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# ROBUST *A PRIORI* ERROR ANALYSIS FOR THE APPROXIMATION OF DEGREE-ONE GINZBURG-LANDAU VORTICES

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**Abstract.** This article discusses the numerical approximation of time dependent Ginzburg-Landau equations. Optimal error estimates which are robust with respect to a large Ginzburg-Landau parameter are established for a semi-discrete in time and a fully discrete approximation scheme. The proofs rely on an asymptotic expansion of the exact solution and a stability result for degree-one Ginzburg-Landau vortices. The error bounds prove that degree-one vortices can be approximated robustly while unstable higher degree vortices are critical.

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## 1. Introduction

A mathematical model due to Ginzburg and Landau [18] for the description of certain superconducting materials involves a minimization of the energy functional

$$G_\kappa(u, A) = \int_\Omega |\nabla u - iAu|^2 \,\mathrm{d}x + \frac{\kappa}{2}\int_\Omega (|u|^2 - 1)^2 \,\mathrm{d}x + \int_\Omega |\mathrm{curl}\, A - h_{ext}|^2 \,\mathrm{d}x$$

where $\Omega \subseteq \mathbb{R}^n$, $n = 2, 3$, is a bounded Lipschitz domain and represents the region occupied by the superconductor. The complex-valued function $u$ is the condensate wave function, $A$ is the magnetic potential associated to the induced magnetic field, $h_{ext}$ represents an external magnetic field, and $\kappa > 0$ is the Ginzburg-Landau parameter. The quantity $|u|^2$ gives the density of cooper pairs of electrons that produce the superconductivity and zeros of $|u|^2$ are called *vortices*. Since a superconducting property is violated in such points it is interesting to predict the number, the location, and the dynamics of vortices.

It is known that the properties of vortices sensitively depend on $\kappa$. The case $\kappa = 1/\sqrt{2}$ has been studied by Jaffe and Taubes [21] and it has been shown that vortices can be located anywhere in $\Omega$ and do not tend to interact. If $\kappa < 1/\sqrt{2}$ then vortices tend to attract each other and concentrate in one place. The case $\kappa > 1/\sqrt{2}$ is called the repulsive case since for such values of $\kappa$ vortices of same sign tend to repulse each other. Materials for which $\kappa < 1/\sqrt{2}$ or $\kappa > 1/\sqrt{2}$ are also known as type-I or type-II conductors, respectively. For details on the qualitative and theoretical study of Ginzburg-Landau vortices we refer the reader to [4, 8, 13, 21, 24–27, 33, 34] and references therein.

*Keywords and phrases.* Ginzburg-Landau equations, numerical approximation, error analysis, spectral estimate, finite element method.

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The parameter $\kappa$ also influences the stability of numerical approximation schemes for superconducting materials. In particular, the dynamics of superconducting materials are often modeled by the gradient flow of the energy functional $G_\kappa$ and numerical results sensitively depend upon $\kappa$. It is not difficult to see that standard error estimates depend exponentially upon $\kappa$ which can make simulations useless if $\kappa$ is large. In this article we aim to design and analyze approximation schemes for which the discretization error grows in $\kappa$ only in a low order polynomial and thereby allow for reliable simulations. We remark that our analysis is inspired by recent work of Feng and Prohl [15–17] on phase field models and we employ several of their arguments. Owing to a lack of appropriate *a priori* information for the vectorial problem considered here, their proofs are however not directly transferable. We remark that we also employ several techniques from [1,36]. Interesting numerical studies of stationary and time-dependent Ginzburg-Landau equations which motivated this work can be found in [7,11,12,20,32]. An a posteriori error analysis based on ideas of this work can be found in [2].

For ease of presentation we will restrict the analysis to a simplified version of $G_\kappa$, in which $A \equiv 0$, $\varepsilon^2 = 1/(2\kappa)$, and $n = 2$, *i.e.*, $\Omega \subseteq \mathbb{R}^2$,

$$J_\varepsilon : H^1(\Omega;\mathbb{C}) \to \mathbb{R}, \quad v \mapsto \frac{1}{2}\int_\Omega |\nabla v|^2 \,\mathrm{d}x + \frac{\varepsilon^{-2}}{4}\int_\Omega (|v|^2-1)^2 \,\mathrm{d}x.$$

In this model, Dirichlet type boundary conditions replace an applied magnetic field. The dynamics of a superconducting material can then – in a greatly simplified manner – be described by the gradient flow of $J_\varepsilon$. In order to define the time-dependent problem, we assume that we are given a parameter $T > 0$, initial data $u_0^\varepsilon \in H^1(\Omega;\mathbb{C})$, and boundary data $g^\varepsilon = u_0^\varepsilon|_{\partial\Omega} \in H^{1/2}(\partial\Omega;\mathbb{C})$. We then aim to solve the following problem:

$$(\mathrm{P}) \qquad \begin{cases} \textit{Find } u^\varepsilon \in H^1(0,T;H^{-1}(\Omega;\mathbb{C})) \cap L^2(0,T;H^1(\Omega;\mathbb{C})) \textit{ such that} \\ \textit{for almost all } t \in (0,T) \textit{ and all } v \in H^1_0(\Omega;\mathbb{C}) \textit{ there holds} \\ \quad \begin{aligned} \langle u_t^\varepsilon; v\rangle + (\nabla u^\varepsilon; \nabla v) + \varepsilon^{-2}(f(u^\varepsilon); v) &= 0, \\ u^\varepsilon|_{\partial\Omega} &= g^\varepsilon, \\ u^\varepsilon(0) &= u_0^\varepsilon. \end{aligned} \end{cases}$$

Here, $f(a) = (|a|^2-1)a$ for $a \in \mathbb{C}$, $(\cdot;\cdot)$ denotes the scalar product in $L^2(\Omega;\mathbb{C})$, $\langle\cdot;\cdot\rangle$ denotes the duality pairing of $H^1(\Omega;\mathbb{C})$ and $H^{-1}(\Omega;\mathbb{C})$, $u_t^\varepsilon$ denotes the time derivative of $u^\varepsilon$, and we use the notation $a \cdot b := (\overline{a}b + a\overline{b})/2$ for $a, b \in \mathbb{C}$. Existence and uniqueness of a solution $u^\varepsilon$ follow from standard techniques. Throughout this work we abbreviate

$$u = u^\varepsilon, \quad u_0 = u_0^\varepsilon \quad \text{and} \quad g = g^\varepsilon$$

but stress that the dependence of the error of numerical approximation schemes upon $\varepsilon$ is the main contribution of this work.

The main results of this work prove that under moderate conditions on the initial data $u_0$ and the domain $\Omega$ and if the time-step size parameter $k > 0$ and the mesh-size $h > 0$ of an implicit in time, lowest order finite element in space discretization of (P) satisfy (up to logarithmic and constant factors) $k \le \varepsilon^8$ and $h \le \varepsilon^4$, then the spatial discretization error in $L^2$ is bounded by $\varepsilon^{-4}(k + h^2)$ for all $t \in (0,T)$. The assumptions involve the condition that vortices of $u_0$ are well separated and of degree one.

The outline of this article is as follows. We state and discuss theoretically and physically motivated assumptions on the solution $u$ of (P) in Section 2. Section 3 gives *a priori* bounds on various norms of the exact solution. Discrete counterparts of those *a priori* bounds are proved in Section 4 for a semi-discrete in time approximation of (P). Optimal error estimates in terms of the time discretization parameter which are robust with respect to the small parameter $\varepsilon$ are established in Section 5. In Section 6 we discuss a fully discrete approximation scheme and prove robust *a priori* error estimates.

## 2. Assumptions on the solution

In order to derive robust *a priori* error estimates for the numerical approximation of (P) we will assume that the solution $u$ of (P) admits vortices only of degree one and allows for an asymptotic expansion. The following lemma is essential for the asymptotic expansion and characterizes degree one-vortices.

**Lemma 2.1** [19]**.** *Let $f_0 : [0,\infty) \to \mathbb{R}$ satisfy $f_0(0) = 0$, $\lim_{s\to\infty} f_0(s) = 1$, $f_0 \geq 0$, and*

$$-f_0'' - s^{-1} f_0' + s^{-2} f_0 = f_0(1 - f_0^2).$$

*Given any real number $\tilde{H} \in \mathbb{R}$ the function*

$$U_{\tilde{H}}(x) := f_0(r(x)/\varepsilon)\mathrm{e}^{i\theta(x)}\mathrm{e}^{-i\tilde{H}},$$

*where $(r(x), \theta(x))$ denote the polar coordinates of $x \in \mathbb{R}^2$, satisfies $U_{\tilde{H}}(0) = 0$ and*

$$-\Delta U_{\tilde{H}} + \varepsilon^{-2} f(U_{\tilde{H}}) = 0 \quad \text{in } \mathbb{R}^2.$$

The following assumption states that in certain neighborhoods of zeros the solution $u$ of (P) is given by small perturbations of functions $U_{\tilde{H}}$ as in the previous lemma. Away from zeros, $|u|$ is assumed to be close to 1.

**Assumption I.** *There exist $1 \geq \varepsilon_0 > 0$, $\delta_0, c_0 > 0$, $c_1 \geq 1$, and an integer $d \geq 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, for all $t \in (0,T]$, and $j = 1,2,...,d$ there exist*

$$a_j^{t,\varepsilon} \in \mathbb{R}^2, \quad p_j^{t,\varepsilon}, q_j^{t,\varepsilon} : B_{\delta_0}(a_j^{t,\varepsilon}) \to \mathbb{C}, \quad H_j^{t,\varepsilon} : B_{\delta_0}(a_j^{t,\varepsilon}) \to \mathbb{R}, \quad \tilde{H}_j^{t,\varepsilon} \in \mathbb{R}$$

*such that*

$$u(t,x) = f_0\big(r(x - a_j^{t,\varepsilon})/\varepsilon\big) e^{i\theta(x-a_j^{t,\varepsilon})} e^{-iH_j^{t,\varepsilon}(x)} + \varepsilon p_j^{t,\varepsilon}(x) + \varepsilon^2 q_j^{t,\varepsilon}(x) \quad \text{for } x \in B_{\delta_0}(a_j^{t,\varepsilon})$$

*and*

$$\big||u(t,x)| - 1\big| \leq c_0\delta_0^{-2}\varepsilon^2 \qquad \text{for } x \in \Omega \setminus \cup_{j=1}^d B_{\delta_0/2}(a_j^{t,\varepsilon}).$$

*The functions $p_j^{t,\varepsilon}, q_j^{t,\varepsilon}, H_j^{t,\varepsilon}$, $j = 1,2,...,d$, satisfy*

$$|H_j^{t,\varepsilon}(x) - \tilde{H}_j^{t,\varepsilon}| \leq c_0\varepsilon^2, \quad \big|f''\big(U_j(x)\big)p_j^{t,\varepsilon}(x)\big| \leq c_0\varepsilon, \quad |q_j^{t,\varepsilon}(x)| \leq c_0 \quad \text{for } x \in B_{\delta_0}(a_j^{t,\varepsilon}),$$

*where for each $j = 1,2,...,d$,*

$$U_j(x) := f_0\big(r(x - a_j^{t,\varepsilon})/\varepsilon\big) e^{i\theta(x-a_j^{t,\varepsilon})} e^{-i\tilde{H}_j^{t,\varepsilon}(x)} \quad \text{for } x \in B_{\delta_0}(a_j^{t,\varepsilon}).$$

*Moreover, for almost all $(t,x) \in (0,T) \times \Omega$ there holds $|u(t,x)| \leq c_1$.*

**Remarks.** (i) Assumption I is motivated by the fact that in the repulsive case, *i.e.*, for small values of $\varepsilon$, vortices tend to repulse each other and that higher degree vortices are unstable [3], *i.e.*, split into degree-one vortices immediately. Therefore, Assumption I is merely an assumption on the initial data $u_0$.
(ii) Assumption I implicitly requires that the vortices are well separated, *i.e.*, have a distance $\delta_0$. Since $f_0(r) = 1 - 1/(2r^2) + \mathcal{O}(1/r^4)$ for $r \to \infty$ the assumed expansion of $u(t,x)$ in a neighborhood of a vortex and the condition $\big||u(t,x)| - 1\big| \leq c_0\delta_0^{-2}\varepsilon^2$ away from the vortices are compatible. Note that $f_0(r) = ar - ar^3/8 + \mathcal{O}(r^5)$ for $r \to 0$ and a constant $a \in \mathbb{R}$ [19].
(iii) Similar assumptions on the solutions of related problems have been made in [13,28] to study the dynamics of vortices, in [6,9] to study the stability of interfaces in phase field equations, and in [15,16,22] to derive robust error estimates for the approximation of Allen-Cahn and Cahn-Hilliard equations.

