ESAIM: M2AN
Vol. 38, N° 3, 2004, pp. 437–455
DOI: 10.1051/m2an:2004021

ESAIM: Mathematical Modelling and Numerical Analysis

A POSTERIORI ERROR ANALYSIS OF THE FULLY DISCRETIZED TIME-DEPENDENT STOKES EQUATIONS*

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Abstract. The time-dependent Stokes equations in two- or three-dimensional bounded domains are discretized by the backward Euler scheme in time and finite elements in space. The error of this discretization is bounded globally from above and locally from below by the sum of two types of computable error indicators, the first one being linked to the time discretization and the second one to the space discretization.

Mathematics Subject Classification. 65N30, 65N15, 65J15.

Received: February 10, 2003.

1. Introduction

The time-dependent Stokes problem in a two- or three-dimensional bounded domain models the laminar flow of a viscous incompressible fluid. The basic discretization of this problem relies on the use of the backward Euler scheme with respect to the time variable and of finite elements with respect to the space variables, and a lot of work has been done concerning its *a priori* analysis, see [9] for preliminary results. The important point is that two discretization parameters are involved: the set of time steps and the mesh at each time step. Moreover, due to the choice of an implicit scheme, these parameters can be chosen in a completely independent way. The aim of this paper is to perform the *a posteriori* analysis of the discretization, more precisely to provide tools that allow for optimizing the choice of each time step when working with adaptive meshes.

Much work has been done concerning the *a posteriori* analysis of parabolic type problems. Part of it (*cf.* [2, 4, 5]) deals only with the space discretization and provides appropriate error indicators for it. Another idea consists in establishing a full time and space variational formulation of the continuous problem and using a discontinuous Galerkin method for the discretization with respect to all variables, see for instance [7,8,16,17,19]. Here, we follow a different approach, according to an idea of [1], which consists in introducing two different types of error indicators, one for the time discretization and one for the space discretization, and in uncoupling as far as possible, the estimates of the time and space errors.

In a first step, we give the space variational formulation of the continuous Stokes problem and also of the time semi-discrete problem derived from the Euler scheme. The finite element discretization in space is then

Keywords and phrases. Time-dependent Stokes equations, *a posteriori* error estimates, backward Euler scheme, finite elements.

* *This paper is dedicated to our friend and colleague Vivette Girault in honour of her 60th birthday.*

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built by applying the Galerkin method to this last problem. We have chosen to work with conforming finite elements for simplicity. Next, we describe the two types of residual error indicators: it must be noted that both of them only depend on the fully discrete solution and the data, so that computing them is easy and not expensive. We prove that the global error in the energy norm is bounded from above by the Hilbertian sum of these indicators and also that the indicators are bounded by the error. Moreover these last estimates are local in time for the first type of indicators, local both in time and space for the second one. So it can be hoped that they provide a good representation of the error and furthermore an efficient tool for choosing each time step in an optimal way and performing mesh adaptivity also at each time step.

The extension of these results to the full nonlinear time-dependent Navier-Stokes equations has been considered and similar results can be proven, at least in the two-dimensional case, by using the abstract result in [13] (Th. 3), see also [15] (Prop. 2.1). However we prefer not to present this extension since first the time error indicators in this case are not so easy to compute in practical situations, and second the time discretization of the Navier-Stokes equation by the backward Euler scheme is not realistic except for very small Reynolds number, hence not for real life simulations.

An outline of the paper is as follows.

- Section 2 is devoted to the description of the continuous, the time semi-discrete and the fully discrete problems. We recall their main properties and some standard *a priori* estimates.
- In Section 3, we perform the *a posteriori* analysis of the time discretization.
- In Section 4, the *a posteriori* analysis of the space discretization is achieved.
- In Section 5, we combine the results of the two previous sections to derive the full estimates.

2. The continuous, semi-discrete and discrete problems

Given a bounded connected domain Ω in $\mathbb{R}^d$, $d = 2$ or 3, with a Lipschitz–continuous boundary and a positive real number T, we consider the following Stokes equations

$$\begin{cases} \partial_t \boldsymbol{u} - \nu\,\Delta \boldsymbol{u} + \mathbf{grad}\, p = \boldsymbol{f} & \text{in } \Omega\times]0,T[, \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega\times]0,T[, \\ \boldsymbol{u} = \boldsymbol{0} & \text{on } \partial\Omega\times]0,T[, \\ \boldsymbol{u}(\cdot,0) = \boldsymbol{u}_0 & \text{in } \Omega. \end{cases} \tag{2.1}$$

Here, the unknowns are the velocity $\boldsymbol{u}$ and the pressure p; the data are the distribution $\boldsymbol{f}$ which represents a density of body forces and the initial velocity $\boldsymbol{u}_0$, while the viscosity ν is a positive constant.

We first give the variational formulation of problem (2.1) and recall its main properties. We next describe the time semi-discretization of this problem and recall the well-posedness of the semi-discrete problem together with some *a priori* error estimates. Finally, we present the fully discrete problem and there also recall its consistency.

The variational formulation

In all that follows, for any t, $0 < t \le T$, and any separable Banach space X provided with the norm $\|\cdot\|_X$, we denote by $L^2(0,t;X)$ the space of measurable functions v from $(0,t)$ in X such that

$$\|v\|_{L^2(0,t;X)} = \left(\int_0^t \|v(\cdot,s)\|_X^2 \,\mathrm{d}s \right)^{\frac{1}{2}} < +\infty.$$

For any positive integer m, we introduce the space $H^m(0,t;X)$ of functions in $L^2(0,t;X)$ such that all their time derivatives of order $\le m$ belong to $L^2(0,t;X)$. We also use the dual space $H^{-1}(0,t;X')$ of the space of all functions in $H^1(0;t;X)$ vanishing in 0 and t, and the space $\mathscr{C}^0(0,t;X)$ of continuous functions v from $[0,t]$ in X. Let $(\cdot,\cdot)$ stand for the scalar product on $L^2(\Omega)$ or $L^2(\Omega)^d$ or $L^2(\Omega)^{d\times d}$ and, by extension, for the duality pairing between $H^{-1}(\Omega)^d$ and $H_0^1(\Omega)^d$. Finally, we introduce the space $L_0^2(\Omega)$ of functions in $L^2(\Omega)$ with zero mean value on Ω.

It can be checked that problem (2.1) admits the variational formulation:

Find $\boldsymbol{u}$ in $L^2(0,T;H_0^1(\Omega)^d)\cap\mathscr{C}^0(0,T;L^2(\Omega)^d)$ and p in $H^{-1}(0,T;L_0^2(\Omega))$, such that

$$\boldsymbol{u}(\cdot,0)=\boldsymbol{u}_0 \qquad a.e. \text{ in } \Omega, \tag{2.2}$$

and that, for *a.e.* t in $(0,T)$ and for all $(\boldsymbol{v},q)$ in $H_0^1(\Omega)^d\times L_0^2(\Omega)$,

$$\begin{aligned}(\partial_t\boldsymbol{u},\boldsymbol{v})+\nu\,(\mathbf{grad}\,\boldsymbol{u},\mathbf{grad}\,\boldsymbol{v})-(\operatorname{div}\boldsymbol{v},p)&=(\boldsymbol{f},\boldsymbol{v}),\\ -(\operatorname{div}\boldsymbol{u},q)&=0.\end{aligned} \tag{2.3}$$

We introduce the following norm on $L^2(0,t;H_0^1(\Omega)^d)\cap\mathscr{C}^0(0,t;L^2(\Omega)^d)$

$$[\boldsymbol{v}](t)=\left(\|\boldsymbol{v}(.,t)\|^2_{L^2(\Omega)^d}+\nu\int_0^t\|\mathbf{grad}\,\boldsymbol{v}(.,s)\|^2_{L^2(\Omega)^{d\times d}}\,\mathrm{d}s\right)^{\frac12}. \tag{2.4}$$

The following existence and stability results can be derived from [9] (Chap. V, Sect. 1.2), and [14] (Chap. III, Th. 1.1).

Proposition 2.1. *For any data $(\boldsymbol{f},\boldsymbol{u}_0)$ in $L^2(0,T;H^{-1}(\Omega)^d)\times L^2(\Omega)^d$ such that $\boldsymbol{u}_0$ is divergence–free in Ω, problem (2.2)-(2.3) has a unique solution $(\boldsymbol{u},p)$, which satisfies for all t, $0<t\le T$,*

$$[\boldsymbol{u}](t)\le\left(\frac{1}{\nu}\,\|\boldsymbol{f}\|^2_{L^2(0,t;H^{-1}(\Omega)^d)}+\|\boldsymbol{u}_0\|^2_{L^2(\Omega)^d}\right)^{\frac12}. \tag{2.5}$$

Moreover this solution is such that $\partial_t\boldsymbol{u}+\mathbf{grad}\,p$ belongs to $L^2(0,T;H^{-1}(\Omega)^d)$ and satisfies for all t, $0<t\le T$,

$$\|\partial_t\boldsymbol{u}+\mathbf{grad}\,p\|_{L^2(0,t;H^{-1}(\Omega)^d)}\le 2\left(\|\boldsymbol{f}\|^2_{L^2(0,t;H^{-1}(\Omega)^d)}+\frac{\nu}{2}\,\|\boldsymbol{u}_0\|^2_{L^2(\Omega)^d}\right)^{\frac12}. \tag{2.6}$$

We finally introduce the kernel

$$V=\big\{\boldsymbol{v}\in H_0^1(\Omega)^d;\ \operatorname{div}\boldsymbol{v}=0 \text{ in } \Omega\big\}, \tag{2.7}$$

which plays an important role in the numerical analysis.

The time semi-discrete problem

In order to describe the time discretization of equation (2.1) with an adaptive choice of local time steps, we introduce a partition of the interval $[0,T]$ into subintervals $[t_{n-1},t_n]$, $1\le n\le N$, such that $0=t_0<t_1<\cdots<t_N=T$. We denote by τ_n the length t_n-t_{n-1}, by τ the N-tuple $(\tau_1,\ldots,\tau_N)$, by $|\tau|$ the maximum of the τ_n, $1\le n\le N$, and finally by σ_τ the regularity parameter

$$\sigma_\tau=\max_{2\le n\le N}\frac{\tau_n}{\tau_{n-1}}\cdot \tag{2.8}$$

For any Banach space X, with each family $(v^n)_{0\le n\le N}$ in X^{N+1} we agree to associate the function v_τ on $[0,T]$ which is affine on each interval $[t_{n-1},t_n]$, $1\le n\le N$, and equal to v^n at t_n, $0\le n\le N$. We denote by $Y_\tau(X)$ the space of such functions.

We now assume that the distribution $\boldsymbol{f}$ belongs to $\mathscr{C}^0(0,T;H^{-1}(\Omega)^d)$ and, for simplicity, we denote by $\boldsymbol{f}^n$ the distribution $\boldsymbol{f}(\cdot,t_n)$. The semi-discrete problem constructed from the backward Euler scheme applied to the variational formulation (2.2)–(2.3) is:

Find $(\boldsymbol{u}^n)_{0\le n\le N}$ in $L^2(\Omega)^d \times \big(H_0^1(\Omega)^d\big)^N$ and $(p^n)_{1\le n\le N}$ in $L_0^2(\Omega)^N$, such that

$$\boldsymbol{u}^0 = \boldsymbol{u}_0 \qquad a.e. \text{ in } \Omega, \tag{2.9}$$

and that, for all n, $1 \le n \le N$, and for all $(\boldsymbol{v}, q)$ in $H_0^1(\Omega)^d \times L_0^2(\Omega)$,

$$\begin{aligned}(\boldsymbol{u}^n, \boldsymbol{v}) + \nu\, \tau_n\, (\mathbf{grad}\, \boldsymbol{u}^n, \mathbf{grad}\, \boldsymbol{v}) - \tau_n\, (\operatorname{div} \boldsymbol{v}, p^n) &= \tau_n\, (\boldsymbol{f}^n, \boldsymbol{v}) + (\boldsymbol{u}^{n-1}, \boldsymbol{v}),\\ -(\operatorname{div} \boldsymbol{u}^n, q) &= 0.\end{aligned} \tag{2.10}$$

Note that, up to a zero-order term, the problem for each n is a stationary Stokes problem, so that the following result is standard. It requires the discrete analogue of the norm introduced in (2.4), which is defined on $Y_\tau(H_0^1(\Omega)^d)$ and for all n, $1 \le n \le N$, by

$$[[\boldsymbol{v}_\tau]](t_n) = \Big(\|\boldsymbol{v}^n\|^2_{L^2(\Omega)^d} + \nu \sum_{m=1}^{n} \tau_m\, \|\mathbf{grad}\, \boldsymbol{v}^m\|^2_{L^2(\Omega)^{d\times d}}\Big)^{\frac12}. \tag{2.11}$$

