

THE FOURTH ORDER ACCURACY DECOMPOSITION SCHEME FOR AN EVOLUTION PROBLEM

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Abstract. In the present work, the symmetrized sequential-parallel decomposition method with the fourth order accuracy for the solution of Cauchy abstract problem with an operator under a split form is presented. The fourth order accuracy is reached by introducing a complex coefficient with the positive real part. For the considered scheme, the explicit *a priori* estimate is obtained.

Mathematics Subject Classification. 65M12, 65M15, 65M55.

Received: July 31, 2003. Revised: May 25, 2004.

INTRODUCTION

It is known that mathematical simulation of processes taking place in the nature frequently leads to consideration of boundary-value problems for partial-differential evolution (nonstationary) equation. These kind of problems can be considered as a Cauchy abstract problem in a Banach space for an evolution equation with an unbounded operator.

Study of approximated schemes for solution of evolution problems leads to the conclusion that a certain operator (solution operator of the discrete problem) corresponds to each approximated scheme. This operator approximates the solution operator (semigroup) of the initial continuous problem. For example, if we use the Rotte scheme for the solution of an evolution problem, then the solution operator of the difference problem thus obtained will be a discrete semigroup and we will have the approximation of a continuous semigroup by discrete semigroups (see [32], Ch. IX). On the other hand, on the basis of the approximation of a continuous semigroup, we can construct an approximated scheme for solution of an evolution problem.

Decomposition formulas approximate a continuous semigroup by means of the combination of the semigroups generated by the addends of the operator generating this semigroup.

The first decomposition formula for an exponential matrix function was constructed by Lie in 1875. Trotter generalized this formula for an exponential operator function-semigroup in 1959 [49]. The resolvent analogue of this formula for the first time was constructed by Chernoff in 1968 [6, 7]. At the same time, in the sixties of the XXth century, in order to elaborate numerical methods for solution of multi-dimensional problems of mathematical physics, the subject of construction of decomposition schemes has naturally raised. Decomposition schemes allow to reduce a solution of multi-dimensional problems to the solution of one-dimensional problems.

Keywords and phrases. Decomposition method, semigroup, operator split method, Trotter formula, Cauchy abstract problem.

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First works dedicated to construction and investigation of decomposition schemes were published in the fifties and sixties of the xxth century (see Andreev [1], Baker [2], Baker and Oliphant [3], Birkhoff and Varga [4], Birkhoff, Varga and Young [5], Douglas [11], Douglas and Rachford [12], Diakonov [9,10], Dryja [13], Fairweather, Gourlay and Mitchell [14], Fryazinov [15], Gordeziani [20], Gourlay and Mitchell [24], Ianenko [25,26], Ianenko and Demidov [27], Konovalov [33], Marchuk and Ianenko [37], Marchuk and Sultangazin [38], Peaceman and Rachford [39], Ilin [30], Samarskii [43–45], and Temam [48]). These works became a basis of the further investigation of decomposition schemes.

We can show that the split problem, obtained by means of a decomposition method, generates the Trotter formula (see Trotter [49]), or the Chernoff formula (see Chernoff [6,7]), or a formula which is a combination of these formulas. Thus, an estimate of decomposition formula's error is equivalent to the study of approximation of continuous semigroup by Lie-Trotter and Lie-Chernoff type formulas. The works of Ichinose and Takanobu [28], Ichinose and Tamura [29], Rogava [41], (see also [42], T. II) are devoted to estimate of error of Lie-Trotter and Lie-Chernoff type formulas.

We call Lie-Trotter type formulas the formulas which approximate a semigroup by a combination of semigroups generated by the addends of the operator generating this semigroup.

We call Lie-Chernoff type formulas the formulas which are obtained from Lie-Trotter type formulas if we replace semigroups by the corresponding resolvents.

Decomposition schemes conditionally can be divided into two groups: differential and difference. Lie-Trotter type formulas correspond to differential decomposition schemes and Lie-Chernoff type formulas to difference schemes.

Decomposition schemes, associated with the Lie and Trotter formulas, allow to split a Cauchy problem for an evolution equation with the operator $A = A_1 + A_2$ into two problems, respectively with the operators A_1 and A_2 , which are solved sequentially on the time interval with the length t/n .

Decomposition schemes associated with the Chernoff formula are known as the fractional-step method (see Ianenko [26]).

Decomposition schemes in view of numerical calculation can be divided into two groups: schemes of sequential account (see for example Marchuk [36]) and schemes of parallel account (Gordeziani and Meladze [22,23], Gordeziani and Samarskii [21], Kuzyk and Makarov [35]). In [42] (see Sect. II), there are obtained explicit *a priori* estimates for decomposition schemes of parallel account considered in [22]. There exist a lot of works devoted to decomposition schemes. For example, see [26,36,46] and the references therein.

The accuracy order of the above-mentioned schemes are first or second. As it is known to us, the high accuracy order decomposition schemes in case of two addends ($A = A_1 + A_2$) for the first time were obtained by Dia and Schatzman (see [8]). Note that the formulas constructed in these works are not automatically stable decomposition formulas. Decomposition formula is called automatically stable if a sum of the absolute values of split coefficients is equal to one. Sheng has proved [47] that, on the real number field, there does not exist such automatically stable splitting of $\exp(-tA)$, the accuracy order of which is higher than two.

The present work is devoted to construction and investigation of the high order accuracy decomposition scheme for an evolution problem.

In this work, by introducing a complex parameter, the fourth order accuracy symmetrized sequential-parallel type differential scheme is constructed for a two-dimensional evolution problem with a split operator. The main operator of the evolution problem conditionally is called the two-dimensional split operator if it represents a sum of two addends ($A = A_1 + A_2$). The formula, corresponding to the constructed scheme, is an automatically stable decomposition formula. For the considered scheme, there is obtained an explicit *a priori* estimate. Under the explicit estimate we mean such *a priori* estimate for the solution error, where constants in the right-hand side do not depend on the solution of the initial continuous problem, *i.e.* are absolute constants. In the works [16–19], we have constructed the third order accuracy decomposition schemes for two- and multi-dimensional homogeneous and inhomogeneous evolution problems.

1. STATEMENT OF THE PROBLEM

Let us consider the Cauchy problem for an evolution equation in the Banach space X :

$$\frac{du(t)}{dt} + Au(t) = 0, \quad t > 0, \quad u(0) = \varphi, \tag{1}$$

where A is a linear closed operator with a definition domain $D(A)$, which is everywhere dense in X , φ is a given element from $D(A)$.

Suppose that the operator $(-A)$ generates a strongly continuous semigroup $\{\exp(-tA)\}_{t \geq 0}$. Then the solution of problem (1) is given by the following formula [1, 36, 37]:

$$u(t) = U(t, A)\varphi, \tag{2}$$

where $U(t, A) = \exp(-tA)$ is a strongly continuous semigroup.

Let $A = A_1 + A_2$, where A_i ($i = 1, 2$) are closed operators, densely defined in X .

