

AN EQUILIBRATED RESIDUAL METHOD WITH A COMPUTABLE ERROR APPROXIMATION FOR A SINGULARLY PERTURBED REACTION-DIFFUSION PROBLEM ON ANISOTROPIC FINITE ELEMENT MESHES

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Abstract. Singularly perturbed reaction-diffusion problems exhibit in general solutions with anisotropic features, *e.g.* strong boundary and/or interior layers. This anisotropy is reflected in a discretization by using meshes with anisotropic elements. The quality of the numerical solution rests on the robustness of the *a posteriori* error estimator with respect to both, the perturbation parameters of the problem and the anisotropy of the mesh. The equilibrated residual method has been shown to provide one of the most reliable error estimates for the reaction-diffusion problem. Its modification suggested by Ainsworth and Babuška has been proved to be robust for the case of singular perturbation. In the present work we investigate the modified method on anisotropic meshes. The method in the form of Ainsworth and Babuška is shown here to fail on anisotropic meshes. We suggest a new modification based on the stretching ratios of the mesh elements. The resulting error estimator is equivalent to the equilibrated residual method in the case of isotropic meshes and is proved to be robust on anisotropic meshes as well. Among others, the equilibrated residual method involves the solution of an infinite dimensional local problem on each element. In practical computations an approximate solution to this local problem was successfully computed. Nevertheless, up to now no rigorous analysis has been done showing the appropriateness of any computable approximation. This demands special attention since an improper approximate solution to the local problem can be fatal for the robustness of the whole method. In the present work we provide one of the desired approximations. We prove that the method is not affected by the approximate solution of the local problem.

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1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^2$ be an open domain with polyhedral boundary $\partial\Omega$. Consider the reaction-diffusion problem with homogeneous Dirichlet boundary conditions

$$-\Delta u + \kappa^2 u = f \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \quad (1)$$

where κ is a non-negative constant.

Keywords and phrases. *A posteriori* error estimation, singular perturbations, reaction-diffusion problem, robustness, anisotropic solution, stretched elements.

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If $\kappa \gg 1$, then we have a singularly perturbed problem. Singularly perturbed problems arise in many fields of application. For instance boundary value problems formulated on thin domains [26], where κ is proportional to the inverse of the domain thickness. They also arise in mathematical models of physical problems, where diffusion is small compared with reaction and convection.

Such problems yield solutions with local anisotropic behavior, *e.g.* boundary and/or interior layers. In these cases a special mesh adaptivity is desirable. Triangles should not only adapt in size but also in shape, in order to fit the function to be approximated better. Standard finite element meshes consist of isotropic elements. In this work we investigate so-called anisotropic elements. They are characterized by a large aspect ratio (the ratio of the diameters of the circumscribed and inscribed circles). The singularly perturbed reaction diffusion problem typically requires triangles stretched along the boundary or in the direction of the interior layer [4, 5, 8].

There is a large variety of a posteriori error estimation techniques. We do not aim at giving an overview of the related works here, instead we refer to [3, 25] and citations therein. Some types of a posteriori error estimation methods have already been generalized for anisotropic meshes. They include the works on residual error estimators [18, 20, 22], hierarchical error estimators [13, 14] and Dirichlet local problem error estimators [19, 21]. The modified equilibrated residual method [1] has been shown to provide one of the most reliable error estimates for the singularly perturbed reaction-diffusion problem. We will refer to this method as Ainsworth-Babuška estimator. The equilibrated residual method dates back to [11] and was further developed in [1, 2, 10, 23]. The main purpose of the current work is to consider the estimator from [1] on anisotropic meshes and to construct upper and lower error bounds. It turns out that the equilibrated residual method fails on anisotropic meshes due to a (potentially unbounded) factor appearing in the lower bound. This factor is $O(1)$ on isotropic meshes, but it can be of size of the maximum aspect ratio on anisotropic meshes. We propose a new modification for anisotropic elements leading to a robust error estimator. The upper error bound of the modification contains an alignment measure which is in accordance with the results by Kunert [21].

Among others, the robustness of the equilibrated residual method relies on the solution of a local residual problem on each element. This problem is infinite dimensional and does not allow an exact solution in general. In practical computations it is therefore solved approximately by means of the finite element method. According to the existing theory, however, the equilibrated residual method is guaranteed to be robust if the exact solution to the local problem can be computed. The potentially fatal role of the approximation of this solution has been poorly investigated up to now, which gives rise to heavy criticism of the whole method. In this work we provide an appropriate basis for the solution of the local problem and show that the whole method is not affected by this approximation.

The paper is organized as follows. After describing the model problem and its discretization in Section 2, we give a brief overview of the standard equilibrated residual method and its modification for the singularly perturbed case in Section 3. Moreover we derive the upper error bound. In Section 4 some properties of the equilibrated residual method on anisotropic meshes are proved. In Section 5 the lower error bound is given with the standard estimator of the modified equilibrated residual method [1]. Furthermore, we introduce in Section 6 the modification of the equilibrated residual method for the anisotropic case and prove that the resulting estimator is robust. In Section 7 a suitable approximate solution of the local problem is proposed and its equivalence to the exact solution is shown. At the end of Section 7 the computable error estimator is defined and the resulting error bounds are stated. Finally, a numerical example and some additional remarks complete the discussion.

2. THE MODEL PROBLEM, ITS DISCRETIZATION AND SOME NOTATION

Let ω be an open subset of $\Omega \subset \mathbb{R}^2$. For any subset $\omega \subset \Omega$ we introduce the usual L_2 inner product $(u, v)_{L_2(\omega)} := \int_{\omega} uv \, dx$ and inner product $B_{\omega}(u, v) := \int_{\omega} (\nabla^{\top} u \nabla v + \kappa^2 uv) \, dx$, which lead to the norms $\|v\|_{L_2(\omega)}^2 = (u, u)_{L_2(\omega)}$ and $\|v\|_{\omega}^2 = B_{\omega}(u, u)$. When $\omega = \Omega$, the subscript will be omitted.

We consider problem (1) and assume $f \in L_2(\Omega)$. The corresponding variational formulation for (1) is

$$\text{Find } u \in H_0^1(\Omega) : \quad B(u, v) = (f, v) \quad \forall v \in H_0^1(\Omega). \quad (2)$$

We utilize a family $\mathcal{F} = \{\mathcal{T}\}$ of conforming triangulations \mathcal{T} of Ω , *i.e.* any two triangles are either disjoint or share a common vertex or an edge. Let $X \subset H_0^1(\Omega)$ be the space of continuous, piecewise linear functions over \mathcal{T} that vanish on $\partial\Omega$. Then the finite element solution $u_X \in X$ is uniquely defined by

$$B(u_X, v_X) = (f, v_X) \quad \forall v_X \in X. \quad (3)$$

Due to the Lax-Milgram Lemma both problems (2) and (3) admit unique solutions.

Let \mathcal{N} be the set of all the nodes in the triangulation \mathcal{T} , then we denote by $\mathcal{N}(K)$ and $\mathcal{N}(\gamma)$ the sets of all vertices of a triangle K and an edge γ respectively. Let $x_n \in \mathcal{N}$ be any node and let θ_n be the Lagrange basis function associated with that node. Let $\tilde{x}_n = \text{supp}\theta_n$ be the patch of elements around vertex x_n . Similarly, \mathcal{E}_n is the set of those edges which have a vertex at x_n . For an edge γ and an element K of the triangulation \mathcal{T} , let $|\gamma| = \text{meas}_1(\gamma)$ and $|K| = \text{meas}_2(K)$ denote the length of γ and area of K , respectively. In what follows ω_K will denote the patch of elements around K that satisfy $K' \subset \omega_K$ if and only if $\overline{K} \cap \overline{K'}$ is nonempty. Analogously we define the patch ω_γ of an edge γ : $K \subset \omega_\gamma$ if and only if $\gamma \subset \partial K$.

We will need an extension operator $F_{ext} : \mathbb{P}^0(\gamma) \mapsto \mathbb{P}^0(K)$ defined by

$$F_{ext}(\varphi)(x) := \varphi|_\gamma \equiv \text{const.}$$

Bubble functions play an important role in the derivation of lower error bounds. We will define them in the usual way, see for example [24]. Denote by $\lambda_{K,1}, \lambda_{K,2}, \lambda_{K,3}$ the barycentric coordinates of an arbitrary triangle K . The *element bubble function* b_K is defined by

$$b_K := 27\lambda_{K,1}\lambda_{K,2}\lambda_{K,3} \text{ on } K.$$

Let $\gamma = \text{int}(\overline{K_1} \cap \overline{K_2})$ be an inner edge of \mathcal{T} . Enumerate the vertices of K_1 and K_2 such that the vertices of γ are numbered first. Define the *edge bubble function* b_γ and the *edge spline function* s_γ by

$$\begin{aligned} b_\gamma &:= 4\lambda_{K_i,1}\lambda_{K_i,2} && \text{on } K_i, i = 1, 2, \\ s_\gamma &:= \frac{18}{\sqrt{3}}\lambda_{K_i,1}\lambda_{K_i,2}(\lambda_{K_i,2} - \lambda_{K_i,1}) && \text{on } K_i, i = 1, 2. \end{aligned}$$

For simplicity we assume that b_K , b_γ and s_γ are extended by zero outside their original domain of definition. It holds that $0 \leq b_K(x), b_\gamma(x) \leq 1$, $-1 \leq s_\gamma(x) \leq 1$ and $\|b_K\|_{L^\infty(K)} = \|b_\gamma\|_{L^\infty(K)} = \|s_\gamma\|_{L^\infty(K)} = 1$.

We use the abbreviation $a \preceq b$ for $a \leq Cb$, with a positive constant C independent of a and b . We also write $a \sim b$ when $a \preceq b$ and $a \succeq b$. All constants used in this work are independent of h , κ and the aspect ratio of the elements.

3. THE EQUILIBRATED RESIDUAL METHOD

In this section a brief overview over the equilibrated residual method is given since we strongly require parts of this method for our subsequent analysis. The equilibrated residual method may be found in [3] and its modification for the singularly perturbed case in [1].

3.1. The equilibrated residual method

Consider the model problem of Section 2. Then the error $e := u - u_X$ belongs to the space $H_0^1(\Omega)$ and satisfies the variational formulation

$$B(e, v) = B(u, v) - B(u_X, v) = (f, v) - B(u_X, v) \quad \forall v \in H_0^1(\Omega). \quad (4)$$

For an element K with boundary ∂K , let n_K be the outer normal vector. Next we introduce a set of boundary fluxes $\{g_K : K \in \mathcal{T}\}$ which approximate the actual fluxes of the exact solution on the element boundaries,

$g_K \approx n_K \cdot \nabla u|_K$. Taking into account that the trace of the true solution is continuous on the edges, we construct the approximate fluxes g_K so that the condition

$$g_K + g_{K'} = 0 \text{ on } \partial K \cap \partial K', \quad K, K' \in \mathcal{T} \quad (5)$$

holds true. With this definition the residual on the right hand side of (4) can be decomposed into contributions from the individual elements

$$(f, v) - B(u_X, v) = \sum_{K \in \mathcal{T}} \left\{ (f, v)_K - B_K(u_X, v) + \int_{\partial K} g_K v \, ds \right\} \quad \forall v \in H_0^1(\Omega). \quad (6)$$

The term in parentheses defines a linear functional on the space of the locally admissible functions

$$V_K = \{v : v = w|_\Omega \text{ for some } w \in H_0^1(\Omega)\}.$$

If the parameter κ in (1) is not zero, there is a unique solution $\phi_K \in V_K$ to the local residual problem

$$B_K(\phi_K, v) = (f, v)_K - B_K(u_X, v) + \int_{\partial K} g_K v \, ds \quad \forall v \in V_K. \quad (7)$$

If κ vanishes then the problem will have a solution if and only if the collection of fluxes $\{g_K : K \in \mathcal{T}\}$ satisfies the so-called *equilibration condition*

$$0 = (f, 1)_K - B_K(u_X, 1) + \int_{\partial K} g_K \, ds. \quad (8)$$

This condition means that the boundary flux g_K is in equilibrium with the interior load. Note that the local problem (7) is infinite dimensional. The solution ϕ_K is treated as an approximation of the true error on the element K . It yields the *a posteriori* error estimation $\|e\|^2 \sim \sum_{K \in \mathcal{T}} \|\phi_K\|_K^2$, which will be shown later.

