A POSTERIORI ERROR ANALYSIS OF THE FULLY DISCRETIZED TIME-DEPENDENT STOKES EQUATIONS

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Abstract. The time-dependent Stokes equations in two- or three-dimensional bounded domains are discretized by the backward Euler scheme in time and finite elements in space. The error of this discretization is bounded globally from above and locally from below by the sum of two types of computable error indicators, the first one being linked to the time discretization and the second one to the space discretization.

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1. Introduction

The time-dependent Stokes problem in a two- or three-dimensional bounded domain models the laminar flow of a viscous incompressible fluid. The basic discretization of this problem relies on the use of the backward Euler scheme with respect to the time variable and of finite elements with respect to the space variables, and a lot of work has been done concerning its a priori analysis, see [9] for preliminary results. The important point is that two discretization parameters are involved: the set of time steps and the mesh at each time step. Moreover, due to the choice of an implicit scheme, these parameters can be chosen in a completely independent way. The aim of this paper is to perform the a posteriori analysis of the discretization, more precisely to provide tools that allow for optimizing the choice of each time step when working with adaptive meshes.

Much work has been done concerning the a posteriori analysis of parabolic type problems. Part of it (cf. [2, 4, 5]) deals only with the space discretization and provides appropriate error indicators for it. Another idea consists in establishing a full time and space variational formulation of the continuous problem and using a discontinuous Galerkin method for the discretization with respect to all variables, see for instance [7,8,16,17,19]. Here, we follow a different approach, according to an idea of [1], which consists in introducing two different types of error indicators, one for the time discretization and one for the space discretization, and in uncoupling as far as possible, the estimates of the time and space errors.

In a first step, we give the space variational formulation of the continuous Stokes problem and also of the time semi-discrete problem derived from the Euler scheme. The finite element discretization in space is then...
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built by applying the Galerkin method to this last problem. We have chosen to work with conforming finite elements for simplicity. Next, we describe the two types of residual error indicators: it must be noted that both of them only depend on the fully discrete solution and the data, so that computing them is easy and not expensive. We prove that the global error in the energy norm is bounded from above by the Hilbertian sum of these indicators and also that the indicators are bounded by the error. Moreover these last estimates are local in time for the first type of indicators, local both in time and space for the second one. So it can be hoped that they provide a good representation of the error and furthermore an efficient tool for choosing each time step in an optimal way and performing mesh adaptivity also at each time step.

The extension of these results to the full nonlinear time-dependent Navier-Stokes equations has been considered and similar results can be proven, at least in the two-dimensional case, by using the abstract result in [13] (Th. 3), see also [15] (Prop. 2.1). However we prefer not to present this extension since first the time error indicators in this case are not so easy to compute in practical situations, and second the time discretization of the Navier-Stokes equation by the backward Euler scheme is not realistic except for very small Reynolds number, hence not for real life simulations.

An outline of the paper is as follows.

• Section 2 is devoted to the description of the continuous, the time semi-discrete and the fully discrete problems. We recall their main properties and some standard a priori estimates.

• In Section 3, we perform the a posteriori analysis of the time discretization.

• In Section 4, the a posteriori analysis of the space discretization is achieved.

• In Section 5, we combine the results of the two previous sections to derive the full estimates.

2. THE CONTINUOUS, SEMI-DISCRETE AND DISCRETE PROBLEMS

Given a bounded connected domain $\Omega$ in $\mathbb{R}^d$, $d = 2$ or $3$, with a Lipschitz–continuous boundary and a positive real number $T$, we consider the following Stokes equations

$$\begin{aligned}
\partial_t u - \nu \Delta u + \text{grad } p &= f \quad \text{in } \Omega \times [0,T[,
\text{div } u &= 0 \quad \text{in } \Omega \times [0,T[,
\mathbf{u} &= 0 \quad \text{on } \partial \Omega \times [0,T[,
\mathbf{u}(\cdot,0) &= \mathbf{u}_0 \quad \text{in } \Omega.
\end{aligned}$$

(2.1)

Here, the unknowns are the velocity $\mathbf{u}$ and the pressure $p$; the data are the distribution $f$ which represents a density of body forces and the initial velocity $\mathbf{u}_0$, while the viscosity $\nu$ is a positive constant.

We first give the variational formulation of problem (2.1) and recall its main properties. We next describe the time semi-discretization of this problem and recall the well-posedness of the semi-discrete problem together with some a priori error estimates. Finally, we present the fully discrete problem and there also recall its consistency.

The variational formulation

In all that follows, for any $t$, $0 < t \leq T$, and any separable Banach space $X$ provided with the norm $\| \cdot \|_X$, we denote by $L^2(0,t;X)$ the space of measurable functions $v$ from $(0,t)$ in $X$ such that

$$\|v\|_{L^2(0,t;X)} = \left( \int_0^t \|v(\cdot,s)\|^2_X \, ds \right)^{\frac{1}{2}} < +\infty.$$ 

For any positive integer $m$, we introduce the space $H^m(0,t;X)$ of functions in $L^2(0,t;X)$ such that all their time derivatives of order $\leq m$ belong to $L^2(0,t;X)$. We also use the dual space $H^{-1}(0,t;X')$ of the space of all functions in $H^1(0,t;X)$ vanishing in 0 and $t$, and the space $C^0([0,t];X)$ of continuous functions $v$ from $[0,t]$ in $X$. Let $(\cdot,\cdot)$ stand for the scalar product on $L^2(\Omega)$ or $L^2(\Omega)^d$ or $L^2(\Omega)^{d \times d}$ and, by extension, for the duality
pairing between $H^{-1}(\Omega)^d$ and $H_0^1(\Omega)^d$. Finally, we introduce the space $L_0^2(\Omega)$ of functions in $L^2(\Omega)$ with zero mean value on $\Omega$.

It can be checked that problem (2.1) admits the variational formulation:

Find $u$ in $L^2(0,T;H_0^1(\Omega)^d) \cap C^0(0,T;L^2(\Omega)^d)$ and $p$ in $H^{-1}(0,T;L_0^2(\Omega))$, such that

$$u(\cdot,0) = u_0 \quad \text{a.e. in } \Omega,$$

and that, for a.e. $t$ in $(0,T)$ and for all $(v,q)$ in $H_0^1(\Omega)^d \times L_0^2(\Omega)$,

$$(\partial_t u, v) + \nu (\nabla u, \nabla v) - (\text{div } v, p) = (f, v), \quad -(\text{div } u, q) = 0. \tag{2.3}$$

We introduce the following norm on $L^2(0,T;H_0^1(\Omega)^d) \cap C^0(0,T;L^2(\Omega)^d)$

$$[v](t) = \left(\|v(\cdot,t)\|_{L^2(\Omega)^d}^2 + \nu \int_0^t \|\nabla v(\cdot,s)\|_{L^2(\Omega)^d}^2 \, ds\right)^{\frac{1}{2}}. \tag{2.4}$$

The following existence and stability results can be derived from [9] (Chap. V, Sect. 1.2), and [14] (Chap. III, Th. 1.1).

**Proposition 2.1.** For any data $(f, u_0)$ in $L^2(0,T;H^{-1}(\Omega)^d) \times L^2(\Omega)^d$ such that $u_0$ is divergence-free in $\Omega$, problem (2.2)-(2.3) has a unique solution $(u, p)$, which satisfies for all $t, 0 < t \leq T$,

$$[u](t) \leq \left(\frac{1}{\nu} \|f\|_{L^2(0,t;H^{-1}(\Omega)^d)}^2 + \|u_0\|_{L^2(\Omega)^d}^2\right)^{\frac{1}{2}}. \tag{2.5}$$

Moreover this solution is such that $\partial_t u + \nabla p$ belongs to $L^2(0,T;H^{-1}(\Omega)^d)$ and satisfies for all $t, 0 < t \leq T$,

$$\|\partial_t u + \nabla p\|_{L^2(0,t;H^{-1}(\Omega)^d)} \leq 2 \left(\|f\|_{L^2(0,t;H^{-1}(\Omega)^d)}^2 + \|u_0\|_{L^2(\Omega)^d}^2\right)^{\frac{1}{2}}. \tag{2.6}$$

We finally introduce the kernel

$$V = \{v \in H_0^1(\Omega)^d; \text{ div } v = 0 \text{ in } \Omega\}, \tag{2.7}$$

which plays an important role in the numerical analysis.

**The time semi-discrete problem**

In order to describe the time discretization of equation (2.1) with an adaptive choice of local time steps, we introduce a partition of the interval $[0,T]$ into subintervals $[t_{n-1}, t_n]$, $1 \leq n \leq N$, such that $0 = t_0 < t_1 < \cdots < t_N = T$. We denote by $\tau_n$ the length $t_n - t_{n-1}$, by $\tau$ the $N$-tuple $(\tau_1, \ldots, \tau_N)$, by $|\tau|$ the maximum of the $\tau_n$, $1 \leq n \leq N$, and finally by $\sigma_\tau$ the regularity parameter

$$\sigma_\tau = \max_{2 \leq n \leq N} \frac{\tau_n}{\tau_{n-1}}. \tag{2.8}$$

For any Banach space $X$, with each family $(v^n)_{0 \leq n \leq N}$ in $X^{N+1}$ we agree to associate the function $v_\tau$ on $[0,T]$ which is affine on each interval $[t_{n-1}, t_n]$, $1 \leq n \leq N$, and equal to $v^n$ at $t_n$, $0 \leq n \leq N$. We denote by $Y_\tau(X)$ the space of such functions.

