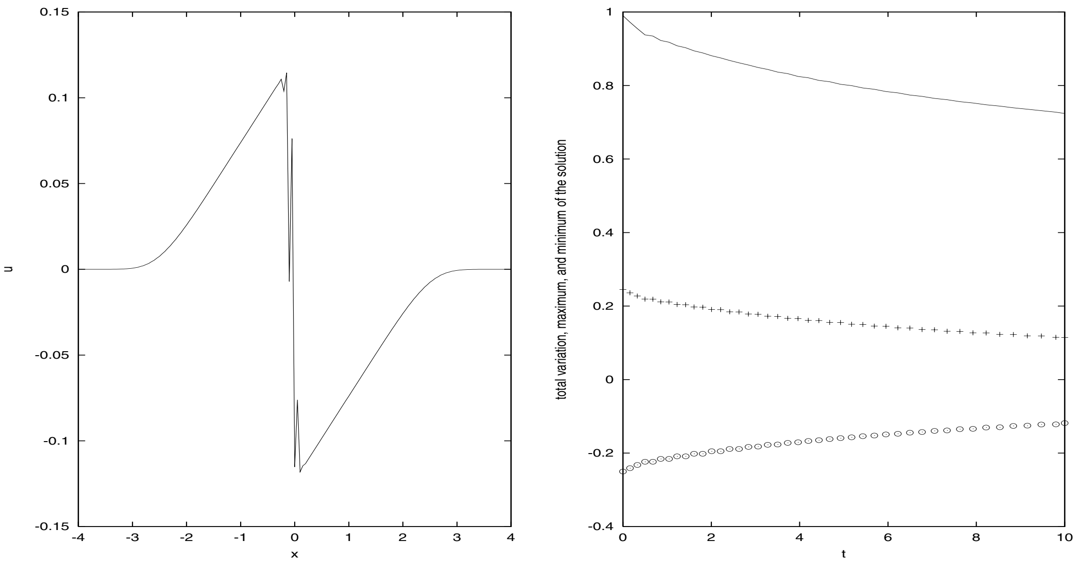
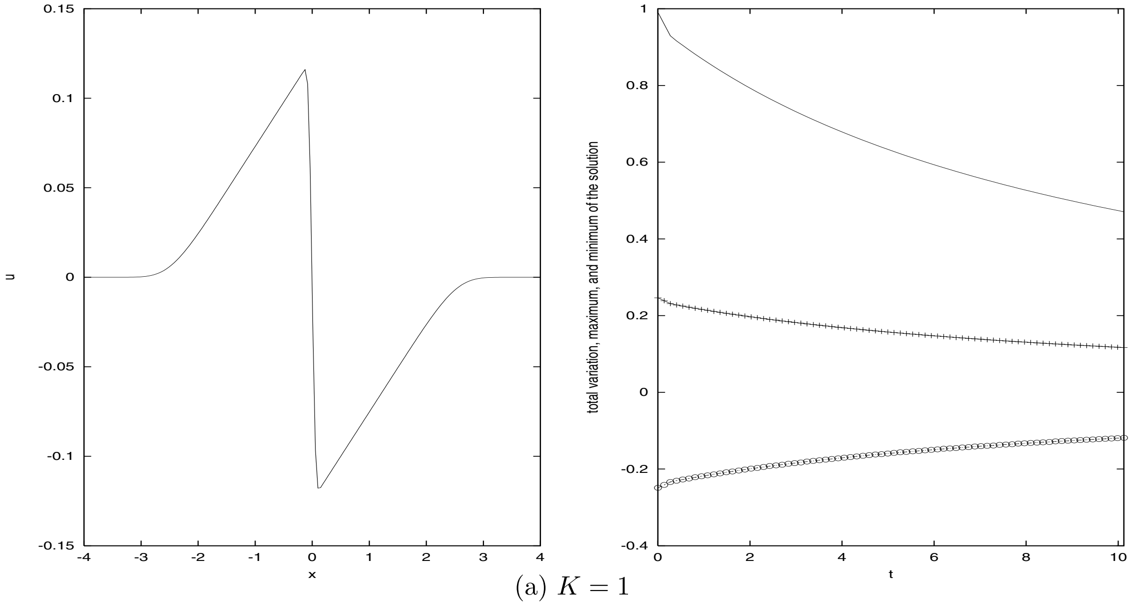
