ESAIM: M2AN
Vol. 39, N° 4, 2005, pp. 637–648
DOI: 10.1051/m2an:2005028

ESAIM: Mathematical Modelling and Numerical Analysis

CHARACTERIZATION OF THE LIMIT LOAD IN THE CASE OF AN UNBOUNDED ELASTIC CONVEX

Adnene Elyacoubi[1] and Taieb Hadhri[1]

Abstract. In this work we consider a solid body $\Omega \subset \mathbb{R}^3$ constituted by a nonhomogeneous elastoplastic material, submitted to a density of body forces λf and a density of forces λg acting on the boundary where the real λ is the loading parameter. The problem is to determine, in the case of an unbounded convex of elasticity, the Limit load denoted by $\bar{\lambda}$ beyond which there is a break of the structure. The case of a bounded convex of elasticity is done in [El-Fekih and Hadhri, *RAIRO: Modél. Math. Anal. Numér.* **29** (1995) 391–419]. Then assuming that the convex of elasticity at the point x of Ω, denoted by K(x), is written in the form of $\mathrm{K}^D(x) + \mathbb{R}\mathrm{I}$, I is the identity of $\mathbb{R}^9{}_{sym}$, and the deviatoric component K^D is bounded regardless of $\mathrm{x} \in \Omega$, we show under the condition "Rot f $\neq 0$ or g is not colinear to the normal on a part of the boundary of Ω", that the Limit Load $\bar{\lambda}$ searched is equal to the inverse of the infimum of the gauge of the Elastic convex translated by stress field equilibrating the unitary load corresponding to $\lambda = 1$; moreover we show that this infimum is reached in a suitable function space.

Mathematics Subject Classification. 74xx.

Received: February 6, 2004.

1. The Hencky's problem for a non-homogeneous elastoplastic structure

Using the notations and the operators given in [5], the Hencky's problem is given by the following system: find a tensor σ and a displacement u such that

$$\left\{\begin{aligned} \operatorname{div}\sigma &= \lambda f \quad a.e. \text{ in } \quad \Omega \\ \sigma\cdot n &= \lambda g \quad \text{on} \quad \Gamma_1 \\ u &= u_0 \quad \text{on} \quad \Gamma_0 \\ \sigma(x) &= \Pi_{K(x)}\left(A^{-1}_{(x)}(\varepsilon(u)(x))\right). \end{aligned}\right. \tag{1}$$

Here:

$$\varepsilon(u) = (\varepsilon_{ij}(u)) \quad \text{and} \quad \varepsilon_{ij}(u) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad \text{for } 1 \le i,j \le 3$$

Keywords and phrases. Elasticity, limit load.

[1] LIM, Polytechnic School of Tunisia. adnene.elyacoubi@ept.rnu.tn; taieb.hadhri@ept.rnu.tn

A_x^{-1} the inverse matrix of A_x, A_x defined for $\eta = (\eta_{ij})_{1\leq i,j\leq 3} \in \mathbb{R}^9$ by:

$$(A_x(\eta))_{ij} = \frac{1}{9K_0(x)}\eta_{kk}(x)\delta_{ij} + \frac{1}{2\mu(x)}\eta_{ij}^D$$

$\mathrm{K}_0(x) = \alpha(x) + \frac{2\mu(x)}{3}$ where α and μ are the Lamé coefficients.

We suppose that:
(H_1) $\Gamma_1 \cup \Gamma_0 = \partial\Omega$: the boundary of Ω with $(\Gamma_1) \neq 0$ and the interiors of Γ_1 and Γ_0 satisfy $\Gamma_1^0 \cap \Gamma_0^0 = \emptyset$
(H_2) $K(x)$ is a closed convex part of $\mathbb{R}^9_{sym}$ and $\exists c > 0$ such that:

$$B(0,c) \subset K(x) \ \textit{a.e.} \text{ in } \ \Omega.$$

Here: $\mathbb{R}^9_{sym} = \{X = (x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}), x_{ij} \in \mathbb{R} \text{ for } 1 \leq i \leq 3;\ 1 \leq j \leq 3 \text{ and } x_{ij} = x_{ji}\}$
(H_3) $g \in (L^\infty(\Gamma_1))^3, f \in (L^4(\Omega))^3$ such that:

$$\exists \tilde{g} \in (L^\infty(\partial\Omega))^3,\ \tilde{g} = g \text{ on } \Gamma_1 \ \text{ and } \ \int_\Omega f\mathrm{d}x + \int_{\partial\Omega} \tilde{g}\mathrm{d}\Gamma = 0$$

(H_4) $K(x) = K^D(x) + \mathbb{R}I$ and $\exists M > 0$ such that:

$$K^D(x) \subset B(o,M) \ \textit{a.e.} \text{ in } \Omega.$$

We define the following set K_{ad}:

$$K_{ad} = \{\eta \in (L^2(\Omega))^9_s \ \text{ such that } \ \eta(x) \in K(x) \quad \textit{a.e.} \text{ in } \Omega\}. \tag{2}$$

It is clear that K_{ad} is a closed convex of $(L^2(\Omega))^9_s$.