A spectral estimate for the linearized Ginzburg-Landau operator about symmetric degree-one vortices is specified in the next theorem and will be one key ingredient for our *a priori* error analysis.

**Theorem 2.2** [3, 23, 29, 30]**.** *Given* $\tilde{H} \in \mathbb{R}$ *let* $U_{\tilde{H}}(x) := f_0(r(x)/\varepsilon)\mathrm{e}^{i\theta(x)}\mathrm{e}^{-i\tilde{H}}$. *Let* $L_{\varepsilon,\tilde{H}} : H_0^1(B_{\delta_0}(0);\mathbb{C}) \to L^2(B_{\delta_0}(0);\mathbb{C})$ *be defined by*

$$L_{\varepsilon,\tilde{H}}v := -\Delta v + \varepsilon^{-2} f'(U_{\tilde{H}})v.$$

*Then, the principal eigenvalue of* $L_{\varepsilon,\tilde{H}}$ *is non-negative, i.e., for all* $v \in H_0^1(B_{\delta_0}(0);\mathbb{C})$ *there holds*

$$(L_{\varepsilon,\tilde{H}}v; v) \geq 0.$$

*Proof.* Proofs of the theorem can be found in [3, 23, 29, 30] for $\tilde{H} = 0$. If $\tilde{H} \neq 0$ one can check that given any $v \in H_0^1(B_{\delta_0}(0);\mathbb{C})$ there holds with $w := \mathrm{e}^{i\tilde{H}}v$

$$(L_{\varepsilon,\tilde{H}}v; v) = (L_{\varepsilon,0}w; w) \geq 0$$

which proves the theorem. □

The theorem can be generalized to small perturbations of degree one vortices as follows.

**Proposition 2.3.** *Suppose that Assumption I holds. Then there exists an* $\varepsilon$*-independent constant* $\lambda_0 \geq 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0)$*, for* $j = 1, 2, ..., d$*, for all* $t \in (0, T]$*, and for all* $v \in H_0^1(B_{\delta_0}(a_j^{t,\varepsilon});\mathbb{C})$ *there holds*

$$(\nabla v; \nabla v) + \varepsilon^{-2}(f'(u)v; v) \geq -\lambda_0 ||v||^2_{L^2(B_{\delta_0}(a_j))}.$$

*Proof.* Let $j \in \{1, 2, ..., d\}$ and $t \in (0, T]$. Throughout this proof we abbreviate

$$a := a_j^{t,\varepsilon}, \quad p := p_j^{t,\varepsilon}, \quad q := q_j^{t,\varepsilon}, \quad H := H_j^{t,\varepsilon}, \quad \tilde{H} := \tilde{H}_j^{t,\varepsilon},$$

where $a_j^{t,\varepsilon}$, $p_j^{t,\varepsilon}$, $q_j^{t,\varepsilon}$, $H_j^{t,\varepsilon}$, and $\tilde{H}_j^{t,\varepsilon}$ are as in Assumption I. For $x \in B_{\delta_0}(a)$ define

$$\begin{aligned} U_{\tilde{H}}(x) &:= f_0\big(r(x-a)/\varepsilon\big)\mathrm{e}^{i\theta(x-a)}\mathrm{e}^{-i\tilde{H}}, \\ U_H(x) &:= f_0\big(r(x-a)/\varepsilon\big)\mathrm{e}^{i\theta(x-a)}\mathrm{e}^{-iH(x)}. \end{aligned}$$

There holds

$$f'(u) = f'(U_{\tilde{H}}) + \big(f'(u) - f'(U_H)\big) + \big(f'(U_H) - f'(U_{\tilde{H}})\big)$$

and owing to the assumed asymptotic expansion of $u$ in $B_{\delta_0}(a)$, a Taylor expansion of the quadratic function $f'$ shows in $B_{\delta_0}(a)$

$$\begin{aligned} |f'(u) - f'(U_H)| &\leq |f''(U_H)(u - U_H)| + \frac{1}{2}|f'''(U_H)||u - u_H|^2 \\ &\leq \varepsilon|f''(U_H)p| + \varepsilon^2|f''(U_H)||q| + \frac{\varepsilon^2}{2}|f'''(U_H)||p + \varepsilon q|^2 \\ &\leq C\varepsilon^2. \end{aligned}$$

Since $|\mathrm{e}^{-iH} - \mathrm{e}^{-i\tilde{H}}| \leq C|H - \tilde{H}| \leq C\varepsilon^2$ in $B_{\delta_0}(a)$ we have

$$|f'(U_H) - f'(U_{\tilde{H}})| \leq |f''(U_{\tilde{H}})||U_H - U_{\tilde{H}}| + \frac{1}{2}|f'''(U_H)||U_H - U_{\tilde{H}}|^2 \leq C\varepsilon^2.$$

Using these estimates we infer with Theorem 2.2 for all $v \in H_0^1(B_{\delta_0}(a);\mathbb{C})$ that

$$(\nabla v;\nabla v)+\varepsilon^{-2}(f'(u)v;v) \geq (\nabla v;\nabla v)+\varepsilon^{-2}(f'(U_{\tilde{H}})v;v) - C\|v\|^2_{L^2(B_{\delta_0}(a))} \geq -C\|v\|^2_{L^2(B_{\delta_0}(a))}.$$

This proves the proposition. □

We conclude this section by a discussion of Assumption I. The first example specifies (P), allows for a solution without vortices, and yields Assumption I.

**Example 2.4** [10]**.** Suppose that $\Omega$ is smooth and $g$ is smooth, independent of $\varepsilon$, and satisfies $|g(s)| = 1$ for all $s \in \partial\Omega$. Assume that $u_0 = u_0^\varepsilon$ is smooth, independent of $\varepsilon$, and satisfies $|u_0(x)| = 1$ for all $x \in \Omega$. Then, there exist $\tilde{\varepsilon}_0 > 0$ and an $\varepsilon$-independent constant $C > 0$ such that for all $\varepsilon \in (0, \tilde{\varepsilon}_0)$, for almost all $(t,x) \in (0,T)\times\Omega$ there holds $\big||u(t,x)| - 1\big| \leq C\varepsilon^2$. In particular, Assumption I holds with $\delta_0 = 1$, $\varepsilon_0 = \tilde{\varepsilon}_0$, $c_0 = C$, and $d = 0$.

Assumption I can be motivated as follows.

**Example 2.5.** Given $t \in (0,T]$ we introduce $y := x/\varepsilon$ and make the ansatz

$$u(t,x) = U(y) + \varepsilon p(y) + \varepsilon^2 q(y)$$

which leads to the three leading order equations

$$\varepsilon^{-2}\big(-\Delta U + (|U|^2-1)U\big) = 0, \tag{2.1}$$

$$\varepsilon^{-1}\big(-\Delta p + |U|^2 p + 2(U\cdot p)U - p\big) = 0, \tag{2.2}$$

$$\varepsilon^{0}\big(-\Delta q + |U|^2 q + |p|^2 U + 2(U\cdot q)U - q\big) = u_t(t,\cdot). \tag{2.3}$$

Equation (2.1) admits a solution $U$ subject to boundary conditions $U|_{\partial\Omega} = g$. Provided that $U(x/\varepsilon)$ is minimal for $J_\varepsilon$ it can be shown [34] that there exists an integer $d \geq 0$, a harmonic function $\psi : \Omega \to \mathbb{R}$, and $a_j^\varepsilon \in \Omega$, $j = 1,2,...,d$, such that

$$\Big\|U(\cdot/\varepsilon) - \mathrm{e}^{i\psi}\prod_{j=1}^{d} f_0\big(r(\cdot - a_j^\varepsilon)/\varepsilon\big)\mathrm{e}^{i\theta(\cdot - a_j^\varepsilon)}\Big\|_{L^\infty(\Omega)} \to 0 \quad \text{for } \varepsilon \to 0.$$

We choose $p \equiv 0$ as a solution for (2.2) and let $q$ be the solution of the linear equation (2.3) subject to $q|_{\partial\Omega} = 0$.

The assertions of the previous example can be made more precise. We only focus on the dominant term in the asymptotic expansion, *i.e.*, on Equation (2.1) in the following example. It gives a precise characterization of the solution of (2.1) in neighborhoods of vortices and shows that $|u|$ is close to 1 away from vortices.

**Example 2.6** [4, 31]**.**
(i) Suppose $U = U^\varepsilon$ satisfies $U(0) = 0$, $\big\|1 - |U|^2\big\|_{L^2(B_{\delta_0/\varepsilon}(0))} \leq C$, solves (2.1) in $B_{\delta_0/\varepsilon}(0)$, and $U(\cdot/\varepsilon)$ is minimal for $J_\varepsilon$ with $\Omega = B_{\delta_0/\varepsilon}(0)$. Assuming that $\varepsilon$ is small enough we may assume that $U$ solves (2.1) in $\mathbb{R}^2$. It can then be shown that there exists $\theta_0 \in \mathbb{R}$ such that for all $y \in \mathbb{R}^2$ there holds $U(y) = f_0(r(y))e^{i(\theta(y)+\theta_0)}$.
(ii) Suppose that $G : \partial B_{\delta_0/\varepsilon}(0) \to \mathbb{C}$ is smooth, satisfies $|G(s)| = 1$ for all $s \in \partial B_{\delta_0/\varepsilon}(0)$, and $\deg(G, \partial B_{\delta_0/\varepsilon}(0)) = 0$. Let $U$ satisfy $U|_{\partial B_{\delta_0/\varepsilon}(0)} = G$, solve (2.1) in $B_{\delta_0/\varepsilon}$, and suppose that $U(\cdot/\varepsilon)$ is minimal for $J_\varepsilon$ with $\Omega = B_\delta(0)$. Then there exists an $\varepsilon$-independent constant $C > 0$ such that for all $y \in B_{\delta_0/\varepsilon}(0)$ there holds $\big||U(y)| - 1\big| \leq C\varepsilon^2$.

## 3. *A priori* bounds

The following *a priori* bounds on a solution $u$ of (P) are independent of Assumption I. They solely assume that the modulus of the solution $u$ is uniformly bounded in $(0,T)\times\Omega$.