Let us introduce a grid set:

$$\bar{\omega}_\tau = \{t_k = k\tau, k = 1, 2, \dots, \tau > 0\}.$$

Together with problem (1), on each interval $[t_{k-1}, t_k]$, we consider a sequence of the following problems:

$$\begin{aligned} \frac{dv_k^{(1)}(t)}{dt} + \frac{\alpha}{2}A_1v_k^{(1)}(t) &= 0, & v_k^{(1)}(t_{k-1}) &= u_{k-1}(t_{k-1}), \\ \frac{dv_k^{(2)}(t)}{dt} + \frac{1}{2}A_2v_k^{(2)}(t) &= 0, & v_k^{(2)}(t_{k-1}) &= v_k^{(1)}(t_k), \\ \frac{dv_k^{(3)}(t)}{dt} + \bar{\alpha}A_1v_k^{(3)}(t) &= 0, & v_k^{(3)}(t_{k-1}) &= v_k^{(2)}(t_k), \\ \frac{dv_k^{(4)}(t)}{dt} + \frac{1}{2}A_2v_k^{(4)}(t) &= 0, & v_k^{(4)}(t_{k-1}) &= v_k^{(3)}(t_k), \\ \frac{dv_k^{(5)}(t)}{dt} + \frac{\alpha}{2}A_1v_k^{(5)}(t) &= 0, & v_k^{(5)}(t_{k-1}) &= v_k^{(4)}(t_k), \\ \\ \frac{dw_k^{(1)}(t)}{dt} + \frac{\alpha}{2}A_2w_k^{(1)}(t) &= 0, & w_k^{(1)}(t_{k-1}) &= u_{k-1}(t_{k-1}), \\ \frac{dw_k^{(2)}(t)}{dt} + \frac{1}{2}A_1w_k^{(2)}(t) &= 0, & w_k^{(2)}(t_{k-1}) &= w_k^{(1)}(t_k), \\ \frac{dw_k^{(3)}(t)}{dt} + \bar{\alpha}A_2w_k^{(3)}(t) &= 0, & w_k^{(3)}(t_{k-1}) &= w_k^{(2)}(t_k), \\ \frac{dw_k^{(4)}(t)}{dt} + \frac{1}{2}A_1w_k^{(4)}(t) &= 0, & w_k^{(4)}(t_{k-1}) &= w_k^{(3)}(t_k), \\ \frac{dw_k^{(5)}(t)}{dt} + \frac{\alpha}{2}A_2w_k^{(5)}(t) &= 0, & w_k^{(5)}(t_{k-1}) &= w_k^{(4)}(t_k), \end{aligned}$$

where α is a complex number with the positive real part, $Re(\alpha) > 0$; $u_0(0) = \varphi$. Suppose that the operators $(-A_j), (-\alpha A_j), (-\bar{\alpha}A_j)$, $j = 1, 2$ generate strongly continuous semigroups.

$u_k(t)$, $k = 1, 2, \dots$, is defined on each interval $[t_{k-1}, t_k]$ as follows:

$$u_k(t) = \frac{1}{2}[v_k^{(5)}(t) + w_k^{(5)}(t)]. \tag{3}$$

We declare function $u_k(t)$ as an approximated solution of problem (1) on each interval $[t_{k-1}, t_k]$.

2. ESTIMATE OF ERROR OF THE APPROXIMATED SOLUTION

We need the natural powers ($A^s, s = 2, 3, 4, 5$) of the operator $A = A_1 + A_2$. They are usually defined as follows:

$$\begin{aligned} A^2 &= (A_1^2 + A_2^2) + (A_1A_2 + A_2A_1), \\ A^3 &= (A_1^3 + A_2^3) + (A_1^2A_2 + \dots + A_2^2A_1) + (A_1A_2A_1 + A_2A_1A_2), \\ A^4 &= (A_1^4 + A_2^4) + (A_1^3A_2 + \dots + A_2^3A_1) + (A_1^2A_2A_1 + \dots + A_2^2A_1A_2) \\ &\quad + (A_1A_2A_1A_2 + A_2A_1A_2A_1), \\ A^5 &= (A_1^5 + A_2^5) + (A_1^4A_2 + \dots + A_2^4A_1) + (A_1^3A_2A_1 + \dots + A_2^3A_1A_2) \\ &\quad + (A_1^2A_2A_1A_2 + A_2^2A_1A_2A_1) + (A_1A_2A_1A_2A_1 + A_2A_1A_2A_1A_2). \end{aligned}$$

It is obvious that the definition domain $D(A^s)$ of the operator A^s represents an intersection of definition domains of its addends.

Let us introduce the following notations:

$$\begin{aligned} \|\varphi\|_A &= \|A_1\varphi\| + \|A_2\varphi\|, \quad \varphi \in D(A); \\ \|\varphi\|_{A^2} &= \|A_1^2\varphi\| + \|A_2^2\varphi\| + \|A_1A_2\varphi\| + \|A_2A_1\varphi\|, \quad \varphi \in D(A^2), \end{aligned}$$

where $\|\cdot\|$ is a norm in X . $\|\varphi\|_{A^s}, (s = 3, 4, 5)$ is defined analogously.

Theorem. *Let the following conditions be fulfilled:*

- (a) $\alpha = \frac{1}{2} \pm i \frac{1}{2\sqrt{3}} \quad (i = \sqrt{-1})$;
- (b) Operators $(-\gamma A_j), \gamma = 1, \alpha, \bar{\alpha} \quad (j = 1, 2)$ and $(-A)$ generate strongly continuous semigroups, for which the following estimates are true:

$$\begin{aligned} \|U(t, \gamma A_j)\| &\leq e^{\omega t}, \\ \|U(t, A)\| &\leq Me^{\omega t}, \quad M = \text{const.} > 0; \end{aligned}$$

- (c) $U(s, A)\varphi \in D(A^5)$ for each fixed $s \geq 0$.

Then the following estimate holds:

$$\|u(t_k) - u_k(t_k)\| \leq ce^{\omega_0 t_k} t_k \tau^4 \sup_{s \in [0, t_k]} \|U(s, A)\varphi\|_{A^5},$$

where c and ω_0 are positive constants.

Proof. According to the following formula (see [32], p. 603):

$$A \int_r^t U(s, A) ds = U(r, A) - U(t, A), \quad 0 \leq r \leq t,$$

we can obtain the expansion:

$$U(t, A) = \sum_{i=0}^{k-1} (-1)^i \frac{t^i}{i!} A^i + R_k(t, A), \tag{4}$$

where

$$R_k(t, A) = (-A)^k \int_0^t \int_0^{s_1} \dots \int_0^{s_{k-1}} U(s, A) ds ds_{k-1} \dots ds_1. \tag{5}$$

From formula (3) we obtain:

$$u_k(t_k) = V^k(\tau) \varphi, \tag{6}$$

where

$$V(\tau) = \frac{1}{2} [V_1(\tau) + V_2(\tau)],$$

and where

$$\begin{aligned} V_1(\tau) &= U\left(\tau, \frac{\alpha}{2} A_1\right) U\left(\tau, \frac{1}{2} A_2\right) U(\tau, \bar{\alpha} A_1) U\left(\tau, \frac{1}{2} A_2\right) U\left(\tau, \frac{\alpha}{2} A_1\right), \\ V_2(\tau) &= U\left(\tau, \frac{\alpha}{2} A_2\right) U\left(\tau, \frac{1}{2} A_1\right) U(\tau, \bar{\alpha} A_2) U\left(\tau, \frac{1}{2} A_1\right) U\left(\tau, \frac{\alpha}{2} A_2\right). \end{aligned}$$

Remark 1. Stability of the considered scheme on each finite time interval follows from the first inequality of the condition (b) of the theorem. In this case, for the solving operator, the following estimate holds:

$$\|V^k(\tau)\| \leq e^{\omega_1 t_k}, \tag{7}$$

where ω_1 is a positive constant.

We introduce the following notations for combinations (sum, product) of semigroups. Let $T(\tau)$ be a combination (sum, product) of the semigroups, which are generated by the operators $(-\gamma A_i)$ ($i = 1, 2$). Let us decompose every semigroup included in operator $T(\tau)$ according to formula (4), multiply these decompositions on each other, add the similar members and, in the decomposition thus obtained, denote coefficients of the members $(-\tau A_i)$, $(\tau^2 A_i A_j)$, $(-\tau^3 A_i A_j A_k)$ and $(\tau^4 A_i A_j A_k A_l)$ ($i, j, k, l = 1, 2$) respectively by $[T(\tau)]_i$, $[T(\tau)]_{i,j}$, $[T(\tau)]_{i,j,k}$ and $[T(\tau)]_{i,j,k,l}$.