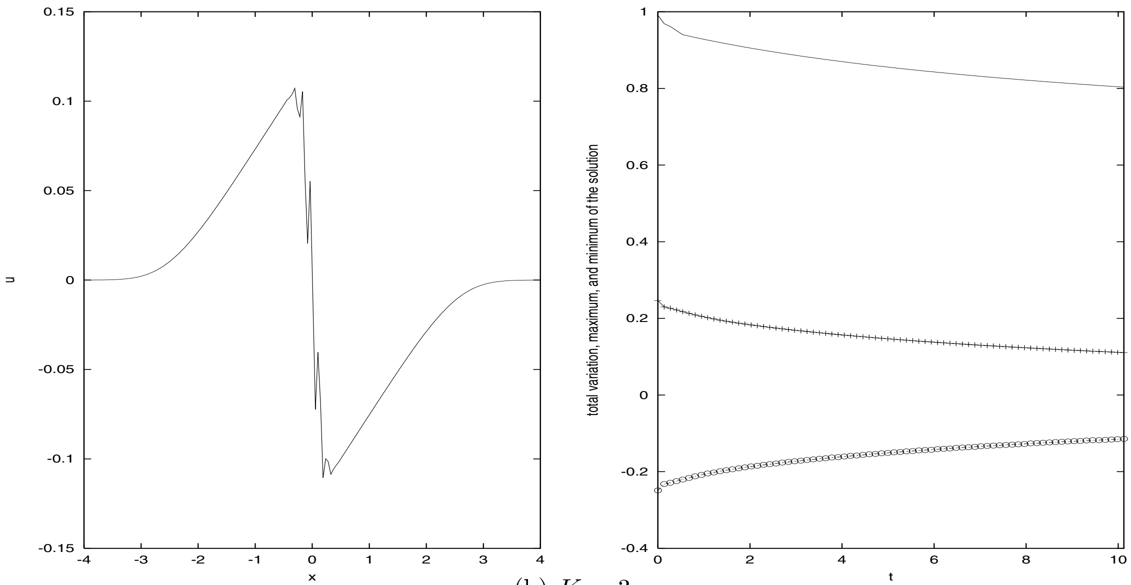