We define now the Quasi-elastic problem:
Find a tensor σ^e and a displacement u^e satisfying:

$$\begin{cases} \operatorname{div} \sigma^e = f & \textit{a.e.} \text{ in } \Omega \\ \sigma^e \cdot n = \tilde{g} & \text{on } \partial\Omega \\ \sigma^e(x) = \left(A_{(x)}^{-1}(\varepsilon(u^e)(x))\right) & \textit{a.e.} \text{ in } \Omega. \end{cases} \tag{3}$$

Referring to [3], the above problem has a solution (σ^e, u^e), which is unique within a rigid body displacement for u^e, since f and g satisfy (H_3); moreover, we have the following proposition.

Proposition 1. *We assume that f and g satisfy (H_3), then we have:*

$$\sigma^e \in L^\infty(\Omega, \mathbb{R}^9_{sym}).$$

Proof. According to [3], we have $\sigma^e \in (W^{1,4}(\Omega))^9$ and according to [1] we conclude that: $\sigma^e \in L^\infty(\Omega, \mathbb{R}^9_{sym})$. □

2. Characterization of the limit load $\bar{\lambda}$

Definition 1. Considering the functional F defined on V_1:

$$\mathrm{V}_1 = \{\eta \in L^2(\Omega, \mathbb{R}^9_{sym}) \ \text{ such that } \ \operatorname{div} \eta = 0 \ \textit{a.e.} \text{ in } \ \Omega \text{ and } \eta \cdot n = 0 \text{ on } \Gamma_1\} \tag{4}$$

by:

$$F(\eta) = \mathrm{J}_{K_{ad}}(\sigma^e - \eta) \tag{5}$$

where, $J_{K_{ad}}$ is the gauge of K_{ad} defined by:

$$J_{K_{ad}}(\alpha) = \inf\{s > 0 \text{ such that } \alpha(x) \in s.K(x) \ a.e. \text{ in } \Omega\}. \tag{6}$$

Then we have:

Proposition 2. *The functional F is l.s.c (lower semicontinuous) on V_1 for the weak topology of $L^2(\Omega, \mathbb{R}^9_{sym})$ and we have: $F(\eta) = F(\eta^D)$.*

Proof. The functional F is l.s.c for the strong topology of $L^2(\Omega, \mathbb{R}^9_{sym})$ and according to [2] we have F is l.s.c on V_1 for the weak topology of $L^2(\Omega, \mathbb{R}^9_{sym})$.

On the other hand K_{ad} is, according to (H_4), unchanged in the direction of the spherical stress, then we have: $F(\eta) = F(\eta^D)$. □

Definition 2. The Limit Load $\bar{\lambda}$ is defined in [5] by:

$$\bar{\lambda} = \sup\{\lambda > 0 \text{ such that } D_\lambda \neq \emptyset\} \tag{7}$$

where

$$D_\lambda = \left\{ \sigma \in L^2(\Omega, \mathbb{R}^9_{sym}) \text{ such that } \left\{ \begin{array}{c} \operatorname{div}\sigma - \lambda f = 0 \quad a.e. \text{ in } \Omega \\ \sigma \cdot n = \lambda g \quad \text{on } \Gamma_1 \\ \sigma(x) \in K(x) \ a.e. \text{ in } \Omega \end{array} \right\} \right\}. \tag{8}$$

Then we have the following theorems:

Theorem 1. *Under the hypotheses $(H_1), (H_2)$ and (H_3) we have:*
(i) *If* $\inf\limits_{\eta \in V_1} F(\eta) = 0$, *then* $\bar{\lambda} = +\infty$.
(ii) *If* $\inf\limits_{\eta \in V_1} F(\eta) \neq 0$, *then* $\bar{\lambda} = \dfrac{1}{\inf\limits_{\eta \in V_1} F(\eta)}$.

Proof.
1. Proof of (i).
We assume that:

$$\inf_{\eta \in V_1} F(\eta) = 0. \tag{9}$$

Let $\lambda > 0$, and let us show that $\bar{\lambda} > \lambda$.