If we decompose all the semigroups included in the operator $V(\tau)$ according to formula (4) from left to right in such a way that each residual term appears of the fifth order, we will obtain the following formula:

$$\begin{aligned} V(\tau) &= I - \tau \sum_{i=1}^2 [V(\tau)]_i A_i + \tau^2 \sum_{i,j=1}^2 [V(\tau)]_{i,j} A_i A_j - \tau^3 \sum_{i,j,k=1}^2 [V(\tau)]_{i,j,k} A_i A_j A_k \\ &\quad + \tau^4 \sum_{i,j,k,l=1}^2 [V(\tau)]_{i,j,k,l} A_i A_j A_k A_l + R_5(\tau). \end{aligned} \tag{8}$$

According to the first inequality of the condition (b) of the Theorem, for $R_5(\tau)$, the following estimate holds:

$$\|R_5(\tau) \varphi\| \leq c e^{\omega_0 \tau} \tau^5 \|\varphi\|_{A^5}, \quad \varphi \in D(A^5), \tag{9}$$

where c and ω_0 are positive constants.

It is obvious that, for the coefficients in formula (8), we have:

$$\begin{aligned} [V(\tau)]_i &= \frac{1}{2} ([V_1(\tau)]_i + [V_2(\tau)]_i), \quad i = 1, 2, \\ [V(\tau)]_{i,j} &= \frac{1}{2} ([V_1(\tau)]_{i,j} + [V_2(\tau)]_{i,j}), \quad i, j = 1, 2, \\ [V(\tau)]_{i,j,k} &= \frac{1}{2} ([V_1(\tau)]_{i,j,k} + [V_2(\tau)]_{i,j,k}), \quad i, j, k = 1, 2, \\ [V(\tau)]_{i,j,k,l} &= \frac{1}{2} ([V_1(\tau)]_{i,j,k,l} + [V_2(\tau)]_{i,j,k,l}), \quad i, j, k, l = 1, 2. \end{aligned}$$

Let us make two remarks which will simplify a calculation of coefficients in decomposition (8):

Remark 2. Operator $V(\tau)$ will not change if we replace with each other the operators A_1 and A_2 in its expression, as in this case $V_1(\tau)$ will coincide with $V_2(\tau)$, and $V_2(\tau)$ - with $V_1(\tau)$. Therefore we have:

$$\begin{aligned} [V(\tau)]_i &= [V(\tau)]_{3-i}, \quad i = 1, 2; \\ [V(\tau)]_{i,j} &= [V(\tau)]_{3-i,3-j}, \quad i, j = 1, 2; \\ [V(\tau)]_{i,j,k} &= [V(\tau)]_{3-i,3-j,3-k}, \quad i, j, k = 1, 2; \\ [V(\tau)]_{i,j,k,l} &= [V(\tau)]_{3-i,3-j,3-k,3-l}, \quad i, j, k, l = 1, 2. \end{aligned}$$

Remark 3. Operators $V_1(\tau)$ and $V_2(\tau)$ are symmetrical in the sense that in their expressions the factors (semigroups) equally remote from the ends coincide with each other. Therefore we have:

$$\begin{aligned} [V(\tau)]_{i,j} &= [V(\tau)]_{j,i}, \quad i, j = 1, 2; \\ [V(\tau)]_{i,j,k} &= [V(\tau)]_{k,j,i}, \quad i, j, k = 1, 2; \\ [V(\tau)]_{i,j,k,l} &= [V(\tau)]_{l,k,j,i}, \quad i, j, k, l = 1, 2. \end{aligned}$$

Let us calculate the coefficients $[V(\tau)]_i$ corresponding to the first order members in formula (8). It is obvious that the members, corresponding to these coefficients, are obtained from the decomposition of only those factors (semigroups) of the operator $V(\tau)$, which are generated by the operators $(-\gamma A_i)$, and from the decomposition of other semigroups only first addends (the members with identical operators) will participate.

On the whole, we have two cases: $i = 1$ and $i = 2$. Let us consider the case $i = 1$. We obviously have:

$$[V_1(\tau)]_1 = [U(\tau, A_1)]_1 = 1$$

and

$$[V_2(\tau)]_1 = [U(\tau, A_1)]_1 = 1.$$

Thus

$$[V(\tau)]_1 = \frac{1}{2} ([V_1(\tau)]_1 + [V_2(\tau)]_1) = 1.$$

According to Remark 2:

$$[V(\tau)]_2 = [V(\tau)]_1 = 1. \tag{10}$$

Let us calculate the coefficients $[V(\tau)]_{i,j}$ ($i, j = 1, 2$) corresponding to the second order members included in formula (8). On the whole we have two cases: $(i, j) = (1, 1), (1, 2), (2, 1), (2, 2)$. Let us consider the case $(i, j) = (1, 1)$. We obviously have:

$$[V_1(\tau)]_{1,1} = [U(\tau, A_1)]_{1,1} = \frac{1}{2}$$

and

$$[V_2(\tau)]_{1,1} = [U(\tau, A_1)]_{1,1} = \frac{1}{2}.$$

Therefore

$$[V(\tau)]_{1,1} = \frac{1}{2} ([V_1(\tau)]_{1,1} + [V_2(\tau)]_{1,1}) = \frac{1}{2}.$$

According to Remark 2:

$$[V(\tau)]_{2,2} = [V(\tau)]_{1,1} = \frac{1}{2}. \tag{11}$$

Let us consider the case $(i, j) = (1, 2)$, we obviously have:

$$\begin{aligned} [V_1(\tau)]_{1,2} &= \left[U\left(\tau, \frac{\alpha}{2} A_1\right) \right]_1 \left[U\left(\tau, \frac{1}{2} A_2\right) \right]_2 + \left[U\left(\tau, \frac{\alpha}{2} A_1\right) \right]_1 \left[U\left(\tau, \frac{1}{2} A_2\right) \right]_2 \\ &\quad + \left[U\left(\tau, \bar{\alpha} A_1\right) \right]_1 \left[U\left(\tau, \frac{1}{2} A_2\right) \right]_2 = \frac{\alpha}{2} \frac{1}{2} + \frac{\alpha}{2} \frac{1}{2} + \bar{\alpha} \frac{1}{2} = \frac{1}{2} \end{aligned}$$

and

$$[V_2(\tau)]_{1,2} = \left[U \left(\tau, \frac{1}{2} A_1 \right) \right]_1 [U(\tau, \bar{\alpha} A_2)]_2 + \left[U \left(\tau, \frac{1}{2} A_1 \right) \right]_1 \left[U \left(\tau, \frac{\alpha}{2} A_2 \right) \right]_2 + \left[U \left(\tau, \frac{1}{2} A_1 \right) \right]_1 \left[U \left(\tau, \frac{\alpha}{2} A_2 \right) \right]_2 = \bar{\alpha} \frac{1}{2} + \frac{\alpha}{2} \frac{1}{2} + \frac{\alpha}{2} \frac{1}{2} = \frac{1}{2}.$$

Thus

$$[V(\tau)]_{1,2} = \frac{1}{2} \left([V_1(\tau)]_{1,2} + [V_2(\tau)]_{1,2} \right) = \frac{1}{2}.$$

According to Remark 2:

$$[V(\tau)]_{2,1} = [V(\tau)]_{1,2} = \frac{1}{2}. \tag{12}$$

Here we used the identity $\alpha + \bar{\alpha} = 1$.