**Proposition 3.1.** *Suppose that there exists an $\varepsilon$-independent constant $c_1 > 0$ such that for almost all $(t,x)\in(0,T)\times\Omega$ there holds $|u(t,x)|\le c_1$. Then, there exists an $\varepsilon$-independent constant $c_2>0$ such that*

$$(i)\qquad \operatorname{ess\,sup}_{s\in(0,T)}\frac{1}{2}||\nabla u||^2_{L^2(\Omega)}+\int_0^T ||u_t(s)||^2_{L^2(\Omega)}\,\mathrm{d}s\le 2J_\varepsilon(u_0),$$

$$(ii)\qquad \operatorname{ess\,sup}_{s\in(0,T)}||u_t||^2_{L^2(\Omega)}+\int_0^T ||\nabla u_t||^2_{L^2(\Omega)}\,\mathrm{d}s \le c_2\big(\varepsilon^{-2}J_\varepsilon(u_0)+||\Delta u_0-\varepsilon^{-2}f(u_0)||^2_{L^2(\Omega)}\big),$$

$$(iii)\qquad \int_0^T ||u_{tt}||^2_{H^{-1}(\Omega;\mathbb{C})}\,\mathrm{d}s\le c_2\big(\varepsilon^{-2}J_\varepsilon(u_0)+||\Delta u_0-\varepsilon^{-2}f(u_0)||^2_{L^2(\Omega)}+\varepsilon^{-4}J_\varepsilon(u_0)\big).$$

*Sketch of proof.* We formally derive the estimates to verify the dependence on $\varepsilon$ and note that the assertions can be proved rigorously by employing techniques from, *e.g.*, [14].
$(i)$ This follows from choosing $v=u_t$ (note $u_t|_{\partial\Omega}\equiv 0$) in (P)

$$||u_t||^2_{L^2(\Omega)}+\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{2}||\nabla u||^2_{L^2(\Omega)}+\varepsilon^{-2}\frac{1}{4}||(|u|^2-1)^2||_{L^1(\Omega)}\Big)=0$$

and integrating this equation over $(0,s)$ for any $0\le s\le T$.
$(ii)$ We formally differentiate the first equation in (P) in time,

$$u_{tt}-\Delta u_t+\varepsilon^{-2}f'(u)u_t=0, \tag{3.1}$$

and test this equation with $u_t$, *i.e.*,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}||u_t||^2_{L^2(\Omega)}=-||\nabla u_t||^2_{L^2(\Omega)}-\varepsilon^{-2}(f'(u)u_t;u_t).$$

Using $||u||_{L^\infty((0,T)\times\Omega)}\le c_1$ and integrating over $(0,s)$ for any $0\le s\le T$ we infer

$$\begin{aligned}||u_t(s)||^2_{L^2(\Omega)}+\int_0^T ||\nabla u_t||^2_{L^2(\Omega)}\,\mathrm{d}s &\le C\varepsilon^{-2}\int_0^T ||u_t||^2_{L^2(\Omega)}\,\mathrm{d}s+||u_t(0)||^2_{L^2(\Omega)}\\ &\le C\varepsilon^{-2}J_\varepsilon(u_0)+||u_t(0)||^2_{L^2(\Omega)}.\end{aligned}$$

We then use $(i)$ and $u_t(0)=\Delta u_0-\varepsilon^{-2}f(u_0)$ to verify the assertion.
$(iii)$ Using the estimate $||f'(u)||_{L^\infty(\Omega)}\le C$ we verify with (3.1)

$$\begin{aligned}||u_{tt}||_{H^{-1}(\Omega;\mathbb{C})}&\le ||\nabla u_t||_{L^2(\Omega)}+\varepsilon^{-2}\sup_{\varphi\in H^1_0(\Omega;\mathbb{C})}\frac{(f'(u)u_t;\varphi)}{||\varphi||_{H^1(\Omega)}}\\ &\le ||\nabla u_t||_{L^2(\Omega)}+C\varepsilon^{-2}||u_t||_{L^2(\Omega)}.\end{aligned}$$

Assertion $(iii)$ is then deduced from squaring this estimate, integrating the resulting equation over $(0,T)$, and employing previous estimates. □

In order to exploit the estimates of the proposition we need to assume bounds on $J_\varepsilon(u_0)=J_\varepsilon(u_0^\varepsilon)$ and $||\Delta u_0-\varepsilon^{-2}f(u_0)||_{L^2(\Omega)}$. In general, it is impossible to bound these quantities independently of $\varepsilon$ in the considered two-dimensional setting.

**Assumption II.** *There exists a positive function* $\gamma : \mathbb{R} \to \mathbb{R}$ *such that* $\gamma(\varepsilon) \le \gamma(\varepsilon)^2$ *and*

$$J_\varepsilon(u_0) \le \gamma_\varepsilon := \gamma(\varepsilon), \qquad ||u_0||^2_{H^2(\Omega)} + ||\Delta u_0 - \varepsilon^{-2} f(u_0)||^2_{L^2(\Omega)} \le \gamma_\varepsilon \varepsilon^{-4}$$

*for all* $\varepsilon \in (0, \varepsilon_0)$.

For an optimal choice of $u_0$, which extends $g$ from $\partial\Omega$ to $\Omega$, we may assume that $\gamma$ grows at most logarithmically for $\varepsilon \to 0$.

**Example 3.2** [4]**.** Assume that $\Omega$ is smooth and simply connected and suppose that $g$ is smooth, independent of $\varepsilon$, and satisfies $|g(s)| = 1$ for all $s \in \partial\Omega$. Then there exist $\tilde{\varepsilon}_0 > 0$, an $\varepsilon$-independent constant $C > 0$, and $\tilde{g} \in H^1(\Omega; \mathbb{C})$ with $\tilde{g}|_{\partial\Omega} = g$ such that for all $\varepsilon \in (0, \tilde{\varepsilon}_0)$ there holds

$$J_\varepsilon(\tilde{g}) \le 2\pi \tilde{d} \log(\varepsilon^{-1}) + C$$

where $\tilde{d} = \deg(g, \partial\Omega)$. The estimate is sharp in the sense that if $\Omega$ is starshaped and $\tilde{d} > 0$, *e.g.*, $\Omega = B_1(0)$ and $g(s) = s$ for $s \in \partial\Omega$, then there holds $\lim_{\varepsilon\to 0}\big\{\min_{v|_{\partial\Omega}=g} J_\varepsilon(v) - \pi\tilde{d}|\log(\varepsilon)|\big\} = C$ for an $\varepsilon$-independent constant $C > 0$.

## 4. Semi-discrete in time approximation scheme

We analyze the following semi-discrete in time approximation ($\mathrm{P}_k$) of (P) which is defined through a time step size parameter $k > 0$.

$$(\mathrm{P}_k) \qquad \begin{cases} \textit{Find } (U_m : 0 \le m \le M) \subseteq H^1(\Omega; \mathbb{C}) \textit{ such that for all} \\ 1 \le m \le M \textit{ and all } V \in H^1_0(\Omega; \mathbb{C}) \textit{ there holds} \\ (d_t U_m; V) + (\nabla U_m; \nabla V) + \varepsilon^{-2}\big(f(U_m); V\big) \quad = 0, \\ \qquad\qquad\qquad\qquad\qquad\qquad U_m|_{\partial\Omega} \quad = g, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad U_0 \quad = u_0. \end{cases}$$

Here, $M$ is the largest integer for which $Mk \le T$ and given any sequence $(a_m : 0 \le m \le M)$ the discrete time derivative $d_t a_m$ is for $1 \le m \le M$ defined by $d_t a_m := (a_m - a_{m-1})/k$. We set $t_m := mk$ for $0 \le m \le M$ and define

$$e_m := u(t_m) - U_m.$$

The following *a priori* bound for a solution $(U_m : 0 \le m \le M)$ is the discrete counterpart of assertion $(i)$ in Proposition 3.1.

**Proposition 4.1.** *Suppose that* $k \le \varepsilon^2$. *There holds*

$$\max_{1\le m\le M}\Big(\frac{1}{2}||\nabla U_m||^2_{L^2(\Omega)} + \frac{\varepsilon^{-2}}{4}\big\||U_m|^2 - 1\big\|^2_{L^2(\Omega)}\Big) \\ + k\sum_{m=1}^{M}\Big(\frac{1}{2}||d_t U_m||^2_{L^2(\Omega)} + \frac{k\varepsilon^{-2}}{4}\big\|d_t(|U_m|^2 - 1)\big\|^2_{L^2(\Omega)}\Big) \le 2J_\varepsilon(u_0).$$

*Proof.* The choice $V = d_t U_m$ for $1 \le m \le M$ in ($\mathrm{P}_k$) yields

$$||d_t U_m||^2_{L^2(\Omega)} + \frac{1}{2k}\big(||\nabla U_m||^2_{L^2(\Omega)} - ||\nabla U_{m-1}||^2_{L^2(\Omega)}\big) + \varepsilon^{-2}\big(f(U_m); d_t U_m\big) \\ \le ||d_t U_m||^2_{L^2(\Omega)} + \big(\nabla U_m; d_t \nabla U_m\big) + \varepsilon^{-2}\big(f(U_m); d_t U_m\big) = 0.$$

We write

$$f(U_m) = \frac{1}{2}\big(|U_m|^2 - 1\big)\big(U_m + U_{m-1} + kd_tU_m\big)$$

and use $d_tU_m \cdot (U_m + U_{m-1}) = d_t\big(|U_m|^2 - 1\big)$ and $\frac{k}{2}(|U_m|^2; |d_tU_m|^2) \geq 0$ to verify

$$\begin{aligned}\big(f(U_m); d_tU_m\big) &= \frac{1}{2}\big((|U_m|^2 - 1); d_t(|U_m|^2 - 1)\big) + \frac{k}{2}\big((|U_m|^2 - 1); |d_tU_m|^2\big) \\ &\geq \frac{1}{2}\big((|U_m|^2 - 1); d_t(|U_m|^2 - 1)\big) - \frac{k}{2}||d_tU_m||^2_{L^2(\Omega)}.\end{aligned}$$

Employing the identity $a \cdot (a - b) = \frac{1}{2}|a - b|^2 + \frac{1}{2}(|a|^2 - |b|^2)$ for $a, b \in \mathbb{C}$ we deduce

$$\begin{aligned}\big((|U_m|^2 - 1); d_t(|U_m|^2 - 1)\big) &= \frac{1}{2k}\big\|(|U_m|^2 - 1) - (|U_{m-1}|^2 - 1)\big\|^2_{L^2(\Omega)} \\ &\quad + \frac{1}{2k}\big(\big\||U_m|^2 - 1\big\|^2_{L^2(\Omega)} - \big\||U_{m-1}|^2 - 1\big\|^2_{L^2(\Omega)}\big) \\ &= \frac{k}{2}\big\|d_t(|U_m|^2 - 1)\big\|^2_{L^2(\Omega)} + \frac{1}{2}d_t\big\||U_m|^2 - 1\big\|^2_{L^2(\Omega)}.\end{aligned}$$

We may therefore estimate

$$\begin{aligned}&\left(1 - \frac{\varepsilon^{-2}k}{2}\right)||d_tU_m||^2_{L^2(\Omega)} + \frac{1}{2k}||\nabla U_m||^2_{L^2(\Omega)} - \frac{1}{2k}||\nabla U_{m-1}||^2_{L^2(\Omega)} \\ &\qquad + \frac{k\varepsilon^{-2}}{4}\big\|d_t(|U_m|^2 - 1)\big\|^2_{L^2(\Omega)} + \frac{\varepsilon^{-2}}{4}d_t\big\||U_m|^2 - 1\big\|^2_{L^2(\Omega)} \leq 0.\end{aligned}$$

Multiplying this estimate with $k$ and taking sum over $m = 1, ..., \ell$ for any $1 \leq \ell \leq M$ we deduce

$$\begin{aligned}&\frac{1}{2}||\nabla U_\ell||^2_{L^2(\Omega)} + \frac{\varepsilon^{-2}}{4}\big\||U_\ell|^2 - 1\big\|^2_{L^2(\Omega)} \\ &\qquad + k\sum_{m=1}^{\ell}\left(\left(1 - \frac{k\varepsilon^{-2}}{2}\right)||d_tU_m||^2_{L^2(\Omega)} + \frac{k\varepsilon^{-2}}{4}\big\|d_t(|U_m|^2 - 1)\big\|^2_{L^2(\Omega)}\right) \\ &\qquad\qquad \leq \frac{1}{2}||\nabla U_0||^2_{L^2(\Omega)} + \frac{\varepsilon^{-2}}{4}|||U_0|^2 - 1||^2_{L^2(\Omega)} = J_\varepsilon(U_0).\end{aligned}$$

Since $U_0 = u_0$ and $1 - k\varepsilon^{-2}/2 \geq 1/2$ we deduce the proposition. □

## 5. Analysis of the semi-discrete in time approximation scheme

In this section we aim to derive a robust error estimate for the approximation scheme $(\mathrm{P}_k)$. The following definition introduces for each time step $t_m > 0$ a partition of unity which is adapted to the neighborhoods of vortices in Assumption I.