FIGURE 4. The solution (left) computed by the LxF scheme for $f(u) = \frac{1}{2}u^2$, and the recorded total variation ("solid line"), maximum ("plus"), and minimum ("circle") of the solutions (right).

Due to nonlinearity, it is still difficult now to give a sufficient condition to guarantee that a generally conservative difference scheme is nonoscillatory. Here, we want to state that the above observed results do not contradict the convergence of a $(2K + 1)$-point conservative monotone scheme to the physically relevant limit solution. Many practical computations have also shown that the solution to (1.1) calculated by a three-point



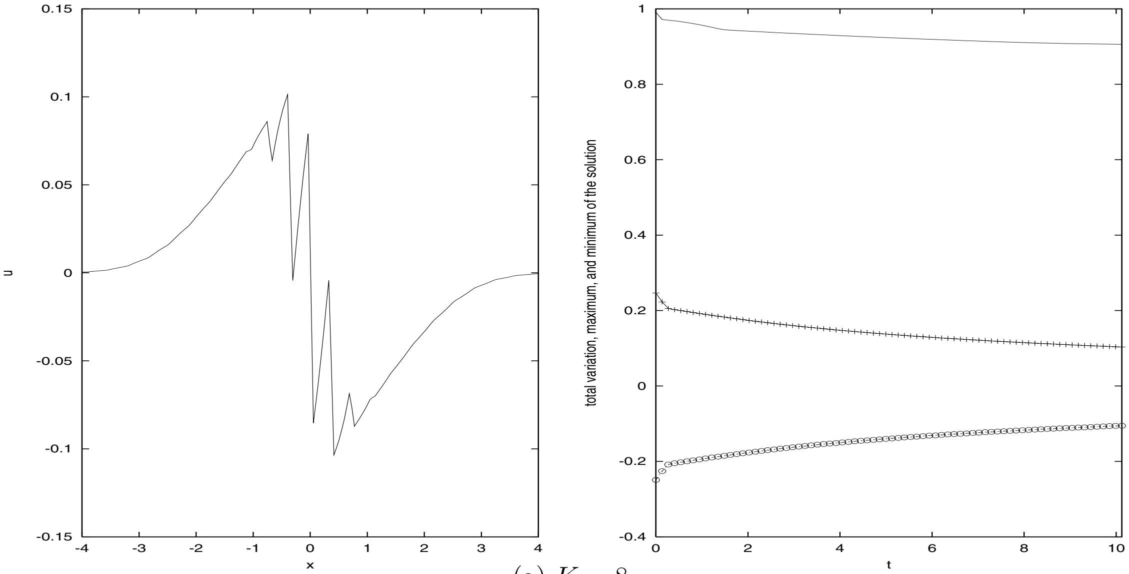
(a) $K = 1$



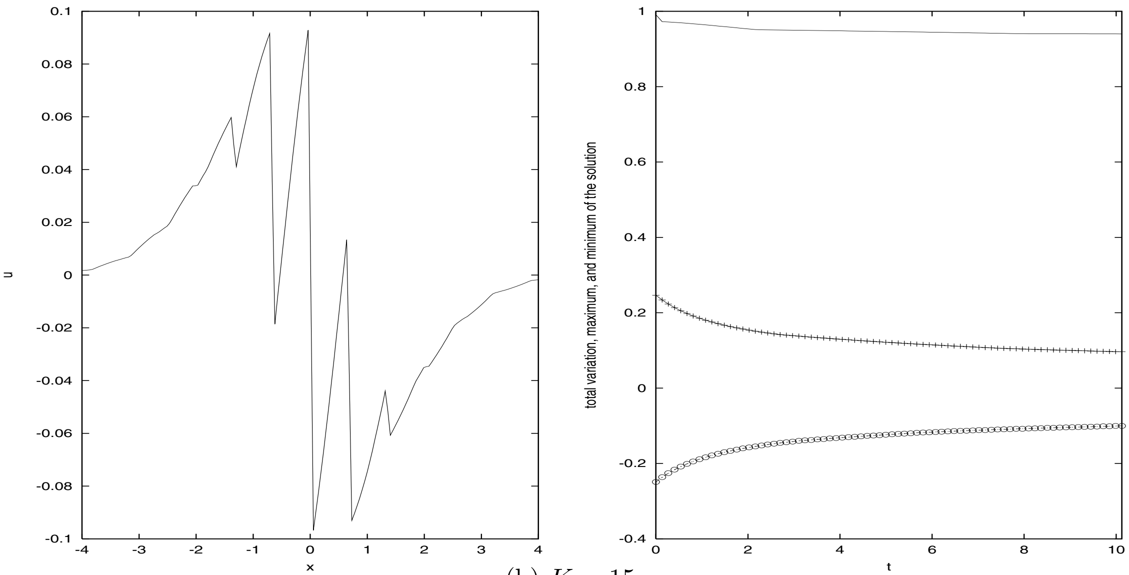
(b) $K = 3$

FIGURE 5. The computed solution (left) and the recorded total variation ("solid line"), maximum ("plus"), and minimum ("circle") of the solutions (right). $f(u) = \frac{1}{2}u^2$.

scheme is "essentially" nonoscillatory. Thus, the three-point monotone scheme can be considered as "essentially" nonoscillatory scheme.