By combining formulas (11) and (12), we will obtain:

$$[V(\tau)]_{i,j} = \frac{1}{2}, \quad i, j = 1, 2. \tag{13}$$

Let us calculate the coefficients $[V(\tau)]_{i,j,k}$ ($i, j, k = 1, 2$) corresponding to the third order members in formula (8). On the whole we have eight cases: $(i, j, k) = (1, 1, 1), (1, 1, 2), (1, 2, 1), (1, 2, 2), (2, 1, 1), (2, 1, 2), (2, 2, 1), (2, 2, 2)$. Let us consider the case $(i, j, k) = (1, 1, 1)$. We obviously have:

$$[V_1(\tau)]_{1,1,1} = [U(\tau, A_1)]_{1,1,1} = \frac{1}{6}$$

and

$$[V_2(\tau)]_{1,1,1} = [U(\tau, A_1)]_{1,1,1} = \frac{1}{6}.$$

Thus:

$$[V(\tau)]_{1,1,1} = \frac{1}{2} \left([V_1(\tau)]_{1,1,1} + [V_2(\tau)]_{1,1,1} \right) = \frac{1}{6}.$$

According to Remark 2:

$$[V(\tau)]_{2,2,2} = [V(\tau)]_{1,1,1} = \frac{1}{6}. \tag{14}$$

Let us calculate the case $(i, j, k) = (1, 1, 2)$. We obviously have:

$$\begin{aligned} [V_1(\tau)]_{1,1,2} &= \left[U \left(\tau, \frac{\alpha}{2} A_1 \right) \right]_{1,1} \left[U \left(\tau, \frac{1}{2} A_2 \right) \right]_2 + \left[U \left(\tau, \frac{\alpha}{2} A_1 \right) \right]_{1,1} \left[U \left(\tau, \frac{1}{2} A_2 \right) \right]_2 \\ &\quad + \left[U \left(\tau, \frac{\alpha}{2} A_1 \right) \right]_1 [U(\tau, \bar{\alpha} A_1)]_1 \left[U \left(\tau, \frac{1}{2} A_2 \right) \right]_2 + [U(\tau, \bar{\alpha} A_1)]_{1,1} \left[U \left(\tau, \frac{1}{2} A_2 \right) \right]_2 \\ &= \frac{1}{2} \frac{\alpha^2}{8} + \frac{1}{2} \frac{\alpha^2}{8} + \frac{\alpha}{2} \bar{\alpha} \frac{1}{2} + \frac{\bar{\alpha}^2}{2} \frac{1}{2} = \frac{1 + \bar{\alpha}^2}{8} \end{aligned}$$

and

$$\begin{aligned} [V_2(\tau)]_{1,1,2} &= \left[U \left(\tau, \frac{1}{2} A_1 \right) \right]_{1,1} [U(\tau, \bar{\alpha} A_2)]_2 + \left[U \left(\tau, \frac{1}{2} A_1 \right) \right]_{1,1} \left[U \left(\tau, \frac{\alpha}{2} A_2 \right) \right]_2 \\ &\quad + \left[U \left(\tau, \frac{1}{2} A_1 \right) \right]_1 \left[U \left(\tau, \frac{1}{2} A_1 \right) \right]_1 \left[U \left(\tau, \frac{\alpha}{2} A_2 \right) \right]_2 + \left[U \left(\tau, \frac{1}{2} A_1 \right) \right]_{1,1} \left[U \left(\tau, \frac{\alpha}{2} A_2 \right) \right]_2 \\ &= \frac{1}{8} \bar{\alpha} + \frac{1}{8} \frac{\alpha}{2} + \frac{1}{2} \frac{1}{2} \frac{\alpha}{2} + \frac{1}{8} \frac{\alpha}{2} = \frac{1 + \alpha}{8}. \end{aligned}$$

Thus

$$[V(\tau)]_{1,1,2} = \frac{1}{2} \left([V_1(\tau)]_{1,1,2} + [V_2(\tau)]_{1,1,2} \right) = \frac{2 + \bar{\alpha}^2 + \alpha}{16} = \frac{1}{6}.$$

Here we used the identities $\alpha + \bar{\alpha} = 1$, $\alpha\bar{\alpha} = \frac{1}{3}$ and $\alpha + \bar{\alpha}^2 = \frac{2}{3}$.

According to Remarks 2 and 3:

$$[V(\tau)]_{1,1,2} = [V(\tau)]_{2,1,1} = [V(\tau)]_{2,2,1} = [V(\tau)]_{1,2,2} = \frac{1}{6}. \quad (15)$$

Let us consider the case $(i, j, k) = (1, 2, 1)$. We obviously have:

$$\begin{aligned} [V_1(\tau)]_{1,2,1} &= \left[U \left(\tau, \frac{\alpha}{2} A_1 \right) \right]_1 \left[U \left(\tau, \frac{1}{2} A_2 \right) \right]_2 [U(\tau, \bar{\alpha} A_1)]_1 \\ &\quad + \left[U \left(\tau, \frac{\alpha}{2} A_1 \right) \right]_1 \left[U \left(\tau, \frac{1}{2} A_2 \right) \right]_2 \left[U \left(\tau, \frac{\alpha}{2} A_1 \right) \right]_1 \\ &\quad + \left[U \left(\tau, \frac{\alpha}{2} A_1 \right) \right]_1 \left[U \left(\tau, \frac{1}{2} A_2 \right) \right]_2 \left[U \left(\tau, \frac{\alpha}{2} A_1 \right) \right]_1 \\ &\quad + [U(\tau, \bar{\alpha} A_1)]_1 \left[U \left(\tau, \frac{1}{2} A_2 \right) \right]_2 \left[U \left(\tau, \frac{\alpha}{2} A_1 \right) \right]_1 \\ &= \frac{\alpha}{2} \frac{1}{2} \bar{\alpha} + \frac{\alpha}{2} \frac{1}{2} \frac{\alpha}{2} + \frac{\alpha}{2} \frac{1}{2} \frac{\alpha}{2} + \bar{\alpha} \frac{1}{2} \frac{\alpha}{2} = \frac{1}{6} + \frac{\alpha^2}{4} \end{aligned}$$

and

$$[V_2(\tau)]_{1,2,1} = \left[U \left(\tau, \frac{1}{2} A_1 \right) \right]_1 [U(\tau, \bar{\alpha} A_2)]_2 \left[U \left(\tau, \frac{1}{2} A_1 \right) \right]_1 = \frac{\bar{\alpha}}{4}.$$

Thus

$$[V(\tau)]_{1,2,1} = \frac{1}{2} \left([V_1(\tau)]_{1,2,1} + [V_2(\tau)]_{1,2,1} \right) = \frac{1}{12} + \frac{\alpha^2 + \bar{\alpha}}{8} = \frac{1}{6}.$$

Here we used the identity $\alpha^2 + \bar{\alpha} = \frac{2}{3}$.