**Definition 5.1.** Given an integer $1 \leq m \leq M$ let $\big(\psi_{m,j} : j = 1, 2, ..., L_m\big) \subseteq C^\infty(\Omega; \mathbb{R})$ be a partition of unity with finite overlap $\alpha > 0$, *i.e.*, for all $x \in \Omega$ there holds

$$\mathrm{card}\big\{j \in \{1, 2, ..., L_m\} : \psi_{m,j}(x) > 0\big\} \leq \alpha,$$

such that for an $\varepsilon$-independent constant $c_3 > 0$ and for $j = 1, 2, ..., L_m$, there holds

$$\psi_{m,j} \geq 0, \quad \sum_{j=1}^{L_m}\psi_{m,j} = 1, \quad \text{and} \quad |\nabla\big(\psi_{m,j}^{1/2}\big)|^2 \leq c_3\delta_0^{-2} \quad \text{in } \Omega,$$

and, for $j = 1, 2, ..., d$,

$$\psi_{m,j} = 1 \quad \text{in } B_{\delta_0/2}(a_j^{t_m,\varepsilon}) \qquad \text{and} \qquad \operatorname{supp}\psi_{m,j} \subseteq B_{\delta_0}(a_j^{t_m,\varepsilon}).$$

The nonlinearity in the error equation defined by (P) and ($P_k$) requires a different treatment in neighborhoods of vortices and in the remaining parts of $\Omega$. Recall that $e_m = u(t_m) - U_m$ for all $0 \le m \le M$.

**Lemma 5.2.** *Suppose that Assumption I holds. For all $\varepsilon \in (0, \varepsilon_0)$, $1 \le m \le M$, and $j = d+1, d+2, ..., L_m$ there holds*

$$\varepsilon^{-2}(f(u(t_m)) - f(U_m); \psi_{m,j} e_m) \ge -c_0(c_1+1)\delta_0^{-2}\|\psi_{m,j}^{1/2}\, e_m\|_{L^2(\Omega)}^2 - \frac{\varepsilon^{-2}}{8}\big\|\psi_{m,j}^{1/4}\, e_m\big\|_{L^4(\Omega)}^4.$$

*Proof.* Given $1 \le m \le M$ we abbreviate $u = u(t_m)$, $U = U_m$, $e = e_m$, and $\psi_j = \psi_{m,j}$ for $j = d+1, d+2, ..., L$. Assumption I guarantees $\big||u|-1\big| \le c_0\delta_0^{-2}\varepsilon^2$ in $\operatorname{supp}\psi_j$ and we may therefore deduce

$$\begin{aligned}(f(u) - f(U); \psi_j e) &= \big(|u|^2-1)u; \psi_j e\big) - \big((|U|^2-1)U; \psi_j e\big)\\ &= \big((|u|^2-1)e; \psi_j e\big) + \big(\{(|u|^2-1) - (|U|^2-1)\}U; \psi_j e\big)\\ &= \big((|u|-1)(|u|+1)e; \psi_j e\big) + \big(\{(|u|^2-1) - (|U|^2-1)\}U; \psi_j e\big)\\ &\ge -c_0\delta_0^{-2}\varepsilon^2(c_1+1)\|\psi_j^{1/2} e\|_{L^2(\Omega)}^2 + \big((|u|^2 - |U|^2)U; \psi_j e\big).\end{aligned}$$

We manipulate the second term in the right-hand side as follows,

$$\begin{aligned}\big((|u|^2 - |U|^2)U; \psi_j e\big) &= \big(\{(u-U)\cdot(u+U)\}U; \psi_j e\big)\\ &= \big(\{e\cdot(e+2U)\}U; \psi_j e\big)\\ &= \big(|e|^2 U; \psi_j e\big) + 2\big((e\cdot U)U; \psi_j e\big)\\ &= \big(|e|^2 U; \psi_j e\big) + 2||\psi_j^{1/2}(e\cdot U)||_{L^2(\Omega)}^2\\ &\ge -\frac{1}{8}\big\|\psi_j^{1/2}|e|^2\big\|_{L^2(\Omega)}^2 - 2\big\|\psi_j^{1/2}(e\cdot U)\big\|_{L^2(\Omega)}^2 + 2||\psi_j^{1/2}(e\cdot U)||_{L^2(\Omega)}^2\\ &= -\frac{1}{8}\big\|\psi_j^{1/2}|e|^2\big\|_{L^2(\Omega)}^2 = -\frac{1}{8}\big\|\psi_j^{1/4} e\big\|_{L^4(\Omega)}^4.\end{aligned}$$

The lemma follows from a combination of the two estimates. □

**Lemma 5.3.** *Suppose that Assumption I holds. For all $\varepsilon \in (0, \varepsilon_0)$, $1 \le m \le M$, and $j = 1, 2, ..., d$ there holds*

$$\begin{aligned}\varepsilon^{-2}\big(f(u(t_m)) - f(U_m); \psi_{m,j} e_m\big) &\ge -\lambda_0(1-\varepsilon^2)||\psi_{m,j}^{1/2}\, e_m||_{L^2(\Omega)}^2\\ &\quad - (1-\varepsilon^2)||\nabla(\psi_{m,j}^{1/2}\, e_m)||_{L^2(\Omega)}^2 - 3c_1\varepsilon^{-2}||\psi_{m,j}^{1/3}\, e_m||_{L^3(\Omega)}^3 - ||\psi_{m,j}^{1/2}\, e_m||_{L^2(\Omega)}^2.\end{aligned}$$

*Proof.* As in the previous proof we abbreviate $u = u(t_m)$, $U = U_m$, $e = e_m$, and $\psi_j = \psi_{m,j}$ for $j = 1, 2, ..., d$. Given any $a, b \in \mathbb{C}$ there holds

$$(f(a) - f(b))\cdot(a-b) \ge f'(a)(a-b)\cdot(a-b) - 3|a||a-b|^3$$

and

$$f'(a)(a-b)\cdot(a-b) \ge -|a-b|^2.$$

We thus have for all $\theta \in [0,1]$,

$$\begin{aligned}\big(f(u)-f(U);\psi_j e\big) &\geq \big(f'(u)e,\psi_j e\big) - 3\big(|u||e|^3;\psi_j\big)\\ &= (1-\theta)\big(f'(u)e,\psi_j e\big) - 3c_1||\psi_j^{1/3}e||_{L^3(\Omega)}^3 + \theta\big(f'(u)e;\psi_j e\big)\\ &\geq (1-\theta)\big(f'(u)e,\psi_j e\big) - 3c_1||\psi_j^{1/3}e||_{L^3(\Omega)}^3 - \theta||\psi_j^{1/2}e||_{L^2(\Omega)}^2.\end{aligned}$$

Applying the spectral estimate of Proposition 2.3 to $v := \psi_j^{1/2}e$ we infer

$$\big(f'(u)e;\psi_j e\big) = \big(f'(u)v;v\big) \geq -\lambda_0\varepsilon^2||v||_{L^2(\Omega)}^2 - \varepsilon^2||\nabla v||_{L^2(\Omega)}^2.$$

Choosing $\theta = \varepsilon^2$ we verify the assertion of the lemma. □

The following lemma gives bounds on the higher norms of the error that arose in the previous estimates.

**Lemma 5.4.** *Suppose that there exists a positive function $\rho : \mathbb{R} \to \mathbb{R}$ such that $\rho \leq \rho^2$ and $||\nabla e_m||_{L^2(\Omega)} \leq \rho_\varepsilon := \rho(\varepsilon)$ for all $1 \leq m \leq M$ and all $\varepsilon \in (0,\varepsilon_0)$. Then there exists an $\varepsilon$-independent constant $c_4 > 0$ such that for all $\varepsilon \in (0,\varepsilon_0)$ and all $1 \leq m \leq M$ there holds*

$$||e_m||_{L^3(\Omega)}^3 + ||e_m||_{L^4(\Omega)}^4 \leq c_4\rho_\varepsilon^2 k^2||d_t e_m||_{L^2(\Omega)}^2 + c_4\rho_\varepsilon||e_{m-1}||_{L^2(\Omega)}||\nabla e_{m-1}||_{L^2(\Omega)}^2.$$

*Proof.* Employing the estimate $||v||_{L^4(\Omega)}^2 \leq C||v||_{L^2(\Omega)}||\nabla v||_{L^2(\Omega)}$ for $v \in H_0^1(\Omega;\mathbb{C})$ and the bounds $C||e_m||_{L^2(\Omega)} \leq ||\nabla e_m||_{L^2(\Omega)} \leq \rho_\varepsilon$ we infer for $1 \leq m \leq M$

$$\begin{aligned}\frac{1}{8}||e_m||_{L^4(\Omega)}^4 &\leq ||e_m - e_{m-1}||_{L^4(\Omega)}^4 + ||e_{m-1}||_{L^4(\Omega)}^4\\ &= k^4||d_t e_m||_{L^4(\Omega)}^4 + ||e_{m-1}||_{L^4(\Omega)}^4\\ &\leq Ck^4||d_t e_m||_{L^2(\Omega)}^2||d_t\nabla e_m||_{L^2(\Omega)}^2 + ||e_{m-1}||_{L^4(\Omega)}^4\\ &= Ck^2||d_t e_m||_{L^2(\Omega)}^2||\nabla(e_m - e_{m-1})||_{L^2(\Omega)}^2 + ||e_{m-1}||_{L^4(\Omega)}^4\\ &\leq C\rho_\varepsilon^2 k^2||d_t e_m||_{L^2(\Omega)}^2 + ||e_{m-1}||_{L^4(\Omega)}^4\\ &\leq C\rho_\varepsilon^2 k^2||d_t e_m||_{L^2(\Omega)}^2 + C||e_{m-1}||_{L^2(\Omega)}^2||\nabla e_{m-1}||_{L^2(\Omega)}^2.\end{aligned}$$

Similarly, using $||v||_{L^3(\Omega)}^3 \leq C||v||_{L^4(\Omega)}^2||v||_{L^2(\Omega)}$ and $||v||_{L^4(\Omega)} \leq C||\nabla v||_{L^2(\Omega)}$, we derive

$$\begin{aligned}\frac{1}{4}||e_m||_{L^3(\Omega)}^3 &\leq ||e_m - e_{m-1}||_{L^3(\Omega)}^3 + ||e_{m-1}||_{L^3(\Omega)}^3\\ &= k^3||d_t e_m||_{L^3(\Omega)}^3 + ||e_{m-1}||_{L^3(\Omega)}^3\\ &\leq Ck^3||d_t e_m||_{L^4(\Omega)}^2||d_t e_m||_{L^2(\Omega)} + ||e_{m-1}||_{L^3(\Omega)}^3\\ &\leq Ck^3||d_t e_m||_{L^2(\Omega)}^2||d_t\nabla e_m||_{L^2(\Omega)} + ||e_{m-1}||_{L^3(\Omega)}^3\\ &\leq C\rho_\varepsilon k^2||d_t e_m||_{L^2(\Omega)}^2 + ||e_{m-1}||_{L^3(\Omega)}^3\\ &\leq C\rho_\varepsilon k^2||d_t e_m||_{L^2(\Omega)}^2 + C||e_{m-1}||_{L^2(\Omega)}||\nabla e_{m-1}||_{L^2(\Omega)}^2.\end{aligned}$$

This proves the lemma. □

The next lemma gives a bound on the time discretization residual.