Proposition 2.2. *For any data $(\boldsymbol{f}, \boldsymbol{u}_0)$ in $\mathscr{C}^0(0,T; H^{-1}(\Omega)^d) \times L^2(\Omega)^d$, problem (2.9)–(2.10) has a unique solution $\big(\boldsymbol{u}^0, (\boldsymbol{u}^n, p^n)_{1\le n\le N}\big)$, which satisfies for all n, $1 \le n \le N$,*

$$[[\boldsymbol{u}_\tau]](t_n) \le \big(\frac{1}{\nu} \sum_{m=1}^{n} \tau_m\, \|\boldsymbol{f}^m\|^2_{H^{-1}(\Omega)^d} + \|\boldsymbol{u}_0\|^2_{L^2(\Omega)^d}\big)^{\frac12}. \tag{2.12}$$

Moreover this solution is such that, for all n, $1 \le n \le N$,

$$\left(\sum_{m=1}^{n} \tau_m\, \|\frac{\boldsymbol{u}^m - \boldsymbol{u}^{m-1}}{\tau_m} + \mathbf{grad}\, p^m\|^2_{H^{-1}(\Omega)^d}\right)^{\frac12} \le 2 \left(\sum_{m=1}^{n} \tau_m\, \|\boldsymbol{f}^m\|^2_{H^{-1}(\Omega)^d} + \frac{\nu}{2}\, \|\boldsymbol{u}_0\|^2_{L^2(\Omega)^d}\right)^{\frac12}. \tag{2.13}$$

Proof. The existence and uniqueness of a solution $(\boldsymbol{u}^n, p^n)$ for each n, $1 \le n \le N$, is derived from the standard arguments for the Stokes problem [9] (Chap. I, Th. 5.1), namely the ellipticity of the form involving the gradients and the inf-sup condition on the form for the divergence. In order to derive estimate (2.12), we take $\boldsymbol{v}$ equal to $\boldsymbol{u}^n$ in the first equation of (2.10) and q equal to p^n in the second line. This yields

$$\begin{aligned}\|\boldsymbol{u}^n\|^2_{L^2(\Omega)^d} + \nu\tau_n\, \|\mathbf{grad}\, \boldsymbol{u}^n\|^2_{L^2(\Omega)^{d\times d}} &= \tau_n\, (\boldsymbol{f}^n, \boldsymbol{u}^n) + (\boldsymbol{u}^{n-1}, \boldsymbol{u}^n)\\ &\le \frac{\nu}{2}\tau_n\, \|\mathbf{grad}\, \boldsymbol{u}^n\|^2_{L^2(\Omega)^{d\times d}} + \frac{1}{2\nu}\tau_n\, \|\boldsymbol{f}^n\|^2_{H^{-1}(\Omega)^d}\\ &\quad + \frac12\, \|\boldsymbol{u}^n\|^2_{L^2(\Omega)^d} + \frac12\, \|\boldsymbol{u}^{n-1}\|^2_{L^2(\Omega)^d},\end{aligned}$$

whence

$$\|\boldsymbol{u}^n\|^2_{L^2(\Omega)^d} + \nu\tau_n\, \|\mathbf{grad}\, \boldsymbol{u}^n\|^2_{L^2(\Omega)^{d\times d}} \le \frac{\tau_n}{\nu}\, \|\boldsymbol{f}^n\|^2_{H^{-1}(\Omega)^d} + \|\boldsymbol{u}^{n-1}\|^2_{L^2(\Omega)^d}$$

Replacing n by m and summing on m gives (2.12). On the other hand, we derive from the first line in (2.10) that

$$\left\|\frac{\boldsymbol{u}^n - \boldsymbol{u}^{n-1}}{\tau_n} + \mathbf{grad}\, p^n\right\|_{H^{-1}(\Omega)^d} = \sup_{\boldsymbol{v}\in H_0^1(\Omega)^d} \frac{(\boldsymbol{f}^n, \boldsymbol{v}) - \nu\, (\mathbf{grad}\, \boldsymbol{u}^n, \mathbf{grad}\, \boldsymbol{v})}{\|\mathbf{grad}\, \boldsymbol{v}\|_{L^2(\Omega)^{d\times d}}},$$

which gives

$$\|\frac{\boldsymbol{u}^n - \boldsymbol{u}^{n-1}}{\tau_n} + \mathbf{grad}\, p^n\|_{H^{-1}(\Omega)^d} \le \|\boldsymbol{f}^n\|_{H^{-1}(\Omega)^d} + \nu\, \|\mathbf{grad}\, \boldsymbol{u}^n\|_{L^2(\Omega)^{d\times d}}.$$

Multiplying the square of this inequality by τ_n, summing on n and using (2.12) leads to (2.13). $\square$

The following lemma is of great use in what follows.

Lemma 2.3. *The following equivalence property holds for any family* $(\boldsymbol{v}^n)_{0\le n\le N}$ *in* $(H^1(\Omega)^d)^{N+1}$ *and for all* n*,* $1 \le n \le N$*,*

$$\frac{1}{4}\,[[\boldsymbol{v}_\tau]]^2(t_n) \le [\boldsymbol{v}_\tau]^2(t_n) \le \frac{1+\sigma_\tau}{2}\,[[\boldsymbol{v}_\tau]]^2(t_n) + \frac{\tau_1}{2}\,\|\mathbf{grad}\,\boldsymbol{v}^0\|^2_{L^2(\Omega)^{d\times d}}. \tag{2.14}$$

Proof. In view of the definitions (2.4) and (2.11), we have to compare the quantities

$$X_m = \int_{t_{m-1}}^{t_m} \|\mathbf{grad}\,\boldsymbol{v}_\tau(.,s)\|^2_{L^2(\Omega)^{d\times d}}\,\mathrm{d}s \qquad \text{and} \qquad Y_m = \tau_m\,\|\mathbf{grad}\,\boldsymbol{v}^m\|^2_{L^2(\Omega)^{d\times d}}.$$

Since $\|\mathbf{grad}\,\boldsymbol{v}_\tau(.,s)\|^2_{L^2(\Omega)^{d\times d}}$ is a quadratic function of s, using for instance the Simpson formula gives

$$X_m = \frac{\tau_m}{3}\,\big(\|\mathbf{grad}\,\boldsymbol{v}^m\|^2_{L^2(\Omega)^{d\times d}} + \|\mathbf{grad}\,\boldsymbol{v}^{m-1}\|^2_{L^2(\Omega)^{d\times d}} + (\mathbf{grad}\,\boldsymbol{v}^m, \mathbf{grad}\,\boldsymbol{v}^{m-1})\big).$$

Using both inequalities $ab \ge -\frac{1}{4}\,a^2 - b^2$ and $ab \le \frac{1}{2}a^2 + \frac{1}{2}b^2$ gives

$$\frac{1}{4}Y_m \le X_m \le \frac{\tau_m}{2}\,\Big(\|\mathbf{grad}\,\boldsymbol{v}^m\|^2_{L^2(\Omega)^{d\times d}} + \|\mathbf{grad}\,\boldsymbol{v}^{m-1}\|^2_{L^2(\Omega)^{d\times d}}\Big).$$

Using the defintion (2.8) of σ_τ and summing on m, yield the desired estimate. □

The following *a priori* error estimate is derived in a standard way, see [9] (Chap. V, Th. 2.1): If the velocity $\boldsymbol{u}$ of problem (2.2)–(2.3) belongs to the space $H^1(0,T;H^1(\Omega)^d) \cap H^2(0,T;H^{-1}(\Omega)^d)$, for $0 \le t \le T$,

$$[\boldsymbol{u} - \boldsymbol{u}_\tau](t) \le c(\boldsymbol{u})\,|\tau|. \tag{2.15}$$

The constant $c(\boldsymbol{u})$ depends on the norm of $\boldsymbol{u}$ in the space given above. This estimate, however, is in general not realistic since the required regularity only holds under very strong additional non local compatibility conditions on the data (*cf.* [10]). Nevertheless, by invoking an appropriate regularization process, it yields the convergence of $\boldsymbol{u}_\tau$ to $\boldsymbol{u}$ in the norm $[\,\cdot\,](t)$ when $|\tau|$ tends to zero.

To conclude, we recall the regularity property of the solution of problem (2.9)–(2.10): If the data $(\boldsymbol{f}, \boldsymbol{u}_0)$ belong to $\mathscr{C}^0(0,T;L^2(\Omega)^d) \times V$, the family $(\boldsymbol{u}^n, p^n)_{1\le n\le N}$ belongs to $(H^{s+1}(\Omega)^d \times H^s(\Omega))^N$ and satisfies for all n, $1 \le n \le N$,

$$\Big(\|\boldsymbol{v}^n\|^2_{H^s(\Omega)^d} + \sum_{m=1}^{n} \tau_m\,\|\mathbf{grad}\,\boldsymbol{v}^m\|^2_{H^s(\Omega)^{d\times d}}\Big)^{\frac{1}{2}} \le c\,\big(\sum_{m=1}^{n} \tau_m\,\|\boldsymbol{f}^m\|^2_{L^2(\Omega)^d} + \|\boldsymbol{u}_0\|^2_{H^1(\Omega)^d}\big)^{\frac{1}{2}}. \tag{2.16}$$

Here, the exponent s is equal to $\frac{1}{2}$ for an arbitrary domain Ω and to 1 for a convex domain Ω. When Ω is a non-convex polygon or a polyhedron, the previous estimate holds for some $s > \frac{1}{2}$ depending on Ω. Moreover it holds for $s = 1$ in $\Omega \setminus \mathcal{V}$ where $\mathcal{V}$ is a neighbourhood of the re-entrant corners or edges of Ω.

The time and space discrete problem

We now describe the space discretization of problem (2.9)–(2.10). For each n, $0 \le n \le N$, let $(\mathcal{T}_{nh})_h$ be a regular family of triangulations of Ω by closed triangles (in dimension $d = 2$) or tetrahedra (in dimension $d = 3$), in the usual sense that

- for each h, $\overline{\Omega}$ is the union of all elements of $\mathcal{T}_{nh}$;
- for each h, the intersection of two different elements of $\mathcal{T}_{nh}$ is either empty, or a vertex, or a whole edge, or a whole face (if $d = 3$) of these elements;

- the maximal ratio of the diameter of an element K in $\mathcal{T}_{nh}$ to the diameter of its inscribed circle or sphere is bounded by a constant independent of n and h.

As usual the discretization parameter h denotes the maximal diameter of the elements of all $\mathcal{T}_{nh}$, $1 \le n \le N$, while, for each n, h_n denotes the maximal diameter of the elements of $\mathcal{T}_{nh}$. In all that follows, c stands for a constant that may vary from a line to the next but which is always independent of h_n and n.

We introduce two finite-dimensional subspaces X_{nh} of $H_0^1(\Omega)^d$ and M_{nh} of $L_0^2(\Omega)$ and we assume that the following properties hold:
(i) the space X_{nh} contains the space Z_{nh}^d, with

$$Z_{nh} = \left\{ v_h \in H_0^1(\Omega);\ \forall K \in \mathcal{T}_{nh},\ v_{h\,|K} \in \mathcal{P}_1(K) \right\}, \tag{2.17}$$

where $\mathcal{P}_1(K)$ denotes the space of restrictions to K of affine functions in $\mathbb{R}^d$;
(ii) for each h and n, $1 \le n \le N$, there exists a constant $\beta_{nh} > 0$ such that the following inf-sup condition holds for all q_h in M_{nh}

$$\sup_{\boldsymbol{v}_h \in X_{nh}} \frac{(\operatorname{div} \boldsymbol{v}_h, q_h)}{\|\mathbf{grad}\, \boldsymbol{v}_h\|_{L^2(\Omega)^{d\times d}}} \ge \beta_{nh}\, \|q_h\|_{L^2(\Omega)}. \tag{2.18}$$

These assumptions are not at all restrictive, since they are satisfied for all the finite elements used for the Stokes problem, see [9] (Chap. II, Sect. 2). Note that the inf-sup condition (2.18) guarantees the well-posedness of the discrete problems and is sufficient for our *a posteriori* error analysis. Optimal *a priori* error estimates, however, can most often be derived only under the stronger condition that the constants β_{nh} are uniformly bounded away from 0.

