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MIXED DISCONTINUOUS GALERKIN APPROXIMATION OF THE MAXWELL OPERATOR: THE INDEFINITE CASE

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Abstract. We present and analyze an interior penalty method for the numerical discretization of the indefinite time-harmonic Maxwell equations in mixed form. The method is based on the mixed discretization of the curl-curl operator developed in [Houston *et al.*, *J. Sci. Comp.* **22** (2005) 325–356] and can be understood as a non-stabilized variant of the approach proposed in [Perugia *et al.*, *Comput. Methods Appl. Mech. Engrg.* **191** (2002) 4675–4697]. We show the well-posedness of this approach and derive optimal *a priori* error estimates in the energy-norm as well as the L^2-norm. The theoretical results are confirmed in a series of numerical experiments.

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1. Introduction

In the series of articles [11–14, 21, 22], we have been concerned with the design and analysis of interior penalty discontinuous Galerkin methods for Maxwell's equations in the frequency-domain; indeed, both the *low-frequency* and *high-frequency* regimes have been considered. In the low-frequency case, we first mention the work [11, 21] where we introduced and analyzed several *hp*-version discontinuous Galerkin methods for low-frequency models where the resulting bilinear forms are coercive. Such models typically arise in conducting materials or after time discretization of the full time-domain Maxwell equations. In order to incorporate the divergence-free constraint on the electric field within insulating materials, we then proposed a Lagrange multiplier approach and analyzed two families of mixed interior penalty methods; see [12, 13]. The scheme in [12] is based on elements of the same order for the approximation of the electric field and the Lagrange multiplier, and on the introduction of a normal jump stabilization term for the electric field. However, this stabilization term is unphysical and has been observed to lead to spurious oscillations in the vicinity of strong singularities in the underlying analytical solution. Fortunately, this stabilization can be avoided altogether by increasing the approximation degree for

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the Lagrange multiplier by one. The resulting mixed interior penalty method has been studied in [13]; it can be viewed as a discontinuous version of the natural pairing that is obtained when Nédélec's second family of elements of degree ℓ and standard nodal elements of degree $\ell + 1$ are employed; *cf.* [18, 20].

While the above interior penalty methods can be immediately extended to the time-harmonic Maxwell equations in the high-frequency regime, their numerical analysis becomes much more involved in this case, due to the indefiniteness of the underlying problem; a discrepancy that also arises for conforming finite element methods. In [22], a first error analysis of a stabilized mixed interior penalty method was carried out for the indefinite Maxwell system. The analysis there heavily relies on the introduction of certain volume stabilization terms, which have been numerically observed to be unnecessary. In fact, much of the efforts in [12, 13] were directed towards reducing the stabilization of [22], though in the context of low-frequency models. In the recent work [14], we developed a new technique for analyzing the interior penalty method for the indefinite Maxwell system in non-mixed form. The approach there is based on a novel approximation result that allows one to find a conforming finite element function close to any discontinuous one, very much in the spirit of the techniques in [15] used for deriving a posteriori estimates for discontinuous Galerkin discretizations of diffusions problems.

In this paper, we revisit the stabilized mixed interior penalty method in [22] and devise and analyze a non-stabilized variant thereof, by using the mixed approach of [13] for the discretization of the curl-curl operator. Thus, we propose a new mixed interior penalty method for the indefinite time-harmonic Maxwell equations, where the stabilization terms of [22] can be avoided altogether (except for the interior penalty terms, of course). Using the recent techniques of [14], we carry out the error analysis of this approach and derive optimal *a priori* error estimates in the energy-norm, as well as in the L^2-norm. As in [14], our analysis employs duality techniques (see [18], Sect. 7.2), and does not cover the case of non-smooth material coefficients. With respect to the direct formulation in [14], the mixed formulation studied here is equally applicable to both the low-frequency and high-frequency regimes, since control of the divergence of the electric field is achieved by the introduction of an appropriate Lagrange multiplier variable. Indeed, the numerical analysis of the corresponding mixed interior penalty method for the principal operator of the time-harmonic Maxwell equations in a heterogeneous insulating medium has already been undertaken in the article [13].

The outline of the paper is as follows: in Section 2 we introduce the mixed form of the indefinite time-harmonic Maxwell equations and, in Section 3, we present their mixed interior penalty discretization and review some basic properties of the discrete scheme. The *a priori* error bounds are stated in Section 4; the proofs of these estimates are carried out in Sections 5–7. The numerical performance of the proposed method is demonstrated in Section 8. Finally, in Section 9 we summarize the work presented in this paper and draw some conclusions.

2. Model problem

In this section, we introduce the model problem we shall consider in this paper. For comprehensive accounts of Maxwell's equations and their finite element discretization, we refer the reader to [10, 18] and the references cited therein.

2.1. Indefinite time-harmonic Maxwell equations

Let $\Omega \subset \mathbb{R}^3$ be a lossless isotropic medium with constant magnetic permeability μ, constant electric permittivity ε and a perfectly conducting boundary $\Gamma = \partial\Omega$. For a given temporal frequency $\omega > 0$, we seek to determine the time-harmonic electric field $\mathcal{E}(t, \mathbf{x}) = \Re\left(\exp(-i\omega t)\mathbf{E}(\mathbf{x})\right)$ whose spatial component $\mathbf{E}$ satisfies the indefinite equations

$$\nabla \times \nabla \times \mathbf{E} - k^2\mathbf{E} = \mathbf{j} \qquad \text{in } \Omega, \tag{1}$$

$$\mathbf{n} \times \mathbf{E} = \mathbf{0} \qquad \text{on } \Gamma. \tag{2}$$

Here, we take Ω to be an open bounded Lipschitz polyhedron with unit outward normal vector $\mathbf{n}$ on Γ. In order to avoid topological complications, we assume that Ω is simply-connected and that Γ is connected.

The parameter $k > 0$ is the wave number given by $k = \omega\sqrt{\varepsilon\mu}$. Throughout, we assume that k^2 is *not* an interior Maxwell eigenvalue, *i.e.*, for any $\mathbf{E} \neq \mathbf{0}$, the pair $(\lambda = k^2, \mathbf{E})$ is *not* an eigensolution of $\nabla \times \nabla \times \mathbf{E} = \lambda\mathbf{E}$ in Ω, $\mathbf{n} \times \mathbf{E} = \mathbf{0}$ on Γ. Finally, the right-hand side $\mathbf{j}$ is a given generic source field in $L^2(\Omega)^3$ corresponding to a time-harmonic excitation.

2.2. Function spaces

For a bounded domain D in $\mathbb{R}^3$, we denote by $H^s(D)$ the standard Sobolev space of order $s \geq 0$ and by $\|\cdot\|_{s,D}$ the usual Sobolev norm (see, *e.g.*, [16]). When $D = \Omega$, we simply write $\|\cdot\|_s$. For $s = 0$, we write $L^2(D)$ in lieu of $H^0(D)$. We also use $\|\cdot\|_{s,D}$ to denote the norm for the space $H^s(D)^3$. $H^1_0(D)$ is the subspace of $H^1(D)$ of functions with zero trace on ∂D. If Λ is a subset of ∂D, we denote by $\|\cdot\|_{0,\Lambda}$ the L^2-norm in $L^2(\Lambda)$ and $L^2(\Lambda)^3$. On the computational domain Ω, we introduce the spaces

$$H(\mathrm{curl};\Omega) = \left\{ \mathbf{v} \in L^2(\Omega)^3 \,:\, \nabla \times \mathbf{v} \in L^2(\Omega)^3 \right\},$$
$$H_0(\mathrm{curl};\Omega) = \left\{ \mathbf{v} \in H(\mathrm{curl};\Omega) \,:\, \mathbf{n} \times \mathbf{v} = \mathbf{0} \text{ on } \Gamma \right\},$$

and endow them with the norm $\|\mathbf{v}\|^2_{\mathrm{curl}} := \|\mathbf{v}\|^2_0 + \|\nabla \times \mathbf{v}\|^2_0$. Similarly, we set

$$H(\mathrm{div};\Omega) = \left\{ \mathbf{v} \in L^2(\Omega)^3 \,:\, \nabla \cdot \mathbf{v} \in L^2(\Omega) \right\},$$
$$H_0(\mathrm{div};\Omega) = \left\{ \mathbf{v} \in H(\mathrm{div};\Omega) \,:\, \mathbf{v} \cdot \mathbf{n} = 0 \text{ on } \Gamma \right\},$$
$$H(\mathrm{div}^0;\Omega) = \left\{ \mathbf{v} \in H(\mathrm{div};\Omega) \,:\, \nabla \cdot \mathbf{v} = 0 \text{ in } \Omega \right\},$$

equipped with the norm $\|\mathbf{v}\|^2_{\mathrm{div}} := \|\mathbf{v}\|^2_0 + \|\nabla \cdot \mathbf{v}\|^2_0$. Finally, we denote by $(\cdot,\cdot)$ the standard inner product in $L^2(\Omega)^3$ given by $(\mathbf{u},\mathbf{v}) := \int_\Omega \mathbf{u} \cdot \mathbf{v}\, d\mathbf{x}$.

2.3. Mixed formulation

The interior penalty method proposed in this article is based on a mixed formulation of (1)–(2) already used in the *hp*-approaches of [1, 8], as well as in the mortar approach [6]. To this end, we decompose the field $\mathbf{E}$ as

$$\mathbf{E} = \mathbf{u} + \nabla\varphi, \tag{3}$$

where φ is scalar function in $H^1_0(\Omega)$ and $\mathbf{u}$ belongs to $H_0(\mathrm{curl};\Omega) \cap H(\mathrm{div}^0;\Omega)$. The decomposition (3) is orthogonal in $L^2(\Omega)^3$, which implies that

$$(\mathbf{u}, \nabla q) = 0 \qquad \forall q \in H^1_0(\Omega); \tag{4}$$

see [9] for details. Thus, upon setting

$$p = k^2\varphi, \tag{5}$$

we are led to consider the following system: find $(\mathbf{u}, p)$ such that

$$\nabla \times \nabla \times \mathbf{u} - k^2\mathbf{u} - \nabla p = \mathbf{j} \qquad \text{in } \Omega, \tag{6}$$
$$\nabla \cdot \mathbf{u} = 0 \qquad \text{in } \Omega, \tag{7}$$
$$\mathbf{n} \times \mathbf{u} = \mathbf{0} \qquad \text{on } \Gamma, \tag{8}$$
$$p = 0 \qquad \text{on } \Gamma. \tag{9}$$

Introducing the spaces $\mathbf{V} = H_0(\mathrm{curl};\Omega)$ and $Q = H^1_0(\Omega)$, the weak formulation of problem (6)–(9) consists in finding $(\mathbf{u}, p) \in \mathbf{V} \times Q$ such that

$$\begin{aligned} a(\mathbf{u},\mathbf{v}) - k^2(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) &= (\mathbf{j},\mathbf{v}), \\ b(\mathbf{u},q) &= 0 \end{aligned} \tag{10}$$

for all $(\mathbf{v}, q) \in \mathbf{V} \times Q$, where the forms a and b are defined, respectively, by

$$a(\mathbf{u}, \mathbf{v}) = (\nabla \times \mathbf{u}, \nabla \times \mathbf{v}), \qquad b(\mathbf{v}, p) = -(\mathbf{v}, \nabla p).$$

We notice that the form a is bilinear, continuous and coercive on the kernel of b, and b is bilinear, continuous, and satisfies the inf-sup condition; see, *e.g.*, [8, 18, 24]. Hence, problem (6)–(9) is well-posed (provided that k^2 is not an interior Maxwell eigenvalue) and there is a positive constant C, depending on Ω and k^2, such that

$$\|\mathbf{u}\|_{\mathrm{curl}} + \|p\|_1 \leq C\|\mathbf{j}\|_0; \tag{11}$$

cf. [22], *Prop.* 1. Moreover, under the foregoing assumptions on Ω, there exists a regularity exponent $\sigma = \sigma(\Omega) > 1/2$, only depending on Ω, such that

$$\mathbf{u} \in H^\sigma(\Omega)^3, \qquad \nabla \times \mathbf{u} \in H^\sigma(\Omega)^3, \quad \text{and} \quad \|\mathbf{u}\|_\sigma + \|\nabla \times \mathbf{u}\|_\sigma \leq C\|\mathbf{j}\|_0, \tag{12}$$

for a constant C depending on Ω and k^2; see [22], *Prop.* 2.

We point out that the regularity exponent $\sigma = \sigma(\Omega) > 1/2$ stems from the embeddings

$$\begin{aligned} &H_0(\mathrm{curl}; \Omega) \cap H(\mathrm{div}; \Omega) \hookrightarrow H^\sigma(\Omega)^3, \\ &H(\mathrm{curl}; \Omega) \cap H_0(\mathrm{div}; \Omega) \hookrightarrow H^\sigma(\Omega)^3; \end{aligned} \tag{13}$$

see [2], *Prop.* 3.7. The maximal value of σ for which the above embeddings hold is closely related to the elliptic regularity properties of the Laplacian in polyhedra and only depends on the opening angles at the corners and edges of the domain, *cf.* [2]. In particular, for a convex domain, the embeddings in (13) hold with $\sigma = 1$.

3. Discretization

In this section, we introduce an interior penalty discretization for the system (6)–(9) and discuss its stability and consistency properties.

3.1. Preliminaries

We consider conforming, shape-regular partitions $\mathcal{T}_h$ of Ω into tetrahedra $\{K\}$; here, h denotes the granularity of the mesh $\mathcal{T}_h$, *i.e.*, $h = \max_{K \in \mathcal{T}_h} h_K$, where $h_K = \mathrm{diam}(K)$ for all $K \in \mathcal{T}_h$. We denote by $\mathcal{F}_h^{\mathcal{I}}$ the set of all interior faces of $\mathcal{T}_h$, by $\mathcal{F}_h^{\mathcal{B}}$ the set of all boundary faces of $\mathcal{T}_h$, and set $\mathcal{F}_h = \mathcal{F}_h^{\mathcal{I}} \cup \mathcal{F}_h^{\mathcal{B}}$.

For piecewise smooth vector- and scalar-valued functions $\mathbf{v}$ and q, respectively, we introduce the following trace operators. Let $F \in \mathcal{F}_h^{\mathcal{I}}$ be an interior face shared by two elements K^+ and K^- with unit outward normal vectors $\mathbf{n}^\pm$, respectively. Denoting by $\mathbf{v}^\pm$ and $q^\pm$ the traces of $\mathbf{v}$ and q on $\partial K^\pm$ taken from within $K^\pm$, respectively, we define the *jumps* across F by

$$[\![\mathbf{v}]\!]_T = \mathbf{n}^+ \times \mathbf{v}^+ + \mathbf{n}^- \times \mathbf{v}^-, \qquad [\![q]\!]_N = q^+\mathbf{n}^+ + q^-\mathbf{n}^-,$$

and the *averages* by

$$\{\!\{\mathbf{v}\}\!\} = (\mathbf{v}^+ + \mathbf{v}^-)/2, \qquad \{\!\{q\}\!\} = (q^+ + q^-)/2.$$

On a boundary face $F \in \mathcal{F}_h^{\mathcal{B}}$, we set analogously $[\![\mathbf{v}]\!]_T = \mathbf{n} \times \mathbf{v}$, $[\![q]\!]_N = q\,\mathbf{n}$, $\{\!\{\mathbf{v}\}\!\} = \mathbf{v}$ and $\{\!\{q\}\!\} = q$.

