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ON THE ACCURACY OF REISSNER–MINDLIN PLATE MODEL FOR STRESS BOUNDARY CONDITIONS*

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Abstract. For a plate subject to stress boundary condition, the deformation determined by the Reissner–Mindlin plate bending model could be bending dominated, transverse shear dominated, or neither (intermediate), depending on the load. We show that the Reissner–Mindlin model has a wider range of applicability than the Kirchhoff–Love model, but it does not always converge to the elasticity theory. In the case of bending domination, both the two models are accurate. In the case of transverse shear domination, the Reissner–Mindlin model is accurate but the Kirchhoff–Love model totally fails. In the intermediate case, while the Kirchhoff–Love model fails, the Reissner–Mindlin solution also has a relative error comparing to the elasticity solution, which does not decrease when the plate thickness tends to zero. We also show that under the conventional definition of the resultant loading functional, the well known shear correction factor 5/6 in the Reissner–Mindlin model should be replaced by 1. Otherwise, the range of applicability of the Reissner–Mindlin model is not wider than that of Kirchhoff–Love's.

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1. Introduction

The Reissner–Mindlin plate bending model is one of the most frequently used dimensionally reduced models for linearly elastic plates. It was originally derived to cope with the boundary condition *paradox* for plates subject to stress boundary conditions [11,18]: the physical intuition leads one to expect three Poisson conditions (that are the resultant transverse shear force, bending moment, and twisting moment), but the forth order Kirchhoff–Love plate bending model can only incorporate two conditions (that are the Kirchhoff contractions of the three Poisson conditions). The Reissner–Mindlin model resolved this paradox by allowing transverse shear deformability. And it can formally represent the three Poisson resultants. The Reissner–Mindlin model is preferred for many reasons including that it seems closer to the three-dimensional elasticity theory than the Kirchhoff–Love biharmonic model. It is often remarked in the engineering literature, based mostly on computational evidence, that the Reissner–Mindlin model is more accurate, particularly for moderately thin plates and when transverse shear plays a significant role, see [12]. However, as far as we know, this fact has not been completely justified [7], especially for plates subject to stress boundary conditions which originally motivated the Reissner–Mindlin model. In [4], it was shown that when a plate is totally clamped, the Reissner–Mindlin model is convergent for the full range of surface loads while the Kirchhoff–Love is divergent if the surface loads induce a significant

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transverse shear. Thus for clamped plates, we can say that the Reissner–Mindlin plate is more accurate when the Kirchhoff–Love fails. But this analysis is in-relevant to the boundary condition paradox. For plates subject to stress boundary conditions, it not clear whether or not the Reissner–Mindlin model represents the Poisson resultants more faithfully than the Kirchhoff–Love, and under what circumstances the Kirchhoff contraction is not appropriate. It is the purpose of this note to show that under the usual asymptotic assumption on the loading forces the Reissner–Mindlin approximation is provably accurate over a wider range of loadings than the Kirchhoff–Love. However, there are cases in which the Reissner–Mindlin model does not converge to the elasticity theory either. In these latter cases, we show that the Reissner–Mindlin can accurately capture the interior shear effect.

The Reissner–Mindlin model incorporates the plate thickness as a parameter and it allows both bending and transverse shear deformabilities. It formerly resembles the Naghdi and Koiter shell models. Following the classical shell classification [20], a plate deformation determined by the Reissner–Mindlin model could be bending dominated, transverse shear dominated, or neither (intermediate), depending on the loads. Under some general assumptions on the lateral surface force and body force densities, we show that in the case of bending domination, the Reissner–Mindlin solution is close to that of Kirchhoff–Love and they both converge to the elasticity solution when the plate thickness tends to zero. In the case of transverse shear domination, the Kirchhoff–Love solution offers no approximation to the elasticity solution. However, the Reissner–Mindlin model accurately captures the shear dominated deformation. In the intermediate case, the Kirchhoff–Love solution is again useless, but the Reissner–Mindlin solution also has a relative error that does not tends to zero with the plate thickness. The classification is rather simple: the Reissner–Mindlin solution is bending dominated if and only if the corresponding Kirchhoff–Love solution is not zero; it is shear dominated if and only if the Kirchhoff–Love solution is zero and the twisting moment on the boundary is zero; it is intermediate if and only if the Kirchhoff–Love solution is zero and the twisting moment is not zero. In the latter case, the twisting moment causes both bending and transverse shear effects which are equal in order of magnitude, in which the shear effect is confined in a boundary layer but the bending effect extends throughout the plate domain. The Reissner–Mindlin model fails to accurately capture either of these. In the intermediate case, it is the global shear effect due to the surface couple that can be accurately captured by the Reissner–Mindlin model. However, there is a possibility to correct the Reissner–Mindlin model by solving another two-dimensional equation to capture the interior bending effect as well, thus to produce an accurate interior approximation to the elasticity theory. But this modification leads to a model that is no longer in the form of Reissner–Mindlin's. These are the results obtained under the usual asymptotic assumption on the loads. Practically, if the Kirchhoff–Love solution is significantly smaller than one expected, then the Reissner–Mindlin would be more accurate if the twisting moment were small. A dominant twisting moment would mean that the Reissner–Mindlin is not accurate either.

When the plate boundary is free (lateral surface force density is zero), there is no intermediate case (due to the zero twisting moment). Thus the Reissner–Mindlin is always accurate, while Kirchhoff–Love fails in the shear dominated case. For free boundary condition, the asymptotic expansions for both the Reissner–Mindlin plate model and three-dimensional elasticity have been thoroughly investigated in, notably, [2] for Reissner–Mindlin and [8, 9] for elasticity, respectively. In terms of the asymptotic expansions, our result just means that when the leading term in the Reissner–Mindlin expansion is not zero, it agrees with the leading term in the elasticity expansion, and this leading term is just the Kirchhoff–Love solution. This is well-known. When the leading term vanishes, the second terms (both the smooth part and boundary layer part) in the two expansions vanish together. In this case, the higher order response yielded by the elasticity theory appears in the third term of the asymptotic expansion, which is the sum of a Kirchhoff–Love field and an odd function in the transverse variable plus a boundary layer [8]. Our result means that this higher order response is accurately captured by the Reissner–Mindlin model. This seems a new result.

The situation for non-zero lateral force density is more complicated, and the asymptotic expansions are not available in the aforementioned work. Such expansions can certainly be worked out by their methods. The significant difference in the expansions from that for free boundary condition would be that when the first terms in the expansions (of the Reissner–Mindlin and elasticity) vanished, the second terms (a Kirchhoff–Love field plus a boundary layer) do not vanish if the twisting moment is non-zero. And they do not agree with each other not only in the boundary layer part but also in the global Kirchhoff–Love fields. Thus there is no consistency between the Reissner–Mindlin and the elasticity in any norm. But in this case, the interior transverse shears that appear in the third terms of the asymptotic expansions agree between the Reissner–Mindlin and the elasticity.

If the plate boundary and the loading forces are infinitely smooth, the convergence in the bending dominated and shear dominated cases could be proved by the asymptotic expansions as well. However, we shall resort to another approach: we use the Prager–Synge theorem (the two energies principle) [17] to prove the convergence under a minimum regularity assumption, and estimate the convergence rate when the plate data is sufficiently smooth. This is well-known method pioneered in [15].