Let Π_h denote a projection operator from $L^2(\Omega)^d$ onto X_{0h}. The fully discrete problem constructed from problem (2.9)–(2.10) by the Galerkin method is the following one:

Find $(\boldsymbol{u}_h^n)_{0\le n\le N}$ in $\prod_{n=0}^N X_{nh}$ and $(p_h^n)_{1\le n\le N}$ in $\prod_{n=1}^N M_{nh}$ such that

$$\boldsymbol{u}_h^0 = \Pi_h \boldsymbol{u}_0 \qquad a.e. \text{ in } \Omega, \tag{2.19}$$

and that, for all n, $1 \le n \le N$, and for all $(\boldsymbol{v}_h, q_h)$ in $X_{nh} \times M_{nh}$,

$$\begin{aligned} (\boldsymbol{u}_h^n, \boldsymbol{v}_h) + \nu\, \tau_n\, (\mathbf{grad}\, \boldsymbol{u}_h^n, \mathbf{grad}\, \boldsymbol{v}_h) - \tau_n\, (\operatorname{div} \boldsymbol{v}_h, p_h^n) &= \tau_n\, (\boldsymbol{f}^n, \boldsymbol{v}_h) + (\boldsymbol{u}_h^{n-1}, \boldsymbol{v}_h), \\ -(\operatorname{div} \boldsymbol{u}_h^n, q_h) &= 0. \end{aligned} \tag{2.20}$$

The same arguments as for problem (2.9)–(2.10) lead to the following statement. We omit the proof since it is exactly the same as that of Proposition 2.2.

Proposition 2.4. *For any data* $(\boldsymbol{f}, \boldsymbol{u}_0)$ *in* $\mathscr{C}^0(0,T;H^{-1}(\Omega)^d) \times L^2(\Omega)^d$*, problem (2.19)–(2.20) has a unique solution* $\big(\boldsymbol{u}_h^0, (\boldsymbol{u}_h^n, p_h^n)_{1\le n\le N}\big)$*, which satisfies for all* n*,* $1 \le n \le N$*,*

$$[[\boldsymbol{u}_{h\tau}]](t_n) \le \left(\frac{1}{\nu} \sum_{m=1}^{n} \tau_m\, \|\boldsymbol{f}^m\|_{H^{-1}(\Omega)^d}^2 + \|\Pi_h \boldsymbol{u}_0\|_{L^2(\Omega)^d}^2 \right)^{\frac{1}{2}}. \tag{2.21}$$

We refer to [10] for the proof of the convergence of the discrete solution towards the exact one and also for *a priori* error estimates which are optimal whenever the constant β_{nh} in (2.18) is independent of h and n.

To conclude, we introduce the discrete kernel

$$V_{nh} = \left\{ \boldsymbol{v}_h \in X_{nh};\ \forall q_h \in M_{nh},\ (\operatorname{div} \boldsymbol{v}_h, q_h) = 0 \right\}, \tag{2.22}$$

and recall that one of the main difficulties of the numerical analysis of the previous problem is that, for most finite elements, V_{nh} is not contained in V.

3. *A posteriori* analysis of the time discretization

In analogy to [1] (see also [11] for the basic idea in a different context and [12] for analogous results for the heat equation), we define for each n, $1 \leq n \leq N$, the error indicator

$$\eta^n = \left(\frac{\tau_n}{3}\,\nu\right)^{\frac{1}{2}} \|\mathbf{grad}\,(\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1})\|_{L^2(\Omega)^{d\times d}}. \tag{3.1}$$

It can be observed that, once the discrete velocity $(\boldsymbol{u}_h^n)_{0\leq n\leq N}$ is known, the previous error indicators are very easy to compute.

From now on we assume that the data $(\boldsymbol{f}, \boldsymbol{u}_0)$ belong to $\mathscr{C}^0(0,T;H^{-1}(\Omega)^d) \times V$. In order to prove the *a posteriori* error estimate, we introduce the operator π_τ: For any Banach space X and any function g continuous from $]0,T]$ into X, $\pi_\tau g$ denotes the step function which is constant and equal to $g(t_n)$ on each interval $]t_{n-1},t_n]$, $1 \leq n \leq N$. Similarly, we denote for any sequence $(\varphi^n)_{1\leq n\leq N}$ in X^N by $\pi_\tau\varphi_\tau$ the step function which is constant and equal to φ^n on each interval $]t_{n-1},t_n]$, $1 \leq n \leq N$. By combining problems (2.2)–(2.3) and (2.9)–(2.10), we observe that the pair $(\boldsymbol{u}-\boldsymbol{u}_\tau, p-\pi_\tau p_\tau)$ satisfies

$$(\boldsymbol{u}-\boldsymbol{u}_\tau)(\cdot,0) = \boldsymbol{0} \qquad \textit{a.e.}\ \text{in}\ \Omega, \tag{3.2}$$

and that, for $1 \leq n \leq N$, for *a.e.* t in $]t_{n-1},t_n]$ and for all $(\boldsymbol{v},q)$ in $H_0^1(\Omega)^d \times L_0^2(\Omega)$,

$$\begin{aligned}(\partial_t(\boldsymbol{u}-\boldsymbol{u}_\tau),\boldsymbol{v}) + \nu\,(\mathbf{grad}\,(\boldsymbol{u}-\boldsymbol{u}_\tau),\mathbf{grad}\,\boldsymbol{v}) - (\operatorname{div}\boldsymbol{v}, p-\pi_\tau p_\tau) &= (\boldsymbol{f}-\pi_\tau\boldsymbol{f},\boldsymbol{v}) + \nu\,(\mathbf{grad}\,(\boldsymbol{u}^n-\boldsymbol{u}_\tau),\mathbf{grad}\,\boldsymbol{v}),\\ -(\operatorname{div}(\boldsymbol{u}-\boldsymbol{u}_\tau),q) &= 0.\end{aligned} \tag{3.3}$$

The *a posteriori* estimate can be derived from this residual equation by quite standard arguments.

Proposition 3.1. *The following a posteriori error estimate holds between the velocity* $\boldsymbol{u}$ *of problem (2.2)–(2.3) and the velocity* $\boldsymbol{u}_\tau$ *associated with the solution* $(\boldsymbol{u}^n)_{0\leq n\leq N}$ *of problem (2.9)–(2.10), for* $1 \leq n \leq N$,

$$[\boldsymbol{u}-\boldsymbol{u}_\tau](t_n) \leq \Big(6\sum_{m=1}^{n}(\eta^m)^2 + 12\,\nu\,\|\boldsymbol{u}_\tau - \boldsymbol{u}_{h\tau}\|^2_{L^2(0,t_n;H^1(\Omega)^d)} + \frac{2}{\nu}\,\|\boldsymbol{f}-\pi_\tau\boldsymbol{f}\|^2_{L^2(0,t_n;H^{-1}(\Omega)^d)}\Big)^{\frac{1}{2}}. \tag{3.4}$$

Proof. By taking $\boldsymbol{v}$ equal to $\boldsymbol{u}-\boldsymbol{u}_\tau$ and q equal to $p-\pi_\tau p_\tau$ in (3.3) and subtracting the second line from the first one, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}-\boldsymbol{u}_\tau\|^2_{L^2(\Omega)^d} + \nu\,\|\mathbf{grad}\,(\boldsymbol{u}-\boldsymbol{u}_\tau)\|^2_{L^2(\Omega)^{d\times d}}\\ &\qquad\leq \left(\frac{1}{\nu^{\frac{1}{2}}}\,\|\boldsymbol{f}-\pi_\tau\boldsymbol{f}\|_{H^{-1}(\Omega)^d} + \nu^{\frac{1}{2}}\,\|\mathbf{grad}\,(\boldsymbol{u}^n-\boldsymbol{u}_\tau)\|_{L^2(\Omega)^{d\times d}}\right)\nu^{\frac{1}{2}}\,\|\mathbf{grad}\,(\boldsymbol{u}-\boldsymbol{u}_\tau)\|_{L^2(\Omega)^{d\times d}},\end{aligned}$$

whence

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}-\boldsymbol{u}_\tau\|^2_{L^2(\Omega)^d} + \nu\,\|\mathbf{grad}\,(\boldsymbol{u}-\boldsymbol{u}_\tau)\|^2_{L^2(\Omega)^{d\times d}} \leq 2\big(\frac{1}{\nu}\,\|\boldsymbol{f}-\pi_\tau\boldsymbol{f}\|^2_{H^{-1}(\Omega)^d} + \nu\,\|\mathbf{grad}\,(\boldsymbol{u}^n-\boldsymbol{u}_\tau)\|^2_{L^2(\Omega)^{d\times d}}\big).$$

By integrating this inequality between t_{n-1} and t_n, summing on n and using (3.2), we derive

$$[\boldsymbol{u}-\boldsymbol{u}_\tau]^2(t_n) \leq 2\Big(\frac{1}{\nu}\,\|\boldsymbol{f}-\pi_\tau\boldsymbol{f}\|^2_{L^2(0,t_n;H^{-1}(\Omega)^d)} + \nu\sum_{m=1}^{n}\int_{t_{m-1}}^{t_m}\|\mathbf{grad}\,(\boldsymbol{u}^m-\boldsymbol{u}_\tau)(.,s)\|^2_{L^2(\Omega)^{d\times d}}\,\mathrm{d}s\Big). \tag{3.5}$$

Next, we note that, for all t in $[t_{m-1}, t_m]$,

$$(\boldsymbol{u}^m - \boldsymbol{u}_\tau)(t) = \frac{t_m - t}{\tau_m}\,(\boldsymbol{u}^m - \boldsymbol{u}^{m-1}),$$

so that

$$\int_{t_{m-1}}^{t_m} \|\mathbf{grad}\,(\boldsymbol{u}^m - \boldsymbol{u}_\tau)(.,s)\|^2_{L^2(\Omega)^{d\times d}}\,\mathrm{d}s = \frac{\tau_m}{3}\,\|\mathbf{grad}\,(\boldsymbol{u}^m - \boldsymbol{u}^{m-1})\|^2_{L^2(\Omega)^{d\times d}}. \tag{3.6}$$

We then use a triangle inequality

$$\frac{\tau_m}{3}\,\|\mathbf{grad}\,(\boldsymbol{u}^m - \boldsymbol{u}^{m-1})\|^2_{L^2(\Omega)^{d\times d}} \le \frac{3}{\nu}\,\eta_m^2 + \tau_m\,\|\mathbf{grad}\,(\boldsymbol{u}^m - \boldsymbol{u}_h^m)\|^2_{L^2(\Omega)^{d\times d}} + \tau_m\,\|\mathbf{grad}\,(\boldsymbol{u}^{m-1} - \boldsymbol{u}_h^{m-1})\|^2_{L^2(\Omega)^{d\times d}}.$$

When applying the arguments used in the proof of Lemma 2.3 to the second and third terms in the right-hand side of this estimate, we obtain

$$\frac{\tau_m}{3}\,\|\mathbf{grad}\,(\boldsymbol{u}^m - \boldsymbol{u}^{m-1})\|^2_{L^2(\Omega)^{d\times d}} \le \frac{3}{\nu}\,\eta_m^2 + 6\int_{t_{m-1}}^{t_m} \|\mathbf{grad}\,(\boldsymbol{u}_\tau - \boldsymbol{u}_{h\tau})(.,s)\|^2_{L^2(\Omega)^{d\times d}}\,\mathrm{d}s. \tag{3.7}$$

Inserting (3.6) and (3.7) into (3.5) yields the desired result. □

A further argument is needed to prove a similar estimate concerning the function $\partial_t(\boldsymbol{u}-\boldsymbol{u}_\tau)+\mathbf{grad}\,(p-\pi_\tau p_\tau)$ in the norm of $H^{-1}(\Omega)$.

