

MIXED FINITE ELEMENT APPROXIMATION OF 3D CONTACT PROBLEMS WITH GIVEN FRICTION: ERROR ANALYSIS AND NUMERICAL REALIZATION

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Abstract. This contribution deals with a mixed variational formulation of 3D contact problems with the simplest model involving friction. This formulation is based on a dualization of the set of admissible displacements and the regularization of the non-differentiable term. Displacements are approximated by piecewise linear elements while the respective dual variables by piecewise constant functions on a dual partition of the contact zone. The rate of convergence is established provided that the solution is smooth enough. The numerical realization of such problems will be discussed and results of a model example will be shown.

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1. INTRODUCTION

In the theory of variational inequalities, problems involving contact and friction conditions are of a permanent interest (we refer to [8] and to [18] for a large review of the main unilateral contact models). Finite elements are the most currently used methods for the approximation of contact problems involving friction (see [14, 17] and the references therein). In the primal variational formulation (where the displacement is the only unknown), discretized non-penetration conditions constitute the key point of the approximation model. These conditions can be relaxed and expressed in a weaker sense [4, 14, 16]. In the mixed variational formulation where the displacement and the normal stresses on the contact zone are treated as independent variables, the unilateral conditions have been expressed by using Lagrange multipliers (see [14]). Both variational formulations lead to a *non-differentiable* problem. To overcome this difficulty one can introduce (see [12]) another set of Lagrange multipliers related to the tangential component of the stress on the contact zone transforming the *non-differentiable* problem into a smooth one.

The purpose of this contribution is to extend this approach to 3D contact problems with given friction by using piecewise constant discretizations of Lagrange multipliers on the contact zone. Unlike to 2D problems, constraints imposed on the “tangential” Lagrange multipliers are now quadratic. This fact complicates not only

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the theoretical analysis but also the practical realization. We establish *a priori* error estimates and propose the efficient numerical method.

The paper is organized as follows. Firstly, we introduce the mathematical model of the Signorini problem with given friction. In Section 2 we present the continuous mixed variational formulation of the problem. In the third section, we propose a well-posed finite element discretization in order to approximate the mixed formulation. In Section 4 *a priori* error estimates for the mixed finite element approximations are established. Finally, in Section 5 we present a numerical approach which is based on the dual variational formulation (*i.e.* the formulation in terms of the Lagrange multipliers) combined with a linearization of the quadratic constraints. Numerical results of a model example will be shown.

2. PRELIMINARY AND NOTATIONS

The Euclidean norm of a point $\mathbf{x} \in \mathbb{R}^n$, $n \geq 2$ will be denoted $\|\mathbf{x}\|$ in what follows.

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with the Lipschitz boundary $\partial\Omega$. The symbol $H^k(\Omega)$, $k \geq 1$ integer, stands for the classical Sobolev space equipped with the norm:

$$\|v\|_{k,\Omega} := \left(\sum_{0 \leq |\alpha| \leq k} \|D^\alpha v\|_{0,\Omega}^2 \right)^{\frac{1}{2}}$$

using the standard multi-index notation (the following convention is adopted: $H^0(\Omega) := L^2(\Omega)$, $\| \cdot \|_{0,\Omega} := \| \cdot \|_{L^2(\Omega)}$). Fractional Sobolev spaces $H^\tau(\Omega)$, $\tau \in \mathbb{R}^+ \setminus \mathbb{N}$ are defined by

$$H^\tau(\Omega) = \{v \in H^k(\Omega) \mid \|v\|_{\tau,\Omega} < +\infty\},$$

where

$$\|v\|_{\tau,\Omega} := \left(\|v\|_{k,\Omega}^2 + \sum_{|\alpha|=k} \int_\Omega \int_\Omega \frac{(D^\alpha v(\mathbf{x}) - D^\alpha v(\mathbf{y}))^2}{|\mathbf{x} - \mathbf{y}|^{2+2\theta}} \, d\mathbf{x}d\mathbf{y} \right)^{\frac{1}{2}}$$

with k being the integer part of τ and θ its decimal part (see [1, 11]).

On any portion $\Gamma \subseteq \partial\Omega$ we introduce the space $H^{\frac{1}{2}}(\Gamma)$ as follows:

$$H^{\frac{1}{2}}(\Gamma) = \left\{ \psi \in L^2(\Gamma) \mid \|\psi\|_{\frac{1}{2},\Gamma} < +\infty \right\},$$

where

$$\|\psi\|_{\frac{1}{2},\Gamma} := \left(\|\psi\|_{0,\Gamma}^2 + \int_\Gamma \int_\Gamma \frac{(\psi(\mathbf{x}) - \psi(\mathbf{y}))^2}{|\mathbf{x} - \mathbf{y}|^2} \, d\Gamma_x \, d\Gamma_y \right)^{\frac{1}{2}}. \tag{2.1}$$

It is well-known that there exists a linear continuous mapping $E \in \mathcal{L}(H^{\frac{1}{2}}(\Gamma), H^1(\Omega))$ such that

$$E\psi|_\Gamma = \psi \quad \text{on } \Gamma,$$

where $E\psi|_\Gamma$ denotes the trace of $E\psi$ on Γ .

The topological dual space of $H^{\frac{1}{2}}(\Gamma)$ is denoted $H^{-\frac{1}{2}}(\Gamma)$, $\langle \cdot, \cdot \rangle_{\frac{1}{2},\Gamma}$ stands for the duality pairing and

$$\|\mu\|_{-\frac{1}{2},\Gamma} := \sup_{\psi \in H^{\frac{1}{2}}(\Gamma), \psi \neq 0} \frac{\langle \mu, \psi \rangle_{\frac{1}{2},\Gamma}}{\|\psi\|_{\frac{1}{2},\Gamma}} \tag{2.2}$$

is the norm in $H^{-\frac{1}{2}}(\Gamma)$.

The Cartesian product of m previous spaces and their elements are denoted by bold characters. The respective norms are introduced as follows:

$$\begin{aligned} \|\mathbf{v}\|_{k,\Omega} &:= \left(\sum_{\ell=1}^m \|v_\ell\|_{k,\Omega}^2 \right)^{1/2}, \quad \mathbf{v} = (v_1, \dots, v_m) \in \mathbf{H}^k(\Omega) \\ \|\boldsymbol{\psi}\|_{\frac{1}{2},\Gamma} &:= \left(\sum_{\ell=1}^m \|\psi_\ell\|_{\frac{1}{2},\Gamma}^2 \right)^{1/2}, \quad \boldsymbol{\psi} = (\psi_1, \dots, \psi_m) \in \mathbf{H}^{\frac{1}{2}}(\Gamma) \\ \|\boldsymbol{\mu}\|_{-\frac{1}{2},\Gamma} &:= \left(\sum_{\ell=1}^m \|\mu_\ell\|_{-\frac{1}{2},\Gamma}^2 \right)^{1/2}, \quad \boldsymbol{\mu} = (\mu_1, \dots, \mu_m) \in \mathbf{H}^{-\frac{1}{2}}(\Gamma). \end{aligned}$$

A special attention will be paid to the case when $k = 1$ and $n = m = 3$, i.e. $\Omega \subset \mathbb{R}^3$ and $\mathbf{H}^1(\Omega) = (H^1(\Omega))^3$. Let $\mathbf{V} \subset \mathbf{H}^1(\Omega)$ be a subspace of functions vanishing on a non-empty portion $\Gamma_{\mathbf{u}}$ open in $\partial\Omega$:

$$\mathbf{V} = \left\{ \mathbf{v} = (v_1, v_2, v_3) \in \mathbf{H}^1(\Omega) \mid v_i = 0 \text{ on } \Gamma_{\mathbf{u}}, i = 1, 2, 3 \right\}.$$

In addition to the classical norm $\|\cdot\|_{1,\Omega}$, we introduce the energetic norm $\|\cdot\|_{1,\Omega}$ in \mathbf{V} corresponding to the scalar product

$$((\mathbf{u}, \mathbf{v}))_{1,\Omega} := \int_{\Omega} \sum_{i,j=1}^3 \varepsilon_{ij}(\mathbf{u}) \varepsilon_{ij}(\mathbf{v}) \, d\mathbf{x},$$

where $\varepsilon_{ij}(\mathbf{u})$ is the i, j th component of the linearized strain tensor

$$\boldsymbol{\varepsilon}(\mathbf{u}) := \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T).$$

From the Korn's inequality it follows that $\|\cdot\|_{1,\Omega}$ and $\|\cdot\|_{1,\Omega}$ are equivalent in \mathbf{V} .

