CHARACTERIZATION OF THE LIMIT LOAD IN THE CASE OF AN UNBOUNDED ELASTIC CONVEX

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Abstract. In this work we consider a solid body $\Omega \subset \mathbb{R}^3$ constituted by a nonhomogeneous elasto-plastic material, submitted to a density of body forces $\lambda f$ and a density of forces $\lambda g$ acting on the boundary where the real $\lambda$ is the loading parameter. The problem is to determine, in the case of an unbounded convex of elasticity, the Limit load denoted by $\bar{\lambda}$ beyond which there is a break of the structure. The case of a bounded convex of elasticity is done in [El-Fekih and Hadhri, RAIRO: Modél. Math. Anal. Numér. 29 (1995) 391–419]. Then assuming that the convex of elasticity at the point $x$ of $\Omega$, denoted by $K(x)$, is written in the form of $K^D(x) + RI$, $I$ is the identity of $\mathbb{R}^9_{sym}$, and the deviatoric component $K^D$ is bounded regardless of $x \in \Omega$, we show under the condition “$\text{Rot} f \neq 0$ or $g$ is not colinear to the normal on a part of the boundary of $\Omega$”, that the Limit Load $\bar{\lambda}$ searched is equal to the inverse of the infimum of the gauge of the Elastic convex translated by stress field equilibrating the unitary load corresponding to $\lambda = 1$; moreover we show that this infimum is reached in a suitable function space.

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1. THE HENCKY’S PROBLEM FOR A NON-HOMOGENEOUS ELASTOPLASTIC STRUCTURE

Using the notations and the operators given in [5], the Hencky’s problem is given by the following system: find a tensor $\sigma$ and a displacement $u$ such that

\[
\begin{cases}
\n\text{div } \sigma = \lambda f & \text{a.e. in } \Omega \\
\sigma \cdot n = \lambda g & \text{on } \Gamma_1 \\
u = u_0 & \text{on } \Gamma_0 \\
\sigma(x) = \Pi_{K(x)} \left( A^{-1}(x) (\varepsilon(u)(x)) \right).
\end{cases}
\]

Here:

\[\varepsilon(u) = (\varepsilon_{ij}(u)) \quad \text{and} \quad \varepsilon_{ij}(u) = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad \text{for } 1 \leq i, j \leq 3\]

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$A^{-1}_x$ the inverse matrix of $A_x$, $A_x$ defined for $\eta = (\eta_{ij})_{1 \leq i,j \leq 3} \in \mathbb{R}^9$ by:

$$(A_x(\eta))_{ij} = \frac{1}{9K_0(x)}\eta_{kk}(x)\delta_{ij} + \frac{1}{2\mu(x)}\eta^{ij}_{x}$$

$K_0(x) = \alpha(x) + \frac{2\mu(x)}{3}$, where $\alpha$ and $\mu$ are the Lamé coefficients.

We suppose that:

$(H_1)$ $\Gamma_1 \cup \Gamma_0 = \partial \Omega$: the boundary of $\Omega$ with $(\Gamma_1) \neq 0$ and the interiors of $\Gamma_1$ and $\Gamma_0$ satisfy $\Gamma_1^0 \cap \Gamma_0^0 = \emptyset$

$(H_2)$ $K(x)$ is a closed convex part of $\mathbb{R}_{sym}^9$ and $\exists \epsilon > 0$ such that:

$$B(0, \epsilon) \subset K(x) \text{ a.e. in } \Omega.$$

Here: $\mathbb{R}_{sym}^9 = \{X = (x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33}), x_{ij} \in \mathbb{R} \text{ for } 1 \leq i \leq 3; 1 \leq j \leq 3 \text{ and } x_{ij} = x_{ji}\}$

$(H_3)$ $g \in (L^2(\Gamma_1))^3, f \in (L^4(\Omega))^3$ such that:

$$\exists \bar{g} \in (L^\infty(\partial \Omega))^3, \bar{g} = g \text{ on } \Gamma_1 \text{ and } \int_{\Omega} f \, dx + \int_{\partial \Omega} \bar{g} \, d\Gamma = 0$$

$(H_4)$ $K(x) = K_D(x) + RI$ and $\exists M > 0$ such that:

$$K_D(x) \subset B(o, M) \text{ a.e. in } \Omega.$$

We define the following set $K_{ad}$:

$$K_{ad} = \{\eta \in (L^2(\Omega))^9 \text{ such that } \eta(x) \in K(x) \text{ a.e. in } \Omega\}. \tag{2}$$

It is clear that $K_{ad}$ is a closed convex of $(L^2(\Omega))^9$.