According to Remark 2:

$$[V(\tau)]_{2,1,2} = [V(\tau)]_{1,2,1} = \frac{1}{6}. \quad (16)$$

By combining formulas (14), (15) and (16), we will obtain:

$$[V(\tau)]_{i,j,k} = \frac{1}{6}, \quad i, j, k = 1, 2. \quad (17)$$

Let us calculate the coefficients $[V(\tau)]_{i,j,k,l}$ ($i, j, k, l = 1, 2$) corresponding to the fourth order members in formula (8). On the whole we have sixteen cases: $(i, j, k, l) = (1, 1, 1, 1), (1, 1, 1, 2), \dots, (2, 2, 2, 1), (2, 2, 2, 2)$. Let us consider the case $(i, j, k, l) = (1, 1, 1, 1)$. We obviously have:

$$[V_1(\tau)]_{1,1,1,1} = [U(\tau, A_1)]_{1,1,1,1} = \frac{1}{24}$$

and

$$[V_2(\tau)]_{1,1,1,1} = [U(\tau, A_1)]_{1,1,1,1} = \frac{1}{24}.$$

Thus:

$$[V(\tau)]_{1,1,1,1} = \frac{1}{2} \left([V_1(\tau)]_{1,1,1,1} + [V_2(\tau)]_{1,1,1,1} \right) = \frac{1}{24}.$$

According to Remark 2:

$$[V(\tau)]_{2,2,2,2} = [V(\tau)]_{1,1,1,1} = \frac{1}{24}. \tag{18}$$

Let us consider the case $(i, j, k, l) = (1, 1, 1, 2)$, we obviously have:

$$\begin{aligned} [V_1(\tau)]_{1,1,1,2} &= \left[U\left(\tau, \frac{\alpha}{2}A_1\right) \right]_{1,1,1} \left[U\left(\tau, \frac{1}{2}A_2\right) \right]_2 + \left[U\left(\tau, \frac{\alpha}{2}A_1\right) \right]_{1,1,1} \left[U\left(\tau, \frac{1}{2}A_2\right) \right]_2 \\ &\quad + \left[U\left(\tau, \frac{\alpha}{2}A_1\right) \right]_{1,1} \left[U(\tau, \bar{\alpha}A_1) \right]_1 \left[U\left(\tau, \frac{1}{2}A_2\right) \right]_2 + \left[U\left(\tau, \frac{\alpha}{2}A_1\right) \right]_1 \left[U(\tau, \bar{\alpha}A_1) \right]_{1,1} \left[U\left(\tau, \frac{1}{2}A_2\right) \right]_2 \\ &\quad + \left[U(\tau, \bar{\alpha}A_1) \right]_{1,1,1} \left[U\left(\tau, \frac{1}{2}A_2\right) \right]_2 \frac{\alpha^3}{48} \frac{1}{2} + \frac{\alpha^3}{48} \frac{1}{2} + \frac{\alpha^2}{8} \bar{\alpha} \frac{1}{2} + \frac{\alpha}{2} \frac{\bar{\alpha}^2}{2} \frac{1}{2} + \frac{\bar{\alpha}^3}{6} \frac{1}{2} \\ &= \frac{\alpha^3 + \alpha + 4\bar{\alpha}^3 + 2\bar{\alpha}}{48} = \frac{1 + 3\bar{\alpha}^3 + \bar{\alpha}}{48} \end{aligned}$$

and

$$\begin{aligned} [V_2(\tau)]_{1,1,1,2} &= \left[U\left(\tau, \frac{1}{2}A_1\right) \right]_{1,1,1} \left[U(\tau, \bar{\alpha}A_2) \right]_2 + \left[U\left(\tau, \frac{1}{2}A_1\right) \right]_{1,1,1} \left[U\left(\tau, \frac{\alpha}{2}A_2\right) \right]_2 \\ &\quad + \left[U\left(\tau, \frac{1}{2}A_1\right) \right]_{1,1} \left[U\left(\tau, \frac{1}{2}A_1\right) \right]_1 \left[U\left(\tau, \frac{\alpha}{2}A_2\right) \right]_2 \\ &\quad + \left[U\left(\tau, \frac{1}{2}A_1\right) \right]_1 \left[U\left(\tau, \frac{1}{2}A_1\right) \right]_{1,1} \left[U\left(\tau, \frac{\alpha}{2}A_2\right) \right]_2 + \left[U\left(\tau, \frac{1}{2}A_1\right) \right]_{1,1,1} \left[U\left(\tau, \frac{\alpha}{2}A_2\right) \right]_2 \\ &= \frac{1}{48} \bar{\alpha} + \frac{1}{48} \frac{\alpha}{2} + \frac{1}{8} \frac{1}{2} \frac{\alpha}{2} + \frac{1}{2} \frac{1}{8} \frac{\alpha}{2} + \frac{1}{48} \frac{\alpha}{2} = \frac{1 + 3\alpha}{48}. \end{aligned}$$

Thus

$$[V(\tau)]_{1,1,1,2} = \frac{1}{2} \left([V_1(\tau)]_{1,1,1,2} + [V_2(\tau)]_{1,1,1,2} \right) = \frac{2 + 3(\bar{\alpha}^3 + \alpha) + \bar{\alpha}}{96} = \frac{1}{24}.$$

Here we used the identities $3(\bar{\alpha}^3 + \alpha) = 2 - \bar{\alpha}$, $\bar{\alpha}^3 + \alpha^3 = 0$.

According to Remarks 2 and 3:

$$[V(\tau)]_{1,1,1,2} = [V(\tau)]_{2,1,1,1} = [V(\tau)]_{1,2,2,2} = [V(\tau)]_{2,2,2,1} = \frac{1}{24}. \tag{19}$$

Let us consider the case $(i, j, k, l) = (1, 1, 2, 1)$. We obviously have:

$$\begin{aligned} [V_1(\tau)]_{1,1,2,1} &= \left[U\left(\tau, \frac{\alpha}{2}A_1\right) \right]_{1,1} \left[U\left(\tau, \frac{1}{2}A_2\right) \right]_2 \left[U(\tau, \bar{\alpha}A_1) \right]_1 + \left[U\left(\tau, \frac{\alpha}{2}A_1\right) \right]_{1,1} \left[U\left(\tau, \frac{1}{2}A_2\right) \right]_2 \left[U\left(\tau, \frac{\alpha}{2}A_1\right) \right]_1 \\ &\quad + \left[U\left(\tau, \frac{\alpha}{2}A_1\right) \right]_{1,1} \left[U\left(\tau, \frac{1}{2}A_2\right) \right]_2 \left[U\left(\tau, \frac{\alpha}{2}A_1\right) \right]_1 \\ &\quad + \left[U\left(\tau, \frac{\alpha}{2}A_1\right) \right]_1 \left[U(\tau, \bar{\alpha}A_1) \right]_1 \left[U\left(\tau, \frac{1}{2}A_2\right) \right]_2 \left[U\left(\tau, \frac{\alpha}{2}A_1\right) \right]_1 \\ &\quad + \left[U(\tau, \bar{\alpha}A_1) \right]_{1,1} \left[U\left(\tau, \frac{1}{2}A_2\right) \right]_2 \left[U\left(\tau, \frac{\alpha}{2}A_1\right) \right]_1 \\ &= \frac{\alpha^2}{8} \frac{1}{2} \bar{\alpha} + \frac{\alpha^2}{8} \frac{1}{2} \frac{\alpha}{2} + \frac{\alpha^2}{8} \frac{1}{2} \frac{\alpha}{2} + \frac{\alpha}{2} \bar{\alpha} \frac{1}{2} \frac{\alpha}{2} + \frac{\bar{\alpha}^2}{2} \frac{1}{2} \frac{\alpha}{2} \\ &= \frac{3\alpha^3 + 3\alpha + 2\bar{\alpha}}{48} = \frac{3\alpha^3 + \alpha + 2}{48} \end{aligned}$$

and

$$[V_2(\tau)]_{1,1,2,1} = \left[U \left(\tau, \frac{1}{2} A_1 \right) \right]_{1,1} [U(\tau, \bar{\alpha} A_2)]_2 \left[U \left(\tau, \frac{1}{2} A_1 \right) \right]_1 = \frac{1}{8} \bar{\alpha} \frac{1}{2} = \frac{1}{16} \bar{\alpha}.$$

Thus

$$[V(\tau)]_{1,1,2,1} = \frac{1}{2} \left([V_1(\tau)]_{1,1,2,1} + [V_2(\tau)]_{1,1,2,1} \right) = \frac{3\alpha^3 + \alpha + 2 + 3\bar{\alpha}}{96} = \frac{3(\alpha^3 + \bar{\alpha}) + \alpha + 2}{96} = \frac{1}{24}.$$

Here we used the identity $\bar{\alpha}^3 + \alpha^3 = 0$.