The non-convergence of the Reissner–Mindlin solution to the elasticity solution in the intermediate case may shed some light on intermediate shells, which represent many shell structures in practice for which the validity of Naghdi or Koiter shell model has not been fully established. It seems unlikely that for intermediate shells these shell models are provably convergent to the elasticity theory, at least in the energy norm, especially for shells subject to stress boundary conditions.

There is an issue about the value of a shear correction factor in the Reissner–Mindlin plate model. The value 5/6 is generally viewed as the best [7,10]. We show that with the conventional definition of the resultant loading functional, the shear correction factor 5/6 in the Reissner–Mindlin model should be replaced by 1. Otherwise, the Reissner–Mindlin model does not converge to the elasticity theory in the transverse shear dominated case. In the bending dominated case, as it is well-known, the shear correction factor can be taken as any positive number without affecting the convergence of the Reissner–Mindlin model, which might be the reason why the value of shear correction factor has been an issue. In the intermediate case, no matter how one chooses the shear correction factor, the Reissner–Mindlin model is not asymptotically consistent with the elasticity theory (in the terminology of [21]). But to capture the interior transverse shear, the shear correction factor must be put to 1. Therefore, we may say that in the Reissner–Mindlin model, as a shear correction factor, 1 is better than 5/6.

In the remainder of this introduction, we introduce the notations, describe the data assumptions, and give a brief summary of our results. Let Ω be a plane domain. For small but positive ϵ, we consider the plate domain $P^\epsilon = \Omega \times (-\epsilon, \epsilon)$, on which the three-dimensional elasticity theory is defined. Throughout the paper, we use the following notations [1]. We indicate tensors in three variables with under-bars. A first-order tensor is written with one under-bar, a second-order tensor with two under-bars, etc. For tensors in two variables we use under-tildes in the same way. By way of illustration, any 3-vector may be expressed in terms of a 2-vector giving its in-plane components and a scalar giving its transverse components, and any 3×3 symmetric matrix may be expressed in terms of a 2×2 symmetric matrix, a 2-vector, and a scalar thus:

$$\underline{v} = \begin{pmatrix} \underset{\sim}{v} \\ v_3 \end{pmatrix}, \quad \underline{\underline{\tau}} = \begin{pmatrix} \underset{\approx}{\tau} & \underset{\sim}{\tau} \\ \underset{\sim}{\tau}^T & \tau_{33} \end{pmatrix}.$$

Under bars and under tildes will be used for tensor-valued functions, operators yielding such functions, and spaces of such functions, as well. Even without explicit mention, all second-order tensors arising in this paper will be assumed symmetric. Thus, for example, the notation $\underset{\approx}{H}^\alpha(\Omega)$ denotes the Sobolev space of order α which consists of all functions on Ω with values in $\mathbb{R}^{2\times 2}_{\text{sym}}$ whose partial derivatives of order at most α are square integrable. We denote the space of square integrable functions on a domain Q by $L^2(Q)$. In the case when $Q = \Omega$ we just write L^2, etc. We use $\underset{\sim}{n}$ to denote the unit outward normal to $\partial\Omega$ and $\underset{\sim}{s}$ the unit counterclockwise tangential. With $P \lesssim Q$ we indicate that there exists a constant C independent of ϵ, P, and Q such that $P \leq CQ$. The notation $P \simeq Q$ means $P \lesssim Q$ and $Q \lesssim P$.

We suppose that for each ϵ we are given surface tractions on the top, bottom, and lateral faces of P^ϵ, and a volume force in P^ϵ. Let $\partial P^\epsilon_L = \partial\Omega \boldsymbol{x}(-\epsilon, \epsilon)$, $\partial P^\epsilon_+ = \Omega \boldsymbol{x}\{\epsilon\}$, and $\partial P^\epsilon_- = \Omega \boldsymbol{x}\{-\epsilon\}$ denote the lateral portion and the top and bottom portions of the plate boundary, respectively, and set $\partial P^\epsilon_\pm = \partial P^\epsilon_+ \cup \partial P^\epsilon_-$. Let $\underline{g}^\epsilon_\pm : \partial P^\epsilon_\pm \to \mathbb{R}^3$

denote the given surface force densities on the top and bottom faces of the plate, $\underline{h}^\epsilon : \partial P_L^\epsilon \to \mathbb{R}^3$ be the surface force density on the lateral face of the plate, and $\underline{f}^\epsilon : P^\epsilon \to \mathbb{R}^3$ the body force density. As usual we define the elasticity tensor as

$$\underline{\underline{\underline{C}}}\,\underline{\underline{\tau}} = 2\mu\tau + \lambda\,\mathrm{tr}(\underline{\underline{\tau}})\underline{\underline{\delta}}$$

where μ and λ are the Lamé coefficients and $\underline{\underline{\delta}}$ is the 3×3 identity map. We denote the compliance tensor (the inverse of $\underline{\underline{\underline{C}}}$) by $\underline{\underline{\underline{A}}}$. By $\underline{\underline{e}}(\underline{u})$ we denote the infinitesimal strain tensor, *i.e.*, the symmetric part of the gradient of $\underline{u}$. Thus the displacement vector $\underline{u}_{3\mathrm{D}}^\epsilon : P^\epsilon \to \mathbb{R}^3$ satisfies the weak equation

$$\int_{P^\epsilon} [\underline{\underline{\underline{C}}}\,\underline{\underline{e}}(\underline{u}_{3\mathrm{D}}^\epsilon) : \underline{\underline{e}}(\underline{v})]\mathrm{d}\underline{x} = \langle \boldsymbol{f}_{3\mathrm{D}}, \underline{v}\rangle \quad , \forall \underline{v} \in \underline{H}^1(P^\epsilon). \tag{1.1}$$

Here the loading functional is defined by

$$\langle \boldsymbol{f}_{3\mathrm{D}}, \underline{v}\rangle = \int_{P^\epsilon} (\underline{f}^\epsilon, \underline{v})\mathrm{d}\underline{x} + \int_{P_\pm^\epsilon} (\underline{g}_\pm^\epsilon, \underline{v})\mathrm{d}\underset{\sim}{x} + \int_{P_L^\epsilon} (\underline{h}^\epsilon, \underline{v})\mathrm{d}S. \tag{1.2}$$

The elasticity equation (1.1) has a solution in $\underline{H}^1(P^\epsilon)$ if and only if the load is compatible, *i.e.*, $\boldsymbol{f}_{3\mathrm{D}}$ annihilates the subspace of infinitesimal rigid body motion, which is a six dimensional subspace. And the solution is determined up to the addition of an arbitrary element of this subspace. We shall assume throughout this paper that the applied loads give rise to *plate bending* only. This means that the in-plane force components $\underset{\sim}{f}^\epsilon$ and $\underset{\sim}{h}^\epsilon$ are odd in x_3, the transverse force components f_3^ϵ and h_3^ϵ are even in x_3, and $\underset{\sim}{g}_+^\epsilon = -\underset{\sim}{g}_-^\epsilon$, $g_{3+}^\epsilon = g_{3-}^\epsilon$. Under this loading assumption, we can assume that the in-plane displacement is an odd function in x_3 and the transverse displacement is even. Such functions form a subspace $\underline{H}_b^1(P^\epsilon)$ of $\underline{H}^1(P^\epsilon)$, in which the subspace of infinitesimal rigid body motion is reduced to a three-dimensional subspace $\mathcal{R} := \{(-ax_3, -bx_3, ax_1 + bx_2 + c); (a,b,c) \in \mathbb{R}^3\}$. And the plate displacement is determined up to the addition of an arbitrary element in $\mathcal{R}$. If $\boldsymbol{f}_{3\mathrm{D}}|_{\mathcal{R}} = 0$, the three-dimensional elasticity equation is well-posed on $H_{3\mathrm{D}} := \underline{H}_b^1(P^\epsilon)/\mathcal{R}$.