3.2. Interior penalty method

For a given partition $\mathcal{T}_h$ of Ω and an approximation order $\ell \geq 1$, we wish to approximate $(\mathbf{u}, p)$ by $(\mathbf{u}_h, p_h)$ in the finite element space $\mathbf{V}_h \times Q_h$, where

$$\mathbf{V}_h := \{\mathbf{v} \in L^2(\Omega)^3 : \mathbf{v}|_K \in \mathcal{P}^\ell(K)^3 \;\; \forall K \in \mathcal{T}_h\},$$
$$Q_h = \{q \in L^2(\Omega) : q|_K \in \mathcal{P}^{\ell+1}(K) \;\; \forall K \in \mathcal{T}_h\},$$

and $\mathcal{P}^m(K)$ denotes the space of polynomials of total degree at most m on K. To this end, we consider the discontinuous Galerkin method: find $\mathbf{u}_h \in \mathbf{V}_h$ and $p_h \in Q_h$ such that

$$\begin{aligned} a_h(\mathbf{u}_h, \mathbf{v}) - k^2(\mathbf{u}_h, \mathbf{v}) + b_h(\mathbf{v}, p_h) &= (\mathbf{j}, \mathbf{v}), \\ b_h(\mathbf{u}_h, q) - c_h(p_h, q) &= 0 \end{aligned} \tag{14}$$

for all $(\mathbf{v}, q) \in \mathbf{V}_h \times Q_h$, with discrete forms $a_h(\cdot,\cdot)$, $b_h(\cdot,\cdot)$ and $c_h(\cdot,\cdot)$ defined by

$$\begin{aligned} a_h(\mathbf{u}, \mathbf{v}) &= (\nabla_h \times \mathbf{u}, \nabla_h \times \mathbf{v}) - \int_{\mathcal{F}_h} [\![\mathbf{u}]\!]_T \cdot \{\!\{\nabla_h \times \mathbf{v}\}\!\} \, \mathrm{d}s \\ &\quad - \int_{\mathcal{F}_h} [\![\mathbf{v}]\!]_T \cdot \{\!\{\nabla_h \times \mathbf{u}\}\!\} \, \mathrm{d}s + \int_{\mathcal{F}_h} \mathtt{a} \, [\![\mathbf{u}]\!]_T \cdot [\![\mathbf{v}]\!]_T \, \mathrm{d}s, \\ b_h(\mathbf{v}, p) &= -(\mathbf{v}, \nabla_h p) + \int_{\mathcal{F}_h} \{\!\{\mathbf{v}\}\!\} \cdot [\![p]\!]_N \, \mathrm{d}s, \\ c_h(p, q) &= \int_{\mathcal{F}_h} \mathtt{c} [\![p]\!]_N \cdot [\![q]\!]_N \, \mathrm{d}s, \end{aligned}$$

respectively. Here, ∇_h is the discrete *"nabla"* operator defined elementwise (*i.e.*, $(\nabla_h \times \mathbf{v})_{|K} = \nabla \times \mathbf{v}_{|K}$ and $(\nabla_h q)_{|K} = \nabla q_{|K}$), and use the convention that

$$\int_{\mathcal{F}_h} \psi \, \mathrm{d}s = \sum_{F \in \mathcal{F}_h} \int_F \psi \, \mathrm{d}s.$$

The functions $\mathtt{a}$ and $\mathtt{c}$ are the so-called interior penalty stabilization functions that are taken as follows:

$$\mathtt{a} = \alpha \, \mathtt{h}^{-1}, \qquad \mathtt{c} = \gamma \, \mathtt{h}^{-1}. \tag{15}$$

Here, $\mathtt{h}$ is the mesh size function given by $\mathtt{h}|_F \equiv h_F = \mathrm{diam}(F)$ for all $F \in \mathcal{F}_h$. Furthermore, α and γ are positive parameters independent of the mesh size.

Remark 3.1. The jumps $[\![\mathbf{v}]\!]_T$ and $[\![q]\!]_N$ are well-defined for elements of $\mathbf{V}_h$ and Q_h, respectively, since the elements of $\mathbf{V}_h$ and Q_h are elementwise polynomials, and therefore elementwise arbitrarily smooth. Moreover, if $q \in Q$, then $[\![q]\!]_N$ is well-defined and equal to zero on any $F \in \mathcal{F}_h$. Similarly, if $\mathbf{v} \in \mathbf{V}$, then the jump condition $\mathbf{n}^+ \times \mathbf{v}^+ + \mathbf{n}^- \times \mathbf{v}^- = \mathbf{0}$ and the boundary condition $\mathbf{n} \times \mathbf{v} = \mathbf{0}$ hold in $H_{00}^{-\frac{1}{2}}(F)^3$ (for the definition of $H_{00}^{-\frac{1}{2}}(F)$, see [16]), and thus also in $L^2(F)^3$ for all $F \in \mathcal{F}_h^{\mathcal{I}}$ and for any $F \in \mathcal{F}_h^{\mathcal{B}}$, respectively; therefore, $[\![\mathbf{v}]\!]_T$ is well-defined and equal to zero on any $F \in \mathcal{F}_h$.

The well-posedness of the method (14) will be established in Corollary 4.3 below.

Remark 3.2. We note that the formulation (14) is a non-stabilized variant of the one proposed in [22]. Furthermore, we point out that the formulation (14) can be easily modified in order to include non-constant material coefficients, see [13, 21, 22]. However, while the subsequent analysis, based on employing duality arguments, can be immediately extended to the case of smooth material coefficients, problems with non-smooth coefficients cannot be dealt with using this approach. Indeed, in this latter case, the error analysis of the proposed interior penalty method remains an open issue.

Remark 3.3. Instead of the interior penalty approach presented here, many other discontinuous Galerkin methods could be employed for the discretization of the curl-curl operator; see [3] for a presentation of different discontinuous Galerkin discretizations of second order operators, and [21] for details on the LDG discretization of the curl-curl operator.

3.3. **Auxiliary forms and error equations**

In order to study the discretization in (14), we first define how a_h and b_h should be understood on the continuous level. To this end, we introduce the spaces $\mathbf{V}(h)$ and $Q(h)$ given by

$$\mathbf{V}(h) = \mathbf{V} + \mathbf{V}_h, \qquad Q(h) = Q + Q_h,$$

and endow them with the following DG-norms:

$$\begin{aligned}\|\mathbf{v}\|^2_{\mathbf{V}(h)} &= \|\mathbf{v}\|_0^2 + \|\nabla_h \times \mathbf{v}\|_0^2 + \|\mathtt{h}^{-\frac{1}{2}} [\![\mathbf{v}]\!]_T\|^2_{0,\mathcal{F}_h},\\ \|q\|^2_{Q(h)} &= \|\nabla_h q\|_0^2 + \|\mathtt{h}^{-\frac{1}{2}} [\![q]\!]_N\|^2_{0,\mathcal{F}_h},\end{aligned}$$

respectively. Note that the jumps $[\![\mathbf{v}]\!]_T$ and $[\![q]\!]_N$ are well-defined for elements of $\mathbf{V}(h)$ and $Q(h)$, respectively, and coincide with the jumps of the components of $\mathbf{v}$ and q in $\mathbf{V}_h$ and Q_h, respectively (see Rem. 3.1).

Here, we use the notation

$$\|\psi\|^2_{0,\mathcal{F}_h} = \sum_{F\in\mathcal{F}_h} \|\psi\|^2_{0,F}.$$

Then, for $\mathbf{v} \in \mathbf{V}(h)$, we define the lifted element $\mathcal{L}(\mathbf{v}) \in \mathbf{V}_h$ by

$$(\mathcal{L}(\mathbf{v}), \mathbf{w}) = \int_{\mathcal{F}_h} [\![\mathbf{v}]\!]_T \cdot \{\!\{\mathbf{w}\}\!\}\, \mathrm{d}s \qquad \forall \mathbf{w} \in \mathbf{V}_h.$$

Similarly, for q in $Q(h)$, we define $\mathcal{M}(q) \in \mathbf{V}_h$ by

$$(\mathcal{M}(q), \mathbf{w}) = \int_{\mathcal{F}_h} \{\!\{\mathbf{w}\}\!\} \cdot [\![q]\!]_N\, \mathrm{d}s \qquad \forall \mathbf{w} \in \mathbf{V}_h.$$

The lifting operators $\mathcal{L}$ and $\mathcal{M}$ are well-defined; see [22], *Prop.* 12.

Next, we introduce the auxiliary forms

$$\begin{aligned}\widetilde{a}_h(\mathbf{u}, \mathbf{v}) &= (\nabla_h \times \mathbf{u}, \nabla_h \times \mathbf{v}) - (\mathcal{L}(\mathbf{u}), \nabla_h \times \mathbf{v})\\ &\quad -(\mathcal{L}(\mathbf{v}), \nabla_h \times \mathbf{u}) + \int_{\mathcal{F}_h} \mathtt{a}\, [\![\mathbf{u}]\!]_T \cdot [\![\mathbf{v}]\!]_T\, \mathrm{d}s,\\ \widetilde{b}_h(\mathbf{v}, p) &= -(\mathbf{v}, \nabla_h p - \mathcal{M}(p)).\end{aligned}$$

Then, we have

$$\widetilde{a}_h = a_h \quad \text{on } \mathbf{V}_h \times \mathbf{V}_h, \qquad \widetilde{a}_h = a \quad \text{on } \mathbf{V} \times \mathbf{V},$$

as well as

$$\widetilde{b}_h = b_h \quad \text{on } \mathbf{V}_h \times Q_h, \qquad \widetilde{b}_h = b \quad \text{on } \mathbf{V} \times Q.$$

Hence, $\widetilde{a}_h$ and $\widetilde{b}_h$ can be viewed as extensions of a_h and b_h, as well as a and b, to the spaces $\mathbf{V}(h) \times \mathbf{V}(h)$ and $\mathbf{V}(h) \times Q(h)$, respectively. With this notation, we may reformulate the discrete problem (14) in the following equivalent way: find $(\mathbf{u}_h, p_h)$ in $\mathbf{V}_h \times Q_h$ such that

$$\begin{aligned} \widetilde{a}_h(\mathbf{u}_h, \mathbf{v}) - k^2(\mathbf{u}_h, \mathbf{v}) + \widetilde{b}_h(\mathbf{v}, p_h) &= (\mathbf{j}, \mathbf{v}), \\ \widetilde{b}_h(\mathbf{u}_h, q) - c_h(p_h, q) &= 0 \end{aligned} \tag{16}$$

for all $(\mathbf{v}, q) \in \mathbf{V}_h \times Q_h$.

Let $(\mathbf{u}, p)$ be the analytical solution of (6)–(9) and $(\mathbf{v}, q) \in \mathbf{V}_h \times Q_h$. We define

$$\begin{aligned} R_1(\mathbf{u}, p; \mathbf{v}) &:= \widetilde{a}_h(\mathbf{u}, \mathbf{v}) - k^2(\mathbf{u}, \mathbf{v}) + \widetilde{b}_h(\mathbf{v}, p) - (\mathbf{j}, \mathbf{v}), \\ R_2(\mathbf{u}; q) &:= \widetilde{b}_h(\mathbf{u}, q) = \widetilde{b}_h(\mathbf{u}, q) - c_h(p, q), \end{aligned}$$

where we have used that $c_h(p, q) = 0$ for any $q \in Q_h$. The functionals R_1 and R_2 measure how well the analytical solution $(\mathbf{u}, p)$ satisfies the formulation in (16). Owing to the regularity properties in (12), it is possible to show that

$$\begin{aligned} R_1(\mathbf{u}, p; \mathbf{v}) &= \int_{\mathcal{F}_h} [\![\mathbf{v}]\!]_T \cdot \{\!\{\nabla \times \mathbf{u} - \mathbf{\Pi}_h(\nabla \times \mathbf{u})\}\!\} \, \mathrm{d}s, \\ R_2(\mathbf{u}; q) &= \int_{\mathcal{F}_h} [\![q]\!]_N \cdot \{\!\{\mathbf{u} - \mathbf{\Pi}_h \mathbf{u}\}\!\} \, \mathrm{d}s, \end{aligned} \tag{17}$$

with $\mathbf{\Pi}_h$ denoting the L^2-projection onto $\mathbf{V}_h$; see [22], *Lem.* 24, for details. In particular, we have that R_1 is independent of p, and that $R_1(\mathbf{u}, p; \mathbf{v}) = 0$ for all $\mathbf{v} \in \mathbf{V}_h \cap \mathbf{V}$, as well as $R_2(\mathbf{u}; q) = 0$ for all $q \in Q_h \cap Q$.

With these definitions, it is obvious that the error $(\mathbf{u} - \mathbf{u}_h, p - p_h)$ between the analytical solution $(\mathbf{u}, p)$ and the mixed DG approximation $(\mathbf{u}_h, p_h)$ satisfies

$$\widetilde{a}_h(\mathbf{u} - \mathbf{u}_h, \mathbf{v}) - k^2(\mathbf{u} - \mathbf{u}_h, \mathbf{v}) + \widetilde{b}_h(\mathbf{v}, p - p_h) = R_1(\mathbf{u}, p; \mathbf{v}) \qquad \forall \mathbf{v} \in \mathbf{V}_h, \tag{18}$$

as well as

$$\widetilde{b}_h(\mathbf{u} - \mathbf{u}_h, q) - c_h(p - p_h, q) = R_2(\mathbf{u}; q) \qquad \forall q \in Q_h. \tag{19}$$

Here, (18) and (19) are referred to as the error equations.

3.4. **Continuity and stability properties**

Next, let us review the main stability results for the forms $\widetilde{a}_h$ and $\widetilde{b}_h$, as well as some crucial properties of the discrete solution $(\mathbf{u}_h, p_h)$. To this end, we first note that the following continuity properties hold.

Proposition 3.4. *There are continuity constants C_A and C_B, independent of the mesh size, such that*

$$\begin{aligned} |\widetilde{a}_h(\mathbf{u}, \mathbf{v})| &\le C_A \|\mathbf{u}\|_{\mathbf{V}(h)} \|\mathbf{v}\|_{\mathbf{V}(h)} \qquad \forall \mathbf{u}, \mathbf{v} \in \mathbf{V}(h), \\ |\widetilde{b}_h(\mathbf{v}, q)| &\le C_B \|\mathbf{v}\|_{\mathbf{V}(h)} \|q\|_{Q(h)} \qquad \forall (\mathbf{v}, q) \in \mathbf{V}(h) \times Q(h). \end{aligned}$$

The linear functional on the right-hand side of the first equation in (16) *satisfies*

$$|(\mathbf{j}, \mathbf{v})| \le \|\mathbf{j}\|_0 \|\mathbf{v}\|_{\mathbf{V}(h)} \qquad \forall \mathbf{v} \in \mathbf{V}_h.$$

Furthermore, there is a constant C_R, independent of the mesh size, such that

$$\begin{aligned} |R_1(\mathbf{u}, p; \mathbf{v})| &\le C_R \|\mathbf{v}\|_{\mathbf{V}(h)} \mathcal{E}_{1,h}(\mathbf{u}) \qquad \forall \mathbf{v} \in \mathbf{V}_h, \\ |R_2(\mathbf{u}; q)| &\le C_R \|q\|_{Q(h)} \mathcal{E}_{2,h}(\mathbf{u}) \qquad \forall q \in Q_h. \end{aligned}$$

Here, we have set

$$\mathcal{E}_{1,h}(\mathbf{u})^2 := \sum_K h_K \|\nabla \times \mathbf{u} - \mathbf{\Pi}_h(\nabla \times \mathbf{u})\|_{0,\partial K}^2,$$
$$\mathcal{E}_{2,h}(\mathbf{u})^2 := \sum_K h_K \|\mathbf{u} - \mathbf{\Pi}_h \mathbf{u}\|_{0,\partial K}^2, \tag{20}$$

where we recall that $\mathbf{\Pi}_h$ *denotes the* L^2*-projection onto* $\mathbf{V}_h$*.*

Proof. For the proof of the first three assertions, we refer the reader to [12], *Prop.* 5.1. The stability bounds for R_1 and R_2 in (20) follow immediately from weighted Cauchy-Schwarz inequalities and the definitions of the norms $\|\cdot\|_{\mathbf{V}(h)}$, $\|\cdot\|_{Q(h)}$ and the parameters $\mathtt{a}$, $\mathtt{c}$ in (15). □

The form $\widetilde{a}_h$ satisfies the following Gårding-type inequality.

Proposition 3.5. *There exists a parameter* $\alpha_{\min} > 0$*, independent of the mesh size, such that for* $\alpha \geq \alpha_{\min}$ *we have*

$$\widetilde{a}_h(\mathbf{v},\mathbf{v}) \geq C_G \|\mathbf{v}\|_{\mathbf{V}(h)}^2 - \|\mathbf{v}\|_0^2 \qquad \forall \mathbf{v} \in \mathbf{V}_h,$$

with a constant $C_G > 0$ *independent of the mesh size.*

Proof. The Gårding-type inequality readily follows from the fact that there is an $\alpha_{\min} > 0$, independent of the mesh size, such that for $\alpha \geq \alpha_{\min}$

$$\widetilde{a}_h(\mathbf{v},\mathbf{v}) \geq C \left[\|\nabla_h \times \mathbf{v}\|_0^2 + \|\mathrm{h}^{-\frac{1}{2}} [\![\mathbf{v}]\!]_T\|_{0,\mathcal{F}_h}^2 \right];$$

see [11, 12] for details. □

Next, let us recall a stability property of the form $\widetilde{b}_h$ on the conforming subspaces underlying $\mathbf{V}_h$ and Q_h. To this end, we set

$$\mathbf{V}_h^c = \mathbf{V}_h \cap \mathbf{V}, \qquad Q_h^c = Q_h \cap Q. \tag{21}$$

Notice that $\mathbf{V}_h^c$ is the Nédélec finite element space of second type (see [20] or [18], Sect. 8.2), with zero tangential trace prescribed on Γ, and Q_h^c is the space of continuous polynomials of degree $\ell+1$, with zero trace prescribed on Γ.