We define now the Quasi-elastic problem:

Find a tensor $\sigma^e$ and a displacement $u^e$ satisfying:

$$\begin{cases}
\text{div } \sigma^e = f & \text{a.e. in } \Omega \\
\sigma^e \cdot n = \bar{g} & \text{on } \partial \Omega \\
\sigma^e(x) = \left(A^{-1}_{ij}(\varepsilon(u^e)(x))\right) & \text{a.e. in } \Omega.
\end{cases} \tag{3}$$

Referring to [3], the above problem has a solution $(\sigma^e, u^e)$, which is unique within a rigid body displacement for $u^e$, since $f$ and $g$ satisfy $(H_3)$; moreover, we have the following proposition.

**Proposition 1.** We assume that $f$ and $g$ satisfy $(H_3)$, then we have:

$$\sigma^e \in L^\infty(\Omega, \mathbb{R}_{sym}^9).$$

**Proof.** According to [3], we have $\sigma^e \in (W^{1,4}(\Omega))^9$ and according to [1] we conclude that: $\sigma^e \in L^\infty(\Omega, \mathbb{R}_{sym}^9).$ \qed

2. **Characterization of the Limit Load $\bar{\lambda}$**

**Definition 1.** Considering the functional $F$ defined on $V_1$:

$$V_1 = \{\eta \in L^2(\Omega, \mathbb{R}_{sym}^9) \text{ such that } \text{div } \eta = 0 \text{ a.e. in } \Omega \text{ and } \eta \cdot n = 0 \text{ on } \Gamma_1\} \tag{4}$$

by:

$$F(\eta) = J_{K_{ad}}(\sigma^e - \eta) \tag{5}$$
where, $J_{K_{ad}}$ is the gauge of $K_{ad}$ defined by:

$$J_{K_{ad}}(\alpha) = \inf\{s > 0 \text{ such that } \alpha(x) \in s.K(x) \text{ a.e. in } \Omega\}. \tag{6}$$

Then we have:

**Proposition 2.** The functional $F$ is l.s.c (lower semicontinuous) on $V_1$ for the weak topology of $L^2(\Omega, \mathbb{R}^9_{\text{sym}})$ and we have: $F(\eta) = F(\eta^D)$.

**Proof.** The functional $F$ is l.s.c for the strong topology of $L^2(\Omega, \mathbb{R}^9_{\text{sym}})$ and according to [2] we have $F$ is l.s.c on $V_1$ for the weak topology of $L^2(\Omega, \mathbb{R}^9_{\text{sym}})$.

On the other hand $K_{ad}$ is, according to (H4), unchanged in the direction of the spherical stress, then we have:

$$F(\eta) = F(\eta^D). \quad \square$$

**Definition 2.** The Limit Load $\bar{\lambda}$ is defined in [5] by:

$$\bar{\lambda} = \sup\{\lambda > 0 \text{ such that } D_\lambda \neq \emptyset\} \tag{7}$$

where

$$D_\lambda = \sigma \in L^2(\Omega, \mathbb{R}^9_{\text{sym}}) \text{ such that } \begin{cases} \text{div } \sigma - \lambda f = 0 \text{ a.e. in } \Omega \\ \sigma \cdot n = \lambda g \text{ on } \Gamma_1 \\ \sigma(x) \in K(x) \text{ a.e. in } \Omega \end{cases}.$$ \tag{8}

Then we have the following theorems:

**Theorem 1.** Under the hypotheses $(H_1), (H_2)$ and $(H_3)$ we have:

(i) If $\inf_{\eta \in V_1} F(\eta) = 0$, then $\bar{\lambda} = +\infty$.

(ii) If $\inf_{\eta \in V_1} F(\eta) \neq 0$, then $\bar{\lambda} = \frac{1}{\inf_{\eta \in V_1} F(\eta)}$.

**Proof.**

1. Proof of (i).

   We assume that:

   $$\inf_{\eta \in V_1} F(\eta) = 0. \tag{9}$$

   Let $\lambda > 0$, and let us show that $\bar{\lambda} > \lambda$.