The substitution of (7) into (6) implies

$$B(e, v) = (f, v) - B(u_X, v) = \sum_{K \in \mathcal{T}} B_K(\phi_K, v), \text{ for all } v \in V.$$

An immediate consequence of this result is the upper bound on the true error. We obtain from the Cauchy-Schwarz inequality that

$$|B(e, v)| \leq \sum_{K \in \mathcal{T}} \|\phi_K\|_K \|v\|_K \leq \left\{ \sum_{K \in \mathcal{T}} \|\phi_K\|_K^2 \right\}^{1/2} \|v\|,$$

and conclude that

$$\|e\| = \sup_{v \in H_0^1(\Omega) : \|v\|=1} B(e, v) \leq \left\{ \sum_{K \in \mathcal{T}} \|\phi_K\|_K^2 \right\}^{1/2}.$$

These developments lead to the following theorem:

Theorem 3.1 (upper error bound). *Let $\{g_K : K \in \mathcal{T}\}$ be any set of boundary fluxes satisfying condition (5). Additionally, if κ vanishes, then (8) is assumed to hold on all elements that do not abut the boundary $\partial\Omega$. Then, the global error in the finite element approximation is bounded by*

$$\|e\|^2 \leq \sum_{K \in \mathcal{T}} \|\phi_K\|_K^2.$$

Proof. For the proof see the argumentation above. □

3.2. Construction of the equilibrated fluxes

For the construction of the equilibrated fluxes we adopt the theory of [3] which we will briefly repeat in this section. It will be assumed that the finite element subspace X is constructed using linear elements on a partition \mathcal{T} of the domain Ω into triangular elements. The key issue of the lower bound of the error is the construction of appropriate approximate fluxes. The procedure that will be developed produces a set of fluxes $\{g_K\}$ that satisfy the *first-order equilibration conditions*:

$$\left. \begin{aligned} (f, \theta_n)_K - B_K(u_X, \theta_n) + \int_{\partial K} g_K \theta_n \, ds &= 0 \quad \forall n \in \mathcal{N}(K) \\ g_K + g_{K'} &= 0 \text{ on } \partial K \cap \partial K' \end{aligned} \right\}. \tag{9}$$

It is convenient to look for $g_K|_\gamma$ belonging to $\text{span}\{\theta_n : n \in \mathcal{N}(\gamma)\}$ on all edges. Ainsworth and Oden [3] suggest to choose the degrees of freedom for the fluxes to be the moments $\mu_{K,n}^\gamma = \int_\gamma g_K \theta_n \, ds$ with respect to the FEM basis functions θ_n associated with $x_n \in \mathcal{N}(\gamma)$. Thereby we avoid a global problem by reducing the construction of fluxes to computation of the moments over local patches of elements.

Let $\mathcal{N}(\gamma) = \{x_l, x_r\}$, then it can be shown that the actual flux may be reconstructed from its moments:

$$g_K|_\gamma = \frac{2}{|\gamma|} \left\{ (2\mu_{K,l}^\gamma - \mu_{K,r}^\gamma)\theta_l + (-\mu_{K,l}^\gamma + 2\mu_{K,r}^\gamma)\theta_r \right\}. \tag{10}$$

Note that (10) could be rewritten in the form

$$g_K|_\gamma = \mu_{K,l}^\gamma \psi_l + \mu_{K,r}^\gamma \psi_r,$$

where ψ_l and ψ_r are the dual basis functions corresponding to θ_l and θ_r , *i.e.* $(\psi_i, \theta_j)_{L_2(\gamma)} = \delta_{ij}$ for $i, j \in \{l, r\}$:

$$\psi_l = \frac{2}{|\gamma|} (2\theta_l - \theta_r), \quad \psi_r = \frac{2}{|\gamma|} (-\theta_l + 2\theta_r).$$

In order to determine the boundary fluxes, it is sufficient to determine the moments of the flux with respect to the basis functions. The first-order equilibration conditions (9) for the flux g_K may be rewritten in terms of the flux moments in the form

$$\left. \begin{aligned} \sum_{\gamma \subset \partial K} \mu_{K,n}^\gamma &= \Delta_K(\theta_n) \quad \forall n \in \mathcal{N}(K) \\ \mu_{K,n}^\gamma + \mu_{K',n}^\gamma &= 0 \quad \forall n \in \mathcal{N}(\gamma), \quad \gamma = \partial K \cap \partial K' \end{aligned} \right\} \tag{11}$$

where

$$\Delta_K(\theta_n) = B_K(u_X, \theta_n) - (f, \theta_n)_K. \tag{12}$$

In (11) we used the convention that $\mu_{K,n}^\gamma = 0$ if $n \notin \mathcal{N}(\gamma)$.

The conditions in (11) have one of two distinct structures depending on the location of the node x_n in the interior or on the boundary of Ω . Here we omit the case of a boundary vertex, see [3] for details. Assume x_n to be an interior vertex. The elements and edges are labeled as shown in Figure 1. The moment equilibration conditions (11) for the elements $K \in \tilde{x}_n$ associated with the node x_n may be rewritten in the form

$$\left. \begin{aligned} \mu_{1,n}^{\gamma_1} + \mu_{1,n}^{\gamma_2} &= \Delta_1(\theta_n) \\ \mu_{2,n}^{\gamma_2} + \mu_{2,n}^{\gamma_3} &= \Delta_2(\theta_n) \\ &\vdots \\ \mu_{N,n}^{\gamma_N} + \mu_{N,n}^{\gamma_1} &= \Delta_N(\theta_n) \end{aligned} \right\}$$

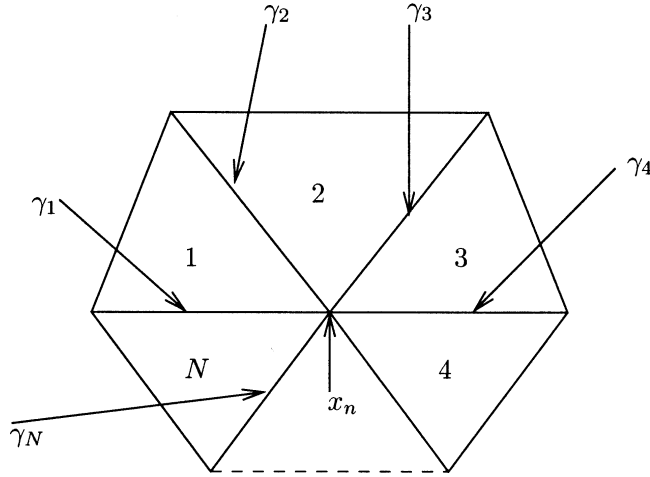


FIGURE 1. The patch of elements influenced by the basis function θ_n .

with constraints on the edges

$$\left. \begin{aligned} \mu_{1,n}^{\gamma_1} + \mu_{N,n}^{\gamma_1} &= 0 \\ \mu_{2,n}^{\gamma_2} + \mu_{1,n}^{\gamma_2} &= 0 \\ &\vdots \\ \mu_{N,n}^{\gamma_N} + \mu_{N-1,n}^{\gamma_N} &= 0 \end{aligned} \right\}.$$

It is shown in [3] that this system of $2N$ equations for $2N$ unknowns has a one-parametric family of solutions. We recall from [3] which solution should be selected.

The ideal situation would be to choose the flux moments $\{g_K\}$ satisfying $\mu_{K,n}^\gamma \approx \int_\gamma \theta_n n_K \cdot \nabla u \, ds$. Since the true fluxes are unknown, the flux moments are selected so that

$$\mu_{K,n}^\gamma \approx \tilde{\mu}_{K,n}^\gamma := \int_\gamma \theta_n n_K \cdot \nabla u_X|_K \, ds. \tag{13}$$

We seek flux moments that minimize the objective

$$\frac{1}{2} \sum_{K \in \tilde{x}_n} \sum_{\gamma \subset \partial K} \left(\mu_{K,n}^\gamma - \tilde{\mu}_{K,n}^\gamma \right)^2. \tag{14}$$

Introducing Lagrange multipliers we come to the optimality condition. The Lagrangian is given by

$$\begin{aligned} \mathcal{L} \left(\{ \tilde{\mu}_{K,n}^\gamma \}, \{ \lambda_\gamma \}, \{ \sigma_K \} \right) &= \frac{1}{2} \sum_{K \in \tilde{x}_n} \sum_{\gamma \subset \partial K} \left(\mu_{K,n}^\gamma - \tilde{\mu}_{K,n}^\gamma \right)^2 \\ &+ \sum_{K \in \tilde{x}_n} \sigma_{K,n} \left(\Delta_K(\theta_n) - \sum_{\gamma \subset \partial K} \mu_{K,n}^\gamma \right) + \sum_{\gamma = \partial K \cap \partial K'} \lambda_{\gamma,n} \left(\mu_{K,n}^\gamma + \mu_{K',n}^\gamma \right). \end{aligned}$$

Here we used the convention that $\lambda_{\gamma,n} = 0$ on $\gamma \subset \partial\Omega$. We conclude that the conditions for a stationary point consist of two parts; the first part is (11), the second part is

$$\mu_{K,n}^\gamma - \tilde{\mu}_{K,n}^\gamma - \sigma_{K,n} + \lambda_{\gamma,n} = 0. \tag{15}$$

Using the second part of (11) we obtain

$$\lambda_{\gamma,n} = \begin{cases} \frac{1}{2} (\sigma_{K,n} + \sigma_{K',n} + \tilde{\mu}_{K,n}^\gamma + \tilde{\mu}_{K',n}^\gamma) & \gamma = \partial K \cap \partial K', \\ 0 & \gamma = \partial K \cap \partial \Omega. \end{cases}$$

Using the last formula together with (15) the flux moments are expressed as

$$\mu_{K,n}^\gamma = \begin{cases} \frac{1}{2} (\sigma_{K,n} - \sigma_{K',n} + \tilde{\mu}_{K,n}^\gamma - \tilde{\mu}_{K',n}^\gamma) & \gamma = \partial K \cap \partial K', \\ \sigma_{K,n} + \tilde{\mu}_{K,n}^\gamma & \gamma = \partial K \cap \partial \Omega. \end{cases} \tag{16}$$

Substituting this into the first equation of (11) we obtain the following conditions for $\{\sigma_{K,n} : K \in \tilde{x}_n\}$:

$$\frac{1}{2} \sum_{\gamma=\partial K \cap \partial K'} (\sigma_{K,n} - \sigma_{K',n}) + \sum_{\gamma \subset \partial K \cap \partial \Omega} \sigma_{K,n} = \tilde{\Delta}_K(\theta_n) \quad \forall K \in \tilde{x}_n, \tag{17}$$

where

$$\tilde{\Delta}_K(\theta_n) := B_K(u_X, \theta_n) - (f, \theta_n)_K - \int_{\partial K} \left\langle \frac{\partial u_X}{\partial n_K} \right\rangle \theta_n \, ds, \tag{18}$$

$$\left\langle \frac{\partial u_X}{\partial n_K} \right\rangle := \begin{cases} \frac{1}{2} n_K \cdot \{(\nabla u_X)_K + (\nabla u_X)_{K'}\} & \text{on } \partial K \cap \partial K' \\ n_K \cdot (\nabla u_X)_K & \text{on } \partial K \cap \partial \Omega. \end{cases} \tag{19}$$

The conditions (17) form a linear system of equations over the element patches \tilde{x}_n with unknowns $\{\sigma_{K,n} : K \in \tilde{x}_n\}$ corresponding to the elements in the patch. The specific form for an interior vertex is

$$\frac{1}{2} \begin{bmatrix} 2 & -1 & \dots & -1 \\ -1 & 2 & -1 & \dots \\ \vdots & & & \vdots \\ 0 & \dots & -1 & 2 & -1 \\ -1 & \dots & & -1 & 2 \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_{N-1} \\ \sigma_N \end{bmatrix} = \begin{bmatrix} \tilde{\Delta}_1(\theta_n) \\ \tilde{\Delta}_2(\theta_n) \\ \vdots \\ \tilde{\Delta}_{N-1}(\theta_n) \\ \tilde{\Delta}_N(\theta_n) \end{bmatrix}.$$

The kernel of this matrix is the vector $\mathbf{1} = [1, 1, \dots, 1]^\top$, which implies that a solution exists if and only if the sum of the components of the right-hand data vanishes. This may be easily verified thanks to the Galerkin property (see [3]).

Since the system (17) is singular the least square solution is selected. As a consequence, there exists a constant C , depending only on the number of elements in the patch \tilde{x}_n surrounding a vertex x_n , such that (for proof see for ex. [15])

$$\sum_{K \in \tilde{x}_n} \sigma_{K,n}^2 \preceq \sum_{K \in \tilde{x}_n} \tilde{\Delta}_K(\theta_n)^2. \tag{20}$$

3.3. Minimum energy extensions

Minimum energy extensions were first introduced in the work of Ainsworth and Babuška [1]. These extensions play a key role in the construction of an estimator which is stable with respect to the perturbation parameter κ . The original equilibrated residual method is described in the work of Ainsworth and Oden [2]. However, as it is shown in [1], it is not stable with respect to κ . Ainsworth and Babuška [1] propose the following modification of the previous method for the singularly perturbed case. The functions θ_n in (17) are replaced by an approximate

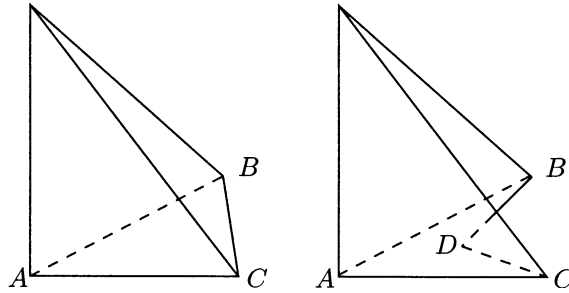


FIGURE 2. Original hat function and the family of functions used to approximate the minimum energy extension $\mathcal{E}\theta$.

minimum energy extension θ_n^* to $\theta_n|_{\partial K}$. The system (17) then is solved in a least-square sense, since it has no solution in general. As in (20), one gets the solution that depends continuously on the data:

$$\sum_{K \in \tilde{x}_n} \sigma_{K,n}^2 \preceq \sum_{K \in \tilde{x}_n} \tilde{\Delta}_K (\theta_n^*)^2. \tag{21}$$

The error estimator we propose is derived from the estimator of the work [1] but differs in two details. Firstly, we pay more attention to the minimization of the appropriate function energy norm and even obtain the minimum. We will develop this in this section. The second modification is described in Section 6.