The following inf-sup condition holds on $\mathbf{V}_h^c \times Q_h^c$; see [13], *Lem.* 1, for details.

Lemma 3.6. *There is a stability constant* C_S*, independent of the mesh size, such that*

$$\inf_{q \in Q_h^c \setminus \{0\}} \sup_{\mathbf{v} \in \mathbf{V}_h^c \setminus \{\mathbf{0}\}} \frac{\widetilde{b}_h(\mathbf{v}, q)}{\|\mathbf{v}\|_{\mathbf{V}(h)} \|q\|_{Q(h)}} \geq C_S > 0. \tag{22}$$

Note that, since $\mathbf{V}_h^c \subset \mathbf{V}_h$, the inf-sup condition (22) in Lemma 3.6 remains valid when $\mathbf{V}_h^c$ is replaced by $\mathbf{V}_h$, with the same inf-sup constant.

Now, define the discrete kernel

$$\mathbf{Z}_h = \{\mathbf{v} \in \mathbf{V}_h : \widetilde{b}_h(\mathbf{v}, q) = 0 \ \forall q \in Q_h^c\}. \tag{23}$$

Lemma 3.7. *Let* $\mathbf{u}$ *be the vector-valued component of the analytical solution of* (6)–(9) *and* $\mathbf{u}_h$ *its discontinuous Galerkin approximation obtained in* (14)*. Then,*

(i) $\mathbf{u}_h \in \mathbf{Z}_h$*;*
(ii) $(\mathbf{u} - \mathbf{u}_h, \nabla q) = 0$ *for all* $q \in Q_h^c$*.*

Proof. Since $c_h(p_h, q) = 0$ for all $q \in Q_h^c$, the first claim follows immediately. Furthermore, in view of (4), $(\mathbf{u} - \mathbf{u}_h, \nabla q) = -(\mathbf{u}_h, \nabla q)$ for all $q \in Q_h^c$. Since $-(\mathbf{u}_h, \nabla q) = \widetilde{b}(\mathbf{u}_h, q)$, the second claim follows from the first one. □

Finally, we will make use of a discrete Helmholtz decomposition: the space $\mathbf{V}_h^c$ can be written as

$$\mathbf{V}_h^c = \mathbf{X}_h \oplus \nabla Q_h^c, \tag{24}$$

with $\mathbf{X}_h$ given by

$$\mathbf{X}_h := \{\mathbf{v} \in \mathbf{V}_h^c : (\mathbf{v}, \nabla q) = 0 \ \forall q \in Q_h^c\}. \tag{25}$$

By construction, the decomposition (24) is orthogonal in $L^2(\Omega)^3$; *cf.* [18], Sect. 8.2.

4. *A priori* error estimates and well-posedness

In this section, we state optimal *a priori* error estimates in the DG energy-norm and the L^2-norm. We further show that the energy error estimates imply the well-posedness of the interior penalty formulation (14); see [23].

The following result addresses the error in the energy-norm.

Theorem 4.1. *Suppose that the analytical solution* $(\mathbf{u}, p)$ *of* (6)–(9) *satisfies*

$$\mathbf{u} \in H^s(\Omega)^3, \qquad \nabla \times \mathbf{u} \in H^s(\Omega)^3, \qquad p \in H^{s+1}(\Omega), \tag{26}$$

for a parameter $s > 1/2$. *Let* $(\mathbf{u}_h, p_h)$ *be the mixed DG approximation obtained by* (14) *with* $\alpha \geq \alpha_{\min}$ *and* $\gamma > 0$. *Then, there exists a mesh size* $h_0 > 0$ *such that*

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{V}(h)} + \|p - p_h\|_{Q(h)} \leq C\, h^{\min\{s,\ell\}} \left[\|\mathbf{u}\|_s + \|\nabla \times \mathbf{u}\|_s + \|p\|_{s+1}\right]$$

for all meshes $\mathcal{T}_h$ *of mesh size* $h < h_0$. *The constant* $C > 0$ *is independent of the mesh size.*

Remark 4.2. We observe that the regularity assumption on p in Theorem 4.1 is automatically fulfilled, with $s = \sigma$, in the case when $\nabla \cdot \mathbf{j} \in L^2(\Omega)$. Here, σ is the embedding parameter from (13).

By proceeding along the lines of [23], well-posedness of the formulation (14) can be established from the *a priori* estimate in Theorem 4.1.

Corollary 4.3. *For stabilization parameters* $\alpha \geq \alpha_{\min} > 0$ *and* $\gamma > 0$, *and mesh sizes* $h < h_0$, *the method* (14) *has a unique solution.*

Proof. If $\mathbf{j} = \mathbf{0}$, then $(\mathbf{u}, p) = (\mathbf{0}, 0)$ and the estimate in Theorem 4.1 implies that $\|\mathbf{u}_h\|_{\mathbf{V}(h)} + \|p_h\|_{Q(h)} \leq 0$ for $h < h_0$. Hence, $(\mathbf{u}_h, p_h) = (\mathbf{0}, 0)$ for $h < h_0$. □

Next, we state an *a priori* bound for the error $\|\mathbf{u} - \mathbf{u}_h\|_0$ and show that the optimal order $\mathcal{O}(h^{\ell+1})$ is obtained for smooth solutions and convex domains.

Theorem 4.4. *Suppose the vector-valued component* $\mathbf{u}$ *of the analytical solution* $(\mathbf{u}, p)$ *of* (6)–(9) *satisfies*

$$\mathbf{u} \in H^{s+\sigma}(\Omega)^3, \qquad \nabla \times \mathbf{u} \in H^s(\Omega)^3,$$

for a parameter $s > 1/2$ *and the embedding exponent* $\sigma \in (1/2, 1]$ *from* (13). *Let* $\mathbf{u}_h$ *be the DG approximation obtained by* (14) *with* $\alpha \geq \alpha_{\min} > 0$ *and* $\gamma > 0$. *Then there is a mesh size* $0 < h_1 < 1$ *such that for all meshes* $\mathcal{T}_h$ *of mesh size* $0 < h < h_1$ *we have*

$$\|\mathbf{u} - \mathbf{u}_h\|_0 \leq C\, h^{\min\{s,\ell\}+\sigma} \left[\|\mathbf{u}\|_{s+\sigma} + \|\nabla \times \mathbf{u}\|_s\right] + C\, h^{\sigma} \left[\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{V}(h)} + \|p - p_h\|_{Q(h)}\right],$$

where the constant $C > 0$ *is independent of the mesh size.*

We note that the minimal mesh sizes h_0 in Theorem 4.1 and h_1 in Theorem 4.4 depend on the wave number k and the regularity exponent σ in (13).
Theorem 4.4 and Theorem 4.1 ensure optimal L^2-error estimates for smooth solutions and convex domains.

Corollary 4.5. *For a convex domain where $\sigma = 1$ and an analytical solution $(\mathbf{u}, p) \in H^{\ell+1}(\Omega)^3 \times H^{\ell+1}(\Omega)$, we obtain for $h < \min\{h_0, h_1\}$ the optimal error bound*

$$\|\mathbf{u} - \mathbf{u}_h\|_0 \leq Ch^{\ell+1}\big[\|\mathbf{u}\|_{\ell+1} + \|p\|_{\ell+1}\big],$$

with a constant $C > 0$ independent of the mesh size.

The proofs of Theorems 4.1 and 4.4 are given in Sections 6 and 7, respectively. Before that, we recall in Section 5 some crucial approximation results.

5. Approximation results

In this section, we collect several approximation results which will be required in the error analysis of the method in (14).

5.1. Conforming approximation of discontinuous Galerkin functions

We start by recalling an approximation result that allows us to find a conforming function close to any discontinuous one.

Theorem 5.1. *There exist approximants $\mathbf{A} : \mathbf{V}_h \to \mathbf{V}_h^c$ and $A : Q_h \to Q_h^c$ such that*

$$\|\mathbf{v} - \mathbf{A}\mathbf{v}\|_0^2 \leq C \int_{\mathcal{F}_h} \mathtt{h} |[\![\mathbf{v}]\!]_T|^2 \, \mathrm{d}s,$$
$$\|\mathbf{v} - \mathbf{A}\mathbf{v}\|_{\mathbf{V}(h)}^2 \leq C \int_{\mathcal{F}_h} \mathtt{h}^{-1} |[\![\mathbf{v}]\!]_T|^2 \, \mathrm{d}s,$$
$$\|q - Aq\|_{Q(h)}^2 \leq C \int_{\mathcal{F}_h} \mathtt{h}^{-1} |[\![q]\!]_N|^2 \, \mathrm{d}s$$

for all $\mathbf{v} \in \mathbf{V}_h$ and $q \in Q_h$. The constant $C > 0$ solely depends on the shape-regularity of the mesh and the polynomial degree ℓ.

For the space $\mathbf{V}_h$, this result has been proved in [14], Appendix A, whereas the result for Q_h can be found in [15], Sect. 2.1. Theorem 5.1 and the definition of the DG-norms $\|\cdot\|_{\mathbf{V}(h)}$ and $\|\cdot\|_{Q(h)}$ immediately imply the following result.

Corollary 5.2. *There is a constant $C > 0$ independent of the mesh size such that*

$$\|\mathbf{v} - \mathbf{A}\mathbf{v}\|_{\mathbf{V}(h)} + \|\mathbf{A}\mathbf{v}\|_{\mathbf{V}(h)} + h^{-1}\|\mathbf{v} - \mathbf{A}\mathbf{v}\|_0 \leq C\|\mathbf{v}\|_{\mathbf{V}(h)},$$
$$\|q - Aq\|_{Q(h)} + \|Aq\|_{Q(h)} \leq C\|q\|_{Q(h)}$$

for all $\mathbf{v} \in \mathbf{V}_h$ and $q \in Q_h$.

We will further need the following consequence of Theorem 5.1, which follows from the fact that $[\![\mathbf{w}]\!]_T = \mathbf{0}$ on $\mathcal{F}_h$, for any $\mathbf{w} \in \mathbf{V}$, and the definition of the DG-norm $\|\cdot\|_{\mathbf{V}(h)}$.

Corollary 5.3. *Let $\mathbf{v} \in \mathbf{V}_h$ and $\mathbf{w} \in \mathbf{V}$. Let $\mathbf{A}$ be the conforming approximant from Theorem 5.1. Then we have*

$$\|\mathbf{v} - \mathbf{A}\mathbf{v}\|_{\mathbf{V}(h)} \leq C\|\mathbf{v} - \mathbf{w}\|_{\mathbf{V}(h)},$$
$$\|\mathbf{v} - \mathbf{A}\mathbf{v}\|_0 \leq Ch\|\mathbf{v} - \mathbf{w}\|_{\mathbf{V}(h)},$$

with a constant $C > 0$ that is independent of the mesh size.

5.2. Standard approximation operators

Next, we introduce standard approximation operators for the space $\mathbf{V}$. We start by recalling the properties of the curl-conforming Nédélec interpolant $\mathbf{\Pi}_{\rm N}$ of the second kind.

Lemma 5.4. *There exists a constant $C > 0$, independent of the mesh size, such that, for any $\mathbf{v} \in \mathbf{V} \cap H^t(\Omega)^3$ with $\nabla \times \mathbf{v} \in H^t(\Omega)^3$, $t > \frac{1}{2}$,*

$$\|\mathbf{v} - \mathbf{\Pi}_{\rm N}\mathbf{v}\|_{\rm curl} \leq C\, h^{\min\{t,\ell\}} \big[\|\mathbf{v}\|_t + \|\nabla \times \mathbf{v}\|_t\big], \tag{27}$$

$$\|\nabla \times (\mathbf{v} - \mathbf{\Pi}_{\rm N}\mathbf{v})\|_0 \leq C\, h^{\min\{t,\ell\}} \|\nabla \times \mathbf{v}\|_t. \tag{28}$$

Moreover, there exists a constant $C > 0$, independent of the mesh size, such that, for any $\mathbf{v} \in \mathbf{V} \cap H^{1+t}(\Omega)^3$ with $t > 0$,

$$\|\mathbf{v} - \mathbf{\Pi}_{\rm N}\mathbf{v}\|_0 \leq C\, h^{\min\{t,\ell\}+1} \|\mathbf{v}\|_{1+t}. \tag{29}$$

A proof of the first two results in (27) and (28) can be found in [18], Theorem 5.41, Remark 5.42 and Theorem 8.15, whereas (29) has been shown in [14], *Lem.* 4.1.

Furthermore, for any $\mathbf{v} \in \mathbf{V}$, we define the Galerkin projection $\mathbf{\Pi}^c\mathbf{v} \in \mathbf{V}_h^c$ by

$$(\nabla \times (\mathbf{v} - \mathbf{\Pi}^c\mathbf{v}), \nabla \times \mathbf{w}) + (\mathbf{v} - \mathbf{\Pi}^c\mathbf{v}, \mathbf{w}) = 0 \qquad \forall \mathbf{w} \in \mathbf{V}_h^c. \tag{30}$$

An immediate consequence of this definition is that

$$\|\mathbf{v} - \mathbf{\Pi}^c\mathbf{v}\|_{\rm curl} = \inf_{\mathbf{w} \in \mathbf{V}_h^c} \|\mathbf{v} - \mathbf{w}\|_{\rm curl}.$$

Thus, from property (27) in Lemma 5.4 we obtain the following approximation result:

Lemma 5.5. *There exists a constant $C > 0$, independent of the mesh size, such that, for any $\mathbf{v} \in \mathbf{V} \cap H^t(\Omega)^3$ with $\nabla \times \mathbf{v} \in H^t(\Omega)^3$, $t > \frac{1}{2}$,*

$$\|\mathbf{v} - \mathbf{\Pi}^c\mathbf{v}\|_{\rm curl} \leq C\, h^{\min\{t,\ell\}} \big[\|\mathbf{v}\|_t + \|\nabla \times \mathbf{v}\|_t\big].$$

Next, recalling that $\mathbf{\Pi}_h$ denotes the L^2-projection onto $\mathbf{V}_h$, we state the following approximation result.

Lemma 5.6. *There exists a constant $C > 0$, independent of the local mesh sizes h_K, such that, for any $\mathbf{v} \in H^t(K)^3$, $K \in \mathcal{T}_h$, $t > \frac{1}{2}$,*

$$\|\mathbf{v} - \mathbf{\Pi}_h\mathbf{v}\|_{0,K}^2 + h_K \|\mathbf{v} - \mathbf{\Pi}_h\mathbf{v}\|_{0,\partial K}^2 \leq C\, h_K^{2\min\{t,\ell+1\}}\, |\mathbf{v}|_{t,K}^2.$$

Proof. The bound on $\|\mathbf{v} - \mathbf{\Pi}_h\mathbf{v}\|_{0,K}$ for any integer $t \geq 0$ is well-known; see [7]. For any fractional $t > 0$, it follows from the standard operator-interpolation theory (see, *e.g.*, [5], Chap. 12) applied to the quotient spaces $H^{\lfloor t \rfloor}(K)/\mathcal{P}^\ell(K)$ and $H^{\lfloor t \rfloor+1}(K)/\mathcal{P}^\ell(K)$ endowed with the corresponding Sobolev seminorms (see [16], Vol. 1, Prop. 13.2), observing that the interpolation norm in $H^t(K)/\mathcal{P}^\ell(K)$ scales, with respect to h_K, like the standard Sobolev $H^t(K)$–seminorm.

The bound for $\|\mathbf{v}-\mathbf{\Pi}_h\mathbf{v}\|_{0,\partial K}$, for any $t \geq \frac{1}{2}$, can be derived by scaling arguments, using the continuity of the trace operator from $H^t(\widehat{K})$ in $L^2(\partial\widehat{K})$, and the stability of the L^2–projection in the $H^t(\widehat{K})$–norm, where $\widehat{K}$ denotes the reference element. □

Finally, we recall the following result that allows us to approximate discretely divergence-free functions by exactly divergence-free ones.

