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ANALYSIS OF TWO-LEVEL DOMAIN DECOMPOSITION PRECONDITIONERS BASED ON AGGREGATION

Marzio Sala[1]

Abstract. In this paper we present two-level overlapping domain decomposition preconditioners for the finite-element discretisation of elliptic problems in two and three dimensions. The computational domain is partitioned into overlapping subdomains, and a coarse space correction is added. We present an algebraic way to define the coarse space, based on the concept of aggregation. This employs a (smoothed) aggregation technique and does not require the introduction of a coarse grid. We consider a set of assumptions on the coarse basis functions, to ensure bound for the resulting preconditioned system. These assumptions only involve geometrical quantities associated to the aggregates, namely their diameter and the overlap. A condition number which depends on the product of the relative overlap among the subdomains and the relative overlap among the aggregates is proved. Numerical experiments on a model problem are reported to illustrate the performance of the proposed preconditioners.

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1. Introduction

In this paper we consider the scalar Poisson problem

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

where Ω is a bounded polyhedral domain in $\mathbb{R}^d, d \in \{2, 3\}$. The finite-element discretisation of this problem leads to a sparse, large linear system, which is typically solved using an iterative solver, like the conjugate gradient. As the condition number of the associated matrix is $\mathcal{O}(h^{-2})$ (see for instance [15]), where h is the grid size of the triangulation, a preconditioner needs to be employed. A possible solution, well-suited for parallel computations, is to resort to preconditioners based on domain decomposition (DD) ideas, see [16, 19].

Here, we consider Schwarz procedures, which can be described as follows. The computational domain Ω is subdivided into M overlapping subdomains $\Omega_i, i = 1, \ldots, M$, and local Dirichlet-type problems are then solved on each Ω_i. The communication between the solutions on the different subdomains is here guaranteed by the overlapping regions.

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[1] CMCS/SB/EPFL, 1015 Lausanne, Switzerland. e-mail: msala@sandia.gov

It is well-known that this procedure, referred to as one-level Schwarz method, is non-scalable. In fact, the information exchange among the subdomains is only through the overlapping regions, while for elliptic problems the domain of dependence is global. A good scalability may be recovered by the addition of a coarse operator: the original problem is solved on a coarse grid, to furnish a way to spread out information among far away subdomains. Thinking as in a multigrid setting, this coarse space will be used to correct the "smooth" part of the error, whereas the local preconditioner is used to damp the "high-frequency" ones.

A typical choice for the coarse problem is a finite-element approximation on a coarse grid. For structured grids, it is relatively easy to find such a coarse triangulation. For unstructured grids, it is not always a trivial task to define the interpolation operator from the fine grid to the coarse grid, and it may be problematic to ensure that the boundary conditions are correctly represented on the coarse level. Moreover, the implementation of this operator can be difficult or computationally expensive, especially for 3D computations.

A possible way to overcome the difficulties induced by the definition of a coarse grid – without losing the power of two-level methods – is to resort to aggregation procedures. As presented in this paper, aggregation (or agglomeration) is of particular advantage when dealing with problems defined on unstructured grids, since it is possible to introduce coarse spaces without the need of introducing coarse triangulations.

The procedure is as follows. First, the differential problem is discretized on the fine grid. Then, the matrix corresponding to the discretisation of the differential operator on the coarse space is build using the elements of the fine-grid matrix. The basis functions of the coarse space are formed by summing up the finite element basis functions of the nodes in each aggregate.

Aggregation procedures have been presented in literature by various authors. An aggregation technique was first introduced in 1951 by Leontief ([12], Chap. 9). Here, it is written that *"in the case of products which comprise final demand, if enough is known to form a group of products whose use is strictly complementary [...], no information is sacrificed by lumping them together. By the same token, products which are substitutable but have dissimilar production functions must be separately identified since a change in their relative consumption will have different effects on the economy"*. Since then, aggregation has been extensively used in Economics; see [13] and the references therein. For multigrid applications, smoothed aggregation techniques are used, for example, in [2, 23], where investigations of the smoothed aggregation properties have been reported. In the framework of domain decomposition methods, the focus is mainly on two-level methods. Results are presented in [14] for the shallow water equations and 2D potential flows, in [6] for 3D potential flow computations, in [8] for groundwater flows, in [9] for multiphase flows, in [10] for discontinuous Galerkin approximation of advection-diffusion problems, and in [17, 18] for the 3D compressible Euler equations on unstructured tetrahedral grids.

The basic theory and the assumptions on the smoothed aggregation procedures here presented are essentially the same as those already proposed in some recent articles; see [1,8,11]. However, in these articles the amount δ of the overlap among the subdomains is assumed to be equal to that among the aggregates, here indicated by δ_0. Also, the size of the subdomains H is assumed to be equal to that H_0 of the aggregates. In this paper, instead, we keep these two ingredients separate, thus extending the previously developed theory. This allows to better underline the different phases of our smoothed aggregation algorithm: the definition of a subdomain partition, and the definition of the aggregates. Effects of these two phases are reflected on the proposed bound of the condition number which involves geometrical quantities on the subdomains (H and δ) and on the aggregates (H_0 and δ_0).

For the sake of completeness, for the special case of $H = H_0$ and $\delta = \delta_0$ we report an improved convergence bound, as proposed in [11]. In this case, the bound depends linearly on H/h, and this seems to be confirmed by our numerical results. However, it was not possible to extend the proof of [11] to our general setting.

The paper outline is as follows. Section 2 introduces the problem and the basic results of two-level Schwarz preconditioners. The definition of the coarse space using the concept of aggregation is considered in Section 3. In Section 4, a theoretical estimate of the condition number for a model problem is given. Section 5 reports the improved convergence bound presented in [11]. Section 6 reports several numerical results for a model problem. Finally, conclusions are drawn in Section 7.

2. Model problem

Let us consider again the Poisson problem (1), whose finite-element approximation reads

$$\begin{cases} \text{Find } u_h \in V_h \text{ such that} \\ a(u_h, v_h) = f(v_h) \quad \forall v_h \in V_h, \end{cases} \tag{2}$$

with $a(u_h, v_h) = \int_\Omega \nabla u_h \cdot \nabla v_h \, \mathrm{d}\Omega$, $f(v_h) = \int_\Omega f v_h \, \mathrm{d}\Omega$. By assumption, $f \in L_2(\Omega)$, and V_h is a family of finite dimensional subspaces of $H_0^1(\Omega)$. Precisely, V_h is the space of linear finite-element functions defined on a conforming, quasi-uniform triangulation $\mathcal{T}_h$ of Ω. By notation, φ_i is a generic basis function of V_h and h the grid size parameter. Existence of a unique solution of (2) is shown, for instance, in [15].