Let K be any element and let $v \in H^{1/2}(\partial K)$. The *minimum energy extension* $\mathcal{E}v$ of v to the interior of the element is characterized by the conditions

$$\mathcal{E}v \in H^1(K) : \mathcal{E}v = v \text{ on } \partial K, \quad B_K(\mathcal{E}v, \omega) = 0 \quad \forall \omega \in H_0^1(K).$$

The definition of the minimum energy extension has an advantageous property. *Let $v \in H^{1/2}(\partial K)$. The minimum energy extension $\mathcal{E}v$ of v to the interior of the element has the minimal energy norm among all functions coinciding with v on the boundary ∂K .* Indeed, consider the energy norm of the function $\mathcal{E}v + \omega$:

$$\|\|\mathcal{E}v + \omega\|\|^2 = \|\|\mathcal{E}v\|\|^2 + \|\|\omega\|\|^2 + 2B_K(\mathcal{E}v, \omega) = \|\|\mathcal{E}v\|\|^2 + \|\|\omega\|\|^2 \geq \|\|\mathcal{E}v\|\|^2. \tag{22}$$

The proof easily follows from (22) observing that $\mathcal{E}v + \omega$ coincides with $\mathcal{E}v$ on the boundary ∂K .

For the one-dimensional case it is possible to find a minimum energy extension explicitly (see [1]).

Consider now the two-dimensional case. We look for an approximation for the minimum energy extension of the first-order basis function. Let the element $K = \triangle ABC$ be a triangle. Consider the basis function θ corresponding to the vertex A . We seek an approximation to the minimum energy extension $\mathcal{E}\theta$ in the following class Λ of functions. Set

$$\Lambda : = \{v \in C^0(K) : v = \theta \text{ on } \partial K, v = 0 \text{ in } \triangle CDB, \\ v \text{ is linear in each triangle } \triangle CAD \text{ and } \triangle BAD, D \in \triangle ABC\}.$$

We obtain now an approximation for the minimum energy extension of this basis function. To this end we put an arbitrary point D in the triangle (see Fig. 2).

Next we choose that function from the set Λ that minimizes energy norm. Our developments here differs from the original paper [1]. There a point $(1/\kappa, 1/\kappa)$ is introduced in the *reference* triangle and D is the image of this point after the corresponding affine transformation. The corresponding function does not necessarily minimize the energy over Λ but it is shown to be sufficiently accurate. For us, however *this is not sufficient* and we instead consider the point D to be in the *actual* triangle in order to obtain the optimal position of this point.

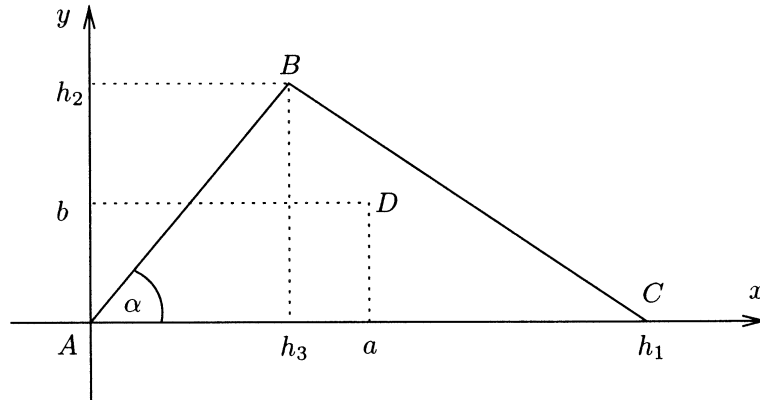


FIGURE 3. Notations for the parameters of an arbitrary triangle.

Introduce a local coordinate system such that the vertex A coincides with the origin and the edge AC lies on the axis Ox . Let $D = (a, b)$, $C = (h_1, 0)$ and $B = (h_3, h_2)$ (see Fig. 3).

Let $\Theta(a, b, x, y) \in \Lambda$ be an admissible function, see Figure 2. The squared energy norm of this function is

$$\begin{aligned} \Phi(a, b) &= B_K(\Theta(a, b, x, y), \Theta(a, b, x, y)) \\ &= \frac{\kappa^2}{12}(b(h_1 - h_3) + ah_2) + \frac{2h_1h_2h_3 - ah_1h_2 - bh_1h_3 + bh_3^2}{2h_1h_3^2} \\ &\quad + \frac{(h_2^2 + h_3^2)(a - h_3)^2}{2h_3^2(ah_2 - bh_3)} + \frac{(a - h_1)^2}{2h_1b}. \end{aligned}$$

For given parameters h_1, h_2, h_3 we want to minimize $\Phi(a, b)$ with respect to a and b . A number of calculations leads to a stationary point of this function

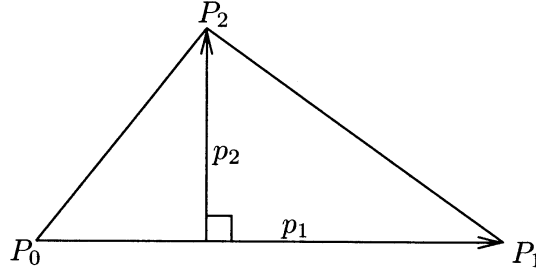
$$\begin{cases} \bar{a}^* = \frac{\sqrt{6h_1}(h_3 + \sqrt{h_2^2 + h_3^2})}{\sqrt{12h_3 + 12\sqrt{h_2^2 + h_3^2} + \kappa^2h_2^2h_1}} \\ \bar{b}^* = \left(-\frac{h_3}{h_2} + \sqrt{\left(\frac{h_3}{h_2}\right)^2 + 1}\right)\bar{a}^*. \end{cases}$$

To prove that this solution is a minimum point it is sufficient to show convexity of the function. Therefore consider the Hessian matrix $D^2\Phi$ of the second-order derivatives. By direct computations one obtains

$$\begin{aligned} \frac{\partial^2\Phi}{\partial a^2} &= \frac{1}{bh_1} + \frac{(h_2^2 + h_3^2)(b - h_2)^2}{(h_2a - h_3b)^3} \geq 0, \\ \det D^2\Phi &= \frac{(h_2^2 + h_3^2)(h_1b - h_3b + h_2a - h_1h_2)^2}{(h_2a - h_3b)^3b^3h_1} \geq 0. \end{aligned}$$

Hence $\Phi(a, b)$ is convex and (\bar{a}^*, \bar{b}^*) is the unique minimum.

We have found the function $\Theta^*(\bar{a}^*, \bar{b}^*, x, y)$ which minimizes the energy norm over the set Λ . However, in practice it is sufficient to take not the exact values of \bar{a}^* and \bar{b}^* , but some values a^*, b^* that are equivalent for

FIGURE 4. Notation of a triangle K .

$\kappa \rightarrow \infty$, namely

$$\begin{cases} a^* &= \frac{\sqrt{6} \left(h_3 + \sqrt{h_2^2 + h_3^2} \right)}{\kappa h_2} \\ b^* &= \frac{\sqrt{6}}{\kappa}. \end{cases}$$

Note that the corresponding point $D = (a^*, b^*)$ lies on the bisector of the angle $\angle BAC =: \alpha$ and $|AD| = \frac{\sqrt{6}}{\kappa \sin \alpha/2}$. The analysis given neglects the fact that (a, b) should be contained in K . Therefore, we construct the function θ^* as follows:

$$\theta^* := \begin{cases} \Theta(a^*, b^*, x, y), & \text{if } (a^*, b^*) \in K, \\ \theta, & \text{otherwise.} \end{cases}$$

Lemma 3.2. *Under the above notations and assumptions the following holds*

$$\|\theta^*\|_{L_2(K)}^2 \leq |K| \min(1, h_{\min, K}^{-1} \kappa^{-1}) \sim \text{meas}(\partial K) \min(h_{\min, K}, \kappa^{-1}),$$

where $h_{\min, K}$ is the height corresponding to the largest edge of the triangle K .

Proof. Consider K for which $(a^*, b^*) \in K$. A short calculation yields $\kappa^{-1} \leq h_{\min, K}$ and $\min(h_{\min, K}, \kappa^{-1}) \sim \kappa^{-1}$. Furthermore one obtains

$$\begin{aligned} \|\theta^*\|_{L_2(K)}^2 &= \frac{\sqrt{6} \left(h_1 + \sqrt{h_2^2 + h_3^2} \right)}{12\kappa} \\ &\leq \text{meas}(\partial K) \kappa^{-1} \sim \text{meas}(\partial K) \min(h_{\min, K}, \kappa^{-1}). \end{aligned}$$

It remains to consider the case $\kappa^{-1} \gg h_{\min, K}$. In this case we have $\min(1, h_{\min, K}^{-1} \kappa^{-1}) = 1$ and θ^* coincides with θ . The estimate

$$\|\theta^*\|_{L_2(K)}^2 = \|\theta\|_{L_2(K)}^2 \sim |K| = |K| \min(1, h_{\min, K}^{-1} \kappa^{-1})$$

completes the proof. \square

4. THEORETICAL BACKGROUND OF THE EQUILIBRATED RESIDUAL METHOD IN ANISOTROPIC CASE

4.1. Notation

Let a triangulation \mathcal{T} be given which satisfies the usual conformity condition (see [12], Chap. 2). Following the notation of Kunert [17], the three vertices of an arbitrary triangle $K \in \mathcal{T}$ are denoted by P_0, P_1, P_2 such that P_0P_1 is the longest edge of K . Additionally define two orthogonal vectors p_i with lengths $h_{i, K} := |p_i|$, see Figure 4. Observe that $h_{1, K} > h_{2, K}$ and set $h_{\min, K} := h_{2, K}$, $h_{\max, K} := h_{1, K}$.

In addition to the usual conformity conditions of the mesh we assume that the following two properties hold.

1. The number of triangles containing a node x_n is bounded uniformly.
2. The dimensions of adjacent triangles must not change rapidly, *i.e.*

$$h_{i,K'} \sim h_{i,K} \quad \forall K, K' \text{ with } \overline{K} \cap \overline{K'} \neq \emptyset, \quad i = 1, 2.$$

Define the matrices A_K and $C_K \in \mathbb{R}^{2 \times 2}$ by

$$A_K := (\overrightarrow{P_0 P_1}, \overrightarrow{P_0 P_2}) \quad \text{and} \quad C_K := (p_1, p_2)$$

and introduce an affine linear mapping

$$F_A(\mathbf{x}) := A_K \cdot \mathbf{x} + \overrightarrow{P_0}, \quad \mathbf{x} \in \mathbb{R}^2.$$

We will use the notion of the *reference triangle* $\hat{K} = F_A^{-1}(K)$ which is uniquely defined by the mapping F_A .

The following mesh dependent energy scalar product and norm will play a crucial role in the modification done in Section 6 to guarantee the robustness of the equilibrated residual method on anisotropic meshes.

Definition 4.1 (Mesh dependent energy scalar product). Let $K \in \mathcal{T}$ be any triangle, $u \in H_0^1(\Omega)$ and $v \in H^1(K)$, then we define the mesh-dependent energy scalar product and norms by

$$\begin{aligned} {}_m B_K(u, v) &:= h_{\min, K}^{-2} (C_K^\top \nabla u, C_K^\top \nabla v)_K + \kappa^2 (u, v)_K, \\ {}_m \|u\|_K &:= {}_m B_K(u, u)^{1/2}, \\ {}_m \|u\| &:= \left(\sum_{K \in \mathcal{T}} {}_m \|u\|_K^2 \right)^{1/2}. \end{aligned}$$

The local mesh-dependent energy norm satisfies the following property

$$\|u\|_K \leq {}_m \|u\|_K \leq \frac{h_{\max, K}}{h_{\min, K}} \|u\|_K.$$

Note that the standard and mesh dependent energy norms are equivalent in the case of isotropic elements.

4.2. Some basic inequalities

This paragraph provides some facts which will be useful in the subsequent analysis for obtaining the lower error bound. The reader who are not interested in details of the theory may omit this paragraph without any affecting the understanding. The following two lemmata are extended versions of the corresponding results from [21].