Let $\Gamma \subseteq \partial\Omega \setminus \overline{\Gamma_{\mathbf{u}}}$ be a portion of $\partial\Omega$ such that $\text{dist}(\Gamma, \Gamma_{\mathbf{u}}) > 0^1$ and denote by $\mathbf{H}_{\Gamma}^{\frac{1}{2}}$ the space of traces $\mathbf{v}|_{\Gamma}$ for $\mathbf{v} \in \mathbf{V}$. Besides the norm $\|\cdot\|_{\frac{1}{2},\Gamma}$ defined by (2.1) we introduce the following quotient norm:

$$\|\boldsymbol{\psi}\|_{\frac{1}{2},\Gamma} := \inf_{\mathbf{v} \in \mathbf{V}, \mathbf{v}=\boldsymbol{\psi} \text{ on } \Gamma} \|\mathbf{v}\|_{1,\Omega}.$$

It is readily seen that

$$\|\boldsymbol{\psi}\|_{\frac{1}{2},\Gamma} = \|\mathbf{w}_{\boldsymbol{\psi}}\|_{1,\Omega},$$

where $\mathbf{w}_{\boldsymbol{\psi}} \in \mathbf{V}$ is the unique solution of the elliptic problem:

$$\begin{cases} ((\mathbf{w}_{\boldsymbol{\psi}}, \mathbf{v}))_{1,\Omega} = 0 & \forall \mathbf{v} \in \mathbf{V}_0 = \{\mathbf{z} \in \mathbf{V} \mid \mathbf{z} = \mathbf{0} \text{ on } \Gamma\}, \\ \mathbf{w}_{\boldsymbol{\psi}} = \boldsymbol{\psi} \text{ on } \Gamma. \end{cases}$$

From the Banach theorem on the isomorphism it follows that $\|\cdot\|_{\frac{1}{2},\Gamma}$ and $\|\cdot\|_{\frac{1}{2},\Gamma}$ are equivalent in $\mathbf{H}_{\Gamma}^{\frac{1}{2}}$.

Let $\mathbf{H}_{\Gamma}^{-\frac{1}{2}}$ be the dual space of $\mathbf{H}_{\Gamma}^{\frac{1}{2}}$ and denote $\|\cdot\|_{-\frac{1}{2},\Gamma}$ the dual norm corresponding to $\|\cdot\|_{\frac{1}{2},\Gamma}$:

$$\|\boldsymbol{\mu}\|_{-\frac{1}{2},\Gamma} := \sup_{\boldsymbol{\psi} \in \mathbf{H}_{\Gamma}^{\frac{1}{2}}, \boldsymbol{\psi} \neq \mathbf{0}} \frac{\langle \boldsymbol{\mu}, \boldsymbol{\psi} \rangle_{\frac{1}{2},\Gamma}}{\|\boldsymbol{\psi}\|_{\frac{1}{2},\Gamma}}.$$

¹This assumption is not necessary but it simplifies the presentation.

It is readily seen that

$$\|\boldsymbol{\mu}\|_{-\frac{1}{2},\Gamma} = \|\mathbf{w}_\mu\|_{1,\Omega}, \tag{2.3}$$

where $\mathbf{w}_\mu \in \mathbf{V}$ is the unique solution of the elliptic problem:

$$((\mathbf{w}_\mu, \mathbf{v}))_{1,\Omega} = \langle \boldsymbol{\mu}, \mathbf{v} \rangle_{\frac{1}{2},\Gamma} \quad \forall \mathbf{v} \in \mathbf{V}. \tag{2.4}$$

In addition, $\|\cdot\|_{-\frac{1}{2},\Gamma}$ and $\|\cdot\|_{-\frac{1}{2},\Gamma}$ are equivalent in $\mathbf{H}_\Gamma^{-\frac{1}{2}}$.

3. VARIATIONAL FORMULATION OF THE SIGNORINI PROBLEM WITH GIVEN FRICTION

Let us consider an elastic body occupying a domain $\Omega \subset \mathbb{R}^3$ with the Lipschitz boundary $\partial\Omega$ which is split into three non-empty, non-overlapping parts $\Gamma_{\mathbf{u}}$, $\Gamma_{\mathbf{g}}$ and Γ_c , $dist(\Gamma_c, \Gamma_{\mathbf{u}}) > 0$: $\partial\Omega = \overline{\Gamma_{\mathbf{u}}} \cup \overline{\Gamma_{\mathbf{g}}} \cup \overline{\Gamma_c}$. For the sake of simplicity of our presentation we shall suppose that $\Omega = (0, a) \times (0, b) \times (0, c)$, where $a, b, c > 0$ are given and $\Gamma_c = (0, a) \times (0, b) \times \{0\}$. The zero displacements are prescribed on $\Gamma_{\mathbf{u}}$ while surface tractions of density $\mathbf{g} \in \mathbf{L}^2(\Gamma_{\mathbf{g}})$ act on $\Gamma_{\mathbf{g}}$. The body is unilaterally supported along Γ_c by the rigid half-space

$$S = \{\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 \leq 0\}.$$

Besides constraints on the deformation of Ω we shall take into account effects of friction. We restrict ourselves to the simplest model, namely the model with *given friction*. Finally the body is subject to volume forces of density $\mathbf{f} \in \mathbf{L}^2(\Omega)$. Our aim is to find an equilibrium state of Ω .

The classical formulation of the previous problem consists in finding a displacement vector $\mathbf{u} = (u_1, u_2, u_3)$ satisfying the equations and conditions (3.1)–(3.4):

$$\begin{cases} \operatorname{div} \boldsymbol{\sigma}(\mathbf{u}) + \mathbf{f} &= \mathbf{0} \text{ in } \Omega, \\ \boldsymbol{\sigma}(\mathbf{u})\mathbf{n} - \mathbf{g} &= \mathbf{0} \text{ on } \Gamma_{\mathbf{g}}, \\ \mathbf{u} &= \mathbf{0} \text{ on } \Gamma_{\mathbf{u}}. \end{cases} \tag{3.1}$$

The symbol $\boldsymbol{\sigma}(\mathbf{u}) = (\sigma_{ij}(\mathbf{u}))_{i,j=1}^3$ stands for a symmetric stress tensor which is related to the linearized strain tensor $\boldsymbol{\varepsilon}(\mathbf{u})$ by means of a linear Hooke’s law:

$$\boldsymbol{\sigma}(\mathbf{u}) = A\boldsymbol{\varepsilon}(\mathbf{u}), \tag{3.2}$$

where A is a fourth order symmetric tensor satisfying the usual ellipticity conditions in Ω . Further \mathbf{n} is the outward unit normal vector to $\partial\Omega$ and div denotes the divergence operator acting on tensor functions (the summation convention is adopted):

$$\operatorname{div} \boldsymbol{\sigma} = \left(\frac{\partial \sigma_{ij}}{\partial x_j} \right)_{i=1}^3.$$

Let $\mathbf{u}_n := u_n \mathbf{n}$, $\mathbf{u}_t := \mathbf{u} - u_n \mathbf{n}$ be the normal, tangential component of the displacement vector \mathbf{u} and $\boldsymbol{\sigma}_n(\mathbf{u}) := \boldsymbol{\sigma}(\mathbf{u})\mathbf{n}$, and $\boldsymbol{\sigma}_t(\mathbf{u}_t) := \boldsymbol{\sigma}(\mathbf{u})\mathbf{n} - \boldsymbol{\sigma}_n(\mathbf{u})\mathbf{n}$ the normal, tangential component of the stress vector $\boldsymbol{\sigma}(\mathbf{u})\mathbf{n}$, respectively, where $u_n := \mathbf{u} \cdot \mathbf{n}$ and $\sigma_n(\mathbf{u}) := (\boldsymbol{\sigma}(\mathbf{u})\mathbf{n}) \cdot \mathbf{n}$. Here \cdot denotes the scalar product of two vectors.

The unilateral and friction conditions prescribed on Γ_c read as follows:

$$u_n \leq 0, \quad \sigma_n(\mathbf{u}) \leq 0, \quad u_n \sigma_n(\mathbf{u}) = 0 \quad \text{on } \Gamma_c, \tag{3.3}$$

$$\begin{cases} \|\boldsymbol{\sigma}_t(\mathbf{u})\| \leq s \text{ a.e. on } \Gamma_c, \\ \text{if } \|\boldsymbol{\sigma}_t(\mathbf{u})(\mathbf{x})\| < s(\mathbf{x}) \text{ then } \mathbf{u}_t(\mathbf{x}) = \mathbf{0}, \quad \mathbf{x} \in \Gamma_c, \\ \text{if } \|\boldsymbol{\sigma}_t(\mathbf{u})(\mathbf{x})\| = s(\mathbf{x}) > 0 \text{ then there exists } \nu(\mathbf{x}) \geq 0 \\ \text{such that } \mathbf{u}_t(\mathbf{x}) = -\nu(\mathbf{x})\boldsymbol{\sigma}_t(\mathbf{u})(\mathbf{x}), \quad \mathbf{x} \in \Gamma_c. \end{cases} \tag{3.4}$$

Here $s \in L^2(\Gamma_c)$, $s \geq 0$ is a given function and $\|\boldsymbol{\sigma}_t(\mathbf{u})\|$ is the Euclidean norm of $\boldsymbol{\sigma}_t(\mathbf{u})$.