Corollary 3.2. *The following a posteriori error estimate holds between the solution $(\boldsymbol{u}, p)$ of problem (2.2)–(2.3) and the pair $(\boldsymbol{u}_\tau, \pi_\tau p_\tau)$ associated with the solution of problem (2.9)–(2.10), for $1 \le n \le N$,*

$$\|\partial_t(\boldsymbol{u} - \boldsymbol{u}_\tau) + \mathbf{grad}\,(p - \pi_\tau p_\tau)\|_{L^2(0,t_n;H^{-1}(\Omega)^d)} \le 3\Big(3\nu \sum_{m=1}^n (\eta^m)^2 + 6\,\nu^2\,\|\boldsymbol{u}_\tau - \boldsymbol{u}_{h\tau}\|^2_{L^2(0,t_n;H^1(\Omega)^d)} + \|\boldsymbol{f} - \pi_\tau \boldsymbol{f}\|^2_{L^2(0,t_n;H^{-1}(\Omega)^d)}\Big)^{\frac{1}{2}}. \tag{3.8}$$

Proof. We have

$$\|\partial_t(\boldsymbol{u} - \boldsymbol{u}_\tau) + \mathbf{grad}\,(p - \pi_\tau p_\tau)\|_{H^{-1}(\Omega)^d} = \sup_{\boldsymbol{v}\in H_0^1(\Omega)^d} \frac{(\partial_t(\boldsymbol{u} - \boldsymbol{u}_\tau), \boldsymbol{v}) - (\mathrm{div}\,\boldsymbol{v}, p - \pi_\tau p_\tau)}{\|\mathbf{grad}\,\boldsymbol{v}\|_{L^2(\Omega)^{d\times d}}}.$$

Using the first equation in (3.3) yields that, for any t in $[t_{n-1}, t_n]$,

$$(\partial_t(\boldsymbol{u} - \boldsymbol{u}_\tau), \boldsymbol{v}) - (\mathrm{div}\,\boldsymbol{v}, p - \pi_\tau p_\tau) = (\boldsymbol{f} - \pi_\tau \boldsymbol{f}, \boldsymbol{v}) + \nu\,(\mathbf{grad}\,(\boldsymbol{u}^n - \boldsymbol{u}_\tau), \mathbf{grad}\,\boldsymbol{v}) - \nu\,(\mathbf{grad}\,(\boldsymbol{u} - \boldsymbol{u}_\tau), \mathbf{grad}\,\boldsymbol{v}),$$

whence

$$\|\partial_t(\boldsymbol{u} - \boldsymbol{u}_\tau) + \mathbf{grad}\,(p - \pi_\tau p_\tau)\|_{L^2(0,t_n;H^{-1}(\Omega)^d)} \le \big(3\|\boldsymbol{f} - \pi_\tau \boldsymbol{f}\|^2_{L^2(0,t_n;H^{-1}(\Omega)^d)} + 3\nu^2 \sum_{m=1}^n \|\mathbf{grad}\,(\boldsymbol{u}^m - \boldsymbol{u}_\tau)\|^2_{L^2(t_{m-1},t_m;L^2(\Omega)^{d\times d})} + 3\nu\,[\boldsymbol{u} - \boldsymbol{u}_\tau]^2(t_n)\big)^{\frac{1}{2}}.$$

The second term in the right-hand side can be estimated by the same arguments as in the proof of Proposition 3.1, see (3.6) and (3.7), and the last one is bounded in (3.4). This concludes the proof. □

We now establish an upper bound for each indicator η^n.

Proposition 3.3. *The following estimate holds for each indicator η^n introduced in (3.1), $1 \leq n \leq N$,*

$$\begin{aligned}\eta^n \leq \nu^{\frac{1}{2}} \|\mathbf{grad}\,(\boldsymbol{u} - \boldsymbol{u}_\tau)\|_{L^2(t_{n-1},t_n;L^2(\Omega)^{d\times d})} + \frac{1}{\nu^{\frac{1}{2}}}\|\partial_t(\boldsymbol{u} - \boldsymbol{u}_\tau) + \mathbf{grad}\,(p - \pi_\tau p_\tau)\|_{L^2(t_{n-1},t_n;H^{-1}(\Omega)^d)}\\ + \frac{1}{\nu^{\frac{1}{2}}}\|\boldsymbol{f} - \pi_\tau \boldsymbol{f}\|_{L^2(t_{n-1},t_n;H^{-1}(\Omega)^d)} + \Big(\frac{\tau_n}{3}\,\nu\Big)^{\frac{1}{2}} \big(\|\mathbf{grad}\,(\boldsymbol{u}^n - \boldsymbol{u}_h^n)\|_{L^2(\Omega)^{d\times d}} + \|\mathbf{grad}\,(\boldsymbol{u}^{n-1} - \boldsymbol{u}_h^{n-1})\|_{L^2(\Omega)^{d\times d}}\big).\end{aligned} \tag{3.9}$$

Proof. Thanks to a triangle inequality, we have

$$\eta^n = \Big(\frac{\tau_n}{3}\,\nu\Big)^{\frac{1}{2}} \big(\|\mathbf{grad}\,(\boldsymbol{u}^n - \boldsymbol{u}^{n-1})\|_{L^2(\Omega)^{d\times d}} + \|\mathbf{grad}\,(\boldsymbol{u}^n - \boldsymbol{u}_h^n)\|_{L^2(\Omega)^{d\times d}} + \|\mathbf{grad}\,(\boldsymbol{u}^{n-1} - \boldsymbol{u}_h^{n-1})\|_{L^2(\Omega)^{d\times d}}\big).$$

In order to bound the first term, we derive from (3.6) that

$$\frac{\tau_n}{3}\,\nu\,\|\mathbf{grad}\,(\boldsymbol{u}^n - \boldsymbol{u}^{n-1})\|^2_{L^2(\Omega)^{d\times d}} = \nu \int_{t_{n-1}}^{t_n} \|\mathbf{grad}\,(\boldsymbol{u}^n - \boldsymbol{u}_\tau)(.,s)\|^2_{L^2(\Omega)^{d\times d}}\,\mathrm{d}s.$$

Next, in the first line of (3.3), we take $\boldsymbol{v}$ equal to $\boldsymbol{u}^n - \boldsymbol{u}_\tau(t)$, and we integrate between t_{n-1} and t_n. This gives

$$\begin{aligned}\nu \int_{t_{n-1}}^{t_n} &\|\mathbf{grad}\,(\boldsymbol{u}^n - \boldsymbol{u}_\tau)(.,s)\|^2_{L^2(\Omega)^{d\times d}}\,\mathrm{d}s\\ &\leq \big(\nu\,\|\mathbf{grad}\,(\boldsymbol{u} - \boldsymbol{u}_\tau)\|_{L^2(t_{n-1},t_n;L^2(\Omega)^{d\times d})}\\ &\quad + \|\partial_t(\boldsymbol{u} - \boldsymbol{u}_\tau) + \mathbf{grad}\,(p - \pi_\tau p_\tau)\|_{L^2(t_{n-1},t_n;H^{-1}(\Omega)^d)}\\ &\quad + \|\boldsymbol{f} - \pi_\tau \boldsymbol{f}\|_{L^2(t_{n-1},t_n;H^{-1}(\Omega)^d)}\big)\\ &\quad \left(\int_{t_{n-1}}^{t_n} \|\mathbf{grad}\,(\boldsymbol{u}^n - \boldsymbol{u}_\tau)(s)\|^2_{L^2(\Omega)^{d\times d}}\,\mathrm{d}s\right)^{\frac{1}{2}},\end{aligned}$$

or equivalently

$$\begin{aligned}\Big(\frac{\tau_n}{3}\,\nu\Big)^{\frac{1}{2}} \|\mathbf{grad}\,(\boldsymbol{u}^n - \boldsymbol{u}^{n-1})\|_{L^2(\Omega)^{d\times d}} &\leq \nu^{\frac{1}{2}} \|\mathbf{grad}\,(\boldsymbol{u} - \boldsymbol{u}_\tau)\|_{L^2(t_{n-1},t_n;L^2(\Omega)^{d\times d})}\\ &\quad + \frac{1}{\nu^{\frac{1}{2}}}\|\partial_t(\boldsymbol{u} - \boldsymbol{u}_\tau) + \mathbf{grad}\,(p - \pi_\tau p_\tau)\|_{L^2(t_{n-1},t_n;H^{-1}(\Omega)^d)}\\ &\quad + \frac{1}{\nu^{\frac{1}{2}}}\|\boldsymbol{f} - \pi_\tau \boldsymbol{f}\|_{L^2(t_{n-1},t_n;H^{-1}(\Omega)^d)}.\end{aligned}$$

This leads to the desired result. □

It follows from Proposition 3.1, Corollary 3.2 and Proposition 3.3 that, if the regularity parameter σ_τ is bounded independently of τ, the full error

$$[\boldsymbol{u} - \boldsymbol{u}_\tau](t_n) + \nu^{-\frac{1}{2}}\,\|\partial_t(\boldsymbol{u} - \boldsymbol{u}_\tau) + \mathbf{grad}\,(p - \pi_\tau p_\tau)\|_{L^2(0,t_n;H^{-1}(\Omega)^d)}$$

is equivalent to the quantity $\big(\sum_{m=1}^n (\eta^m)^2\big)^{\frac{1}{2}}$, up to some terms involving the approximation of the data, namely the distance of $\boldsymbol{f}$ to $\pi_\tau \boldsymbol{f}$ in an appropriate norm, and the spatial error $[\boldsymbol{u}_\tau - \boldsymbol{u}_{h\tau}](t_n)$. Moreover the equivalence constants are explicitly known and estimate (3.9) is local with respect to the time variable.

4. *A posteriori* analysis of the space discretization

In order to describe the family of space error indicators, we first need some notation. For each n and each K in $\mathcal{T}_{nh}$, we introduce the set $\mathcal{E}_K$ of edges ($d=2$) or faces ($d=3$) of K which are not contained in $\partial\Omega$. With any e in $\mathcal{E}_K$ we associate a unit vector $\boldsymbol{n}_e$ orthogonal to e and denote by $[.]_e$ the jump across e in the direction $\boldsymbol{n}_e$. Note that $[.]_e$ depends on the orientation of $\boldsymbol{n}_e$ but that quantities like $[\boldsymbol{v}\cdot\boldsymbol{n}_e]_e$ for any vector field $\boldsymbol{v}$ are independent thereof. For each K, we denote by $\boldsymbol{n}_K$ the unit outward normal to K on ∂K. Finally, h_K stands for the diameter of K and, for each e in $\mathcal{E}_K$, h_e denotes its length ($d=2$) or diameter ($d=3$).

For each n, $1\le n\le N$, we also introduce an approximation $\boldsymbol{f}_h^n$ of the data $\boldsymbol{f}^n=\boldsymbol{f}(\cdot,t_n)$, such that its restrictions to all K in $\mathcal{T}_{nh}$ are polynomials of a fixed degree. The error indicators are now defined by analogy with the stationary Stokes problem, see [3] (Sect. 8.4.1), or [15] (Sect. 3.5). For each n, $1\le n\le N$, and for each K in $\mathcal{T}_{nh}$, we define the error indicator

$$\eta_K^n=\nu^{-\frac{1}{2}}\Big(h_K\,\|\boldsymbol{f}_h^n-\frac{\boldsymbol{u}_h^n-\boldsymbol{u}_h^{n-1}}{\tau_n}+\nu\,\Delta\boldsymbol{u}_h^n-\mathbf{grad}\,p_h^n\|_{L^2(K)^d}\\
+\sum_{e\in\mathcal{E}_K}h_e^{\frac{1}{2}}\,\|\,[\nu\,\partial_{n_e}\boldsymbol{u}_h^n-p_h^n\,\boldsymbol{n}_e]_e\,\|_{L^2(e)^d}+\nu\,\|\mathrm{div}\,\boldsymbol{u}_h^n\|_{L^2(K)}\Big).\tag{4.1}$$

For any n, $1\le n\le N$, and any $(\boldsymbol{v},q)$ in $H_0^1(\Omega)^d\times L_0^2(\Omega)$, we obtain from problems (2.9)–(2.10) and (2.19)–(2.20) the following residual equation

$$\begin{aligned}(\boldsymbol{u}^n-\boldsymbol{u}_h^n,\boldsymbol{v})+\nu\,\tau_n\,(\mathbf{grad}\,(\boldsymbol{u}^n-\boldsymbol{u}_h^n),\mathbf{grad}\,\boldsymbol{v})-\tau_n\,(\mathrm{div}\,\boldsymbol{v},p^n-p_h^n)&=(\boldsymbol{F}^n,\boldsymbol{v})+(\boldsymbol{u}^{n-1}-\boldsymbol{u}_h^{n-1},\boldsymbol{v}),\\
-(\mathrm{div}\,(\boldsymbol{u}^n-\boldsymbol{u}_h^n),q)&=(\mathrm{div}\,\boldsymbol{u}_h^n,q).\end{aligned}\tag{4.2}$$

Here, the residual $\boldsymbol{F}^n$ belongs to $H^{-1}(\Omega)^d$ and is given by

$$(\boldsymbol{F}^n,\boldsymbol{v})=\tau_n\,(\boldsymbol{f}^n,\boldsymbol{v})-(\boldsymbol{u}_h^n-\boldsymbol{u}_h^{n-1},\boldsymbol{v})-\nu\,\tau_n\,(\mathbf{grad}\,\boldsymbol{u}_h^n,\mathbf{grad}\,\boldsymbol{v})+\tau_n\,(\mathrm{div}\,\boldsymbol{v},p_h^n).\tag{4.3}$$

Moreover it can be observed that, for any $(\boldsymbol{v}_h,q_h)$ in $X_{nh}\times M_{nh}$,

$$(\boldsymbol{F}^n,\boldsymbol{v})=(\boldsymbol{F}^n,\boldsymbol{v}-\boldsymbol{v}_h)\qquad\text{and}\qquad(\mathrm{div}\,\boldsymbol{u}_h^n,q)=(\mathrm{div}\,\boldsymbol{u}_h^n,q-q_h).\tag{4.4}$$

A further lemma is needed to handle the non-zero right-hand side of the second line in equation (4.2). It requires the following assumption which is satisfied by all the finite elements used for the Stokes problem.