(a) $K = 8$

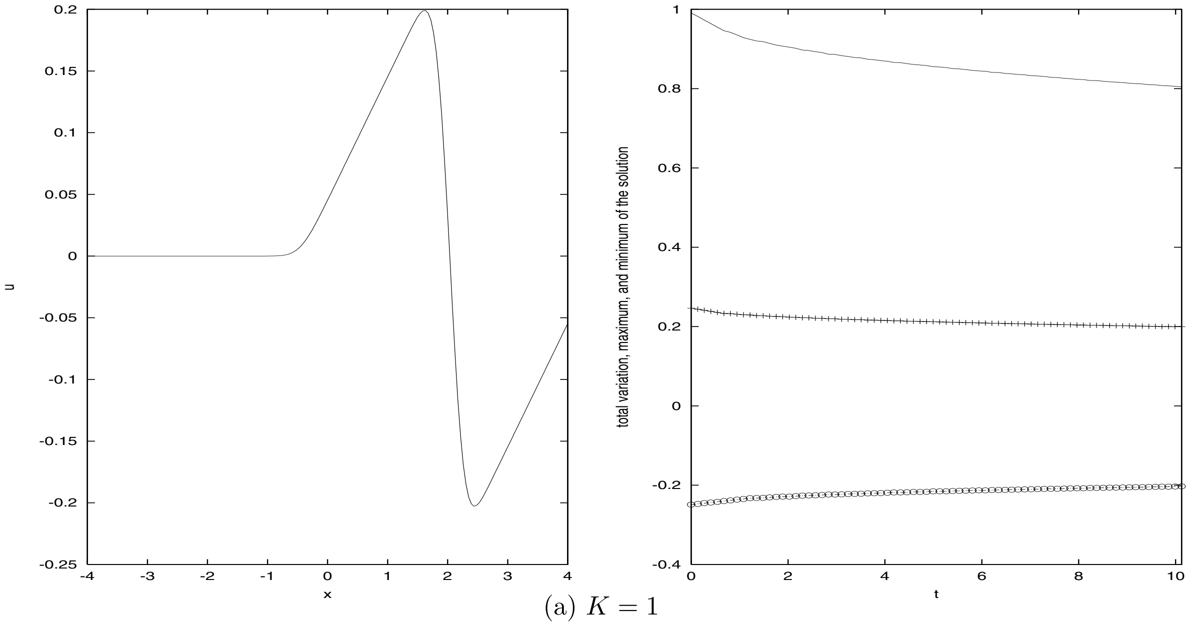


(b) $K = 15$

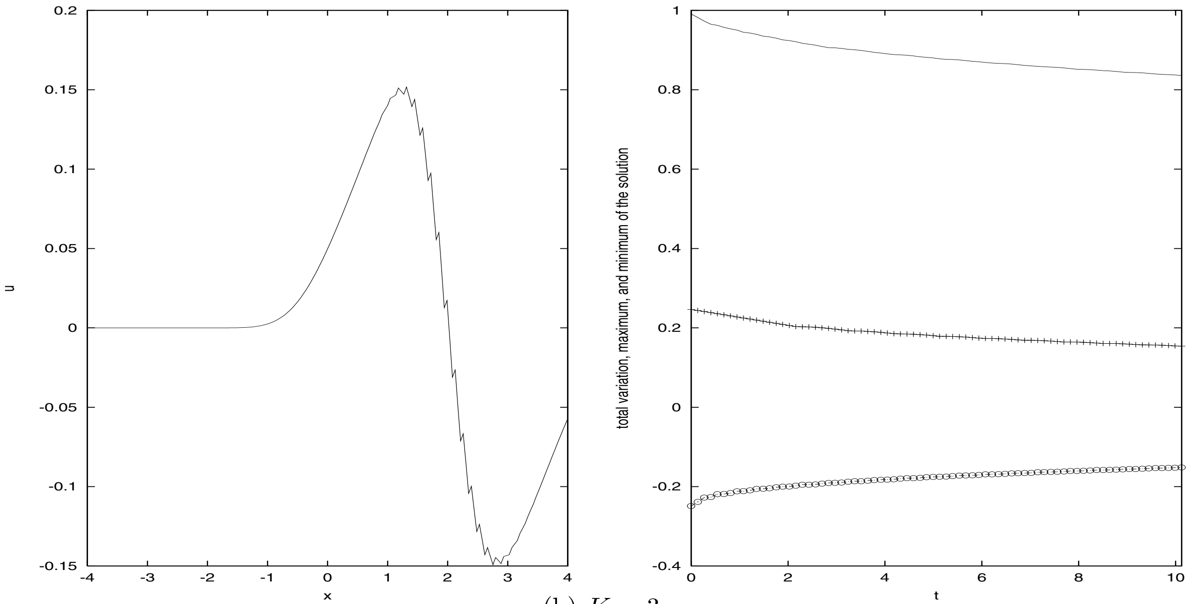
FIGURE 6. Same as Figure 5.