We now assume that the distribution $f$ belongs to $C^0(0,T;H^{-1}(\Omega)^d)$ and, for simplicity, we denote by $f^n$ the distribution $f(\cdot, t_n)$. The semi-discrete problem constructed from the backward Euler scheme applied to the variational formulation (2.2)-(2.3) is:
Find \((u^n)_{0 \leq n \leq N}\) in \(L^2(\Omega)^d \times (H^1_0(\Omega))^d\) and \((p^n)_{1 \leq n \leq N}\) in \(L^2(\Omega)^N\), such that
\[
u = \nabla u^0 \quad \text{a.e. in } \Omega, \tag{2.9}\]
and that, for all \(n, 1 \leq n \leq N\), and for all \((v, q)\) in \(H^1_0(\Omega)^d \times L^2_0(\Omega),\)
\[
(u^n, v) + \nu \tau_n (\nabla u^n, \nabla v) - \tau_n (\text{div} \ v, p^n) = \tau_n (f^n, v) + (u^{n-1}, v),
-(\text{div} u^n, q) = 0. \tag{2.10}\]

Note that, up to a zero-order term, the problem for each \(n\) is a stationary Stokes problem, so that the following result is standard. It requires the discrete analogue of the norm introduced in (2.4), which is defined on \(Y_T(\Omega)^d\) and for all \(n, 1 \leq n \leq N\), by
\[
\|v_T^n\|_n = \left(\|v^n\|_{L^2(\Omega)^d}^2 + \nu \sum_{m=1}^n \tau_m \|\nabla v^m\|_{L^2(\Omega)^{d \times d}}^2\right)^{1/2}. \tag{2.11}\]

**Proposition 2.2.** For any data \((f, u_0)\) in \(\mathcal{K}^0(0, T; H^{-1}(\Omega)^d) \times L^2(\Omega)^d\), problem (2.9)–(2.10) has a unique solution \((u^n, (p^n)_{1 \leq n \leq N})\), which satisfies for all \(n, 1 \leq n \leq N,\)
\[
\|u_T^n\|_n \leq \left(\frac{1}{L} \sum_{m=1}^n \tau_m \|f^m\|_{H^{-1}(\Omega)^d}^2 + \|u_0\|_{L^2(\Omega)^d}^2\right)^{1/2}. \tag{2.12}\]
Moreover this solution is such that, for all \(n, 1 \leq n \leq N,\)
\[
\left(\sum_{m=1}^n \tau_m \left\|\frac{u^m - u^{m-1}}{\tau_m} + \nabla p^m\right\|^2_{H^{-1}(\Omega)^d}\right)^{1/2} \leq 2 \left(\sum_{m=1}^n \tau_m \|f^m\|_{H^{-1}(\Omega)^d}^2 + \frac{\nu}{2} \|u_0\|_{L^2(\Omega)^d}^2\right)^{1/2}. \tag{2.13}\]

**Proof.** The existence and uniqueness of a solution \((u^n, p^n)\) for each \(n, 1 \leq n \leq N,\) is derived from the standard arguments for the Stokes problem \([9]\) (Chap. I, Th. 5.1), namely the ellipticity of the form involving the gradients and the inf-sup condition on the form for the divergence. In order to derive estimate (2.12), we take \(v\) equal to \(u^n\) in the first equation of (2.10) and \(q\) equal to \(p^n\) in the second line. This yields
\[
\|u^n\|_{L^2(\Omega)^d}^2 + \nu \tau_n \|\nabla u^n\|_{L^2(\Omega)^{d \times d}}^2 = \tau_n (f^n, u^n) + (u^{n-1}, u^n)
\leq \frac{\nu}{2} \tau_n \|\nabla u^n\|_{L^2(\Omega)^{d \times d}}^2 + \frac{1}{2} \tau_n \|f^n\|_{H^{-1}(\Omega)^d}^2,
+ \frac{1}{2} \|u^n\|_{L^2(\Omega)^d}^2 + \frac{1}{2} \|u^{n-1}\|_{L^2(\Omega)^d}^2,
\]
whence
\[
\|u^n\|_{L^2(\Omega)^d}^2 + \nu \tau_n \|\nabla u^n\|_{L^2(\Omega)^{d \times d}}^2 \leq \frac{\tau_n}{\nu} \|f^n\|_{H^{-1}(\Omega)^d}^2 + \|u^{n-1}\|_{L^2(\Omega)^d}^2,
\]
Replacing \(n\) by \(m\) and summing on \(m\) gives (2.12). On the other hand, we derive from the first line in (2.10) that
\[
\left\|\frac{u^n - u^{n-1}}{\tau_n} + \nabla p^n\right\|_{H^{-1}(\Omega)^d} = \sup_{v \in H^1_0(\Omega)^d} \frac{(f^n, v) - \nu (\nabla u^n, \nabla v)}{\|\nabla v\|_{L^2(\Omega)^{d \times d}}},
\]
which gives
\[
\left\|\frac{u^n - u^{n-1}}{\tau_n} + \nabla p^n\right\|_{H^{-1}(\Omega)^d} \leq \|f^n\|_{H^{-1}(\Omega)^d} + \nu \|\nabla u^n\|_{L^2(\Omega)^{d \times d}}.
\]
Multiplying the square of this inequality by \(\tau_n\), summing on \(n\) and using (2.12) leads to (2.13). \(\square\)
The following lemma is of great use in what follows.

**Lemma 2.3.** The following equivalence property holds for any family \((u^n)_{0 \leq n \leq N}\) in \((H^1(\Omega)^d)^{N+1}\) and for all \(n, 1 \leq n \leq N\),

\[
\frac{1}{4} [\|v\|^2(t_n) \leq [v_{\tau}]^2(t_n) \leq \frac{1 + \sigma_{\tau}}{2} [\|v\|^2(t_n) + \tau] \|\text{grad } v^0\|_{L^2(\Omega)^d}^2. \tag{2.14}
\]

**Proof.** In view of the definitions (2.4) and (2.11), we have to compare the quantities

\[
X_m = \int_{t_{m-1}}^{t_m} \|\text{grad } v(\cdot, s)\|_{L^2(\Omega)^{d \times d}}^2 \, ds \quad \text{and} \quad Y_m = \tau_m \|\text{grad } v^m\|_{L^2(\Omega)^{d \times d}}^2.
\]

Since \(\|\text{grad } v(\cdot, s)\|_{L^2(\Omega)^{d \times d}}^2\) is a quadratic function of \(s\), using for instance the Simpson formula gives

\[
X_m = \frac{7}{3} m \left(\|\text{grad } v^m\|_{L^2(\Omega)^{d \times d}}^2 + \|\text{grad } v^{m-1}\|_{L^2(\Omega)^{d \times d}}^2 + \langle \text{grad } v^m, \text{grad } v^{m-1}\rangle\right).
\]

Using both inequalities \(ab \geq -\frac{1}{4} a^2 - b^2\) and \(ab \leq \frac{1}{4} a^2 + \frac{1}{2} b^2\) gives

\[
\frac{1}{4} Y_m \leq X_m \leq \frac{7}{2} m \left(\|\text{grad } v^m\|_{L^2(\Omega)^{d \times d}}^2 + \|\text{grad } v^{m-1}\|_{L^2(\Omega)^{d \times d}}^2\right).
\]

The following a priori error estimate is derived in a standard way, see [9] (Chap. V, Th. 2.1): If the velocity \(u\) of problem (2.2)--(2.3) belongs to the space \(H^1(0, T; \Sigma \Omega) \cap H^2(0, T; \Sigma \Omega)^d\), for \(0 \leq t \leq T\),

\[
[u - \hat{u}]_{T}\{(t) \leq c(u) |T|.
\]

The constant \(c(u)\) depends on the norm of \(u\) in the space given above. This estimate, however, is in general not realistic since the required regularity only holds under very strong additional non local compatibility conditions on the data (cf. [10]). Nevertheless, by invoking an appropriate regularization process, it yields the convergence of \(u_{\tau}\) to \(u\) in the norm \(|\cdot|/\tau|\) when \(\|\tau\|\) tends to zero.

To conclude, we recall the regularity property of the solution of problem (2.9)--(2.10): If the data \((f, u_0)\) belong to \(C^{0}(0, T; \Sigma \Omega)^d \times \Sigma\), the family \((u^n, p^n)_{1 \leq n \leq N}\) belongs to \((H^{1+1}(\Omega)^d \times H^4(\Omega))^N\) and satisfies for all \(n, 1 \leq n \leq N\),

\[
\left(\|u^n\|_{H^1(\Omega)^d}^2 + \sum_{m=1}^{n} \tau_m \|\text{grad } v^m\|_{L^2(\Omega)^{d \times d}}^2\right)^{\frac{1}{2}} \leq c \left(\sum_{m=1}^{n} \tau_m \|f^m\|_{L^2(\Omega)^{d}}^2 + \|u_0\|_{H^4(\Omega)^d}^2\right)^{\frac{1}{2}}.
\]

Here, the exponent \(s\) is equal to \(\frac{1}{2}\) for an arbitrary domain \(\Omega\) and to 1 for a convex domain \(\Omega\). When \(\Omega\) is a non-convex polygon or a polyhedron, the previous estimate holds for some \(s > \frac{1}{2}\) depending on \(\Omega\). Moreover it holds for \(s = 1\) in \(\Omega \setminus \mathcal{V}\) where \(\mathcal{V}\) is a neighbourhood of the re-entrant corners or edges of \(\Omega\).