The computational domain Ω is then partitioned into M overlapping subdomains Ω_i. On each Ω_i we have, by construction, $\mathrm{diam}(\Omega_i) \leq CH$. Here and in the following, C is a constant independent of H, h, the overlap among the subdomains δ, and the coarse space parameters H_0, δ_0, and n_0 defined below.

The algebraic formulation of problem (2) reads

$$A\mathbf{u} = \mathbf{f}, \tag{3}$$

where $A \in \mathbb{R}^{n \times n}$ is a symmetric positive definite matrix, $\mathbf{u} \in \mathbb{R}^n$ the solution vector, and $\mathbf{f} \in \mathbb{R}^n$ the discretisation of force term f.

Let $V_i \subset H_0^1(\Omega_i)$ be a subspace of V_h containing the finite element functions defined on each subdomain Ω_i, $i = 1, \ldots, M$. We can now define the matrices $B_i = R_i^T A_i^{-1} R_i$, where $A_i = R_i A R_i^T$ and R_i^T is the discrete representation of the interpolation from V_i to V_h. With this notation, the one-level additive Schwarz method can be regarded as preconditioned iterative method (ideally, the preconditioned conjugate gradient method if A is symmetric and positive definite) for solving (3) with a preconditioner P_S such that

$$P_S^{-1} = \sum_{i=1}^{M} B_i. \tag{4}$$

In a two-level Schwarz method, a further term $B_0 = R_0^T A_0^{-1} R_0$ is added to the preconditioner, obtaining

$$P_{S,C,\mathrm{add}}^{-1} = \sum_{i=0}^{M} B_i = \sum_{i=0}^{M} R_i^T A_i^{-1} R_i. \tag{5}$$

A_0 corresponds to the solution of the original variational problem in the space V_0, which is "coarse" in the sense that it contains a limited number of degrees of freedom, in order to make the "exact" inversion of A_0 computationally acceptable – if n_0 is the dimension of the coarse space, one must have $n_0 \ll n$.

An alternative two-level method, derived from a three-step Richardson method, leads to the following preconditioned matrix:

$$P_{C,\mathrm{hybrid}}^{-1} A = I - (I - B_0 A)\left(I - P_S^{-1} A\right)(I - B_0 A). \tag{6}$$

This approach is very close to the so-called balancing Neumann-Neumann method for non-overlapping decompositions [20]. If the roles of B_0 and P_S are interchanged, the resulting preconditioner is close to the multigrid approach, with P_S playing the role of the smoother.

Remark 1. We always assume exact solvers for both the local and the coarse problems; approximate solvers could be considered as well with minor modifications.

We assume that there exists a set of interpolation operators $\mathcal{I}_i : V_i \to V_h$, and that a continuous and coercive bilinear form $a_i(\cdot,\cdot)$ that approximates $a(\cdot,\cdot)$ is defined on each V_i. Finally, let the projection-like operators $\tilde{\mathcal{I}}_i : V_h \to V_i$ be defined as

$$a_i\left(\tilde{\mathcal{I}}_i u, v\right) = a(u, \mathcal{I}_i v) \quad \forall v \in V_i, i = 1, \ldots, M. \tag{7}$$

The properties of additive Schwarz preconditioner (5) can be studied using an abstract convergence theory, as presented in [5]; see also [19] and the references therein. This theory is based on the following 3 assumptions.

Assumption 1. *Let C_0 be the minimum constant such that for all $u_h \in V_h$ there exists a representation $u_h = \sum_{i=0}^{M} \mathcal{I}_i u_i, u_i \in V_i$ such that*

$$\sum_{i=0}^{M} a_i(u_i, u_i) \leq C_0^2 a(u_h, u_h). \tag{8}$$

Assumption 2. *Define $0 \leq \epsilon_{i,j} \leq 1$ to be the minimal values that satisfy*

$$|a(\mathcal{I}_i u_i, \mathcal{I}_j u_j)| \leq \epsilon_{i,j} a(\mathcal{I}_i u_i, \mathcal{I}_i u_i)^{1/2} a(\mathcal{I}_j u_j, \mathcal{I}_j u_j)^{1/2} \qquad \forall u_i \in V_i, \quad \forall u_j \in V_j, i, j = 1, \ldots, M.$$

We define $\rho(\epsilon)$ to be the spectral radius of a matrix ϵ whose entries are the $\epsilon_{i,j}$.

Assumption 3. *Let ω be the minimum constant such that*

$$a(\mathcal{I}_i u_i, \mathcal{I}_i u_i) \leq \omega a_i(u_i, u_i) \quad \forall u_i \in V_i, \quad i = 0, \ldots, M \tag{9}$$

where we assume that the $a_i(\cdot,\cdot)$ are suitably scaled.

The following theorem holds. By notation, $\kappa(B)$ denotes the the spectral condition number of the square matrix B. For the proof, we refer to [5].

Lemma 1. *The abstract additive Schwarz preconditioner satisfies*

$$\kappa(P_{S,C,\text{add}}^{-1} A) \leq \omega \left[1 + \rho(\epsilon)\right] C_0^2. \tag{10}$$

In particular, $1/C_0^2$ is a sharp lower bound on the smallest eigenvalue of $P_{S,C,\text{add}}^{-1} A$ and $\omega\left[1 + \rho(\epsilon)\right]$ is a bound on the largest eigenvalue of $P_{S,C,\text{add}}^{-1} A$.

3. Definition of the coarse space

The spectral properties (and the parallel performance) of the two-level Schwarz preconditioner will depend on the definition of the coarse space V_0. There are virtually unlimited choices of the coarse grid correction that may be used. Convergence of the entire scheme will depend on the particular interpolation and coarse grid operator used. When possible, this coarse space V_0 may be itself embedded into V_h. If a coarse grid is used to define V_0, the following result holds [19].

Theorem 1. *For the additive two-level overlapping Schwarz method, when the overlap is uniform of width $\mathcal{O}(\delta)$, the coarse grid space V_0 corresponds to the finite-element functions on elements of $\mathcal{O}(H)$, and $V_0 \subset V_h$, we have:*

$$\kappa(P_{S,C,\text{add}}^{-1} A) \leq C\left(1 + \frac{H}{\delta}\right), \tag{11}$$

where C is a constant independent of h, H and δ.