Remark 3.1. In the classical Coulomb's law of friction [2, 6] the given slip bound s is replaced by the term $\mathbf{F}|\sigma_n(\mathbf{u})|$ which is not known *a priori*. The symbol \mathbf{F} stands for the coefficient of Coulomb friction.

Remark 3.2. Taking into account the fact that $\mathbf{n} = (0, 0, -1)$ on Γ_c , we have:

$$u_n = -u_3, \sigma_n(\mathbf{u}) = \sigma_{33}(\mathbf{u}),$$

$$\boldsymbol{\sigma}_t(\mathbf{u}) = (-\sigma_{13}(\mathbf{u}), -\sigma_{23}(\mathbf{u}), 0).$$

Remark 3.3. Let $\mathbf{t}_1(\mathbf{x}), \mathbf{t}_2(\mathbf{x}), \mathbf{x} \in \Gamma_c$ be two unit vectors such that the triplet $\{\mathbf{n}(\mathbf{x}), \mathbf{t}_1(\mathbf{x}), \mathbf{t}_2(\mathbf{x})\}$ forms a local orthonormal basis in \mathbb{R}^3 with origin at $\mathbf{x} \in \Gamma_c$. Then any vector function $\mathbf{w} : \Gamma_c \rightarrow \mathbb{R}^3$ can be represented in the local coordinate system $\{\mathbf{n}(\mathbf{x}), \mathbf{t}_1(\mathbf{x}), \mathbf{t}_2(\mathbf{x})\}$ as follows:

$$\mathbf{w}(\mathbf{x}) = (w_n(\mathbf{x}), \mathbf{w}_t(\mathbf{x})) \in \mathbb{R} \times \mathbb{R}^2,$$

where $w_n(\mathbf{x}) = \mathbf{w}(\mathbf{x}) \cdot \mathbf{n}$, $\mathbf{w}_t(\mathbf{x}) = (w_{t_1}(\mathbf{x}), w_{t_2}(\mathbf{x}))$, $w_{t_j}(\mathbf{x}) = \mathbf{w}(\mathbf{x}) \cdot \mathbf{t}_j(\mathbf{x})$, $j = 1, 2$. In particular, we shall use this representation for the trace of displacement vectors \mathbf{v} on Γ_c in what follows, *i.e.* $\mathbf{v}(\mathbf{x}) = (v_n(\mathbf{x}), \mathbf{v}_t(\mathbf{x})) \in \mathbb{R} \times \mathbb{R}^2$, $\mathbf{x} \in \Gamma_c$ and

$$\|\mathbf{v}_t\| = \left(|v_{t_1}|^2 + |v_{t_2}|^2 \right)^{\frac{1}{2}} \quad a.e. \quad \text{on } \Gamma_c.$$

To give the variational formulation of our problem we first introduce the Hilbert space \mathbf{V} :

$$\mathbf{V} = \left\{ \mathbf{v} \in \mathbf{H}^1(\Omega) \mid \mathbf{v} = \mathbf{0} \text{ on } \Gamma_u \right\},$$

and its closed convex subset \mathbf{K} of *kinematically admissible displacements*:

$$\mathbf{K} = \left\{ \mathbf{v} \in \mathbf{V} \mid v_n \leq 0 \text{ on } \Gamma_c \right\}.$$

Next, we define:

$$a(\mathbf{u}, \mathbf{v}) := \int_{\Omega} A \boldsymbol{\varepsilon}(\mathbf{u}) \boldsymbol{\varepsilon}(\mathbf{v}) \, d\mathbf{x},$$

$$L(\mathbf{v}) := \int_{\Omega} \mathbf{f} \cdot \mathbf{v} \, d\mathbf{x} + \int_{\Gamma_g} \mathbf{g} \cdot \mathbf{v} \, d\Gamma,$$

$$j(\mathbf{v}) := \int_{\Gamma_c} s \|\mathbf{v}_t\| \, d\Gamma.$$

The *primal variational formulation* of the contact problem with given friction reads as follows:

$$\left\{ \begin{array}{l} \text{Find } \mathbf{u} \in \mathbf{K} \text{ such that} \\ \mathbf{J}(\mathbf{u}) \leq \mathbf{J}(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{K}, \end{array} \right. \tag{3.5}$$

where

$$\mathbf{J}(\mathbf{v}) = \frac{1}{2} a(\mathbf{v}, \mathbf{v}) - L(\mathbf{v}) + j(\mathbf{v})$$

is the total potential energy functional.

On the basis of the previous assumptions we have:

Theorem 3.1. *Problem (3.5) has a unique solution.*

In order to release the unilateral constraint $v_n \leq 0$ on Γ_c and to regularize the non-differentiable term j we use a duality approach. To this end we introduce several notations. Let $\gamma_n : \mathbf{V} \rightarrow L^2(\Gamma_c)$ be the trace mapping defined by

$$\gamma_n \mathbf{v} := v_n \text{ on } \Gamma_c, \mathbf{v} \in \mathbf{V}.$$

Further let

$$\begin{aligned} H_{\Gamma_c}^{\frac{1}{2}} &:= \gamma_n \mathbf{V}, \quad H_{\Gamma_c}^{-\frac{1}{2}} := \text{the dual of } H_{\Gamma_c}^{\frac{1}{2}}, \\ \mathbf{H}_{\Gamma_c}^{\frac{1}{2}} &:= (H_{\Gamma_c}^{\frac{1}{2}})^3, \quad \mathbf{H}_{\Gamma_c}^{-\frac{1}{2}} := (H_{\Gamma_c}^{-\frac{1}{2}})^3, \\ M_n &= \left\{ \mu_n \in H_{\Gamma_c}^{-\frac{1}{2}}, \quad \mu_n \geq 0 \text{ on } \Gamma_c \right\}, \end{aligned}$$

$$\begin{aligned} \mathbf{M}_t &= \left\{ \boldsymbol{\mu}_t = (\mu_{t_1}, \mu_{t_2}) \in \mathbf{L}^2(\Gamma_c), \quad \|\boldsymbol{\mu}_t\| \leq s \quad \text{a.e. on } \Gamma_c \right\} \\ &= \left\{ \boldsymbol{\mu}_t \in \mathbf{L}^2(\Gamma_c) \mid \int_{\Gamma_c} \boldsymbol{\mu}_t \cdot \boldsymbol{\psi} \, d\Gamma - \int_{\Gamma_c} s \|\boldsymbol{\psi}\| \, d\Gamma \leq 0 \quad \forall \boldsymbol{\psi} \in \mathbf{L}^2(\Gamma_c) \right\}. \end{aligned}$$

Remark 3.4. In accordance with the previous section, the spaces $H_{\Gamma_c}^{\frac{1}{2}}, H_{\Gamma_c}^{-\frac{1}{2}}$ will be endowed with two equivalent norms $\|\cdot\|_{\frac{1}{2}, \Gamma_c}, \|\cdot\|_{-\frac{1}{2}, \Gamma_c}$ and $\|\cdot\|_{-\frac{1}{2}, \Gamma_c}, \|\cdot\|_{\frac{1}{2}, \Gamma_c}$, respectively.

It is easy to verify that

$$\min_{\mathbf{K}} \mathbf{J}(\mathbf{v}) = \min_{\mathbf{V}} \sup_{M_n \times \mathbf{M}_t} \mathbf{L}(\mathbf{v}, \mu_n, \boldsymbol{\mu}_t),$$

where $\mathbf{L} : \mathbf{V} \times M_n \times \mathbf{M}_t \rightarrow \mathbb{R}$ is the Lagrangian defined by

$$\mathbf{L}(\mathbf{v}, \mu_n, \boldsymbol{\mu}_t) = \mathbf{J}(\mathbf{v}) + \langle \mu_n, v_n \rangle_{\frac{1}{2}, \Gamma_c} + \int_{\Gamma_c} \boldsymbol{\mu}_t \cdot \mathbf{v}_t \, d\Gamma,$$

and $\langle \cdot, \cdot \rangle_{\frac{1}{2}, \Gamma_c}$ stands for the duality pairing between $H_{\Gamma_c}^{-\frac{1}{2}}$ and $H_{\Gamma_c}^{\frac{1}{2}}$.