**The time and space discrete problem**

We now describe the space discretization of problem (2.9)--(2.10). For each \(n, 0 \leq n \leq N\), let \((T_{nh})_h\) be a regular family of triangulations of \(\Omega\) by closed triangles (in dimension \(d = 2\)) or tetrahedra (in dimension \(d = 3\)), in the usual sense that

- for each \(h\), \(\widehat{\Omega}\) is the union of all elements of \(T_{nh}\);
- for each \(h\), the intersection of two different elements of \(T_{nh}\) is either empty, or a vertex, or a whole edge, or a whole face (if \(d = 3\)) of these elements;
the maximal ratio of the diameter of an element $K$ in $T_{nh}$ to the diameter of its inscribed circle or sphere is bounded by a constant independent of $n$ and $h$.

As usual the discretization parameter $h$ denotes the maximal diameter of the elements of all $T_{nh}$, $1 \leq n \leq N$, while, for each $n$, $h_n$ denotes the maximal diameter of the elements of $T_{nh}$. In all that follows, $c$ stands for a constant that may vary from a line to the next but which is always independent of $h_n$ and $n$.

We introduce two finite-dimensional subspaces $X_{nh}$ of $H^1_0(\Omega)^d$ and $M_{nh}$ of $L^2_0(\Omega)$ and we assume that the following properties hold:

(i) the space $X_{nh}$ contains the space $Z^d_{nh}$, with

\[ Z_{nh} = \{ v_h \in H^1_0(\Omega); \forall K \in T_{nh}, \ v_h|_K \in P_1(K) \}, \]  

(2.17)

where $P_1(K)$ denotes the space of restrictions to $K$ of affine functions in $\mathbb{R}^d$. (ii) for each $h$ and $n$, $1 \leq n \leq N$, there exists a constant $\beta_{nh} > 0$ such that the following inf-sup condition holds for all $q_h$ in $M_{nh}$

\[ \sup_{v_h \in X_{nh}} \frac{(\text{div} \, v_h, q_h)}{\| \text{grad} \, v_h \|_{L^2(\Omega)^{d \times d}}} \geq \beta_{nh} \| q_h \|_{L^2(\Omega)}. \]  

(2.18)

These assumptions are not at all restrictive, since they are satisfied for all the finite elements used for the Stokes problem, see [9] (Chap. II, Sect. 2). Note that the inf-sup condition (2.18) guarantees the well-posedness of the discrete problems and is sufficient for our a posteriori error analysis. Optimal a priori error estimates, however, can most often be derived only under the stronger condition that the constants $\beta_{nh}$ are uniformly bounded away from $0$.

Let $\Pi_h$ denote a projection operator from $L^2(\Omega)^d$ onto $X_{0h}$. The fully discrete problem constructed from problem (2.9)–(2.10) by the Galerkin method is the following one:

Find $(u^n_h)_{0 \leq n \leq N}$ in $\prod_{n=0}^N X_{nh}$ and $(p^n_h)_{1 \leq n \leq N}$ in $\prod_{n=1}^N M_{nh}$ such that

\[ u^n_h = \Pi_h u^n_0 \quad \text{a.e. in } \Omega, \]  

(2.19)

and that, for all $n$, $1 \leq n \leq N$, and for all $(v_h, q_h)$ in $X_{nh} \times M_{nh}$,

\[ (u^n_h, v_h) + \nu \tau_n (\text{grad} \, u^n_h, \text{grad} \, v_h) - \tau_n (\text{div} \, u^n_h, p^n_h) = \tau_n (f^n, v_h) + (u^{n-1}_h, v_h), \]

\[ -(\text{div} \, u^n_h, q_h) = 0. \]  

(2.20)

The same arguments as for problem (2.9)–(2.10) lead to the following statement. We omit the proof since it is exactly the same as that of Proposition 2.2.

**Proposition 2.4.** For any data $(f, u_0)$ in $C^0(0,T;H^{-1}(\Omega)^d) \times L^2(\Omega)^d$, problem (2.19)–(2.20) has a unique solution $(u^n_h, (u^n_h, p^n_h))_{1 \leq n \leq N}$, which satisfies for all $n$, $1 \leq n \leq N$,

\[ \|[u_{nh}](t_n)\| \leq \left( \frac{1}{\nu} \sum_{m=1}^n \tau_m \| f^m \|_{H^{-1}(\Omega)^d} + \| \Pi_h u_0 \|_{L^2(\Omega)^d}^2 \right)^{\frac{1}{2}}. \]  

(2.21)

We refer to [10] for the proof of the convergence of the discrete solution towards the exact one and also for a priori error estimates which are optimal whenever the constant $\beta_{nh}$ in (2.18) is independent of $h$ and $n$.

To conclude, we introduce the discrete kernel

\[ V_{nh} = \{ v_h \in X_{nh}; \forall q_h \in M_{nh}, (\text{div} \, v_h, q_h) = 0 \}, \]  

(2.22)

and recall that one of the main difficulties of the numerical analysis of the previous problem is that, for most finite elements, $V_{nh}$ is not contained in $V$. 
3. A POSTERIORI ANALYSIS OF THE TIME DISCRETIZATION

In analogy to [1] (see also [11] for the basic idea in a different context and [12] for analogous results for the heat equation), we define for each \( n, 1 \leq n \leq N \), the error indicator

\[
\eta^n = \left(\frac{\tau_n}{3} \nu\right)^{1/2} \| \text{grad} (u^n_h - u_h^n) \|_{L^2(\Omega)^d}.
\]

(3.1)

It can be observed that, once the discrete velocity \((u_h^n)_{0 \leq n \leq N}\) is known, the previous error indicators are very easy to compute.

From now on we assume that the data \((f, u_0)\) belong to \( H^0(0,T; H^{-1}(\Omega)^d) \times V \). In order to prove the a posteriori error estimate, we introduce the operator \( \pi_r \): For any Banach space \( X \) and any function \( g \) continuous from \([0,T]\) into \( X \), \( \pi_r g \) denotes the step function which is constant and equal to \( g(t_n) \) on each interval \([t_{n-1}, t_n] \), \( 1 \leq n \leq N \). Similarly, we denote for any sequence \((\varphi^n)_{1 \leq n \leq N}\) in \( X_N \) by \( \pi_r \varphi \) the step function which is constant and equal to \( \varphi^n \) on each interval \([t_{n-1}, t_n] \), \( 1 \leq n \leq N \). By combining problems (2.2)–(2.3) and (2.9)–(2.10), we observe that the pair \((u - u_r, p - \pi_r p_r)\) satisfies

\[
(u - u_r)(\cdot, 0) = 0 \quad \text{a.e. in } \Omega,
\]

(3.2)

and that, for \( 1 \leq n \leq N \), for a.e. \( t \) in \([t_{n-1}, t_n] \) and for all \((v, q)\) in \( H^0_0(\Omega)^d \times L^2(\Omega), \)

\[
(\partial_t (u - u_r), v) + \nu (\text{grad} (u - u_r), \text{grad} v) - (\text{div} v, p - \pi_r p_r) = (f - \pi_r f, v) + \nu (\text{grad} (u^n - u_r), \text{grad} v),
\]

\[
-(\text{div} (u - u_r), q) = 0.
\]

(3.3)

The a posteriori estimate can be derived from this residual equation by quite standard arguments.

**Proposition 3.1.** The following a posteriori error estimate holds between the velocity \( u \) of problem (2.2)–(2.3) and the velocity \( u_r \) associated with the solution \((u^n)_{0 \leq n \leq N}\) of problem (2.9)–(2.10), for \( 1 \leq n \leq N \),

\[
[u - u_r](t_n) \leq \left( \frac{6}{n} \sum_{m=1}^{n} (\eta^m)^2 + 12 \nu \| u_r - u_{hR} \|_{L^2(0,t_n; H^1(\Omega)^d)}^2 + \frac{2}{\nu} \| f - \pi_r f \|_{L^2(0,t_n; H^{-1}(\Omega)^d)}^2 \right)^{1/2}.
\]

(3.4)

**Proof.** By taking \( v \) equal to \( u - u_r \) and \( q \) equal to \( p - \pi_r p_r \) in (3.3) and subtracting the second line from the first one, we obtain

\[
\frac{1}{2} \frac{d}{dt} \| u - u_r \|_{L^2(\Omega)^d}^2 + \nu \| \text{grad} (u - u_r) \|_{L^2(\Omega)^{d \times d}}^2 \\
\leq \left( \frac{1}{\nu^2} \| f - \pi_r f \|_{H^{-1}(\Omega)^d}^2 + \nu \| \text{grad} (u^n - u_r) \|_{L^2(\Omega)^{d \times d}}^2 \right) \nu^{1/2} \| \text{grad} (u - u_r) \|_{L^2(\Omega)^{d \times d}},
\]

whence

\[
\frac{d}{dt} \| u - u_r \|_{L^2(\Omega)^d}^2 + \nu \| \text{grad} (u - u_r) \|_{L^2(\Omega)^{d \times d}}^2 \leq 2 \left( \frac{1}{\nu^2} \| f - \pi_r f \|_{H^{-1}(\Omega)^d}^2 + \nu \| \text{grad} (u^n - u_r) \|_{L^2(\Omega)^{d \times d}}^2 \right).
\]

By integrating this inequality between \( t_{n-1} \) and \( t_n \), summing on \( n \) and using (3.2), we derive

\[
[u - u_r]^2(t_n) \leq 2 \left( \frac{1}{\nu} \| f - \pi_r f \|_{L^2(0,t_n; H^{-1}(\Omega)^d)} + \nu \sum_{m=1}^{n} \int_{t_{m-1}}^{t_m} \| \text{grad} (u^n - u_r)(s) \|_{L^2(\Omega)^{d \times d}} ds \right).
\]

(3.5)
Next, we note that, for all \( t \) in \([t_{m-1}, t_m]\),
\[
(u^m - u^*)(t) = \frac{t_m - t}{\tau_m} (u^m - u^{m-1}),
\]
so that
\[
\int_{t_{m-1}}^{t_m} \| \nabla (u^m - u^*)(., s) \|_{L^2(\Omega)^d \times d}^2 \, ds = \frac{\tau_m}{3} \| \nabla (u^m - u^{m-1}) \|_{L^2(\Omega)^d \times d}^2.
\]  
(3.6)

We then use a triangle inequality
\[
\frac{\tau_m}{3} \| \nabla (u^m - u^{m-1}) \|_{L^2(\Omega)^d \times d}^2 \leq \frac{3}{\nu} \eta_m^2 + \tau_m \| \nabla (u^m - u_h^m) \|_{L^2(\Omega)^d \times d}^2 + \tau_m \| \nabla (u^{m-1} - u_h^{m-1}) \|_{L^2(\Omega)^d \times d}^2.
\]

When applying the arguments used in the proof of Lemma 2.3 to the second and third terms in the right-hand side of this estimate, we obtain
\[
\frac{\tau_m}{3} \| \nabla (u^m - u^{m-1}) \|_{L^2(\Omega)^d \times d}^2 \leq \frac{3}{\nu} \eta_m^2 + 6 \int_{t_{m-1}}^{t_m} \| \nabla (u^m - u^*)(., s) \|_{L^2(\Omega)^d \times d}^2 \, ds.
\]  
(3.7)

Inserting (3.6) and (3.7) into (3.5) yields the desired result.