Assumption 4.1. The space M_{nh} contains either M_{nh}^0 or M_{nh}^1, with

$$\begin{aligned}M_{nh}^0&=\big\{q_h\in L_0^2(\Omega);\ \forall K\in\mathcal{T}_{nh},\ q_{h\,|K}\in\mathcal{P}_0(K)\big\},\\
M_{nh}^1&=\big\{v_h\in H^1(\Omega)\cap L_0^2(\Omega);\ \forall K\in\mathcal{T}_{nh},\ q_{h\,|K}\in\mathcal{P}_1(K)\big\},\end{aligned}\tag{4.5}$$

where $\mathcal{P}_0(K)$ denotes the space of constant functions on K.

Let now Π denote the operator defined from $H_0^1(\Omega)^d$ into itself as follows: For each $\boldsymbol{v}$ in $H_0^1(\Omega)^d$, $\Pi\boldsymbol{v}$ denotes the velocity $\boldsymbol{w}$ of the unique weak solution $(\boldsymbol{w},r)$ in $H_0^1(\Omega)^d\times L_0^2(\Omega)$ of the Stokes problem

$$\begin{cases}-\Delta\boldsymbol{w}+\mathbf{grad}\,r=\mathbf{0} & \text{in }\Omega,\\ \mathrm{div}\,\boldsymbol{w}=\mathrm{div}\,\boldsymbol{v} & \text{in }\Omega,\\ \boldsymbol{w}=\mathbf{0} & \text{on }\partial\Omega.\end{cases}$$

The next lemma states some properties of the operator Π and refers to the constant $\beta > 0$ which is defined by

$$\beta = \inf_{q \in L^2_0(\Omega)\setminus\{0\}} \sup_{\boldsymbol{v} \in H^1_0(\Omega)^d} \frac{(\operatorname{div} \boldsymbol{v}, q)}{\|\mathbf{grad}\, \boldsymbol{v}\|_{L^2(\Omega)^{d\times d}} \|q\|_{L^2(\Omega)}}. \tag{4.6}$$

Lemma 4.2. *The operator Π has the following properties:*

(i) *For all $\boldsymbol{v}$ in V, $\Pi \boldsymbol{v}$ is zero.*

(ii) *The following estimates hold for all $\boldsymbol{v}$ in $H^1_0(\Omega)^d$,*

$$\|\mathbf{grad}\,(\boldsymbol{v} - \Pi\boldsymbol{v})\|_{L^2(\Omega)^{d\times d}} \le \|\mathbf{grad}\, \boldsymbol{v}\|_{L^2(\Omega)^{d\times d}} \quad \text{and} \quad \|\mathbf{grad}\, \Pi\boldsymbol{v}\|_{L^2(\Omega)^{d\times d}} \le \frac{1}{\beta} \|\operatorname{div} \boldsymbol{v}\|_{L^2(\Omega)}. \tag{4.7}$$

(iii) *If Assumption 4.1 is satisfied, the following estimate holds for $1 \le n \le N$ and for any $\boldsymbol{v}_h$ in V_{nh},*

$$\|\Pi \boldsymbol{v}_h\|_{L^2(\Omega)^d} \le c\, h_n^\alpha \|\operatorname{div} \boldsymbol{v}_h\|_{L^2(\Omega)}, \tag{4.8}$$

where the constant α equals 1 if Ω is convex, and equals $\frac{1}{2}$ otherwise.

Proof. Part (i) of the Lemma is obvious. Moreover, since $\boldsymbol{v} - \Pi\boldsymbol{v}$ has vanishing divergence, we conclude from the weak form of the Stokes problem that

$$(\mathbf{grad}\, \Pi\boldsymbol{v}, \mathbf{grad}\,(\boldsymbol{v} - \Pi\boldsymbol{v})) = (\operatorname{div}(\boldsymbol{v} - \Pi\boldsymbol{v}), r) = 0.$$

This proves the first estimate in (4.7). Similarly, we obtain

$$\|\mathbf{grad}\, \Pi\boldsymbol{v}\|^2_{L^2(\Omega)^{d\times d}} = (\mathbf{grad}\, \Pi\boldsymbol{v}, \mathbf{grad}\, \Pi\boldsymbol{v}) = (\operatorname{div} \Pi\boldsymbol{v}, r) = (\operatorname{div} \boldsymbol{v}, r) \le \|\operatorname{div} \boldsymbol{v}\|_{L^2(\Omega)} \|r\|_{L^2(\Omega)},$$

and

$$\begin{aligned} \beta\, \|r\|_{L^2(\Omega)} &\le \sup_{\boldsymbol{z} \in H^1_0(\Omega)^d} \frac{(\operatorname{div} \boldsymbol{z}, r)}{\|\mathbf{grad}\, \boldsymbol{z}\|_{L^2(\Omega)^{d\times d}}} = \sup_{\boldsymbol{z} \in H^1_0(\Omega)^d} \frac{(\mathbf{grad}\, \Pi\boldsymbol{v}, \mathbf{grad}\, \mathbf{z})}{\|\mathbf{grad}\, \boldsymbol{z}\|_{L^2(\Omega)^{d\times d}}} \\ &\le \|\mathbf{grad}\, \Pi\boldsymbol{v}\|_{L^2(\Omega)^{d\times d}}. \end{aligned}$$

This proves the second part of (4.7).

To derive the last estimate, for any $\boldsymbol{v}_h$ in V_{nh}, we use a duality argument, that relies on the Stokes problem

$$\begin{cases} -\Delta\boldsymbol{\varphi} + \mathbf{grad}\, \rho = \Pi\boldsymbol{v}_h & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{\varphi} = 0 & \text{in } \Omega, \\ \boldsymbol{\varphi} = \mathbf{0} & \text{on } \partial\Omega. \end{cases}$$

This problem has a unique solution $(\boldsymbol{\varphi}, \rho)$ too. Moreover, for the α introduced in the statement of the lemma, this solution belongs to the space $H^{\alpha+1}(\Omega)^d \times H^\alpha(\Omega)$ and satisfies

$$\|\boldsymbol{\varphi}\|_{H^{\alpha+1}(\Omega)^d} + \|\rho\|_{H^\alpha(\Omega)} \le c\, \|\Pi\boldsymbol{v}_h\|_{L^2(\Omega)^d}. \tag{4.9}$$

By combining the two previous problems, we have

$$\|\Pi\boldsymbol{v}_h\|^2_{L^2(\Omega)^d} = (\Pi\boldsymbol{v}_h, \Pi\boldsymbol{v}_h) = (\mathbf{grad}\, \boldsymbol{\varphi}, \mathbf{grad}\, \Pi\boldsymbol{v}_h) - (\operatorname{div} \Pi\boldsymbol{v}_h, \rho) = (\operatorname{div} \boldsymbol{\varphi}, r) - (\operatorname{div} \Pi\boldsymbol{v}_h, \rho).$$

Since $\operatorname{div} \boldsymbol{\varphi}$ is zero and $\operatorname{div} \Pi\boldsymbol{v}_h$ is equal to $\operatorname{div} \boldsymbol{v}_h$, this yields

$$\|\Pi\boldsymbol{v}_h\|^2_{L^2(\Omega)^d} = -(\operatorname{div} \boldsymbol{v}_h, \rho).$$

Using the definition of V_{nh} gives, for any ρ_h in M_{nh},

$$\|\Pi \boldsymbol{v}_h\|^2_{L^2(\Omega)^d} = -(\operatorname{div} \boldsymbol{v}_h, \rho - \rho_h) \le \|\operatorname{div} \boldsymbol{v}_h\|_{L^2(\Omega)} \|\rho - \rho_h\|_{L^2(\Omega)}.$$

Thanks to Assumption 4.1, standard approximation properties combined with (4.9) lead to the desired estimate. □

Remark 4.3. When Ω is not convex, estimate (4.8) can be improved. Choose in the proof of Lemma 4.2 ρ_h equal to the image of ρ under a Clément type regularization operator (*cf. e.g.* [6], [18]). Denote by $\mathcal{V}$ a fixed neighbourhood of the re-entrant corners or edges of Ω. Then, in estimate (4.8), the function $\Pi \boldsymbol{v}_h$ satisfies

$$\|\Pi \boldsymbol{v}_h\|_{L^2(\Omega)^d} \le c \left(\sum_{K \in \mathcal{T}_{nh}} h_K^{2\alpha_K} \|\operatorname{div} \boldsymbol{v}_h\|^2_{L^2(K)} \right)^{\frac{1}{2}}, \tag{4.10}$$

with α_K equal to 1 if K does not intersect $\mathcal{V}$, and equal to $\frac{1}{2}$ otherwise. Moreover, if Ω is a polygon, the α_K for the K intersecting $\mathcal{V}$ can be computed explicitly as a function of the largest angle of Ω in a neighbourhood of K.

In the next lemma, we evaluate the quantity $(\boldsymbol{F}^n, \boldsymbol{v} - \boldsymbol{v}_h)$ which appears in (4.4).

Lemma 4.4. *For any function $\boldsymbol{v}$ in $H_0^1(\Omega)^d$, the following estimate holds*

$$\inf_{\boldsymbol{v}_h \in X_{nh}} (\boldsymbol{F}^n, \boldsymbol{v} - \boldsymbol{v}_h) \le c\, \tau_n \left(\sum_{K \in \mathcal{T}_{nh}} \left(\nu\, (\eta_K^n)^2 + h_K^2\, \|\boldsymbol{f}^n - \boldsymbol{f}_h^n\|^2_{L^2(K)^d} \right) \right)^{\frac{1}{2}} \|\mathbf{grad}\, \boldsymbol{v}\|_{L^2(\Omega)^{d \times d}}. \tag{4.11}$$

Proof. By integrating by parts on each K, we obtain

$$(\boldsymbol{F}^n, \boldsymbol{v} - \boldsymbol{v}_h) = \tau_n \sum_{K \in \mathcal{T}_{nh}} \left\{ \int_K (\boldsymbol{f}^n - \frac{\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}}{\tau_n} + \nu\, \Delta \boldsymbol{u}_h^n - \mathbf{grad}\, p_h^n)(\boldsymbol{v} - \boldsymbol{v}_h)\, \mathrm{d}\boldsymbol{x} \right.$$
$$\left. - \sum_{e \in \mathcal{E}_K} \int_e (\nu\, \partial_{n_K} \boldsymbol{u}_h^n - p_h^n\, \boldsymbol{n}_K)(\boldsymbol{v} - \boldsymbol{v}_h)\, \mathrm{d}\tau \right\}$$

or, equivalently,

$$(\boldsymbol{F}^n, \boldsymbol{v} - \boldsymbol{v}_h) = \tau_n \sum_{K \in \mathcal{T}_{nh}} \left\{ \int_K (\boldsymbol{f}^n - \frac{\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}}{\tau_n} + \nu\, \Delta \boldsymbol{u}_h^n - \mathbf{grad}\, p_h^n)(\boldsymbol{v} - \boldsymbol{v}_h)\, \mathrm{d}\boldsymbol{x} \right.$$
$$\left. + \frac{1}{2} \sum_{e \in \mathcal{E}_K} \int_e [\nu\, \partial_{n_e} \boldsymbol{u}_h^n - p_h^n\, \boldsymbol{n}_e]_e (\boldsymbol{v} - \boldsymbol{v}_h)\, \mathrm{d}\tau \right\}.$$