By a *mixed formulation* of (3.5) we call a problem of finding a *saddle-point* of \mathbf{L} on $\mathbf{V} \times M_n \times \mathbf{M}_t$:

$$\left\{ \begin{array}{l} \text{Find } (\mathbf{w}, \lambda_n, \boldsymbol{\lambda}_t) \in \mathbf{V} \times M_n \times \mathbf{M}_t \text{ such that} \\ \mathbf{L}(\mathbf{w}, \mu_n, \boldsymbol{\mu}_t) \leq \mathbf{L}(\mathbf{w}, \lambda_n, \boldsymbol{\lambda}_t) \leq \mathbf{L}(\mathbf{v}, \lambda_n, \boldsymbol{\lambda}_t) \quad \forall (\mathbf{v}, \mu_n, \boldsymbol{\mu}_t) \in \mathbf{V} \times M_n \times \mathbf{M}_t, \end{array} \right.$$

or equivalently

$$\left\{ \begin{array}{l} \text{Find } (\mathbf{w}, \boldsymbol{\lambda}) \in \mathbf{V} \times \mathbf{M} \text{ such that} \\ a(\mathbf{w}, \mathbf{v}) + b(\boldsymbol{\lambda}, \mathbf{v}) = L(\mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{V}, \\ b(\boldsymbol{\mu} - \boldsymbol{\lambda}, \mathbf{w}) \leq 0, \quad \forall \boldsymbol{\mu} \in \mathbf{M}, \end{array} \right. \tag{3.6}$$

where $\boldsymbol{\lambda} := (\lambda_n, \boldsymbol{\lambda}_t)$, $\mathbf{M} := M_n \times \mathbf{M}_t$ and

$$b(\boldsymbol{\mu}, \mathbf{v}) := \langle \mu_n, v_n \rangle_{\frac{1}{2}, \Gamma_c} + \int_{\Gamma_c} \boldsymbol{\mu}_t \cdot \mathbf{v}_t \, d\Gamma, \quad \boldsymbol{\mu} \in \mathbf{M}, \mathbf{v} \in \mathbf{V}.$$

The relation between (3.5) and (3.6) follows from the next theorem.

Theorem 3.2. *There exists a unique solution $(\mathbf{w}, \boldsymbol{\lambda})$ of (3.6). In addition,*

$$\mathbf{w} = \mathbf{u} \quad \text{in } \Omega, \quad \lambda_n = -\sigma_n(\mathbf{u}), \quad \boldsymbol{\lambda}_t = -\boldsymbol{\sigma}_t(\mathbf{u}) \quad \text{on } \Gamma_c,$$

where $\mathbf{u} \in \mathbf{K}$ is the solution to (3.5).

Proof is obvious.

Convention: the solution to (3.6) will be denoted by $(\mathbf{u}, \lambda_n, \boldsymbol{\lambda}_t)$ in what follows.

4. FINITE ELEMENT APPROXIMATION

The present section is devoted to a finite element approximation of the saddle-point problem (3.6). The key point lies in the finite element discretization of the closed convex set \mathbf{M} of Lagrange multipliers which leads to a well-posed discrete problem and gives also a good convergence rate for approximate solutions.

We start with discretizations of \mathbf{V} , M_n and \mathbf{M}_t .

Let $\{\mathbf{T}_h\}$, $h \rightarrow 0+$ be a regular family (see [5]) of partitions of $\bar{\Omega}$ into tetrahedrons κ . With any $\mathbf{T}_h \in \{\mathbf{T}_h\}$ we associate the following space of continuous *piecewise linear* functions:

$$V_h = \left\{ v_h \in \mathbf{C}(\bar{\Omega}) \mid v_{h|\kappa} \in \mathbb{P}_1(\kappa) \quad \forall \kappa \in \mathbf{T}_h, \quad v_h = 0 \quad \text{on } \Gamma_{\mathbf{u}} \right\},$$

$$\mathbf{V}_h = (V_h)^3.$$

Further, let $\{\mathcal{R}_H\}$, $H \rightarrow 0+$ be a strongly regular family of rectangulations of $\bar{\Gamma}_c$ with $H > 0$ being the norm of \mathcal{R}_H . On every $\mathcal{R}_H \in \{\mathcal{R}_H\}$ we construct the space of *piecewise constant* functions:

$$L_H = \left\{ \mu_H \in L^2(\Gamma_c) \mid \mu_{H|\mathcal{R}} \in \mathbb{P}_0(\mathcal{R}) \quad \forall \mathcal{R} \in \mathcal{R}_H \right\},$$

$$\mathbf{L}_H = (L_H)^3$$

and the following convex sets:

$$M_{Hn} = \left\{ \mu_{Hn} \in L_H \mid \mu_{Hn} \geq 0 \quad \text{a.e. on } \Gamma_c \right\}, \tag{4.1}$$

$$\mathbf{M}_{Ht} = \left\{ \boldsymbol{\mu}_{Ht} \in (L_H)^2 \mid \int_{\Gamma_c} \boldsymbol{\mu}_{Ht} \cdot \boldsymbol{\psi}_H \, d\Gamma - \int_{\Gamma_c} s \|\boldsymbol{\psi}_H\| \, d\Gamma \leq 0 \quad \forall \boldsymbol{\psi}_H \in (L_H)^2 \right\}. \tag{4.2}$$

Recall that $\|\cdot\|$ stands for the Euclidean norm in \mathbb{R}^2 . The set \mathbf{M}_{Ht} is the *external* approximation of \mathbf{M}_t , while M_{Hn} is the *internal* approximation of M_n .

The discretization of (3.6) is defined in a standard way:

$$\left\{ \begin{array}{l} \text{Find } \mathbf{u}_h \in \mathbf{V}_h \text{ and } \boldsymbol{\lambda}_H \in \mathbf{M}_H \text{ such that} \\ a(\mathbf{u}_h, \mathbf{v}_h) + b(\boldsymbol{\lambda}_H, \mathbf{v}_h) = L(\mathbf{v}_h), \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \\ b(\boldsymbol{\mu}_H - \boldsymbol{\lambda}_H, \mathbf{u}_h) \leq 0, \quad \forall \boldsymbol{\mu}_H \in \mathbf{M}_H, \end{array} \right. \tag{4.3}$$

where $\boldsymbol{\lambda}_H := (\lambda_{Hn}, \boldsymbol{\lambda}_{Ht})$ and $\mathbf{M}_H := M_{Hn} \times \mathbf{M}_{Ht}$. To ensure the existence and uniqueness of a solution to (4.3), the following stability condition is needed:

$$\left\{ \boldsymbol{\mu}_H \in \mathbf{L}_H \mid b(\boldsymbol{\mu}_H, \mathbf{v}_h) = 0 \quad \forall \mathbf{v}_h \in \mathbf{V}_h \right\} = \{\mathbf{0}\}. \tag{4.4}$$

In the forthcoming convergence analysis, we will need more information on the compatibility between the spaces \mathbf{L}_H and \mathbf{V}_h . We shall introduce a stronger assumption, namely the *Ladyzhenskaya-Babuska-Brezzi* condition which in our particular case takes the form

$$\sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{b(\boldsymbol{\mu}_H, \mathbf{v}_h)}{\|\mathbf{v}_h\|_{1,\Omega}} \geq \beta \|\boldsymbol{\mu}_H\|_{-\frac{1}{2}, \Gamma_c} \quad \forall \boldsymbol{\mu}_H \in \mathbf{L}_H, \quad (4.5)$$

where $\beta > 0$ is independent of h and H . The dual norm $\|\cdot\|_{-\frac{1}{2}, \Gamma_c}$ in (4.5) is evaluated by means of (2.3). Next we shall suppose that the auxiliary problem (2.4) is *regular* in the following sense:

$$\left\{ \begin{array}{l} \text{there exists } \varepsilon \in (0, 1/2) \text{ such that for every } \boldsymbol{\mu} \in \mathbf{H}_{\Gamma_c}^{-\frac{1}{2}+\varepsilon} := \text{the dual of } \mathbf{H}_{\Gamma_c}^{\frac{1}{2}-\varepsilon} \\ \text{the solution } \mathbf{w}_\mu \text{ of (2.4) belongs to } \mathbf{V} \cap \mathbf{H}^{1+\varepsilon}(\Omega) \text{ and} \\ \|\mathbf{w}_\mu\|_{1+\varepsilon, \Omega} \leq C(\varepsilon) \|\boldsymbol{\mu}\|_{-\frac{1}{2}+\varepsilon, \Gamma_c}, \end{array} \right.$$

holds with a positive constant $C(\varepsilon)$ which depends solely on $\varepsilon > 0$. If it is so then using exactly the same approach as in [12] one can prove the next lemma.