A further argument is needed to prove a similar estimate concerning the function \( \partial_t (u - u^*) + \nabla (p - \pi^p) \) in the norm of \( H^{-1}(\Omega) \).

**Corollary 3.2.** The following a posteriori error estimate holds between the solution \((u, p)\) of problem (2.2)–(2.3) and the pair \((u^*, \pi^p)\) associated with the solution of problem (2.9)–(2.10), for \( 1 \leq n \leq N \),
\[
\| \partial_t (u - u^*) + \nabla (p - \pi^p) \|_{L^2(0, t_n; H^{-1}(\Omega)^d)} \leq 3 \left( 3 \nu \sum_{m=1}^{n} (\eta_m^m)^2 + 6 \nu^2 \| u^m - u_h^m \|^2_{L^2(0, t_n; H^1(\Omega)^d)} + \| f - \pi^p f \|^2_{L^2(0, t_n; H^{-1}(\Omega)^d)} \right)^{\frac{1}{2}}.
\]  
(3.8)

**Proof.** We have
\[
\| \partial_t (u - u^*) + \nabla (p - \pi^p) \|_{H^{-1}(\Omega)^d} = \sup_{v \in H^1(\Omega)^d} \frac{\langle \partial_t (u - u^*), v \rangle - \langle \nabla v, p - \pi^p \rangle}{\| \nabla v \|_{L^2(\Omega)^d \times d}}.
\]

Using the first equation in (3.3) yields that, for any \( t \) in \([t_{n-1}, t_n]\),
\[
\langle \partial_t (u - u^*), v \rangle - \langle \nabla v, p - \pi^p \rangle = (f - \pi^p f, v) + \nu \langle \nabla (u^n - u^*), \nabla v \rangle - \nu \langle \nabla (u - u^*), \nabla v \rangle,
\]
whence
\[
\| \partial_t (u - u^*) + \nabla (p - \pi^p) \|_{L^2(0, t_n; H^{-1}(\Omega)^d)} \leq (3 \| f - \pi^p f \|^2_{L^2(0, t_n; H^{-1}(\Omega)^d)})^{\frac{1}{2}} + 3 \nu^2 \sum_{m=1}^{n} \| \nabla (u^m - u^*) \|^2_{L^2(t_m-1, t_m; L^2(\Omega)^d \times d)} + 3 \nu \| u - u^* \|^2_{L^2(t_n)}.
\]

The second term in the right-hand side can be estimated by the same arguments as in the proof of Proposition 3.1, see (3.6) and (3.7), and the last one is bounded in (3.4). This concludes the proof. \( \square \)
We now establish an upper bound for each indicator $\eta^n$.

**Proposition 3.3.** The following estimate holds for each indicator $\eta^n$ introduced in (3.1), $1 \leq n \leq N$,

$$
\eta^n \leq \nu_2^\frac{1}{2} \| \nabla (u - u_\tau) \|_{L^2(t_{n-1}, t_n; L^2(\Omega)^{d \times d})} + \frac{1}{\nu_2^2} \| \partial_t (u - u_\tau) + \nabla (p - \pi_\tau p_\tau) \|_{L^2(t_{n-1}, t_n; H^{-1}(\Omega)^d} \\
+ \frac{1}{\nu_2^2} \| f - \pi_\tau f \|_{L^2(t_{n-1}, t_n; H^{-1}(\Omega)^d)} + \left( \frac{\tau_n}{3} \nu \right)^\frac{1}{2} \left( \| \nabla (u^n - u^{n-1}_h) \|_{L^2(\Omega)^{d \times d}} + \| \nabla (u^n - u^{n-1}_h) \|_{L^2(\Omega)^{d \times d}} + \| \nabla (u^n - u^{n-1}_h) \|_{L^2(\Omega)^{d \times d}} \right).
$$

(3.9)

**Proof.** Thanks to a triangle inequality, we have

$$
\eta^n = \left( \frac{\tau_n}{3} \nu \right)^\frac{1}{2} \left( \| \nabla (u^n - u^{n-1}) \|_{L^2(\Omega)^{d \times d}} + \| \nabla (u^n - u^{n-1}_h) \|_{L^2(\Omega)^{d \times d}} + \| \nabla (u^n - u^{n-1}_h) \|_{L^2(\Omega)^{d \times d}} \right).
$$

In order to bound the first term, we derive from (3.6) that

$$
\frac{\tau_n}{3} \nu \| \nabla (u^n - u^{n-1}) \|_{L^2(\Omega)^{d \times d}} = \nu \int_{t_{n-1}}^{t_n} \| \nabla (u^n - u_\tau)(\cdot, s) \|_{L^2(\Omega)^{d \times d}} ds.
$$

Next, in the first line of (3.3), we take $v$ equal to $u^n - u_\tau(t)$, and we integrate between $t_{n-1}$ and $t_n$. This gives

$$
\nu \int_{t_{n-1}}^{t_n} \| \nabla (u^n - u_\tau)(\cdot, s) \|_{L^2(\Omega)^{d \times d}} ds \leq \left( \nu \| \nabla (u^n - u_\tau) \|_{L^2(t_{n-1}, t_n; L^2(\Omega)^{d \times d})} \\
+ \| \partial_t (u - u_\tau) + \nabla (p - \pi_\tau p_\tau) \|_{L^2(t_{n-1}, t_n; H^{-1}(\Omega)^d)} \\
+ \| f - \pi_\tau f \|_{L^2(t_{n-1}, t_n; H^{-1}(\Omega)^d)} \right) \left( \int_{t_{n-1}}^{t_n} \| \nabla (u^n - u_\tau)(s) \|_{L^2(\Omega)^{d \times d}} ds \right)^\frac{1}{2},
$$

or equivalently

$$
\left( \frac{\tau_n}{3} \nu \right)^\frac{1}{2} \| \nabla (u^n - u^{n-1}) \|_{L^2(\Omega)^{d \times d}} \leq \nu_2^\frac{1}{2} \| \nabla (u - u_\tau) \|_{L^2(t_{n-1}, t_n; L^2(\Omega)^{d \times d})} \\
+ \frac{1}{\nu_2^2} \| \partial_t (u - u_\tau) + \nabla (p - \pi_\tau p_\tau) \|_{L^2(t_{n-1}, t_n; H^{-1}(\Omega)^d)} \\
+ \frac{1}{\nu_2^2} \| f - \pi_\tau f \|_{L^2(t_{n-1}, t_n; H^{-1}(\Omega)^d)}.
$$

This leads to the desired result. \qed

It follows from Proposition 3.1, Corollary 3.2 and Proposition 3.3 that, if the regularity parameter $\sigma_\tau$ is bounded independently of $\tau$, the full error

$$
|u - u_\tau(t_n) + \nu_2^\frac{1}{2} \| \partial_t (u - u_\tau) + \nabla (p - \pi_\tau p_\tau) \|_{L^2(\Omega, t_n; H^{-1}(\Omega)^d)}
$$

is equivalent to the quantity $\left( \sum_{n=1}^{N} (\eta^n)^2 \right)^\frac{1}{2}$, up to some terms involving the approximation of the data, namely the distance of $f$ to $\pi_\tau f$ in an appropriate norm, and the spatial error $|u_\tau - u_h\tau(t_n)$. Moreover the equivalence constants are explicitly known and estimate (3.9) is local with respect to the time variable.
4. A POSTERIORI ANALYSIS OF THE SPACE DISCRETIZATION

In order to describe the family of space error indicators, we first need some notation. For each \( n \) and each \( K \) in \( T_{nh} \), we introduce the set \( \mathcal{E}_K \) of edges \((d = 2)\) or faces \((d = 3)\) of \( K \) which are not contained in \( \partial \Omega \). With any \( e \in \mathcal{E}_K \) we associate a unit vector \( \mathbf{n}_e \) orthogonal to \( e \) and denote by \([.]_e\) the jump across \( e \) in the direction \( \mathbf{n}_e \). Note that \([.]_e\) depends on the orientation of \( \mathbf{n}_e \) but that quantities like \([\mathbf{v} \cdot \mathbf{n}_e]_e\) for any vector field \( \mathbf{v} \) are independent thereof. For each \( K \), we denote by \( n_K \) the unit outward normal to \( K \) on \( \partial K \). Finally, \( h_K \) stands for the diameter of \( K \) and, for each \( e \in \mathcal{E}_K \), \( h_e \) denotes its length \((d = 2)\) or diameter \((d = 3)\).