Thanks to the Cauchy–Schwarz inequality, this yields

$$(\boldsymbol{F}^n, \boldsymbol{v} - \boldsymbol{v}_h) \le \tau_n \sum_{K \in \mathcal{T}_{nh}} \left(\left\| \boldsymbol{f}^n - \frac{\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}}{\tau_n} + \nu\, \Delta \boldsymbol{u}_h^n - \mathbf{grad}\, p_h^n \right\|_{L^2(K)^d} \|\boldsymbol{v} - \boldsymbol{v}_h\|_{L^2(K)^d} \right.$$
$$\left. + \frac{1}{2} \sum_{e \in \mathcal{E}_K} \|\, [\nu\, \partial_{n_e} \boldsymbol{u}_h^n - p_h^n\, \boldsymbol{n}_e]_e\, \|_{L^2(e)^d} \|\boldsymbol{v} - \boldsymbol{v}_h\|_{L^2(e)^d} \right).$$

Next, we introduce a Clément type regularization operator R_{nh} which has the following properties, see [6] or [18]: For any function v in $H_0^1(\Omega)$, $R_{nh}v$ belongs to the space Z_{nh} of continuous affine finite elements (*cf.* (2.17)) and satisfies for any K in $\mathcal{T}_{nh}$ and e in $\mathcal{E}_K$,

$$\|v - R_{nh}v\|_{L^2(K)} \leq c\, h_K\, |\mathbf{grad}\, v|_{H^1(\Delta_K)^d} \quad \text{and} \quad \|v - R_{nh}v\|_{L^2(e)} \leq c\, h_e^{\frac{1}{2}}\, |\mathbf{grad}\, v|_{H^1(\Delta_e)^d}. \tag{4.12}$$

Here Δ_K, resp. Δ_e, denote the union of elements of $\mathcal{T}_{nh}$ that share at least a vertex with K, resp. with e. Since X_{nh} contains Z_{nh}^d, taking $\boldsymbol{v}_h$ equal to $R_{nh}\boldsymbol{v}$ gives

$$\begin{aligned}(\boldsymbol{F}^n, \boldsymbol{v} - \boldsymbol{v}_h) \leq c\,\tau_n \sum_{K\in\mathcal{T}_h} \Bigg\{ h_K \left\| \boldsymbol{f}^n - \frac{\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}}{\tau_n} + \nu\,\Delta \boldsymbol{u}_h^n - \mathbf{grad}\, p_h^n \right\|_{L^2(K)^d} \\ + \frac{1}{2} \sum_{e\in\mathcal{E}_K} h_e^{\frac{1}{2}}\, \|\, [\nu\,\partial_{n_e}\boldsymbol{u}_h^n - p_h^n\,\boldsymbol{n}_e]_e\,\|_{L^2(e)^d} \Bigg\} \|\mathbf{grad}\,\boldsymbol{v}\|_{L^2(\Delta_K)^{d\times d}}.\end{aligned}$$

The desired estimate follows from a triangle inequality by adding and subtracting $\boldsymbol{f}_h^n$ and noting that, when summing on the K, each element of $\mathcal{T}_{nh}$ only belongs to a finite number of Δ_K's, this number being bounded as a function of the regularity parameter of the family of triangulations. □

We are now in a position to prove the first *a posteriori* estimate. It requires the following parameter

$$\lambda_{h\tau} = \sup_{1\leq n\leq N} \frac{\sup_{K\in\mathcal{T}_{nh}} h_K^{2\alpha_K}}{\nu\tau_n}. \tag{4.13}$$

Proposition 4.5. *If Assumption* 4.1 *is satisfied, the following a posteriori error estimate holds between the velocity* $\boldsymbol{u}_\tau$ *associated with the solution* $(\boldsymbol{u}^n)_{0\leq n\leq N}$ *of problem (2.9)–(2.10) and the velocity* $\boldsymbol{u}_{h\tau}$ *associated with the solution* $(\boldsymbol{u}_h^n)_{0\leq n\leq N}$ *of problem (2.19)–(2.20), for* $1 \leq n \leq N$,

$$\begin{aligned}[[\boldsymbol{u}_\tau - \boldsymbol{u}_{h\tau}]](t_n) \leq c \left(\sum_{m=1}^{n} \tau_m \sum_{K\in\mathcal{T}_{mh}} \left((1+\lambda_{h\tau})\,(\eta_K^m)^2 + \frac{h_K^2}{\nu}\,\|\boldsymbol{f}^m - \boldsymbol{f}_h^m\|_{L^2(K)^d}^2 \right) \right)^{\frac{1}{2}} \\ + c'\,\|\boldsymbol{u}_0 - \Pi_h \boldsymbol{u}_0\|_{L^2(\Omega)^d}.\end{aligned} \tag{4.14}$$

Proof. For abbreviation we set

$$\boldsymbol{e}^n = \boldsymbol{u}^n - \boldsymbol{u}_h^n, \quad 0 \leq n \leq N, \qquad \text{and} \qquad \varepsilon^n = p^n - p_h^n, \quad 1 \leq n \leq N.$$

For any n, $1 \leq n \leq N$, we then have

$$\begin{aligned}\frac{1}{2}\|\boldsymbol{e}^n\|_{L^2(\Omega)^d}^2 - \frac{1}{2}\|\boldsymbol{e}^{n-1}\|_{L^2(\Omega)^d}^2 + \frac{1}{2}\|\boldsymbol{e}^n - \boldsymbol{e}^{n-1}\|_{L^2(\Omega)^d}^2 + \nu\tau_n\|\mathbf{grad}\,\boldsymbol{e}^n\|_{L^2(\Omega)^{d\times d}}^2 \\ = (\boldsymbol{e}^n - \boldsymbol{e}^{n-1}, \boldsymbol{e}^n) + \nu\tau_n(\mathbf{grad}\,\boldsymbol{e}^n, \mathbf{grad}\,\boldsymbol{e}^n).\end{aligned} \tag{4.15}$$

Observing that $\mathrm{div}(\boldsymbol{e}^n - \Pi\boldsymbol{e}^n) = 0$, we obtain

$$\begin{aligned}(\boldsymbol{e}^n - \boldsymbol{e}^{n-1}, \boldsymbol{e}^n) + \nu\tau_n(\mathbf{grad}\,\boldsymbol{e}^n, \mathbf{grad}\,\boldsymbol{e}^n) &= (\boldsymbol{e}^n - \boldsymbol{e}^{n-1}, \Pi\boldsymbol{e}^n) + \nu\tau_n(\mathbf{grad}\,\boldsymbol{e}^n, \mathbf{grad}\,\Pi\boldsymbol{e}^n) \\ &\quad + (\boldsymbol{e}^n - \boldsymbol{e}^{n-1}, \boldsymbol{e}^n - \Pi\boldsymbol{e}^n) + \nu\tau_n(\mathbf{grad}\,\boldsymbol{e}^n, \mathbf{grad}\,(\boldsymbol{e}^n - \Pi\boldsymbol{e}^n)) \\ &\quad - \tau_n(\mathrm{div}(\boldsymbol{e}^n - \Pi\boldsymbol{e}^n), \varepsilon^n).\end{aligned}$$

By inserting $\boldsymbol{v} = \boldsymbol{e}^n - \Pi \boldsymbol{e}^n$ in equation (4.2), this yields

$$(\boldsymbol{e}^n - \boldsymbol{e}^{n-1}, \boldsymbol{e}^n) + \nu\tau_n(\mathbf{grad}\,\boldsymbol{e}^n, \mathbf{grad}\,\boldsymbol{e}^n) = (\boldsymbol{e}^n - \boldsymbol{e}^{n-1}, \Pi\boldsymbol{e}^n) + \nu\tau_n(\mathbf{grad}\,\boldsymbol{e}^n, \mathbf{grad}\,\Pi\boldsymbol{e}^n) + (\mathbf{F}^n, \boldsymbol{e}^n - \Pi\boldsymbol{e}^n). \tag{4.16}$$

Next, we bound the three terms on the right-hand side of (4.16) separately. Taking into account that $\Pi\boldsymbol{e}^n = -\Pi\boldsymbol{u}_h^n$ and using Lemma 4.2 and Remark 4.3, we obtain for the first term

$$\begin{aligned}(\boldsymbol{e}^n - \boldsymbol{e}^{n-1}, \Pi\boldsymbol{e}^n) &\le \frac{1}{2}\|\boldsymbol{e}^n - \boldsymbol{e}^{n-1}\|^2_{L^2(\Omega)^d} + \frac{1}{2}\|\Pi\boldsymbol{e}^n\|^2_{L^2(\Omega)^d} \\ &\le \frac{1}{2}\|\boldsymbol{e}^n - \boldsymbol{e}^{n-1}\|^2_{L^2(\Omega)^d} + c\left(\sum_{K\in\mathcal{T}_{nh}} h_K^{2\alpha_K}\,\|\mathrm{div}\,\boldsymbol{u}_h^n\|^2_{L^2(K)}\right),\end{aligned}$$

whence

$$(\boldsymbol{e}^n - \boldsymbol{e}^{n-1}, \Pi\boldsymbol{e}^n) \le \frac{1}{2}\|\boldsymbol{e}^n - \boldsymbol{e}^{n-1}\|^2_{L^2(\Omega)^d} + c\,\lambda_{h\tau}\,\nu\tau_n\|\mathrm{div}\boldsymbol{u}_h^n\|^2_{L^2(\Omega)}. \tag{4.17}$$

Similarly, we derive from (4.7) the estimate for the second term

$$\begin{aligned}\nu\tau_n(\mathbf{grad}\,\boldsymbol{e}^n, \mathbf{grad}\,\Pi\boldsymbol{e}^n) &\le \frac{\nu\tau_n}{4}\,\|\mathbf{grad}\,\boldsymbol{e}^n\|^2_{L^2(\Omega)^{d\times d}} + \nu\tau_n\,\|\mathbf{grad}\,\Pi\boldsymbol{e}^n\|^2_{L^2(\Omega)^{d\times d}} \\ &\le \frac{\nu\tau_n}{4}\,\|\mathbf{grad}\,\boldsymbol{e}^n\|^2_{L^2(\Omega)^{d\times d}} + \frac{\nu\tau_n}{\beta}\,\|\mathrm{div}\boldsymbol{u}_h^n\|^2_{L^2(\Omega)}.\end{aligned} \tag{4.18}$$

Finally, Lemmas 4.2 and 4.4 and the first equation in (4.4) imply the following bound for the third term

$$\begin{aligned}(\mathbf{F}^n, \boldsymbol{e}^n - \Pi\boldsymbol{e}^n) &\le c\,\tau_n\Big(\sum_{K\in\mathcal{T}_{nh}}\Big(\nu\,(\eta_K^n)^2 + h_K^2\,\|\boldsymbol{f}^n - \boldsymbol{f}_h^n\|^2_{L^2(K)^d}\Big)\Big)^{\frac{1}{2}}\|\mathbf{grad}\,(\boldsymbol{e}^n - \Pi\boldsymbol{e}^n)\|_{L^2(\Omega)^{d\times d}} \\ &\le c'\,\tau_n\left(\sum_{K\in\mathcal{T}_{nh}}\left((\eta_K^n)^2 + \frac{h_K^2}{\nu}\,\|\boldsymbol{f}^n - \boldsymbol{f}_h^n\|^2_{L^2(K)^d}\right)\right) + \frac{\nu\tau_n}{4}\,\|\mathbf{grad}\,\boldsymbol{e}^n\|^2_{L^2(\Omega)^{d\times d}}.\end{aligned} \tag{4.19}$$

Combining equations (4.15) to (4.19) and multiplying the resulting equation by 2, we arrive at

$$\|\boldsymbol{e}^n\|^2_{L^2(\Omega)^d} - \|\boldsymbol{e}^{n-1}\|^2_{L^2(\Omega)^d} + \nu\tau_n\,\|\mathbf{grad}\,\boldsymbol{e}^n\|^2_{L^2(\Omega)^{d\times d}} \le c\,\tau_n\Big(\sum_{K\in\mathcal{T}_{nh}}\left((1+\lambda_{h\tau})(\eta_K^n)^2 + \frac{h_K^2}{\nu}\,\|\boldsymbol{f}^n - \boldsymbol{f}_h^n\|^2_{L^2(K)^d}\right)\Big).$$

Summing with respect to n yields the desired estimate. □

As in Corollary 3.2, we now prove an estimate for the second part of the error, which combines the time derivative of the velocity and the gradient of the pressure.