The Reissner–Mindlin model of the three-dimensional elasticity theory (1.1) determines a two-dimensional vector field $\underset{\sim}{\theta}^\epsilon$ and a scalar field w^ϵ, both of which are defined on Ω. The model seeks $(\underset{\sim}{\theta}^\epsilon, w^\epsilon) \in \underset{\sim}{H}^1 \times H^1$ such that

$$\epsilon^2 \frac{1}{3}\int_\Omega [\underset{\approxeq}{C}^* \underset{\approx}{e}(\underset{\sim}{\theta}^\epsilon) : \underset{\approx}{e}(\underset{\sim}{\phi})]\mathrm{d}\underset{\sim}{x} + \mu\int_\Omega (\underset{\sim}{\theta}^\epsilon - \underset{\sim}{\nabla} w^\epsilon, \underset{\sim}{\phi} - \underset{\sim}{\nabla} z)\mathrm{d}\underset{\sim}{x} = \langle \boldsymbol{f}_{\mathrm{RM}}, (\underset{\sim}{\phi}, z)\rangle \quad \forall(\underset{\sim}{\phi}, z) \in \underset{\sim}{H}^1 \times H^1. \tag{1.3}$$

Here

$$\underset{\approxeq}{C}^* \underset{\approx}{\tau} = 2\mu\underset{\approx}{\tau} + \lambda^*\,\mathrm{tr}(\underset{\approx}{\tau})\underset{\approx}{\delta}$$

with $\lambda^* = 2\mu\lambda/(2\mu + \lambda)$. And $\underset{\approx}{e}(\underset{\sim}{\phi})$ is the symmetric part of the gradient of $\underset{\sim}{\phi}$. Note that there is no shear correction factor in front of the transverse shear term (that is the second term in the left hand side).

The loading functional is defined as follows. For $(\underset{\sim}{\phi}, z) \in \underset{\sim}{H}^1 \times H^1$, we define $(-x_3\underset{\sim}{\phi}, z) \in \underline{H}_b^1(P^\epsilon)$ (in this way, the space $\underset{\sim}{H}^1 \times H^1$ is identified with a subspace of $\underline{H}_b^1(P^\epsilon)$). We define

$$\langle \boldsymbol{f}_{\mathrm{RM}}, (\underset{\sim}{\phi}, z)\rangle = \frac{1}{2\,\epsilon}\langle \boldsymbol{f}_{3\mathrm{D}}, (-x_3\underset{\sim}{\phi}, z)\rangle. \tag{1.4}$$

This is the conventional way to define the resultant loading functional in the plate model [13, 16]. It is clear that the subspace $\mathcal{R}_{\mathrm{RM}} = \{((a,b), ax_1 + bx_2 + c)\,; (a,b,c) \in \mathbb{R}^3\}$ of $\underset{\sim}{H}^1 \times H^1$ is identified with $\mathcal{R}$. And we see that if $\boldsymbol{f}_{3\mathrm{D}}|_{\mathcal{R}} = 0$ we have $\boldsymbol{f}_{\mathrm{RM}}|_{\mathcal{R}_{\mathrm{RM}}} = 0$. Since the bilinear form in the Reissner–Mindlin model is continuous and coercive on $H_{\mathrm{RM}} := (\underset{\sim}{H}^1 \times H^1)/\mathcal{R}_{\mathrm{RM}}$, the Reissner–Mindlin model is well-posed on this quotient space.

The Kirchoff–Love model determines an approximate transverse deflection. It seeks $w^0 \in H^2$ such that

$$\epsilon^2\,\frac{1}{3}\int_\Omega [\underset{\approxeq}{C}^*(\underset{\approx}{e}\underset{\sim}{\nabla} w^0) : \underset{\approx}{e}\underset{\sim}{\nabla} z]\mathrm{d}\underset{\sim}{x} = \langle \boldsymbol{f}_{\mathrm{KL}}, z\rangle,\ \forall z \in H^2. \tag{1.5}$$

For any $z \in H^2$, we can determine $(\underset{\sim}{\nabla} z, z) \in \underset{\sim}{H}^1 \times H^1$. By doing so, we identify H^2 with a subspace of $\underset{\sim}{H}^1 \times H^1$. In the same way, we identify the space of linear functions $\mathcal{L} = \{ax_1 + bx_2 + c|(a, b, c) \in \mathbb{R}^3\}$ with $\mathcal{R}_{\mathrm{RM}}$. The loading functional in the Kirchhoff–Love model is then defined by

$$\langle \boldsymbol{f}_{\mathrm{KL}}, z\rangle = \langle \boldsymbol{f}_{\mathrm{RM}}, (\underset{\sim}{\nabla} z, z)\rangle, \forall z \in H^2. \tag{1.6}$$

Since $\boldsymbol{f}_{\mathrm{RM}}|_{\mathcal{R}_{\mathrm{RM}}} = 0$, we have $\boldsymbol{f}_{\mathrm{KL}}|_{\mathcal{L}} = 0$. Therefore, the Kirchhoff–Love model is well posed on $H_{\mathrm{KL}} := H^2/\mathcal{L}$, which is identified with a subspace of H_{RM} in an obvious way.

We are concerned with how well the elasticity model (1.1) is approximated by the Reissner–Mindlin model (1.3) and the Kirchhoff–Love model (1.5). For this purpose, we define displacement fields on the plate domain P^ϵ in terms of the two model solutions [7]. Based on the Reissner–Mindlin solution $(\underset{\sim}{\theta}^\epsilon, w^\epsilon)$, with a higher order correction on w^ϵ, we define a displacement field $\underline{u}^\epsilon_{\mathrm{RM}}$ on P^ϵ, see (4.1). Based on the Kirchhoff–Love solution w^0, we defined a displacement field $\underline{u}^\epsilon_{\mathrm{KL}}$ on the plate similarly, see (4.9). We shall prove that $\underline{u}^\epsilon_{\mathrm{3D}}$ can be approximated by $\underline{u}^\epsilon_{\mathrm{RM}}$ and $\underline{u}^\epsilon_{\mathrm{KL}}$ under certain conditions. And $\underline{u}^\epsilon_{\mathrm{RM}}$ has a wider range of applicability than $\underline{u}^\epsilon_{\mathrm{KL}}$. To make the statements more precise, we consider a sequence of plates P^ϵ with varying thickness $2\,\epsilon$. As usual [7], we specify the dependence of the loads on ϵ by supposing that