Moreover it can be observed that, for any \((u, v)\) \( \in \mathcal{E}_K \), \( f \) belongs to \( H^{-1}(\Omega) \) and it is given by

\[
\eta^n_K = \nu^{-\frac{1}{2}} \left( h_K \| f^n - \frac{u^n - u^n_{n-1}}{\tau_n} \| + \nu \Delta u^n K_{\mathcal{E}_K} \| L_2(K) \right) + \sum_{e \in \mathcal{E}_K} h_e^{1/2} \| [\nu \cdot \mathbf{n}_e]_{u^n - p^n} \| L_2(e) \| + \nu \| \mathbf{u}^n \| L_2(K) \right). \tag{4.1}
\]

For any \( n, 1 \leq n \leq N \), and any \((\mathbf{v}, q)\) \( \in H^1_0(\Omega)^d \times L^2_0(\Omega) \), we obtain from problems (2.9)–(2.10) and (2.19)–(2.20) the following residual equation

\[
(u^n - u^n_{h}, \mathbf{v}) + \nu \tau_n (\mathbf{u}^n_{h}, \mathbf{v}) - \tau_n (\mathbf{v} \cdot \mathbf{u}^n - p^n, \mathbf{v}) = (F^n, \mathbf{v}) + (\mathbf{u}^{n-1} - u^n - v^n, \mathbf{v}),
\]

\[
-(\mathbf{v} \cdot \mathbf{u}^n - u^n_{h}, \mathbf{v}) = (\mathbf{v} \cdot \mathbf{u}^n_{h}, \mathbf{v}). \tag{4.2}
\]

Here, the residual \( F^n \) belongs to \( H^{-1}(\Omega)^d \) and is given by

\[
(F^n, \mathbf{v}) = \tau_n (F^n, \mathbf{v}) - (u^n_{h} - u^n_{h-1}, \mathbf{v}) - \nu \tau_n (\mathbf{u}^n_{h}, \mathbf{v}) + \tau_n (\mathbf{v} \cdot \mathbf{u}^n_{h}, \mathbf{v}). \tag{4.3}
\]

Moreover it can be observed that, for any \((\mathbf{v}_h, q_h)\) \( \in X_{nh} \times M_{nh} \),

\[
(F^n, \mathbf{v}) = (F^n, \mathbf{v} - \mathbf{v}_h) \quad \text{and} \quad (\mathbf{v} \cdot \mathbf{u}^n_{h}, q) = (\mathbf{v} \cdot \mathbf{u}^n_{h}, q - q_h). \tag{4.4}
\]

A further lemma is needed to handle the non-zero right-hand side of the second line in equation (4.2). It requires the following assumption which is satisfied by all the finite elements used for the Stokes problem.

**Assumption 4.1.** The space \( M_{nh} \) contains either \( M_{nh}^0 \) or \( M_{nh}^1 \), with

\[
M_{nh}^0 = \{ q_h \in L^2_0(\Omega); \forall K \in T_{nh}, q_h|_K \in P_0(K) \},
\]

\[
M_{nh}^1 = \{ v_h \in H^1(\Omega) \cap L^2_0(\Omega); \forall K \in T_{nh}, v_h|_K \in P_1(K) \},
\]

where \( P_0(K) \) denotes the space of constant functions on \( K \).

Let now \( \Pi \) denote the operator defined from \( H^1_0(\Omega)^d \) into itself as follows: For each \( \mathbf{v} \) \( \in H^1_0(\Omega)^d \), \( \Pi \mathbf{v} \) denotes the velocity \( \mathbf{w} \) of the unique weak solution \((\mathbf{w}, r)\) \( \in H^1_0(\Omega)^d \times L^2_0(\Omega) \) of the Stokes problem

\[
\begin{aligned}
-\Delta \mathbf{w} + \nabla r &= 0 \quad \text{in } \Omega, \\
\text{div } \mathbf{w} &= \text{div } \mathbf{v} \quad \text{in } \Omega, \\
\mathbf{w} &= \mathbf{0} \quad \text{on } \partial \Omega.
\end{aligned}
\]
The next lemma states some properties of the operator \( \Pi \) and refers to the constant \( \beta > 0 \) which is defined by

\[
\beta = \inf_{q \in L^2_0(\Omega)} \sup_{\nu \in H^1(\Omega)^d} \frac{(\text{div} \nu, q)}{\|\nabla \nu\|_{L^2(\Omega)^{d \times d}} \|q\|_{L^2(\Omega)}}. 
\] (4.6)

**Lemma 4.2.** The operator \( \Pi \) has the following properties:

(i) For all \( \nu \) in \( V \), \( \Pi \nu \) is zero.

(ii) The following estimates hold for all \( \nu \) in \( H^1_0(\Omega)^d \),

\[
\|\nabla (\nu - \Pi \nu)\|_{L^2(\Omega)^{d \times d}} \leq \|\nabla \nu\|_{L^2(\Omega)^{d \times d}} \quad \text{and} \quad \|\nabla \Pi \nu\|_{L^2(\Omega)^{d \times d}} \leq \frac{1}{\beta} \|\text{div} \nu\|_{L^2(\Omega)},
\] (4.7)

(iii) If Assumption 4.1 is satisfied, the following estimate holds for \( 1 \leq n \leq N \) and for any \( \nu_h \) in \( V_{nh} \),

\[
\|\Pi \nu_h\|_{L^2(\Omega)^d} \leq c \|\nabla \nu_h\|_{L^2(\Omega)},
\] (4.8)

where the constant \( \alpha \) equals 1 if \( \Omega \) is convex, and equals \( \frac{1}{2} \) otherwise.

**Proof.** Part (i) of the Lemma is obvious. Moreover, since \( \nu - \Pi \nu \) has vanishing divergence, we conclude from the weak form of the Stokes problem that

\[
(\nabla \Pi \nu, \nabla (\nu - \Pi \nu)) = (\text{div} (\nu - \Pi \nu), r) = 0.
\]

This proves the first estimate in (4.7). Similarly, we obtain

\[
\|\nabla \Pi \nu\|_{L^2(\Omega)^{d \times d}}^2 = (\nabla \Pi \nu, \nabla \Pi \nu) = (\text{div} \nu, r) = \|\text{div} \nu\|_{L^2(\Omega)} \|r\|_{L^2(\Omega)},
\]

and

\[
\beta \|r\|_{L^2(\Omega)} \leq \sup_{z \in H^1_0(\Omega)^d} \frac{(\text{div} z, r)}{\|\nabla z\|_{L^2(\Omega)^{d \times d}}} = \sup_{z \in H^1_0(\Omega)^d} \frac{(\nabla \Pi \nu, \nabla z)}{\|\nabla \Pi \nu\|_{L^2(\Omega)^{d \times d}}} \|\nabla z\|_{L^2(\Omega)^{d \times d}}.
\]

This proves the second part of (4.7).

To derive the last estimate, for any \( \nu_h \) in \( V_{nh} \), we use a duality argument, that relies on the Stokes problem

\[
\begin{cases}
-\Delta \varphi + \nabla \rho = \Pi \nu_h & \text{in } \Omega, \\
\text{div } \varphi = 0 & \text{in } \Omega, \\
\varphi = 0 & \text{on } \partial \Omega.
\end{cases}
\]

This problem has a unique solution \( (\varphi, \rho) \) too. Moreover, for the \( \alpha \) introduced in the statement of the lemma, this solution belongs to the space \( H^{\alpha+1}(\Omega)^d \times H^\alpha(\Omega) \) and satisfies

\[
\|\varphi\|_{H^{\alpha+1}(\Omega)^d} + \|\rho\|_{H^\alpha(\Omega)} \leq c \|\Pi \nu_h\|_{L^2(\Omega)^d}.
\] (4.9)

By combining the two previous problems, we have

\[
\|\Pi \nu_h\|_{L^2(\Omega)^d}^2 = (\Pi \nu_h, \Pi \nu_h) = (\nabla \varphi, \nabla \Pi \nu_h) - (\text{div} \Pi \nu_h, \rho) = (\text{div} \varphi, r) - (\text{div} \Pi \nu_h, \rho).
\]

Since \( \text{div} \varphi \) is zero and \( \text{div} \Pi \nu_h \) is equal to \( \text{div} \nu_h \), this yields

\[
\|\Pi \nu_h\|_{L^2(\Omega)^d}^2 = - (\text{div} \nu_h, \rho).
\]
Using the definition of $V_{nh}$ gives, for any $\rho_h$ in $M_{nh}$,
\[
\|\Pi v_h\|_{L^2(\Omega)^d}^2 = -(\text{div} \, v_h, \rho - \rho_h) \leq \|\text{div} \, v_h\|_{L^2(\Omega)} \|\rho - \rho_h\|_{L^2(\Omega)}.
\]

Thanks to Assumption 4.1, standard approximation properties combined with (4.9) lead to the desired estimate.

\[
\text{Remark 4.3. When } \Omega \text{ is not convex, estimate (4.8) can be improved. Choose in the proof of Lemma 4.2 } \rho_h \text{ equal to the image of } \rho \text{ under a Clément type regularization operator (cf. e.g. [6], [18]). Denote by } \mathcal{V} \text{ a fixed neighbourhood of the re-entrant corners or edges of } \Omega. \text{ Then, in estimate (4.8), the function } \Pi v_h \text{ satisfies}
\]
\[
\|\Pi v_h\|_{L^2(\Omega)^d} \leq c \left( \sum_{K \in T_{nh}} h_{K}^{\alpha_K} \|\text{div} \, v_h\|_{L^2(K)}^2 \right)^{\frac{1}{2}},
\]
with $\alpha_K$ equal to 1 if $K$ does not intersect $\mathcal{V}$, and equal to $\frac{1}{2}$ otherwise. Moreover, if $\Omega$ is a polygon, the $\alpha_K$ for the $K$ intersecting $\mathcal{V}$ can be computed explicitly as a function of the largest angle of $\Omega$ in a neighbourhood of $K$.