There exists $\eta \in V_1$ such that $F(\eta) < \frac{1}{\lambda}$ and according to (5) we have:

$$J_{K_{ad}}(\sigma^e - \eta) < \frac{1}{\lambda}.$$

Using (6) we obtain:

$$(\sigma^e - \eta) \in \frac{1}{\lambda} K_{ad}.$$

and then

$$\lambda(\sigma^e - \eta)(x) \in K(x) \ a.e. \text{ in } \Omega. \tag{10}$$

On the other hand, we have: $\operatorname{div}(\lambda(\sigma^e - \eta)) = \lambda(\operatorname{div}\sigma^e - \operatorname{div}\eta)$.

Using (3) and (4) we conclude:

$$\operatorname{div}(\lambda(\sigma^e - \eta)) = \lambda f \quad \text{in } \Omega, \tag{11}$$

and

$$\lambda(\sigma^e - \eta) \cdot n = \lambda g \quad \text{on } \Gamma_1. \tag{12}$$

From (10)–(12) we obtain: $\lambda(\sigma^e - \eta) \in D_\lambda$, and according to (7) it is clear that:

$$\bar{\lambda} > \lambda.$$

We finally conclude that:

$$\bar{\lambda} = +\infty.$$

2. Using the same idea, we prove (ii). □

We now distinguish these two cases:

Theorem 2. *Under the hypotheses* $(H_1), (H_2), (H_3)$ *and* (H_4) *the following statements* (i) *and* (ii) *are equivalent:*
(i) $\exists \eta \in V_1$ *such that* $F(\eta) = 0$.
(ii) *The following problem* (P_2) *has at least one solution*

$$(P_2) \begin{cases} \textit{Find } \alpha \textit{ in } W^{1,2}(\Omega) \textit{ satisfying:} \\ \nabla\alpha = -f \quad \textit{in } \Omega \\ \alpha n = -g \quad \textit{on } \Gamma_1. \end{cases} \tag{13}$$

Proof.
1. Assume there exists $\eta \in V_1$ such that $F(\eta) = 0$, then according to (5) we have

$$\mathrm{J}_{K_{ad}}(\sigma^e - \eta) = 0, \tag{14}$$

which implies, using (6):

$$\inf\{s > 0 \text{ such that } (\sigma^e - \eta)(x) \in sK(x) \ \ a.e. \text{ in } \Omega\} = 0;$$

and using (H_4) we have:

$$\inf\{s > 0 \text{ such that } (\sigma^e - \eta)^D(x) \in sK^D(x) \ \ a.e. \text{ in } \Omega\} = 0,$$

which implies:

$$\exists s_n > 0; (s_n)_{n\in\mathbb{N}} \text{ independent of } x \text{ and such that: } \begin{cases} s_n \xrightarrow[n\to+\infty]{} 0 \\ \text{and} \\ (\sigma^e - \eta)^D(x) \in s_n K^D(x) \ a.e. \text{ in } \Omega. \end{cases}$$

Then we can write:

$$(\sigma^e)^D(x) = \eta^D(x) \ \ a.e. \text{ in } \Omega, \tag{15}$$

and then:

$$\eta(x) = (\sigma^e)^D(x) + \frac{1}{3}\mathrm{tr}\eta I,$$

and so:

$$\mathrm{div}\eta = \mathrm{div}(\sigma^e)^D + \mathrm{div}\left(\frac{1}{3}\mathrm{tr}\eta I\right).$$

But $\eta \in V_1$, then divη =0, which gives using (3):

$$\nabla\alpha = -f, \tag{16}$$

where $\alpha = \frac{1}{3}(\mathrm{tr}\eta - \mathrm{tr}\sigma^e)$.

It is clear that $\alpha \in L^2(\Omega)$ and $\nabla\alpha \in (L^2(\Omega))^3$ which implies that $\alpha \in W^{1,2}(\Omega)$.
On the other hand, using (4) we obtain:

$$\eta \cdot n = 0 \quad \text{on } \Gamma_1,$$

then

$$\left(\eta^D + \frac{1}{3}\mathrm{tr}\eta I\right) \cdot n = 0 \quad \text{on } \Gamma_1,$$

which can be written using (15) as:

$$\left((\sigma^e)^D + \frac{1}{3}\mathrm{tr}\eta I\right) \cdot n = 0 \quad \text{on } \Gamma_1$$

or

$$\left((\sigma^e) + \frac{1}{3}(\mathrm{tr}\eta - \mathrm{tr}\sigma^e)I\right) \cdot n = 0 \quad \text{on } \Gamma_1$$