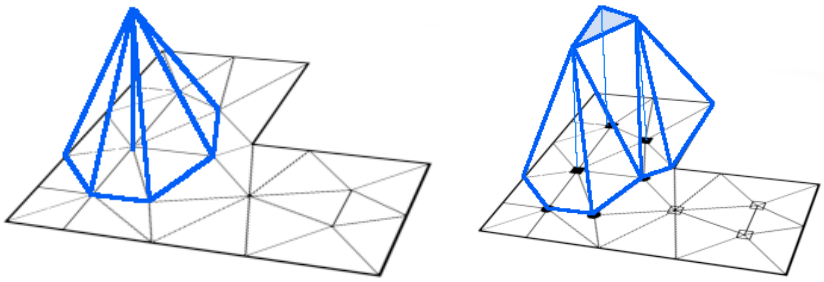

FIGURE 1. Example of basis function for the fine grid (on the left), and non-smoothed basis function for the coarse space (on the right).

In aggregation procedures, instead, the definition of the coarse space does not require the explicit construction of a coarse triangulation, since each element of V_0 is actually composed by a linear combination of elements of V_h. More precisely, we will group the nodes of the fine grid into n_0 sets of contiguous nodes, called aggregates. Then, the basis functions of the coarse space V_0 will be formed by summing up the finite element basis functions of all the nodes belonging to each aggregate. This defines the restriction operator R_0. The coarse matrix is computed as $A_0 = R_0 A R_0^T$.

To define R_0, we start by constructing a tentative, non-smoothed restriction $\tilde{R}_0$, so that each fine grid node is included in just one aggregate. This decomposition can be obtained using a graph partitioning algorithm (using for instance a vertex-oriented decomposition of the grid). The number of aggregates n_0 will represent the dimension of the coarse space V_0, since each aggregate will be given a single coarse grid basis function. We will indicate with $\tilde{\vartheta}_i$ the set of nodes that form the non-smoothed aggregate i. The entries of $\tilde{R}_0$ are thus as follows:

$$\tilde{R}_0(i,j) = \begin{cases} 1 & \text{if } j \in \tilde{\vartheta}_i \\ 0 & \text{otherwise.} \end{cases} \tag{12}$$

$\tilde{R}_0$ can be viewed as a simple grid transfer operator corresponding to piecewise constant interpolation. With this technique $V_0 \subset V_h$ and the non-smoothed basis functions $\{\tilde{\Phi}_i\}$ of V_0 are defined as

$$\tilde{\Phi}_i(\mathbf{x}) = \sum_{j \in \tilde{\vartheta}_i} \varphi_j(\mathbf{x}). \tag{13}$$

A 2D example is reported in Figure 1.

Once $\tilde{R}_0$ has been defined, the second step (that may actually be avoided) consists of applying a prolongator smoother $\tilde{S}_0$ to produce the final prolongator operator $R_0^T = \tilde{R}_0^T \tilde{S}_0$. The idea is to smooth out the $\{\tilde{\Phi}_i\}$, to reduce their H_1-norm (this also results in an increased support), using the stencil of suitable polynomials in A. The final coarse space reads

$$V_0 := \operatorname{span}\{\Phi_i\} = \operatorname{span}\left\{\tilde{S}_0 \tilde{\Phi}_i\right\}.$$

A possible choice for the smoother is the Richardson smoother [10,11] (where A may need to be suitably scaled):

$$\tilde{S} = \tilde{S}_0(\varpi_k, k) = (I - \varpi_k A)^k, \tag{14}$$

where $\varpi_k \in (0,1]$ is a real parameter and $k \in \mathbb{N}_0$. Unless A has particular properties (for example, it is a M-matrix), the spectral radius of a non-damped Richardson method is greater than one, hence the smoother requires the definition of a proper value for ϖ. In this case, ϖ must be chosen such that $\varpi < 2/\rho(A)$. Usually, it is not too difficult to obtain a good estimate of $\rho(A)$, using for instance the Gershgorin theorem, a small number

of Lanczos or conjugate gradient or power method iterations. It is sometimes useful to scale the estimated value by a small factor, because most computational methods give lower bounds to the largest eigenvalue.

Other smoothers proposed in literature are a recursive Richardson smoother [2, 22], or a SPAI smoother [3]. For a general comparison of these smoothers the reader is addressed to [10] or [11], where it is shown that, for the considered model problem, the effect of the above mentioned smoothers is quite similar.

4. A convergence bound

In order to provide a bound for the two-level Schwarz preconditioner with aggregation coarse space, we need to make precise some geometrical information about the used decomposition. This is done in the following properties. By notation, we indicate with ϑ_j the set of nodes belonging to the j-smoothed aggregate, with $\Theta_j = \text{int}\left(\cup_{i\in\vartheta_j} \text{supp}(\varphi_i)\right)$ the support of $\Phi_j, j = 1, \ldots, n_0$, and with $H_0 = \max_j\{\text{diam}(\Theta_j)\}$. In the following we will use the term aggregate for both ϑ_j and Θ_j, depending on the context.

Property 1 (partition)**.** *There exist two constants C_1 and C_2 so that, for each aggregate $\Theta_i, i = 1, \ldots, n_0$, we have:*

a. $\text{diam}(\Theta_i) \leq C_1 H_0$*;*
b. the Lebesgue measure $|\Theta_i|$ of Θ_i satisfies $|\Theta_i| \geq C_2 H_0^d$;
c. the overlap among the aggregates is of order δ_0.

Property 1 states that the aggregates have diameter of comparable size H_0 and are shape-regular. The following Property 2 requires a certain regularity on the coarse space basis function Φ_i.

Property 2 (coarse space)**.** *We assume that the basis functions $\{\Phi_i\}$ of the coarse space satisfy*

a. $|\Phi_i|^2_{H^1(\Omega)} \leq C\frac{H_0^{d-1}}{\delta_0}$*;*
b. $||\Phi_i||^2_{L^2(\Omega)} \leq CH_0^d$*;*
c. there is a domain $\hat{\Omega} \subset \Omega$ such that $\sum_i \Phi_i(x) = 1$ for every $x \in \hat{\Omega}$ and $\sup_{x\in\Omega\setminus\hat{\Omega}} \text{dist}(x, \partial\Omega) \leq CH_0$.

We can now prove the following lemma.

Lemma 2. *Let us define the operator $Q_0 : V_h \to V_0$ by*

$$Q_0 u = \sum_{i=1}^{n_0} \alpha_i \Phi_i, \quad \alpha_i = \frac{1}{|\Theta_i|}\int_{\Theta_i} u(\mathbf{x})\mathrm{d}\Omega. \tag{15}$$

If Properties 1 and 2 are satisfied, then there exists a constant $C > 0$ independent of H_0, δ_0 and h such that $\forall u \in V_h$

a. $||u - Q_0 u||^2_{L^2(\Omega)} \leq CH_0^2 |u|^2_{H^1(\Omega)}$*;*
b. $|Q_0 u|^2_{H^1(\Omega)} \leq C\frac{H_0}{\delta_0}|u|^2_{H^1(\Omega)}$.