Lemma 4.1. *Let us suppose that (2.4) is regular, $\{\mathcal{R}_H\}$ is a strongly regular family of rectangulations of $\bar{\Gamma}_c$ and the ratio h/H is sufficiently small. Then the Ladyzhenskaya-Babuska-Brezzi condition (4.5) is satisfied.*

5. ERROR ANALYSIS

In this section we establish *a priori* error estimates for the mixed finite element approximation (4.3). We start with the following lemma [3]:

Lemma 5.1. *Let $(\mathbf{u}, \boldsymbol{\lambda})$, $(\mathbf{u}_h, \boldsymbol{\lambda}_H)$ be the solution of (3.6), (4.3), respectively. Then for any $\mathbf{v}_h \in \mathbf{V}_h$ and $\boldsymbol{\mu}_H \in \mathbf{M}_H$ it holds:*

$$\begin{aligned} a(\mathbf{u} - \mathbf{u}_h, \mathbf{u} - \mathbf{u}_h) &\leq a(\mathbf{u} - \mathbf{u}_h, \mathbf{u} - \mathbf{v}_h) + b(\boldsymbol{\lambda} - \boldsymbol{\mu}_H, \mathbf{u}_h - \mathbf{u}) + b(\boldsymbol{\lambda} - \boldsymbol{\lambda}_H, \mathbf{u} - \mathbf{v}_h) \\ &\quad + b(\boldsymbol{\lambda} - \boldsymbol{\mu}_H, \mathbf{u}) + b(\boldsymbol{\lambda}_H - \boldsymbol{\lambda}, \mathbf{u}). \end{aligned} \quad (5.1)$$

Proof. Let \mathbf{v}_h be an element of \mathbf{V}_h . Then

$$a(\mathbf{u} - \mathbf{u}_h, \mathbf{u} - \mathbf{u}_h) = a(\mathbf{u} - \mathbf{u}_h, \mathbf{u} - \mathbf{v}_h) + a(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h - \mathbf{u}_h).$$

From the equations in (3.6) and (4.3), we obtain:

$$\begin{aligned} a(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h - \mathbf{u}_h) &= L(\mathbf{v}_h - \mathbf{u}_h) - b(\boldsymbol{\lambda}, \mathbf{v}_h - \mathbf{u}_h) - L(\mathbf{v}_h - \mathbf{u}_h) + b(\boldsymbol{\lambda}_H, \mathbf{v}_h - \mathbf{u}_h) \\ &= b(\boldsymbol{\lambda}, \mathbf{u}_h - \mathbf{v}_h) + b(\boldsymbol{\lambda}_H, \mathbf{v}_h - \mathbf{u}_h). \end{aligned}$$

Further

$$a(\mathbf{u} - \mathbf{u}_h, \mathbf{u} - \mathbf{u}_h) = a(\mathbf{u} - \mathbf{u}_h, \mathbf{u} - \mathbf{v}_h) + b(\boldsymbol{\lambda}, \mathbf{u}_h - \mathbf{v}_h) + b(\boldsymbol{\lambda}_H, \mathbf{v}_h - \mathbf{u}_h),$$

and finally

$$\begin{aligned} a(\mathbf{u} - \mathbf{u}_h, \mathbf{u} - \mathbf{u}_h) &= a(\mathbf{u} - \mathbf{u}_h, \mathbf{u} - \mathbf{v}_h) + b(\boldsymbol{\mu}_H - \boldsymbol{\lambda}, \mathbf{u} - \mathbf{u}_h) + b(\boldsymbol{\lambda} - \boldsymbol{\lambda}_H, \mathbf{u} - \mathbf{v}_h) \\ &\quad + b(\boldsymbol{\lambda} - \boldsymbol{\mu}_H, \mathbf{u}) + b(\boldsymbol{\lambda}_H - \boldsymbol{\lambda}, \mathbf{u}) + b(\boldsymbol{\mu}_H - \boldsymbol{\lambda}_H, \mathbf{u}_h). \end{aligned}$$

The inequality in (4.3) implies that $b(\boldsymbol{\mu}_H - \boldsymbol{\lambda}_H, \mathbf{u}_h) \leq 0$ for any $\boldsymbol{\mu}_H \in \mathbf{M}_H$. From this the error estimate (5.1) follows. \square

We now derive an upper bound for $\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega}$.

Lemma 5.2. *Let $(\mathbf{u}, \boldsymbol{\lambda})$, $(\mathbf{u}_h, \boldsymbol{\lambda}_H)$ be the solution of (3.6), (4.3), respectively. Suppose that $\mathbf{u} \in \mathbf{H}^2(\Omega)$ and the fourth order tensor A defining the Hooke's law is sufficiently smooth. Then*

$$\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega}^2 \leq C \left[h \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_H\|_{-\frac{1}{2},\Gamma_c} + H^{\frac{1}{2}+\varepsilon} \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_H\|_{-\frac{1}{2}+\varepsilon,\Gamma_c} + H^{\frac{3}{2}} + H \right], \tag{5.2}$$

where $\varepsilon > 0$ is arbitrarily small and $C > 0$ is a positive constant which does not depend on h, H .

Proof. We shall estimate each term on the right hand side of (5.1). But before proving these estimates, let us recall some approximation results. Let \mathbf{I}_h be the Lagrange interpolation operator with values in \mathbf{V}_h . There exists a constant $C > 0$ which does not depend on h and such that for all $\mathbf{v} \in \mathbf{H}^2(\Omega)$ (see [5]) it holds:

$$\|\mathbf{v} - \mathbf{I}_h \mathbf{v}\|_{1,\Omega} \leq Ch \|\mathbf{v}\|_{2,\Omega}. \tag{5.3}$$

Next, we introduce the projection operator $\boldsymbol{\Pi}_H : \mathbf{L}^2(\Gamma_c) \rightarrow \mathbf{L}_H$ by

$$\boldsymbol{\Pi}_H \boldsymbol{\varphi} \in \mathbf{L}_H, \quad \int_{\Gamma_c} (\boldsymbol{\Pi}_H \boldsymbol{\varphi} - \boldsymbol{\varphi}) \cdot \boldsymbol{\mu}_H \, d\Gamma = 0, \quad \forall \boldsymbol{\mu}_H \in \mathbf{L}_H. \tag{5.4}$$

The operator $\boldsymbol{\Pi}_H$ has the following approximation property (see [19]): there exists a positive constant C independent of H such that $\forall \boldsymbol{\varphi} \in \mathbf{H}_{\Gamma_c}^\nu$, $\nu = \frac{1}{2}, 1$ and any $\boldsymbol{\mu} \in [0, \frac{1}{2})$, it holds that

$$\|\boldsymbol{\varphi} - \boldsymbol{\Pi}_H \boldsymbol{\varphi}\|_{\boldsymbol{\mu},\Gamma_c} \leq C H^{\nu-\boldsymbol{\mu}} \|\boldsymbol{\varphi}\|_{\nu,\Gamma_c}. \tag{5.5}$$

Moreover, if $\boldsymbol{\varphi} \in \mathbf{L}^2(\Gamma_c)$ then

$$\|\boldsymbol{\varphi} - \boldsymbol{\Pi}_H \boldsymbol{\varphi}\|_{-\frac{1}{2}+\varepsilon,\Gamma_c} \leq C H^{\frac{1}{2}-\varepsilon} \|\boldsymbol{\varphi} - \boldsymbol{\Pi}_H \boldsymbol{\varphi}\|_{0,\Gamma_c}, \quad \varepsilon \in [0, \frac{1}{2}). \tag{5.6}$$

Finally, let us note that the trace theorem implies that

$$\|\boldsymbol{\lambda}\|_{\frac{1}{2},\Gamma_c} \leq C \|\mathbf{u}\|_{2,\Omega}, \tag{5.7}$$

provided that the tensor A is sufficiently smooth (the constant $C > 0$ depends only on A).

Next we estimate each term appearing on the right hand side of (5.1) by choosing: $\mathbf{v}_h = \mathbf{I}_h \mathbf{u}$ and $\boldsymbol{\mu}_H = \boldsymbol{\Pi}_H \boldsymbol{\lambda} = (\pi_{Hn} \lambda_n, \pi_{Ht} \lambda_t)$, where π_{Hn} and π_{Ht} are the components of $\boldsymbol{\Pi}_H$ in L_H and $(L_H)^2$, respectively. It is readily seen that $\boldsymbol{\Pi}_H \boldsymbol{\lambda} \in \mathbf{M}_H$ as follows from the definition of \mathbf{M}, \mathbf{M}_H and (5.4).