In the next lemma, we evaluate the quantity $(F^n, v - v_h)$ which appears in (4.4).

\[
\text{Lemma 4.4. For any function } v \in H^1_0(\Omega)^d, \text{ the following estimate holds}
\]
\[
\inf_{v_h \in X_{nh}} (F^n, v - v_h) \leq c \tau_n \left( \sum_{K \in T_{nh}} (\nu(n_K^n)^2 + h_K^2 \|f^n - f_h^n\|_{L^2(K)^d}^2) \right)^{\frac{1}{2}} \|\text{grad} \, v\|_{L^2(\Omega)^d}. \tag{4.11}
\]

\text{Proof. By integrating by parts on each } K, \text{ we obtain}
\[
(F^n, v - v_h) = \tau_n \sum_{K \in T_{nh}} \left\{ \int_K (f^n - u_h^n - u_h^{n-1}) + \nu \Delta u_h^n - \text{grad} \, p_h^n)(v - v_h) \, dx \right.
\]
\[
- \sum_{e \in E_K} \int_e (\nu \partial_n u_h^n - p_h^n n_K)(v - v_h) \, d\tau \right\}
\]
or, equivalently,
\[
(F^n, v - v_h) = \tau_n \sum_{K \in T_{nh}} \left\{ \int_K (f^n - u_h^n - u_h^{n-1}) + \nu \Delta u_h^n - \text{grad} \, p_h^n)(v - v_h) \, dx \right.
\]
\[
+ \frac{1}{2} \sum_{e \in E_K} \int_e [\nu \partial_n u_h^n - p_h^n n_e](v - v_h) \, d\tau \right\}.
\]

Thanks to the Cauchy–Schwarz inequality, this yields
\[
(F^n, v - v_h) \leq \tau_n \sum_{K \in T_{nh}} \left( \|f^n - u_h^n - u_h^{n-1}\|_{L^2(K)^d} + \nu \Delta u_h^n - \text{grad} \, p_h^n \right\|_{L^2(K)^d} \|v - v_h\|_{L^2(K)^d}
\]
\[
+ \frac{1}{2} \sum_{e \in E_K} \|\nu \partial_n u_h^n - p_h^n n_e\|_{L^2(e)^d} \|v - v_h\|_{L^2(e)^d} \right).
Next, we introduce a Clément type regularization operator $R_{n\tau}$ which has the following properties, see [6] or [18]: For any function $v$ in $H^1_0(\Omega)$, $R_{n\tau}v$ belongs to the space $Z_{n\tau}^d$ of continuous affine finite elements (cf. (2.17)) and satisfies for any $K \in T_{n\tau}$ and $e \in E_K$,

$$
\|v - R_{n\tau}v\|_{L^2(K)^d} \leq c h_K \|\text{grad} v\|_{H^1(\Delta_K)^d} \quad \text{and} \quad \|v - R_{n\tau}v\|_{L^2(\Omega)^d} \leq c h_\tau^{\frac{1}{2}} \|\text{grad} v\|_{H^1(\Delta_\tau)^d}.
$$

(4.12)

Here $\Delta_K$, resp. $\Delta_\tau$, denote the union of elements of $T_{n\tau}$ that share at least a vertex with $K$, resp. with $e$. Since $X_{n\tau}$ contains $Z_{n\tau}^d$, taking $v_h$ equal to $R_{n\tau}v$ gives

$$
(F^n, v - v_h) \leq c \tau_n \sum_{K \in T_{n\tau}} \left\{ h_K \| f^n - \frac{u_h^n - u_h^{n-1}}{\tau_n} + \nu \Delta u_h^n - \text{grad} p_h^n \|_{L^2(K)^d}
+ \frac{1}{2} \sum_{e \in E_K} h_e^{\frac{1}{2}} \| \nu \partial_n u_h^n - p_h^n n_e \|_{L^2(\tau)^d} \right\} \|\text{grad} v\|_{L^2(\Delta_K)^d \times d}.
$$

The desired estimate follows from a triangle inequality by adding and subtracting $f_h^n$ and noting that, when summing on the $K$, each element of $T_{n\tau}$ only belongs to a finite number of $\Delta_K$'s, this number being bounded as a function of the regularity parameter of the family of triangulations.

We are now in a position to prove the first a posteriori estimate. It requires the following parameter

$$
\lambda_{n\tau} = \sup_{1 \leq n \leq N} \frac{\sup_{K \in T_{n\tau}} h_{20K}^n}{\nu \tau_n}.
$$

(4.13)

**Proposition 4.5.** If Assumption 4.1 is satisfied, the following a posteriori error estimate holds between the velocity $u_\tau$ associated with the solution $(u^n)_{0 \leq n \leq N}$ of problem (2.9)–(2.10) and the velocity $u_{n\tau}$ associated with the solution $(u_h^n)_{0 \leq n \leq N}$ of problem (2.19)–(2.20), for $1 \leq n \leq N$,

$$
\|u_\tau - u_{n\tau}\|_0(\tau_n) \leq c \left( \sum_{m=1}^{n} \tau_n \sum_{K \in T_{n\tau}} \left( 1 + \lambda_{n\tau} (\eta_K)^2 \right)^{\frac{1}{2}} \| f_m^n - f_h^n \|_{L^2(\tau)^d} + \| \tau_n \|_{L^2(\tau)^d} \right)^{\frac{1}{2}}
+ c' \| u_0 - \Pi_h u_0 \|_0(\Omega).
$$

(4.14)

**Proof.** For abbreviation we set

$$
e^n = u^n - u_h^n, \quad 0 \leq n \leq N, \quad \text{and} \quad \epsilon^n = p^n - p_h^n, \quad 1 \leq n \leq N.
$$

For any $n$, $1 \leq n \leq N$, we then have

$$
\frac{1}{2} \| e^n \|_0(\Omega)^d = \frac{1}{2} \| e^{n-1} \|_0(\Omega)^d + \frac{1}{2} \| e^n - e^{n-1} \|_0(\Omega)^d + \nu \tau_n \|\text{grad} e^n \|_0(\Omega)^d \times d
= (e^n - e^{n-1}, e^n) + \nu \tau_n (\text{grad} e^n, \text{grad} e^n).
$$

(4.15)

Observing that $\text{div}(e^n - \Pi e^n) = 0$, we obtain

$$
(e^n - e^{n-1}, e^n) + \nu \tau_n (\text{grad} e^n, \text{grad} e^n) = (e^n - e^{n-1}, \Pi e^n) + \nu \tau_n (\text{grad} e^n, \text{grad} \Pi e^n)
+ (e^n - e^{n-1}, e^n - \Pi e^n) + \nu \tau_n (\text{grad} e^n, \text{grad} (e^n - \Pi e^n))
- \tau_n (\text{div}(e^n - \Pi e^n), e^n).
$$
By inserting \( v = e^n - \Pi e^n \) in equation (4.2), this yields

\[
(e^n - e^{n-1}, e^n) + \nu \tau_n (\text{grad } e^n, \text{grad } e^n) = (e^n - e^{n-1}, \Pi e^n) + \nu \tau_n (\text{grad } e^n, \text{grad } \Pi e^n) + (F^n, e^n - \Pi e^n).
\]

(4.16)

Next, we bound the three terms on the right-hand side of (4.16) separately. Taking into account that \( \Pi e^n = -\Pi u_h^n \) and using Lemma 4.2 and Remark 4.3, we obtain for the first term

\[
(e^n - e^{n-1}, \Pi e^n) \leq \frac{1}{2} \| e^n - e^{n-1} \|^2_{L^2(\Omega)^d} + \frac{1}{2} \| \Pi e^n \|^2_{L^2(\Omega)^d}
\]

\[
\leq \frac{1}{2} \| e^n - e^{n-1} \|^2_{L^2(\Omega)^d} + c \left( \sum_{K \in T_{n_h}} h_K^{2\alpha_K} \| \text{div } u_h^n \|^2_{L^2(K)} \right),
\]

whence

\[
(e^n - e^{n-1}, \Pi e^n) \leq \frac{1}{2} \| e^n - e^{n-1} \|^2_{L^2(\Omega)^d} + c \lambda_{h \nu} \nu \tau_n \| \text{div } u_h^n \|^2_{L^2(\Omega)}.
\]

(4.17)

Similarly, we derive from (4.7) the estimate for the second term

\[
\nu \tau_n (\text{grad } e^n, \text{grad } \Pi e^n) \leq \frac{\nu \tau_n}{4} \| \text{grad } e^n \|^2_{L^2(\Omega)^{d \times d}} + \nu \tau_n \| \text{grad } \Pi e^n \|^2_{L^2(\Omega)^{d \times d}}
\]

\[
\leq \frac{\nu \tau_n}{4} \| \text{grad } e^n \|^2_{L^2(\Omega)^{d \times d}} + \frac{\nu \tau_n}{\beta} \| \text{div } u_h^n \|^2_{L^2(\Omega)}.
\]

(4.18)

Finally, Lemmas 4.2 and 4.4 and the first equation in (4.4) imply the following bound for the third term

\[
(F^n, e^n - \Pi e^n) \leq c \tau_n \left( \sum_{K \in T_{n_h}} \left( \nu (\eta_K^m)^2 + h_K^2 \| F^n - f_h^n \|^2_{L^2(K)^d} \right) \right)^{\frac{1}{2}} \| \text{grad } (e^n - \Pi e^n) \|_{L^2(\Omega)^{d \times d}}
\]

\[
\leq c' \tau_n \left( \sum_{K \in T_{n_h}} \left( (\eta_K^m)^2 + \frac{h_K^2}{\nu} \| F^n - f_h^n \|^2_{L^2(K)^d} \right) \right)^{\frac{1}{2}} \| \text{grad } e^n \|^2_{L^2(\Omega)^{d \times d}}.
\]

(4.19)

Combining equations (4.15) to (4.19) and multiplying the resulting equation by 2, we arrive at

\[
\| e^n \|^2_{L^2(\Omega)^d} - \| e^{n-1} \|^2_{L^2(\Omega)^d} + \nu \tau_n \| \text{grad } e^n \|^2_{L^2(\Omega)^{d \times d}} \leq c \tau_n \left( \sum_{K \in T_{n_h}} \left( (1 + \lambda_{h \nu}) (\eta_K^m)^2 + \frac{h_K^2}{\nu} \| F^n - f_h^n \|^2_{L^2(K)^d} \right) \right).
\]

Summing with respect to \( n \) yields the desired estimate. \( \square \)

As in Corollary 3.2, we now prove an estimate for the second part of the error, which combines the time derivative of the velocity and the gradient of the pressure.