Proof. The proof extends a similar proof proposed in [8]. It is split into 2 parts. First, we prove Lemma 2 in $\Omega \setminus \hat{\Omega}$; then, in $\hat{\Omega}$, where $\hat{\Omega}$ is the domain introduced in Property 2. Throughout the proof, C represents a generic positive constant independent of h, H, H_0, δ, and δ_0.

First, we note that, for the operator Q_0 defined by (15), we have

$$\sum_{i=1}^{n_0} \alpha_i^2(u) \leq CH_0^{-d}||u||^2_{L^2(\Omega)}, \tag{16}$$

which easily follows by the Cauchy-Schwarz inequality.

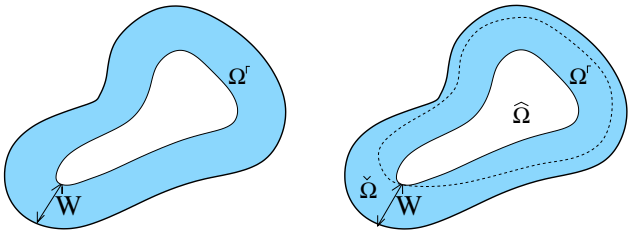

FIGURE 2. On the left, in shaded colour, Ω^Γ. On the right, in shaded colour, $\check{\Omega}$, and in white, $\hat{\Omega}$.

Now, let us consider Lemma 2a, and set $\check{\Omega} = \Omega \setminus \hat{\Omega}$. Further, define

$$\mathcal{B} = \left\{ i \,:\, \Theta_i \cap \check{\Omega} \neq \emptyset \right\}, \quad \check{\Xi} = \bigcup_{i \in \mathcal{B}} \Theta_i, \quad W = \sup_{x \in \check{\Omega}} \inf_{y \in \partial\Omega} \operatorname{dist}(x, y)$$

and set $\Omega^\Gamma = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) \leq W\}$. Figure 2 illustrates the domains $\check{\Omega}$ and Ω^Γ.

From Property 1c it follows that $W \leq CH_0$, and therefore the Poincaré inequality yields

$$\|u\|_{L_2(\check{\Omega})} \leq \|u\|_{L^2(\Omega^\Gamma)} \leq CH_0 |u|_{H^1(\Omega^\Gamma)}. \tag{17}$$

The restriction of $Q_0 u$ onto $\check{\Omega}$ can be expressed as

$$(Q_0 u)\,|_{\check{\Omega}} = \sum_{i \in \mathcal{B}} \alpha_i(u) \Phi_i(x)|_{\check{\Omega}}.$$

Further, let us set

$$\mathcal{N}_i = \{ j \mid \Theta_j \cap \Theta_i \neq \emptyset \}. \tag{18}$$

We have

$$\begin{aligned}
\|Q_0 u\|^2_{L^2(\check{\Omega})} &= \sum_{i \in \mathcal{B}} \sum_{j \in \mathcal{N}_i \cap \mathcal{B}} (\alpha_i(u)\Phi_i, \alpha_j(u)\Phi_j)_{L^2(\check{\Omega})} \\
&\leq \sum_{i \in \mathcal{B}} \sum_{j \in \mathcal{N}_i \cap \mathcal{B}} |\alpha_i(u)| \cdot |\alpha_j(u)| \cdot \|\Phi_i\|_{L^2(\Omega)} \|\Phi_j\|_{L^2(\Omega)} \\
\text{[by Prop. 2b]} \qquad &\leq CH_0^d \sum_{i \in \mathcal{B}} \sum_{j \in \mathcal{N}_i \cap \mathcal{B}} \frac{1}{2} \left(\alpha_i^2(u) + \alpha_j^2(u) \right) \\
&\leq CH_0^d \max\{\operatorname{card} \mathcal{N}_i\} \sum_{i \in \mathcal{B}} \alpha_i^2(u) \\
\text{[by (16)]} \qquad &\leq C \|u\|^2_{L^2(\Omega^\Gamma)}.
\end{aligned}$$

Using the last inequality together with inequality (17) gives

$$\|u - Q_0 u\|_{L^2(\check{\Omega})} \leq C \|u\|_{L^2(\check{\Omega})} \leq CH_0 |u|_{H^1(\Omega^\Gamma)}.$$

In an analogous way, we can estimate 2b in $\check{\Omega}$ exploiting Property 2a, obtaining

$$|Q_0 u|^2_{H^1(\check{\Omega})} \leq CH_0^{-1}/\delta_0 \, \|u\|^2_{L^2(\Omega^\Gamma)} \leq CH_0/\delta_0 \, |u|^2_{H^1(\Omega^\Gamma)}. \tag{19}$$

Now, we turn our attention to $\hat{\Omega}$.

For every function $u \in V_h$, let us consider an extension u_E satisfying

$$\|u_E\|_{H^1(\mathbb{R}^d)} \leq \|u\|_{H^1(\Omega)}, \quad u_E = u \text{ in } \Omega.$$

For $j = 1, \ldots, n_0$, we define $\Theta_j^\star = \cup_{i \in \mathcal{N}_j} \Theta_i$ with $\mathcal{N}_i$ given by (18), and Ξ_j to be the ball circumscribing $\Theta_j^\star$. From property 1 is follows that $\text{diam}(\Xi_j) \leq C H_0$. We now use the Friedrichs inequality in the form

$$\|u\|_{L^2(\Xi_j)} \leq C H_0 \, |u|_{H^1(\Xi_j)} \ , \ \forall u \in \left\{ v \in H^1(\Xi_j) \mid \int_{\Xi_j} v \, \mathrm{d}x = 0 \right\}. \tag{20}$$

For every $j = 1, \ldots, n_0$ we define

$$c_j = \int_{\Xi_j} u_E \, \mathrm{d}x, \quad \bar{u}_j = u_E - c_j.$$

Then, the Friedrichs inequality holds for every $\bar{u}_j$. Due to Property 2c, for every $x \in \Theta_j \cap \hat{\Omega}$ it holds

$$\begin{aligned} (Q_0 u)(x) &= (Q_0 \bar{u}_j)(x) + Q_0 c_j \\ &= \sum_{i \in \mathcal{N}_j} \alpha_i(\bar{u}_j) \Phi_i(x) + c_j \sum_{i \in \mathcal{N}_j} \Phi_i(x) \\ \text{[by Prop. 2c]} \qquad &= (Q_0 \bar{u}_j)(x) + c_j. \end{aligned}$$