   There exists $\eta \in V_1$ such that $F(\eta) < \frac{1}{\lambda}$ and according to (5) we have:

   $$J_{K_{ad}}(\sigma^e - \eta) < \frac{1}{\lambda}.$$  

   Using (6) we obtain:

   $$\sigma^e - \eta \in \frac{1}{\lambda} K_{ad}.$$  

   and then

   $$\lambda(\sigma^e - \eta)(x) \in K(x) \text{ a.e. in } \Omega. \tag{10}$$

   On the other hand, we have: $\text{div}(\lambda(\sigma^e - \eta)) = \lambda(\text{div}\sigma^e - \text{div}\eta)$.

   Using (3) and (4) we conclude:

   $$\text{div}(\lambda(\sigma^e - \eta)) = \lambda f \text{ in } \Omega, \tag{11}$$

   and

   $$\lambda(\sigma^e - \eta) \cdot n = \lambda g \text{ on } \Gamma_1. \tag{12}$$
From (10)–(12) we obtain: \( \lambda (\sigma - \eta) \in D_\lambda \), and according to (7) it is clear that:
\[
\ddot{\lambda} > \lambda.
\]

We finally conclude that:
\[
\ddot{\lambda} = +\infty.
\]

2. Using the same idea, we prove (ii).

\[\square\]

We now distinguish these two cases:

**Theorem 2.** Under the hypotheses \((H_1), (H_2), (H_3)\) and \((H_4)\) the following statements (i) and (ii) are equivalent:

(i) \( \exists \eta \in V_1 \) such that \( F(\eta) = 0 \).

(ii) The following problem \((P_2)\) has at least one solution

\[
(P_2) \begin{cases}
F \alpha \text{ in } W^{1,2}(\Omega) \text{ satisfying:} \\
\nabla \alpha = -f \text{ in } \Omega \\
\alpha n = -g \text{ on } \Gamma_1.
\end{cases}
\]

Proof.
1. Assume there exists \( \eta \in V_1 \) such that \( F(\eta) = 0 \), then according to (5) we have

\[
J_{K_\eta}(\sigma - \eta) = 0,
\]

which implies, using (6):

\[
\inf \left\{ s > 0 \text{ such that } (\sigma - \eta)(x) \in sK(x) \text{ a.e. in } \Omega \right\} = 0;
\]

and using \((H_4)\) we have:

\[
\inf \left\{ s > 0 \text{ such that } (\sigma - \eta)^D(x) \in sK^D(x) \text{ a.e. in } \Omega \right\} = 0,
\]

which implies:

\[
\exists s_n > 0; (s_n)_{n \in \mathbb{N}} \text{ independent of } x \text{ and such that: } \begin{cases}
s_n \underset{n \to +\infty}{\to} 0 \\
(\sigma^e - \eta)^D(x) \in s_nK^D(x) \text{ a.e. in } \Omega.
\end{cases}
\]

Then we can write:

\[
(\sigma^e)^D(x) = \eta^D(x) \text{ a.e. in } \Omega,
\]

and then:

\[
\eta(x) = (\sigma^e)^D(x) + \frac{1}{3} \text{tr} \eta I,
\]

and so:

\[
\text{div} \eta = \text{div}(\sigma^e)^D + \text{div} \left( \frac{1}{3} \text{tr} \eta I \right).
\]

But \( \eta \in V_1 \), then \( \text{div} \eta = 0 \), which gives using (3):

\[
\nabla \alpha = -f,
\]

(16)
where \( \alpha = \frac{1}{3}(\text{tr} \eta - \text{tr} \sigma^e) \).

It is clear that \( \alpha \in L^2(\Omega) \) and \( \nabla \alpha \in (L^2(\Omega))^3 \) which implies that \( \alpha \in W^{1,2}(\Omega) \).