According to Remarks 2 and 3:

$$[V(\tau)]_{1,1,2,1} = [V(\tau)]_{2,2,1,2} = [V(\tau)]_{1,2,1,1} = [V(\tau)]_{2,1,2,2} = \frac{1}{24}. \quad (20)$$

Let us consider the case $(i, j, k, l) = (1, 1, 2, 2)$. We obviously have:

$$\begin{aligned} [V_1(\tau)]_{1,1,2,2} &= \left[U \left(\tau, \frac{\alpha}{2} A_1 \right) \right]_{1,1} \left[U \left(\tau, \frac{1}{2} A_2 \right) \right]_{2,2} + \left[U \left(\tau, \frac{\alpha}{2} A_1 \right) \right]_{1,1} \left[U \left(\tau, \frac{1}{2} A_2 \right) \right]_2 \left[U \left(\tau, \frac{1}{2} A_2 \right) \right]_2 \\ &\quad + \left[U \left(\tau, \frac{\alpha}{2} A_1 \right) \right]_{1,1} \left[U \left(\tau, \frac{1}{2} A_2 \right) \right]_{2,2} + \left[U \left(\tau, \frac{\alpha}{2} A_1 \right) \right]_1 [U(\tau, \bar{\alpha} A_1)]_1 \left[U \left(\tau, \frac{1}{2} A_2 \right) \right]_{2,2} \\ &\quad + [U(\tau, \bar{\alpha} A_1)]_{1,1} \left[U \left(\tau, \frac{1}{2} A_2 \right) \right]_{2,2} \\ &= \frac{\alpha^2}{8} \frac{1}{8} + \frac{\alpha^2}{8} \frac{1}{2} \frac{1}{2} + \frac{\alpha^2}{8} \frac{1}{8} + \frac{\alpha}{2} \bar{\alpha} \frac{1}{8} + \frac{\bar{\alpha}^2}{2} \frac{1}{8} \\ &= \frac{\alpha^2 + \alpha \bar{\alpha} + \bar{\alpha}^2}{16} = \frac{1}{24} \end{aligned}$$

and

$$\begin{aligned} [V_2(\tau)]_{1,1,2,2} &= \left[U \left(\tau, \frac{1}{2} A_1 \right) \right]_{1,1} [U(\tau, \bar{\alpha} A_2)]_{2,2} \left[U \left(\tau, \frac{1}{2} A_1 \right) \right]_{1,1} [U(\tau, \bar{\alpha} A_2)]_2 \left[U \left(\tau, \frac{\alpha}{2} A_2 \right) \right]_2 \\ &\quad + \left[U \left(\tau, \frac{1}{2} A_1 \right) \right]_{1,1} \left[U \left(\tau, \frac{\alpha}{2} A_2 \right) \right]_{2,2} + \left[U \left(\tau, \frac{1}{2} A_1 \right) \right]_1 \left[U \left(\tau, \frac{1}{2} A_1 \right) \right]_1 \left[U \left(\tau, \frac{\alpha}{2} A_2 \right) \right]_{2,2} \\ &\quad + \left[U \left(\tau, \frac{1}{2} A_1 \right) \right]_{1,1} \left[U \left(\tau, \frac{\alpha}{2} A_2 \right) \right]_{2,2} \\ &= \frac{1}{8} \bar{\alpha}^2 + \frac{1}{8} \bar{\alpha} \frac{\alpha}{2} + \frac{1}{8} \frac{\alpha^2}{8} + \frac{1}{2} \frac{1}{2} \frac{\alpha^2}{8} + \frac{1}{8} \frac{\alpha^2}{8} \\ &= \frac{\bar{\alpha}^2 + \bar{\alpha} \alpha + \alpha^2}{16} = \frac{1}{24}. \end{aligned}$$

Thus

$$[V(\tau)]_{1,1,2,2} = \frac{1}{2} \left([V_1(\tau)]_{1,1,2,2} + [V_2(\tau)]_{1,1,2,2} \right) = \frac{1}{24}.$$

According to Remark 2:

$$[V(\tau)]_{1,1,2,2} = [V(\tau)]_{2,2,1,1} = \frac{1}{24}. \quad (21)$$

Let us consider the case $(i, j, k, l) = (1, 2, 2, 1)$. We obviously have:

$$\begin{aligned} [V_1(\tau)]_{1,2,2,1} &= \left[U\left(\tau, \frac{\alpha}{2}A_1\right) \right]_1 \left[U\left(\tau, \frac{1}{2}A_2\right) \right]_{2,2} [U(\tau, \bar{\alpha}A_1)]_1 \\ &\quad + \left[U\left(\tau, \frac{\alpha}{2}A_1\right) \right]_1 \left[U\left(\tau, \frac{1}{2}A_2\right) \right]_{2,2} \left[U\left(\tau, \frac{\alpha}{2}A_1\right) \right]_1 \\ &\quad + \left[U\left(\tau, \frac{\alpha}{2}A_1\right) \right]_1 \left[U\left(\tau, \frac{1}{2}A_2\right) \right]_2 \left[U\left(\tau, \frac{1}{2}A_2\right) \right]_2 \left[U\left(\tau, \frac{\alpha}{2}A_1\right) \right]_1 \\ &\quad + \left[U\left(\tau, \frac{\alpha}{2}A_1\right) \right]_1 \left[U\left(\tau, \frac{1}{2}A_2\right) \right]_{2,2} \left[U\left(\tau, \frac{\alpha}{2}A_1\right) \right]_1 \\ &\quad + [U(\tau, \bar{\alpha}A_1)]_1 \left[U\left(\tau, \frac{1}{2}A_2\right) \right]_{2,2} \left[U\left(\tau, \frac{\alpha}{2}A_1\right) \right]_1 \\ &= \frac{\alpha}{2} \frac{1}{8} \bar{\alpha} + \frac{\alpha}{2} \frac{1}{8} \frac{\alpha}{2} + \frac{\alpha}{2} \frac{1}{2} \frac{1}{2} \frac{\alpha}{2} + \frac{\alpha}{2} \frac{1}{8} \frac{\alpha}{2} + \bar{\alpha} \frac{1}{8} \frac{\alpha}{2} = \frac{\alpha^2 + \alpha \bar{\alpha}}{8} \end{aligned}$$

and

$$[V_2(\tau)]_{1,2,2,1} = \left[U\left(\tau, \frac{1}{2}A_1\right) \right]_1 [U(\tau, \bar{\alpha}A_2)]_{2,2} \left[U\left(\tau, \frac{1}{2}A_1\right) \right]_1 = \frac{1}{2} \frac{\bar{\alpha}^2}{2} \frac{1}{2} = \frac{\bar{\alpha}^2}{8}.$$

Thus

$$[V(\tau)]_{1,2,2,1} = \frac{1}{2} \left([V_1(\tau)]_{1,2,2,1} + [V_2(\tau)]_{1,2,2,1} \right) = \frac{\alpha^2 + \alpha \bar{\alpha} + \bar{\alpha}^2}{16} = \frac{1}{24}.$$