Therefore,

$$\begin{aligned} \|u - Q_0 u\|^2_{L^2(\hat{\Omega})} \leq &= \sum_i \|(I - Q_0)(\bar{u}_i + c_i)\|^2_{L^2(\Xi_i \cap \hat{\Omega})} \\ &\leq \ 2 \sum_i \left(\|\bar{u}_i\|^2_{L^2(\Xi_i \cap \hat{\Omega})} + \|Q_0 \bar{u}_i\|^2_{L^2(\Xi_i \cap \hat{\Omega})} \right). \end{aligned} \tag{21}$$

Further,

$$\begin{aligned} \|Q_0 \bar{u}_i\|^2_{L^2(\Xi_i \cap \hat{\Omega})} &\leq \left\| \sum_{j \in \mathcal{N}_i} \alpha_j(\bar{u}_i) \Phi_j \right\|^2_{L^2(\Omega)} \\ &\leq \left(\sum_{j \in \mathcal{N}_i} |\alpha_j(\bar{u}_i)| \cdot \|\Phi_j\|_{L^2(\Omega)} \right)^2 \\ &\leq \text{card}\{\mathcal{N}_j\} \sum_{j \in \mathcal{N}_i} \alpha_j^2 \, (\bar{u}_i) \, \|\Phi_j\|^2_{L^2(\Xi_i \cap \check{\Omega})} \\ \text{[by 2b and (16)]} \qquad &\leq C \, \|\bar{u}_i\|^2_{L^2(\Xi_i \cap \check{\Omega})}. \end{aligned}$$

Substituting the last inequality into (21), using the Friedrichs inequality (20) and exploiting the bounded intersections of balls $\{\Xi_i\}_{i=1}^{n_0}$, we get

$$\|u - Q_0 u\|^2_{L^2(\hat{\Omega})} \leq C \sum_{i=1}^{n_0} \|\bar{u}_i\|^2_{L^2(\Xi_i)} \leq C H_0^2 \, |u|^2_{H^1(\Omega)}.$$

We have therefore proved Lemma 2a, since

$$\begin{aligned}\|u - Q_0 u\|^2_{L^2(\Omega)} &= \|u - Q_0 u\|^2_{L^2(\check{\Omega})} + \|u - Q_0 u\|^2_{L^2(\hat{\Omega})} \\ &\leq CH_0^2 \, |u|^2_{H^1(\Omega^\Gamma)} + CH_0^2 \, |u|^2_{H^1(\Omega)} \leq CH_0^2 \, |u|^2_{H^1(\Omega)} \, .\end{aligned}$$

To finish the proof, we turn to 2b in $\hat{\Omega}$. We have

$$\begin{aligned}|Q_0 u|^2_{H^1(\hat{\Omega})} &\leq \sum_{i=1}^{n_0} |Q_0 u|^2_{H^1(\Theta_i \cap \hat{\Omega})} \\ \text{[by Prop. 2a and (16)]} \qquad &\leq C \frac{H_0^{d-1}}{\delta_0} H_0^{-d} \sum_{i=1}^{n_0} \sum_{j \in \mathcal{N}_i} \|\bar{u}_i\|^2_{L^2(\Theta_i)} \\ \text{[by (20)]} \qquad &\leq C \frac{H_0}{\delta_0} \sum_{i=1}^{n_0} |\bar{u}_i|^2_{H^1(\Xi_i)} \\ &\leq C \frac{H_0}{\delta_0} \, |u|^2_{H^1(\Omega)} \, . \end{aligned}$$

□

For our main result we need the following two lemmas.

Lemma 3. *Let $\Omega_i \subset \mathbb{R}^d, d = 2, 3$, be a rectangle of diameter H, and let Γ_{δ_i} be a strip along its boundary of width $\delta > 0$. Then, for any function $u \in H^1(\Omega_i)$,*

$$\|u\|^2_{L^2(\Gamma_{\delta_i})} \leq C\delta^2 \left[\left(1 + \frac{H}{\delta} \right) |u|^2_{H^1(\Omega_i)} + \frac{1}{H\delta} \|u\|^2_{L^2(\Omega_i)} \right] . \tag{22}$$

Proof. See [4]. □

Lemma 4. *Under Properties 1 and 2 and for every finite element function $u_h \in V_h$, there exists a decomposition $\{u_i \in V_i\}, i = 0, \ldots, M$, such that*

$$u_h = \sum_{i=0}^{M} u_i, \tag{23}$$

and

$$\sum_{i=0}^{M} |u_i|^2_{H^1(\Omega)} \leq C \left(1 + \frac{H}{\delta} \right) \left(1 + \frac{H_0}{\delta_0} \right) |u_h|^2_{H^1(\Omega)},$$

with $H_0 \leq H$ and $\delta_0 \geq \delta$.

Proof. Define I_h to be the fine grid operator $I_h : V \to V_h$ such that

$$I_h(u_h) = \sum_{i=1}^{n} u_h(\mathbf{x}_i) \varphi_i$$

where φ_i is a generic finite element basis on the fine grid, and $\mathbf{x}_i$'s are the fine grid nodal points. Let η_i be a partition of unity such that $\eta_i \in C_0^\infty(\Omega_i)$ and $0 \leq \eta_i \leq 1$. We then define

$$u_0 = Q_0 u_h, \quad w = u_h - u_0, \quad u_i = I_h(\eta_i w).$$

Equation (23) is verified by construction. Because of the definitions on the overlap, we can ensure that the gradients of η_i are well behaved. That is, we can construct η_i so that $|\nabla \eta_i|^2_{L^\infty(\Omega)} \leq C/\delta^2$.

Over a single element K and using the inverse inequality, we easily get

$$\begin{aligned}|u_i|^2_{H^1(K)} &= |I_h\left[\bar{\eta}_i w + (\eta_i - \bar{\eta}_i)w\right]|^2_{H^1(K)} \\ &\leq 2\,|\bar{\eta}_i w|^2_{H^1(K)} + 2\,|I_h(\eta_i - \bar{\eta}_i)w|^2_{H^1(K)} \\ &\leq 2\,|\bar{\eta}_i w|^2_{H^1(K)} + Ch^{-2}\,\|I_h(\eta_i - \bar{\eta}_i)w\|^2_{L^2(K)} \\ &\leq 2\,|\bar{\eta}_i w|^2_{H^1(K)} + Ch^{-2}\,\|\eta_i - \bar{\eta}_i\|^2_{L^\infty(K)}\,\|I_h w\|^2_{L^2(K)}\,,\end{aligned}$$

where $\bar{\eta}_i$ being the average of η_i on element K. Now, we sum up over all the elements. The last term is identically zero for all elements K in the interior of Ω_i. Therefore, when we take the sum over all the elements, the last term only includes those elements in the overlapping region. Moreover, since a finite number, bounded independently of h, δ, H and H_0, of u_i is non-zero for any element K, we obtain summing over i,

$$\sum_{i=1}^{M} |u_i|^2_{H^1(\Omega)} \leq C\,|w|^2_{H^1(\Omega)}\,C\delta^{-2}\sum_{i=1}^{M}\|u_i\|^2_{L^2(\Gamma_{\delta_i})}\,.$$