**Lemma 5.5.** *For* $1 \le m \le M$ *let* $\mathcal{R}(u_{tt};m) := u_t(t_m) - d_t u(t_m)$. *There holds*

$$k \sum_{m=1}^{M} ||\mathcal{R}(u_{tt};m)||^2_{H^{-1}(\Omega;\mathbb{C})} \le \frac{1}{3} k^2 \int_0^T ||u_{tt}||^2_{H^{-1}(\Omega;\mathbb{C})} \, \mathrm{d}s.$$

*Proof.* Repeated integration by parts shows

$$\mathcal{R}(u_{tt};m) = u_t(t_m) - d_t u(t_m) = \frac{1}{k} \int_{t_{m-1}}^{t_m} (s - t_{m-1}) u_{tt}(s) \, \mathrm{d}s.$$

Therefore, we have

$$\begin{aligned} k \sum_{m=1}^{M} ||\mathcal{R}(u_{tt};m)||^2_{H^{-1}(\Omega;\mathbb{C})} &= k \sum_{m=1}^{M} \Big\| \frac{1}{k} \int_{t_{m-1}}^{t_m} (s - t_{m-1}) u_{tt}(s) \, \mathrm{d}s \Big\|^2_{H^{-1}(\Omega;\mathbb{C})} \\ &\le \frac{1}{k} \sum_{m=1}^{M} \Big( \int_{t_{m-1}}^{t_m} (s - t_{m-1})^2 \, \mathrm{d}s \Big) \Big( \int_{t_{m-1}}^{t_m} \|u_{tt}(s)\|^2_{H^{-1}(\Omega;\mathbb{C})} \, \mathrm{d}s \Big) \\ &= \frac{1}{3} k^2 \sum_{m=1}^{M} \int_{t_{m-1}}^{t_m} \|u_{tt}(s)\|^2_{H^{-1}(\Omega;\mathbb{C})} \, \mathrm{d}s \le \frac{1}{3} k^2 \int_0^T \|u_{tt}(s)\|^2_{H^{-1}(\Omega;\mathbb{C})} \, \mathrm{d}s. \end{aligned}$$

This proves the lemma. □

We are now in position to prove the main result of this section.

**Theorem 5.6.** *Suppose that Assumptions I and II hold and assume that*

$$k \le \min\Big\{ \frac{1}{96\sqrt{2} c_1 c_2^{1/2} c_4} \varepsilon^7 \gamma_\varepsilon^{-1} \exp(2C_{\delta_0} T)^{-3/2}, \frac{1}{4C_{\delta_0}}, \frac{1}{192 c_1 c_4} \varepsilon^2 \gamma_\varepsilon^{-1}, \varepsilon^2 \Big\},$$

*where* $C_{\delta_0} = (\lambda_0 + c_3 \alpha \delta_0^{-2}) + 1 + c_0(c_1 + 1)\delta_0^{-2}$. *For all* $\varepsilon \in (0, \varepsilon_0)$ *and* $1 \le \ell \le M$ *there holds*

$$\begin{aligned} \frac{1}{2} \max_{m=1,2,\ldots,\ell} ||e_m||^2_{L^2(\Omega)} + k \sum_{m=1}^{\ell} \Big( \frac{1}{2} k ||d_t e_m||^2_{L^2(\Omega)} + \varepsilon^2 ||\nabla e_m||^2_{L^2(\Omega)} \Big) \\ \le 2 c_2 k^2 \varepsilon^{-6} \gamma_\varepsilon \exp(2 C_{\delta_0} t_{\ell-1}). \end{aligned} \tag{5.1}$$

*Proof.*
**Step 1**. *Verification of the assumptions of the employed lemmas.* A combination of Propositions 3.1 and 4.1 with Assumption II yields

$$\|\nabla e_m\|^2_{L^2(\Omega)} \le 2 \|\nabla u(t_m)\|^2_{L^2(\Omega)} + 2 \|\nabla U_m\|^2_{L^2(\Omega)} \le 16 J_\varepsilon(u_0) \le 16 \gamma_\varepsilon$$

so that we may choose $\rho_\varepsilon = 4 \gamma_\varepsilon^{1/2}$ in Lemma 5.4. Assumption II, Proposition 3.1, and Lemma 5.5 imply (since $\varepsilon \le \varepsilon_0 \le 1$)

$$k \sum_{m=0}^{M} ||\mathcal{R}(u_{tt};m)||^2_{H^{-1}(\Omega;\mathbb{C})} \le \frac{1}{3} k^2 c_2 (\varepsilon^{-2} \gamma_\varepsilon + 2 \varepsilon^{-4} \gamma_\varepsilon) \le c_2 k^2 \varepsilon^{-4} \gamma_\varepsilon. \tag{5.2}$$

**Step 2**. *Derivation of an error equation.* Choosing $v = V = e_m$ for $1 \le m \le M$ and subtracting the first equation in ($\mathrm{P}_k$) from the first equation in (P) we deduce

$$(d_t e_m; e_m) + ||\nabla e_m||^2_{L^2(\Omega)} + \varepsilon^{-2} \big( f(u(t_m)) - f(U_m); e_m \big) = -\big( \mathcal{R}(u_{tt};m); e_m \big).$$

We use the identity

$$(d_t e_m; e_m) = \frac{1}{2} d_t ||e_m||^2_{L^2(\Omega)} + \frac{1}{2} k ||d_t e_m||^2_{L^2(\Omega)},$$

the estimate

$$-\big(\mathcal{R}(u_{tt}; m); e_m\big) \le \frac{\varepsilon^2}{2} ||\nabla e_m||^2_{L^2(\Omega)} + \frac{\varepsilon^{-2}}{2} ||\mathcal{R}(u_{tt}; m)||^2_{H^{-1}(\Omega;\mathbb{C})}, \tag{5.3}$$

and employ the partition of unity $(\psi_{m,j} : j = 1, 2, ..., L_m)$ to verify

$$\frac{1}{2} d_t ||e_m||^2_{L^2(\Omega)} + \frac{1}{2} k ||d_t e_m||^2_{L^2(\Omega)} + \big(1 - \frac{\varepsilon^2}{2}\big) ||\nabla e_m||^2_{L^2(\Omega)} \\ + \sum_{j=1}^{L_m} \varepsilon^{-2} \big(f(u(t_m)) - f(U_m); \psi_{m,j} e_m\big) \le \frac{\varepsilon^{-2}}{2} ||\mathcal{R}(u_{tt}; m)||^2_{H^{-1}(\Omega;\mathbb{C})}.$$

Employing Lemmas 5.2 and 5.3, $||\nabla(\psi^{1/2}_{m,j})||^2_{L^\infty(\Omega)} \le c_3 \delta_0^{-2}$, and the finite overlap $\alpha$ of $(\psi_{m,j} : j = 1, 2, ..., L_m)$ we verify with $\sum_{j=1}^{L_m} \nabla \psi_{m,j} = 0$ almost everywhere in $\Omega$,

$$\begin{aligned} \sum_{j=1}^{L_m} \varepsilon^{-2} \big(f(u(t_m)) - f(U_m); \psi_{m,j} e_m\big) &\ge -\big(\lambda_0(1 - \varepsilon^2) + 1 + c_0(c_1 + 1)\delta_0^{-2}\big) ||e_m||^2_{L^2(\Omega)} \\ &\quad - 3c_1 \varepsilon^{-2} ||e_m||^3_{L^3(\Omega)} - \frac{\varepsilon^{-2}}{8} \big\|e_m\big\|^4_{L^4(\Omega)} - (1 - \varepsilon^2) \sum_{j=1}^{L_m} ||\nabla(\psi^{1/2}_{m,j} e_m)||^2_{L^2(\Omega)} \\ &= -\big(\lambda_0(1 - \varepsilon^2) + 1 + c_0(c_1 + 1)\delta_0^{-2}\big) ||e_m||^2_{L^2(\Omega)} - 3c_1 \varepsilon^{-2} ||e_m||^3_{L^3(\Omega)} \\ &\quad - \frac{\varepsilon^{-2}}{8} \big\|e_m\big\|^4_{L^4(\Omega)} - (1 - \varepsilon^2) ||\nabla e_m||^2_{L^2(\Omega)} - (1 - \varepsilon^2) \sum_{j=1}^{L_m} ||e_m \nabla(\psi^{1/2}_{m,j})||^2_{L^2(\Omega)} \\ &\ge -\big((\lambda_0 + c_3 \alpha \delta_0^{-2})(1 - \varepsilon^2) + 1 + c_0(c_1 + 1)\delta_0^{-2}\big) ||e_m||^2_{L^2(\Omega)} - 3c_1 \varepsilon^{-2} ||e_m||^3_{L^3(\Omega)} \\ &\quad - \frac{\varepsilon^{-2}}{8} \big\|e_m\big\|^4_{L^4(\Omega)} - (1 - \varepsilon^2) ||\nabla e_m||^2_{L^2(\Omega)}. \end{aligned}$$

Estimating

$$(\lambda_0 + c_3 \alpha \delta_0^{-2})(1 - \varepsilon^2) + 1 + c_0(c_1 + 1)\delta_0^{-2} \le C_{\delta_0},$$

we deduce from the previous estimates

$$d_t ||e_m||^2_{L^2(\Omega)} + k ||d_t e_m||^2_{L^2(\Omega)} + \varepsilon^2 ||\nabla e_m||^2_{L^2(\Omega)} \le \varepsilon^{-2} ||\mathcal{R}(u_{tt}; m)||^2_{H^{-1}(\Omega;\mathbb{C})} \\ + 2C_{\delta_0} ||e_m||^2_{L^2(\Omega)} + \varepsilon^{-2} \max\{6c_1, 1/4\} \big(||e_m||^3_{L^3(\Omega)} + \|e_m\|^4_{L^4(\Omega)}\big).$$

Since $c_1 \ge 1$ we have $\max\{6c_1, 1/4\} = 6c_1$. Lemma 5.4 leads to

$$d_t ||e_m||^2_{L^2(\Omega)} + k ||d_t e_m||^2_{L^2(\Omega)} + \varepsilon^2 ||\nabla e_m||^2_{L^2(\Omega)} \\ \le \varepsilon^{-2} ||\mathcal{R}(u_{tt}; m)||^2_{H^{-1}(\Omega;\mathbb{C})} + 2C_{\delta_0} ||e_m||^2_{L^2(\Omega)} + 6c_1 c_4 \rho_\varepsilon^2 k^2 \varepsilon^{-2} ||d_t e_m||^2_{L^2(\Omega)} \\ + 6c_1 c_4 \rho_\varepsilon \varepsilon^{-2} ||e_{m-1}||_{L^2(\Omega)} ||\nabla e_{m-1}||^2_{L^2(\Omega)}.$$