On the other hand, using (4) we obtain:

\[
\eta \cdot n = 0 \quad \text{on } \Gamma_1,
\]

then

\[
\left( \eta^D + \frac{1}{3} \text{tr} \eta I \right) \cdot n = 0 \quad \text{on } \Gamma_1,
\]

which can be written using (15) as:

\[
\left( (\sigma^e)^D + \frac{1}{3} \text{tr} \eta I \right) \cdot n = 0 \quad \text{on } \Gamma_1
\]

or

\[
\left( (\sigma^e) + \frac{1}{3}(\text{tr} \eta - \text{tr} \sigma^e)I \right) \cdot n = 0 \quad \text{on } \Gamma_1
\]

and using (3), we have:

\[
\left( \frac{1}{3}(\text{tr} \eta - \text{tr} \sigma^e)I \right) \cdot n = -g \quad \text{on } \Gamma_1,
\]

that means:

\[
\alpha \cdot n = -g \quad \text{on } \Gamma_1
\]

and using (16) we conclude the first implication.

2. Assume now that \((P_2)\) has at least one solution in \( W^{1,2}(\Omega) \), then there exists \( \alpha \in W^{1,2}(\Omega) \) such that:

\[
\begin{cases}
\nabla \alpha = -f \quad \text{in } \Omega \\
\alpha \cdot n = -g \quad \text{on } \Gamma_1.
\end{cases}
\]

(17)

Then, let

\[
\eta = \sigma^e + \alpha I.
\]

(18)

So, we get:

\[
\eta \in L^2(\Omega, \mathbb{R}^{9}_{\text{sym}}) \text{ and } \text{div} \eta = \text{div} \sigma^e + \nabla \alpha.
\]

(19)

According to (17) and (3) we obtain:

\[
\text{div} \eta = 0 \quad \text{in } \Omega.
\]

(20)

Let us show now that \( \eta \cdot n = 0 \) on \( \Gamma_1 \).

Let \( \phi \in W^{1,2}(\Omega) \) be such that \( \phi/(\partial \Omega)\setminus \Gamma_1 = 0 \), we have for \( i \in \{1, 2, 3\} \)

\[
\int_{\Omega} \eta_i \nabla \phi dx = -\int_{\Omega} \text{div} \eta_i \phi dx + \int_{\Gamma_1} \eta_i \cdot n_i \phi d\Gamma
\]

where \( \eta_i \) is the vector line of \( \eta \).

Using (18) and (20) we obtain:

\[
\int_{\Gamma_1} \eta_i \cdot n_i \phi d\Gamma = \int_{\Omega} (\sigma^e_i) \nabla \phi dx + \int_{\Omega} (\alpha_i I) \nabla \phi dx.
\]
Then since \( \phi/((\partial \Omega) \setminus \Gamma_1) = 0 \),

\[
\int_{\Gamma_1} \eta^i \cdot n \phi d\Gamma = - \int_{\Omega} (\text{div} \sigma^e)^i \phi dx + \int_{\Gamma_1} (\sigma^e)^i \cdot n \phi d\Gamma - \int_{\Omega} (\nabla \alpha)^i \phi dx + \int_{\Gamma_1} (\alpha n_i) \phi d\Gamma.
\]

And then according to (3) and (17) we have:

\[
\int_{\Gamma_1} (\eta \cdot n) \phi d\Gamma = 0.
\]  

(21)

The statements (19)–(21) prove that \( \eta \in V_1 \); moreover, we have:

\[
F(\eta) = F(\eta^D) = F((\sigma^e)^D) = F(\sigma^e) = 0.
\]

Then

\[
\eta \in V_1 \quad \text{and} \quad F(\eta) = 0.
\]

□

Moreover, we have the following theorem:

**Theorem 3.** Under the hypotheses of Theorem 2, we have:

If \( \text{Rot} f \neq 0 \), then \( \inf_{\eta \in V_1} F(\eta) \neq 0 \).

**Proof.** Assume that there exists a sequence \((\eta_n)_n \in V_1\) such that \( F(\eta_n) \xrightarrow{n \to +\infty} 0 \).

Then, according to (5) we have:

\[
J_{K_{ad}}(\sigma^e - \eta_n) \xrightarrow{n \to +\infty} 0
\]

and then \( \exists \alpha_n > 0 (\alpha_n \text{ independent of } x) \) such that:

\[
\begin{cases}
\alpha_n \xrightarrow{n \to +\infty} 0 \\
(\sigma^e - \eta_n)^D(x) \in \alpha_n K^D(x) \quad \text{a.e. in } \Omega.
\end{cases}
\]

This gives, according to \( (H_4) \):

\[
\eta_n^D \xrightarrow{n \to +\infty} (\sigma^e)^D \quad \text{a.e. in } \Omega
\]

and

\[
\eta_n^D \xrightarrow{n \to +\infty} (\sigma^e)^D \quad \text{in } D'(\Omega).
\]