$$\underset{\sim}{g}^\epsilon_+(\underset{\sim}{x}) = -\underset{\sim}{g}^\epsilon_-(\underset{\sim}{x}) = \underset{\sim}{g}(\underset{\sim}{x}), \quad g^\epsilon_3(\underset{\sim}{x}) = \epsilon g_3(\underset{\sim}{x}), \quad \underline{f}^\epsilon_3(\underline{x}) = \underline{f}_3(\underset{\sim}{x}, x_3/\,\epsilon), \tag{1.7}$$

$$\underset{\sim}{h}^\epsilon(\underline{x}) = \epsilon^{-1}\,\underset{\sim}{h}(\underset{\sim}{x}, x_3/\,\epsilon), \quad h^\epsilon_3(\underline{x}) = h_3(\underset{\sim}{x}, x_3/\,\epsilon), \tag{1.8}$$

for some functions $\underline{g}$, $\underline{h}$, and $\underline{f}$ independent of ϵ. For each ϵ, we consider the approximatibility in the energy norm defined by $\|\underline{u}\|_{E^\epsilon} := \left(\int_{P^\epsilon}[\underline{\underline{\underline{C}}}\,\underline{\underline{e}}(\underline{u}) : \underline{\underline{e}}(\underline{u})]\mathrm{d}\underline{x}\right)^{1/2}$ for $\underline{u} \in \underline{H}^1(P^\epsilon)$. Since the true solution $\underline{u}^\epsilon_{\mathrm{3D}}$ is varying with ϵ, we consider the relative errors

$$\frac{\|\underline{u}^\epsilon_{\mathrm{3D}} - \underline{u}^\epsilon_{\mathrm{KL}}\|_{E^\epsilon}}{\|\underline{u}^\epsilon_{\mathrm{3D}}\|_{E^\epsilon}}, \quad \frac{\|\underline{u}^\epsilon_{\mathrm{3D}} - \underline{u}^\epsilon_{\mathrm{RM}}\|_{E^\epsilon}}{\|\underline{u}^\epsilon_{\mathrm{3D}}\|_{E^\epsilon}}.$$

We then say that the plate model is convergent (or convergent with order p) with respect to ϵ and this sequence of plates and loadings, if this relative error quantity tends to zero with ϵ (with order p).

We shall assume that the plate boundary and the applied force functions are sufficiently smooth. However, such smoothness is often not necessary, and we will give some remarks on the results without smoothness assumptions. Besides the assumptions on smoothness and the transverse variance (1.7) and (1.8), we make the following assumptions on the dependence on x_3 of $\underline{f}^\epsilon$ and $\underline{h}^\epsilon$. We assume that the body force density $\underline{f}^\epsilon$ is constant transversely, so in our bending state, $\underset{\sim}{f}^\epsilon = 0$ and f^ϵ_3 is constant in x_3. This assumption is just for brevity of presentation, which can be released. For example, we could have assumed that $\underset{\sim}{f}^\epsilon$ is linear and f^ϵ_3 quadratic in x_3 and incorporated the moment of such body force in the model. We will assume the lateral surface force density $\underline{h}^\epsilon$ is quadratic in x_3. In our assumed bending state, $\underset{\sim}{h}^\epsilon$ is then an odd linear function and h^ϵ_3 is an even quadratic function in x_3. Such assumptions are required by our method of proof. They are reasonable in the sense that 1) when a continuity condition is satisfied by the surface forces on the plate edge, this kind of lateral force density function can be exactly represented by its Poisson resultants, so that we can avoid the issue of justifying the Poisson resultants, 2) our main concern is whether or not the Reissner–Mindlin model better represents the Poisson resultants than Kirchhoff–Love, and 3) there are many theories on the validity of Poisson resultants, see [22] and the references therein.

Under these assumptions, the loading functional (1.4) in the Reissner–Mindlin model is independent of ϵ and it can be written as

$$\langle \boldsymbol{f}_{\mathrm{RM}}, (\underset{\sim}{\phi}, z)\rangle = \int_\Omega [(-\underset{\sim}{p}, \underset{\sim}{\phi}) + p_3 z]\mathrm{d}\underset{\sim}{x} + \int_{\partial\Omega}(-q_n\phi_n - q_s\phi_s + q_3 z)\mathrm{d}s, \tag{1.9}$$

in which $\phi_s = \underset{\sim}{\phi} \cdot \underset{\sim}{s}$ and $\phi_n = \underset{\sim}{\phi} \cdot \underset{\sim}{n}$ are the tangential and normal components of $\underset{\sim}{\phi}$, respectively. And

$$\underset{\sim}{p} = \underset{\sim}{g}, \quad p_3 = g_3 + f_3, \quad q_s = \underset{\sim}{q} \cdot \underset{\sim}{s}, \quad q_n = \underset{\sim}{q} \cdot \underset{\sim}{n}, \tag{1.10}$$

where

$$\underset{\sim}{q} = \frac{1}{2}\int_{-1}^{1} X_3 \underset{\sim}{h}(\,\cdot\,, X_3)\mathrm{d}X_3, \quad q_3 = \frac{1}{2}\int_{-1}^{1} h_3(\,\cdot\,, X_3)\mathrm{d}X_3. \tag{1.11}$$

These resultant loading functions are the one usually found in the literature, in which $\underset{\sim}{p}$ is the resultant couple, p_3 the resultant transverse load, q_n the bending moment, q_s the twisting moments, and q_3 the resultant transverse shear force on the lateral boundary. The latter three are the so-called Poisson resultants. The loading functional (1.6) in the Kirchhoff–Love model can be written as

$$\langle \boldsymbol{f}_{\mathrm{KL}}, z\rangle = \int_\Omega [(-\underset{\sim}{p}, \underset{\sim}{\nabla} z) + p_3 z]\mathrm{d}\underset{\sim}{x} + \int_{\partial\Omega} (-q_n)\partial_n z \mathrm{d}s + \int_{\partial\Omega} (q_3 + \partial_s q_s) z \mathrm{d}s. \tag{1.12}$$

The loading functions on Ω, *i.e.*, $-\underset{\sim}{p}$ and p_3, are the same as what found by asymptotic analysis, see [7] in which the lateral surface force was assumed to be zero. The two functions on the boundary $\partial\Omega$, *i.e.*, $-q_n$ and $q_3 + \partial_s q_s$ are the well-known Kirchhoff contractions of the three Poisson resultants [11, 14].