The assumption $6c_1c_4\rho_\varepsilon^2 k\varepsilon^{-2} \le 1/2$ (where we substituted $\rho_\varepsilon^2 = 16\gamma_\varepsilon$) allows to absorb $(1/2)k||d_te_m||^2_{L^2(\Omega)}$ on the right-hand side of the previous estimate and implies

$$d_t||e_m||^2_{L^2(\Omega)} + \frac{1}{2}k||d_te_m||^2_{L^2(\Omega)} + \varepsilon^2||\nabla e_m||^2_{L^2(\Omega)} \le \varepsilon^{-2}||\mathcal{R}(u_{tt};m)||^2_{H^{-1}(\Omega;\mathbb{C})} + 2C_{\delta_0}||e_m||^2_{L^2(\Omega)} + 6c_1c_4\rho_\varepsilon\varepsilon^{-2}||e_{m-1}||_{L^2(\Omega)}||\nabla e_{m-1}||^2_{L^2(\Omega)}.$$

Multiplying the last estimate with $k$ and summing over $m = 1, 2, ..., \ell$ for any $1 \le \ell \le M$ and abbreviating $C' := 6c_1c_4$ yields with 5.2 and $e_0 = 0$

$$||e_\ell||^2_{L^2(\Omega)} + k\sum_{m=1}^{\ell}\Big(\frac{1}{2}k||d_te_m||^2_{L^2(\Omega)} + \varepsilon^2||\nabla e_m||^2_{L^2(\Omega)}\Big) \le c_2\gamma_\varepsilon k^2\varepsilon^{-6} + 2C_{\delta_0}k\sum_{m=1}^{\ell}||e_m||^2_{L^2(\Omega)} + C'\rho_\varepsilon\varepsilon^{-2}k\sum_{m=1}^{\ell}||e_{m-1}||_{L^2(\Omega)}||\nabla e_{m-1}||^2_{L^2(\Omega)}.$$

The assumption $2C_{\delta_0}k \le 1/2$ allows to absorb $||e_\ell||^2_{L^2(\Omega)}/2$ on the right-hand side. Employing once more $e_0 = 0$ we verify

$$\begin{aligned}\frac{1}{2}||e_\ell||^2_{L^2(\Omega)} + k\sum_{m=1}^{\ell}\Big(\frac{1}{2}k||d_te_m||^2_{L^2(\Omega)} + \varepsilon^2||\nabla e_m||^2_{L^2(\Omega)}\Big) &\le c_2\gamma_\varepsilon k^2\varepsilon^{-6} + 2C_{\delta_0}k\sum_{m=1}^{\ell-1}||e_m||^2_{L^2(\Omega)} \\ &\quad + \sqrt{2}C'\rho_\varepsilon\varepsilon^{-4}\Big(\frac{1}{\sqrt{2}}\max_{m=1,2,\dots,\ell-1}||e_m||_{L^2(\Omega)}\Big)\,k\sum_{m=1}^{\ell-1}\varepsilon^2||\nabla e_m||^2_{L^2(\Omega)}. \end{aligned} \tag{5.4}$$

**Step 3**. *Proof of* (5.1) *by induction over* $\ell$. With $\ell = 1$ estimate (5.4) reads

$$\frac{1}{2}||e_1||^2_{L^2(\Omega)} + k\Big(\frac{1}{2}k||d_te_1||^2_{L^2(\Omega)} + \varepsilon^2||\nabla e_1||^2_{L^2(\Omega)}\Big) \le c_2\gamma_\varepsilon k^2\varepsilon^{-6}$$

and proves (5.1). Suppose that (5.1) holds with $\ell$ replaced by $\ell - 1$. Using that estimate to bound the last term in the right-hand side of (5.4) we verify that

$$\frac{1}{2}||e_\ell||^2_{L^2(\Omega)} + k\sum_{m=1}^{\ell}\Big(\frac{1}{2}k||d_te_m||^2_{L^2(\Omega)} + \varepsilon^2||\nabla e_m||^2_{L^2(\Omega)}\Big) \le c_2\gamma_\varepsilon k^2\varepsilon^{-6} + 2C_{\delta_0}k\sum_{m=1}^{\ell-1}||e_m||^2_{L^2(\Omega)} + \sqrt{2}C'\rho_\varepsilon\varepsilon^{-4}\big(2c_2k^2\varepsilon^{-6}\gamma_\varepsilon\exp(2C_{\delta_0}t_{\ell-2})\big)^{3/2}.$$

Since the assumptions on $k$ imply

$$\sqrt{2}C'\rho_\varepsilon\varepsilon^{-4}\big(2c_2k^2\varepsilon^{-6}\gamma_\varepsilon\exp(2C_{\delta_0}t_{\ell-2})\big)^{3/2} \le c_2\gamma_\varepsilon k^2\varepsilon^{-6} \tag{5.5}$$

we have

$$\frac{1}{2}||e_\ell||^2_{L^2(\Omega)} + k\sum_{m=1}^{\ell}\Big(\frac{1}{2}k||d_t e_m||^2_{L^2(\Omega)} + \varepsilon^2||\nabla e_m||^2_{L^2(\Omega)}\Big) \leq 2c_2\gamma_\varepsilon k^2\varepsilon^{-6} + 2C_{\delta_0}k\sum_{m=1}^{\ell-1}||e_m||^2_{L^2(\Omega)}.$$

A discrete Gronwall inequality proves (5.1). □

**Remark.** Employing only Assumption II and using a local Lipschitz estimate to bound the nonlinear term (provided that $||u(t_m)||_{L^\infty(\Omega)} + ||U_m||_{L^\infty(\Omega)} \leq C$),

$$\varepsilon^{-2}\big(f(u(t_m)) - f(U_m); e_m) \geq -C_f\varepsilon^{-2}||e_m||^2_{L^2(\Omega)},$$

one can prove the error estimate

$$\frac{1}{2}\max_{m=1,2,\ldots,\ell}||e_m||^2_{L^2(\Omega)} + k\sum_{m=1}^{\ell}\Big(\frac{1}{2}k||d_t e_m||^2_{L^2(\Omega)} + \frac{1}{2}||\nabla e_m||^2_{L^2(\Omega)}\Big) \leq Ck^2\varepsilon^{-4}\exp(C_f\varepsilon^{-2}t_{\ell-1}) \tag{5.6}$$

for $1 \leq \ell \leq M$. This estimate is, owing to its exponential dependence on $\varepsilon^{-2}$, useful only if $\varepsilon$ is large or if $t_{\ell-1} = \mathcal{O}(\varepsilon^2)$. The estimate may be used to impose Assumption I only for $t \in [C\varepsilon^2, T]$. This is of interest if the initial data involve higher degree vortices which split into well separated degree-one vortices within a time $\mathcal{O}(\varepsilon^2)$.

## 6. Discussion of a fully discrete approximation scheme

For a time and space discretization of (P) we assume that $\Omega$ is polygonal and let $\mathcal{T}$ be a quasi-uniform regular triangulation of $\Omega$ with maximal mesh-size $h$. We let $\mathcal{S}^1(\mathcal{T};\mathbb{C})$ denote the lowest order finite element space which consists of all $\mathcal{T}$-elementwise affine, globally continuous functions. The subset $\mathcal{S}^1_0(\mathcal{T};\mathbb{C})$ is defined as $\mathcal{S}^1_0(\mathcal{T};\mathbb{C}) := \{v_h \in \mathcal{S}^1(\mathcal{T};\mathbb{C}) : v_h|_{\partial\Omega} = 0\}$. We let $u_{h,0}$ be the nodal interpolant of $u_0$, and denote by $g_h$ the restriction of $u_{h,0}$ to $\partial\Omega$. As for the semi-discrete problem ($\text{P}_k$) we assume that we are given a time step size $k > 0$. With the notation of Section 4, the fully discrete problem reads:

$$(\text{P}_{k,h}) \qquad \begin{cases} \textit{Find } (U_{h,m} : 0 \leq m \leq M) \subseteq \mathcal{S}^1(\mathcal{T};\mathbb{C}) \textit{ such that for all} \\ 1 \leq m \leq M \textit{ and all } V_h \in \mathcal{S}^1_0(\mathcal{T};\mathbb{C}) \textit{ there holds} \\ (d_t U_{h,m}; V_h) + (\nabla U_{h,m}; \nabla V_h) + \varepsilon^{-2}\big(f(U_{h,m}); V_h) = 0, \\ \qquad U_{h,m}|_{\partial\Omega} = g_h, \\ \qquad U_{h,0} = u_{h,0}. \end{cases}$$

Existence of a unique solution $(U_{h,m})$ holds if $k \leq \varepsilon^2$. For $0 \leq m \leq M$ we define

$$E_m := u(t_m) - U_{h,m}.$$

The following *a priori* bound for the solution of ($\text{P}_{k,h}$) is proved as Proposition 4.1.

**Proposition 6.1.** *Suppose that $k \le \varepsilon^2$. There holds*

$$\max_{1\le m\le M}\left(\frac{1}{2}||\nabla U_{h,m}||^2_{L^2(\Omega)} + \frac{\varepsilon^{-2}}{4}\big\||U_{h,m}|^2-1\big\|^2_{L^2(\Omega)}\right) + k\sum_{m=1}^{M}\left(\frac{1}{2}||d_tU_{h,m}||^2_{L^2(\Omega)} + \frac{k\varepsilon^{-2}}{4}\big\|d_t(|U_{h,m}|^2-1)\big\|^2_{L^2(\Omega)}\right) \le 2J_\varepsilon(u_{h,0}). \quad \square$$

Additional regularity of $u$ is needed for the *a priori* error analysis of $(\mathrm{P}_{k,h})$. The proofs need further assumptions.

**Assumption III.** *The domain $\Omega$ is convex, there holds $|g(s)| = 1$ for all $s \in \partial\Omega$, and there exist an $\varepsilon$-independent constant $c_{10} > 0$ and an integer $\sigma \ge 0$ such that*

$$\lim_{s\to 0^+} ||\nabla u_t(s)||^2_{L^2(\Omega)} \le c_{10}\varepsilon^{-\sigma} \tag{6.1}$$

*for all $\varepsilon \in (0, \varepsilon_0)$.*

**Proposition 6.2.** *Suppose that there exists an $\varepsilon$-independent constant $c_1 > 0$ such that for almost all $(t,x) \in (0,T)\times\Omega$ there holds $|u(t,x)| \le c_1$ and suppose that Assumption III holds. There exists an $\varepsilon$-independent constant $c_{11} > 0$ such that*

(*i*) $\displaystyle\int_0^T ||u_{tt}||^2_{L^2(\Omega)}\,\mathrm{d}s \le c_{11}\big(\varepsilon^{-4}J_\varepsilon(u_0) + \varepsilon^{-\sigma}\big)$;

(*ii*) $\displaystyle\int_0^T ||u_t||^2_{H^2(\Omega)}\,\mathrm{d}s \le c_{11}\big(\varepsilon^{-4}J_\varepsilon(u_0) + \varepsilon^{-\sigma}\big)$;

(*iii*) $\operatorname{ess\,sup}_{s\in(0,T)}||u||^2_{H^2(\Omega)} \le c_{11}\big(\varepsilon^{-2}J_\varepsilon(u_0) + ||\Delta u_0 - \varepsilon^{-2}f(u_0)||^2_{L^2(\Omega)} + ||u_0||^2_{H^2(\Omega)}\big)$.

*Proof.* The proof of the proposition is similar to the proof of Proposition 2 in [16]. □

Some basic properties of approximation operators are summarized in the following proposition. For proofs we refer the reader to [5, 16, 35, 36].