Then:

\[
\text{div} \eta_n^D \xrightarrow{n \to +\infty} \text{div}(\sigma^e)^D \quad \text{in } D'(\Omega),
\]

which can be written:

\[
\text{div} \left( \eta_n - \frac{1}{3} \text{tr} \eta_n I \right) \xrightarrow{n \to +\infty} \text{div}(\sigma^e)^D \quad \text{in } D'(\Omega);
\]

but according to (4) we have \( \text{div} \eta_n = 0 \), then:

\[
\text{div} \left( -\frac{1}{3} \text{tr} \eta_n I \right) \xrightarrow{n \to +\infty} \text{div}(\sigma^e)^D \quad \text{in } D'(\Omega),
\]

or also

\[
\nabla \left( -\frac{1}{3} (\text{tr} \eta_n - \text{tr} \sigma^e) \right) \xrightarrow{n \to +\infty} \text{div}(\sigma^e) \quad \text{in } D'(\Omega),
\]
which implies:
\[
\text{Rot} \nabla \left( -\frac{1}{3}(\text{tr}(\eta_n - \text{tr} \sigma)) \right) \longrightarrow [n \to +\infty] \text{Rot} \left( \text{div} \sigma \right) \quad \text{in } D'(\Omega).
\]

Using Rot\nabla \delta = 0 \forall \delta \in \text{in } D'(\Omega) we obtain Rotf = 0, which concludes the proof.

**Corollary 1.** Under the hypotheses \((H_1), (H_2), (H_3)\) and \((H_4)\) we have:
\[
\text{If } Rot f \neq 0, \text{ then } \bar{\lambda} = \frac{1}{\inf_{\eta \in V_1} F(\eta)}.
\]

**Proof.** Results from Theorems 1 and 3.

In Theorem 3 we have characterized the Limit load \(\bar{\lambda}\); and we have shown that Rot\(f \neq 0\) is a sufficient condition to prove that \(\bar{\lambda} = \frac{1}{\inf_{\eta \in V_1} F(\eta)}\); but this condition is not always satisfied by the volumic force \(f\). In the case where Rot\(f = 0\), we introduce in the following a condition on the boundary force \(g\) to show the same characterization of the Limit load.

**Theorem 4.** Under the hypotheses \((H_1), (H_2), (H_3), (H_4)\) and if we assume that Rot\( f \neq 0\) or \(g\) satisfies:
\[
C_g : \exists B \subset \Gamma_1, \text{meas}(B) \neq 0 \text{ such that } g \land n \neq 0 \text{ on } B
\]
(22)

(which means that \(g\) is not colinear to the normal on \(B\)).
Then, we have:
\[
F(\eta) \neq 0 \quad \forall \eta \in V_1.
\]

**Proof.** The result is deduced from Theorem 2.

In the following part, we will prove, by adding a condition on the open set \(\Omega\), that \(\bar{\lambda} = \frac{1}{\inf_{\eta \in V_1} F(\eta)}\) under hypothesis Rot\( f \neq 0\) or \(g\) satisfying \(C_g\) given above and that \(\inf_{\eta \in V_1} F(\eta)\) is reached on \(V_1\).

### 3. An existence result obtained by extension of \(\Omega\)

#### 3.1. Problem obtained by extension of \(\Omega\)

We assume that \(\Omega\) satisfies:
\[
\begin{cases}
\text{There exists an open set } \Omega_0 \subset \mathbb{R}^3 \text{ such that:} \\
\Omega \subset \Omega_0 \\
\Omega_0 \text{ is convex ; } \Omega_0 \setminus \Omega \text{ is connex} \\
\partial \Omega \cap (\partial(\Omega_0 \setminus \Omega)) = \Gamma_1 \\
\text{and} \\
\forall \phi \in W^{1,2}(\Omega), \exists \phi_1 \in W^{1,2}(\Omega_0 \setminus \Omega) \text{ such that: } \phi = \phi_1 \text{ on } \Gamma_1.
\end{cases}
\]

(23)