and using (3), we have:

$$\left(\frac{1}{3}(\mathrm{tr}\eta - \mathrm{tr}\sigma^e)I\right) \cdot n = -g \quad \text{on } \Gamma_1,$$

that means:

$$\alpha \cdot n = -g \quad \text{on } \Gamma_1$$

and using (16) we conclude the first implication.
2. Assume now that (P_2) has at least one solution in $W^{1,2}(\Omega)$, then there exists $\alpha \in W^{1,2}(\Omega)$ such that:

$$\begin{cases} \nabla\alpha = -f \quad \text{in } \Omega \\ \text{and} \\ \alpha \cdot n = -g \quad \text{on } \Gamma_1. \end{cases} \tag{17}$$

Then, let

$$\eta = \sigma^e + \alpha I. \tag{18}$$

So, we get:

$$\eta \in L^2(\Omega, \mathbb{R}^9_{sym}) \text{ and } \mathrm{div}\eta = \mathrm{div}\sigma^e + \nabla\alpha. \tag{19}$$

According to (17) and (3) we obtain:

$$\mathrm{div}\eta = 0 \quad \text{in } \Omega. \tag{20}$$

Let us show now that $\eta \cdot n = 0$ on Γ_1.
Let $\phi \in W^{1,2}(\Omega)$ be such that $\phi/((\partial\Omega)\backslash\Gamma_1) = 0$, we have for $i \in \{1, 2, 3\}$

$$\int_\Omega \eta^i \nabla\phi \mathrm{d}x = -\int_\Omega \mathrm{div}\eta^i \phi \mathrm{d}x + \int_{\Gamma_1} \eta^i.n.\phi \mathrm{d}\Gamma$$

where η^i is the vector line of η.
Using (18) and (20) we obtain:

$$\int_{\Gamma_1} \eta^i.n.\phi \mathrm{d}\Gamma = \int_\Omega (\sigma^e)^i.\nabla\phi \mathrm{d}x + \int_\Omega (\alpha.I)^i \nabla\phi \mathrm{d}x.$$

Then since $\phi/((\partial\Omega)\backslash\Gamma_1) = 0$,

$$\int_{\Gamma_1} \eta^i.n.\phi \mathrm{d}\Gamma = -\int_\Omega (\mathrm{div}\sigma^e)^i \phi \mathrm{d}x + \int_{\Gamma_1} (\sigma^e)^i.n.\phi \mathrm{d}\Gamma - \int_\Omega (\nabla\alpha)^i \phi \mathrm{d}x + \int_{\Gamma_1} (\alpha.n_i)\phi \mathrm{d}\Gamma.$$

And then according to (3) and (17) we have:

$$\int_{\Gamma_1} (\eta.n)\phi \mathrm{d}\Gamma = 0. \tag{21}$$

The statements (19)–(21) prove that $\eta \in V_1$; moreover, we have:

$$F(\eta) = F(\eta^D) = F((\sigma^e)^D)) = F(\sigma^e) = 0.$$

Then

$$\eta \in V_1 \text{ and } F(\eta) = 0. \qquad \square$$

Moreover, we have the following theorem:

Theorem 3. *Under the hypotheses of Theorem 2, we have:*

$$\textit{If Rotf} \neq 0, \quad \textit{then} \quad \inf_{\eta\in V_1} F(\eta) \neq 0.$$

Proof. Assume that there exists a sequence $(\eta_n)_n \in V_1$ such that $F(\eta_n) \xrightarrow[n\to+\infty]{} 0$.

Then, according to (5) we have:

$$\mathrm{J}_{K_{ad}}(\sigma^e - \eta_n) \xrightarrow[n\to+\infty]{} 0$$

and then $\exists\alpha_n > 0$ (α_n independent of x) such that: $\begin{cases} \alpha_n \xrightarrow[n\to+\infty]{} 0 \\ \text{and} \\ (\sigma^e - \eta_n)^D(x) \in \alpha_n K^D(x) \quad a.e. \text{ in } \Omega. \end{cases}$

This gives, according to (H_4):

$$\eta_n^D \xrightarrow[n\to+\infty]{} (\sigma^e)^D \quad a.e. \text{ in } \Omega$$

and

$$\eta_n^D \xrightarrow[n\to+\infty]{} (\sigma^e)^D \quad \text{in } \mathrm{D}'(\Omega).$$