Using Lemma 3 to bound the last term and Lemma 2, we get

$$\begin{aligned}\sum_{i=1}^{M} |u_i|^2_{H^1(\Omega)} &\leq C\,|w|^2_{H^1(\Omega)} + C\sum_{i=1}^{M}\left[\left(1+\frac{H}{\delta}\right)|u_i|^2_{H^1(\Omega)} + \frac{1}{H\delta}\,\|u_i\|^2_{L^2(\Omega)}\right] \\ &\leq C\,|w|^2_{H^1(\Omega)} + C\left(1+\frac{H}{\delta}\right)|w|^2_{H^1(\Omega)} + C\frac{1}{H\delta}\,\|w\|^2_{L^2(\Omega)} \\ &\leq C\left(1+\frac{H}{\delta}\right)|u - Q_0 u|^2_{H^1(\Omega)} + C\frac{H_0^2}{H\delta}\,|u|^2_{H^1(\Omega)} \\ &\leq C\left(1+\frac{H}{\delta}\right)\left(1+\frac{H_0}{\delta_0}\right)|u|^2_{H^1(\Omega)}\,.\end{aligned}$$

Note that we have used the fact that $\delta_0 \geq \delta$ and $H_0 \leq H$. This latter inequality states that the coarse space must be sufficiently rich with respect to the number of subdomains. □

The following theorem states the main result of this paper.

Theorem 2 (aggregation coarse space)**.** *Let the Properties 1 and 2 hold. Then, for the additive two-level overlapping Schwarz method, when the overlap is uniform of width $\mathcal{O}(\delta)$ and $V_0 = \operatorname{span}\{\Phi_i, i = 1, \ldots, n_0\}$, there exists $C > 0$ such that*

$$\kappa(P^{-1}_{S,C,\mathrm{add}}A) \leq C\left(1+\frac{H_0}{\delta_0}\right)\left(1+\frac{H}{\delta}\right)\cdot \tag{24}$$

Proof. The proof follows easily using Lemma 1. As verified by Lemma 4, the first parameter C_0^2 of Definition 1 is bounded by

$$C_0^2 \leq \left(1+\frac{H}{\delta}\right)\left(1+\frac{H_0}{\delta_0}\right)\cdot$$

The second parameter can be estimated by using a coloring argument. We know that we can colour the subdomains with N_c colours, independently of h and H. Taking the coarse space into account we have therefore that we can group the T_i into $N_c + 1$ classes. Hence, $\rho(\epsilon) \leq N_c + 1$. Since $\omega = 1$ as we suppose to use exact solvers on the subdomains, the thesis follows from Lemma 1. □

Next, we report two lemmas, which prove that both the non-smoothed and the smoothed aggregation procedure outlined in Section 3 satisfy Properties 1 and 2.

Lemma 5 (non-smoothed aggregation)**.** *The non-smoothed functions* $\{\tilde{\Phi}_i\}$ *defined by equation (13) satisfy Property 2 with* $H_0 = \tilde{H}_0$ *and* $\delta_0 = h$.

Proof. Property 2a is proved by noting that $\nabla\tilde{\Phi}_i$ is zero outside the overlapping part $\Gamma_{\tilde{\Theta}_i}$ of $\tilde{\Theta}_i$ of width h, while it is bounded by $1/h$ on $\Gamma_{\tilde{\Theta}_i}$, and that $\Gamma_{\tilde{\Theta}_i}$, where the gradient of $\tilde{\Phi}_i$ is non-zero, has a Lebesgue measure of order $\mathcal{O}(h\tilde{H}_0^{d-1})$ and the grid is quasi-uniform. Property 2b is shown noting that $||\tilde{\Phi}_i||^2_{L^2(\Omega)} \leq C|\tilde{\Theta}_i| \leq C\tilde{H}_0^d$. where $\tilde{\Theta}_i$ is the support of $\tilde{\Phi}_i$. Property 2c easily follows. □

Lemma 6 (smoothed aggregation)**.** *Let* $\tilde{S}_0$ *be defined as in equation (14), and let* $\tilde{\Phi}_i, i = 1, \ldots, n_0$ *satisfy Properties 1 and 2 with* H_0 *replaced by* $\tilde{H}_0$ *and* δ_0 *by* $\tilde{\delta}_0$. *Then, for a fixed, sufficiently small, degree* k *of the smoother, the coarse functions* $\Phi_i = \tilde{S}_0(k)\tilde{\Phi}_i$ *satisfy Properties 1 and 2.*

Proof. The effect of the smoother is to increase the diameter of the subdomains and the overlap. The application of the smoother results in a diameter of the aggregates of $\mathcal{O}(\tilde{H}_0 + kh)$ and an overlap of order $\mathcal{O}(h + kh)$. Since the triangulation is quasi-uniform, we have that, for a fixed k,

$$\operatorname{diam}(\Theta_i) = C(\tilde{H}_0 + kh) \leq CH_0.$$

This verifies Property 1a. Using similar arguments, Property 1b is verified.

Now, let us prove that Property 2a holds. We have

$$|\Phi_i|^2_{H^1(\Omega)} = \left|\tilde{S}_0\tilde{\Phi}_i\right|^2_{H^1(\Omega)} \leq \rho(\tilde{S}_0)^2 \left|\tilde{\Phi}_i\right|^2_{H^1(\Omega)} \leq C\rho(\tilde{S}_0)^2 \frac{H_0^{d-1}}{\delta_0},$$

since $\tilde{H}_0 \leq H_0$ and $\delta_0 = h + kh \geq C\tilde{\delta}_0$ for a fixed k.