Lemma 5.7. *For any function $\mathbf{v} \in \mathbf{X}_h$, define $\mathbf{H}\mathbf{v} \in \mathbf{V} \cap H(\mathrm{div}^0;\Omega)$ by $\nabla \times \mathbf{H}\mathbf{v} = \nabla \times \mathbf{v}$. Then, there exists a constant $C > 0$, independent of the mesh size, such that*

$$\|\mathbf{v}-\mathbf{H}\mathbf{v}\|_0 \leq Ch^\sigma \|\nabla \times \mathbf{v}\|_0,$$

with the parameter σ from (13)*. Moreover, we have that $\|\mathbf{H}\mathbf{v}\|_0 \leq \|\mathbf{v}\|_0$.*

The result in Lemma 5.7 is obtained by proceeding as in [10], *Lem.* 4.5 and [18], *Lem.* 7.6, using Nédélec's second family of elements. The L^2-stability of $\mathbf{H}$ is a consequence of the L^2-orthogonality of the continuous Helmholtz decomposition.

6. Proof of Theorem 4.1 (energy norm error estimate)

In this section, we prove the result of Theorem 4.1 by proceeding along the lines of [14], Sect. 5, [19] and [18], Sect. 7.2. To this end, we define

$$\mathcal{D}_h(\mathbf{u}-\mathbf{u}_h) := \sup_{\mathbf{0}\neq\mathbf{v}\in\mathbf{V}_h} \frac{(\mathbf{u}-\mathbf{u}_h,\mathbf{v})}{\|\mathbf{v}\|_{\mathbf{V}(h)}}. \tag{31}$$

We start by proving a preliminary energy norm error bound in terms of $\mathcal{E}_{1,h}(\mathbf{u})$, $\mathcal{E}_{2,h}(\mathbf{u})$ and $\mathcal{D}_h(\mathbf{u}-\mathbf{u}_h)$. Then, we estimate these quantities separately; in particular, a duality argument will be used for bounding $\mathcal{D}_h(\mathbf{u}-\mathbf{u}_h)$.

6.1. Preliminary error bound

We first prove the following error bound.

Proposition 6.1. *Let $(\mathbf{u},p)$ be the analytical solution of* (6)–(9)*, and let $(\mathbf{u}_h,p_h)$ be the solution of* (14) *obtained with $\alpha \geq \alpha_{\min} > 0$ and $\gamma > 0$. Then we have that*

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{V}(h)} + \|p-p_h\|_{Q(h)} \leq C\big[&\|\mathbf{u}-\mathbf{v}\|_{\mathbf{V}(h)} + \|p-q\|_{Q(h)} \\ &+ \mathcal{E}_{1,h}(\mathbf{u}) + \mathcal{E}_{2,h}(\mathbf{u}) + \mathcal{D}_h(\mathbf{u}-\mathbf{u}_h)\big]\end{aligned}$$

for all $\mathbf{v} \in \mathbf{V}_h^c$ and all $q \in Q_h^c$, with $\mathcal{E}_{1,h}$, $\mathcal{E}_{2,h}$ and $\mathcal{D}_h$ defined in (20) *and* (31)*, respectively. Here, the constant $C > 0$ is independent of the mesh size.*

Proof. We decompose $\mathbf{u}_h$ and p_h into a conforming part and a remainder by setting

$$\mathbf{u}_h = \mathbf{u}_h^c + \mathbf{u}_h^\perp, \qquad p_h = p_h^c + p_h^\perp, \tag{32}$$

where $\mathbf{u}_h^c = \mathbf{A}\mathbf{u}_h$, $\mathbf{u}_h^\perp = \mathbf{u}_h - \mathbf{A}\mathbf{u}_h$, $p_h^c = Ap_h$, $p_h^\perp = p_h - Ap_h$, $\mathbf{A}$ and A being the approximants from Theorem 5.1. We now proceed in three steps.

Step 1. *Estimate of $\|p_h^\perp\|_{Q(h)}$ and $\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{V}(h)}$.* We claim that

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{V}(h)} + \|p_h^\perp\|_{Q(h)} \leq C\big[&\|\mathbf{u}-\mathbf{v}\|_{\mathbf{V}(h)} + \|p-q\|_{Q(h)} \\ &+ \mathcal{E}_{1,h}(\mathbf{u}) + \mathcal{E}_{2,h}(\mathbf{u}) + \mathcal{D}_h(\mathbf{u}-\mathbf{u}_h)\big]\end{aligned} \tag{33}$$

for all $\mathbf{v} \in \mathbf{V}_h^c$ and $q \in Q_h^c$, with a positive constant C independent of the mesh size.

We start by showing that (33) holds for any $\mathbf{v} \in \mathbf{V}_h^c \cap \mathbf{Z}_h$ and $q \in Q_h^c$. To this end, fix $\mathbf{v} \in \mathbf{V}_h^c \cap \mathbf{Z}_h$ and $q \in Q_h^c$, then

$$C\|p_h^\perp\|_{Q(h)}^2 \le c_h(p_h^\perp, p_h^\perp) = c_h(p_h - q, p_h - q).$$

This, together with the Gårding-type inequality in *Prop.* 3.5, gives the bound

$$\begin{aligned} \min\{C_G, C\}\big[\|\mathbf{u}_h - \mathbf{v}\|_{\mathbf{V}(h)}^2 + \|p_h^\perp\|_{Q(h)}^2\big] &\le C_G\|\mathbf{u}_h - \mathbf{v}\|_{\mathbf{V}(h)}^2 + C\|p_h^\perp\|_{Q(h)}^2 \\ &\le \widetilde{a}_h(\mathbf{u}_h - \mathbf{v}, \mathbf{u}_h - \mathbf{v}) + c_h(p_h - q, p_h - q) + (\mathbf{u}_h - \mathbf{v}, \mathbf{u}_h - \mathbf{v}) \\ &\equiv T_1 + T_2 + T_3, \end{aligned} \tag{34}$$

where

$$\begin{aligned} T_1 &= \widetilde{a}_h(\mathbf{u}_h - \mathbf{v}, \mathbf{u}_h - \mathbf{v}) - k^2(\mathbf{u}_h - \mathbf{v}, \mathbf{u}_h - \mathbf{v}) + \widetilde{b}_h(\mathbf{u}_h - \mathbf{v}, p_h - q), \\ T_2 &= -\widetilde{b}_h(\mathbf{u}_h - \mathbf{v}, p_h - q) + c_h(p_h - q, p_h - q), \\ T_3 &= (k^2 + 1)(\mathbf{u}_h - \mathbf{v}, \mathbf{u}_h - \mathbf{v}). \end{aligned}$$

We now proceed to bound the three terms T_1, T_2, and T_3.

For T_1, the error equation (18) and the continuity properties in *Prop.* 3.4 yield

$$\begin{aligned} T_1 &= -R_1(\mathbf{u}, p; \mathbf{u}_h - \mathbf{v}) + \widetilde{a}_h(\mathbf{u} - \mathbf{v}, \mathbf{u}_h - \mathbf{v}) \\ &\quad - k^2(\mathbf{u} - \mathbf{v}, \mathbf{u}_h - \mathbf{v}) + \widetilde{b}_h(\mathbf{u}_h - \mathbf{v}, p - q) \\ &\le \|\mathbf{u}_h - \mathbf{v}\|_{\mathbf{V}(h)}\big[C_R\mathcal{E}_{1,h}(\mathbf{u}) + (C_A + k^2)\|\mathbf{u} - \mathbf{v}\|_{\mathbf{V}(h)} + C_B\|p - q\|_{Q(h)}\big]. \end{aligned} \tag{35}$$

Similarly, using the error equation in (19), term T_2 can be written as

$$\begin{aligned} T_2 &= R_2(\mathbf{u}; p_h - q) - \widetilde{b}_h(\mathbf{u} - \mathbf{v}, p_h - q) + c_h(p - q, p_h - q) \\ &= R_2(\mathbf{u}; p_h - q) - \widetilde{b}_h(\mathbf{u} - \mathbf{v}, p_h - q), \end{aligned}$$

where we also have used the fact that $c_h(p - q, p_h - q) = 0$ (since $p - q \in Q$). Then, we observe that

$$\widetilde{b}_h(\mathbf{u} - \mathbf{v}, p_h - q) = \widetilde{b}_h(\mathbf{u} - \mathbf{v}, p_h^c - q) + \widetilde{b}_h(\mathbf{u} - \mathbf{v}, p_h^\perp) = \widetilde{b}_h(\mathbf{u} - \mathbf{v}, p_h^\perp),$$

since $\mathbf{u}$ is divergence-free, see (4), and $\mathbf{v}$ belongs to the kernel $\mathbf{Z}_h$. Furthermore, we conclude from (17) that $R_2(\mathbf{u}; p_h - q) = R_2(\mathbf{u}; p_h^\perp)$. Hence, we obtain

$$T_2 = R_2(\mathbf{u}; p_h^\perp) - \widetilde{b}_h(\mathbf{u} - \mathbf{v}, p_h^\perp),$$

and the continuity properties in *Prop.* 3.4 yield

$$T_2 \le \|p_h^\perp\|_{Q(h)}\big[C_R\mathcal{E}_{2,h}(\mathbf{u}) + C_B\|\mathbf{u} - \mathbf{v}\|_{\mathbf{V}(h)}\big]. \tag{36}$$

Term T_3 can be estimated in a similar fashion:

$$\begin{aligned} T_3 &= (k^2 + 1)(\mathbf{u}_h - \mathbf{u}, \mathbf{u}_h - \mathbf{v}) + (k^2 + 1)(\mathbf{u} - \mathbf{v}, \mathbf{u}_h - \mathbf{v}) \\ &\le (k^2 + 1)\|\mathbf{u}_h - \mathbf{v}\|_{\mathbf{V}(h)}\big[\mathcal{D}_h(\mathbf{u} - \mathbf{u}_h) + \|\mathbf{u} - \mathbf{v}\|_{\mathbf{V}(h)}\big]. \end{aligned} \tag{37}$$

By combining (34) with the estimates in (35)–(37), and by dividing the resulting inequality by $(\|\mathbf{u}_h - \mathbf{v}\|_{\mathbf{V}(h)}^2 + \|p_h^\perp\|_{Q(h)}^2)^{\frac{1}{2}}$, we obtain that

$$\begin{aligned} \|\mathbf{u}_h - \mathbf{v}\|_{\mathbf{V}(h)} + \|p_h^\perp\|_{Q(h)} \le C\big[&(2k^2 + 1 + C_A + C_B)\|\mathbf{u} - \mathbf{v}\|_{\mathbf{V}(h)} + C_B\|p - q\|_{Q(h)} \\ &+ C_R\mathcal{E}_{1,h}(\mathbf{u}) + C_R\mathcal{E}_{2,h}(\mathbf{u}) + (k^2 + 1)\mathcal{D}_h(\mathbf{u} - \mathbf{u}_h)\big]. \end{aligned}$$

This bound and the triangle inequality

$$\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{V}(h)} \leq \|\mathbf{u}-\mathbf{v}\|_{\mathbf{V}(h)} + \|\mathbf{u}_h-\mathbf{v}\|_{\mathbf{V}(h)}$$

result in

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{V}(h)} + \|p_h^\perp\|_{Q(h)} \leq C\big[&\|\mathbf{u}-\mathbf{v}\|_{\mathbf{V}(h)} + \|p-q\|_{Q(h)}\\ &+ \mathcal{E}_{1,h}(\mathbf{u}) + \mathcal{E}_{2,h}(\mathbf{u}) + \mathcal{D}_h(\mathbf{u}-\mathbf{u}_h)\big].\end{aligned}$$

This shows (33) for all $\mathbf{v} \in \mathbf{V}_h^c \cap \mathbf{Z}_h$ and all $q \in Q_h^c$.

In order to complete the proof of (33), it remains to show that the estimate (33) is also valid for any $\mathbf{v} \in \mathbf{V}_h^c$. To this end, fix $\mathbf{v} \in \mathbf{V}_h^c$ and choose $\mathbf{r} \in \mathbf{V}_h^c$ such that

$$\begin{aligned}&\widetilde{b}_h(\mathbf{r}, s) = \widetilde{b}_h(\mathbf{u}-\mathbf{v}, s) \quad \forall s \in Q_h^c,\\ &\|\mathbf{r}\|_{\mathbf{V}(h)} \leq C_S^{-1} C_B \|\mathbf{u}-\mathbf{v}\|_{\mathbf{V}(h)};\end{aligned}$$

the existence of such a $\mathbf{r}$ is guaranteed by the inf-sup condition in Lemma 3.6. We set $\mathbf{w} := \mathbf{r}+\mathbf{v}$; by construction, $\mathbf{w} \in \mathbf{V}_h^c \cap \mathbf{Z}_h$. Thereby,

$$\|\mathbf{u}-\mathbf{w}\|_{\mathbf{V}(h)} \leq \|\mathbf{u}-\mathbf{v}\|_{\mathbf{V}(h)} + \|\mathbf{r}\|_{\mathbf{V}(h)} \leq (1 + C_S^{-1} C_B)\|\mathbf{u}-\mathbf{v}\|_{\mathbf{V}(h)},$$

from which (33) follows.

Step 2. *Estimate of* $\|p-p_h\|_{Q(h)}$. Next, we address the error in the multiplier p and show that, for any $q \in Q_h^c$,

$$\|p-p_h\|_{Q(h)} \leq C\big[\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{V}(h)} + \|p-q\|_{Q(h)} + \|p_h^\perp\|_{Q(h)} + k^2\mathcal{D}_h(\mathbf{u}-\mathbf{u}_h)\big]. \tag{38}$$

To prove (38), fix $q \in Q_h^c$. From the triangle inequality and the decomposition $p_h = p_h^c + p_h^\perp$, we have

$$\|p-p_h\|_{Q(h)} \leq \|p-q\|_{Q(h)} + \|q-p_h^c\|_{Q(h)} + \|p_h^\perp\|_{Q(h)}. \tag{39}$$

The inf-sup condition (22) in Lemma 3.6 implies that

$$\begin{aligned}C_S\,\|q-p_h^c\|_{Q(h)} &\leq \sup_{\mathbf{0}\neq\mathbf{v}\in\mathbf{V}_h^c} \frac{\widetilde{b}_h(\mathbf{v}, q-p_h^c)}{\|\mathbf{v}\|_{\mathbf{V}(h)}}\\ &= \sup_{\mathbf{0}\neq\mathbf{v}\in\mathbf{V}_h^c} \frac{\widetilde{b}_h(\mathbf{v}, q-p) + \widetilde{b}_h(\mathbf{v}, p-p_h) + \widetilde{b}_h(\mathbf{v}, p_h^\perp)}{\|\mathbf{v}\|_{\mathbf{V}(h)}}.\end{aligned}$$

Notice that the error equation (18) yields, for $\mathbf{v} \in \mathbf{V}_h^c$,

$$\widetilde{b}_h(\mathbf{v}, p-p_h) = -\widetilde{a}_h(\mathbf{u}-\mathbf{u}_h, \mathbf{v}) + k^2(\mathbf{u}-\mathbf{u}_h, \mathbf{v}),$$

where we have used the fact that $R_1(\mathbf{u}, p; \mathbf{v}) = 0$ for $\mathbf{v} \in \mathbf{V}_h^c$. Hence,

$$C_S\,\|q-p_h^c\|_{Q(h)} \leq \sup_{\mathbf{0}\neq\mathbf{v}\in\mathbf{V}_h^c} \frac{\widetilde{b}_h(\mathbf{v}, q-p) - \widetilde{a}_h(\mathbf{u}-\mathbf{u}_h, \mathbf{v}) + k^2(\mathbf{u}-\mathbf{u}_h, \mathbf{v}) + \widetilde{b}_h(\mathbf{v}, p_h^\perp)}{\|\mathbf{v}\|_{\mathbf{V}(h)}}.$$

Then, the continuity properties of $\widetilde{a}_h$ and $\widetilde{b}_h$ in *Prop.* 3.4 and (31) yield the bound

$$C_S\,\|q-p_h^c\|_{Q(h)} \leq C_B\,\|p-q\|_{Q(h)} + C_A\,\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{V}(h)} + C_B\,\|p_h^\perp\|_{Q(h)} + k^2\mathcal{D}_h(\mathbf{u}-\mathbf{u}_h);$$

substituting this estimate into (39), we deduce (38).