Then:

$$\mathrm{div}\eta_n^D \xrightarrow[n\to+\infty]{} \mathrm{div}(\sigma^e)^D \quad \text{in } \mathrm{D}'(\Omega),$$

which can be written:

$$\mathrm{div}\left(\eta_n - \frac{1}{3}\mathrm{tr}\eta_n.I\right) \longrightarrow [n\to+\infty]\mathrm{div}(\sigma^e)^D \quad \text{in } \mathrm{D}'(\Omega);$$

but according to (4) we have $\mathrm{div}(\eta_n) = 0$, then:

$$\mathrm{div}\left(-\frac{1}{3}\mathrm{tr}\eta_n.I\right) \xrightarrow[n\to+\infty]{} \mathrm{div}(\sigma^e)^D \quad \text{in } \mathrm{D}'(\Omega),$$

or also

$$\nabla\left(-\frac{1}{3}(\mathrm{tr}(\eta_n - \mathrm{tr}\sigma^e)\right) \longrightarrow [n\to+\infty]\mathrm{div}(\sigma^e) \quad \text{in } \mathrm{D}'(\Omega),$$

which implies:

$$\mathrm{Rot}\nabla\left(-\frac{1}{3}(\mathrm{tr}(\eta_n - \mathrm{tr}\sigma^e)\right) \longrightarrow [n \to +\infty]\mathrm{Rot}(\mathrm{div}(\sigma^e)) \quad \text{in } \mathrm{D}'(\Omega).$$

Using $\mathrm{Rot}\nabla\delta = 0 \quad \forall\delta \in$ in $\mathrm{D}'(\Omega)$ we obtain $\mathrm{Rotf} = 0$, which concludes the proof. □

Corollary 1. *Under the hypotheses $(H_1), (H_2), (H_3)$ and (H_4) we have:*

$$\textit{If Rotf} \neq 0, \ \textit{then} \quad \bar{\lambda} = \frac{1}{\inf\limits_{\eta\in V_1} F(\eta)}.$$

Proof. Results from Theorems 1 and 3. □

In Theorem 3 we have characterized the Limit load $\bar{\lambda}$; and we have shown that Rotf $\neq 0$ is a sufficient condition to prove that $\bar{\lambda} = \frac{1}{\inf_{\eta\in V_1} F(\eta)}$; but this condition is not always satisfied by the volumic force f. In the case where $\mathrm{Rotf} = 0$, we introduce in the following a condition on the boundary force g to show the same characterization of the Limit load.

Theorem 4. *Under the hypotheses $(H_1), (H_2), (H_3), (H_4)$ and if we assume that Rot $f \neq 0$ or g satisfies:*

$$C_g \quad : \quad \exists B \subset \Gamma_1, \ meas(B) \neq 0 \ \ such\ that \ \ g \wedge n \neq 0 \ \ on\ B \tag{22}$$

(which means that g is not colinear to the normal on B).

Then, we have:

$$F(\eta) \neq 0 \quad \forall\eta \in V_1.$$

Proof. The result is deduced from Theorem 2. □

In the following part, we will prove, by adding a condition on the open set Ω, that $\bar{\lambda} = \frac{1}{\inf_{\eta\in V_1} F(\eta)}$ under hypothesis Rot f $\neq 0$ or g satisfying C_g given above and that $\inf_{\eta\in V_1} F(\eta)$ is reached on V_1.

3. An existence result obtained by extension of Ω

3.1. Problem obtained by extension of Ω

We assume that Ω satisfies:

$$\begin{cases} \text{There exists an open set } \ \Omega_0 \subset \mathbb{R}^3 \text{ such that:} \\ \Omega \subset \Omega_0 \\ \Omega_0 \text{ is convex ; } \Omega_0\backslash\Omega \text{ is connex} \\ \partial\Omega \cap (\partial(\Omega_0\backslash\Omega)) = \Gamma_1 \\ \text{and} \\ \forall\phi \in \mathrm{W}^{1,2}(\Omega), \exists\phi_1 \in \mathrm{W}^{1,2}(\Omega_0\backslash\Omega) \text{ such that: } \phi = \phi_1 \text{ on } \Gamma_1. \end{cases} \tag{23}$$

Let now $(\eta_n)_n$ be a minimizing sequence of $F(\eta)$ on V_1 and let $\tilde{\eta}_n$ be defined by:

$$\tilde{\eta}_n = \begin{cases} \eta_n & \text{in} \quad \Omega \\ 0 & \text{in} \quad \Omega_0\backslash\Omega. \end{cases} \tag{24}$$

Then we have the following results:

Lemma 1. *We assume that* (23) *is satisfied, then we have:*

$$\mathrm{div}\ \tilde{\eta}_n = 0 \qquad in \quad \Omega_0.$$

Proof. Let $\phi \in \mathrm{D}(\Omega_0)$, we have:

$$\int_{\Omega_0} (\mathrm{div}\tilde{\eta}_n^i)\phi \mathrm{d}x = -\int_{\Omega_0} \tilde{\eta}_n^i \nabla\phi \mathrm{d}x.$$

This means:

$$\int_{\Omega_0} (\mathrm{div}\ \tilde{\eta}_n^i)\phi \mathrm{d}x = -\int_{\Omega} \tilde{\eta}_n^i \nabla\phi \mathrm{d}x - \int_{\Omega_0\backslash\Omega} \tilde{\eta}_n^i \nabla\phi \mathrm{d}x$$

and according to (24) we obtain:

$$\int_{\Omega_0} (\mathrm{div}\tilde{\eta}_n^i)\phi \mathrm{d}x = \int_{\Omega} \mathrm{div}{\eta_n}^i \phi \mathrm{d}x - \int_{\partial\Omega} ({\eta_n}^i.n)\phi \mathrm{d}\Gamma.$$

But ϕ belongs to $\mathrm{D}(\Omega_0)$ then $\phi = 0$ on $\partial\Omega\backslash\Gamma_1$ and $(\eta_n)_n \in V_1$, and we get:

$$\int_{\Omega_0} \mathrm{div}\tilde{\eta}_n^i \phi \mathrm{d}x = 0, \text{ for all } \phi \in \mathrm{D}(\Omega_0) \text{ and for all } 1 \leq i \leq 3,$$

which allows us to conclude:

$$\mathrm{div}\tilde{\eta}_n = 0 \qquad \text{in} \quad \Omega_0.$$

□

Remark 1. Let $(\eta_n)_{n\in\mathbb{N}}$ be a minimizing sequence of $F(\eta)$ on V_1, then we have:

$$\mathrm{J}_{K_{ad}}(\sigma^e - \eta_n) \leq \mathrm{Const.}$$

Using Proposition 2 and the property (H_4), we obtain:

$$\|\ \eta_n^D\ \|_{L^\infty(\Omega,\mathbb{R}^9_{sym})} \leq \mathrm{Cte} \text{ for all } n \in \mathbb{N}. \tag{25}$$

Then, let:

$$(\tilde{\eta}_n)^D = \begin{cases} \eta_n^D & \text{in } \Omega \\ 0 & \text{in } \Omega_0\backslash\Omega. \end{cases} \tag{26}$$

We obtain:

$(\tilde{\eta}_n^D)_n$ is bounded in $L^\infty(\Omega_0, \mathbb{R}^9_{sym})$, then there exists a subsequence of $(\tilde{\eta}_n^D)_n$ (denoted also $(\tilde{\eta}_n^D)_n$) and there exists $\sigma_0 \in L^2(\Omega_0, \mathbb{R}^9_{sym})$ such that:

$$(\tilde{\eta}_n^D) \longrightarrow \sigma_0 \quad \text{weakly in } \mathrm{L}^2(\Omega_0, \mathbb{R}^9_{sym}). \tag{27}$$

Lemma 2. *Under the hypotheses* (H_1), (H_2), (H_3), (H_4) *and assuming that* Ω *satisfies* (23), *we have:*

$$\mathrm{Rot}(\mathrm{div}\sigma_0) = 0 \quad in \quad \Omega_0,$$

where σ_0 *is the element of* $L^2(\Omega_0, \mathbb{R}^9_{sym})$ *given by* (27).

Proof. Using (27) we get:

$$(\tilde{\eta}_n)^D \xrightarrow[n\to+\infty]{\sigma}{}_0 \quad \text{in } \mathrm{D}'(\Omega_0),$$

which implies:

$$\mathrm{div}(\tilde{\eta}_n)^D \xrightarrow[n\to+\infty]{} \mathrm{div}\sigma_0 \quad \text{in } \mathrm{D}'(\Omega_0)$$

and

$$\mathrm{Rot}(\mathrm{div}(\tilde{\eta}_n)^D) \xrightarrow[n\to+\infty]{} \mathrm{Rot}(\mathrm{div}\sigma_0) \quad \text{in } \mathrm{D}'(\Omega_0), \tag{28}$$

but we have:

$$\mathrm{div}(\tilde{\eta}_n)^D = \mathrm{div}\tilde{\eta}_n - \frac{1}{3}\nabla(\mathrm{tr}\tilde{\eta}_n).$$