Lemma 4.2. *Assume that $\varphi_K \in \mathbb{P}^0(K)$ and $\varphi_\gamma \in \mathbb{P}^0(\gamma)$. Then*

$$\|b_K^{1/2} \varphi_K\|_{L_2(K)} \sim \|\varphi_K\|_{L_2(K)} \tag{23}$$

$$\|b_K \varphi_K\|_{L_2(K)} \sim \|\varphi_K\|_{L_2(K)} \tag{24}$$

$$\|\nabla(b_K \varphi_K)\|_{L_2(K)} \leq h_{\min, K}^{-1} \|\varphi_K\|_{L_2(K)} \tag{25}$$

$$\|C_K^\top \nabla(b_K \varphi_K)\|_{L_2(K)} \leq \|\varphi_K\|_{L_2(K)} \tag{26}$$

$$\|b_K \varphi_K\|_K \leq \min(h_{\min, K}, \kappa^{-1})^{-1} \|\varphi_K\|_{L_2(K)} \tag{27}$$

$${}_m \|b_K \varphi_K\|_K \leq \min(h_{\min, K}, \kappa^{-1})^{-1} \|\varphi_K\|_{L_2(K)}. \tag{28}$$

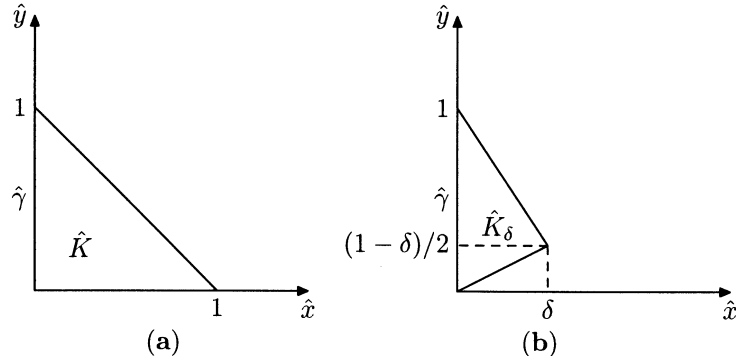


FIGURE 5. Definition of squeezed edge bubble functions: (a) – reference triangle \hat{K} , (b) – squeezed reference triangle \hat{K}_δ .

Proof. Inequalities (23) and (25) are copied from [17] (p. 27, Lem. 2.7) without changes. Inequality (24) can be obtained analogously to (23). Inequality (26) is a refined version of (25) obtained by avoiding additional estimation on page 26, lines 3–4 from above in [17]. Another refined inequality (28) may be obtained in the same manner. Finally, combining (24) and (25) we show (27):

$$\|b_K \varphi_K\|_K^2 = \|\nabla(b_K \varphi_K)\|_{L_2(K)}^2 + \kappa^2 \|b_K \varphi_K\|_{L_2(K)}^2 \preceq (h_{\min, K}^{-2} + \kappa^2) \|\varphi_K\|_{L_2(K)}^2 \preceq \min(h_{\min, K}, \kappa^{-1})^{-2} \|\varphi_K\|_{L_2(K)}^2.$$

□

Again following [21], we define the squeezed edge bubble functions, extend the definition also for the squeezed spline functions and state the corresponding inverse inequalities. The definitions are given first for the reference triangle \hat{K} and then for the actual triangle K .

Consider the reference triangle \hat{K} and an edge $\hat{\gamma}$ thereof. Without loss of generality, assume that it lies on the axis $O\hat{y}$. By γ we denote the corresponding edge on the boundary of actual triangle K . For a real number $\delta \in (0, 1]$ define a linear mapping $F_\delta : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by

$$F_\delta(\mathbf{x}) := B_\delta \cdot \mathbf{x} \quad \text{with } B_\delta = \begin{pmatrix} \delta & 0 \\ \frac{1-\delta}{2} & 1 \end{pmatrix} \in \mathbb{R}^{2 \times 2}.$$

Set $\hat{K}_\delta := F_\delta(\hat{K})$, i.e. it is the triangle with the edge $\hat{\gamma}$ and a vertex at $(\delta, \frac{1-\delta}{2})$, see Figure 5.

Let $b_{\hat{\gamma}}$ be the usual edge bubble function of $\hat{\gamma}$ on \hat{K} . Define the squeezed bubble function $b_{\hat{\gamma}, \delta}$ by $b_{\hat{\gamma}, \delta} := b_{\hat{\gamma}} \circ F_\delta^{-1}$, i.e. $b_{\hat{\gamma}, \delta}$ is the usual face bubble function of $\hat{\gamma}$ on the triangle \hat{K}_δ . For clarity we recall that $b_{\hat{\gamma}, \delta} = 0$ on $\hat{K} \setminus \hat{K}_\delta$.

Consider now an actual triangle K . The squeezed edge bubble function $b_{\gamma, \delta} \in H^1(K)$ of an edge γ of K is defined by $b_{\gamma, \delta} := b_{\hat{\gamma}, \delta} \circ F_A^{-1}$. Analogously we can define the squeezed edge spline function $s_{\gamma, \delta} := s_{\hat{\gamma}} \circ F_\delta^{-1} \circ F_A^{-1}$. The actual value of parameter δ will be specified later. The usual and squeezed edge bubble/spline functions are drawn in Figure 6.

We are interested in a particular value of parameter δ depending on the edge γ for which the squeezed functions are defined. From now on we let

$$\delta_\gamma := \min\left(1, \kappa^{-1} h_{\min, \gamma}^{-1}\right) = h_{\min, \gamma}^{-1} \min(h_{\min, \gamma}, \kappa^{-1}).$$

Lemma 4.3 (inverse inequalities for squeezed edge bubble/spline functions). *Let γ be an arbitrary face of K . Assume that $\varphi_\gamma \in \mathbb{P}^0(\gamma)$, $\mu_{\gamma, \delta_\gamma} \in \{b_{\gamma, \delta_\gamma}, s_{\gamma, \delta_\gamma}\}$ and $\nu_{\gamma, \delta_\gamma} \in \text{span}\{b_{\gamma, \delta_\gamma}, s_{\gamma, \delta_\gamma}\}$. Then the following inverse*

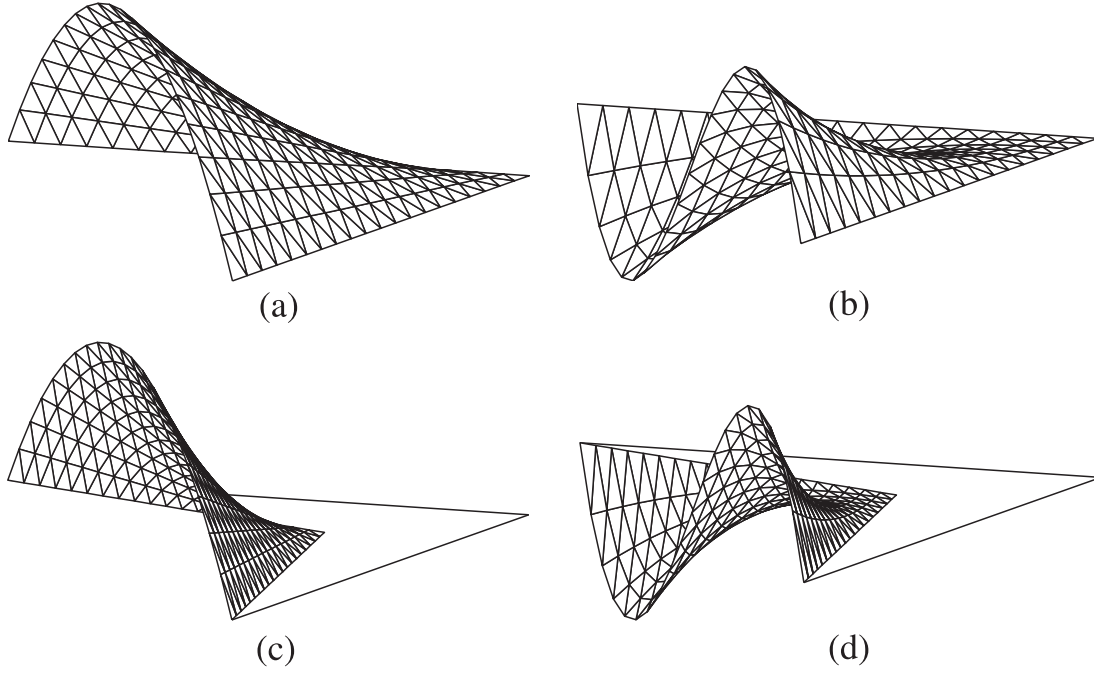


FIGURE 6. Bubble and spline functions: (a) – edge bubble function b_γ , (b) – edge spline function s_γ , (c) – squeezed edge bubble function $b_{\gamma,\delta}$, (d) – squeezed edge spline function $s_{\gamma,\delta}$.

inequalities hold:

$$\|b_{\gamma,\delta}^{1/2} \varphi_\gamma\|_{L_2(\gamma)} \sim \|\varphi_\gamma\|_{L_2(\gamma)} \quad (29)$$

$$\|\mu_{\gamma,\delta} \varphi_\gamma\|_{L_2(\gamma)} \sim \|\varphi_\gamma\|_{L_2(\gamma)} \quad (30)$$

$$\|\mu_{\gamma,\delta} F_{ext}(\varphi_\gamma)\|_{L_2(K)} \lesssim \left(\frac{|K|}{|\gamma|}\right)^{1/2} h_{\min,K}^{-1/2} \min(h_{\min,K}, \kappa^{-1})^{1/2} \|\varphi_\gamma\|_{L_2(\gamma)} \quad (31)$$

$$\|\nabla(\mu_{\gamma,\delta} F_{ext}(\varphi_\gamma))\|_{L_2(K)} \lesssim \left(\frac{|K|}{|\gamma|}\right)^{1/2} h_{\min,K}^{-1/2} \min(h_{\min,K}, \kappa^{-1})^{-1/2} \|\varphi_\gamma\|_{L_2(\gamma)} \quad (32)$$

$$h_{\min,K}^{-1} \|C_K^\top \nabla(\mu_{\gamma,\delta} F_{ext}(\varphi_\gamma))\|_{L_2(K)} \lesssim \left(\frac{|K|}{|\gamma|}\right)^{1/2} h_{\min,K}^{-1/2} \min(h_{\min,K}, \kappa^{-1})^{-1/2} \|\varphi_\gamma\|_{L_2(\gamma)} \quad (33)$$

$$\|\mu_{\gamma,\delta} F_{ext}(\varphi_\gamma)\|_K \lesssim \left(\frac{|K|}{|\gamma|}\right)^{1/2} h_{\min,K}^{-1/2} \min(h_{\min,K}, \kappa^{-1})^{-1/2} \|\varphi_\gamma\|_{L_2(\gamma)} \quad (34)$$

$$m \|\mu_{\gamma,\delta} F_{ext}(\varphi_\gamma)\|_K \lesssim \left(\frac{|K|}{|\gamma|}\right)^{1/2} h_{\min,K}^{-1/2} \min(h_{\min,K}, \kappa^{-1})^{-1/2} \|\varphi_\gamma\|_{L_2(\gamma)} \quad (35)$$

$$\|\nu_{\gamma,\delta}\|_{L_2(K)} \lesssim \left(\frac{|K|}{|\gamma|}\right)^{1/2} h_{\min,K}^{-1/2} \min(h_{\min,K}, \kappa^{-1})^{1/2} \|\nu_{\gamma,\delta}\|_{L_2(\gamma)} \quad (36)$$

$$m \|\nu_{\gamma,\delta}\|_K \lesssim \left(\frac{|K|}{|\gamma|}\right)^{1/2} h_{\min,K}^{-1/2} \min(h_{\min,K}, \kappa^{-1})^{-1/2} \|\nu_{\gamma,\delta}\|_{L_2(\gamma)}. \quad (37)$$

Proof. We observe first that (29) is identical to (2.26) in [17] (p. 27, Lem. 2.7) and (30) can be obtained analogously. Inequalities (31) and (32) follow directly from the corresponding inequalities in [21] (p. 247, Lem. 3.7) extended also to the spline functions $s_{\gamma, \delta_\gamma}$. Inequality (34) follows from (31) and (32) in the similar way as (27) in Lemma 4.2. The proof of refined estimates (33) and (35) follows the lines of the corresponding proof of (26). In order to show (36) we express $\nu_{\gamma, \delta_\gamma}$ in the form $\nu_{\gamma, \delta_\gamma} = C_b b_{\gamma, \delta_\gamma} + C_s s_{\gamma, \delta_\gamma}$, where C_b and C_s are two constants, and show (36) utilizing the triangle inequality (31), and (30) subsequently:

$$\begin{aligned} \|\nu_{\gamma, \delta_\gamma}\|_{L_2(K)}^2 &= \|C_b b_{\gamma, \delta_\gamma} + C_s s_{\gamma, \delta_\gamma}\|_{L_2(K)}^2 \leq \|C_b b_{\gamma, \delta_\gamma}\|_{L_2(K)}^2 + \|C_s s_{\gamma, \delta_\gamma}\|_{L_2(K)}^2 \\ &\preceq \frac{|K|}{|\gamma|} h_{\min, K}^{-1} \min(h_{\min, K}, \kappa^{-1}) \left(\|C_b\|_{L_2(\gamma)}^2 + \|C_s\|_{L_2(\gamma)}^2 \right) \\ &\sim \frac{|K|}{|\gamma|} h_{\min, K}^{-1} \min(h_{\min, K}, \kappa^{-1}) \left(\|C_b b_{\gamma, \delta_\gamma}\|_{L_2(\gamma)}^2 + \|C_s s_{\gamma, \delta_\gamma}\|_{L_2(\gamma)}^2 \right) \\ &= \frac{|K|}{|\gamma|} h_{\min, K}^{-1} \min(h_{\min, K}, \kappa^{-1}) \|C_b b_{\gamma, \delta_\gamma} + C_s s_{\gamma, \delta_\gamma}\|_{L_2(\gamma)}^2, \end{aligned}$$

where we used the orthogonality of functions $b_{\gamma, \delta_\gamma}$ and $s_{\gamma, \delta_\gamma}$ over edge γ . Estimate (37) may be obtained analogously. \square

From a heuristic point of view one should stretch the triangle in that direction where the (directional) derivative of the function shows little change. The better the anisotropic mesh \mathcal{T} is aligned with the anisotropic function v , the more accurate one would expect the error estimates to be. In order to measure the alignment of \mathcal{T} with v , Kunert [17, 18] has introduced the *alignment measure* $m_1(v, \mathcal{T})$ which is defined as follows.