**Corollary 4.6.** If Assumption 4.1 is satisfied, the following a posteriori error estimate holds between the pair \((u_T, \pi_T p_T)\) associated with the solution of problem (2.9)–(2.10) and the pair \((u_{hT}, \pi_T p_{hT})\) associated with the solution of problem (2.19)–(2.20), for \( 1 \leq n \leq N \),

\[
\| \partial_t (u_T - u_{hT}) + \text{grad } \pi_T (p_T - p_{hT}) \|_{L^2(0, t_n; H^{-1}(\Omega)^d)}
\]

\[
\leq c \left( \sum_{m=1}^n \tau_m \sum_{K \in T_{n_h}} \left( (1 + \lambda_{h \nu}) (\eta_K^m)^2 + \frac{h_K^2}{\nu} \| F^m - f_h^m \|^2_{L^2(K)^d} \right) \right)^{\frac{1}{2}}
\]

\[
+ c' \nu \| u_0 - \Pi_h u_0 \|_{L^2(\Omega)^d}.
\]

(4.20)
Proof. We have at each time \( t \), \( 0 \leq t \leq t_n \),

\[
\| \partial_t (u_t - u_{h_t}) + \nabla \pi_t (p_t - p_{h_t}) \|_{H^{-1}(\Omega)^d} = \sup_{v \in H_0^1(\Omega)^d} \frac{\langle \partial_t (u_t - u_{h_t}), v \rangle - (\text{div} \, v, \pi_t (p_t - p_{h_t}))}{\| \nabla v \|_{L^2(\Omega)^d}^2}.
\]

Equations (4.2) and (4.4) yield that, for all \( t \), \( t_{n-1} \leq t \leq t_n \), and for any \( v_h \) in \( X_{nh} \),

\[
(\partial_t (u_t - u_{h_t}), v) - (\text{div} \, v, \pi_t (p_t - p_{h_t})) = \frac{1}{\tau_n} \langle F^n, v - v_h \rangle - \nu \langle \nabla (u^n - u^n_h), \nabla v \rangle.
\]

Thus Lemma 4.4 leads to

\[
\| \partial_t (u_t - u_{h_t}) + \nabla \pi_t (p_t - p_{h_t}) \|_{H^{-1}(\Omega)^d} \leq c \left( \sum_{K \in T_{nh}} \left( \nu \| \pi^n_K \|_1^2 + \frac{\nu}{\tau_n} \| F^n - f^n_h \|_{L^2(K)^d}^2 \right) \right)^{\frac{1}{2}} + \frac{\nu}{\tau_n} \| \nabla (u^n - u^n_h) \|_{L^2(\Omega)^d}.
\]

Note that the right-hand side of this inequality is independent of \( t \), for \( t_{n-1} \leq t \leq t_n \). So integrating the square of it between \( t_{n-1} \) and \( t_n \), summing on the \( n \), and using Proposition 4.5 give the desired estimate. \( \square \)

We now establish an upper bound for each indicator \( \eta^n_K \). We only give an abridged proof of this result, since the arguments are very similar to those used for the steady Stokes problem, see [15] (Prop. 3.19) or [3] (Prop. 8.6). Note however that this result requires the following assumption, which is not restrictive in the context of mesh adaptation.

Assumption 4.7. For \( 1 \leq n \leq N \), there exists a family of triangulations \((\tilde{T}_{nh})_h \), such that, for all \( h \) and \( n \), \( 1 \leq n \leq N \), each element \( K \) of \( T_{n-1,h} \) or of \( T_{nh} \) is the union of elements \( \tilde{K} \) of \( \tilde{T}_{nh} \) such that the ratio of the diameter of \( \tilde{K} \) and of the diameter of the largest ball inscribed into each \( \tilde{K} \) is bounded uniformly with respect to \( h \) and \( n \).

Proposition 4.8. Assume that there exists an integer \( k \) such that, for all \( h \) and for \( 1 \leq n \leq N \), the restrictions to any \( K \) in \( T_{nh} \) of all functions in \( X_{nh} \) and \( M_{nh} \) are polynomials of degree \( \leq k \). If Assumption 4.7 is satisfied, the following estimate holds for each indicator \( \eta^n_K \) introduced in (4.1), \( 1 \leq n \leq N \) and \( K \in T_{nh} \),

\[
\eta^n_K \leq c \left( \nu \| \nabla (u^n - u^n_h) \|_{L^2(\omega_K)^d} + \nu \| (u^n - u^{n-1}_h) - (u^{n-1}_h - u^{n-1}_h) \|_{\tau_n} + \nu \| (p^n - p^n_h) \|_{H^{-1}(\omega_K)^d} \right)
\]

where \( \omega_K \) denotes the union of the elements of \( T_{nh} \) that share at least an edge \( (d = 2) \) or a face \( (d = 3) \) with \( K \).

Proof. For any domain \( \Delta \) contained in \( \Omega \), let \( R(\Delta) \) denote the right hand-side of (4.21) with \( \omega_K \) replaced by \( \Delta \). Since this estimate relies on equation (4.2), we first note by integrating by parts of each \( K \) that

\[
(F^n, v) = \tau_n \sum_{K \in T_{nh}} \left\{ \left( f_h^n - \frac{u^n_h - u^{n-1}_h}{\tau_n} \right) + \nu \Delta u^n_h = \nabla p^n_h, v \right\} + (f^n - f^n_h, v)
\]

\[
\frac{1}{2} \sum_{e \in E_K} \int_{\tau} [\nu \partial_{\nu e} (u^n_h - p^n_h \nu e |_e (\tau)) \cdot v(\tau)] d\tau.
\]
We now prove a bound successively for each of the three terms in $v^K_h$.

(1) Set
\[
v_K = \begin{cases} 
(f_h^n - \frac{u_h^n - u_h^{n-1}}{\tau_n} + \nu \Delta u_h^n - \text{grad } p_h^n) \psi_K & \text{on } K, \\
0 & \text{on } \Omega \setminus K,
\end{cases}
\]
where $\psi_K$ denotes the bubble function on $K$, equal to $(d+1)^{d+1}$ times the product of the $(d+1)$ barycentric coordinates associated with the vertices of $K$. Inserting $v = v_K$ and $q = 0$ in equation (4.2), we obtain
\[
\tau_n \left\| \left( f_h^n - \frac{u_h^n - u_h^{n-1}}{\tau_n} + \nu \Delta u_h^n - \text{grad } p_h^n \right) \psi_K \right\|_{L^2(K)^d}^2 
\leq \tau_n \left\| \frac{(u_h^n - u_h^{n-1})}{\tau_n} \right\|_{L^2(K)^d}^2 \leq \left\| \nu \Delta u_h^n - \text{grad } p_h^n \right\|_{H^{-1}(K)^d} \left\| \text{grad } v_K \right\|_{L^2(K)^{d \times d}} \left\| \text{grad } v_K \right\|_{L^2(K)^{d \times d}} + \tau_n \left\| f_h^n - f_h^n \right\|_{L^2(K)^d} \left\| v_K \right\|_{L^2(K)^d}.
\]
We divide this inequality by $\tau_n$ and we multiply it by $\nu - \frac{1}{2}$ $h_K$. Finally, we observe that $\psi_K$ takes its values in $[0, 1]$ and we use the inverse inequalities on the elements $\tilde{K}$ of $\tilde{T}_{nh}$ contained in the element $K$ of $T_{nh}$, which are valid for any fixed integer $m$ and any polynomial $w$ of degree $\leq m$,

\[
\left\| \text{grad } w \right\|_{L^2(\tilde{K})^d} \leq c h_{K}^{-1} \left\| w \right\|_{L^2(K)}, \quad \left\| w \right\|_{L^2(K)} \leq c \left\| w \psi_K \right\|_{L^2(\tilde{K})},
\]
and, thanks to Assumption 4.7, we note that $h_K$ is $\leq c h_{\tilde{K}}$, where the constant $c$ only depends on the regularity parameters of $(T_{nh})_h$ and $(\tilde{T}_{nh})_h$. This leads to
\[
\nu - \frac{1}{2} h_K \left\| f_h^n - \frac{u_h^n - u_h^{n-1}}{\tau_n} + \nu \Delta u_h^n - \text{grad } p_h^n \right\|_{L^2(K)^d} \leq c R(K).
\]
Note also that, since the extension by zero is continuous from $H^1_0(K)$ into $H^1_0(\Delta)$ for any domain $\Delta$ containing $K$, the norm of $H^{-1}(K)$ which appears in $R(K)$ can be replaced by the norm of $H^{-1}(\Delta)$.