Then according to Lemma 1.1, we obtain:

$$\mathrm{Rotdiv}(\tilde{\eta}_n)^D = -\frac{1}{3}\mathrm{rot}\nabla(\mathrm{tr}\tilde{\eta}_n) \quad \forall n \in \mathbb{N}.$$

Finally, we conclude from (28) that:

$$\mathrm{Rot}(\mathrm{div}\sigma_0) = 0 \quad \text{in } \mathrm{D}'(\Omega_0).$$ □

3.2. An existence result

Firstly, we begin by the following result:

Lemma 3. *Let Ω_0 be an open convex set of $\mathbb{R}^n$ and let $v \in (H^{-1}(\Omega_0))^3$ satisfying:*

$$\mathrm{Rot}\ v = 0 \quad in \quad \Omega_0.$$

Then there exists a unique $q \in L^2(\Omega_0)/\mathbb{R}$ such that:

$$\nabla q = v.$$

Proof. See [5]. □

Remark 2. We remark that $\mathrm{div}\sigma_0$ cannot always be equal to 0 and on the other hand we search a stress σ which achieves $\inf\limits_{\eta\in V_1} F(\eta)$, so we will change the spherical component to have a stress σ satisfying $\mathrm{div}\sigma = 0$. That is the purpose of the following paragraph.

Lemma 4. *Under the hypotheses of Lemma 2 we have:*

$$\exists\sigma \in L^2(\Omega_0, \mathbb{R}^9_{sym}) \text{ such that } \begin{cases} \mathrm{div}\,\sigma = 0 & in \quad \Omega_0 \\ \sigma^D = \sigma_0^D & in \quad \Omega_0 \\ \sigma = 0 & in \quad \Omega_0\backslash\Omega. \end{cases} \tag{29}$$

Proof. Using Lemmas 2 and 3 we get: $\exists q \in L^2(\Omega_0)/\mathbb{R}$ such that :

$$\mathrm{div}\sigma_0 = -\nabla q \quad \text{in } \mathrm{D}'(\Omega_0). \tag{30}$$

And according to (24) and (27), we deduce:

$$\sigma_0 = 0 \text{ on } \quad \Omega_0\backslash\Omega, \tag{31}$$

then

$$\mathrm{div}\sigma_0 = 0 \text{ on } \Omega_0\backslash\Omega.$$

This proves that:

$$\nabla q = 0 \quad \text{on} \quad \Omega_0 \backslash \Omega.$$

We have, using (23), $\Omega_0 \backslash \Omega$ is connex, and then:

$$q = C_1 \text{ on } \quad \Omega_0 \backslash \Omega \quad (C_1 \in \mathbb{R}),$$

so we can choose $C_1 = 0$.

Then, let $\sigma \in L^2(\Omega_0, \mathbb{R}^9_{sym})$ be defined by:

$$\sigma = \sigma_0 + qI; \tag{32}$$

we have

$$\sigma = 0 \text{ on } \quad \Omega_0 \backslash \Omega. \tag{33}$$

From (30), (32) and (33) we conclude the result. □

We can now prove the main theorem:

Theorem 5. *Under the hypotheses* $(H_1), (H_2), (H_3), (H_4)$ *and assuming that* Ω *satisfies* (23), *there exists* $\sigma_1 \in V_1$ *such that*

$$F(\sigma_1) = \inf_{\eta \in V_1} F(\eta),$$

so the infimum of F *is achieved on* V_1.

Proof. Let $(\eta_n)_{n \in \mathbb{N}}$ be a minimizing sequence of F on V_1, σ_0 given in the Remark 1.

Let $\sigma_1 = \sigma/\Omega$ where σ is given by (29); we have:

$$\sigma_1 \in (L^2(\Omega))^9_s, \tag{34}$$

and:

$$\text{div}\sigma_1 = (\text{div}\sigma)/\Omega = 0 \quad \text{in} \quad \Omega.$$

So,

$$\text{div}\sigma_1 = 0 \quad \text{in} \quad \Omega. \tag{35}$$

Let us prove that $\sigma_1.n = 0 \quad \text{in } \Gamma_1$, it is equivalent to prove:

$$\int_{\Gamma_1} (\sigma_1.n)\phi \mathrm{d}\Gamma = 0 \; \forall \phi \in W^{1,2}(\Omega) \text{ be such that } \phi/(\partial\Omega \backslash \Gamma_1) = 0. \tag{36}$$