(i) The first term is evaluated by using continuity of $a(\cdot, \cdot)$ and (5.3):

$$a(\mathbf{u} - \mathbf{u}_h, \mathbf{u} - \mathbf{v}_h) \leq Ch \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega}.$$

(ii) The second is estimated by using (5.5), (5.6), (5.7) and the trace theorem:

$$b(\boldsymbol{\lambda} - \boldsymbol{\mu}_H, \mathbf{u}_h - \mathbf{u}) \leq C \|\boldsymbol{\lambda} - \boldsymbol{\Pi}_H \boldsymbol{\lambda}\|_{-\frac{1}{2},\Gamma_c} \|\mathbf{u}_h - \mathbf{u}\|_{1,\Omega} \leq CH \|\mathbf{u}_h - \mathbf{u}\|_{1,\Omega}.$$

(iii) The third term is estimated as follows:

$$b(\boldsymbol{\lambda} - \boldsymbol{\lambda}_H, \mathbf{u} - \mathbf{v}_h) \leq C \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_H\|_{-\frac{1}{2},\Gamma_c} \|\mathbf{u} - \mathbf{v}_h\|_{1,\Omega} \leq Ch \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_H\|_{-\frac{1}{2},\Gamma_c}.$$

(iv) To evaluate the fourth term, we invoke the definition of the L^2 -projection operators:

$$\begin{aligned}
b(\boldsymbol{\lambda} - \boldsymbol{\mu}_H, \mathbf{u}) &= \int_{\Gamma_c} (\lambda_n - \pi_{Hn} \lambda_n) u_n d\Gamma + \int_{\Gamma_c} (\boldsymbol{\lambda}_t - \boldsymbol{\pi}_{Ht} \boldsymbol{\lambda}_t) \cdot \mathbf{u}_t d\Gamma \\
&= \int_{\Gamma_c} (\lambda_n - \pi_{Hn} \lambda_n) (u_n - \pi_{Hn} u_n) d\Gamma + \int_{\Gamma_c} (\boldsymbol{\lambda}_t - \boldsymbol{\pi}_{Ht} \boldsymbol{\lambda}_t) \cdot (\mathbf{u}_t - \boldsymbol{\pi}_{Ht} \mathbf{u}_t) d\Gamma \\
&\leq \|\lambda_n - \pi_{Hn} \lambda_n\|_{0, \Gamma_c} \|u_n - \pi_{Hn} u_n\|_{0, \Gamma_c} \\
&\quad + \|\boldsymbol{\lambda}_t - \boldsymbol{\pi}_{Ht} \boldsymbol{\lambda}_t\|_{0, \Gamma_c} \|\mathbf{u}_t - \boldsymbol{\pi}_{Ht} \mathbf{u}_t\|_{0, \Gamma_c} \\
&\leq CH^{3/2},
\end{aligned}$$

making use of (5.5) and (5.7).

(v) To estimate the last term, let us observe that the complementarity condition $\lambda_n u_n = 0$ *a.e.* on Γ_c yields:

$$b(\boldsymbol{\lambda}_H - \boldsymbol{\lambda}, \mathbf{u}) = \int_{\Gamma_c} \lambda_{Hn} u_n d\Gamma + \int_{\Gamma_c} (\boldsymbol{\lambda}_{Ht} - \boldsymbol{\lambda}_t) \cdot \mathbf{u}_t d\Gamma.$$

Since $u_n \leq 0$ and $\lambda_{Hn} \geq 0$ *a.e.* on Γ_c , we have

$$\int_{\Gamma_c} \lambda_{Hn} u_n d\Gamma \leq 0.$$

Therefore

$$b(\boldsymbol{\lambda}_H - \boldsymbol{\lambda}, \mathbf{u}) \leq \int_{\Gamma_c} (\boldsymbol{\lambda}_{Ht} - \boldsymbol{\lambda}_t) \cdot \mathbf{u}_t d\Gamma. \quad (5.8)$$

We now shall estimate (5.8). Noticing that

$$\int_{\Gamma_c} \boldsymbol{\mu}_{Ht} \cdot \boldsymbol{\psi}_H d\Gamma - \int_{\Gamma_c} s \|\boldsymbol{\psi}_H\| d\Gamma \leq 0 \quad \forall \boldsymbol{\psi}_H \in (L_H)^2,$$

and using that $-\boldsymbol{\lambda}_t \cdot \mathbf{u}_t + s \|\mathbf{u}_t\| = 0$ *a.e.* on Γ_c (see (3.4) and Th. 3.2), we have:

$$\begin{aligned}
\int_{\Gamma_c} (\boldsymbol{\lambda}_{Ht} - \boldsymbol{\lambda}_t) \cdot \mathbf{u}_t d\Gamma &= \int_{\Gamma_c} (\boldsymbol{\lambda}_{Ht} - \boldsymbol{\lambda}_t) \cdot (\mathbf{u}_t - \boldsymbol{\pi}_{Ht}(\mathbf{u}_t)) d\Gamma + \int_{\Gamma_c} (\boldsymbol{\lambda}_{Ht} - \boldsymbol{\lambda}_t) \cdot \boldsymbol{\pi}_{Ht}(\mathbf{u}_t) d\Gamma \\
&\quad + \int_{\Gamma_c} \boldsymbol{\lambda}_t \cdot \mathbf{u}_t d\Gamma - \int_{\Gamma_c} s \|\mathbf{u}_t\| d\Gamma \\
&\leq \int_{\Gamma_c} (\boldsymbol{\lambda}_{Ht} - \boldsymbol{\lambda}_t) \cdot (\mathbf{u}_t - \boldsymbol{\pi}_{Ht}(\mathbf{u}_t)) d\Gamma + \int_{\Gamma_c} s (\|\boldsymbol{\pi}_{Ht}(\mathbf{u}_t)\| - \|\mathbf{u}_t\|) d\Gamma \\
&\quad + \int_{\Gamma_c} \boldsymbol{\lambda}_t \cdot (\mathbf{u}_t - \boldsymbol{\pi}_{Ht}(\mathbf{u}_t)) d\Gamma \\
&\leq \int_{\Gamma_c} (\boldsymbol{\lambda}_{Ht} - \boldsymbol{\lambda}_t) \cdot (\mathbf{u}_t - \boldsymbol{\pi}_{Ht}(\mathbf{u}_t)) d\Gamma + \int_{\Gamma_c} s (\|\boldsymbol{\pi}_{Ht}(\mathbf{u}_t) - \mathbf{u}_t\|) d\Gamma \\
&\quad + \int_{\Gamma_c} \boldsymbol{\lambda}_t \cdot (\mathbf{u}_t - \boldsymbol{\pi}_{Ht}(\mathbf{u}_t)) d\Gamma \\
&\leq \|\boldsymbol{\lambda}_{Ht} - \boldsymbol{\lambda}_t\|_{-\frac{1}{2} + \varepsilon, \Gamma_c} \|\mathbf{u}_t - \boldsymbol{\pi}_{Ht}(\mathbf{u}_t)\|_{\frac{1}{2} - \varepsilon, \Gamma_c} \\
&\quad + \|s\|_{0, \Gamma_c} \|\mathbf{u}_t - \boldsymbol{\pi}_{Ht}(\mathbf{u}_t)\|_{0, \Gamma_c} + \|\boldsymbol{\lambda}_t\|_{0, \Gamma_c} \|\mathbf{u}_t - \boldsymbol{\pi}_{Ht}(\mathbf{u}_t)\|_{0, \Gamma_c} \\
&\leq CH^{\frac{1}{2} + \varepsilon} \|\boldsymbol{\lambda}_H - \boldsymbol{\lambda}\|_{-\frac{1}{2} + \varepsilon, \Gamma_c} + CH
\end{aligned} \quad (5.9)$$

making use of (5.5) and

$$\|\boldsymbol{\mu}_t\|_{(H^j_{\Gamma_c})^2} \leq \|\boldsymbol{\mu}\|_{(H^j_{\Gamma_c})^3} \quad \text{for any } \boldsymbol{\mu} = (\boldsymbol{\mu}_n, \boldsymbol{\mu}_t) \in \mathbf{H}^j_{\Gamma_c}, j = \pm \frac{1}{2}.$$

From (i)–(v) and the \mathbf{V} -ellipticity of the bilinear form $a(\cdot, \cdot)$, we easily arrive at (5.2). □

The next lemma gives the error estimate for the Lagrange multiplier.

Lemma 5.3. *Let all the assumptions of Lemma 5.2 together with (4.5) be satisfied. Then*

$$\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_H\|_{-\frac{1}{2}, \Gamma_c} \leq C\|\mathbf{u} - \mathbf{u}_h\|_{1, \Omega} + CH, \tag{5.10}$$

$$\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_H\|_{-\frac{1}{2} + \varepsilon, \Gamma_c} \leq CH^{-\varepsilon}\|\mathbf{u} - \mathbf{u}_h\|_{1, \Omega} + CH^{1-\varepsilon}, \tag{5.11}$$

where $\varepsilon > 0$ is arbitrarily small and C is a positive constant which does not depend on h and H .