Definition 4.4 (alignment measure m_1). Let $v \in H^1(\Omega)$ be an arbitrary non-constant function, and \mathcal{F} be a family of triangulations of Ω . Define the matching function $m_1(\cdot, \cdot) : H^1(\Omega) \times \mathcal{F} \rightarrow \mathbb{R}$ by

$$m_1(v, \mathcal{T}) := \frac{\left(\sum_{K \in \mathcal{T}} h_{\min, K}^{-2} \cdot \|C_K^\top \nabla v\|_{L_2(K)}^2 \right)^{1/2}}{\|\nabla v\|}. \quad (38)$$

Furthermore the *local* matching function $m_1(\cdot, \cdot) : H^1(\Omega) \times \mathcal{T} \rightarrow \mathbb{R}$ is obviously defined by

$$m_1(v, K) := h_{\min, K}^{-1} \frac{\|C_K^\top \nabla v\|_{L_2(K)}}{\|\nabla v\|_{L_2(K)}}.$$

The alignment measure satisfies the following property:

$$1 \leq m_1(v, \mathcal{T}) \leq C \max_{K \in \mathcal{T}} \frac{h_{\max, K}}{h_{\min, K}}.$$

The definition implies that a mesh \mathcal{T} which is well aligned with an anisotropic function v , results in a small alignment measure $m_1(v, \mathcal{T})$.

Further we will need the anisotropic trace inequality and some more facts concerning approximation properties on an anisotropic triangle.

Lemma 4.5 (anisotropic trace inequality). Let K be an arbitrary triangle and γ be an edge of it. For $v \in H^1(K)$ the following trace inequality holds:

$$\|v\|_{L_2(\gamma)}^2 \preceq \frac{|\gamma|}{|K|} \|v\|_{L_2(K)} \left(\|v\|_{L_2(K)} + \|C_K^\top \nabla v\|_{L_2(K)} \right).$$

Lemma 4.6 (anisotropic approximation properties). *Let K be any triangle, $\gamma \subset \partial K$ be any edge thereof and $v \in H^1(K)$. Denote by $\bar{v} = \frac{1}{|K|} \int_K v$ the mean value of v over an element K . Then*

$$\|v - \bar{v}\|_{L_2(K)} \leq \|v\|_{L_2(K)}, \quad (39)$$

$$\|v - \bar{v}\|_{L_2(K)} \leq \|C_K^\top \nabla v\|_{L_2(K)}, \quad (40)$$

$$\|v - \bar{v}\|_{L_2(K)} \leq \min(h_{\min,K}, \kappa^{-1})_m \|v\|_K \leq \min(h_{\min,K}, \kappa^{-1}) m_1(v, K) \|v\|_K, \quad (41)$$

$$\|v - \bar{v}\|_{L_2(\gamma)} \leq \left(\frac{|\gamma|}{|K|}\right)^{1/2} \|C_K^\top \nabla v\|_{L_2(K)}, \quad (42)$$

$$\begin{aligned} \|v - \bar{v}\|_{L_2(\gamma)} &\leq \left(\frac{|\gamma|}{|K|}\right)^{1/2} h_{\min,K}^{1/2} \min(h_{\min,K}, \kappa^{-1})^{1/2}_m \|v\|_K \\ &\leq \left(\frac{|\gamma|}{|K|}\right)^{1/2} h_{\min,K}^{1/2} \min(h_{\min,K}, \kappa^{-1})^{1/2} m_1(v, K) \|v\|_K. \end{aligned} \quad (43)$$

Proof. Estimate (39) is obvious. For estimate (40) see for instance [18]. Estimate (41) evidently follows from (39) and (40). Combining Lemma 4.5 and estimate (39) of the current lemma and observing that $\|v\|_{L_2(K)} \|C_K^\top \nabla v\|_{L_2(K)} \leq \|v\|_{L_2(K)}^2 + \|C_K^\top \nabla v\|_{L_2(K)}^2$ we verify (42). In order to show (43) we use (42) to obtain

$$\|v - \bar{v}\|_{L_2(\gamma)}^2 \leq \frac{|\gamma|}{|K|} \|C_K^\top \nabla v\|_{L_2(K)}^2 \leq \frac{|\gamma|}{|K|} h_{\min,K}^2 m \|v\|_K^2.$$

Furthermore, with the aid of Lemma 4.5 and estimates (39), (40) we get the following:

$$\|v - \bar{v}\|_{L_2(\gamma)}^2 \leq \frac{|\gamma|}{|K|} \kappa^{-1} \sqrt{\kappa^2 \|v\|_{L_2(K)}^2 h_{\min,K}} \sqrt{h_{\min,K}^{-2} \|C_K^\top \nabla v\|_{L_2(K)}^2} \leq \frac{|\gamma|}{|K|} h_{\min,K} \kappa^{-1}_m \|v\|_K^2.$$

Combining the two previous estimates we get the result claimed. \square

4.3. Estimates for element and face residuals in the anisotropic case

In this section we prove two lemmas which we will need later. Namely, we derive the upper bounds for interior and face residuals. The jump discontinuity in the approximation of the normal flux at an interelement boundary is defined by

$$\left[\frac{\partial u_X}{\partial n} \right] := n_K \cdot (\nabla u_X)_K + n_{K'} \cdot (\nabla u_X)_{K'},$$

and the usual interior and boundary residuals r and R are given by

$$r := f + \Delta u_X - \kappa^2 u_X$$

and

$$R := \begin{cases} - \left[\frac{\partial u_X}{\partial n} \right] & \text{on } \partial K \cap \partial K' \\ 0 & \text{on } \partial K \cap \partial \Omega. \end{cases}$$

Lemma 4.7 (residuals estimates). *Let $K \in \mathcal{T}$ and γ be any interior edge. Then*

$$\|r\|_{L_2(K)} \leq \min(h_{\min,K}, \kappa^{-1})^{-1} \|e\|_K + \|r - \bar{r}\|_{L_2(K)}, \quad (44)$$

$$\|R\|_{L_2(\gamma)} \leq \sum_{K' \in \omega_\gamma} \left(\frac{|K'|}{|\gamma|}\right)^{1/2} h_{\min,K'}^{-1/2} \min(h_{\min,K'}, \kappa^{-1})^{-1/2} (\|e\|_{K'} + \min(h_{\min,K'}, \kappa^{-1}) \|r - \bar{r}\|_{L_2(K')}). \quad (45)$$

Proof. Let $v \in H_0^1(\Omega)$. Integrating by parts on each element yields

$$B(e, v) = \sum_{K \in \mathcal{T}} \int_K rv \, dx - \sum_{\gamma \in \partial \mathcal{T}} \int_{\gamma} Rv \, ds, \quad (46)$$

where $\partial \mathcal{T}$ denotes the collection of interelement edges. Hence for any $v \in H_0^1(\Omega)$

$$B(e, v) = \sum_{K \in \mathcal{T}} \int_K \bar{r}v \, dx - \sum_{\gamma \in \partial \mathcal{T}} \int_{\gamma} Rv \, ds + \sum_{K \in \mathcal{T}} \int_K (r - \bar{r})v \, dx.$$

Now, choosing $v := b_K \bar{r}$ in the previous equality gives

$$\int_K b_K \bar{r}^2 \, dx = B_K(e, b_K \bar{r}) - \int_K (r - \bar{r})b_K \bar{r} \, dx.$$

Using (23), with the aid of Cauchy-Schwarz inequality we obtain

$$\|\bar{r}\|_{L_2(K)}^2 \preceq \int_K b_K \bar{r}^2 \, dx \leq \|e\|_K \|b_K \bar{r}\|_K + \|r - \bar{r}\|_{L_2(K)} \|b_K \bar{r}\|_{L_2(K)}.$$

Estimates (24), (27) together with the triangle inequality imply (44).

We now show (45). Let $\gamma \in \partial \mathcal{T}$. Suppose that $\gamma = \overline{K}_1 \cap \overline{K}_2$. Then $\omega_\gamma = \text{int}(\overline{K}_1 \cup \overline{K}_2)$. Choosing $v := F_{ext}(R)b_{\gamma, \delta_\gamma} \in H_0^1(\Omega)$ in (46) implies

$$\int_{\gamma} b_{\gamma, \delta_\gamma} R^2 \, ds = \sum_{K \subset \omega_\gamma} \int_K r F_{ext}(R)b_{\gamma, \delta_\gamma} \, dx - B_{\omega_\gamma}(e, F_{ext}(R)b_{\gamma, \delta_\gamma}).$$

Furthermore, applying the Cauchy-Schwarz inequality and (29), one obtains

$$\|R\|_{L_2(\gamma)}^2 \preceq \int_{\gamma} b_{\gamma, \delta_\gamma} R^2 \, ds \leq \sum_{K \subset \omega_\gamma} \|r\|_{L_2(K)} \|F_{ext}(R)b_{\gamma, \delta_\gamma}\|_{L_2(K)} + \|e\|_{\omega_\gamma} \|F_{ext}(R)b_{\gamma, \delta_\gamma}\|_{\omega_\gamma}.$$

The desired inequality (45) follows now from (31), (34) and the first result of the current lemma (44). \square

Recall that we use the procedure for finding approximate fluxes described in Section 3.2 with the functions θ_n replaced by θ_n^* in the system (17). In the singularly perturbed case and using anisotropic elements we have the following theorem.

Theorem 4.8 (stability of the approximate fluxes). *Suppose that the finite element subspace X is constructed using first-order (linear) elements on a partition \mathcal{T} of the domain Ω into triangular elements. Let $\{g_K\}$ be the set of approximate fluxes, produced by the algorithm described in Section 3.2 with the functions θ_n replaced by θ_n^* , $n \in \mathcal{N}$. Then, for each edge γ of any element K ,*

$$\begin{aligned} \left\| g_K - \left\langle \frac{\partial u_X}{\partial n_K} \right\rangle \right\|_{L_2(\gamma)} &\preceq \sum_{K' \subset \omega_K} \left(\frac{|K'|}{|\gamma|} \right)^{1/2} h_{\min, K'}^{-1/2} \min(h_{\min, K'}, \kappa^{-1})^{-1/2} \\ &\quad \times (\|e\|_{K'} + \min(h_{\min, K'}, \kappa^{-1}) \|r - \bar{r}\|_{L_2(K')}). \end{aligned}$$

Proof. Let $K \in \mathcal{T}$ be a fixed element and $\gamma \subset K$ be an edge thereof. Then

$$\left(g_K - \left\langle \frac{\partial u_X}{\partial n_K} \right\rangle \right) \Big|_{\gamma} \in \mathbb{P}_1(\gamma).$$

Following Section 3.2 the moments of this quantity are

$$\mu_{K,n}^{*\gamma} = \int_{\gamma} \left(g_K - \left\langle \frac{\partial u_X}{\partial n_K} \right\rangle \right) \theta_n \, ds.$$

By analogy with (10),

$$\left(g_K - \left\langle \frac{\partial u_X}{\partial n_K} \right\rangle \right) \Big|_{\gamma} = \mu_{K,l}^{*\gamma} \psi_l + \mu_{K,r}^{*\gamma} \psi_r.$$

Therefore,

$$\left\| g_K - \left\langle \frac{\partial u_X}{\partial n_K} \right\rangle \right\|_{L_2(\gamma)} \leq \left| \mu_{K,l}^{*\gamma} \right| \|\psi_l\|_{L_2(\gamma)} + \left| \mu_{K,r}^{*\gamma} \right| \|\psi_r\|_{L_2(\gamma)}$$

and since

$$\|\psi_l\|_{L_2(\gamma)}^2 = \|\psi_r\|_{L_2(\gamma)}^2 = C|\gamma|^{-1},$$

it follows that

$$\left\| g_K - \left\langle \frac{\partial u_X}{\partial n_K} \right\rangle \right\|_{L_2(\gamma)}^2 \leq |\gamma|^{-1} \sum_{n \in \mathcal{N}(\gamma)} \left| \mu_{K,n}^{*\gamma} \right|^2. \tag{47}$$

With the aid of (19), we conclude that

$$\int_{\gamma} \left\langle \frac{\partial u_X}{\partial n_K} \right\rangle \theta_n \, ds = \begin{cases} \frac{1}{2} (\tilde{\mu}_{K,n}^{\gamma} - \tilde{\mu}_{K',n}^{\gamma}) & \text{on } \gamma = \partial K \cap \partial K' \\ \tilde{\mu}_{K,n}^{\gamma} & \text{on } \gamma = \partial K \cap \partial \Omega \end{cases}$$

with $\tilde{\mu}_{K,n}^{\gamma}$ defined in (13). Hence, thanks to (16),

$$\mu_{K,n}^{*\gamma} = \begin{cases} \frac{1}{2} (\sigma_{K,n} - \sigma_{K',n}) & \text{on } \gamma = \partial K \cap \partial K' \\ \sigma_{K,n} & \text{on } \gamma = \partial K \cap \partial \Omega \end{cases}$$

where the unknowns $\{\sigma_{K,n}\}$ are determined from conditions (17) and satisfy (20). It follows that

$$\left| \mu_{K,n}^{*\gamma} \right|^2 \leq \sum_{K' \in \tilde{x}_n} \sigma_{K',n}^2 \leq \sum_{K' \in \tilde{x}_n} \tilde{\Delta}_{K'}(\theta_n^*)^2. \tag{48}$$