According to Remark 2:

$$[V(\tau)]_{1,2,2,1} = [V(\tau)]_{2,1,1,2} = \frac{1}{24}. \tag{22}$$

Let us consider the case $(i, j, k, l) = (1, 2, 1, 2)$. We obviously have:

$$[V_1(\tau)]_{1,2,1,2} = \left[U\left(\tau, \frac{\alpha}{2}A_1\right) \right]_1 \left[U\left(\tau, \frac{1}{2}A_2\right) \right]_2 [U(\tau, \bar{\alpha}A_1)]_1 \left[U\left(\tau, \frac{1}{2}A_2\right) \right]_2 = \frac{\alpha}{2} \frac{1}{2} \frac{\bar{\alpha}}{2} \frac{1}{2} = \frac{1}{24}$$

and

$$[V_2(\tau)]_{1,2,1,2} = \left[U\left(\tau, \frac{1}{2}A_1\right) \right]_1 [U(\tau, \bar{\alpha}A_2)]_2 \left[U\left(\tau, \frac{1}{2}A_1\right) \right]_1 \left[U\left(\tau, \frac{\alpha}{2}A_1\right) \right]_2 = \frac{1}{2} \bar{\alpha} \frac{1}{2} \frac{\alpha}{2} = \frac{1}{24}.$$

Thus

$$[V(\tau)]_{1,2,1,2} = \frac{1}{2} \left([V_1(\tau)]_{1,2,1,2} + [V_2(\tau)]_{1,2,1,2} \right) = \frac{1}{24},$$

According to Remark 2:

$$[V(\tau)]_{1,2,1,2} = [V(\tau)]_{2,1,2,1} = \frac{1}{24}. \tag{23}$$

By combining formulas (18)–(23), we will obtain:

$$[V(\tau)]_{i,j,k,l} = \frac{1}{24}, \quad i, j, k, l = 1, 2. \tag{24}$$

From equality (8), taking into account formulas (10), (13), (17) and (24), we will obtain:

$$\begin{aligned}
 V(\tau) &= I - \tau \sum_{i=1}^2 A_i + \frac{1}{2} \tau^2 \sum_{i,j=1}^2 A_i A_j - \frac{1}{6} \tau^3 \sum_{i,j,k=1}^2 A_i A_j A_k + \frac{1}{24} \tau^4 \sum_{i,j,k,l=1}^2 A_i A_j A_k A_l + R_5(\tau) \\
 &= I - \tau \sum_{i=1}^2 A_i + \frac{1}{2} \tau^2 \left(\sum_{i=1}^2 A_i \right)^2 - \frac{1}{6} \tau^3 \left(\sum_{i=1}^2 A_i \right)^3 + \frac{1}{24} \tau^4 \left(\sum_{i=1}^2 A_i \right)^4 + R_5(\tau) \\
 &= I - \tau A + \frac{1}{2} \tau^2 A^2 - \frac{1}{6} \tau^3 A^3 + \frac{1}{24} \tau^4 A^4 + R_5(\tau).
 \end{aligned} \tag{25}$$

According to formula (4):

$$U(\tau, A) = I - \tau A + \frac{1}{2} \tau^2 A^2 - \frac{1}{6} \tau^3 A^3 + \frac{1}{24} \tau^4 A^4 + R_5(\tau, A). \tag{26}$$

According to condition (b) of the second inequality of the theorem, for $R_5(\tau, A)$, the following estimate holds:

$$\|R_5(\tau, A) \varphi\| \leq c e^{\omega \tau} \tau^5 \|A^5 \varphi\| \leq c e^{\omega \tau} \tau^5 \|\varphi\|_{A^5}. \tag{27}$$

According to equalities (25) and (26):

$$U(\tau, A) - V(\tau) = R_5(\tau, A) - R_5(\tau).$$

From here, taking into account inequalities (9) and (27), we will obtain the following estimate:

$$\|[U(\tau, A) - V(\tau)] \varphi\| \leq c e^{\omega_2 \tau} \tau^5 \|\varphi\|_{A^5}. \tag{28}$$

From equalities (2) and (6), taking into account inequalities (7) and (28), we will obtain:

$$\begin{aligned}
 \|u(t_k) - u_k(t_k)\| &= \|[U(t_k, A) - V^k(\tau)] \varphi\| = \|[U^k(\tau, A) - V^k(\tau)] \varphi\| \\
 &= \left\| \left[\sum_{i=1}^k V^{k-i}(\tau) [U(\tau, A) - V(\tau)] U((i-1)\tau, A) \right] \varphi \right\| \\
 &\leq \sum_{i=1}^k \|V(\tau)\|^{k-i} \|[U(\tau, A) - V(\tau)] U((i-1)\tau, A) \varphi\| \\
 &\leq \sum_{i=1}^k e^{\omega_1(k-i)\tau} c e^{\omega_2 \tau} \tau^5 \|U((i-1)\tau, A) \varphi\|_{A^5} \\
 &\leq c e^{\omega_0 t_k} \tau^5 \sum_{i=1}^k \|U((i-1)\tau, A) \varphi\|_{A^5} \leq k c e^{\omega_0 t_k} \tau^5 \sup_{s \in [0, t_k]} \|U(s, A) \varphi\|_{A^5} \\
 &\leq c e^{\omega_0 t_k} t_k \tau^4 \sup_{s \in [0, t_k]} \|U(s, A) \varphi\|_{A^5}. \quad \square
 \end{aligned}$$

Remark 4. In case of a Hilbert space, if A_1, A_2 and $A_1 + A_2$ are self adjoint nonnegative operators, then ω_0 will be replaced by 0 in the estimate of the theorem. In addition, for the solution operator of the split problem, the following estimate holds: $\|V^k(\tau)\| \leq 1$.

Remark 5. In case of a Hilbert space, if A_1, A_2 and $A_1 + A_2$ are self adjoint positive defined operators, then ω_0 will be replaced by $(-\alpha_0)$, $\alpha_0 > 0$ in the estimate of the Theorem. In addition, for the solution operator of the split problem, the following estimate holds: $\|V^k(\tau)\| \leq e^{-\alpha_1 t_k}$, $\alpha_1 > 0$.

Remark 6. According to the classical theorem of Hille-Philips-Iosida (see [40]), if the operator $(-A)$ generates a strongly continuous semigroup, then the second inequality of condition (b) of the Theorem is automatically satisfied. The proof of this inequality is based on the uniform boundedness principle, due to which constants M and ω exist, but can not be explicitly constructed (following to method of the proof). For this reason we demand a satisfaction of the second inequality of the condition (b) of the theorem.

3. CONNECTION BETWEEN DECOMPOSITION FORMULAS WITH DIFFERENT ACCURACIES

It is interesting if there exists a certain regularity, on the basis of which it is available to construct automatically stable decomposition formulas with accuracy of any order. Concerning the above-mentioned let us consider the concrete first and second order accuracy decomposition formulas and see whether there exists a connection between them.

$$V^{(1)}(\tau) = U(\tau, A_1)U(\tau, A_2), \tag{29}$$

$$V^{(2)}(\tau) = U\left(\tau, \frac{1}{2}A_1\right)U(\tau, A_2)U\left(\tau, \frac{1}{2}A_1\right). \tag{30}$$

In this formula and the formulas given below, the upper indices of the operator V denote the order of the corresponding decomposition formula. Formula (29) represents the first order accuracy decomposition formula (see [49]), while formula (30) represents the second order accuracy decomposition formula (see [3]). In order to show more clearly the connection between them, let us rewrite formula (30) in the following form:

$$\begin{aligned} V^{(2)}(\tau) &= \left[U\left(\tau, \frac{1}{2}A_1\right)U\left(\tau, \frac{1}{2}A_2\right) \right] \left[U\left(\tau, \frac{1}{2}A_2\right)\left(\tau, \frac{1}{2}A_1\right) \right] \\ &= V^{(1)}\left(\frac{1}{2}\tau\right)\overline{V^{(1)}}\left(\frac{1}{2}\tau\right). \end{aligned}$$

In this formula and the formulas given below, we denote by \overline{V} the multiplication of factors of the operator V in the reverse order.