The existing high-resolution TVD schemes, *e.g.* Harten's 5-point TVD scheme [2], can also give a numerical solution to many practical problem that is "essentially" nonoscillatory, because they may be actually considered as a three-point like scheme, for example, we can write them in the following incremental form:

$$u_j^{n+1} = u_j^n + C_{j+\frac{1}{2}}^n \Delta u_j^n - D_{j-\frac{1}{2}}^n \Delta u_{j-1}^n. \tag{2.23}$$



(a) $K = 1$



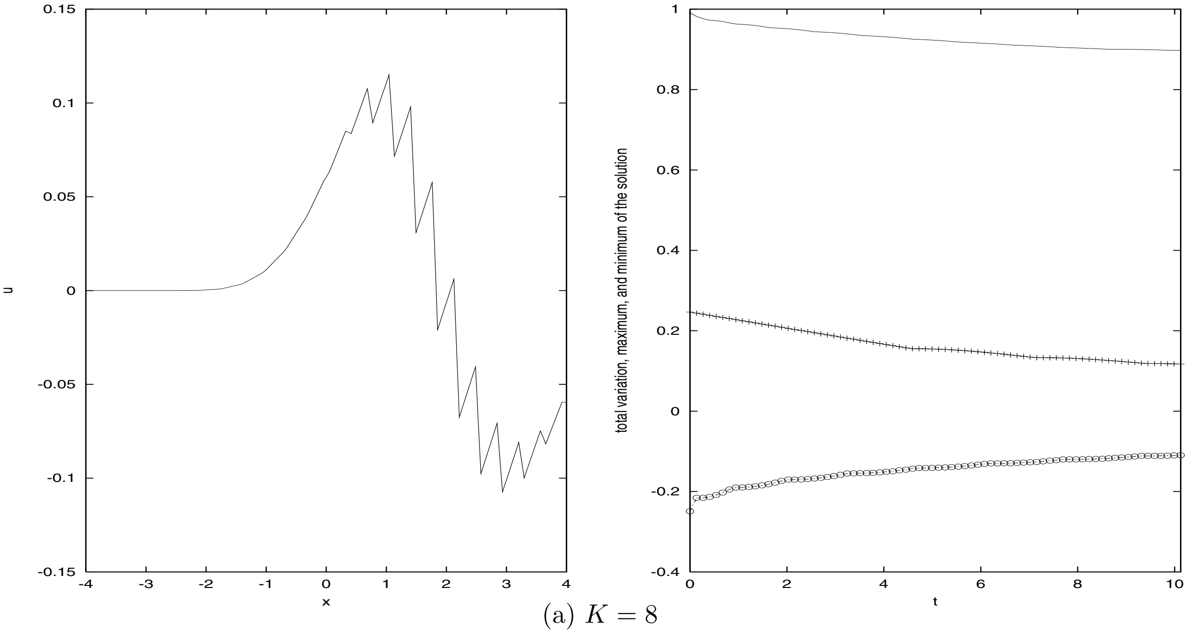
(b) $K = 3$

FIGURE 7. The computed solution (left) and the recorded total variation ("solid line"), maximum ("plus"), and minimum ("circle") of the solutions (right). $f(u) = 0.2u$.