The terms appearing on the right-hand side may be bounded by first recalling (18),

$$\tilde{\Delta}_{K'}(\theta_n^*) = B_{K'}(u_X, \theta_n^*) - (f, \theta_n^*)_{K'} - \int_{\partial K'} \left\langle \frac{\partial u_X}{\partial n_{K'}} \right\rangle \theta_n^* \, ds;$$

then, integrating by parts reveals that

$$\tilde{\Delta}_{K'}(\theta_n^*) = -(r, \theta_n^*)_{K'} - \int_{\partial K'} R \theta_n^* \, ds.$$

We proceed applying the Cauchy-Schwarz inequality and using Lemma 3.2 and Lemma 4.7:

$$\begin{aligned} \left| \tilde{\Delta}_{K'}(\theta_n^*) \right| &\leq \|r\|_{L_2(K')} \|\theta_n^*\|_{L_2(K')} + \sum_{\gamma' \subset \partial K' \cap \mathcal{E}_n} \|R\|_{L_2(\gamma')} \|\theta_n^*\|_{L_2(\gamma')} \\ &\leq (\min(h_{\min,K'}, \kappa^{-1})^{-1} \|e\|_{K'} + \|r - \bar{r}\|_{L_2(K')}) |K'|^{1/2} h_{\min,K'}^{-1/2} \min(h_{\min,K'}, \kappa^{-1})^{1/2} \\ &+ \sum_{\gamma' \subset \partial K' \cap \mathcal{E}_n} \sum_{K'' \subset \omega_{\gamma'}} |K''|^{1/2} h_{\min,K''}^{-1/2} \min(h_{\min,K''}, \kappa^{-1})^{-1/2} (\|e\|_{K''} + \min(h_{\min,K''}, \kappa^{-1}) \|r - \bar{r}\|_{L_2(K'')}) \\ &\leq \sum_{K'' \in \tilde{\mathcal{X}}_n} |K''|^{1/2} h_{\min,K''}^{-1/2} \left(\min(h_{\min,K''}, \kappa^{-1})^{-1/2} \|e\|_{K''} + \min(h_{\min,K''}, \kappa^{-1})^{1/2} \|r - \bar{r}\|_{L_2(K'')} \right). \end{aligned}$$

Hence,

$$\sum_{K' \in \tilde{\mathcal{X}}_n} \left| \tilde{\Delta}_{K'}(\theta_n^*) \right|^2 \leq \sum_{K' \in \tilde{\mathcal{X}}_n} |K'| h_{\min,K'}^{-1} \left(\min(h_{\min,K'}, \kappa^{-1})^{-1} \|e\|_{K'}^2 + \min(h_{\min,K'}, \kappa^{-1}) \|r - \bar{r}\|_{L_2(K')}^2 \right). \tag{49}$$

Combining (47), (48) and (49) leads to the result claimed. □

5. LOWER ERROR BOUND OF THE ORIGINAL AINSWORTH-BABUŠKA ESTIMATOR IN THE ANISOTROPIC SINGULARLY PERTURBED CASE

Describing in Section 3 the equilibrated residual method, we derived the upper error bound. The original analysis of the lower error bound for *isotropic* triangles dates back to the work by Ainsworth and Babuška [1]. Here we analyse the *anisotropic* case. It turns out that the original error estimator described in [1] has degenerating lower error bound.

The right hand side of the local problem (7) is originally defined as a linear functional only for the functions $v \in V_K$. We will need, however, to apply this functional also to the functions outside of V_K , namely to those not preserving the Dirichlet boundary conditions. To this end we introduce a new notation for the residual functional on the right hand side of (7):

$$\mathcal{B}_K(v) := (f, v)_K - B_K(u_X, v) + \int_{\partial K} g_K v \, ds \quad \forall v \in H^1(K). \tag{50}$$

From this notation it is clear that $\mathcal{B}_K(v) = B_K(\phi_K, v)$ for all $v \in V_K$, but the domain of definition of the functional \mathcal{B}_K is larger for the elements K touching the boundary. The next lemma states some stability properties of the residual functional \mathcal{B}_K which we will require in the lower error bound estimates.

Lemma 5.1. *Let ϕ_K denote the solution of the local residual problem (7) for the error estimator on element K . Then, for any $v \in H^1(K)$,*

$$|\mathcal{B}_K(v - \bar{v})| \leq m_1(v, K) (\|e\|_{\omega_K} + \min(h_{\min,K}, \kappa^{-1}) \|r - \bar{r}\|_{L_2(\omega_K)}) \|v\|_K. \tag{51}$$

Furthermore, if $\kappa \geq h_{\min,K}^{-1}$, then

$$\begin{aligned} |\bar{\phi}_K \mathcal{B}_K(1)| &\leq (\|e\|_{\omega_K} + \kappa^{-1} \|r - \bar{r}\|_{L_2(\omega_K)}) m \|\phi_K\|_K \\ &\leq m_1(\phi_K, K) (\|e\|_{\omega_K} + \kappa^{-1} \|r - \bar{r}\|_{L_2(\omega_K)}) \|\phi_K\|_K. \end{aligned} \tag{52}$$

Proof. 1. Integrating (50) by parts yields

$$\mathcal{B}_K(v - \bar{v}) = \int_K r(v - \bar{v}) \, dx + \frac{1}{2} \int_{\partial K} R(v - \bar{v}) \, ds + \int_{\partial K} \left(g_K - \left\langle \frac{\partial u_X}{\partial n_K} \right\rangle \right) (v - \bar{v}) \, ds.$$

and it therefore follows that

$$\begin{aligned} |\mathcal{B}_K(v - \bar{v})| &\leq \sum_{\gamma \subset \partial K} \left\| g_K - \left\langle \frac{\partial u_X}{\partial n_K} \right\rangle \right\|_{L_2(\gamma)} \|v - \bar{v}\|_{L_2(\gamma)} \\ &+ \|r\|_{L_2(K)} \|v - \bar{v}\|_{L_2(K)} + \frac{1}{2} \sum_{\gamma \subset \partial K} \|R\|_{L_2(\gamma)} \|v - \bar{v}\|_{L_2(\gamma)}. \end{aligned} \tag{53}$$

Combining results from Lemma 4.7, Theorem 4.8 and Lemma 4.6 we get (51).

2. Suppose that $\kappa \geq h_{\min,K}^{-1}$. Then

$$\mathcal{B}_K(1) = (f, 1)_K - B_K(u_X, 1) + \int_{\partial K} g_K \, ds.$$

Integrating by parts, applying the Cauchy-Schwarz inequality, and estimating each term using Lemma 4.7 and Theorem 4.8 yield

$$\begin{aligned} |\mathcal{B}_K(1)| &\leq |K|^{1/2} \|r\|_{L_2(K)} + \frac{1}{2} \sum_{\gamma \in \partial K} |\gamma|^{1/2} \|R\|_{L_2(\gamma)} + \sum_{\gamma \in \partial K} |\gamma|^{1/2} \left\| g_K - \left\langle \frac{\partial u_X}{\partial n_K} \right\rangle \right\|_{L_2(\gamma)} \\ &\leq \kappa |K|^{1/2} (\|e\|_{\omega_K} + \kappa^{-1} \|r - \bar{r}\|_{L_2(\omega_K)}), \end{aligned}$$

where the inequality $\min(h_{\min,K}, \kappa^{-1}) \leq \kappa^{-1}$ has been used. Inequality (52) now can be easily obtained

$$\begin{aligned} |\bar{\phi}_K \mathcal{B}_K(1)| &\leq \kappa |K|^{1/2} |\bar{\phi}_K| (\|e\|_{\omega_K} + \kappa^{-1} \|r - \bar{r}\|_{L_2(\omega_K)}) \\ &\leq \kappa \|\phi_K\|_{L_2(K)} (\|e\|_{\omega_K} + \kappa^{-1} \|r - \bar{r}\|_{L_2(\omega_K)}) \leq (\|e\|_{\omega_K} + \kappa^{-1} \|r - \bar{r}\|_{L_2(\omega_K)}) m \|\phi_K\|_K. \quad \square \end{aligned}$$

For the lower bound we have the following result.

Theorem 5.2 (Lower error bound). *Let g_K be the set of fluxes produced by the algorithm described in Section 3.2 with the functions θ_n replaced by θ_n^* , and let $\phi_K \in V_K$ denote the solution of the local residual problem (7). Then,*

$$\|\phi_K\|_K \leq m_1(\phi_K, K) (\|e\|_{\omega_K} + \min(h_{\min,K}, \kappa^{-1}) \|r - \bar{r}\|_{L_2(\omega_K)}).$$

If κ vanishes, then $\min(h_{\min,K}, \kappa^{-1})$ is replaced by $h_{\min,K}$.

Proof. Observe that for any $v \in V_K$,

$$B_K(\phi_K, v) = \mathcal{B}_K(v) = \mathcal{B}_K(v - \bar{v}) + \bar{v} \mathcal{B}_K(1). \tag{54}$$

First, suppose $\kappa h_{\min,K} \geq 1$ so that, in particular, κ is positive and $\min(h_{\min,K}, \kappa^{-1})^{-1} \sim \kappa$. Therefore, with the aid of Lemma 5.1,

$$|\mathcal{B}_K(\phi_K - \bar{\phi}_K)| \leq m_1(\phi_K, K) \{ \|e\|_{\omega_K} + \min(h_{\min,K}, \kappa^{-1}) \|r - \bar{r}\|_{L_2(\omega_K)} \} \|\phi_K\|_K.$$

Choosing v to be equal to ϕ_K in (54), together with the above estimate, proves that the result holds for all elements K satisfying $\kappa h_{\min,K} \geq 1$.

The remaining elements satisfy $\kappa h_{\min,K} \ll 1$. Thanks to the assumptions on the partition, the condition $\kappa h_{K'} \ll 1$ is satisfied by all elements K' contained in the patch ω_K . Therefore, Lemma 3.2 reveals that the modified basis functions reduce to the standard basis functions on the patch. Consequently, the approximate fluxes will actually satisfy the equilibration conditions (9) exactly. Moreover, since

$$\mathcal{B}_K(1) = (f, 1)_K - B_K(u_X, 1) + \int_{\partial K} g_K \, ds = 0, \tag{55}$$

the second term in (54) vanishes. The first estimate in Lemma 5.1 then completes the proof. \square

Theorem 5.2 gives the lower error bound of the true error. The main danger for reliability of the estimator is the function $m_1(\phi_K, K)$ presented on the right hand side. One cannot guarantee that the approximation for the error ϕ_K is aligned as well as the true error e . Unfortunately, it may happen so that the alignment of the approximation ϕ_K on the element K is much worse than e : $m_1(\phi_K, K) \gg m_1(e, K)$. To avoid this problem a modification is proposed in the next paragraph.

6. MODIFIED EQUILIBRATED RESIDUAL METHOD

For finding the equilibrated fluxes we use again the equilibrated residual method described in Section 3.2. In this paragraph, we propose an alternative method by changing the local problem, namely, instead of (7) we use

$${}_m B_K(\phi_K, v) = (f, v)_K - B_K(u_X, v) + \int_{\partial K} g_K v \, ds \quad \forall v \in V_K. \quad (56)$$

This local problem differs from the original local problem (7) only in the scalar product ${}_m B_K(u, v)$ on the left hand side. The quantity ϕ_K is then not equivalent to the error e , but we will show that the ${}_m \|\phi_K\|_K$ is related to $\|e\|_K$. The following two theorems give upper and lower bounds for the error.

Theorem 6.1 (reliability). *Let $\{g_K : K \in \mathcal{T}\}$ be any set of boundary fluxes satisfying condition (5). In addition, if the absolute term κ vanishes, then it is assumed that the fluxes satisfy the equilibration condition (8) on all elements that do not abut the boundary $\partial\Omega$. Then, the global error residual may be decomposed into local contributions*

$$B(e, v) = L(v) - B(u_X, v) = \sum_{K \in \mathcal{T}} {}_m B_K(\phi_K, v), \quad v \in H^1(K),$$

where $\phi_K \in V_K$ is the solution of the local problem (56). The global error in the finite element approximation may be bounded by

$$\|e\|^2 \leq m_1(e, \mathcal{T})^2 \sum_{K \in \mathcal{T}} {}_m \|\phi_K\|_K^2,$$

where $m_1(e, \mathcal{T})$ is the matching function introduced by (38).