Let now \((\eta_n)_n\) be a minimizing sequence of \(F(\eta)\) on \(V_1\) and let \(\tilde{\eta}_n\) be defined by:
\[
\tilde{\eta}_n = \begin{cases}
\eta_n & \text{in } \Omega \\
0 & \text{in } \Omega_0 \setminus \Omega.
\end{cases}
\]

(24)
Then we have the following results:

**Lemma 1.** We assume that (23) is satisfied, then we have:

\[
\text{div } \tilde{\eta}_n = 0 \quad \text{in } \Omega_0.
\]

**Proof.** Let \( \phi \in D(\Omega_0) \), we have:

\[
\int_{\Omega_0} (\text{div} \tilde{\eta}_n^i) \phi \, dx = -\int_{\Omega_0} \tilde{\eta}_n^i \nabla \phi \, dx.
\]

This means:

\[
\int_{\Omega_0} (\text{div} \tilde{\eta}_n^i) \phi \, dx = -\int_{\Omega} \tilde{\eta}_n^i \nabla \phi \, dx - \int_{\Omega_0 \setminus \Omega} \tilde{\eta}_n^i \nabla \phi \, dx
\]

and according to (24) we obtain:

\[
\int_{\Omega_0} (\text{div} \tilde{\eta}_n^i) \phi \, dx = \int_{\Omega} \text{div} \eta_n^i \phi \, dx - \int_{\partial \Omega} (\eta_n^i \cdot n) \phi \, d\Gamma.
\]

But \( \phi \) belongs to \( D(\Omega_0) \) then \( \phi = 0 \) on \( \partial \Omega \setminus \Gamma_1 \) and \( (\eta_n)_n \in \mathcal{V} \), and we get:

\[
\int_{\Omega_0} \text{div} \tilde{\eta}_n^i \phi \, dx = 0, \quad \text{for all } \phi \in D(\Omega_0) \text{ and for all } 1 \leq i \leq 3,
\]

which allows us to conclude:

\[
\text{div } \tilde{\eta}_n = 0 \quad \text{in } \Omega_0.
\]

**Remark 1.** Let \( (\eta_n)_n \in \mathbb{N} \) be a minimizing sequence of \( F(\eta) \) on \( \mathcal{V} \), then we have:

\[
J_{K_{ad}} (\sigma^e - \eta_n) \leq \text{Const}.
\]

Using Proposition 2 and the property \( (H_4) \), we obtain:

\[
\| \eta_n^D \|_{L^\infty(\Omega, \mathbb{R}_9^{sym})} \leq \text{Cte} \text{ for all } n \in \mathbb{N}.
\]

Then, let:

\[
(\tilde{\eta}_n)^D = \begin{cases} 
\eta_n^D & \text{in } \Omega \\
0 & \text{in } \Omega_0 \setminus \Omega.
\end{cases}
\]

We obtain:

\[
(\tilde{\eta}_n^D)_n \text{ is bounded in } L^\infty(\Omega_0, \mathbb{R}_9^{sym}), \text{ then there exists a subsequence of } (\tilde{\eta}_n^D)_n \text{ (denoted also } (\tilde{\eta}_n^D)_n) \text{ and there exists } \sigma_0 \in L^2(\Omega_0, \mathbb{R}_9^{sym}) \text{ such that:}
\]

\[
(\tilde{\eta}_n^D) \rightharpoonup \sigma_0 \text{ weakly in } L^2(\Omega_0, \mathbb{R}_9^{sym}).
\]

**Lemma 2.** Under the hypotheses \( (H_1), (H_2), (H_3), (H_4) \) and assuming that \( \Omega \) satisfies (23), we have:

\[
\text{Rot} (\text{div} \sigma_0) = 0 \quad \text{in } \Omega_0,
\]

where \( \sigma_0 \) is the element of \( L^2(\Omega_0, \mathbb{R}_9^{sym}) \) given by (27).
Proof. Using (27) we get:

\[(\tilde{\eta}_n)^D \xrightarrow{n \to +\infty} 0 \text{ in } D'(\Omega_0),\]

which implies:

\[\text{div}(\tilde{\eta}_n)^D \xrightarrow{n \to +\infty} \text{div}\sigma_0 \text{ in } D'(\Omega_0)\]

and

\[\text{Rot}(\text{div}(\tilde{\eta}_n)^D) \xrightarrow{n \to +\infty} \text{Rot}(\text{div}\sigma_0) \text{ in } D'(\Omega_0),\]  