Proof. Let us consider the equations in (3.6) and (4.3). Since $\mathbf{V}_h \subset \mathbf{V}$, we have

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}_h) + b(\boldsymbol{\lambda}, \mathbf{v}_h) &= L(\mathbf{v}_h) & \forall \mathbf{v}_h \in \mathbf{V}_h, \\ a(\mathbf{u}_h, \mathbf{v}_h) + b(\boldsymbol{\lambda}_H, \mathbf{v}_h) &= L(\mathbf{v}_h) & \forall \mathbf{v}_h \in \mathbf{V}_h. \end{aligned}$$

Therefore

$$a(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h) + b(\boldsymbol{\lambda} - \boldsymbol{\lambda}_H, \mathbf{v}_h) = 0 \quad \forall \mathbf{v}_h \in \mathbf{V}_h,$$

and

$$\begin{aligned} b(\boldsymbol{\lambda}_H - \boldsymbol{\Pi}_H \boldsymbol{\lambda}, \mathbf{v}_h) &= a(\mathbf{u} - \mathbf{u}_h, \mathbf{v}_h) + b(\boldsymbol{\lambda} - \boldsymbol{\Pi}_H \boldsymbol{\lambda}, \mathbf{v}_h) \\ &\leq C\|\mathbf{u} - \mathbf{u}_h\|_{1, \Omega} \|\mathbf{v}_h\|_{1, \Omega} + CH\|\boldsymbol{\lambda}\|_{\frac{1}{2}, \Gamma_c} \|\mathbf{v}_h\|_{1, \Omega} \quad \forall \mathbf{v}_h \in \mathbf{V}_h. \end{aligned}$$

The inf-sup condition (4.5) yields:

$$\tilde{\beta} \|\boldsymbol{\lambda}_H - \boldsymbol{\Pi}_H \boldsymbol{\lambda}\|_{-\frac{1}{2}, \Gamma_c} \leq \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{b(\boldsymbol{\lambda}_H - \boldsymbol{\Pi}_H \boldsymbol{\lambda}, \mathbf{v}_h)}{\|\mathbf{v}_h\|_{1, \Omega}} \leq C\|\mathbf{u} - \mathbf{u}_h\|_{1, \Omega} + CH\|\boldsymbol{\lambda}\|_{\frac{1}{2}, \Gamma_c}. \tag{5.12}$$

Here we used the Korn’s inequality and the fact that the dual norms $\|\cdot\|_{-\frac{1}{2}, \Gamma_c}$ and $\|\cdot\|_{-\frac{1}{2}, \Gamma_c}$ are equivalent in $\mathbf{H}^{-\frac{1}{2}}_{\Gamma_c}$. The constant $\tilde{\beta} > 0$ appearing on the left of (5.12) is independent of h and H .

The triangle inequality

$$\|\boldsymbol{\lambda} - \boldsymbol{\lambda}_H\|_{-\frac{1}{2}, \Gamma_c} \leq \|\boldsymbol{\lambda} - \boldsymbol{\Pi}_H \boldsymbol{\lambda}\|_{-\frac{1}{2}, \Gamma_c} + \|\boldsymbol{\Pi}_H \boldsymbol{\lambda} - \boldsymbol{\lambda}_H\|_{-\frac{1}{2}, \Gamma_c}$$

together with (5.12) and (5.5) proves (5.10).

Let us prove (5.11). The triangle and inverse inequality yield:

$$\begin{aligned} \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_H\|_{-\frac{1}{2} + \varepsilon, \Gamma_c} &\leq \|\boldsymbol{\lambda} - \boldsymbol{\Pi}_H \boldsymbol{\lambda}\|_{-\frac{1}{2} + \varepsilon, \Gamma_c} + \|\boldsymbol{\Pi}_H \boldsymbol{\lambda} - \boldsymbol{\lambda}_H\|_{-\frac{1}{2} + \varepsilon, \Gamma_c} \\ &\leq C\left(H^{1-\varepsilon} + H^{-\varepsilon}\|\boldsymbol{\Pi}_H \boldsymbol{\lambda} - \boldsymbol{\lambda}_H\|_{-\frac{1}{2}, \Gamma_c}\right), \end{aligned}$$

making use of (5.6). The rest of the proof now follows from (5.12) □

We finally obtain the following global result giving an upper bound for our mixed finite element approximation.

Theorem 5.1. *Let $(\mathbf{u}, \boldsymbol{\lambda}), (\mathbf{u}_h, \boldsymbol{\lambda}_H)$ be the solution to (3.6), (4.3), respectively. Suppose that $\mathbf{u} \in \mathbf{H}^2(\Omega)$, the fourth order tensor A defining the Hooke's law is sufficiently smooth and the Ladyzhenskaya-Babuska-Brezzi condition (4.5) is satisfied. Then*

$$\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} + \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_H\|_{-\frac{1}{2},\Gamma_c} \leq CH^{\frac{1}{2}}, \tag{5.13}$$

where the positive constant C does not depend on h and H .

Proof. Assembling (5.2), (5.10) and (5.11) we arrive at (5.13) making use that $h \leq CH$, where $C > 0$ does not depend on h and H . □

Remark 5.1. Let us suppose that the slip bound s is a positive constant. Then the error estimate (5.13) can be improved. Indeed, instead of (4.2) we define the set \mathbf{M}_{Ht} as follows:

$$\mathbf{M}_{Ht} = \left\{ \boldsymbol{\mu}_{Ht} \in (L_H)^2 \mid \|\boldsymbol{\mu}_{Ht}\| \leq s \text{ a.e. on } \Gamma_c \right\}.$$

Then \mathbf{M}_{Ht} is the internal approximation of \mathbf{M}_t so that

$$\int_{\Gamma_c} \boldsymbol{\mu}_{Ht} \cdot \boldsymbol{\psi} \, d\Gamma - s \int_{\Gamma_c} \|\boldsymbol{\psi}\| \, d\Gamma \leq 0 \quad \forall \boldsymbol{\psi} \in \mathbf{L}^2(\Gamma_c). \tag{5.14}$$

It is very easy to verify that if $\boldsymbol{\mu}_t \in \mathbf{M}_t$ then $\boldsymbol{\pi}_{Ht} \boldsymbol{\mu}_t \in \mathbf{M}_{Ht}$ so that the estimates (i)–(iv) of Lemma 5.2 remain valid. Since $-\boldsymbol{\lambda}_t \cdot \mathbf{u}_t + s\|\mathbf{u}_t\| = 0$ a.e. on Γ_c , the last term can be written as follows (see (5.8)):

$$b(\boldsymbol{\lambda}_H - \boldsymbol{\lambda}, \mathbf{u}) \leq \int_{\Gamma_c} (\boldsymbol{\lambda}_{Ht} \cdot \mathbf{u}_t - s\|\mathbf{u}_t\|) \, d\Gamma \leq 0$$

making use of (5.14) with $\boldsymbol{\mu}_{Ht} := \boldsymbol{\lambda}_{Ht}$ and $\boldsymbol{\psi} := \mathbf{u}_t$. Therefore this term which is responsible for the lower order of convergence can be omitted in (5.1). Under the assumptions of Theorem 5.1 we have

$$\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} + \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_H\|_{-\frac{1}{2},\Gamma_c} \leq CH^{\frac{3}{4}}.$$

6. NUMERICAL REALIZATION

In this section we describe in brief the numerical method for the realization of 3D contact problems with given friction. This method is based on the *dual formulation* of (3.5).

Let $(\mathbf{u}, \lambda_n, \boldsymbol{\lambda}_t)$ be the saddle-point of \mathbf{L} on $\mathbf{V} \times M_n \times \mathbf{M}_t$ with the notation introduced in Section 2. Then

$$\begin{aligned} \mathbf{L}(\mathbf{u}, \lambda_n, \boldsymbol{\lambda}_t) &= \min_{\mathbf{V}} \sup_{M_n \times \mathbf{M}_t} \mathbf{L}(\mathbf{v}, \mu_n, \boldsymbol{\mu}_t) \\ &= \max_{M_n \times \mathbf{M}_t} \inf_{\mathbf{V}} \mathbf{L}(\mathbf{v}, \mu_n, \boldsymbol{\mu}_t). \end{aligned}$$

Denote

$$\mathbf{S}(\mu_n, \boldsymbol{\mu}_t) := - \inf_{\mathbf{V}} \mathbf{L}(\mathbf{v}, \mu_n, \boldsymbol{\mu}_t)$$

the dual functional. The dual variational formulation to (3.5) reads as follows:

$$\left\{ \begin{array}{l} \text{Find } (\mu_n^*, \boldsymbol{\mu}_t^*) \in M_n \times \mathbf{M}_t \text{ such that} \\ \mathbf{S}(\mu_n^*, \boldsymbol{\mu}_t^*) \leq \mathbf{S}(\mu_n, \boldsymbol{\mu}_t) \quad \forall (\mu_n, \boldsymbol{\mu}_t) \in M_n \times \mathbf{M}_t. \end{array} \right. \tag{6.1}$$

It is well-known [9] that (6.1) has a unique solution and, in addition $(\mu_n^*, \boldsymbol{\mu}_t^*) = (\lambda_n, \boldsymbol{\lambda}_t)$, where $(\lambda_n, \boldsymbol{\lambda}_t)$ is a part of the solution to (3.6). It is also readily seen [13] that \mathbf{S} is a quadratic functional. Therefore (6.1) is a

quadratic programming problem for the dual variable (μ_n, μ_t) defined on Γ_c and subject to the linear, quadratic constraints $\mu_n \geq 0, \|\mu_t\|^2 \leq s^2$, respectively.

Let us recall that all constraints appearing in the dual formulation of 2D contact problems with given friction are *linear* [14]. This fact is important from the practical point of view: one can use [7] efficient CG type algorithms for solving (6.1). To be able to use the same algorithms in 3D problems, a linearization of the quadratic constraints will be necessary. This is what we shall do now.