Step 3. *Conclusion.* The statement of the proposition readily follows from (33) and (38) in step 1 and step 2, respectively. □

6.2. Estimate of $\mathcal{D}_h(\mathbf{u}-\mathbf{u}_h)$

To estimate $\mathcal{D}_h$, we proceed along the same lines as in the proof of [22], *Prop.* 4.2, and [14], *Prop.* 5.2; to this end, the following result holds.

Proposition 6.2. *There exists $C>0$, independent of the mesh size, such that*

$$\mathcal{D}_h(\mathbf{u}-\mathbf{u}_h) \leq C\,h^\sigma\big[\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{V}(h)} + \|p-p_h\|_{Q(h)}\big],$$

with the parameter $\sigma \in (1/2,1]$ from (12).

Proof. Fix $\mathbf{v}\in\mathbf{V}_h$, and let $\mathbf{v}^c = \mathbf{A}\mathbf{v} \in \mathbf{V}_h^c$ be the conforming approximation of $\mathbf{v}$ from Theorem 5.1. Employing the Helmholtz decomposition (24), we decompose $\mathbf{v}^c$ as

$$\mathbf{v}^c = \mathbf{v}_0^c + \nabla r, \tag{40}$$

with $\mathbf{v}_0^c \in \mathbf{X}_h$ and $r\in Q_h^c$. Employing (40), we obtain

$$\begin{aligned}(\mathbf{u}-\mathbf{u}_h,\mathbf{v}) &= (\mathbf{u}-\mathbf{u}_h,\mathbf{v}-\mathbf{v}^c) + (\mathbf{u}-\mathbf{u}_h,\mathbf{v}^c)\\ &= (\mathbf{u}-\mathbf{u}_h,\mathbf{v}-\mathbf{v}^c) + (\mathbf{u}-\mathbf{u}_h,\mathbf{v}_0^c)\\ &= (\mathbf{u}-\mathbf{u}_h,\mathbf{v}-\mathbf{v}^c) + (\mathbf{u}-\mathbf{u}_h,\mathbf{v}_0^c-\mathbf{H}\mathbf{v}_0^c) + (\mathbf{u}-\mathbf{u}_h,\mathbf{H}\mathbf{v}_0^c)\\ &\equiv T_1+T_2+T_3,\end{aligned}$$

with $\mathbf{H}\mathbf{v}_0^c$ from Lemma 5.7. Here, we have used the orthogonality property of the error $\mathbf{u}-\mathbf{u}_h$ in Lemma 3.7. We now proceed to estimate each of the terms T_1, T_2 and T_3 below.

Exploiting the Cauchy-Schwarz inequality and the approximation result in Corollary 5.2 yields

$$|T_1| \leq \|\mathbf{u}-\mathbf{u}_h\|_0\|\mathbf{v}-\mathbf{v}^c\|_0 \leq Ch\|\mathbf{u}-\mathbf{u}_h\|_0\|\mathbf{v}\|_{\mathbf{V}(h)}. \tag{41}$$

Similarly, using the Cauchy-Schwarz inequality and the approximation results stated in Lemma 5.7 and Corollary 5.2, we obtain

$$\begin{aligned}|T_2| &\leq \|\mathbf{u}-\mathbf{u}_h\|_0\|\mathbf{v}_0^c-\mathbf{H}\mathbf{v}_0^c\|_0 \leq Ch^\sigma\|\mathbf{u}-\mathbf{u}_h\|_0\|\nabla\times\mathbf{v}_0^c\|_0\\ &= Ch^\sigma\|\mathbf{u}-\mathbf{u}_h\|_0\|\nabla\times\mathbf{v}^c\|_0 \leq Ch^\sigma\|\mathbf{u}-\mathbf{u}_h\|_0\|\mathbf{v}^c\|_{\mathbf{V}(h)}\\ &\leq Ch^\sigma\|\mathbf{u}-\mathbf{u}_h\|_0\|\mathbf{v}\|_{\mathbf{V}(h)}.\end{aligned} \tag{42}$$

Next, we prove the bound

$$|T_3| \leq Ch^\sigma\|\mathbf{v}\|_{\mathbf{V}(h)}\big[\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{V}(h)} + \|p-p_h\|_{Q(h)}\big], \tag{43}$$

by employing a duality approach.

To this end, we set $\mathbf{w}=\mathbf{H}\mathbf{v}_0^c$ and let $\mathbf{z}$ denote the solution of the following problem:

$$\begin{aligned}\nabla\times\nabla\times\mathbf{z} - k^2\mathbf{z} &= \mathbf{w} \qquad \text{in } \Omega,\\ \mathbf{n}\times\mathbf{z} &= \mathbf{0} \qquad \text{on } \Gamma.\end{aligned} \tag{44}$$

Since $\mathbf{w} \in H(\mathrm{div}^0; \Omega)$, the solution $\mathbf{z}$ belongs to $H_0(\mathrm{curl}; \Omega) \cap H(\mathrm{div}^0; \Omega)$. As in [18], *Lem.* 7.7, we obtain from the embeddings in (13) that $\mathbf{z} \in H^\sigma(\Omega)^3$, $\nabla \times \mathbf{z} \in H^\sigma(\Omega)^3$ and

$$\|\mathbf{z}\|_\sigma + \|\nabla \times \mathbf{z}\|_\sigma \leq C\|\mathbf{w}\|_0, \tag{45}$$

for a stability constant $C > 0$ and the parameter $\sigma \in (1/2, 1]$ in (13).

Hence, multiplying the dual problem (44) with $\mathbf{e}_h := \mathbf{u} - \mathbf{u}_h$ and integrating by parts, since $\nabla \times \mathbf{z} \in H(\mathrm{curl}; \Omega)$, we obtain

$$(\mathbf{e}_h, \mathbf{w}) = (\nabla_h \times \mathbf{e}_h, \nabla \times \mathbf{z}) - k^2(\mathbf{e}_h, \mathbf{z}) - \int_{\mathcal{F}_h} [\![\mathbf{e}_h]\!]_T \cdot \{\!\{\nabla \times \mathbf{z}\}\!\}\,\mathrm{d}s.$$

Then, using the definitions of $\widetilde{a}_h$, $\widetilde{b}_h$, $\mathcal{L}$, $\mathcal{M}$, the properties of the L^2–projection $\mathbf{\Pi}_h$, integration by parts and the fact that $\mathbf{z} \in H_0(\mathrm{curl}; \Omega) \cap H(\mathrm{div}^0; \Omega)$, we obtain

$$\begin{aligned}
(\mathbf{e}_h, \mathbf{w}) &= \widetilde{a}_h(\mathbf{e}_h, \mathbf{z}) - k^2(\mathbf{e}_h, \mathbf{z}) + \widetilde{b}_h(\mathbf{z}, p - p_h) + (\mathbf{z}, \nabla_h(p - p_h) - \mathcal{M}(p - p_h)) \\
&\quad + (\mathcal{L}(\mathbf{e}_h), \nabla \times \mathbf{z}) - \int_{\mathcal{F}_h} [\![\mathbf{e}_h]\!]_T \cdot \{\!\{\nabla \times \mathbf{z}\}\!\}\,\mathrm{d}s \\
&= \widetilde{a}_h(\mathbf{e}_h, \mathbf{z}) - k^2(\mathbf{e}_h, \mathbf{z}) + \widetilde{b}_h(\mathbf{z}, p - p_h) \\
&\quad + \int_{\mathcal{F}_h} [\![p - p_h]\!]_N \cdot \{\!\{\mathbf{z} - \mathbf{\Pi}_h \mathbf{z}\}\!\}\,\mathrm{d}s \\
&\quad - \int_{\mathcal{F}_h} [\![\mathbf{e}_h]\!]_T \cdot \{\!\{\nabla \times \mathbf{z} - \mathbf{\Pi}_h(\nabla \times \mathbf{z})\}\!\}\,\mathrm{d}s.
\end{aligned}$$

We now define $\mathbf{z}_h = \mathbf{\Pi}_{\mathrm{N}} \mathbf{z} \in \mathbf{V}_h^{\mathrm{c}}$ to be the Nédélec interpolant of the second kind of $\mathbf{z}$, according to Lemma 5.4. Owing to the error equation (18) and the fact that $R_1(\mathbf{u}, p; \mathbf{z}_h) = 0$ (since $\mathbf{z}_h \in \mathbf{V}_h^{\mathrm{c}}$), we have

$$\begin{aligned}
(\mathbf{e}_h, \mathbf{w}) &= \widetilde{a}_h(\mathbf{e}_h, \mathbf{z} - \mathbf{z}_h) - k^2(\mathbf{e}_h, \mathbf{z} - \mathbf{z}_h) + \widetilde{b}_h(\mathbf{z} - \mathbf{z}_h, p - p_h) \\
&\quad + \int_{\mathcal{F}_h} [\![p - p_h]\!]_N \cdot \{\!\{\mathbf{z} - \mathbf{\Pi}_h \mathbf{z}\}\!\}\,\mathrm{d}s - \int_{\mathcal{F}_h} [\![\mathbf{e}_h]\!]_T \cdot \{\!\{\nabla \times \mathbf{z} - \mathbf{\Pi}_h(\nabla \times \mathbf{z})\}\!\}\,\mathrm{d}s.
\end{aligned}$$

Employing the weighted Cauchy-Schwarz inequality, the approximation properties in Lemma 5.6 and the stability bound (45), we get

$$\begin{aligned}
\left| \int_{\mathcal{F}_h} [\![p - p_h]\!]_N \cdot \{\!\{\mathbf{z} - \mathbf{\Pi}_h \mathbf{z}\}\!\}\,\mathrm{d}s \right| &\leq C \left(\int_{\mathcal{F}_h} \mathtt{h}^{-1} |[\![p - p_h]\!]_N|^2\,\mathrm{d}s \right)^{\frac{1}{2}} \left(\sum_{K \in \mathcal{T}_h} h_K \|\mathbf{z} - \mathbf{\Pi}_h \mathbf{z}\|_{0,\partial K}^2 \right)^{\frac{1}{2}} \\
&\leq C h^\sigma \|p - p_h\|_{Q(h)} \|\mathbf{z}\|_\sigma \leq C h^\sigma \|p - p_h\|_{Q(h)} \|\mathbf{w}\|_0.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\left| \int_{\mathcal{F}_h} [\![\mathbf{e}_h]\!]_T \cdot \{\!\{\nabla \times \mathbf{z} - \mathbf{\Pi}_h(\nabla \times \mathbf{z})\}\!\}\,\mathrm{d}s \right| &\leq C \left(\int_{\mathcal{F}_h} \mathtt{h}^{-1} |[\![\mathbf{e}_h]\!]_T|^2\,\mathrm{d}s \right)^{\frac{1}{2}} \left(\sum_{K \in \mathcal{T}_h} h_K \|\nabla \times \mathbf{z} - \mathbf{\Pi}_h(\nabla \times \mathbf{z})\|_{0,\partial K}^2 \right)^{\frac{1}{2}} \\
&\leq C h^\sigma \|\mathbf{e}_h\|_{\mathbf{V}(h)} \|\nabla \times \mathbf{z}\|_\sigma \leq C h^\sigma \|\mathbf{e}_h\|_{\mathbf{V}(h)} \|\mathbf{w}\|_0.
\end{aligned}$$

Furthermore, the continuity of $\widetilde{a}_h$ and $\widetilde{b}_h$ in *Prop.* 3.4, the approximation property (27) in Lemma 5.4 and the stability estimate (45) give

$$\widetilde{a}_h(\mathbf{e}_h, \mathbf{z} - \mathbf{z}_h) - k^2(\mathbf{e}_h, \mathbf{z} - \mathbf{z}_h) + \widetilde{b}_h(\mathbf{z} - \mathbf{z}_h, p - p_h) \leq C h^\sigma \|\mathbf{w}\|_0 \big[\|\mathbf{e}_h\|_{\mathbf{V}(h)} + \|p - p_h\|_{Q(h)}\big].$$

Hence, the above bounds yield

$$(\mathbf{u} - \mathbf{u}_h, \mathbf{w}) \le C h^{\sigma} \|\mathbf{w}\|_0 \big[\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{V}(h)} + \|p - p_h\|_{Q(h)}\big].$$

Since $\|\mathbf{w}\|_0 \le \|\mathbf{v}_0^c\|_0 \le \|\mathbf{v}^c\|_0 \le C\|\mathbf{v}\|_{\mathbf{V}(h)}$, in view of Lemma 5.7, the $L^2(\Omega)^3$–orthogonality of the Helmholtz decomposition (40), and Corollary 5.2, we conclude that (43) holds.

By combining (41), (42) and (43), we obtain

$$|(\mathbf{u} - \mathbf{u}_h, \mathbf{v})| \le C\, h^{\sigma} \|\mathbf{v}\|_{\mathbf{V}(h)} \big[\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{V}(h)} + \|p - p_h\|_{Q(h)}\big]$$

for all $\mathbf{v} \in \mathbf{V}_h$, which immediately implies the desired bound for $\mathcal{D}_h(\mathbf{u} - \mathbf{u}_h)$. □

6.3. Conclusion of the Proof of Theorem 4.1

From the abstract estimate in *Prop.* 6.1 and the bound on $\mathcal{D}_h(\mathbf{u}-\mathbf{u}_h)$ in *Prop.* 6.2, we have that there exists $h_0 > 0$ such that, for any $h < h_0$,

$$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{V}(h)} + \|p - p_h\|_{Q(h)} \le C\big[\|\mathbf{u} - \mathbf{v}\|_{\mathbf{V}(h)} + \|p - q\|_{Q(h)} + \mathcal{E}_{1,h}(\mathbf{u}) + \mathcal{E}_{2,h}(\mathbf{u})\big] \tag{46}$$

for all $\mathbf{v} \in \mathbf{V}_h^c$ and all $q \in Q_h^c$, with a constant $C > 0$ independent of the mesh size. Notice that h_0 also depends on the wave number and on the regularity exponent σ.

Let us now suppose that the analytical solution $(\mathbf{u}, p)$ satisfies (26). First, we use the Nédélec interpolant of the second kind in Lemma 5.4 to obtain

$$\inf_{\mathbf{v}\in\mathbf{V}_h^c} \|\mathbf{u} - \mathbf{v}\|_{\mathbf{V}(h)} \le \|\mathbf{u} - \mathbf{\Pi}_N \mathbf{u}\|_{\mathbf{V}(h)} \le C h^{\min\{s,\ell\}} \left[\|\mathbf{u}\|_s + \|\nabla \times \mathbf{u}\|_s\right].$$

Then, standard approximation properties for Q_h^c give

$$\inf_{q\in Q_h^c} \|p - q\|_{Q(h)} \le C h^{\min\{s,\ell\}} \|p\|_{s+1}.$$

Finally, using Lemma 5.6, we conclude that

$$\begin{aligned} \mathcal{E}_{1,h}(\mathbf{u}) &\le C\, h^{\min\{s,\ell+1\}} \|\nabla \times \mathbf{u}\|_s, \\ \mathcal{E}_{2,h}(\mathbf{u}) &\le C\, h^{\min\{s,\ell+1\}} \|\mathbf{u}\|_s. \end{aligned} \tag{47}$$

Inserting these bounds into (46) completes the proof of Theorem 4.1.

7. Proof of Theorem 4.4 (error estimate in the L^2-norm)

In this section, we present the proof of Theorem 4.4. Our analysis proceeds along the lines of the proof of [14], *Thm.* 3.5 which, in turn, relies on the ideas of [17], Sect. 4, where an L^2-error estimate is derived for conforming discretizations of the indefinite time-harmonic Maxwell equations.