To classify the behavior of the Reissner–Mindlin solution when $\epsilon \to 0$, we fit the model (1.3) into the abstract framework for shell models [20]: let H be a Hilbert space, and $\boldsymbol{f} \in H^*$, the dual of H. For given $\epsilon > 0$, we seek $\boldsymbol{u}^\epsilon \in H$ such that

$$\epsilon^2\, a(\boldsymbol{u}^\epsilon, \boldsymbol{v}) + b(\boldsymbol{u}^\epsilon, \boldsymbol{v}) = \langle \boldsymbol{f}, \boldsymbol{v}\rangle,\ \forall \boldsymbol{v} \in H. \tag{1.13}$$

Here, $a(\,\cdot\,,\,\cdot\,)$ and $b(\,\cdot\,,\,\cdot\,)$ are symmetric nonnegative bilinear forms on H such that $\|\boldsymbol{v}\|_H^2 \simeq a(\boldsymbol{v}, \boldsymbol{v}) + b(\boldsymbol{v}, \boldsymbol{v})$ for all $\boldsymbol{v} \in H$. We shall slightly change this abstract problem to an equivalent form that better serves our purpose. By properly defining Hilbert spaces U and V, and linear operators $B : H \to U$ and $S : H \to V$, we write the problem (1.13) in the form: find $\boldsymbol{u}^\epsilon \in H$, such that

$$\epsilon^2 (B\boldsymbol{u}^\epsilon, B\boldsymbol{v})_U + (S\boldsymbol{u}^\epsilon, S\boldsymbol{v})_V = \langle \boldsymbol{f}, \boldsymbol{v}\rangle,\ \forall \boldsymbol{v} \in H. \tag{1.14}$$

We also need to use the Hilbert space $W = S(H) \subset V$, equipped with a norm such that S defines an isomorphism from $H/\ker S$ to W. Translating the classification of the behavior of the solution of (1.13) to the problem (1.14), the behavior of $\boldsymbol{u}^\epsilon$ is then classified as follows. If $\boldsymbol{f}|_{\ker S} \neq 0$, the behavior of $\boldsymbol{u}^\epsilon$ is B-dominated. In this case we have $\lim_{\epsilon\to 0} \frac{\epsilon^2 (B\boldsymbol{u}^\epsilon, B\boldsymbol{u}^\epsilon)_U}{(S\boldsymbol{u}^\epsilon, S\boldsymbol{u}^\epsilon)_V} = \infty$. If $\boldsymbol{f}|_{\ker S} = 0$, it follows that there exists a $\boldsymbol{\xi}^* \in W^*$ such that

$$\langle \boldsymbol{\xi}^*, S\boldsymbol{v}\rangle = \langle \boldsymbol{f}, \boldsymbol{v}\rangle,\ \forall \boldsymbol{v} \in H. \tag{1.15}$$

If $\boldsymbol{\xi}^* \in V^*$, the behavior of $\boldsymbol{u}^\epsilon$ is S-dominated. And we have $\lim_{\epsilon\to 0} \frac{\epsilon^2 (B\boldsymbol{u}^\epsilon, B\boldsymbol{u}^\epsilon)_U}{(S\boldsymbol{u}^\epsilon, S\boldsymbol{u}^\epsilon)_V} = 0$. The remaining case of $\boldsymbol{f}|_{\ker S} = 0$ and $\boldsymbol{\xi}^* \notin V^*$ is intermediate.

In terms of the Reissner–Mindlin model (1.3), the operator B is the bending strain operator and the space U is just $\underset{\approx}{L}^2$ with the norm equivalently modified. The operator S is the shear strain operator and V is equal to $\underset{\sim}{L}^2$ with an equivalent norm. Clearly, $\ker S = H_{\mathrm{KL}}$, identified with a subspace of H_{RM}. Thus the Reissner–Mindlin solution is bending(B)-dominated if and only if $\boldsymbol{f}_{\mathrm{KL}} \neq 0$, *i.e.*, the Kirchhoff–Love solution $w^0 \neq 0$. The condition $\boldsymbol{f}_{\mathrm{KL}} = 0$ is equivalent to

$$\operatorname{div} \underset{\sim}{p} + p_3 = 0 \text{ in } \Omega, \tag{1.16}$$

$$q_n = 0 \text{ on } \partial\Omega, \tag{1.17}$$

$$q_3 + \partial_s q_s - \underset{\sim}{p} \cdot \underset{\sim}{n} = 0 \text{ on } \partial\Omega. \tag{1.18}$$

Under this condition the Reissner–Mindlin loading functional becomes

$$\langle \boldsymbol{f}_{\mathrm{RM}}, (\underset{\sim}{\phi}, z)\rangle = \int_\Omega (\underset{\sim}{p}, \underset{\sim}{\nabla} z - \underset{\sim}{\phi})\mathrm{d}\underset{\sim}{x} + \langle q_s, \partial_s z - \phi_s\rangle_{\partial\Omega}, \ \forall\ (\underset{\sim}{\phi}, z) \in H_{\mathrm{RM}}. \tag{1.19}$$

This is an explicit version of the condition (1.15). The condition $\boldsymbol{\xi}^* \in V^*$ of shear(S)-domination now is equivalent to $q_s = 0$. In summary, the Reissner–Mindlin solution is bending dominated if any one of the three equations (1.16), (1.17), and (1.18) is not satisfied. It is shear dominated if all the three equations are satisfied and the twisting moment $q_s = 0$ on $\partial\Omega$. It is intermediate if all the three equations are satisfied but $q_s \neq 0$.

Our justification of the two-dimensional models is based on the Prager–Synge theorem. For this purpose, we need to construct a statically admissible stress field whose components are essentially two-dimensional functions. The existence of such admissible stress field requires the continuity condition of the surface loads along the plate edges:

$$\underset{\sim}{g}^\epsilon_+ \cdot \underset{\sim}{n} = -\underset{\sim}{g}^\epsilon_- \cdot \underset{\sim}{n} = h^\epsilon_3(\,\cdot\,, \epsilon) = h^\epsilon_3(\,\cdot\,, -\epsilon) \ \text{ on } \partial\Omega, \tag{1.20}$$

as assumed in [11]. Without this condition, there would be a corner singularity around the plate edge in the elasticity solution. In [7–9], the Kirchoff–Love model was justified for free boundary condition without assuming this continuity condition. We shall adopt this continuity assumption in this paper, but we note that this assumption is automatically satisfied in the shear dominated case and when h^ϵ_3 is constant in x_3 (for example, when the plate is free on its lateral boundary), since it is redundant to (1.18) in which the shear domination requires $q_s = 0$. Thus, without the continuity (1.20), if we assume that h^ϵ_3 is constant in x_3 we would still have our main result that in the bending dominated case the Reissner–Mindlin is as good as the Kirchhoff–Love and in the shear dominated case the Reissner–Mindlin is accurate while the Kirchhoff–Love only yields a zero solution. If h^ϵ_3 is quadratic in x_3, as we assumed, then shear domination does not lead to (1.20). Without assuming the continuity condition, we can not prove the accuracy of Reissner–Mindlin solution in the shear dominated case. In this case, there is only a weak boundary layer in the Reissner–Mindlin solution, which fails to capture the corner singularity in the elasticity solution, as shown in [8, 9].