Proof. Using the representation of $B(e, v)$ in the local terms and subsequently applying the Cauchy-Schwarz inequality and the definition of the matching function, we have:

$$\begin{aligned} |B(e, v)| &= \left| \sum_{K \in \mathcal{T}} \{(f, v)_K - B_K(u_X, v)\} \right| \\ &= \left| \sum_{K \in \mathcal{T}} \{(f, v)_K - B_K(u_X, v) + \int_{\partial K} g_K v \, ds\} \right| \\ &= \left| \sum_{K \in \mathcal{T}} {}_m B_K(\phi_K, v) \right| \\ &\leq \sum_{K \in \mathcal{T}} {}_m \|\phi_K\|_K m \|v\|_K \\ &\leq \sqrt{\sum_{K \in \mathcal{T}} {}_m \|\phi_K\|_K^2} \cdot \sqrt{\sum_{K \in \mathcal{T}} (h_{\min, K}^{-2} \|C_K^\top \nabla v\|_{L_2(K)}^2 + \kappa^2 \|v\|_{L_2(K)}^2)} \\ &\leq \sqrt{\sum_{K \in \mathcal{T}} {}_m \|\phi_K\|_K^2} \cdot \sqrt{m_1(v, \mathcal{T})^2 \|\nabla v\|_{L_2(\Omega)}^2 + \kappa^2 \|v\|_{L_2(\Omega)}^2} \\ &\leq m_1(v, \mathcal{T}) \|v\| \sqrt{\sum_{K \in \mathcal{T}} {}_m \|\phi_K\|_K^2}. \end{aligned}$$

The substitution $v := e$ completes the proof. \square

Theorem 6.1 gives the usual result for anisotropic error estimators. See for instance [18, 21].

Theorem 6.2 (efficiency). *Let g_K be the set of approximate fluxes produced by the algorithm described in Section 3.2 with the functions θ replaced by θ^* , and let $\phi_K \in V_K$ denote the solution of the local residual problem (56). Then,*

$${}_m \|\phi_K\|_K \preceq \|e\|_{\tilde{K}} + \min(h_{\min,K}, \kappa^{-1}) \|r - \bar{r}\|_{L_2(\tilde{K})}.$$

Proof. The proof follows the same lines as the proof of the Theorem 5.2. □

These two theorems are part of the main result of this work and guarantee the reliability and efficiency of the estimator, assuming an exact solution of the local problems.

7. COMPUTABLE APPROXIMATION FOR THE SOLUTION OF THE LOCAL PROBLEM

Up to this time we considered the infinite dimensional local problems (7) and (56). The author has not found any result in the literature saying that some computable approximation $\tilde{\phi}_K$ is equivalent to ϕ_K in the energy norm,

$$\|\tilde{\phi}_K\| \sim \|\phi_K\|, \tag{57}$$

even not for isotropic elements. In the current section we construct an approximation $\tilde{\phi}_K$ so that (57) holds. To this end we restrict the space V_K to the space of bubbles/splines $V_b(K)$ defined in the following way:

$$V_b(K) := \text{span}\{b_K, b_{\gamma, \delta_\gamma}, s_{\gamma, \delta_\gamma} : \gamma \in \partial K \setminus \partial\Omega\} \subset V_K.$$

By means of the space $V_b(K)$ we can define the function $\tilde{\phi}_K \in V_b(K)$ as the solution of the local finite dimensional problem

$${}_m B_K(\tilde{\phi}_K, v) = (f, v)_K - B_K(u_X, v) + \int_{\partial K} g_K v \, ds \quad \forall v \in V_b(K). \tag{58}$$

We prove the lower bound for ϕ_K first.

Theorem 7.1. *Let ϕ_K and $\tilde{\phi}_K$ be the solutions to the problems (56) and (58), respectively. Then*

$${}_m \|\tilde{\phi}_K\|_K \leq {}_m \|\phi_K\|_K.$$

Proof. Substituting $v = \tilde{\phi}_K \in V_b(K) \subset V_K$ in (58) and subsequently utilizing the Cauchy-Schwarz inequality we estimate ${}_m B_K(\tilde{\phi}_K, \tilde{\phi}_K)$ as follows:

$$\begin{aligned} {}_m B_K(\tilde{\phi}_K, \tilde{\phi}_K) &= (f, \tilde{\phi}_K)_K - B_K(u_X, \tilde{\phi}_K) + \int_{\partial K} g_K \tilde{\phi}_K \, ds \\ &= {}_m B_K(\phi_K, \tilde{\phi}_K) \leq {}_m \|\phi_K\|_K {}_m \|\tilde{\phi}_K\|_K. \end{aligned}$$

Dividing both sides by ${}_m \|\tilde{\phi}_K\|_K$ we get the result claimed. □

For further investigations we need the following preliminary result.

Lemma 7.2. *Let ϕ_K denote the solution to the local residual problem (56) for the error estimator on the element K . If $\kappa \succeq h_{\min,K}^{-1}$, then*

$$|\bar{\phi}_K \mathcal{B}_K(1)| \preceq \left(\kappa^{-1} \|r\|_{L_2(K)} + \sum_{\gamma \subset \partial K} \kappa^{-1} \left(\frac{|\gamma|}{|K|} \right)^{1/2} \left\| g_K - \frac{\partial u_x}{\partial n_K} \right\|_{L_2(\gamma)} \right) {}_m \|\phi_K\|_K, \tag{59}$$

where \mathcal{B}_K is the residual functional defined in (50).

Proof. The proof is done analogously to the proof of (52) in Lemma 5.1. \square

For the upper bound we employ the technique that is usually used in obtaining the lower error bound in the residual a posteriori error estimation (see e.g. [9]).

Lemma 7.3. *Let ϕ_K be the solution of (56) and let $\bar{\phi}_K$ be its mean value over the triangle K . Then the following estimate holds:*

$$\begin{aligned} \mathcal{B}_K(\phi_K - \bar{\phi}_K) &\preceq m \|\phi_K\|_K \left(\min(h_{\min,K}, \kappa^{-1}) \|r\|_{L_2(K)} \right. \\ &\quad \left. + \sum_{\gamma \subset \partial K} h_{\min,K}^{1/2} \left(\frac{|\gamma|}{|K|} \right)^{1/2} \min(h_{\min,K}, \kappa^{-1})^{1/2} \left\| g_K - \frac{\partial u_X}{\partial n_K} \right\|_{L_2(\gamma)} \right). \end{aligned}$$

Proof. We use the definition of the residual functional (50) and apply subsequently the Cauchy-Schwarz inequality and the anisotropic approximation properties (41) and (43) obtained in Lemma 4.6:

$$\begin{aligned} \mathcal{B}_K(\phi_K - \bar{\phi}_K) &= (f, \phi_K - \bar{\phi}_K)_K - B_K(u_X, \phi_K - \bar{\phi}_K) + \int_{\partial K} g_K(\phi_K - \bar{\phi}_K) \, ds \\ &= (r, \phi_K - \bar{\phi}_K)_K + \int_{\partial K} \left(g_K - \frac{\partial u_X}{\partial n_K} \right) (\phi_K - \bar{\phi}_K) \, ds \\ &\leq \|r\|_{L_2(K)} \|\phi_K - \bar{\phi}_K\|_{L_2(K)} + \sum_{\gamma \subset \partial K} \left\| g_K - \frac{\partial u_X}{\partial n_K} \right\|_{L_2(\gamma)} \|\phi_K - \bar{\phi}_K\|_{L_2(\gamma)} \\ &\preceq m \|\phi_K\|_K \min(h_{\min,K}, \kappa^{-1}) \|r\|_{L_2(K)} \\ &\quad + m \|\phi_K\|_K \sum_{\gamma \subset \partial K} h_{\min,K}^{1/2} \left(\frac{|\gamma|}{|K|} \right)^{1/2} \min(h_{\min,K}, \kappa^{-1})^{1/2} \left\| g_K - \frac{\partial u_X}{\partial n_K} \right\|_{L_2(\gamma)}, \end{aligned}$$

which completes the proof. \square

Lemma 7.4. *Let ϕ_K be the solution to (56). Then the following estimate holds:*

$$m \|\phi_K\|_K \preceq \min(h_{\min,K}, \kappa^{-1}) \|r\|_{L_2(K)} + \sum_{\gamma \subset \partial K} h_{\min,K}^{1/2} \left(\frac{|\gamma|}{|K|} \right)^{1/2} \min(h_{\min,K}, \kappa^{-1})^{1/2} \left\| g_K - \frac{\partial u_X}{\partial n_K} \right\|_{L_2(\gamma)}$$

Proof. We represent $m \|\phi_K\|_K^2$ as a sum of two terms:

$$m \|\phi_K\|_K^2 = m B_K(\phi_K, \phi_K) = \mathcal{B}_K(\phi_K) = \mathcal{B}_K(\phi_K - \bar{\phi}_K) + \mathcal{B}_K(\bar{\phi}_K). \quad (60)$$

In Lemma 7.3 we constructed already the estimate from above for the first term. We proceed estimating the second term analogously to the proof of Theorem 5.2.

Consider the case $\kappa h_{\min,K} \geq 1$. Thus, the second assertion of Lemma 5.1 holds, *i.e.*

$$\begin{aligned} \mathcal{B}_K(\bar{\phi}_K) &\leq m \|\phi_K\|_K \left(\kappa^{-1} \|r\|_{L_2(K)} + \sum_{\gamma \subset \partial K} \left(\frac{|\gamma|}{|K|} \right)^{1/2} \kappa^{-1} \left\| g_K - \frac{\partial u_X}{\partial n_K} \right\|_{L_2(\gamma)} \right) \\ &\leq m \|\phi_K\|_K \left(\min(h_{\min,K}, \kappa^{-1}) \|r\|_{L_2(K)} \right. \\ &\quad \left. + \sum_{\gamma \subset \partial K} h_{\min,K}^{1/2} \left(\frac{|\gamma|}{|K|} \right)^{1/2} \min(h_{\min,K}, \kappa^{-1})^{1/2} \left\| g_K - \frac{\partial u_X}{\partial n_K} \right\|_{L_2(\gamma)} \right), \end{aligned}$$

where we used $\kappa \geq h_{\min,K}^{-1}$.

The remaining elements satisfy $\kappa h_{\min,K} \ll 1$. With the same arguments as in the proof of Theorem 5.2 we verify that for these elements (55) holds, and thus, the second term in (60) vanishes.

Summing up the contributions from the two terms on the right hand side of representation (60) and dividing the concluding inequality by $m \|\phi_K\|_K$ we get the result claimed. \square

In order to prove the main theorem of this section we need some additional elementary facts.

Lemma 7.5. *Let $\phi \in P_1([-1,1])$ be a linear function. For the L_2 -projection operator $I : P_1([-1,1]) \rightarrow \text{span}\{1 - x^2, x(1 - x^2)\}$ the following inequality holds:*

$$\sqrt{\frac{6}{5}} \|I\phi\|_{L_2([-1,1])} \leq \|\phi\|_{L_2([-1,1])} \leq \sqrt{\frac{10}{7}} \|I\phi\|_{L_2([-1,1])}.$$

Proof. Denote $\psi_1 := 1 - x$ and $\psi_2 := 1 + x$. The desired constants are the square roots of the maximal and minimal eigenvalues of the eigenvalue problem $A \cdot \mathbf{x} = \lambda B \cdot \mathbf{x}$, where $A = \left\{ \int_{-1}^1 \psi_i \psi_j \, dx \right\} \in \mathbb{R}^{2 \times 2}$, $B = \left\{ \int_{-1}^1 I\psi_i I\psi_j \, dx \right\} \in \mathbb{R}^{2 \times 2}$ and $\mathbf{x} \in \mathbb{R}^2$. \square

Lemma 7.6. *Let γ be an edge of a triangle K . We define the operator $I_\gamma : P_1(\gamma) \rightarrow \text{span}\{b_{\gamma,\delta_\gamma}, s_{\gamma,\delta_\gamma}\}$ so that for any function $\phi \in P_1(\gamma)$ the restriction of the resulting function $I_\gamma \phi|_\gamma$ to the edge γ is the L_2 projection of ϕ to the space $\text{span}\{b_{\gamma,\delta_\gamma}|_\gamma, s_{\gamma,\delta_\gamma}|_\gamma\}$. In other words, we project a function ϕ onto the space of two functions on the edge γ and then take the corresponding constant to produce the function $I_\gamma \phi$ inside the triangle K . For the operator I_γ and any function $\phi \in P_1(\gamma)$ the following estimate holds:*

$$\sqrt{\frac{6}{5}} \|I_\gamma \phi\|_{L_2(\gamma)} \leq \|\phi\|_{L_2(\gamma)} \leq \sqrt{\frac{10}{7}} \|I_\gamma \phi\|_{L_2(\gamma)}.$$

Proof. The functions $1 - x^2$ and $x(1 - x^2)$ coincide with the functions $b_{\gamma,\delta_\gamma}|_\gamma$ and $s_{\gamma,\delta_\gamma}|_\gamma$ whenever $\gamma = [-1, 1]$. Thus, the assertion follows from Lemma 7.5 and standard transformation techniques. \square

We proceed with the main result of this section, that guarantees the estimate from above for the solution of the local problem (56).

Theorem 7.7. *Let ϕ_K and $\tilde{\phi}_K$ be the solutions to the problems (56) and (58) respectively. Then:*

$$m \|\phi_K\|_K \leq m \|\tilde{\phi}_K\|_K + \min(h_{\min,K}, \kappa^{-1}) \|r - \bar{r}\|_{L_2(K)}.$$

Proof. We will essentially use the estimate of Lemma 7.4 and bound the terms on the right hand side by $m \|\tilde{\phi}_K\|_K$.