Then let $\phi_1 \in W^{1,2}(\Omega_0 \backslash \Omega)$ be such that $\phi_1/\Gamma_1 = \phi/\Gamma_1$, and let $\tilde{\phi}$ be defined by:

$$\tilde{\phi} = \begin{cases} \phi & \text{in} \quad \Omega \\ \phi_1 & \text{in} \quad \Omega_0 \backslash \Omega. \end{cases}$$

We have, due to the trace theorem for $\tilde{\phi}$ on the two sides of $\Gamma_1 = \partial\Omega \cap (\partial(\Omega_0 \backslash \Omega))$:

$$\tilde{\phi} \in W^{1,2}(\Omega_0).$$

On the other hand, we have:

$$\int_{\Omega_0} \sigma.\varepsilon(\tilde{\phi})\mathrm{d}x = \int_{\Omega} \sigma.\varepsilon(\phi)\mathrm{d}x + \int_{\Omega_0 \backslash \Omega} \sigma.\varepsilon(\tilde{\phi})\mathrm{d}x,$$

which implies, according to (33):

$$\int_{\Omega_0} \sigma.\varepsilon(\tilde{\phi})\mathrm{d}x = \int_{\Omega} \sigma.\varepsilon(\phi)\mathrm{d}x$$
$$= -\int_{\Omega} (\mathrm{div}\sigma).\phi \mathrm{d}x + \int_{\partial\Omega} (\sigma_1.n)\phi \mathrm{d}\Gamma.$$

Using $\mathrm{div}\sigma = 0$ in Ω_0, we obtain:

$$\int_{\Omega_0} \sigma.\varepsilon(\tilde{\phi})\mathrm{d}x = \int_{\partial\Omega} (\sigma_1.n)\phi \mathrm{d}\Gamma.$$

This means according to (36) that:

$$\int_{\Omega_0} \sigma.\varepsilon(\tilde{\phi})\mathrm{d}x = \int_{\Gamma_1} (\sigma_1.n)\phi \mathrm{d}\Gamma. \tag{37}$$

We have on the other hand:

$$\int_{\Omega_0} \sigma.\varepsilon(\tilde{\phi})\mathrm{d}x = \int_{\Omega_0} (\mathrm{div}\sigma).\tilde{\phi} \mathrm{d}x + \int_{\partial\Omega_0} (\sigma.n)\tilde{\phi} \mathrm{d}\Gamma.$$

So, using properties (29) and (36), we deduce that:

$$\int_{\Omega_0} \sigma.\varepsilon(\tilde{\phi})\mathrm{d}x = 0. \tag{38}$$

From (37) and (38), we conclude that:

$$\int_{\Gamma_1} (\sigma_1.n)\phi \mathrm{d}\Gamma = 0 \ \ \forall\phi \in W^{1,2}(\Omega) \text{ such that } \phi/(\partial\Omega\backslash\Gamma_1) = 0.$$

This proves that:

$$\sigma_1.n = 0 \quad \text{on } \Gamma_1. \tag{39}$$

We conclude from (34), (35) and (39) that:

$$\sigma_1 \in V_1. \tag{40}$$

Using Proposition 2 and $({\eta_n}^D)_n$ converge to ${\sigma_0}^D$ weakly in $L^2(\Omega, {\mathbb{R}^9}_{sym})$, we have:

$$F(\sigma_1) \leq \underline{\lim} F(\eta_n^D),$$

or also:

$$F(\sigma_1) \leq \inf_{\eta \in V_1} F(\eta).$$

We finally conclude from (40):

$$F(\sigma_1) = \inf_{\eta \in V_1} F(\eta).$$ □

Corollary 2. *Under the hypotheses of Theorem 5 we have:*

If Rotf $\neq 0$ *in* Ω *or* (22) *is satisfied by g* *then*

$$\exists \sigma_1 \in V_1 \text{ such that: } \bar{\lambda} = \frac{1}{\inf\limits_{\eta \in V_1} F(\eta)} = \frac{1}{F(\sigma_1)}\cdot$$

Proof. The result is deduced from Theorems 4 and 5. □

References

[1] R. Adams, *Sobolev Spaces.* Academic Press, New York (1975).
[2] H. Brezis, *Analyse Fonctionnelle.* Masson, Paris (1983).
[3] P.G. Ciarlet, *Lectures on the three-dimensional elasticity.* Tata Institute of Fundamental Research, Bombay (1983).
[4] H. El-Fekih and T. Hadhri, Calcul des charges limites d'une structure élastoplastique en contraintes planes. *RAIRO: Modél. Math. Anal. Numér.* **29** (1995) 391–419.
[5] R. Temam, *Mathematical Problems in Plasticity.* Bordas, Paris (1985).

To access this journal online:
www.edpsciences.org