7.1. The bound in Theorem 4.4

To derive the bound in Theorem 4.4, we start by splitting $\mathbf{u}_h$ into $\mathbf{u}_h = \mathbf{u}_h^c + \mathbf{u}_h^{\perp}$, where $\mathbf{u}_h^c := \mathbf{A}\mathbf{u}_h \in \mathbf{V}_h^c$ is the conforming approximation from Theorem 5.1 and $\mathbf{u}_h^{\perp} = \mathbf{u}_h - \mathbf{A}\mathbf{u}_h$. We further recall that $\mathbf{\Pi}_N \mathbf{u}$ denotes the curl-conforming Nédélec interpolant of the second kind, and write

$$\|\mathbf{u} - \mathbf{u}_h\|_0^2 = (\mathbf{u} - \mathbf{u}_h, \mathbf{u} - \mathbf{\Pi}_{\mathrm{N}}\mathbf{u}) + (\mathbf{u} - \mathbf{u}_h, \mathbf{\Pi}_{\mathrm{N}}\mathbf{u} - \mathbf{u}_h^c) - (\mathbf{u} - \mathbf{u}_h, \mathbf{u}_h^{\perp}).$$

By using the triangle inequality, the Cauchy-Schwarz inequality and the result in Corollary 5.3, we have

$$\|\mathbf{u}-\mathbf{u}_h\|_0 \le \|\mathbf{u}-\mathbf{\Pi}_{\rm N}\mathbf{u}\|_0 + Ch\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{V}(h)} + \frac{|(\mathbf{u}-\mathbf{u}_h, \mathbf{\Pi}_{\rm N}\mathbf{u}-\mathbf{u}_h^c)|}{\|\mathbf{u}-\mathbf{u}_h\|_0}, \tag{48}$$

with $C>0$ independent of the mesh size.

Defining

$$T := \frac{|(\mathbf{u}-\mathbf{u}_h, \mathbf{\Pi}_{\rm N}\mathbf{u}-\mathbf{u}_h^c)|}{\|\mathbf{u}-\mathbf{u}_h\|_0},$$

we claim that, for a sufficiently small mesh size,

$$T \le C\|\mathbf{u}-\mathbf{\Pi}_{\rm N}\mathbf{u}\|_0 + Ch^\sigma\big[\|\mathbf{u}-\mathbf{\Pi}_{\rm N}\mathbf{u}\|_{\rm curl} + \|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{V}(h)} + \|p-p_h\|_{Q(h)}\big], \tag{49}$$

with $C>0$ independent of the mesh size, and $\sigma \in (1/2, 1]$ denoting the embedding parameter in (13). Combining (48), (49) and using the approximation property (29) for $\mathbf{\Pi}_{\rm N}$ in Lemma 5.4 yields

$$\begin{aligned}\|\mathbf{u}-\mathbf{u}_h\|_0 \le & Ch^{\min\{s+\sigma,\ell+1\}}\|\mathbf{u}\|_{s+\sigma} + Ch^{\min\{s,\ell\}+\sigma}\big[\|\mathbf{u}\|_s + \|\nabla\times\mathbf{u}\|_s\big]\\ & + Ch^\sigma\big[\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{V}(h)} + \|p-p_h\|_{Q(h)}\big].\end{aligned}$$

Since $\|\mathbf{u}\|_s \le \|\mathbf{u}\|_{s+\sigma}$ and $\min\{s+\sigma, \ell+1\} \ge \min\{s,\ell\}+\sigma$, the error estimate in Theorem 4.4 follows from Theorem 4.1. It remains to prove the bound in (49); this is undertaken in the following section.

7.2. Proof of the auxiliary bound in (49)

In order to prove (49), we invoke the Helmholtz decomposition in (24) and write

$$\mathbf{\Pi}_{\rm N}\mathbf{u} - \mathbf{u}_h^c = \mathbf{w}_0^c + \nabla\varphi, \tag{50}$$

with $\mathbf{w}_0^c \in \mathbf{X}_h$ and $\varphi \in Q_h^c$.

We then let $\mathbf{w} = \mathbf{H}\mathbf{w}_0^c$ be the exactly divergence-free approximation of $\mathbf{w}_0^c$ from Lemma 5.7. From (50) and the orthogonality property (ii) in Lemma 3.7, we obtain

$$(\mathbf{u}-\mathbf{u}_h, \mathbf{\Pi}_{\rm N}\mathbf{u}-\mathbf{u}_h^c) = (\mathbf{u}-\mathbf{u}_h, \mathbf{w}_0^c) = (\mathbf{u}-\mathbf{u}_h, \mathbf{w}_0^c - \mathbf{w}) + (\mathbf{u}-\mathbf{u}_h, \mathbf{w}).$$

Hence,

$$\frac{|(\mathbf{u}-\mathbf{u}_h, \mathbf{\Pi}_{\rm N}\mathbf{u}-\mathbf{u}_h^c)|}{\|\mathbf{u}-\mathbf{u}_h\|_0} \le \|\mathbf{w}_0^c - \mathbf{w}\|_0 + \|\mathbf{w}\|_0. \tag{51}$$

Therefore, it is sufficient to estimate $\|\mathbf{w}_0^c - \mathbf{w}\|_0$ and $\|\mathbf{w}\|_0$.

Step 1. *Estimate of* $\|\mathbf{w}_0^c - \mathbf{w}\|_0$. We claim that

$$\|\mathbf{w}_0^c - \mathbf{w}\|_0 \le Ch^\sigma\big[\|\mathbf{u}-\mathbf{\Pi}_{\rm N}\mathbf{u}\|_{\rm curl} + \|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{V}(h)}\big], \tag{52}$$

with a constant $C>0$ independent of the mesh size.

To see this, note that, in view of the definition of $\mathbf{H}$ and (50), we have

$$\nabla\times\mathbf{w} = \nabla\times\mathbf{w}_0^c = \nabla\times(\mathbf{\Pi}_{\rm N}\mathbf{u}-\mathbf{u}_h^c). \tag{53}$$

Thus, the result in Lemma 5.7, the triangle inequality and Corollary 5.3 yield

$$\begin{aligned}\|\mathbf{w}_0^c - \mathbf{w}\|_0 &\le Ch^\sigma\|\nabla\times(\mathbf{\Pi}_{\rm N}\mathbf{u}-\mathbf{u}_h^c)\|_0\\ &\le Ch^\sigma\big[\|\nabla\times(\mathbf{\Pi}_{\rm N}\mathbf{u}-\mathbf{u})\|_0 + \|\nabla_h\times(\mathbf{u}-\mathbf{u}_h)\|_0 + \|\nabla_h\times(\mathbf{u}_h-\mathbf{u}_h^c)\|_0\big]\\ &\le Ch^\sigma\big[\|\mathbf{u}-\mathbf{\Pi}_{\rm N}\mathbf{u}\|_{\rm curl} + \|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{V}(h)}\big],\end{aligned}$$

which proves (52).

Step 2. *Estimate of* $\|\mathbf{w}\|_0$. Next, we claim that, for a sufficiently small mesh size,

$$\|\mathbf{w}\|_0 \leq C\|\mathbf{u} - \mathbf{\Pi}_{\mathrm{N}}\mathbf{u}\|_0 + Ch^\sigma \left[\|\mathbf{u} - \mathbf{\Pi}_{\mathrm{N}}\mathbf{u}\|_{\mathrm{curl}} + \|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{V}(h)} + \|p - p_h\|_{Q(h)}\right], \tag{54}$$

with a constant $C > 0$ independent of the mesh size.

To prove (54) we employ a duality approach. To this end, let $\mathbf{z}$ be the solution of the dual problem (44) with right-hand side $\mathbf{w} = \mathbf{H}\mathbf{w}_0^c$. Again, $\mathbf{w} \in H(\mathrm{div}^0; \Omega)$ so that $\mathbf{z} \in H^\sigma(\Omega)^3$, $\nabla \times \mathbf{z} \in H^\sigma(\Omega)^3$, with $\sigma \in (1/2, 1]$ in (13), and the bound (45) holds. The dual problem (44) can be written in mixed formulation as

$$\nabla \times \nabla \times \mathbf{z} - k^2\mathbf{z} + \nabla r = \mathbf{w} \qquad \text{in } \Omega, \tag{55}$$

$$\nabla \cdot \mathbf{z} = 0 \qquad \text{in } \Omega, \tag{56}$$

$$\mathbf{n} \times \mathbf{z} = \mathbf{0} \qquad \text{on } \Gamma, \tag{57}$$

$$r = 0 \qquad \text{on } \Gamma. \tag{58}$$

Since $\mathbf{w} \in H(\mathrm{div}^0; \Omega)$, we actually have $r \equiv 0$.

Let us denote by $(\mathbf{z}_h, r_h) \in \mathbf{V}_h \times Q_h$ the discontinuous Galerkin approximation of (55)–(58) given by:

$$\begin{aligned} \widetilde{\mathcal{A}}_h(\mathbf{z}_h, \mathbf{v}) - \widetilde{b}_h(\mathbf{v}, r_h) &= (\mathbf{w}, \mathbf{v}), \\ \widetilde{b}_h(\mathbf{z}_h, q) - c_h(r_h, q) &= 0 \end{aligned} \tag{59}$$

for all $(\mathbf{v}, q) \in \mathbf{V}_h \times Q_h$. Here and in the following, we use the notation

$$\widetilde{\mathcal{A}}_h(\mathbf{z}, \mathbf{v}) = \widetilde{a}_h(\mathbf{z}, \mathbf{v}) - k^2(\mathbf{z}, \mathbf{v}).$$

Up to a sign change, the formulation (59) is of the same form as the one in (16). It can be readily seen that Theorem 4.1 and Corollary 4.3 apply to (59). Hence, for a sufficiently small mesh size, the discrete solution $(\mathbf{z}_h, r_h)$ exists and is unique. Moreover, the following *a priori* error bound holds:

$$\|\mathbf{z} - \mathbf{z}_h\|_{\mathbf{V}(h)} + \|r_h\|_{Q(h)} \leq Ch^\sigma \left[\|\mathbf{z}\|_\sigma + \|\nabla \times \mathbf{z}\|_\sigma\right] \leq Ch^\sigma \|\mathbf{w}\|_0. \tag{60}$$

Here, we have taken into account that $r \equiv 0$ and have also used the stability bound in (45).

After these preliminary considerations, we multiply the equation in (44) by $\mathbf{w}$ and integrate by parts. Recalling the equivalence of the forms a and $\widetilde{a}_h$ on $\mathbf{V} \times \mathbf{V}$, we obtain

$$\|\mathbf{w}\|_0^2 = \widetilde{\mathcal{A}}_h(\mathbf{z}, \mathbf{w}) = \widetilde{\mathcal{A}}_h(\mathbf{z} - \mathbf{\Pi}^c\mathbf{z}, \mathbf{w}) + \widetilde{\mathcal{A}}_h(\mathbf{\Pi}^c\mathbf{z}, \mathbf{w}), \tag{61}$$

with $\mathbf{\Pi}^c$ denoting the Galerkin projection from (30).

By the definition of the projection $\mathbf{\Pi}^c$ and the property $\nabla \times \mathbf{w} = \nabla \times \mathbf{w}_0^c$, we conclude that

$$\begin{aligned} \widetilde{\mathcal{A}}_h(\mathbf{z} - \mathbf{\Pi}^c\mathbf{z}, \mathbf{w}) &= -(\mathbf{z} - \mathbf{\Pi}^c\mathbf{z}, \mathbf{w}_0^c) - k^2(\mathbf{z} - \mathbf{\Pi}^c\mathbf{z}, \mathbf{w}) \\ &= -(\mathbf{z} - \mathbf{\Pi}^c\mathbf{z}, \mathbf{w}_0^c - \mathbf{w}) - (1 + k^2)(\mathbf{z} - \mathbf{\Pi}^c\mathbf{z}, \mathbf{w}). \end{aligned}$$

The approximation result for $\mathbf{\Pi}^c$ in Lemma 5.5 and the bound in (45) yield

$$\|\mathbf{z} - \mathbf{\Pi}^c\mathbf{z}\|_0 \leq \|\mathbf{z} - \mathbf{\Pi}^c\mathbf{z}\|_{\mathrm{curl}} \leq Ch^\sigma \|\mathbf{w}\|_0. \tag{62}$$

For later use, we further point out that the stability of $\mathbf{\Pi}^c$ in the norm $\|\cdot\|_{\mathrm{curl}}$ and the bound in (45) give

$$\|\mathbf{\Pi}^c\mathbf{z}\|_0 \leq C\|\mathbf{w}\|_0. \tag{63}$$

Hence, by using the Cauchy-Schwarz inequality and the estimates (52) and (62) we conclude that

$$\begin{aligned}|\widetilde{\mathcal{A}}_h(\mathbf{z}-\mathbf{\Pi}^c\mathbf{z},\mathbf{w})| &\leq \|\mathbf{z}-\mathbf{\Pi}^c\mathbf{z}\|_0\|\mathbf{w}-\mathbf{w}_0^c\|_0 + C\|\mathbf{z}-\mathbf{\Pi}^c\mathbf{z}\|_0\|\mathbf{w}\|_0 \\ &\leq Ch^{2\sigma}\|\mathbf{w}\|_0\big[\|\mathbf{u}-\mathbf{\Pi}_{\mathrm{N}}\mathbf{u}\|_{\mathrm{curl}} + \|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{V}(h)}\big] + Ch^{\sigma}\|\mathbf{w}\|_0^2. \end{aligned} \tag{64}$$

It remains to bound the term $\widetilde{\mathcal{A}}_h(\mathbf{\Pi}^c\mathbf{z},\mathbf{w})$ in (61). To this end, in view of (53) and (50), we first note that

$$\begin{aligned}\widetilde{\mathcal{A}}_h(\mathbf{\Pi}^c\mathbf{z},\mathbf{w}) &= (\nabla\times\mathbf{\Pi}^c\mathbf{z},\nabla\times\mathbf{w}) - k^2(\mathbf{\Pi}^c\mathbf{z},\mathbf{w}) \\ &= (\nabla\times\mathbf{\Pi}^c\mathbf{z},\nabla\times(\mathbf{\Pi}_{\mathrm{N}}\mathbf{u}-\mathbf{u}_h^c)) - k^2(\mathbf{\Pi}^c\mathbf{z},\mathbf{w}-\mathbf{w}_0^c) - k^2(\mathbf{\Pi}^c\mathbf{z},\mathbf{w}_0^c) \\ &= (\nabla\times\mathbf{\Pi}^c\mathbf{z},\nabla\times(\mathbf{\Pi}_{\mathrm{N}}\mathbf{u}-\mathbf{u}_h^c)) - k^2(\mathbf{\Pi}^c\mathbf{z},\mathbf{w}-\mathbf{w}_0^c) - k^2(\mathbf{\Pi}^c\mathbf{z},\mathbf{\Pi}_{\mathrm{N}}\mathbf{u}-\mathbf{u}_h^c) \\ &= \widetilde{\mathcal{A}}_h(\mathbf{\Pi}^c\mathbf{z},\mathbf{\Pi}_{\mathrm{N}}\mathbf{u}-\mathbf{u}_h^c) - k^2(\mathbf{\Pi}^c\mathbf{z},\mathbf{w}-\mathbf{w}_0^c).\end{aligned}$$

Here, we have used that

$$(\mathbf{\Pi}^c\mathbf{z},\nabla\varphi) = (\mathbf{z},\nabla\varphi) = 0, \tag{65}$$

which follows readily from the definition of $\mathbf{\Pi}^c$ and the fact that $\mathbf{z}\in H(\mathrm{div}^0;\Omega)$. Employing (63) and (52) gives

$$\begin{aligned}|\widetilde{\mathcal{A}}_h(\mathbf{\Pi}^c\mathbf{z},\mathbf{w})| &\leq |\widetilde{\mathcal{A}}_h(\mathbf{\Pi}^c\mathbf{z},\mathbf{\Pi}_{\mathrm{N}}\mathbf{u}-\mathbf{u}_h^c)| + C\|\mathbf{\Pi}^c\mathbf{z}\|_0\|\mathbf{w}-\mathbf{w}_0^c\|_0 \\ &\leq |\widetilde{\mathcal{A}}_h(\mathbf{\Pi}^c\mathbf{z},\mathbf{\Pi}_{\mathrm{N}}\mathbf{u}-\mathbf{u}_h^c)| + Ch^{\sigma}\|\mathbf{w}\|_0\left[\|\mathbf{u}-\mathbf{\Pi}_{\mathrm{N}}\mathbf{u}\|_{\mathrm{curl}} + \|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{V}(h)}\right].\end{aligned} \tag{66}$$

In order to estimate $|\widetilde{\mathcal{A}}_h(\mathbf{\Pi}^c\mathbf{z},\mathbf{\Pi}_{\mathrm{N}}\mathbf{u}-\mathbf{u}_h^c)|$, we consider the expansion

$$\begin{aligned}\widetilde{\mathcal{A}}_h(\mathbf{\Pi}^c\mathbf{z},\mathbf{\Pi}_{\mathrm{N}}\mathbf{u}-\mathbf{u}_h^c) &= \widetilde{\mathcal{A}}_h(\mathbf{\Pi}^c\mathbf{z},\mathbf{\Pi}_{\mathrm{N}}\mathbf{u}-\mathbf{u}) + \widetilde{\mathcal{A}}_h(\mathbf{\Pi}^c\mathbf{z},\mathbf{u}-\mathbf{u}_h) + \widetilde{\mathcal{A}}_h(\mathbf{\Pi}^c\mathbf{z},\mathbf{u}_h-\mathbf{u}_h^c) \\ &\equiv T_1 + T_2 + T_3,\end{aligned}$$

and estimate the terms T_1, T_2, and T_3 individually.