(28)

but we have:

\[\text{div}(\tilde{\eta}_n)^D = \text{div}\tilde{\eta}_n - \frac{1}{3}\nabla(\text{tr}\tilde{\eta}_n).\]

Then according to Lemma 1.1, we obtain:

\[\text{Rotdiv}(\tilde{\eta}_n)^D = -\frac{1}{3}\text{rot}\nabla(\text{tr}\tilde{\eta}_n) \quad \forall n \in \mathbb{N}.\]

Finally, we conclude from (28) that:

\[\text{Rot(div}\sigma_0) = 0 \text{ in } D'(\Omega_0).\]  

\[\square\]

3.2. An existence result

Firstly, we begin by the following result:

Lemma 3. Let \(\Omega_0\) be an open convex set of \(\mathbb{R}^n\) and let \(v \in (H^{-1}(\Omega_0))^3\) satisfying:

\[\text{Rot } v = 0 \quad \text{in } \Omega_0.\]

Then there exists a unique \(q \in L^2(\Omega_0)/\mathbb{R}\) such that:

\[\nabla q = v.\]

Proof. See [5].

\[\square\]

Remark 2. We remark that \(\text{div}\sigma_0\) cannot always be equal to 0 and on the other hand we search a stress \(\sigma\) which achieves \(\inf \eta \in V_1 F(\eta)\), so we will change the spherical component to have a stress \(\sigma\) satisfying \(\text{div}\sigma = 0\). That is the purpose of the following paragraph.

Lemma 4. Under the hypotheses of Lemma 2 we have:

\[
\exists \sigma \in L^2(\Omega_0, \mathbb{R}_{sym}) \text{ such that } \begin{cases} 
\text{div } \sigma = 0 & \text{in } \Omega_0 \\
\sigma^D = \sigma^D_0 & \text{in } \Omega_0 \\
\sigma = 0 & \text{in } \Omega_0 \setminus \Omega. 
\end{cases}
\]  

(29)

Proof. Using Lemmas 2 and 3 we get: \(\exists q \in L^2(\Omega_0)/\mathbb{R}\) such that :

\[\text{div}\sigma_0 = -\nabla q \text{ in } D'(\Omega_0).\]

(30)

And according to (24) and (27), we deduce:

\[\sigma_0 = 0 \text{ on } \Omega_0 \setminus \Omega,\]

(31)

then

\[\text{div}\sigma_0 = 0 \text{ on } \Omega_0 \setminus \Omega.\]
This proves that:
\[ \nabla q = 0 \text{ on } \Omega_0 \backslash \Omega. \]
We have, using (23), \( \Omega_0 \backslash \Omega \) is connex, and then:
\[ q = C_1 \text{ on } \Omega_0 \backslash \Omega \quad (C_1 \in \mathbb{R}), \]
so we can choose \( C_1 = 0. \)

Then, let \( \sigma \in L^2(\Omega_0, \mathbb{R}^9_{sym}) \) be defined by:
\[ \sigma = \sigma_0 + qI; \quad (32) \]
we have
\[ \sigma = 0 \text{ on } \Omega_0 \backslash \Omega. \quad (33) \]
From (30), (32) and (33) we conclude the result.

We can now prove the main theorem:

**Theorem 5.** Under the hypotheses \((H_1), (H_2), (H_3), (H_4)\) and assuming that \( \Omega \) satisfies (23), there exists \( \sigma_1 \in V_1 \) such that
\[ F(\sigma_1) = \inf_{\eta \in V_1} F(\eta), \]
so the infimum of \( F \) is achieved on \( V_1 \).

**Proof.** Let \( (\eta_n)_{n\in\mathbb{N}} \) be a minimizing sequence of \( F \) on \( V_1 \), \( \sigma_0 \) given in the Remark 1.