Performing the partial integration we can rewrite the finite dimensional local problem (58) as

$$\begin{aligned} {}_m B_K(\tilde{\phi}_K, v) &= (r, v)_K + \int_{\partial K} \left(g_K - \frac{\partial u_X}{\partial n_K} \right) v \, ds \\ &= (r_K, v)_K + \int_{\partial K} \left(g_K - \frac{\partial u_X}{\partial n_K} \right) v \, ds + (r - r_K, v)_K. \end{aligned} \quad (61)$$

Since (61) holds for all $v \in V_b(K)$, we substitute $v := b_K \bar{r}$,

$$\int_K b_K \bar{r}^2 \, dx = {}_m B_K(\phi_K, b_K \bar{r}) - (r - r_K, b_K \bar{r})_K.$$

Using (23), we obtain with the aid of Cauchy-Schwarz inequality

$$\|\bar{r}\|_{L_2(K)}^2 \leq m \|\tilde{\phi}_K\|_K \|b_K \bar{r}\|_K + \|r - \bar{r}\|_{L_2(K)} \|b_K \bar{r}\|_{L_2(K)}.$$

Applying estimates (24) and (28) of Lemma 4.2 and dividing both parts by $\min(h_{\min, K}, \kappa^{-1})^{-1} \|\bar{r}\|_{L_2(K)}$, we get

$$\min(h_{\min, K}, \kappa^{-1}) \|\bar{r}\|_{L_2(K)} \leq m \|\tilde{\phi}_K\|_K + \min(h_{\min, K}, \kappa^{-1}) \|r - \bar{r}\|_{L_2(K)},$$

or, with the aid of the triangle inequality we derive the upper bound for the term involving the element residual:

$$\min(h_{\min, K}, \kappa^{-1}) \|r\|_{L_2(K)} \leq m \|\tilde{\phi}_K\|_K + \min(h_{\min, K}, \kappa^{-1}) \|r - \bar{r}\|_{L_2(K)}.$$

Suppose $\gamma \subset \partial K \setminus \partial \Omega$ is one of the edges of K , which is not a Dirichlet edge. We choose now $v := I_\gamma(g_K - \frac{\partial u_X}{\partial n_K}) \in V_b(K)$ in (61), where the operator I_γ was defined in Lemma 7.6.

$${}_m B_K\left(\tilde{\phi}_K, I_\gamma\left(g_K - \frac{\partial u_X}{\partial n_K}\right)\right) = \left(r, I_\gamma\left(g_K - \frac{\partial u_X}{\partial n_K}\right)\right)_K + \int_{\partial K} \left(g_K - \frac{\partial u_X}{\partial n_K}\right) \cdot I_\gamma\left(g_K - \frac{\partial u_X}{\partial n_K}\right) \, ds. \quad (62)$$

Since I_γ is the L_2 -projection operator over γ , we rewrite the last equality in the form:

$$\left\| I_\gamma\left(g_K - \frac{\partial u_X}{\partial n_K}\right) \right\|_{L_2(\gamma)}^2 = {}_m B_K\left(\tilde{\phi}_K, I_\gamma\left(g_K - \frac{\partial u_X}{\partial n_K}\right)\right) - \left(r, I_\gamma\left(g_K - \frac{\partial u_X}{\partial n_K}\right)\right)_K, \quad (63)$$

and utilizing the Cauchy-Schwarz inequality, we obtain:

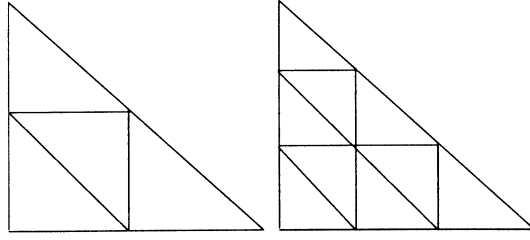
$$\left\| I_\gamma\left(g_K - \frac{\partial u_X}{\partial n_K}\right) \right\|_{L_2(\gamma)}^2 \leq m \|\tilde{\phi}_K\|_K \left\| I_\gamma\left(g_K - \frac{\partial u_X}{\partial n_K}\right) \right\|_K + \|r\|_{L_2(K)} \left\| I_\gamma\left(g_K - \frac{\partial u_X}{\partial n_K}\right) \right\|_{L_2(K)}. \quad (64)$$

Evaluating (36) and (37) of Lemma 4.3 together with Lemma 7.6, we complete the proof. \square

To complete the discussion of the equilibrated residual method it only remains to give a definition for the error estimator that can be used in practical computations and to give the resulting bounds provided by the preceding theory.

Definition 7.8. We define the local estimator of the equilibrated residual method and the local higher order term corresponding to the element K by

$$\begin{aligned} \eta_{ER, K} &:= m \|\tilde{\phi}_K\|_K, \\ \zeta_K &:= \min(h_{\min, K}, \kappa^{-1}) \|r - \bar{r}\|_{L_2(K)}, \end{aligned}$$

FIGURE 7. Triangle subdivisions. $n = 2$ and $n = 3$, respectively.

with the global counterparts

$$\eta_{ER} := \left(\sum_{K \in \mathcal{T}} \eta_{ER,K}^2 \right)^{1/2} \quad \text{and}$$

$$\zeta := \left(\sum_{K \in \mathcal{T}} \zeta_K^2 \right)^{1/2},$$

respectively.

Combining the bounds from Theorems 6.1, 6.2, 7.1 and 7.7 we arrive at the robustness result for the error estimator defined above.

Theorem 7.9. *In notation of Definition 7.8 the upper and lower error bounds hold*

$$\|e\|^2 \leq m_1^2(e, \mathcal{T}) (\eta_{ER}^2 + \zeta^2),$$

$$\eta_{ER,K}^2 \leq \|e\|_{\omega_K}^2 + \sum_{K \subset \omega_K} \zeta_K^2.$$

This theorem is the final result of this work and guarantees the reliability and efficiency of the estimator. The bounds are in accordance with those provided by Kunert [21] for the Dirichlet local problem error estimator for the singularly perturbed reaction-diffusion equation.

8. NUMERICAL EXPERIMENTS

In the previous section we gave an example for the bases for the local problem (56) consisting of the very specially squeezed functions. We have good experience, however, in solving this problem with a finite element method, where we choose the nodal basis corresponding to a division of the triangles into n^2 parts, see Figure 7.

Let us consider the 2D model problem

$$-\Delta u + \kappa^2 u = 0 \text{ in } \Omega := [0, 1]^2, \quad u = u_0 \text{ on } \partial\Omega.$$

Prescribe the exact solution

$$u = e^{-\kappa x} + e^{-\kappa y}$$

which displays typical boundary layers along the sides $x = 0$ and $y = 0$. The Dirichlet boundary data u_0 are chosen accordingly.

We use a sequence of finite element meshes generated by the algorithm described in [7]. The idea of adaptive procedure is that the choice of a refinement direction is done according to the components of energy norm of

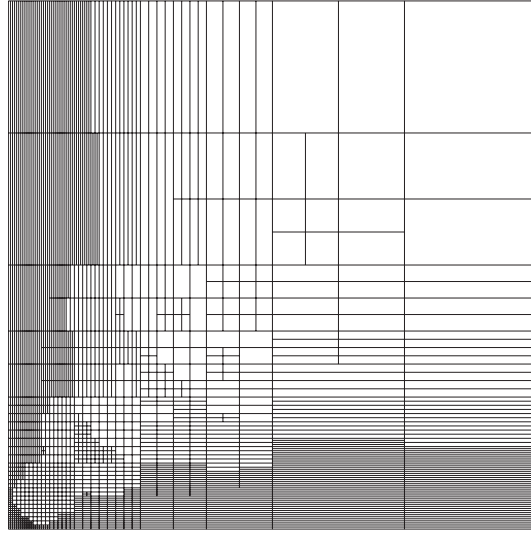


FIGURE 8. Mesh refinement.

Iteration N	Unknowns N	Maximal aspect ratio	$\frac{\text{AB error}}{\text{exact err}}$	$\frac{\text{AB er(mod)}}{\text{exact err}}$
1	25	71	1.392	1.102
2	51	71	1.301	0.896
3	97	71	1.565	0.962
4	116	142	1.903	1.032
5	157	285	2.457	1.094
6	217	571	3.151	1.153
7	382	1142	4.264	1.169
8	714	2284	5.955	1.167
9	1481	4568	8.903	1.157
10	3274	9137	14.475	1.130
11	6847	18 273	23.967	1.112
12	15 187	36 547	44.111	1.090
13	35 536	73 095	81.237	1.062
14	106 819	146 191	138.711	1.005

TABLE 1. Results for $n = 4$, $\kappa = 1000$. The fourth column represents the ratio between the Ainsworth and Babuška estimator and the energy norm of the true solution, while the fifth column represents the similar ratio for the estimator defined in the current work.

an error $\|\frac{\partial e}{\partial x}\|_{L_2(K)}$, $\|\frac{\partial e}{\partial y}\|_{L_2(K)}$, and $\kappa^2\|e\|_{L_2(K)}$. One of the resulting meshes of this program is displayed in Figure 8.

Tables 1 and 2 show the behavior of the estimators in the singularly perturbed case on anisotropic meshes. We observe that the new error estimator is robust while the original one overestimates the true error when the aspect ratio is large enough.

Iteration N	Unknowns N	Maximal aspect ratio	$\frac{\text{AB error}}{\text{exact err}}$	$\frac{\text{AB er(mod)}}{\text{exact err}}$
1	25	541	2.665	2.150
2	51	541	2.177	1.587
3	120	541	1.802	1.198
4	143	541	1.995	1.187
5	192	1083	2.410	1.243
6	217	2167	3.062	1.302
7	283	4335	4.273	1.392
8	446	8669	6.207	1.436
9	814	17 339	9.748	1.405
10	1553	34 679	16.646	1.411
11	3053	69 359	30.149	1.433
12	5809	138 718	54.363	1.429
13	11 357	277 436	101.47	1.420
14	23 376	554 873	211.02	1.407
15	104 916	1 109 745	423.10	1.383

TABLE 2. Results for $n = 4$, $\kappa = 10\,000$.

9. SUMMARY AND ADDITIONAL REMARKS

We consider the singularly perturbed reaction-diffusion equation $-\Delta u + \kappa^2 u = f$. This work has been aiming at *a posteriori* equilibrated residual-like error estimators suitable for anisotropic triangular grids.

The Ainsworth-Babuška estimator is shown to be reliable in the anisotropic case. Unfortunately, the lower error bound fails on anisotropic meshes.

The introduced modification leads to an estimator which is robust with respect to the anisotropy of the mesh as well as to the singular perturbation. Upper and lower error bounds are proved. The factor which made the original error estimator fail does not appear in the lower bound any more, which leads to the efficiency of the modified estimator. The upper error bound of the modified estimator contains the factor $m_1(e, \mathcal{T})$ which is in accordance with the results made by Kunert in [21]. Furthermore, an appropriate basis for the approximate solution of the local problem is provided and the whole method is shown not to be affected by this approximation.

The numerical experiments verify the theory. The modified estimator yields a useful and reliable error bound not only in an asymptotic sense but also for meshes with moderate number of elements.

Remark 9.1. All the proofs are suitable for 3D case. The only two questions we should answer are about the topology matrices and the minimum energy extension of the first-order basis function. For the minimum energy extension of the first-order basis function we construct the approximation by analogy with Section 3.3. A point D in this case may be chosen on the intersection line of the bisection planes of the corresponding cone with the distance $1/\kappa$ from each face. The topology matrices are constructed in the way analogous to Section 3, but seem to be much more complicated.

Remark 9.2. Neumann boundary conditions can be also considered as well as quadrilateral elements. In both cases an additional term corresponding to the face residual appears in the lower bound for the error:

$$\begin{aligned} \|\phi_K\|_K &\preceq \|e\|_{\tilde{K}} + \min(h_{\min,K}, \kappa^{-1}) \|r - \bar{r}\|_{L_2(\tilde{K})} \\ &+ \sum_{\gamma \subset \tilde{K}} \min(h_{\min,K}, \kappa^{-1})^{1/2} \|R - \bar{R}\|_{L_2(\gamma)}. \end{aligned}$$

Remark 9.3. If we solve our FEM problem with polynomials of p -th order then we have to talk about the p -th order equilibration. For details see [3].

Remark 9.4. Consider the problem in another formulation used by some authors (see [6, 8, 16, 21]), namely

$$-\varepsilon^2 \Delta u + u = f \text{ in } \Omega \subset \mathbb{R}^2, \quad u = 0 \text{ on } \partial\Omega.$$

The estimator remains the same as well as its upper error bound:

$$\|e\|^2 \leq m_1(e, \mathcal{T})^2 \sum_{K \in \mathcal{T}} m \|\phi_K\|_K^2,$$

The lower error bound changes to the following:

$$m \|\phi_K\|_K \preceq \|e\|_{\tilde{K}} + \min(h_{\min, K} \varepsilon^{-1}, 1) \|r - \bar{r}\|_{L_2(\tilde{K})}.$$

We should note that this is exactly the same estimate as for another error estimator introduced by Kunert in [21].

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