Under the above assumptions on the loading force functions, we prove the following results. If the plate is loaded in such a way that the Reissner–Mindlin solution is bending dominated, we show that the aforementioned relative error converges to zero for both Reissner–Mindlin and Kirchhoff–Love solutions. And if the loading functions and the plate boundary are sufficiently smooth, the relative error is bounded by $O(\epsilon^{1/2})$ for both the models. In the shear dominated case, the Kirchhoff–Love solution is zero, so it offers no approximation to $\underline{u}^\epsilon_{\mathrm{3D}}$. In this case we prove that the $\underline{u}^\epsilon_{\mathrm{RM}}$ converges to $\underline{u}^\epsilon_{\mathrm{3D}}$ in the relative energy norm. And when the plate boundary and the loading functions are sufficiently smooth, the relative error is bounded by $O(\epsilon)$. However, if in this case the shear correction factor $5/6$ were added in the Reissner–Mindlin model (1.3), the relative error of $\underline{u}^\epsilon_{\mathrm{RM}}$ would converge to $1/6$ rather than 0. In the intermediate case, the Kirchhoff–Love solution is still zero, but the relative error of the Reissner–Mindlin solution is greater than a finite number that does not converge to zero. In this case, the situation is more complicated. There will be a dominating boundary layer singularity involved in the solution. And we shall consider interior accuracy of the Reissner–Mindlin model. The non-convergence of the Reissner–Mindlin plate bending model in the intermediate case indicates that the convergence rate $O(\epsilon^{1/2})$ is sharp in the bending dominated case, a question raised for simply supported plates in [6].

The paper is organized as follows. In Section 2, we give a brief derivation for the Reissner–Mindlin model. The derivation is based on the Hellinger–Reissner variational principle. The reason for us to include this derivation is that it yields a statically admissible stress field, which will play a crucial role in the analysis. This derivation leads to a two-dimensional model which we will modify to obtain the Reissner–Mindlin model (1.3). In Section 3, we give estimates on the solutions of the Reissner–Mindlin model and some related equations. These estimates are based on an abstract analysis of the ϵ-dependent problem (1.14), which we briefly describe. We also classify the behavior of the Reissner–Mindlin model solution in this section. In Section 4, we analyze the accuracy of the Reissner–Mindlin model and the Kirchhoff–Love model.

2. The plate bending models

We first give a brief derivation of a two-dimensional model based on the Hellinger–Reissner variational principle. This derivation yields a statically admissible stress field that will play an important role in our proof of the convergence theorem. We shall modify the two-dimensional model to obtain the Reissner–Mindlin model (1.3). Since the method is similar to that in [1] where a complete derivation is described for clamped plates, we skip all the details.

We define

$$\underline{\underline{\Sigma}} = \{\underline{\underline{\tau}} \in \underline{\underline{H}}(\underline{\operatorname{div}}, P^\epsilon) |\ \underline{\underline{\tau}}\underline{n} = \underline{g}^\epsilon \text{ on } \partial P^\epsilon_\pm, \underline{\underline{\tau}}\underline{n} = \underline{h}^\epsilon \text{ on } \partial P^\epsilon_L\}, \quad \underline{U} = \underline{L}^2(P^\epsilon).$$

The space $\underline{\underline{H}}(\underline{\operatorname{div}}, P^\epsilon)$ is the space of square integrable symmetric matrix valued functions on P^ϵ with square integrable divergence. When $\underline{f}^\epsilon \in \underline{L}^2(P^\epsilon)$, the Hellinger–Reissner principle characterizes the displacement $\underline{u}^\epsilon_{3\mathrm{D}}$ and stress $\underline{\underline{\sigma}}^\epsilon_{3\mathrm{D}} = \underline{\underline{\underline{\underline{C}}}}\,\underline{\underline{e}}(\underline{u}^\epsilon_{3\mathrm{D}})$ determined by the 3D elasticity theory (1.1) as the unique critical point of the functional

$$J(\underline{\underline{\tau}}, \underline{v}) = \frac{1}{2}\int_{P^\epsilon} [\underline{\underline{\underline{\underline{A}}}}\,\underline{\underline{\tau}} : \underline{\underline{\tau}}]\mathrm{d}\underline{x} + \int_{P^\epsilon} (\underline{\operatorname{div}}\,\underline{\underline{\tau}}, \underline{v})\mathrm{d}\underline{x} + \int_{P^\epsilon} (\underline{f}^\epsilon, \underline{v})\mathrm{d}\underline{x}$$

on $\underline{\underline{\Sigma}} \times \underline{U}$. By restricting J to subspaces of $\underline{\underline{\Sigma}}$ and $\underline{U}$ with specified polynomial dependence on x_3 we obtain a variety of plate models [1].

As we have assumed in the introduction, the force functions are $\underset{\sim}{f}^\epsilon = 0$, f^ϵ_3 is constant in x_3, $\underset{\sim}{h}^\epsilon$ linear in x_3, and h^ϵ_3 quadratic in x_3. We also adopt the usual assumption (1.7) and (1.8) on the dependence of these loading functions on ϵ. Under these assumptions, we can express the force densities in terms of their resultants (1.11):

$$\begin{aligned} &\underset{\sim}{h}^\epsilon(\,\cdot\,, x_3) = \frac{3}{\epsilon^2}\underset{\sim}{q}x_3, \quad h^\epsilon_3(\,\cdot\,, x_3) = \hat{h}_3 + \frac{3}{2}\left(1 - \frac{x_3^2}{\epsilon^2}\right)[q_3 - \hat{h}_3], \\ &\underset{\sim}{g}^\epsilon_+(\underset{\sim}{x}) = -\underset{\sim}{g}^\epsilon_-(\underset{\sim}{x}) = \underset{\sim}{g}(\underset{\sim}{x}), \quad g^\epsilon_3(\underset{\sim}{x}) = \epsilon g_3(\underset{\sim}{x}), \quad \underline{f}^\epsilon(\underline{x}) = (0, 0, f_3(\underset{\sim}{x})). \end{aligned} \tag{2.1}$$

Here, $\hat{h}_3 = h_3(\,\cdot\,, 1) = h_3(\,\cdot\,, -1)$, and due to the continuity assumption (1.20) we have $\hat{h}_3 = \underset{\sim}{g} \cdot \underset{\sim}{n}$ on $\partial\Omega$.

We shall use the following three polynomials in x_3.

$$t(x_3) = 1 - \frac{x_3^2}{\epsilon^2}, \quad s(x_3) = t(x_3)\frac{x_3}{\epsilon}, \quad r(x_3) = \frac{1}{5} - \frac{x_3^2}{\epsilon^2}.$$

We choose the subspace of $\underline{\underline{\Sigma}}$ as stress fields of components of the form

$$\underline{\underline{\tau}} = \begin{pmatrix} x_3\underset{\approx}{\sigma} & \underset{\sim}{g} + t(x_3)\underset{\sim}{\sigma} \\ (\underset{\sim}{g} + t(x_3)\underset{\sim}{\sigma})^T & g_3x_3 + s(x_3)\sigma \end{pmatrix} \tag{2.2}$$

such that

$$\frac{1}{3}\,\epsilon^2\,\underset{\approx}{\sigma}\underset{\sim}{n} = \underset{\sim}{q}, \quad \frac{2}{3}\underset{\sim}{\sigma} \cdot \underset{\sim}{n} = q_3 - \hat{h}_3 \ \text{ on } \partial\Omega. \tag{2.3}$$

Here $\underset{\approx}{\sigma} \in \underset{\approx}{H}(\underset{\sim}{\operatorname{div}})$, $\underset{\sim}{\sigma} \in \underset{\sim}{H}(\operatorname{div})$, and $\sigma \in L^2$ are functions depending on $\underset{\sim}{x}$ only. The meaning of these spaces is evident. Then it is easy to see that the stress boundary conditions on the whole face of P^ϵ are satisfied.