By further expanding T_1, we have

$$T_1 = \widetilde{\mathcal{A}}_h(\mathbf{\Pi}^c\mathbf{z}-\mathbf{z},\mathbf{\Pi}_{\mathrm{N}}\mathbf{u}-\mathbf{u}) + \widetilde{\mathcal{A}}_h(\mathbf{z},\mathbf{\Pi}_{\mathrm{N}}\mathbf{u}-\mathbf{u}).$$

Employing the variational formulation of the dual problem (44), we bound the second term as follows:

$$\widetilde{\mathcal{A}}_h(\mathbf{z},\mathbf{\Pi}_{\mathrm{N}}\mathbf{u}-\mathbf{u}) = (\mathbf{w},\mathbf{\Pi}_{\mathrm{N}}\mathbf{u}-\mathbf{u}) \leq \|\mathbf{w}\|_0\|\mathbf{\Pi}_{\mathrm{N}}\mathbf{u}-\mathbf{u}\|_0.$$

Hence, by Lemma 5.5 and (45), T_1 can be estimated by

$$\begin{aligned}|T_1| &\leq C\|\mathbf{\Pi}_{\mathrm{N}}\mathbf{u}-\mathbf{u}\|_{\mathrm{curl}}\|\mathbf{\Pi}^c\mathbf{z}-\mathbf{z}\|_{\mathrm{curl}} + \|\mathbf{w}\|_0\|\mathbf{\Pi}_{\mathrm{N}}\mathbf{u}-\mathbf{u}\|_0 \\ &\leq Ch^{\sigma}\|\mathbf{w}\|_0\|\mathbf{\Pi}_{\mathrm{N}}\mathbf{u}-\mathbf{u}\|_{\mathrm{curl}} + \|\mathbf{w}\|_0\|\mathbf{\Pi}_{\mathrm{N}}\mathbf{u}-\mathbf{u}\|_0.\end{aligned} \tag{67}$$

For T_2, we claim that

$$|T_2| = |\widetilde{b}_h(\mathbf{\Pi}^c\mathbf{z},p-p_h)| \leq Ch^{\sigma}\|\mathbf{w}\|_0\|p-p_h\|_{Q(h)}. \tag{68}$$

Indeed, using the symmetry of $\widetilde{\mathcal{A}}_h$ and the error equation (18), together with the fact that by (17) the residual $R_1(\mathbf{u},p;\mathbf{\Pi}^c\mathbf{z})$ vanishes, we have

$$|\widetilde{\mathcal{A}}_h(\mathbf{\Pi}^c\mathbf{z},\mathbf{u}-\mathbf{u}_h)| = |\widetilde{b}_h(\mathbf{\Pi}^c\mathbf{z},p-p_h)| \leq |\widetilde{b}_h(\mathbf{\Pi}^c\mathbf{z}-\mathbf{z},p-p_h)| + |\widetilde{b}_h(\mathbf{z},p-p_h)|.$$

The continuity of $\widetilde{b}$ from *Prop.* 3.4 and (62) then yield

$$|\widetilde{b}_h(\mathbf{\Pi}^c\mathbf{z}-\mathbf{z},p-p_h)| \leq C\|\mathbf{\Pi}^c\mathbf{z}-\mathbf{z}\|_{\mathrm{curl}}\|p_h-p_h\|_{Q(h)} \leq Ch^{\sigma}\|\mathbf{w}\|_0\|p-p_h\|_{Q(h)}.$$

Estimating the residual $R_2(\mathbf{z};q)$ of the dual problem as in *Prop.* 3.4 and (47) and using the bound in (45), results in

$$\begin{aligned}|\widetilde{b}_h(\mathbf{z},p-p_h)| = |R_2(\mathbf{z};p-p_h)| &\le C_R\mathcal{E}_{2,h}(\mathbf{z})\|p-p_h\|_{Q(h)}\\ &\le Ch^\sigma\|\mathbf{z}\|_\sigma\|p-p_h\|_{Q(h)} \le Ch^\sigma\|\mathbf{w}\|_0\|p-p_h\|_{Q(h)},\end{aligned}$$

which completes the proof of the bound (68) for T_2.

Finally, to bound T_3, we use the continuity property in *Prop.* 3.4, the discrete formulation (59) and the Cauchy-Schwarz inequality:

$$\begin{aligned}|T_3| &\le |\widetilde{\mathcal{A}}_h(\mathbf{\Pi}^c\mathbf{z}-\mathbf{z},\mathbf{u}_h-\mathbf{u}_h^c)| + |\widetilde{\mathcal{A}}_h(\mathbf{z}-\mathbf{z}_h,\mathbf{u}_h-\mathbf{u}_h^c)| + |\widetilde{\mathcal{A}}_h(\mathbf{z}_h,\mathbf{u}_h-\mathbf{u}_h^c)|\\ &\le C\|\mathbf{u}_h-\mathbf{u}_h^c\|_{\mathbf{V}(h)}\big[\|\mathbf{\Pi}^c\mathbf{z}-\mathbf{z}\|_{\mathrm{curl}}+\|\mathbf{z}-\mathbf{z}_h\|_{\mathbf{V}(h)}\big] + \|\mathbf{w}\|_0\|\mathbf{u}_h-\mathbf{u}_h^c\|_0 + |\widetilde{b}_h(\mathbf{u}_h-\mathbf{u}_h^c,r_h)|,\end{aligned} \tag{69}$$

with $\mathbf{z}_h$ denoting the first component of the approximation in (59). From Corollary 5.3 we have

$$\|\mathbf{u}_h-\mathbf{u}_h^c\|_{\mathbf{V}(h)} \le C\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{V}(h)}, \qquad \|\mathbf{u}_h-\mathbf{u}_h^c\|_0 \le Ch\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{V}(h)}.$$

This, combined with the continuity of $\widetilde{b}_h$ from *Prop.* 3.4, the fact that $r\equiv 0$, the energy estimate from Theorem 4.1 applied to (59), and the stability bound in (45), yields the following estimate for the last term in (69):

$$\begin{aligned}|\widetilde{b}_h(\mathbf{u}_h-\mathbf{u}_h^c,r_h)| &\le C\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{V}(h)}\|r-r_h\|_{Q(h)}\\ &\le Ch^\sigma\big[\|\mathbf{z}\|_\sigma+\|\nabla\times\mathbf{z}\|_\sigma+\|r\|_{\sigma+1}\big]\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{V}(h)}\\ &\le Ch^\sigma\|\mathbf{w}\|_0\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{V}(h)}.\end{aligned}$$

Therefore, again by applying Corollary 5.3 and Theorem 4.1 to (59), the stability estimate (45), and equation (62) we conclude that

$$|T_3| \le h^\sigma\|\mathbf{w}\|_0\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{V}(h)}. \tag{70}$$

Gathering the estimates (66)–(68), and (70) gives

$$|\widetilde{\mathcal{A}}(\mathbf{\Pi}^c\mathbf{z},\mathbf{w})| \le Ch^\sigma\|\mathbf{w}\|_0\big[\|\mathbf{u}-\mathbf{\Pi}_{\mathrm{N}}\mathbf{u}\|_{\mathrm{curl}}+\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{V}(h)}+\|p-p_h\|_{Q(h)}\big] + \|\mathbf{w}\|_0\|\mathbf{u}-\mathbf{\Pi}_{\mathrm{N}}\mathbf{u}\|_0. \tag{71}$$

Inserting (64) and (71) into (61) then shows that

$$\|\mathbf{w}\|_0 \le Ch^\sigma\|\mathbf{w}\|_0 + C\|\mathbf{u}-\mathbf{\Pi}_{\mathrm{N}}\mathbf{u}\|_0 + Ch^\sigma\big[\|\mathbf{u}-\mathbf{\Pi}_{\mathrm{N}}\mathbf{u}\|_{\mathrm{curl}}+\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{V}(h)}+\|p-p_h\|_{Q(h)}\big]. \tag{72}$$

Hence, for a sufficiently small mesh size, we obtain the result in (54).

Step 3. *Conclusion.* The bound (49) now follows from (51), (52) and (54). □

TABLE 1. Example 1. Convergence of $\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{V}(h)}$ with $k=1$.

	$\ell=1$		$\ell=2$		$\ell=3$	
Elements	$\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{V}(h)}$	r	$\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{V}(h)}$	r	$\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{V}(h)}$	r
26	1.876e-1	-	2.009e-2	-	5.045e-4	-
104	9.135e-2	1.04	5.004e-3	2.01	6.471e-5	2.96
416	4.456e-2	1.04	1.250e-3	2.00	8.131e-6	2.99
1664	2.194e-2	1.02	3.123e-4	2.00	1.017e-6	3.00

TABLE 2. Example 1. Convergence of $\|p-p_h\|_{Q(h)}$ with $k=1$.

	$\ell=1$		$\ell=2$		$\ell=3$	
Elements	$\|p-p_h\|_{Q(h)}$	r	$\|p-p_h\|_{Q(h)}$	r	$\|p-p_h\|_{Q(h)}$	r
26	9.226e-2	-	1.213e-2	-	2.728e-4	-
104	2.715e-2	1.76	1.332e-3	3.19	1.489e-5	4.19
416	6.774e-3	2.00	1.551e-4	3.10	8.132e-7	4.19
1664	1.609e-3	2.07	1.867e-5	3.05	4.638e-8	4.13

TABLE 3. Example 1. Convergence of $\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{V}(h)}$ with $k=2$.

	$\ell=1$		$\ell=2$		$\ell=3$	
Elements	$\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{V}(h)}$	r	$\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{V}(h)}$	r	$\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{V}(h)}$	r
26	1.131	-	1.265e-1	-	1.243e-2	-
104	5.405e-1	1.06	3.217e-2	1.98	1.582e-3	2.97
416	2.635e-1	1.04	8.078e-3	1.99	1.985e-4	2.99
1664	1.302e-1	1.02	2.022e-3	2.00	2.483e-5	3.00

8. Numerical experiments

In this section we present a series of numerical experiments to highlight the practical performance of the mixed DG method introduced and analyzed in this article for the numerical approximation of the indefinite time-harmonic Maxwell equations (6)–(9). For simplicity, we restrict ourselves to two-dimensional model problems; additionally, we note that throughout this section we select the constants appearing in the interior penalty stabilization functions defined in (15) as follows:

$$\alpha = 10\,\ell^2 \quad \text{and} \quad \gamma = 1.$$

The dependence of α on the polynomial degree ℓ has been chosen in order to guarantee the Gårding-type inequality stated in *Prop.* 3.5 holds independently of ℓ, *cf.* [11], for example.

8.1. **Example 1**

In this first example we select $\Omega \subset \mathbb{R}^2$ to be the square domain $(-1,1)^2$. Furthermore, we set $\mathbf{j}=\mathbf{0}$ and select suitable non-homogeneous boundary conditions for $\mathbf{u}$, *i.e.*, $\mathbf{n}\times\mathbf{u}=\mathbf{g}$, where $\mathbf{g}$ is a given tangential trace, so that the analytical solution to the two-dimensional analogue of (6)–(9) is given by the smooth field

$$\mathbf{u}(x,y) = (\sin(ky), \sin(kx))^T\,, \quad p = 0.$$

TABLE 4. Example 1. Convergence of $\|p - p_h\|_{Q(h)}$ with $k = 2$.

	$\ell = 1$		$\ell = 2$		$\ell = 3$	
Elements	$\|p - p_h\|_{Q(h)}$	r	$\|p - p_h\|_{Q(h)}$	r	$\|p - p_h\|_{Q(h)}$	r
26	5.745e-1	-	7.298e-2	-	6.752e-3	-
104	1.700e-1	1.76	8.377e-3	3.12	3.652e-4	4.21
416	4.232e-2	2.01	9.933e-4	3.08	2.026e-5	4.17
1664	1.002e-2	2.08	1.209e-4	3.04	1.174e-6	4.11

TABLE 5. Example 1. Convergence of $\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{V}(h)}$ with $k = 4$.

	$\ell = 1$		$\ell = 2$		$\ell = 3$	
Elements	$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{V}(h)}$	r	$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{V}(h)}$	r	$\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{V}(h)}$	r
26	3.902	-	1.276	-	1.429e-1	-
104	2.017	0.95	2.971e-1	2.10	2.289e-2	2.64
416	9.871e-1	1.03	7.401e-2	2.01	2.952e-3	2.96
1664	4.864e-1	1.02	1.849e-2	2.00	3.715e-4	2.99

TABLE 6. Example 1. Convergence of $\|p - p_h\|_{Q(h)}$ with $k = 4$.

	$\ell = 1$		$\ell = 2$		$\ell = 3$	
Elements	$\|p - p_h\|_{Q(h)}$	r	$\|p - p_h\|_{Q(h)}$	r	$\|p - p_h\|_{Q(h)}$	r
26	2.077	-	6.953e-1	-	5.923e-2	-
104	5.961e-1	1.80	6.828e-2	3.35	4.982e-3	3.57
416	1.541e-1	1.95	7.722e-3	3.14	3.105e-4	4.00
1664	3.796e-2	2.02	9.207e-4	3.07	1.909e-5	4.02

Here, the boundary conditions for $\mathbf{u}$ are enforced in the usual DG manner by adding boundary terms into the formulation (14); more precisely, the right–hand side of the first equation in (14) is replaced by the term

$$f_h(\mathbf{v}) = (\mathbf{j}, \mathbf{v}) - \int_{\mathcal{F}_h^{\mathcal{B}}} \mathbf{g} \cdot \nabla_h \times \mathbf{v} \, \mathrm{d}s + \int_{\mathcal{F}_h^{\mathcal{B}}} \mathtt{a} \, \mathbf{g} \cdot (\mathbf{n} \times \mathbf{v}) \, \mathrm{d}s,$$

see [11, 12] for details.

We investigate the asymptotic convergence of the mixed DG method on a sequence of successively finer (quasi-uniform) unstructured triangular meshes for $\ell = 1, 2, 3$ as the wave number k increases. To this end, in Tables 1 and 2, Tables 3 and 4, and Tables 5 and 6 we present numerical experiments for $k = 1, 2, 4$, respectively. For each wave number k we show the number of elements in the computational mesh, the corresponding DG-norms of the error in the numerical approximation to both $\mathbf{u}$ and p, and the numerical rate of convergence r. Here, we observe that (asymptotically) $\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{V}(h)}$ converges to zero at the optimal rate $\mathcal{O}(h^{\ell})$, for each fixed ℓ and each k, as h tends to zero, as predicted by Theorem 4.1. On the other hand, for this mixed–order method, $\|p - p_h\|_{Q(h)}$ converges to zero at the rate $\mathcal{O}(h^{\ell+1})$, for each ℓ and k, as h tends to zero; this rate is indeed optimal, though this is not reflected by Theorem 4.1, *cf.* [13]. In particular, we make two key observations: firstly, we note that for a given fixed mesh and fixed polynomial degree, an increase in the wave number k leads to an increase in the DG-norm of the error in the approximation to both $\mathbf{u}$ and p. Indeed, as pointed out in [14] and [1], where interior penalty and curl-conforming finite element methods, respectively, were employed for the numerical approximation of (1)–(2), the pre-asymptotic region increases as k increases. Secondly, we observe that the DG-norm of the error decreases when either the mesh is refined, or the polynomial degree is increased as we would expect for this smooth problem.