Let \( \sigma_1 = \sigma/\Omega \) where \( \sigma \) is given by (29); we have:
\[ \sigma_1 \in (L^2(\Omega))^9, \quad (34) \]
and:
\[ \text{div} \sigma_1 = (\text{div} \sigma)/\Omega = 0 \text{ in } \Omega. \quad (35) \]
So,
\[ \text{div} \sigma_1 = 0 \text{ in } \Omega. \]

Let us prove that \( \sigma_1.n = 0 \) in \( \Gamma_1 \), it is equivalent to prove:
\[ \int_{\Gamma_1} (\sigma_1.n)\phi d\Gamma = 0 \quad \forall \phi \in W^{1,2}(\Omega) \text{ be such that } \phi/(\partial \Omega \cap \Gamma_1) = 0. \quad (36) \]

Then let \( \phi_1 \in W^{1,2}(\Omega_0 \backslash \Omega) \) be such that \( \phi_1/\Gamma_1 = \phi/\Gamma_1 \), and let \( \tilde{\phi} \) be defined by:
\[ \tilde{\phi} = \begin{cases} 
\phi & \text{in } \Omega \\
\phi_1 & \text{in } \Omega_0 \backslash \Omega.
\end{cases} \]
We have, due to the trace theorem for \( \tilde{\phi} \) on the two sides of \( \Gamma_1 = \partial \Omega \cap (\partial (\Omega_0 \backslash \Omega)): \)
\[ \tilde{\phi} \in W^{1,2}(\Omega_0). \]

On the other hand, we have:
\[ \int_{\Omega_0} \sigma.\varepsilon(\tilde{\phi})dx = \int_{\Omega} \sigma.\varepsilon(\tilde{\phi})dx + \int_{\Omega_0 \backslash \Omega} \sigma.\varepsilon(\tilde{\phi})dx, \]
which implies, according to (33):

\[ \int_{\Omega_0} \sigma \varepsilon(\tilde{\phi}) \, dx = \int_{\Omega} \sigma \varepsilon(\phi) \, dx \]

\[ = - \int_{\Omega} (\text{div} \sigma) \phi \, dx + \int_{\partial \Omega} (\sigma_1 \cdot n) \phi \, d\Gamma. \]

Using \( \text{div} \sigma = 0 \) in \( \Omega_0 \), we obtain:

\[ \int_{\Omega_0} \sigma \varepsilon(\tilde{\phi}) \, dx = \int_{\partial \Omega} (\sigma_1 \cdot n) \phi \, d\Gamma. \]

This means according to (36) that:

\[ \int_{\Omega_0} \sigma \varepsilon(\tilde{\phi}) \, dx = \int_{\Gamma_1} (\sigma_1 \cdot n) \phi \, d\Gamma. \tag{37} \]

We have on the other hand:

\[ \int_{\Omega_0} \sigma \varepsilon(\tilde{\phi}) \, dx = \int_{\Omega_0} (\text{div} \sigma) \cdot \tilde{\phi} \, dx + \int_{\partial \Omega} (\sigma \cdot n) \tilde{\phi} \, d\Gamma. \]

So, using properties (29) and (36), we deduce that:

\[ \int_{\Omega_0} \sigma \varepsilon(\tilde{\phi}) \, dx = 0. \tag{38} \]

From (37) and (38), we conclude that:

\[ \int_{\Gamma_1} (\sigma_1 \cdot n) \phi \, d\Gamma = 0 \quad \forall \phi \in W^{1,2}(\Omega) \text{ such that } \phi/(\partial \Omega \setminus \Gamma_1) = 0. \]

This proves that:

\[ \sigma_1 \cdot n = 0 \quad \text{on } \Gamma_1. \tag{39} \]

We conclude from (34), (35) and (39) that:

\[ \sigma_1 \in V_1. \tag{40} \]

Using Proposition 2 and \((\eta_n^D) \) converge to \( \sigma_0^D \) weakly in \( L^2(\Omega, \mathbb{R}^{3 \times 3}_{\text{sym}}) \), we have:

\[ F(\sigma_1) \leq \liminf \ F(\eta_n^D), \]

or also:

\[ F(\sigma_1) \leq \inf_{\eta \in V_1} F(\eta). \]

We finally conclude from (40):

\[ F(\sigma_1) = \inf_{\eta \in V_1} F(\eta). \]

\[ \square \]

**Corollary 2.** Under the hypotheses of Theorem 5 we have:

If \( \text{Rot} f \neq 0 \) in \( \Omega \) or (22) is satisfied by \( g \)
then

\[ \exists \sigma_1 \in V_1 \text{ such that: } \bar{\lambda} = \frac{1}{\inf_{\eta \in V_1} F(\eta)} = \frac{1}{F(\sigma_1)}. \]

Proof. The result is deduced from Theorems 4 and 5. \qed

REFERENCES