Remark 2.1. We need the minimum regularity that $\underset{\sim}{g} \in \underset{\sim}{H}(\operatorname{div})$ and $g_3 \in L^2$ so that $\underline{\underline{\tau}} \in \underline{\underline{H}}(\underline{\operatorname{div}}, P^\epsilon)$. We also need that $f_3 \in L^2$ so that $\underline{f}^\epsilon \in \underline{L}^2(P^\epsilon)$, as required by the Hellinger–Reissner variational principle. Henceforth, we shall assume so.

For the subspace of $\underline{U}$, we choose displacement fields of the form

$$\underline{v} = \begin{pmatrix} -x_3\underset{\sim}{\theta} \\ w + \epsilon^2\,r(x_3)w_2 \end{pmatrix}. \tag{2.4}$$

Here $\underset{\sim}{\theta}$, w, and w_2 are L^2 functions of $\underset{\sim}{x}$. Note that the stress subspace and displacement subspace are closed in $\underline{\underline{\Sigma}}$ and $\underline{U}$, respectively. We also see that the divergence operator $\underline{\mathrm{div}}$ maps the subspace of $\underline{\underline{\Sigma}}$ onto the subspace of $\underline{U}$. According to Brezzi's theory, the critical point of J on such defined subspace of $\underline{\underline{\Sigma}} \times \underline{U}$ is uniquely determined. This critical point is determined as follows. For the displacement part, we have $(\underset{\sim}{\theta}, w) \in \underset{\sim}{H}^1 \times H^1$ determined by the weak equation

$$\epsilon^2 \frac{1}{3} \int_\Omega (\underset{\approx}{C}^* \underset{\approx}{e}(\underset{\sim}{\theta}) : \underset{\approx}{e}(\underset{\sim}{\phi})) \mathrm{d}\underset{\sim}{x} + \frac{5}{6}\mu \int_\Omega (\underset{\sim}{\theta} - \underset{\sim}{\nabla} w, \underset{\sim}{\phi} - \underset{\sim}{\nabla} z) \mathrm{d}\underset{\sim}{x} = \langle \boldsymbol{f}_0, (\underset{\sim}{\phi}, z) \rangle + \epsilon^2 \langle \boldsymbol{f}_2, (\underset{\sim}{\phi}, z) \rangle, \ \forall\ (\underset{\sim}{\phi}, z) \in \underset{\sim}{H}^1 \times H^1. \quad (2.5)$$

The right hand side functional is defined by

$$\langle \boldsymbol{f}_0, (\underset{\sim}{\phi}, z) \rangle = \int_\Omega \left[\frac{5}{6} (\underset{\sim}{g}, \underset{\sim}{\nabla} z - \underset{\sim}{\phi}) + (\mathrm{div}\, \underset{\sim}{g} + g_3 + f_3) z \right] \mathrm{d}\underset{\sim}{x} + \int_{\partial\Omega} [(-\underset{\sim}{q}, \underset{\sim}{\phi}) + (q_3 - \hat{h}_3) z]\, \mathrm{d}s \quad (2.6)$$

and

$$\langle \boldsymbol{f}_2, (\underset{\sim}{\phi}, z) \rangle = -\frac{\lambda}{3(2\mu + \lambda)} \int_\Omega \left[g_3 + \frac{1}{5}(g_3 + f_3 + \mathrm{div}\, \underset{\sim}{g}) \right] \mathrm{div}\, \underset{\sim}{\phi}\, \mathrm{d}\underset{\sim}{x}.$$

We will show that when the applied forces on the plate is compatible, the equation (2.5) with the above defined loading functional has a unique solution in H_{RM}. We denote this solution by $(\underset{\sim}{\bar{\theta}}^\epsilon, \bar{w}^\epsilon)$. The function w_2 is defined by

$$\bar{w}_2^\epsilon = \frac{\lambda}{2(2\mu + \lambda)} [\mathrm{div}\, \underset{\sim}{\bar{\theta}}^\epsilon - g_3] - \frac{10\mu(2\mu + 3\lambda) + 3\lambda^2}{70\mu(2\mu + \lambda)(2\mu + 3\lambda)} (g_3 + f_3 + \mathrm{div}\, \underset{\sim}{g}). \quad (2.7)$$

For the stress part, we have

$$\begin{aligned} \underset{\approx}{\sigma} &= \underset{\approx}{C}^* \underset{\approx}{e}(\underset{\sim}{\bar{\theta}}^\epsilon) + \frac{\lambda}{2\mu + \lambda} [g_3 + \frac{1}{5}(g_3 + f_3 + \mathrm{div}\, \underset{\sim}{g})] \underset{\approx}{\delta}, \\ \underset{\sim}{\sigma} &= \frac{5}{4} [\mu (\nabla \bar{w}^\epsilon - \underset{\sim}{\bar{\theta}}^\epsilon) - \underset{\sim}{g}], \\ \sigma &= \frac{\epsilon}{2}(g_3 + f_3 + \mathrm{div}\, \underset{\sim}{g}). \end{aligned} \quad (2.8)$$

We denote the stress field defined by (2.2) and (2.8) by $\underline{\underline{\sigma}}^\epsilon$. It is important to note that $\underline{\underline{\sigma}}^\epsilon$ is statically admissible, *i.e.*, besides all the surface stress conditions, the equilibrium equation $\underline{\mathrm{div}} \underline{\underline{\sigma}}^\epsilon + \underline{f}^\epsilon = 0$ is also satisfied in P^ϵ (actually, we can let $\underset{\sim}{f}^\epsilon$ be linear and f_3^ϵ be quadratic in x_3. Then the stress field arising from the above derivation is still statically admissible [1]). We use $\underline{\bar{u}}^\epsilon$ to denote the displacement field defined by (2.4) in terms of $\underset{\sim}{\bar{\theta}}^\epsilon$, $\bar{w}^\epsilon$, and $\bar{w}_2^\epsilon$. It is certainly kinematically admissible, because there is no displacement boundary condition imposed on the plate. The residual of the constitutive equation $\underline{\underline{\varrho}} = \underline{\underline{\underline{\underline{A}}}} \underline{\underline{\sigma}}^\epsilon - \underline{\underline{e}}(\underline{\bar{u}}^\epsilon)$, which will be used below, have the following expressions

$$\begin{aligned} \underset{\approx}{\varrho} &= \frac{\lambda}{2\mu(2\mu + 3\lambda)} \left[\frac{2x_3}{5\,\epsilon} - s(x_3) \right] \sigma \underset{\approx}{\delta}, \\ \underset{\sim}{\varrho} &= \frac{5}{8\mu} r(x_3) \left[\mu (\nabla \bar{w}^\epsilon - \underset{\sim}{\bar{\theta}}^\epsilon) - \underset{\sim}{g} - \epsilon^2 \nabla \bar{w}_2^\epsilon \right], \\ \varrho_{33} &= \frac{\lambda + \mu}{2\mu(2\mu + 3\lambda)} \left[s(x_3) - \frac{8x_3}{7\,\epsilon} \right] \sigma. \end{aligned} \quad (2.9)$$

The equation (2.5) is very close to the Reissner–Mindlin model (1.3). The bilinear form in (2.5) is exactly that of the Reissner–Mindlin model with the shear correction factor 5/6 that is often found in the literature. However, the loading functional $\boldsymbol{f}_0 + \epsilon^2 \boldsymbol{f}_2$ appears quite different from the conventional definition (1.9). We shall prove that the displacement field $\underline{\bar{u}}^\epsilon$ converges in the relative energy norm to $\underline{u}_{3\mathrm{D}}^\epsilon$ and the stress field $\underline{\underline{\sigma}}^\epsilon$ converges to

$\underline{\underline{\sigma}}^{\epsilon}_{3\mathrm{D}}$ in the relative L^2 norm in both the bending dominated and transverse shear dominated cases (as classified in terms of the Reissner–Mindlin model (1.3)) at certain rate. We shall make some changes in the equation (2.5) to obtain the Reissner–Mindlin model (1.3) so that the convergence property of (2.5) is preserved.