The regularity of the same type exists between the third and fourth order accuracy decomposition formulas, constructed by us (see [16, 18, 19]). In order to show this, let us introduce the following notations:

$$V^{(3)}(\tau) = \frac{1}{2} \left[V_1^{(3)}(\tau) + V_2^{(3)}(\tau) \right], \tag{31}$$

$$V_1^{(3)}(\tau) = U(\tau, \alpha A_1)U(\tau, A_2)U(\tau, \overline{\alpha}A_1),$$

$$V_2^{(3)}(\tau) = U(\tau, \alpha A_2)U(\tau, A_1)U(\tau, \overline{\alpha}A_2),$$

and

$$V^{(4)}(\tau) = \frac{1}{2} \left[V_1^{(4)}(\tau) + V_2^{(4)}(\tau) \right], \tag{32}$$

$$V_1^{(4)}(\tau) = U\left(\tau, \frac{\alpha}{2}A_1\right)U\left(\tau, \frac{1}{2}A_2\right)U(\tau, \overline{\alpha}A_1)U\left(\tau, \frac{1}{2}A_2\right)U\left(\tau, \frac{\alpha}{2}A_1\right),$$

$$V_2^{(4)}(\tau) = U\left(\tau, \frac{\alpha}{2}A_2\right)U\left(\tau, \frac{1}{2}A_1\right)U(\tau, \overline{\alpha}A_2)U\left(\tau, \frac{1}{2}A_1\right)U\left(\tau, \frac{\alpha}{2}A_2\right).$$

In order to reveal the connection between formulas (31) and (32), let rewrite the addends of formula (32) in the following form:

$$\begin{aligned} V_1^{(4)}(\tau) &= \left[U\left(\tau, \frac{\alpha}{2}A_1\right) U\left(\tau, \frac{1}{2}A_2\right) U\left(\tau, \frac{\bar{\alpha}}{2}A_1\right) \right] \left[U\left(\tau, \frac{\bar{\alpha}}{2}A_1\right) U\left(\tau, \frac{1}{2}A_2\right) U\left(\tau, \frac{\alpha}{2}A_1\right) \right] \\ &= V_1^{(3)}\left(\frac{1}{2}\tau\right) \overline{V_1^{(3)}\left(\frac{1}{2}\tau\right)}, \\ V_2^{(4)}(\tau) &= \left[U\left(\tau, \frac{\alpha}{2}A_2\right) U\left(\tau, \frac{1}{2}A_1\right) U\left(\tau, \frac{\bar{\alpha}}{2}A_2\right) \right] \left[U\left(\tau, \frac{\bar{\alpha}}{2}A_2\right) U\left(\tau, \frac{1}{2}A_1\right) U\left(\tau, \frac{\alpha}{2}A_2\right) \right] \\ &= V_2^{(3)}\left(\frac{1}{2}\tau\right) \overline{V_2^{(3)}\left(\frac{1}{2}\tau\right)}. \end{aligned}$$

Finally we obtain:

$$V^{(4)}(\tau) = \frac{1}{2} \left[V_1^{(3)}\left(\frac{1}{2}\tau\right) \overline{V_1^{(3)}\left(\frac{1}{2}\tau\right)} + V_2^{(3)}\left(\frac{1}{2}\tau\right) \overline{V_2^{(3)}\left(\frac{1}{2}\tau\right)} \right].$$

Unfortunately, the following formula constructed by the same rule:

$$\begin{aligned} V^{(5)}(\tau) &= \frac{1}{2} \left[V_1^{(4)}\left(\frac{1}{2}\tau\right) \overline{V_1^{(4)}\left(\frac{1}{2}\tau\right)} + V_2^{(4)}\left(\frac{1}{2}\tau\right) \overline{V_2^{(4)}\left(\frac{1}{2}\tau\right)} \right] \\ &= \frac{1}{2} \left[U\left(\tau, \frac{\alpha}{4}A_1\right) U\left(\tau, \frac{1}{4}A_2\right) U\left(\tau, \frac{\bar{\alpha}}{2}A_1\right) U\left(\tau, \frac{1}{4}A_2\right) U\left(\tau, \frac{\alpha}{2}A_1\right) \right. \\ &\quad \times U\left(\tau, \frac{1}{4}A_2\right) U\left(\tau, \frac{\bar{\alpha}}{2}A_1\right) U\left(\tau, \frac{1}{4}A_2\right) U\left(\tau, \frac{\alpha}{4}A_1\right) \\ &\quad + U\left(\tau, \frac{\alpha}{4}A_2\right) U\left(\tau, \frac{1}{4}A_1\right) U\left(\tau, \frac{\bar{\alpha}}{2}A_2\right) U\left(\tau, \frac{1}{4}A_1\right) U\left(\tau, \frac{\alpha}{2}A_2\right) \\ &\quad \left. \times U\left(\tau, \frac{1}{4}A_1\right) U\left(\tau, \frac{\bar{\alpha}}{2}A_2\right) U\left(\tau, \frac{1}{4}A_1\right) U\left(\tau, \frac{\alpha}{4}A_2\right) \right]. \end{aligned}$$

does not represent the fifth order accuracy decomposition formula. To check this out, it is sufficient to calculate, for example, the coefficient $[V^{(5)}(\tau)]_{1,2,1,2,1}$. We see that

$$[V^{(5)}(\tau)]_{1,2,1,2,1} \neq \frac{1}{5!}.$$

In our opinion, it is interesting and important to find the general regularity, by means of which it will be available to construct recurrently an automatically stable decomposition formula with accuracy of any order, or to prove that, on the complex number field, there does not exist an automatically stable decomposition formula with accuracy of order more than four (as well as on the real number field there does not exist an automatically stable decomposition formula with accuracy of order more than two). In addition, it is not excluded that, to obtain the higher order accuracy, it will be necessary to use as split parameters, for example, quaternions instead of complex numbers,

In our opinion, these questions are very interesting and difficult, and we work in this direction, but we have not yet obtain actual results.

4. CONCLUSION

In case when operators A_1, A_2 are matrices, it is obvious that conditions of the theorem are automatically satisfied. Also conditions of the Theorem are satisfied if A_1, A_2 and A are self-adjoint, positive definite operators.

The requirement, that the operators αA and $\bar{\alpha} A$ ($\alpha = 1/\sqrt{3}(\cos 30^\circ + i \sin 30^\circ)$) must generate a strongly continuous semigroup, puts the condition on the spectrum of A . Namely, the spectrum of A must be placed within a sector with the angle less than 120 degrees, because in case of turning of spectrum by ± 30 degrees (this is caused by multiplying of A on the parameters α and $\bar{\alpha}$) the spectrum area will stay in the positive (right) half-plane.

The fourth order accuracy is reached by introducing a complex parameter. For this reason, each equation of the given decomposed system is replaced by a pair of real equations, unlike the lower order accuracy schemes. To solve the concrete problem, (for example) the matrix factorization can be used, where coefficients are the matrices of the second order, while in the lower order accuracy schemes the common factorization may be used.

It must be noted that a sum of the absolute values of coefficients of the addends of the transition operator $V(\tau)$ equals to one, unlike the high order accuracy decomposition schemes considered in [8]. Hence, the considered scheme is stable for any bounded operators A_1, A_2 .

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