Corollary 4.6. *If Assumption* 4.1 *is satisfied, the following a posteriori error estimate holds between the pair* $(\boldsymbol{u}_\tau, \pi_\tau p_\tau)$ *associated with the solution of problem (2.9)–(2.10) and the pair* $(\boldsymbol{u}_{h\tau}, \pi_\tau p_{h\tau})$ *associated with the solution of problem (2.19)–(2.20), for* $1 \le n \le N$,

$$\begin{aligned}&\|\partial_t(\boldsymbol{u}_\tau - \boldsymbol{u}_{h\tau}) + \mathbf{grad}\,\pi_\tau(p_\tau - p_{h\tau})\|_{L^2(0,t_n;H^{-1}(\Omega)^d)} \\ &\qquad \le c\left(\sum_{m=1}^n \tau_m \sum_{K\in\mathcal{T}_{mh}}\big(\nu\,(1+\lambda_{h\tau})\,(\eta_K^m)^2 + h_K^2\,\|\boldsymbol{f}^m - \boldsymbol{f}_h^m\|^2_{L^2(K)^d}\big)\right)^{\frac{1}{2}} \\ &\qquad\qquad + c'\,\nu^{\frac{1}{2}}\,\|\boldsymbol{u}_0 - \Pi_h\boldsymbol{u}_0\|_{L^2(\Omega)^d}.\end{aligned} \tag{4.20}$$

Proof. We have at each time t, $0 \le t \le t_n$,

$$\|\partial_t(\boldsymbol{u}_\tau - \boldsymbol{u}_{h\tau}) + \mathbf{grad}\, \pi_\tau(p_\tau - p_{h\tau})\|_{H^{-1}(\Omega)^d} = \sup_{\boldsymbol{v}\in H_0^1(\Omega)^d} \frac{(\partial_t(\boldsymbol{u}_\tau - \boldsymbol{u}_{h\tau}), \boldsymbol{v}) - (\mathrm{div}\, \boldsymbol{v}, \pi_\tau(p_\tau - p_{h\tau}))}{\|\mathbf{grad}\, \boldsymbol{v}\|_{L^2(\Omega)^{d\times d}}}.$$

Equations (4.2) and (4.4) yield that, for all t, $t_{n-1} \le t \le t_n$, and for any $\boldsymbol{v}_h$ in X_{nh},

$$(\partial_t(\boldsymbol{u}_\tau - \boldsymbol{u}_{h\tau}), \boldsymbol{v}) - (\mathrm{div}\, \boldsymbol{v}, \pi_\tau(p_\tau - p_{h\tau})) = \frac{1}{\tau_n}\,(\boldsymbol{F}^n, \boldsymbol{v} - \boldsymbol{v}_h) - \nu\,(\mathbf{grad}\,(\boldsymbol{u}^n - \boldsymbol{u}_h^n), \mathbf{grad}\, \boldsymbol{v}).$$

Thus Lemma 4.4 leads to

$$\|\partial_t(\boldsymbol{u}_\tau - \boldsymbol{u}_{h\tau}) + \mathbf{grad}\, \pi_\tau(p_\tau - p_{h\tau})\|_{H^{-1}(\Omega)^d} \le c \left(\sum_{K\in\mathcal{T}_{nh}} \left(\nu\,(\eta_K^n)^2 + h_K^2\, \|\boldsymbol{f}^n - \boldsymbol{f}_h^n\|^2_{L^2(K)^d} \right) \right)^{\frac12} + \nu\, \|\mathbf{grad}\,(\boldsymbol{u}^n - \boldsymbol{u}_h^n)\|_{L^2(\Omega)^{d\times d}}.$$

Note that the right-hand side of this inequality is independent of t, for $t_{n-1} \le t \le t_n$. So integrating the square of it between t_{n-1} and t_n, summing on the n, and using Proposition 4.5 give the desired estimate. ☐

We now establish an upper bound for each indicator η_n^K. We only give an abridged proof of this result, since the arguments are very similar to those used for the steady Stokes problem, see [15] (Prop. 3.19) or [3] (Prop. 8.6). Note however that this result requires the following assumption, which is not restrictive in the context of mesh adaptation.

Assumption 4.7. For $1 \le n \le N$, there exists a family of triangulations $(\widetilde{\mathcal{T}}_{nh})_h$, such that, for all h and n, $1 \le n \le N$, each element K of $\mathcal{T}_{n-1,h}$ or of $\mathcal{T}_{nh}$ is the union of elements $\tilde{K}$ of $\widetilde{\mathcal{T}}_{nh}$ such that the ratio of the diameter of K and of the diameter of the largest ball inscribed into each $\tilde{K}$ is bounded uniformly with respect to h and n.

Proposition 4.8. *Assume that there exists an integer k such that, for all h and for $1 \le n \le N$, the restrictions to any K in $\mathcal{T}_{nh}$ of all functions in X_{nh} and M_{nh} are polynomials of degree $\le k$. If Assumption 4.7 is satisfied, the following estimate holds for each indicator η_K^n introduced in (4.1), $1 \le n \le N$ and $K \in \mathcal{T}_{nh}$,*

$$\eta_K^n \le c \left(\nu^{\frac12}\, \|\mathbf{grad}\,(\boldsymbol{u}^n - \boldsymbol{u}_h^n)\|_{L^2(\omega_K)^{d\times d}} + \nu^{-\frac12}\, \| \frac{(\boldsymbol{u}^n - \boldsymbol{u}_h^n) - (\boldsymbol{u}^{n-1} - \boldsymbol{u}_h^{n-1})}{\tau_n} + \mathbf{grad}\,(p^n - p_h^n)\|_{H^{-1}(\omega_K)^d} + \nu^{-\frac12} h_K\, \|\boldsymbol{f}^n - \boldsymbol{f}_h^n\|_{L^2(\omega_K)^d} \right), \tag{4.21}$$

where ω_K denotes the union of the elements of $\mathcal{T}_{nh}$ that share at least an edge ($d = 2$) or a face ($d = 3$) with K.

Proof. For any domain Δ contained in Ω, let $R(\Delta)$ denote the right hand-side of (4.21) with ω_K replaced by Δ. Since this estimate relies on equation (4.2), we first note by integrating by parts of each K that

$$(\boldsymbol{F}^n, \boldsymbol{v}) = \tau_n \sum_{K\in\mathcal{T}_{nh}} \Big\{ (\boldsymbol{f}_h^n - \frac{\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}}{\tau_n} + \nu\, \Delta \boldsymbol{u}_h^n - \mathbf{grad}\, p_h^n, \boldsymbol{v}) + (\boldsymbol{f}^n - \boldsymbol{f}_h^n, \boldsymbol{v}) - \frac12 \sum_{e\in\mathcal{E}_K} \int_e [\nu\, \partial_{n_e} \boldsymbol{u}_h^n - p_h^n\, \boldsymbol{n}_e]_e(\boldsymbol{\tau}) \cdot \boldsymbol{v}(\boldsymbol{\tau})\, \mathrm{d}\boldsymbol{\tau} \Big\}.$$

We now prove a bound successively for each of the three terms in η_n^K.
(1) Set

$$\boldsymbol{v}_K = \begin{cases} (\boldsymbol{f}_h^n - \frac{\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}}{\tau_n} + \nu\,\Delta \boldsymbol{u}_h^n - \mathbf{grad}\, p_h^n)\,\psi_K & \text{on } K, \\ \mathbf{0} & \text{on } \Omega \setminus K, \end{cases}$$

where ψ_K denotes the bubble function on K, equal to $(d+1)^{d+1}$ times the product of the $(d+1)$ barycentric coordinates associated with the vertices of K. Inserting $\boldsymbol{v} = \boldsymbol{v}_K$ and $q = 0$ in equation (4.2), we obtain

$$\begin{aligned} \tau_n \left\| (\boldsymbol{f}_h^n - \frac{\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}}{\tau_n} + \nu\,\Delta \boldsymbol{u}_h^n - \mathbf{grad}\, p_h^n)\,\psi_K^{\frac{1}{2}} \right\|_{L^2(K)^d}^2 \\ \leq \tau_n \left\| \frac{(\boldsymbol{u}^n - \boldsymbol{u}_h^n) - (\boldsymbol{u}^{n-1} - \boldsymbol{u}_h^{n-1})}{\tau_n} + \mathbf{grad}\,(p^n - p_h^n) \right\|_{H^{-1}(K)^d} \|\mathbf{grad}\, \boldsymbol{v}_K\|_{L^2(K)^{d\times d}} \\ + \nu\tau_n \,\|\mathbf{grad}\,(\boldsymbol{u}^n - \boldsymbol{u}_h^n)\|_{L^2(K)^{d\times d}} \|\mathbf{grad}\, \boldsymbol{v}_K\|_{L^2(K)^{d\times d}} + \tau_n \,\|\boldsymbol{f}^n - \boldsymbol{f}_h^n\|_{L^2(K)^d} \|\boldsymbol{v}_K\|_{L^2(K)^d}. \end{aligned}$$

We divide this inequality by τ_n and we multiply it by $\nu^{-\frac{1}{2}}\, h_K$. Finally, we observe that ψ_K takes its values in $[0,1]$ and we use the inverse inequalities on the elements $\tilde{K}$ of $\widetilde{\mathcal{T}}_{nh}$ contained in the element K of $\mathcal{T}_{nh}$, which are valid for any fixed integer m and any polynomial w of degree $\leq m$,

$$\|\mathbf{grad}\, w\|_{L^2(\tilde{K})^d} \leq c\, h_{\tilde{K}}^{-1}\, \|w\|_{L^2(\tilde{K})}, \qquad \|w\|_{L^2(\tilde{K})} \leq c\, \|w\, \psi_K^{\frac{1}{2}}\|_{L^2(\tilde{K})},$$

and, thanks to Assumption 4.7, we note that h_K is $\leq c\, h_{\tilde{K}}$, where the constant c only depends on the regularity parameters of $(\mathcal{T}_{nh})_h$ and $(\widetilde{\mathcal{T}}_{nh})_h$. This leads to

$$\nu^{-\frac{1}{2}}\, h_K \left\| \boldsymbol{f}_h^n - \frac{\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}}{\tau_n} + \nu\,\Delta \boldsymbol{u}_h^n - \mathbf{grad}\, p_h^n \right\|_{L^2(K)^d} \leq c\, R(K). \tag{4.22}$$

Note also that, since the extension by zero is continuous from $H_0^1(K)$ into $H_0^1(\Delta)$ for any domain Δ containing K, the norm of $H^{-1}(K)$ which appears in $R(K)$ can be replaced by the norm of $H^{-1}(\Delta)$.
(2) For any e in $\mathcal{E}_K$, let K' denote the other element of $\mathcal{T}_{nh}$ containing e. Let also $\mathcal{L}_K$ and $\mathcal{L}_{K'}$ denote lifting operators from polynomials on e into polynomials on K and K', respectively, which are built by affine transformation from a fixed lifting operator on the reference element. We now define $\boldsymbol{v}_e$ by

$$\boldsymbol{v}_e = \begin{cases} \mathcal{L}_K([\nu\, \partial_{n_e} \boldsymbol{u}_h^n - p_h^n\, \boldsymbol{n}_e]_e)\, \psi_e & \text{on } K, \\ \mathcal{L}_{K'}([\nu\, \partial_{n_e} \boldsymbol{u}_h^n - p_h^n\, \boldsymbol{n}_e]_e)\, \psi_e & \text{on } K', \\ \mathbf{0} & \text{on } \Omega \setminus (K \cup K'), \end{cases}$$

where ψ_e denotes d^d times the product of the d barycentric coordinates associated with the vertices of e. By inserting $\boldsymbol{v} = \boldsymbol{v}_e$ and $q = 0$ in equation (4.2), the same arguments as previously (see [15] (Lem. 1.3) for details) lead to the estimate

$$\nu^{-\frac{1}{2}}\, h_e^{\frac{1}{2}}\, \|\, [\nu\, \partial_{n_e} \boldsymbol{u}_h^n - p_h^n\, \boldsymbol{n}_e]_e \,\|_{L^2(e)^d} \leq c\, R(K \cup K'). \tag{4.23}$$

(3) Finally, we insert $\boldsymbol{v} = \mathbf{0}$ and $q = q_K$ in equation (4.2), with q_K defined by

$$q_K = \operatorname{div} \boldsymbol{u}_h^n\, \chi_K,$$

where χ_K denotes the characteristic function of K. This yields in an obvious way

$$\nu^{\frac{1}{2}}\, \|\operatorname{div} \boldsymbol{u}_h^n\|_{L^2(K)} \leq \nu\, d^{\frac{1}{2}}\, \|\mathbf{grad}\,(\boldsymbol{u}^n - \boldsymbol{u}_h^n)\|_{L^2(K)^{d\times d}}. \tag{4.24}$$

Combining estimates (4.22), (4.23) and (4.24) leads to the desired result. □

When comparing Proposition 4.5 and Corollary 4.6 to Proposition 4.8, we observe that, up to some terms involving the approximation of the data, the full error

$$[[\boldsymbol{u}_\tau - \boldsymbol{u}_{h\tau}]](t_n) + \nu^{-\frac{1}{2}} \,\|\partial_t(\boldsymbol{u}_\tau - \boldsymbol{u}_{h\tau}) + \mathbf{grad}\, \pi_\tau(p_\tau - p_{h\tau})\|_{L^2(0,t_n;H^{-1}(\Omega)^d)}$$

is smaller than a constant times the quantity

$$\left(\sum_{m=1}^{n} \tau_m \sum_{K \in \mathcal{T}_{mh}} (\eta_K^m)^2 \right)^{\frac{1}{2}},$$

but the result is not fully optimal since this constant depends on the parameter $\lambda_{h\tau}$ introduced in (4.13). Note however that:

- when Ω is convex, this constant is bounded independently of n and h_n if
$$\tau_n \geq \mu\, h_n^2,$$
for a fixed constant μ. This condition is not too restrictive;
- when Ω is not convex, the h_K are most often smaller in a neighbourhood of the re-entrant corners or edges, so that the conditions for $\lambda_{h\tau}$ to be bounded are not more restrictive than in the previous case.