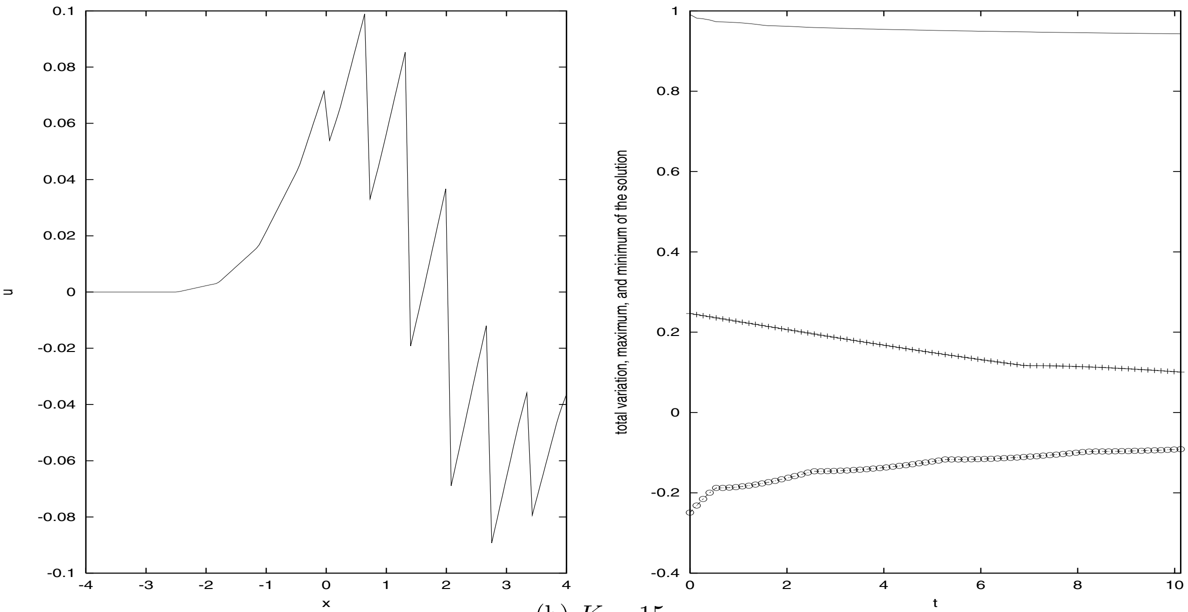
Under a suitable CFL restriction, the incremental coefficients satisfy:

$$C^n_{j+\frac{1}{2}} \geq 0, \quad D^n_{j-\frac{1}{2}} \geq 0, \quad C^n_{j+\frac{1}{2}} + D^n_{j-\frac{1}{2}} \leq 1, \tag{2.24}$$

and hence the corresponding schemes satisfy the local maximum principle (2.22).



(a) $K = 8$



(b) $K = 15$

FIGURE 8. Same as Figure 7.

In the literature, there have existed some uniformly high order nonoscillatory schemes for hyperbolic conservation laws, for example, Harten and Osher's UNO scheme [3], and Liu and Tadmor's third order non-oscillatory central scheme [7]. However, construction of their nonoscillatory schemes is based on the exact solution of the Riemann problem. In fact, their schemes proceed in three steps: first, reconstructing the solution out of its approximate cell-averages to the appropriate accuracy; second, solving exactly local Riemann problem; and finally, taking cell averages of the solution given in the second step. The second and third steps are also nonoscillatory. Thus, if the reconstruction is also nonoscillatory, then the approximate solution at a new time level is also nonoscillatory.

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