(2) For any $e$ in $E_K$, let $K'$ denote the other element of $T_{nh}$ containing $e$. Let also $\mathcal{L}_K$ and $\mathcal{L}_{K'}$ denote lifting operators from polynomials on $e$ into polynomials on $K$ and $K'$, respectively, which are built by affine transformation from a fixed lifting operator on the reference element. We now define $v_e$ by
\[
v_e = \begin{cases} 
\mathcal{L}_K([\nu \partial_{n_e} u_h^n - p_h^n n_e]_e) \psi_e & \text{on } K, \\
\mathcal{L}_{K'}([\nu \partial_{n_e} u_h^n - p_h^n n_e]_e) \psi_e & \text{on } K', \\
0 & \text{on } \Omega \setminus (K \cup K'),
\end{cases}
\]
where $\psi_e$ denotes $d$ times the product of the $d$ barycentric coordinates associated with the vertices of $e$. By inserting $v = v_e$ and $q = 0$ in equation (4.2), the same arguments as previously (see [15] (Lem. 1.3) for details) lead to the estimate
\[
\nu - \frac{1}{2} h_K^e \left\| [\nu \partial_{n_e} u_h^n - p_h^n n_e]_e \right\|_{L^2(K)^d} \leq c R(K \cup K').
\]
(3) Finally, we insert $v = 0$ and $q = q_K$ in equation (4.2), with $q_K$ defined by
\[
q_K = \text{div } u_h^n \chi_K,
\]
where $\chi_K$ denotes the characteristic function of $K$. This yields in an obvious way
\[
\nu - \frac{1}{2} \left\| \text{div } u_h^n \right\|_{L^2(K)} \leq \nu d^2 \left\| \text{grad } (u_h^n - u_h^n) \right\|_{L^2(K)^{d \times d}}.
\]
Combining estimates (4.22), (4.23) and (4.24) leads to the desired result. □
When comparing Proposition 4.5 and Corollary 4.6 to Proposition 4.8, we observe that, up to some terms involving the approximation of the data, the full error
\[ \|u - u_{\eta} \| + \nu^{-\frac{1}{2}} \| \partial_t (u - u_{\eta}) + \nabla \pi (p - p_{\eta}) \|_{L^2(0,t_n;H^{-1}(\Omega)^d)} \]
is smaller than a constant times the quantity
\[ \left( \sum_{m=1}^{n} \tau_m \sum_{K \in T_{mh}} \left( \eta_K^m \right)^2 \right)^{\frac{1}{2}}, \]
but the result is not fully optimal since this constant depends on the parameter \( \lambda_{h_{\pi}} \) introduced in (4.13). Note however that:
- when \( \Omega \) is convex, this constant is bounded independently of \( n \) and \( h_{\eta} \) if
  \[ \tau_n \geq \mu h_{n}^2, \]
  for a fixed constant \( \mu \). This condition is not too restrictive;
- when \( \Omega \) is not convex, the \( h_{K} \) are most often smaller in a neighbourhood of the re-entrant corners or edges, so that the conditions for \( \lambda_{h_{\pi}} \) to be bounded are not more restrictive than in the previous case. Moreover, since each \( \omega_{K} \) contains at most \( d + 2 \) elements of \( T_{nh} \), estimate (4.21) is local with respect to both time and space variables.

5. PUTTING ALL TOGETHER

For \( 1 \leq n \leq N \), we define the full error, linked to both time and space discretizations,
\begin{equation}
E(t_n) = [u - u_{\eta}](t_n) + [u - u_{h_{\pi}}](t_n) + \nu^{-\frac{1}{2}} \| \partial_t (u - u_{\eta}) + \nabla \pi (p - p_{\eta}) \|_{L^2(0,t_n;H^{-1}(\Omega)^d)} + \nu^{-\frac{1}{2}} \| \partial_t (u - u_{h_{\pi}}) + \nabla \pi (p - p_{h_{\eta}}) \|_{L^2(0,t_n;H^{-1}(\Omega)^d)}. \tag{5.1}
\end{equation}

Note that, by a triangle inequality,
\begin{equation}
E(t_n) \geq [u - u_{h_{\pi}}](t_n) + \nu^{-\frac{1}{2}} \| \partial_t (u - u_{h_{\pi}}) + \nabla \pi (p - p_{h_{\eta}}) \|_{L^2(0,t_n;H^{-1}(\Omega)^d)}. \tag{5.2}
\end{equation}

However, since we intend to perform separately time and space adaptivity, we are led to work with the quantity \( E(t_n) \). Thanks to the results of the previous sections, we are in a position to compare it with the sum of the error indicators.

**Theorem 5.1.** The following a posteriori estimates hold for the error \( E(t_n), 1 \leq n \leq N \):
(i) If Assumption 4.1 is satisfied,
\begin{align}
E(t_n) & \leq c \left( \sum_{m=1}^{n} \left( \frac{1}{2} \| f - \pi f \|_{L^2(0,t_n;H^{-1}(\Omega)^d)} + \tau_{m} \right) \right)^{\frac{1}{2}} \\
& + c' \left( \frac{1}{2} \| f - \pi f \|_{L^2(0,t_n;H^{-1}(\Omega)^d)} + \frac{1}{2} \sum_{m=1}^{n} \tau_{m} \right) \sum_{K \in T_{mh}} h_{K}^2 \| f - f_{K} \|_{L^2(K)}^{2} \right)^{\frac{1}{2}} \\
& + c'' \left( \frac{1}{2} \| u_0 - \Pi_{h} u_0 \|_{L^2(\Omega)^d} + \tau_1 \| \nabla (u_0 - \Pi_{h} u_0) \|_{L^2(\Omega)^{d \times d}} \right). \tag{5.3}
\end{align}
If there exists an integer \( k \) such that, for all \( h \) and for \( 1 \leq n \leq N \), the restrictions to any \( K \) in \( T_{nh} \) of all functions in \( X_{nh} \) and \( M_{nh} \) are polynomials of degree \( \leq k \), and if Assumption 4.7 is satisfied,

\[
\left( \sum_{m=1}^{n} \left( (\eta^m)^2 + \tau_m \sum_{K \in T_{nh}} (\eta^m_K)^2 \right) \right)^{\frac{1}{2}} \leq c E(t_n)
\]

\[
+ c' \left( \frac{1}{\nu} \left\| f - \pi f \right\|_{L^2(\Omega)}^2 + \sum_{m=1}^{n} \frac{\tau_m}{\nu} \sum_{K \in T_{nh}} h_K^2 \left\| f^m - f_h^m \right\|_{L^2(K)}^2 \right)^{\frac{1}{2}}.
\]  

(5.4)

**Proof.** We establish successively the two estimates.

1. By combining the second inequality in (2.14) with Proposition 4.5, we obtain an estimate for \( |u - u_{h\tau}|(t_n) \).

This estimate, combined with Proposition 3.1 and Corollary 3.2, respectively, leads to the upper bounds for the terms

\[ [u - u_{h\tau}](t_n) \quad \text{and} \quad \| \partial_t(u - u_{h\tau}) + \text{grad} (p - \pi p_{h\tau}) \|_{L^2(\Omega)} \]

respectively. Finally, the quantity \( \| \partial_t(u - u_{h\tau}) + \text{grad} \pi p_{h\tau} \|_{L^2(\Omega)} \) is bounded in Corollary 4.6.

2. To derive the second estimate, we first sum the square of inequality (3.9) on \( n \) and use the first inequality in (2.14). A further argument is needed to bound the Hilbertian sum of the \( \eta_K \), since the Hilbertian sum of the norms \( \| \cdot \|_{H^{-1}(\Omega)} \) which appear in (4.21) is not bounded as a function of \( \| \cdot \|_{H^{-1}(\Omega)} \); however the proof is similar to that of Proposition 4.8. We now take \( v \) equal to \( v_{nh} \) (instead of \( v_K \)) and \( q = 0 \) in (4.2), with \( v_{nh} \) defined by

\[ v_{nh}|_K = (f_h^n - u_h^n - u_h^{n-1}) + \nu \Delta u_h^n + \text{grad} p_h^n \]  

and a similar choice for replacing \( v_e \). This gives the global estimate (5.4).

Let \( H_n \) denote the quantity

\[
H_n = \left( \sum_{m=1}^{n} \left( (\eta^m)^2 + \tau_m \sum_{K \in T_{nh}} (\eta^m_K)^2 \right) \right)^{\frac{1}{2}}.
\]  

(5.5)

Theorem 5.1 mainly states that, up to the terms involving the data, the full error \( E(t_n) \) for \( 1 \leq n \leq N \) is equivalent to \( H_n \): It is smaller than \( c \frac{1 + \lambda}{\lambda} (1 + h_{h\tau}) \) times \( H_n \) and larger than \( c' \) times \( H_n \), where both constants \( c \) and \( c' \) only depend on the geometry of \( \Omega \) (indeed, the dependence on \( \nu \) has been investigated in all the estimates). So the result of Theorem 5.1 is optimal whenever \( \sigma_h \) is bounded independently of \( \nu \) and \( \lambda_{h\tau} \) is bounded independently of \( h \) and \( \tau \). Moreover all the quantities \( \eta^m, \eta^m_K, \sigma_h \) and \( \lambda_{h\tau} \) are easy to compute once the discrete solution is known. It can also be observed that the assumptions in both parts of Theorem 5.1 are satisfied by all the finite elements currently used for the Stokes problem.

**References**


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