Moreover, since each ω_K contains at most $d+2$ elements of $\mathcal{T}_{nh}$, estimate (4.21) is local with respect to both time and space variables.

5. Putting all together

For $1 \leq n \leq N$, we define the full error, linked to both time and space discretizations,

$$\begin{aligned} E(t_n) = [\boldsymbol{u} - \boldsymbol{u}_\tau](t_n) + [\boldsymbol{u}_\tau - \boldsymbol{u}_{h\tau}](t_n) + \nu^{-\frac{1}{2}} \,\|\partial_t(\boldsymbol{u} - \boldsymbol{u}_\tau) + \mathbf{grad}\,(p - \pi_\tau p_\tau)\|_{L^2(0,t_n;H^{-1}(\Omega)^d)} \\ + \nu^{-\frac{1}{2}} \,\|\partial_t(\boldsymbol{u}_\tau - \boldsymbol{u}_{h\tau}) + \mathbf{grad}\, \pi_\tau(p_\tau - p_{h\tau})\|_{L^2(0,t_n;H^{-1}(\Omega)^d)}. \end{aligned} \tag{5.1}$$

Note that, by a triangle inequality,

$$E(t_n) \geq [\boldsymbol{u} - \boldsymbol{u}_{h\tau}](t_n) + \nu^{-\frac{1}{2}} \,\|\partial_t(\boldsymbol{u} - \boldsymbol{u}_{h\tau}) + \mathbf{grad}\,(p - \pi_\tau p_{h\tau})\|_{L^2(0,t_n;H^{-1}(\Omega)^d)}. \tag{5.2}$$

However, since we intend to perform separately time and space adaptivity, we are led to work with the quantity $E(t_n)$. Thanks to the results of the previous sections, we are in a position to compare it with the sum of the error indicators.

Theorem 5.1. *The following a posteriori estimates hold for the error $E(t_n)$, $1 \leq n \leq N$:*
(i) If Assumption 4.1 *is satisfied,*

$$\begin{aligned} E(t_n) \leq c \left(\sum_{m=1}^{n} \left((\eta^m)^2 + \frac{1+\sigma_\tau}{2}\, \tau_m \sum_{K \in \mathcal{T}_{mh}} (1 + \lambda_{h\tau})\, (\eta_K^m)^2 \right) \right)^{\frac{1}{2}} \\ + c' \left(\frac{1}{\nu} \,\|\boldsymbol{f} - \pi_\tau \boldsymbol{f}\|^2_{L^2(0,t_n;H^{-1}(\Omega)^d)} + \frac{1+\sigma_\tau}{2} \sum_{m=1}^{n} \frac{\tau_m}{\nu} \sum_{K \in \mathcal{T}_{mh}} h_K^2 \,\|\boldsymbol{f}^m - \boldsymbol{f}_h^m\|^2_{L^2(K)^d} \right)^{\frac{1}{2}} \\ + c'' \left(\frac{1+\sigma_\tau}{2} \,\|\boldsymbol{u}_0 - \Pi_h \boldsymbol{u}_0\|_{L^2(\Omega)^d} + \tau_1 \,\|\mathbf{grad}\,(\boldsymbol{u}_0 - \Pi_h \boldsymbol{u}_0)\|_{L^2(\Omega)^{d\times d}} \right). \end{aligned} \tag{5.3}$$

(ii) If there exists an integer k such that, for all h and for $1 \leq n \leq N$, the restrictions to any K in $\mathcal{T}_{nh}$ of all functions in X_{nh} and M_{nh} are polynomials of degree $\leq k$, and if Assumption 4.7 *is satisfied,*

$$\Big(\sum_{m=1}^{n}\big((\eta^m)^2+\tau_m\sum_{K\in\mathcal{T}_{mh}}(\eta_K^m)^2\big)\Big)^{\frac12}\leq c\,E(t_n)$$
$$+\,c'\left(\frac{1}{\nu}\,\|\boldsymbol{f}-\pi_\tau\boldsymbol{f}\|^2_{L^2(0,t_n;H^{-1}(\Omega)^d)}+\sum_{m=1}^{n}\frac{\tau_m}{\nu}\sum_{K\in\mathcal{T}_{mh}}h_K^2\,\|\boldsymbol{f}^m-\boldsymbol{f}_h^m\|^2_{L^2(K)^d}\right)^{\frac12}. \quad (5.4)$$

Proof. We establish successively the two estimates.
(1) By combining the second inequality in (2.14) with Proposition 4.5, we obtain an estimate for $[\boldsymbol{u}_\tau-\boldsymbol{u}_{h\tau}](t_n)$. This estimate, combined with Proposition 3.1 and Corollary 3.2, respectively, leads to the upper bounds for the terms

$$[\boldsymbol{u}-\boldsymbol{u}_\tau](t_n) \qquad\text{and}\qquad \|\partial_t(\boldsymbol{u}-\boldsymbol{u}_\tau)+\mathbf{grad}\,(p-\pi_\tau p_\tau)\|_{L^2(0,t_n;H^{-1}(\Omega)^d)},$$

respectively. Finally, the quantity $\|\partial_t(\boldsymbol{u}_\tau-\boldsymbol{u}_{h\tau})+\mathbf{grad}\,\pi_\tau(p_\tau-p_{h\tau})\|_{L^2(0,t_n;H^{-1}(\Omega)^d)}$ is bounded in Corollary 4.6.
(2) To derive the second estimate, we first sum the square of inequality (3.9) on n and use the first inequality in (2.14). A further argument is needed to bound the Hilbertian sum of the η_n^K, since the Hilbertian sum of the norms $\|\cdot\|_{H^{-1}(\omega_K)}$ which appear in (4.21) is not bounded as a function of $\|\cdot\|_{H^{-1}(\Omega)}$; however the proof is similar to that of Proposition 4.8. We now take $\boldsymbol{v}$ equal to $\boldsymbol{v}_{nh}$ (instead of $\boldsymbol{v}_K$) and $q=0$ in (4.2), with $\boldsymbol{v}_{nh}$ defined by

$$\boldsymbol{v}_{nh\,|K}=\big(\boldsymbol{f}_h^n-\frac{\boldsymbol{u}_h^n-\boldsymbol{u}_h^{n-1}}{\tau_n}+\nu\,\Delta\boldsymbol{u}_h^n-\mathbf{grad}\,p_h^n\big)\,h_K^2\,\psi_K, \qquad K\in\mathcal{T}_{nh},$$

and a similar choice for replacing $\boldsymbol{v}_e$. This gives the global estimate (5.4). □

Let H_n denote the quantity

$$H_n=\left(\sum_{m=1}^{n}\left((\eta^m)^2+\tau_m\sum_{K\in\mathcal{T}_{mh}}(\eta_K^m)^2\right)\right)^{\frac12}. \quad (5.5)$$

Theorem 5.1 mainly states that, up to the terms involving the data, the full error $E(t_n)$ for $1\leq n\leq N$ is equivalent to H_n: It is smaller than $c\,\frac{1+\sigma_\tau}{2}\,(1+\lambda_{h\tau})$ times H_n and larger than c' times H_n, where both constants c and c' only depend on the geometry of Ω (indeed, the dependence on ν has been investigated in all the estimates). So the result of Theorem 5.1 is optimal whenever σ_τ is bounded independently of τ and $\lambda_{h\tau}$ is bounded independently of h and τ. Moreover all the quantities η^m, η_K^m, σ_τ and $\lambda_{h\tau}$ are easy to compute once the discrete solution is known. It can also be observed that the assumptions in both parts of Theorem 5.1 are satisfied by all the finite elements currently used for the Stokes problem.

References

[1] A. Bergam, C. Bernardi and Z. Mghazli, *A posteriori* analysis of the finite element discretization of some parabolic equations. *Math. Comput.* (to appear).
[2] C. Bernardi and B. Métivet, Indicateurs d'erreur pour l'équation de la chaleur. *Rev. Européenne Élém. Finis* **9** (2000) 425–438.
[3] C. Bernardi, B. Métivet and R. Verfürth, Analyse numérique d'indicateurs d'erreur, in *Maillage et adaptation.* P.-L. George Ed., Hermès (2001) 251–278.
[4] M. Bieterman and I. Babuška, The finite element method for parabolic equations. I. *A posteriori* error estimation. *Numer. Math.* **40** (1982) 339–371.
[5] M. Bieterman and I. Babuška, The finite element method for parabolic equations. II. *A posteriori* error estimation and adaptive approach. *Numer. Math.* **40** (1982) 373–406.
[6] P. Clément, Approximation by finite element functions using local regularization. *RAIRO Anal. Numér.* **9** (1975) 77–84.

[7] K. Eriksson and C. Johnson, Adaptive finite element methods for parabolic problems. I. A linear model problem. *SIAM J. Numer. Anal.* **28** (1991) 43–77.

[8] K. Eriksson and C. Johnson, Adaptive finite element methods for parabolic problems. IV. Nonlinear problems. *SIAM J. Numer. Anal.* **32** (1995) 1729–1749.

[9] V. Girault and P.-A. Raviart, Finite Element Approximation of the Navier–Stokes Equations. Springer-Verlag, *Lect. Notes Math.* **749** (1979).

[10] J.G. Heywood and R. Rannacher, Finite-element approximation of the nonstationary Navier–Stokes problem. Part IV: Error analysis for second-order time discretization. *SIAM J. Numer. Anal.* **27** (1990) 353–384.

[11] C. Johnson, Y.-Y. Nie and V. Thomée, An *a posteriori* error estimate and adaptive timestep control for a backward Euler discretization of a parabolic problem. *SIAM J. Numer. Anal.* **27** (1990) 277–291.

[12] M. Picasso, Adaptive finite elements for a linear parabolic problem. *Comput. Methods Appl. Mech. Engrg.* **167** (1998) 223–237.

[13] J. Pousin and J. Rappaz, Consistency, stability, *a priori* and *a posteriori* errors for Petrov–Galerkin methods applied to nonlinear problems. *Numer. Math.* **69** (1994) 213–231.

[14] R. Temam, *Theory and Numerical Analysis of the Navier–Stokes Equations.* North-Holland (1977).

[15] R. Verfürth, *A Review of A Posteriori Error Estimation and Adaptive Mesh-Refinement Techniques.* Wiley & Teubner (1996).

[16] R. Verfürth, *A posteriori* error estimates for nonlinear problems: $L^r(0,T;W^{1,\rho}(\Omega))$–error estimates for finite element discretizations of parabolic equations. *Numer. Methods Partial Differential Equations* **14** (1998) 487–518.

[17] R. Verfürth, *A posteriori* error estimates for nonlinear problems. $L^r(0,T;L^\rho(\Omega))$–error estimates for finite element discretizations of parabolic equations. *Math. Comp.* **67** (1998) 1335–1360.

[18] R. Verfürth, Error estimates for some quasi-interpolation operators. *ESAIM: M2AN* **33** (1999) 695–713.

[19] R. Verfürth, *A posteriori* error estimation techniques for non-linear elliptic and parabolic pdes, *Rev. Européenne Élém. Finis* **9** (2000) 377–402.