**Proposition 6.3.**
(*i*) *Given $v \in H^1(\Omega;\mathbb{C})$ let $P_hv \in \mathcal{S}^1_0(\mathcal{T};\mathbb{C})$ be defined by*

$$\big(\nabla(P_hv - v); \nabla w_h\big) = 0$$

*for all $w_h \in \mathcal{S}^1_0(\mathcal{T};\mathbb{C})$. For $0 \le m \le M$ set $\phi_m := P_hE_m \in \mathcal{S}^1_0(\mathcal{T};\mathbb{C})$ and $\theta_m := E_m - \phi_m$. There exists an $\varepsilon$-independent constant $c_{12} > 0$ such that*

$$\begin{aligned} ||\theta_m||^2_{L^2(\Omega)} + h^2||\nabla\theta_m||^2_{L^2(\Omega)} &\le c_{12}h^4||u(t_m)||^2_{H^2(\Omega)} \\ ||\theta_m||_{L^\infty(\Omega)} &\le c_{12}h|\log h|^{1/2}||u(t_m)||_{H^2(\Omega)} \\ k\sum_{m=1}^{M}||d_t\theta_m||^2_{L^2(\Omega)} &\le c_{12}h^4\int_0^T ||u_t||^2_{H^2(\Omega)}\,\mathrm{d}s. \end{aligned} \tag{6.2}$$

(*ii*) *There exists an $\varepsilon$-independent constant $c_{13} > 0$ such that*

$$\begin{aligned} J_\varepsilon(u_{h,0}) &\le J_\varepsilon(u_0) + c_{13}h||u_0||^2_{H^2(\Omega)}, \\ ||\phi_0||^2_{L^2(\Omega)} + h^2||\nabla\phi_0||^2_{L^2(\Omega)} &\le c_{13}h^4||u_0||^2_{H^2(\Omega)}. \end{aligned}$$

The following estimates are minor modifications of estimates given in Section 5. Notice that $\phi_m + U_m$ approximates $u(t_m)$.

**Proposition 6.4.** *Suppose that Assumption I holds, let* $1 \le m \le M$*, and suppose that* $||\theta_m||_{L^\infty(\Omega)} \le \varepsilon^2$*. There exist* $\varepsilon$*-independent constants* $\tilde\lambda_0, \tilde c_0, \tilde c_4, c_{14} > 0$ *and* $\tilde c_1 > 1$ *such that*
$(i)$ *for* $j = 1, 2, ..., d$ *and all* $v \in H^1_0(B_{\delta_0}(a_j); \mathbb{C})$ *there holds*

$$(\nabla v; \nabla v) + \varepsilon^{-2}(f'(\phi_m + U_{h,m})v; v) \ge -\tilde\lambda_0 ||v||^2_{L^2(B_{\delta_0}(a_j))};$$

$(ii)$ *for* $j = d+1, d+2, ..., L_m$ *there holds*

$$\varepsilon^{-2}(f(\phi_m + U_{h,m}) - f(U_{h,m}); \psi_{m,j}\phi_m) \ge -\tilde c_0(\tilde c_1 + 1)\delta_0^{-2} \|\psi_{m,j}^{1/2}\, \phi_m\|^2_{L^2(\Omega)} - \frac{\varepsilon^{-2}}{8} \|\psi_{m,j}^{1/4}\, \phi_m\|^4_{L^4(\Omega)};$$

$(iii)$ *for* $j = 1, 2, ..., d$ *there holds*

$$\varepsilon^{-2}\big(f(\phi_m + U_{h,m}) - f(U_{h,m}); \psi_{m,j}\phi_m\big) \ge -\tilde\lambda_0(1-\varepsilon^2)||\psi_{m,j}^{1/2}\, \phi_m||^2_{L^2(\Omega)} - (1-\varepsilon^2)||\nabla(\psi_{m,j}^{1/2}\, \phi_m)||^2_{L^2(\Omega)} - 3\tilde c_1 \varepsilon^{-2}||\psi_{m,j}^{1/3}\, \phi_m||^3_{L^3(\Omega)} - ||\psi_{m,j}^{1/2}\, \phi_m||^2_{L^2(\Omega)};$$

$(iv)$ *there holds*

$$||\phi_m||^3_{L^3(\Omega)} + ||\phi_m||^4_{L^4(\Omega)} \le \tilde c_4 \tilde\rho_\varepsilon^2 k^2 ||d_t\phi_m||^2_{L^2(\Omega)} + \tilde c_4 \tilde\rho_\varepsilon ||\phi_{m-1}||_{L^2(\Omega)} ||\nabla\phi_{m-1}||^2_{L^2(\Omega)},$$

*provided* $\max_{j=1,2,...,M} ||\nabla\phi_j||_{L^2(\Omega)} \le \tilde\rho_\varepsilon$ *for some* $\tilde\rho : \mathbb{R} \to \mathbb{R}$ *with* $\tilde\rho_\varepsilon := \tilde\rho(\varepsilon) \le \tilde\rho(\varepsilon)^2$*;*
$(v)$ *there holds*

$$k \sum_{m=1}^{M} ||\mathcal{R}(u_{tt}; m)||^2_{L^2(\Omega)} \le \frac{1}{3} k^2 \int_0^T ||u_{tt}||^2_{L^2(\Omega)}\, \mathrm{d}s;$$

$(vi)$ *there holds*

$$||f(\phi_m + U_{h,m}) - f(u(t_m))||^2_{L^2(\Omega)} \le c_{14} ||\theta_m||^2_{L^2(\Omega)}.$$

*Proof.* Notice $||(\phi_m + U_{h,m}) - u(t_m)||_{L^\infty(\Omega)} = ||\theta_m||_{L^\infty(\Omega)} \le \varepsilon^2$. A local Lipschitz estimate for $f'$ yields

$$\varepsilon^{-2}\big(f'(\phi_m + U_{h,m})v; v\big) \ge \big(f'(u(t_m))v; v\big) - C||v||^2_{L^2(\Omega)}$$

and the proof of $(i)$ follows from Proposition 2.3. The proofs of $(ii)$-$(v)$ follow the lines of the proofs of Lemmas 5.2, 5.3, 5.4, 5.5, respectively. Assertion $(vi)$ follows from uniform bounds for $|u(t_m)|$ and $|\phi_m + U_{h,m}|$ and local Lipschitz continuity of $f$. □

The previous estimates allow to prove a robust *a priori* error estimate for the approximation scheme $(\mathrm{P}_{k,h})$.

**Theorem 6.5.** *Suppose that Assumptions I, II, and III hold and assume that*

$$h \le \min\{1, \varepsilon^4\}, \quad h|\log h|^{1/2} \le \frac{1}{3c_{11}c_{13}} \varepsilon^4 \gamma_\varepsilon^{-1/2}, \quad \textit{and} \quad h^4 \le \frac{(\tilde C' \tilde C'')^2}{c_{13}^3} \tilde\rho_\varepsilon^{-2} \gamma_\varepsilon^{-1},$$
$$k \le \min\Big\{\frac{1}{12\tilde c_1 \tilde c_4} \varepsilon^2 \tilde\rho_\varepsilon^{-2}, \frac{1}{6 + 4\tilde C_{\delta_0}}, \varepsilon^2\Big\},$$
$$k + h^2 \le \frac{1}{\sqrt{2}\tilde C'' 4(\tilde C')^{1/2}} \varepsilon^4 \big(\max\{\varepsilon^{-\sigma}, \varepsilon^{-8}\gamma_\varepsilon\}\big)^{-1/2} \exp((3 + 2\tilde C_{\delta_0})T)^{-3/2},$$

*where* $\tilde{C}'' = 6\tilde{c}_1\tilde{c}_4$, $\tilde{\rho}_\varepsilon = (16+8c_{13})^{1/2}\gamma_\varepsilon^{1/2}$, *and*

$$\tilde{C}_{\delta_0} = (\tilde{\lambda}_0 + \tilde{c}_3\alpha\delta_0^{-2}) + 1 + \tilde{c}_0(\tilde{c}_1+1)\delta_0^{-2}, \quad \tilde{C}' = 2(c_{13} + 2c_{11}c_{12} + 3c_{11}c_{13}c_{14} + c_{11}).$$

*For all* $\varepsilon \in (0,\varepsilon_0)$ *and* $1 \le \ell \le M$ *there holds*

$$\frac{1}{4}\max_{m=1,2,\dots,\ell} ||E_m||^2_{L^2(\Omega)} + k\sum_{m=1}^{\ell} \varepsilon^2 h^2 ||\nabla E_m||^2_{L^2(\Omega)} \le 4\tilde{C}'(k^2+h^4)\max\{\varepsilon^{-\sigma}, \varepsilon^{-8}\gamma_\varepsilon\}\exp((3+2C_{\delta_0})t_{\ell-1}) + 3c_{11}c_{12}h^4\varepsilon^{-4}\gamma_\varepsilon(1+2\varepsilon^2 T). \quad (6.3)$$

*Proof.*
**Step 1**. *Verification of the assumptions of the employed lemmas.* Propositions 6.3 and 6.2, Assumption II and the assumption on $h$ imply

$$||\theta_m||_{L^\infty(\Omega)} \le c_{13}h|\log h|^{1/2} 3c_{11}\varepsilon^{-2}\gamma_\varepsilon^{1/2} \le \varepsilon^2.$$

$H^1$ stability of $P_h$, Propositions 3.1 and 6.3, Assumption II, and the condition $h \le \varepsilon^4$ prove

$$\begin{aligned}||\nabla\phi_m||^2_{L^2(\Omega)} &\le ||\nabla E_m||^2_{L^2(\Omega)} \le 2||\nabla u(t_m)||^2_{L^2(\Omega)} + 2||\nabla U_{h,m}||^2_{L^2(\Omega)} \\ &\le 8J_\varepsilon(u_0) + 8J_\varepsilon(u_{h,0}) \le 16J_\varepsilon(u_0) + 8c_{13}h||u_0||^2_{H^2(\Omega)} \le 16\gamma_\varepsilon + 8c_{13}\gamma_\varepsilon\end{aligned}$$

so that we may choose $\tilde{\rho}_\varepsilon = (16+8c_{13})^{1/2}\gamma_\varepsilon^{1/2}$ in Proposition 6.4.
**Step 2**. *Derivation of an error equation.* Choosing $v = V_h = \phi_m$ for $1 \le m \le M$ and subtracting the first equation in $(\mathrm{P}_{k,h})$ from the first equation in (P) we deduce

$$(d_t E_m;\phi_m) + (\nabla e_m;\nabla\phi_m) + \varepsilon^{-2}\big(f(u(t_m)) - f(U_{h,m});\phi_m\big) = -\big(\mathcal{R}(u_{tt};m);\phi_m\big).$$

We use the identities

$$(d_t E_m;\phi_m) = (d_t\theta_m;\phi_m) + \frac{1}{2}d_t||\phi_m||^2_{L^2(\Omega)} + \frac{1}{2}k||d_t\phi_m||^2_{L^2(\Omega)}$$

and $\big(\nabla(E_m-\phi_m);\nabla\phi_m\big) = 0$, Cauchy inequalities, and $(vi)$ of Proposition 6.4 to verify

$$\begin{aligned}&\frac{1}{2}d_t||\phi_m||^2_{L^2(\Omega)} + \frac{1}{2}k||d_t\phi_m||^2_{L^2(\Omega)} + ||\nabla\phi_m||^2_{L^2(\Omega)} + \varepsilon^{-2}\big(f(\phi_m+U_{h,m}) - f(U_{h,m});\phi_m\big) \\ &\quad = -(d_t\theta_m;\phi_m) - \big(\mathcal{R}(u_{tt};m);\phi_m\big) + \varepsilon^{-2}\big(f(\phi_m+U_{h,m}) - f(u(t_m));\phi_m\big) \\ &\quad \le \frac{1}{2}||d_t\theta_m||^2_{L^2(\Omega)} + \frac{3}{2}||\phi_m||^2_{L^2(\Omega)} + \frac{1}{2}||\mathcal{R}(u_{tt};m)||^2_{L^2(\Omega)} + c_{14}\frac{\varepsilon^{-4}}{2}||\theta_m||^2_{L^2(\Omega)}.\end{aligned}$$