Let $\mathbf{M}(r)$ be the ball in \mathbb{R}^2 of radius $r \geq 0$ with the center at origin:

$$\mathbf{M}(r) = \left\{ \vec{\lambda} \in \mathbb{R}^2 \mid \|\vec{\lambda}\|^2 \leq r^2 \right\}.$$

Next we construct a piecewise linear approximation of $\mathbf{M}(r)$ which will be realized by the intersection of N squares rotated around the origin on a constant angle $\alpha = \pi/(2N)$:

$$\mathbf{M}(r) \sim \cap_{i=1}^N \mathbf{M}_i(r),$$

where

$$\mathbf{M}_i(r) = \left\{ \vec{\lambda} \in \mathbb{R}^2 \mid \|\vec{\lambda}^{\alpha_i}\|_{max} \leq r, \vec{\lambda}^{\alpha_i} := \mathbf{m}_{\alpha_i} \vec{\lambda} \right\},$$

with

$$\mathbf{m}_{\alpha_i} = \begin{pmatrix} \cos \alpha_i & \sin \alpha_i \\ -\sin \alpha_i & \cos \alpha_i \end{pmatrix}, \quad \alpha_i = i\alpha, \quad i = 1, \dots, N$$

being the rotation matrix and $\|\vec{\lambda}\|_{max} := \max_{i=1,2} |\lambda_i|$.

Instead of (6.1) we shall consider the new problem:

$$\left\{ \begin{array}{l} \text{Find } (\hat{\lambda}_n, \hat{\lambda}_t) \in M_n \times \cap_{i=1}^N \mathbf{M}_t^i \text{ such that} \\ \mathbf{S}(\hat{\lambda}_n, \hat{\lambda}_t) \leq \mathbf{S}(\mu_n, \mu_t) \quad \forall (\mu_n, \mu_t) \in M_n \times \cap_{i=1}^N \mathbf{M}_t^i, \end{array} \right. \tag{6.2}$$

where

$$\mathbf{M}_t^i = \left\{ \mu_t \in \mathbf{L}^2(\Gamma_c), \mid \mu_t(\mathbf{x}) \in \mathbf{M}_i(s(\mathbf{x})) \text{ a.e. on } \Gamma_c \right\}.$$

It is known (see [10]) that minimization problems on the intersection of convex sets can be transformed into a sequence of minimization problems over each subset of this intersection. In what follows we show how to proceed in our particular case.

We use the following notations:

$$\mu^i := (\mu_n, \mu_t) \in M_n \times \mathbf{M}_t^i, \quad \mu = (\mu^1, \dots, \mu^N) \in \prod_{i=1}^N M_n \times \mathbf{M}_t^i, \\ \mathbf{S}_N(\mu) := \frac{1}{N} \sum_{i=1}^N \mathbf{S}(\mu^i).$$

Then it is easy to see that (6.2) is equivalent to

$$\left\{ \begin{array}{l} \text{Minimize } \mathbf{S}_N(\mu) \\ \text{subject to } \mu^i \in M_n \times \mathbf{M}_t^i, \quad i = 1, \dots, N \\ \mu^i = \mu^{i+1}, \quad i = 1, \dots, N - 1. \end{array} \right. \tag{6.3}$$

Let us observe that the inclusion $\mu^i \in M_n \times \mathbf{M}_t^i$ can be easily expressed by means of box constraints after an appropriate rotation of μ^i , *i.e.* we formally obtain the same situation as in 2D contact problems. The discretized version of (6.3) together with the augmented Lagrangian formulation has been used in (see [15]) for the numerical realization of 3D contact problems with friction, including Coulomb's model of friction.

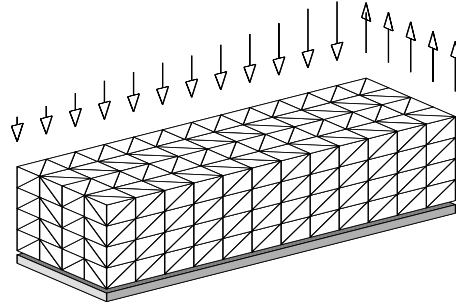


FIGURE 1. The loaded body.

The previous numerical method has a drawback, too, namely the number of variables is now equal to $N \times q$, where N is the number of squares and $q = \dim \mathbf{L}_H$. This drawback however can be compensated by using coarser partitions of $\bar{\Gamma}_c$ when constructing \mathbf{L}_H (another reason for the use of coarser partitions is mentioned in Lem. 4.1). It is worth noticing that one has to find a good compromise between the mesh sizes h and H . Larger H leads to a smaller number of the dual variables on the one hand, but to the poorer approximation of the non-penetration conditions on $\bar{\Gamma}_c$ on the other hand. In the following example the relative error between the “exact” and computed displacements for different ratios H/h will be shown. By the “exact” solution we mean the displacement field computed by the same algorithm but using the “pointwise” Lagrange multipliers (algebraic multipliers) which act at each contact node.

The elastic body is represented by the brick $\Omega = (0, 3) \times (0, 1) \times (0, 1)$ which is made of an elastic material characterized by Young modulus $E = 21.19 \times 10^{10}$ and Poisson’s ratio $\nu = 0.277$. The body is fixed along the face $\Gamma_{\mathbf{u}} = \{0\} \times (0, 1) \times (0, 1)$ and surface tractions \mathbf{g} act on $\Gamma_{\mathbf{g}}^1 = (0, 3) \times (0, 1) \times \{1\}$ and $\Gamma_{\mathbf{g}}^2 = \{3\} \times (0, 1) \times (0, 1)$, where

$$\mathbf{g} = (0, 0, -2 \times 10^6 x_1) \text{ on } \Gamma_{\mathbf{g}}^1$$

and

$$\mathbf{g} = (0, 0, 10^6) \text{ on } \Gamma_{\mathbf{g}}^2.$$

The density of volume forces \mathbf{f} is equal to zero and the slip bound $s = 5.25 \times 10^4$ on Γ_c . The brick is cut into small cubes of size h and each cube is divided into five tetrahedrons (see Fig. 1). Results for the ratios $H/h = 2, 4$ and the number of squares $N = 2, 4$ are summarized in Tables 1–4. All computations were done on IBM SP2. From these results we see that even for higher ratios H/h (i.e. for a less accurate satisfaction of non-penetration conditions) one can get a good approximation of the displacement field. In Table 5 the relative errors between the computed and “exact” solution for $n_x = 36$, $n_y = n_z = 12$ and $N = 2$ on the first 5 grid levels above the contact surface are shown. The largest error occurs just on the contact (first line) surface while decreases when moving inside of the brick.

TABLE 1. Piecewise constant multipliers, $H/h = 2$, $N = 2$.

n_x, n_y, n_z	n_λ	err_u
6,2,2	9N	0.0202
12,4,4	36N	0.0178
18,6,6	81N	0.0064
24,8,8	144N	0.000691
30,10,10	225N	0.002201
36,12,12	324N	0.000685
42,14,14	441N	0.000323

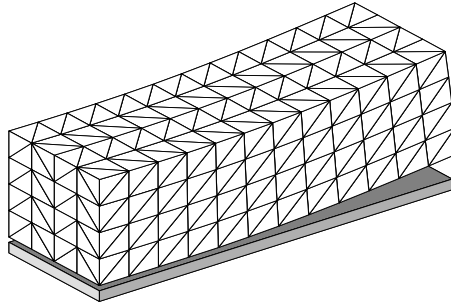


FIGURE 2. The body after deformation.

TABLE 2. Piecewise constant multipliers, $H/h = 4$, $N = 2$.

n_x, n_y, n_z	n_λ	err_u
12,4,4	9N	0.0107
24,8,8	36N	0.0249
36,12,12	54N	0.0045

TABLE 3. Piecewise constant multipliers, $H/h = 2$, $N = 4$.

n_x, n_y, n_z	n_λ	err_u
6,2,2	9N	0.0201
12,4,4	36N	0.0179
18,6,6	81N	0.0064
24,8,8	144N	0.000699
30,10,10	225N	0.002203
36,12,12	324N	0.000677

TABLE 4. Piecewise constant multipliers, $H/h = 4$, $N = 4$.

n_x, n_y, n_z	n_λ	err_u
12,4,4	9N	0.0106
24,8,8	36N	0.0249
36,12,12	54N	0.0044

- $n_x, n_y, n_z \dots$ the numbers of cubes of size h in the x, y, z -directions;
- $n_u \dots$ the number of the primal variables;
- $n_\lambda \dots$ the number of the dual variables;
- $N \dots$ the number of squares approximating the circle;
- $err_u \dots$ the Euclidean norm of the relative error of the primal variable.

TABLE 5. The relative error on different levels of the finite element mesh.

$H/h=2$	$H/h=4$
0.00083178	0.004496
0.00066490	0.004283
0.00062992	0.004184
0.00061946	0.004132
0.00061324	0.004097

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