As regards Property 2b, we have:

$$\|\Phi_i\|^2_{L^2(\Omega)} = \left\|\tilde{S}_0\tilde{\Phi}_i\right\|^2_{L^2(\Omega)} \leq \rho(\tilde{S}_0)^2 \left\|\tilde{\Phi}_i\right\|^2_{L^2(\Omega)} \leq CH_0^d.$$

To prove Property 2c, we define the function $u(x) = \sum_{i=1}^{n_0} \tilde{\Phi}_i(x)$, which is equal to one $\forall \mathbf{x} \in \hat{\Omega}$, and consequently at every $x \in \Omega$ outside a strip of width $\mathcal{O}(\tilde{\delta}_0)$ around $\partial\Omega$. Thus, we obtain

$$\sum_{i=1}^{n_0} \Phi_i(x) = \left(\tilde{S}_0 \sum_{i=1}^{n_0} \tilde{\Phi}_i\right)(x) = (\tilde{S}_0 u)(x) = 1$$

at every $x \in \Omega$ outside a strip of width $\mathcal{O}(h) + kh = \mathcal{O}(\delta_0)$ around $\partial\Omega$. □

We conclude this section with two remarks about the parallel implementation of SA techniques.

Remark 2. In general, one let each processor build the aggregate corresponding to its piece of the grid (corresponding to the triangulation of the computational domain), ignoring the connections among subdomains. That is, each processor is assigned a subgrid of the entire grid (that corresponds to the triangulation of the subdomain given to that processor); then, a serial aggregation algorithm is used on each subgrid. This approach, sometimes referred to as decoupled aggregation, is efficient if the load balance of the grid data among the processors is satisfactory. In general, when the ratio between the number of nodes and the number of aggregates is large enough, like for the two-level methods here presented, the decoupled aggregation offers good partitioning. If a large number of aggregates are required (like, for instance, in multilevel methods), decoupled aggregation may result in a somewhat irregular decomposition, and in this case it is usually worth to re-equilibrate the partitioning among the subdomains to minimize the dependency of the resulting algorithm on the subdomain decomposition; see [21].

Remark 3. If $\delta_0 = \delta = \eta h$, $\eta \geq 1$, and the diameter of the aggregates Θ_i is bounded by H/χ, where $\chi \geq 1$, that is, $\chi H_0 = H$, we have

$$\kappa\left(P_{S,C,\text{add}}^{-1} A\right) \leq C\left(1+\frac{H}{\eta h}\right)\left(1+\frac{H}{\chi \eta h}\right) \leq \frac{C}{\chi \eta^2}\left(\frac{H}{h}\right)^2.$$

Equation (25) states that we can reduce the condition number by increasing the number of aggregates and the overlap. Computationally, it is expensive to use wider overlap (a minimal-overlap is often preferred), whereas larger values of χ have influence only on the coarse problem, whose size is remarkably smaller than that of the global problem.

5. An improved convergence bound

A possible improvement of the estimate given by Theorem 2 can be obtained considering that each subdomain defines an aggregate. This bound is presented in [11]. To establish this result, the following property is required.

Property 3. *We assume that the basis functions* $\{\Phi_i\}, i = 0, \ldots, M$ *of the coarse space* V_0 *satisfy*

(1) $\|\Phi_i\|^2_{L^\infty(\Omega)} \leq C$;
(2) $\|\nabla \Phi_i\|^2_{L^\infty(\Omega)} \leq C/\delta^2$;
(3) $\sum_{i=1}^{M} \Phi(x) = 1, \ \forall x \in \bar{\Omega}$;
(4) $\text{supp}\,(\Phi_i) \subseteq \bar{\Omega}_i$.

The following theorem holds (see [11], Lem. 7).

Theorem 3. *Let Properties 1 (with* $H_0 = H$*) and 3 hold. Then, there exists a constant* $C > 0$ *such that*

$$\kappa(P_{S,C,\text{add}}^{-1} A) \leq C\left(1+\frac{H}{\delta}\right).$$

6. Numerical experiments

In this section we report some numerical results for problem (1) with $\Omega = (0,1) \times (0,1)$. The grid is built by dividing Ω into n^2 equal squares and subdividing them into two triangles. Thus, we obtain a triangulation with $h = \frac{1}{n}$. As regards the decomposition into subdomains, we consider overlapping squares Ω_i of area H^2. We use linear finite-elements, and solve the linear system (3) by the conjugate gradient method. Aggregates are build, of square shape, by grouping fine-grid nodes into non-overlapping sets. As a result, the overlap among the aggregates is one element (that is, $\delta_0 = h$).

The tables report the estimated condition number for the preconditioned system; see for instance [7].

We have used minimal overlap among the subdomains, that is, $\delta = h$. Equation (12) is used to define $\tilde{R}_0$. The smoother is the Richardson smoother (14), with $\varpi = 1.5/\rho(\tilde{A}_0)$, where 1.5 is a chosen correction factor to minimize the spectral radius of the coarse problem. Possible under-estimate in the value of $\rho(A)$ could be handled by lowering this value.

Table 1 gives the condition number for the one-level Schwarz preconditioner (4). The condition number grows as $\mathcal{O}(1/hH)$ as theory predicts [16]. The following Tables 2 and 3, which are about the two-level Schwarz preconditioner (5) and (6) with a coarse space built using a coarse grid, confirm the bound provided by Theorem 1. Hybrid preconditioners behave only slightly better than additive ones, therefore their use for these kind of problems seems unjustified. Tables 4 and 5 report the influence of a non-smoothed aggregation procedure. The effect of the parameter χ of equation (25) can be clearly appreciated. $P_{S,C,\text{hybrid}}^{-1}$ shows the same convergence

TABLE 1. Estimated condition number for $P_S^{-1}A$ using the one-level Schwarz preconditioner P_S with minimal overlap ($\delta = h$).

P_S	$H = 1/2$	$H = 1/4$	$H = 1/8$	$H = 1/16$
$h = 1/16$	15.95	27.09	52.08	-
$h = 1/32$	31.69	54.52	104.85	207.67
$h = 1/64$	63.98	109.22	210.07	416.09
$h = 1/128$	127.99	218.48	420.04	832.57

TABLE 2. Estimated condition number for $P_{S,C,\text{add}}^{-1}A$, with a coarse space built using a coarse grid (standard coarse space).

$P_{S,C,\text{add}}$	$H = 1/4$	$H = 1/8$	$H = 1/16$	$H = 1/32$
$h = 1/32$	7.03	4.94	-	-
$h = 1/64$	12.73	7.59	4.98	-
$h = 1/128$	23.62	13.17	7.66	4.99
$h = 1/256$	45.33	24.34	13.28	-

TABLE 3. Estimated condition number for $P_{S,C,\text{hybrid}}^{-1}A$, with a coarse built using a coarse grid (standard coarse space).

$P_{S,C,\text{hybrid}}$	$H = 1/4$	$H = 1/8$	$H = 1/16$	$H = 1/32$
$h = 1/32$	6.11	3.56	-	-
$h = 1/64$	11.47	6.24	3.58	-
$h = 1/128$	22.26	11.71	6.27	3.58
$h = 1/256$	43.86	22.71	11.77	-

TABLE 4. Estimated condition number for $P_{S,C,\text{add}}^{-1}A$, with a coarse space built using non-smoothed aggregation.