We argue as in the proof of Theorem 5.6 to show with $(ii)$ and $(iii)$ of Proposition 6.4

$$\begin{aligned}\varepsilon^{-2}\big(f(\phi_m+U_{h,m}) - f(U_{h,m});\phi_m\big) &= \sum_{j=1}^{L_m}\varepsilon^{-2}\big(f(\phi_m+U_{h,m}) - f(U_{h,m});\psi_{m,j}\phi_m\big) \\ &\ge -\tilde{C}_{\delta_0}||\phi_m||^2_{L^2(\Omega)} - 3\tilde{c}_1\varepsilon^{-2}||\phi_m||^3_{L^3(\Omega)} - \frac{\varepsilon^{-2}}{8}\big\|\phi_m\big\|^4_{L^4(\Omega)} - (1-\varepsilon^2)||\nabla\phi_m||^2_{L^2(\Omega)}.\end{aligned}$$

Estimate $(iv)$ of Proposition 6.4 yields

$$3\tilde{c}_1\varepsilon^{-2}||\phi_m||^3_{L^3(\Omega)} + \frac{\varepsilon^{-2}}{8}\big\|\phi_m\big\|^4_{L^4(\Omega)} \le \varepsilon^{-2}\max\{3\tilde{c}_1, 1/8\}\tilde{c}_4\left(\tilde{\rho}_\varepsilon^2 k^2||d_t\phi_m||^2_{L^2(\Omega)} + \tilde{\rho}_\varepsilon||\phi_{m-1}||_{L^2(\Omega)}||\nabla\phi_{m-1}||^2_{L^2(\Omega)}\right).$$

A combination of the last three estimates shows

$$
\begin{aligned}
&d_t||\phi_m||^2_{L^2(\Omega)} + k||d_t\phi_m||^2_{L^2(\Omega)} + 2\varepsilon^2||\nabla\phi_m||^2_{L^2(\Omega)} \\
&\qquad \le ||d_t\theta_m||^2_{L^2(\Omega)} + c_{14}\varepsilon^{-4}||\theta_m||^2_{L^2(\Omega)} + (3+2\tilde{C}_{\delta_0})||\phi_m||^2_{L^2(\Omega)} + ||\mathcal{R}(u_{tt};m)||^2_{L^2(\Omega)} \\
&\qquad\qquad + \varepsilon^{-2}\max\{6\tilde{c}_1, 1/4\}\tilde{c}_4\big(\tilde{\rho}_\varepsilon^2 k^2||d_t\phi_m||^2_{L^2(\Omega)} + \tilde{\rho}_\varepsilon||\phi_{m-1}||_{L^2(\Omega)}||\nabla\phi_{m-1}||^2_{L^2(\Omega)}\big).
\end{aligned}
$$

We note $\max\{6\tilde{c}_1, 1/4\} = 6\tilde{c}_1$, multiply the last estimate with $k$, and sum over $m = 1, 2, ..., \ell$ to verify

$$
\begin{aligned}
&||\phi_\ell||^2_{L^2(\Omega)} + k\sum_{m=1}^{\ell}\left(\frac{k}{2}||d_t\phi_m||^2_{L^2(\Omega)} + 2\varepsilon^2||\nabla\phi_m||^2_{L^2(\Omega)}\right) \\
&\qquad \le ||\phi_0||^2_{L^2(\Omega)} + k\sum_{m=1}^{\ell}||d_t\theta_m||^2_{L^2(\Omega)} + c_{14}\varepsilon^{-4}k\sum_{m=1}^{\ell}||\theta_m||^2_{L^2(\Omega)} + k\sum_{m=1}^{\ell}||\mathcal{R}(u_{tt};m)||^2_{L^2(\Omega)} \\
&\qquad\qquad + (3+2\tilde{C}_{\delta_0})k\sum_{m=1}^{\ell}||\phi_m||^2_{L^2(\Omega)} + 6\tilde{c}_1\tilde{c}_4\varepsilon^{-2}\tilde{\rho}_\varepsilon k\sum_{m=1}^{\ell}||\phi_{m-1}||_{L^2(\Omega)}||\nabla\phi_{m-1}||^2_{L^2(\Omega)},
\end{aligned}
$$

where we used $k6\tilde{c}_1\tilde{c}_4\varepsilon^{-2}\tilde{\rho}_\varepsilon^2 \le 1/2$ to absorb a sum over $k||d_t\phi_m||^2_{L^2(\Omega)}$ on the right hand side. Using estimates of Propositions 6.3 and 6.2 and Assumption II to bound the first four terms on the right-hand side and employing $(3+2\tilde{C}_{\delta_0})k \le 1/2$ to absorb $\frac{1}{2}||\phi_\ell||^2_{L^2(\Omega)}$ on the right-hand side we find

$$
\begin{aligned}
&\frac{1}{2}||\phi_\ell||^2_{L^2(\Omega)} + k\sum_{m=1}^{\ell}\left(\frac{k}{2}||d_t\phi_m||^2_{L^2(\Omega)} + 2\varepsilon^2||\nabla\phi_m||^2_{L^2(\Omega)}\right) \\
&\le \big(c_{13}\varepsilon^{-4}\gamma_\varepsilon + c_{12}c_{11}(\varepsilon^{-4}\gamma_\varepsilon + \varepsilon^{-\sigma}) + c_{14}\varepsilon^{-4}c_{13}3c_{12}\varepsilon^{-4}\gamma_\varepsilon\big)h^4 + \frac{k^2}{3}c_{11}(\varepsilon^{-4}\gamma_\varepsilon + \varepsilon^{-\sigma}) \\
&\qquad + (3+2\tilde{C}_{\delta_0})k\sum_{m=1}^{\ell-1}||\phi_m||^2_{L^2(\Omega)} + 6\tilde{c}_1\tilde{c}_4\varepsilon^{-2}\tilde{\rho}_\varepsilon k\sum_{m=0}^{\ell-1}||\phi_m||_{L^2(\Omega)}||\nabla\phi_m||^2_{L^2(\Omega)} \\
&\le \tilde{C}'(h^4+k^2)\max\{\varepsilon^{-\sigma}, \varepsilon^{-8}\gamma_\varepsilon\} \\
&\qquad + (3+2\tilde{C}_{\delta_0})k\sum_{m=1}^{\ell-1}||\phi_m||^2_{L^2(\Omega)} + \tilde{C}''\varepsilon^{-2}\tilde{\rho}_\varepsilon k\sum_{m=0}^{\ell-1}||\phi_m||_{L^2(\Omega)}||\nabla\phi_m||^2_{L^2(\Omega)}.
\end{aligned}
$$

Proposition 6.3 implies

$$
\begin{aligned}
\tilde{C}''\varepsilon^{-2}\tilde{\rho}_\varepsilon k||\phi_0||_{L^2(\Omega)}||\nabla\phi_0||^2_{L^2(\Omega)} &\le \tilde{C}''\varepsilon^{-2}\tilde{\rho}_\varepsilon k h^4 c_{13}^{3/2}\varepsilon^{-6}\gamma_\varepsilon^{3/2} \\
&\le \varepsilon^{-8}\gamma_\varepsilon\big(\tilde{C}'k^2 + (\tilde{C}')^{-1}\tilde{\rho}_\varepsilon^2 h^8 c_{13}^3(\tilde{C}'')^2\gamma_\varepsilon\big) \\
&\le \tilde{C}'(h^4+k^2)\max\{\varepsilon^{-\sigma}, \varepsilon^{-8}\gamma_\varepsilon\},
\end{aligned}
$$

where we used the assumption on $h^4$ in the last estimate, so that we may deduce

$$\frac{1}{2}||\phi_\ell||^2_{L^2(\Omega)} + k\sum_{m=1}^{\ell}\left(\frac{k}{2}||d_t\phi_m||^2_{L^2(\Omega)} + 2\varepsilon^2||\nabla\phi_m||^2_{L^2(\Omega)}\right)$$
$$\leq 2\tilde{C}'(h^4+k^2)\max\{\varepsilon^{-\sigma},\varepsilon^{-8}\gamma_\varepsilon\} + (3+2\tilde{C}_{\delta_0})k\sum_{m=1}^{\ell-1}||\phi_m||^2_{L^2(\Omega)}$$
$$+\sqrt{2}\tilde{C}''\varepsilon^{-4}\tilde{\rho}_\varepsilon\left(\frac{1}{\sqrt{2}}\max_{m=1,2,\ldots,\ell-1}||\phi_m||_{L^2(\Omega)}\right)k\sum_{m=1}^{\ell-1}\varepsilon^2||\nabla\phi_m||^2_{L^2(\Omega)}.$$

**Step 3**. *Induction over $\ell$.* An inductive argumentation with a discrete Gronwall inequality as in Step 3 in the proof of Theorem 5.6 leads to

$$\frac{1}{2}||\phi_\ell||^2_{L^2(\Omega)} + k\sum_{m=1}^{\ell}\left(\frac{k}{2}||d_t\phi_m||^2_{L^2(\Omega)} + 2\varepsilon^2||\nabla\phi_m||^2_{L^2(\Omega)}\right)$$
$$\leq 4\tilde{C}'(h^4+k^2)\max\{\varepsilon^{-\sigma},\varepsilon^{-8}\gamma_\varepsilon\}\exp\big((3+2\tilde{C}_{\delta_0})t_{\ell-1}\big) \quad (6.4)$$

for all $1\leq\ell\leq M$ provided that

$$\sqrt{2}\tilde{C}''\varepsilon^{-4}\tilde{\rho}_\varepsilon\Big(4\tilde{C}'(h^4+k^2)\max\{\varepsilon^{-\sigma},\varepsilon^{-8}\gamma_\varepsilon\}\exp\big((3+2\tilde{C}_{\delta_0})T\big)\Big)^{3/2}$$
$$\leq 2\tilde{C}'(h^4+k^2)\max\{\varepsilon^{-\sigma},\varepsilon^{-8}\gamma_\varepsilon\}$$

which is guaranteed by the assumptions on $h^2+k$.
**Step 4**. *Proof of* (6.3). The estimate follows from a combination of (6.4) together with

$$\frac{1}{4}||E_m||^2_{L^2(\Omega)} \leq \frac{1}{2}||\phi_m||^2_{L^2(\Omega)} + \frac{1}{2}||\theta_m||^2_{L^2(\Omega)},$$
$$||\nabla E_m||^2_{L^2(\Omega)} \leq 2||\nabla\phi_m||^2_{L^2(\Omega)} + 2||\nabla\theta_m||^2_{L^2(\Omega)},$$

and the estimates in (6.2). □

**Remarks.** (i) As for the semi-discrete in time approximation scheme, imposing Assumptions II and III only, one can derive an error estimate with exponential dependence on $\varepsilon^{-2}$ which may be used to weaken Assumption I (see remark below Theorem 5.6).
(ii) The assumption on $h^2+k$ can be weakened resulting in a higher power of $\varepsilon^{-1}$ in (6.3).

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