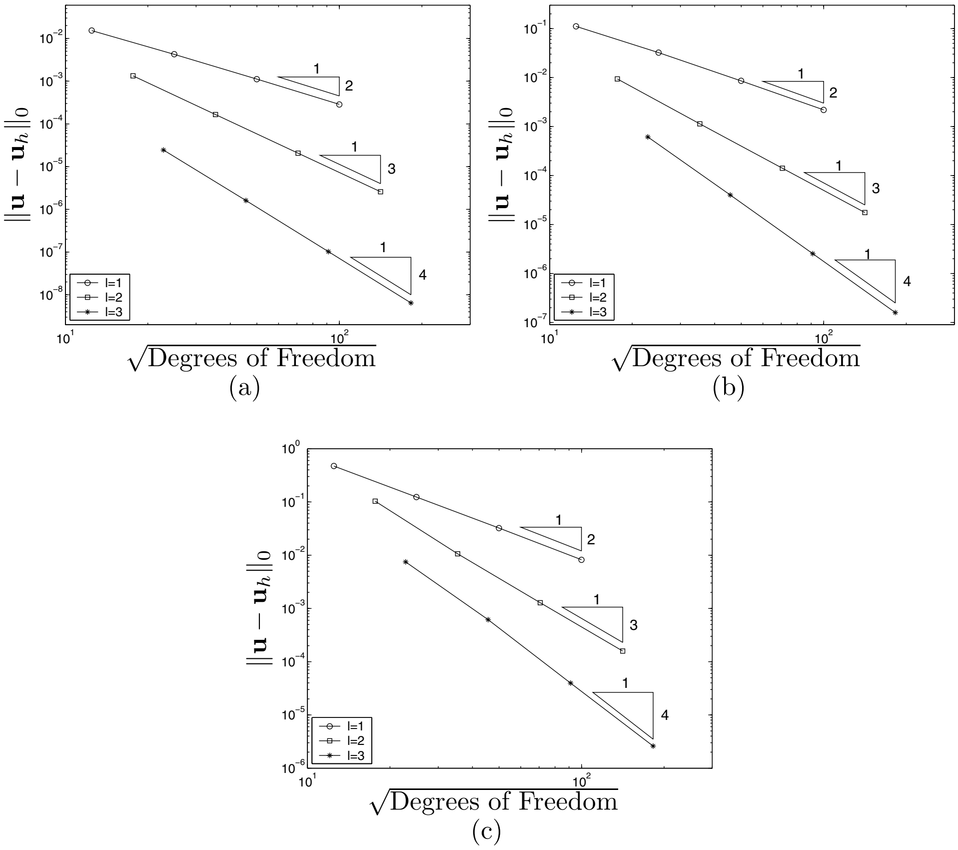

FIGURE 1. Example 1. Convergence of $\|\mathbf{u}-\mathbf{u}_h\|_0$ for: (a) $k=1$; (b) $k=2$; (c) $k=4$.

Finally, in Figure 1 we present a comparison of the $L^2(\Omega)^2$-norm of the error in the approximation to $\mathbf{u}$, with the square root of the number of degrees of freedom in the finite element space $\mathbf{V}_h$. Here, we observe that (asymptotically) $\|\mathbf{u}-\mathbf{u}_h\|_0$ converges to zero at the rate $\mathcal{O}(h^{\ell+1})$, for each fixed ℓ and each k, as h tends to zero. This is in full agreement with the optimal rate predicted by Corollary 4.5. Numerical experiments also indicate that the $L^2(\Omega)$-norm of the error in the approximation to p converges to zero at the optimal rate $\mathcal{O}(h^{\ell+2})$, for each fixed ℓ and each k, as h tends to zero; for brevity, these results have been omitted.

8.2. **Example 2**

In this second example, we investigate the performance of the mixed DG method (14) for a problem with a non-smooth solution. To this end, let Ω be the L-shaped domain $(-1,1)^2 \setminus [0,1) \times (-1,0]$ and select $\mathbf{j}$ (and suitable non-homogeneous boundary conditions for $\mathbf{u}$) so that the analytical solution $(\mathbf{u}, p)$ to the two-dimensional analogue of (6)–(9) is given, in terms of the polar coordinates (r,ϑ), by

$$\mathbf{u}(x,y) = \nabla S(r,\vartheta), \quad p = 0, \tag{73}$$

where

$$S(r,\vartheta) = (kr)^{2/3}\sin(2\vartheta/3).$$

The analytical solution given by (73) then contains a singularity at the re-entrant corner located at the origin of Ω; in particular, we note that $\mathbf{u}$ lies in the Sobolev space $H^{2/3-\varepsilon}(\Omega)^2$, $\varepsilon > 0$.

TABLE 7. Example 2. Convergence of $\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{V}(h)}$ with $k=1$.

	$\ell=1$		$\ell=2$		$\ell=3$	
Elements	$\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{V}(h)}$	r	$\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{V}(h)}$	r	$\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{V}(h)}$	r
24	7.871e-1	-	7.339e-1	-	6.536e-1	-
96	5.073e-1	0.63	4.144e-1	0.83	3.504e-1	0.90
384	2.613e-1	0.96	1.980e-1	1.07	1.620e-1	1.11
1536	1.187e-1	1.14	8.652e-2	1.19	6.945e-2	1.22
6144	5.188e-2	1.19	3.504e-2	1.30	2.495e-2	1.48

TABLE 8. Example 2. Convergence of $\|p-p_h\|_{Q(h)}$ with $k=1$.

	$\ell=1$		$\ell=2$		$\ell=3$	
Elements	$\|p-p_h\|_{Q(h)}$	r	$\|p-p_h\|_{Q(h)}$	r	$\|p-p_h\|_{Q(h)}$	r
24	1.460	-	1.918	-	2.337	-
96	1.408	0.05	1.713	0.16	1.982	0.24
384	1.127	0.32	1.291	0.41	1.446	0.46
1536	7.938e-1	0.51	8.807e-1	0.55	9.718e-1	0.57
6144	5.241e-1	0.60	5.744e-1	0.62	6.305e-1	0.62

TABLE 9. Example 2. Convergence of $\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{V}(h)}$ with $k=4$.

	$\ell=1$		$\ell=2$		$\ell=3$	
Elements	$\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{V}(h)}$	r	$\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{V}(h)}$	r	$\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{V}(h)}$	r
24	8.206e-1	-	7.812e-1	-	7.175e-1	-
96	3.611e-1	1.18	3.429e-1	1.19	3.011e-1	1.25
384	1.830e-1	0.98	1.525e-1	1.17	1.289e-1	1.22
1536	1.059e-1	0.79	7.225e-2	1.08	5.741e-2	1.17
6144	6.808e-2	0.64	5.129e-2	0.49	3.827e-2	0.59

In this example we again consider the convergence of the mixed DG method (14) on a sequence of successively finer (quasi-uniform) unstructured triangular meshes for $\ell = 1, 2, 3$ as the wave number k increases. To this end, in Tables 7 and 8 and Tables 9 and 10 we present numerical experiments for $k = 1, 4$, respectively. Here, we observe that for $k = 1$, the error $\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{V}(h)}$ converges to zero at a slightly superior rate than the optimal one of $\mathcal{O}(h^{2/3})$, for each ℓ, as h tends to zero, predicted by Theorem 4.1, *cf.* Table 7. We remark that analogous behavior is also observed when the interior penalty mixed DG method is applied to the low-frequency problem studied in [13]. However, for the higher wave number of $k = 4$, we now see that the rate of convergence of $\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{V}(h)}$ does seem to be slowly tending towards the optimal predicted one, *cf.* Table 9. On the other hand, from Tables 8 and 10 we see that $\|p-p_h\|_{Q(h)}$ converges to zero at the optimal rate of $\mathcal{O}(h^{2/3})$, for each ℓ and each k, as h tends to zero, predicted by Theorem 4.1, though now, the rate of convergence tends to the optimal one from below at the smaller wave number of $k = 1$. As in the previous example, we see that the DG-norm of the error in the approximation to both $\mathbf{u}$ and p increases as the wave number k increases for a fixed mesh size and polynomial degree. However, for a fixed mesh and wave number, while an increase in the polynomial degree leads to a decrease in $\|\mathbf{u}-\mathbf{u}_h\|_{\mathbf{V}(h)}$, the opposite behavior is observed for the error in the approximation to p; indeed, we observe that for both $k = 1, 4$, an increase in ℓ leads to an increase of $\|p-p_h\|_{Q(h)}$ on a given (fixed) mesh.

TABLE 10. Example 2. Convergence of $\|p - p_h\|_{Q(h)}$ with $k = 4$.

	$\ell = 1$		$\ell = 2$		$\ell = 3$	
Elements	$\|p - p_h\|_{Q(h)}$	r	$\|p - p_h\|_{Q(h)}$	r	$\|p - p_h\|_{Q(h)}$	r
24	7.661	-	9.056	-	10.00	-
96	5.293	0.53	5.847	0.63	6.404	0.64
384	3.410	0.63	3.715	0.65	4.058	0.66
1536	2.156	0.66	2.336	0.67	2.559	0.67
6144	1.364	0.66	1.475	0.66	1.607	0.67

TABLE 11. Example 2. Convergence of $\|\mathbf{u} - \mathbf{u}_h\|_0$ with $k = 1$.

	$\ell = 1$		$\ell = 2$		$\ell = 3$	
Elements	$\|\mathbf{u} - \mathbf{u}_h\|_0$	r	$\|\mathbf{u} - \mathbf{u}_h\|_0$	r	$\|\mathbf{u} - \mathbf{u}_h\|_0$	r
24	5.091e-1	-	4.651e-1	-	4.137e-1	-
96	3.248e-1	0.65	2.634e-1	0.82	2.223e-1	0.90
384	1.687e-1	0.95	1.270e-1	1.05	1.035e-1	1.10
1536	7.867e-2	1.10	5.667e-2	1.16	4.512e-2	1.20
6144	3.631e-2	1.12	2.421e-2	1.23	1.719e-2	1.39

TABLE 12. Example 2. Convergence of $\|\mathbf{u} - \mathbf{u}_h\|_0$ with $k = 4$.

	$\ell = 1$		$\ell = 2$		$\ell = 3$	
Elements	$\|\mathbf{u} - \mathbf{u}_h\|_0$	r	$\|\mathbf{u} - \mathbf{u}_h\|_0$	r	$\|\mathbf{u} - \mathbf{u}_h\|_0$	r
24	3.606e-1	-	2.725e-1	-	2.213e-1	-
96	2.180e-1	0.73	1.506e-1	0.86	1.144e-1	0.95
384	1.351e-1	0.69	8.801e-2	0.78	6.426e-2	0.83
1536	8.411e-2	0.68	5.294e-2	0.73	3.797e-2	0.76
6144	5.339e-2	0.66	3.423e-2	0.63	2.469e-2	0.62

Finally, we end this section by considering the rate of convergence of the error in the approximation to $\mathbf{u}$ measured in terms of the $L^2(\Omega)^2$-norm. To this end, in Tables 11 and 12 we present numerical experiments for $k = 1, 4$, respectively. The regularity assumptions required in the statement of Theorem 4.4 do not hold; as a consequence, the only proven result is $\|\mathbf{u} - \mathbf{u}_h\|_0 \leq \|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{V}(h)} = \mathcal{O}(h^{2/3})$, for each ℓ and k, as h tends to zero. The results obtained for the wave number $k = 4$ indicate that the convergence rate is asymptotically optimal in this case, whereas the results for $k = 1$ point to a convergence rate like $\mathcal{O}(h^{2\times 2/3})$.

9. Concluding remarks

In this paper, we have introduced and analyzed a new interior penalty method for the indefinite time–harmonic Maxwell equations written in mixed form. The proposed scheme can be viewed as a non-stabilized variant of the mixed DG method proposed in [22]; in particular, except for the standard interior penalty stabilization terms, here we exclude all the additional stabilization terms introduced in the DG formulation analyzed in [22]. Employing the recent techniques developed in [14], we have derived optimal *a priori* estimates for the error measured in terms of both the energy-norm, as well as the L^2-norm. The current analysis relies on exploiting duality techniques, and thereby only holds in the case of smooth material coefficients. The extension of this work to problems with non-smooth coefficients, by extending more general analysis approaches for conforming methods (such as the ones in [4] or [10]) to the discontinuous Galerkin context, is currently under investigation.

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References

[1] M. Ainsworth and J. Coyle, Hierarchic *hp*-edge element families for Maxwell's equations on hybrid quadrilateral/triangular meshes. *Comput. Methods Appl. Mech. Engrg.* **190** (2001) 6709–6733.

[2] C. Amrouche, C. Bernardi, M. Dauge and V. Girault, Vector potentials in three-dimensional non-smooth domains. *Math. Models Appl. Sci.* **21** (1998) 823–864.

[3] D.N. Arnold, F. Brezzi, B. Cockburn and L.D. Marini, Unified analysis of discontinuous Galerkin methods for elliptic problems. *SIAM J. Numer. Anal.* **39** (2001) 1749–1779.

[4] D. Boffi and L. Gastaldi, Edge finite elements for the approximation of Maxwell resolvent operator. *ESAIM: M2AN* **36** (2002) 293–305.

[5] S.C. Brenner and L.R. Scott, *The Mathematical Theory of Finite Element Methods*, *Texts in Applied Mathematics* **15**, Springer-Verlag, New York (1994).

[6] Z. Chen, Q. Du and J. Zou, Finite element methods with matching and nonmatching meshes for Maxwell equations with discontinuous coefficients. *SIAM J. Numer. Anal.* **37** (2000) 1542–1570.

[7] P.G. Ciarlet, *The finite element method for elliptic problems.* North–Holland, Amsterdam (1978).

[8] L. Demkowicz and L. Vardapetyan, Modeling of electromagnetic absorption/scattering problems using *hp*–adaptive finite elements. *Comput. Methods Appl. Mech. Engrg.* **152** (1998) 103–124.

[9] P. Fernandes and G. Gilardi, Magnetostatic and electrostatic problems in inhomogeneous anisotropic media with irregular boundary and mixed boundary conditions. *Math. Models Methods Appl. Sci.* **7** (1997) 957–991.

[10] R. Hiptmair, Finite elements in computational electromagnetism. *Acta Numerica* **11** (2002) 237–339.

[11] P. Houston, I. Perugia and D. Schötzau, *hp*-DGFEM for Maxwell's equations, in *Numerical Mathematics and Advanced Applications ENUMATH 2001*, F. Brezzi, A. Buffa, S. Corsaro, and A. Murli, Eds., Springer-Verlag (2003) 785–794.

[12] P. Houston, I. Perugia and D. Schötzau, Mixed discontinuous Galerkin approximation of the Maxwell operator. *SIAM J. Numer. Anal.* **42** (2004) 434–459.

[13] P. Houston, I. Perugia and D. Schötzau, Mixed discontinuous Galerkin approximation of the Maxwell operator: Non-stabilized formulation. *J. Sci. Comput.* **22** (2005) 325–356.

[14] P. Houston, I. Perugia, A. Schneebeli and D. Schötzau, *Interior penalty method for the indefinite time-harmonic Maxwell equations. Numer. Math.* **100** (2005) 485–518.

[15] O.A. Karakashian and F. Pascal, *A posteriori* error estimation for a discontinuous Galerkin approximation of second order elliptic problems. *SIAM J. Numer. Anal.* **41** (2003) 2374–2399.

[16] J. L. Lions and E. Magenes, *Problèmes aux Limites Non-Homogènes et Applications.* Dunod, Paris (1968).

[17] P. Monk, A finite element method for approximating the time-harmonic Maxwell equations. *Numer. Math.* **63** (1992) 243–261.

[18] P. Monk, *Finite element methods for Maxwell's equations.* Oxford University Press, New York (2003).

[19] P. Monk, A simple proof of convergence for an edge element discretization of Maxwell's equations, in *Computational electromagnetics*, C. Carstensen, S. Funken, W. Hackbusch, R. Hoppe and P. Monk, Eds., Springer-Verlag, *Lect. Notes Comput. Sci. Engrg.* **28** (2003) 127–141.

[20] J.C. Nédélec, A new family of mixed finite elements in $\mathbb{R}^3$. *Numer. Math.* **50** (1986) 57–81.

[21] I. Perugia and D. Schötzau, The *hp*-local discontinuous Galerkin method for low-frequency time-harmonic Maxwell equations. *Math. Comput.* **72** (2003) 1179–1214.

[22] I. Perugia, D. Schötzau and P. Monk, Stabilized interior penalty methods for the time-harmonic Maxwell equations. *Comput. Methods Appl. Mech. Engrg.* **191** (2002) 4675–4697.

[23] A. Schatz, An observation concerning Ritz-Galerkin methods with indefinite bilinear forms. *Math. Comp.* **28** (1974) 959–962.

[24] L. Vardapetyan and L. Demkowicz, *hp*-adaptive finite elements in electromagnetics. *Comput. Methods Appl. Mech. Engrg.* **169** (1999) 331–344.