$P_{S,C,\text{add}}$	χ	$H = 1/4$	$H = 1/8$	$H = 1/16$
$h = 1/16$	1	13.37	8.87	-
$h = 1/32$	1	26.93	17.71	9.82
$h = 1/64$	1	54.33	35.21	19.70
$h = 1/128$	1	109.39	70.22	39.07
$h = 1/32$	2	13.13	7.78	-
$h = 1/64$	2	27.18	15.28	9.96
$h = 1/32$	3	7.61	-	-
$h = 1/64$	3	17.13	8.39	-

rate of the additive version, although the estimated condition number is sensibly smaller. Note that two matrix-vector products are needed to apply $P_{S,C,\text{hybrid}}^{-1}$ to a given vector, thus making its application computationally more expensive than that of $P_{S,C,\text{add}}^{-1}$. However, differently from "classical" coarse spaces, hybrid preconditioners perform much better than their additive versions, with significant reduction in the condition number.

Finally, Tables 6–10 show the influence of the smoother (14) for different values of k. Note that, for all the smoothers, the hybrid version performs significantly better than the additive one.

By comparing Tables 4 and 6, one can see that one step of smoother (14) results in small reductions in the condition number, using an additive preconditioner. This reduction is even smaller for the hybrid version,

TABLE 5. Estimated condition number for $P_{S,C,\text{hybrid}}^{-1}A$, with a coarse space built using non-smoothed aggregation, with $\chi = 1$.

$P_{S,C,\text{hybrid}}$	$H = 1/4$	$H = 1/8$	$H = 1/16$
$h = 1/16$	5.24	2.89	-
$h = 1/32$	10.64	5.66	2.97
$h = 1/64$	21.60	11.34	5.79
$h = 1/128$	43.65	22.77	11.55

TABLE 6. Estimated condition number for $P_{S,C,\text{add}}^{-1}A$, with a coarse space built using aggregation and smoother (14) with $k = 1$ and $\chi = 1$.

$P_{S,C,\text{add}}$	$H = 1/4$	$H = 1/8$	$H = 1/16$	$H = 1/32$
$h = 1/16$	11.91	6.02	-	-
$h = 1/32$	25.59	14.95	6.28	-
$h = 1/64$	50.03	32.64	16.23	6.36
$h = 1/128$	108.13	67.75	35.81	16.631
$h = 1/256$	218.57	137.91	74.55	-

TABLE 7. Estimated condition number for $P_{S,C,\text{hybrid}}^{-1}A$, with a coarse space built using aggregation and smoother (14) with $k = 1$ and $\chi = 1$.

$P_{S,C,\text{hybrid}}$	$H = 1/4$	$H = 1/8$	$H = 1/16$	$H = 1/32$
$h = 1/16$	5.09	2.86	-	-
$h = 1/32$	10.49	5.63	2.96	-
$h = 1/64$	21.46	11.31	5.77	2.99
$h = 1/128$	43.51	22.75	11.54	5.82

TABLE 8. Estimated condition number for $P_{S,C,\text{add}}^{-1}A$, with a coarse space built using aggregation and smoother (14) with $k = 2$ and $\chi = 1$.

$P_{S,C,\text{add}}$	$H = 1/4$	$H = 1/8$	$H = 1/16$	$H = 1/32$
$h = 1/16$	10.71	5.70	-	-
$h = 1/32$	24.28	12.78	5.92	-
$h = 1/64$	51.77	30.35	19.70	10.12
$h = 1/128$	106.89	65.42	32.97	13.82

as shown by Tables 5 and 7. This suggests that, at least for the considered problem and two-level hybrid preconditioners, non-smoothed aggregation can be preferred to smoothed aggregation. By comparing Tables 4, 6, 8, and 10, one can note that high-order polynomials can be used to improve the condition number (even if, for $k > 1$, better strategies to define the damping parameters need to be studied).

7. Conclusions

In this paper we have presented aggregation procedures to construct the coarse space for two-level Schwarz preconditioners. The basis functions of V_0 are built as a linear combination of the basis functions of the fine

TABLE 9. Estimated condition number for $P_{S,C,\text{hybrid}}^{-1}A$, with a coarse space built using aggregation and smoother (14) with $k = 2$ and $\chi = 1$.

$P_{S,C,\text{hybrid}}$	$H = 1/4$	$H = 1/8$	$H = 1/16$	$H = 1/32$
$h = 1/16$	5.00	2.82	-	-
$h = 1/32$	10.39	5.611	.	-
$h = 1/64$	21.36	11.29	5.78	2.98

TABLE 10. Estimated condition number for $P_{S,C,\text{add}}^{-1}A$, with a coarse space build using aggregation, using smoother (14) with $k = 3$ and $\chi = 1$.

$P_{S,C,\text{add}}$	$H = 1/4$	$H = 1/8$	$H = 1/16$	$H = 1/32$
$h = 1/16$	9.77	5.59	-	-
$h = 1/32$	23.12	11.08	5.88	-
$h = 1/64$	50.56	28.31	11.55	-

space, and the use of a coarse triangulation is not required. For this reason, aggregation procedures can be easily applied to problems defined on complex geometries without losing the power of two-level methods. The coarse matrix can be constructed automatically and for any computational grid with no input from the user, except for the linear system matrix A and the dimension of the coarse space. Moreover, the computational complexity of the aggregation procedure is smaller since the method is simpler to implement. This simplicity has its origin in the way the restriction and interpolation operators are defined.

A theoretical analysis is reported for an elliptic model problem, discretized on a quasi-uniform grid with finite-elements. Our theory extends results presented in literature; see [8, 11]. There, the authors assume equal overlap among the subdomains δ and the aggregates δ_0, and equal size of the subdomains H and the aggregates H_0. Instead, in the bound of Theorem 2, the influence of the subdomain decomposition and the aggregates are kept separate.

For the special case $H = H_0$ and $\delta = \delta_0$, Theorem 3 furnishes an improved convergence bound, which depends linearly on H/δ. However, the basis functions of the coarse space must satisfy different properties: Property 2 for Theorem 2 and Property 3 for Theorem 3. It can be proved that the presented smoother satisfies Property 2, while, to our knowledge, there are no proofs that it satisfies Property 3. (We note however that this property can be verified for special cases.) Theorem 2 is more general and allows aggregates and subdomains of different shapes and overlap, but numerical results (at least for our model problem) seem to satisfy Theorem 3.

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