First, we remove the factor 5/6 in the expression (2.6) of $\boldsymbol{f}_0$ to obtain the functional

$$\int_\Omega [(\underset{\sim}{g}, \underset{\sim}{\nabla} z) + (-\underset{\sim}{g}, \underset{\sim}{\phi}) + (\operatorname{div} \underset{\sim}{g} + g_3 + f_3)z]\mathrm{d}\underset{\sim}{x} + \int_{\partial\Omega} [-(\underset{\sim}{q}, \underset{\sim}{\phi}) + (q_3 - \hat{h}_3)z]\mathrm{d}s. \tag{2.10}$$

Performing integration by parts on the term $\int_\Omega (\underset{\sim}{g}, \underset{\sim}{\nabla} z)\mathrm{d}\underset{\sim}{x}$, and using the continuity condition (1.20), we see that this functional is exactly the conventional loading functional for the Reissner–Mindlin model (1.9):

$$\langle \boldsymbol{f}_{\mathrm{RM}}, (\underset{\sim}{\phi}, z)\rangle = \int_\Omega [(-\underset{\sim}{p}, \underset{\sim}{\phi}) + p_3 z]\mathrm{d}\underset{\sim}{x} + \int_{\partial\Omega} (-q_n\phi_n - q_s\phi_s + q_3 z)\mathrm{d}s. \tag{2.11}$$

Recall that

$$\underset{\sim}{p} = \underset{\sim}{g}, \quad p_3 = g_3 + f_3, \quad q_s = \underset{\sim}{q} \cdot \underset{\sim}{s}, \quad q_n = \underset{\sim}{q} \cdot \underset{\sim}{n}.$$

And $\underset{\sim}{q}$ is the resultant moment and q_3 is the resultant transverse shear force on the lateral boundary defined by (1.11). However, if we just simply replace the functional $\boldsymbol{f}_0$ in (2.5) by (2.10), the solution will diverge in the shear dominated case, although the convergence will not be affected in the bending dominated case. To preserve the convergence of (2.5) in the shear dominated case, we then have to delete the factor 5/6 in the left hand side of (2.5). Then the convergence will be preserved for both the bending dominated and shear dominated cases. Lastly, we ignore the higher order term $\epsilon^2 \boldsymbol{f}_2$ in the loading functional, at the price of missing some higher order response of the elasticity theory to some very special loads, which the Reissner–Mindlin model could otherwise accurately capture. See Remark 3.1 below. We thus get the Reissner–Mindlin model: seek $(\underset{\sim}{\theta}^\epsilon, w^\epsilon) \in \underset{\sim}{H}^1 \times H^1$, such that

$$\epsilon^2 \frac{1}{3}\int_\Omega (\underset{\approx}{C}^* \underset{\approx}{e}(\underset{\sim}{\theta}^\epsilon) : \underset{\approx}{e}(\underset{\sim}{\phi}))\mathrm{d}\underset{\sim}{x} + \mu \int_\Omega (\underset{\sim}{\theta}^\epsilon - \underset{\sim}{\nabla} w^\epsilon, \underset{\sim}{\phi} - \underset{\sim}{\nabla} z)\mathrm{d}\underset{\sim}{x} = \langle \boldsymbol{f}_{\mathrm{RM}}, (\underset{\sim}{\phi}, z)\rangle, \ \forall\ (\underset{\sim}{\phi}, z) \in \underset{\sim}{H}^1 \times H^1, \tag{2.12}$$

with the loading functional defined by (2.11).

The Kirchhoff–Love plate bending model is obtained by restricting the Reissner–Mindlin model on the subspace $\{(\underset{\sim}{\nabla} z, z); z \in H^2\}$ of $\underset{\sim}{H}^1 \times H^1$. The restriction of the Reissner–Mindlin loading functional $\boldsymbol{f}_{\mathrm{RM}}$ on this subspace is the Kirchhoff–Love loading functional:

$$\begin{aligned}\langle \boldsymbol{f}_{\mathrm{KL}}, z\rangle = \langle \boldsymbol{f}_{\mathrm{RM}}, (\underset{\sim}{\nabla} z, z)\rangle &= \int_\Omega [(-\underset{\sim}{p}, \underset{\sim}{\nabla} z) + p_3 z]\mathrm{d}\underset{\sim}{x} + \int_{\partial\Omega} (-q_n\partial_n z - q_s\partial_s z + q_3 z)\mathrm{d}s \\ &= \int_\Omega [(-\underset{\sim}{p}, \underset{\sim}{\nabla} z) + p_3 z]\mathrm{d}\underset{\sim}{x} + \int_{\partial\Omega} [-q_n\partial_n z + (q_3 + \partial_s q_s)z]\mathrm{d}s \quad \forall z \in H^2. \end{aligned} \tag{2.13}$$

The Kirchhoff–Love model determines $w^0 \in H^2$ such that

$$\epsilon^2 \frac{1}{3}\int_\Omega [\underset{\approx}{C}^* (\underset{\approx}{e}\underset{\sim}{\nabla} w^0) : \underset{\approx}{e}\underset{\sim}{\nabla} z]\mathrm{d}\underset{\sim}{x} = \langle \boldsymbol{f}_{\mathrm{KL}}, z\rangle, \ \forall z \in H^2. \tag{2.14}$$

We give some remarks on the well-posedness of the models (2.5), (2.12), and (2.14). As we mentioned in the introduction, it is easy to verify that for $(\underset{\sim}{\phi}, z) \in \underset{\sim}{H}^1 \times H^1$, we can define $(-x_3\underset{\sim}{\phi}, z) \in \underline{H}^1_b(P^\epsilon)$, and we have $\langle \boldsymbol{f}_{\mathrm{RM}}, (\underset{\sim}{\phi}, z)\rangle = \frac{1}{2\epsilon}\langle \boldsymbol{f}_{3\mathrm{D}}, (-x_3\underset{\sim}{\phi}, z)\rangle$. Since the applied forces on the plate was assumed to be compatible, *i.e.*, $\boldsymbol{f}_{3\mathrm{D}}|_{\mathcal{R}} = 0$, we have $\boldsymbol{f}_{\mathrm{RM}}|_{\mathcal{R}_{\mathrm{RM}}} = 0$. So $\boldsymbol{f}_{\mathrm{KL}}|_{\mathcal{L}} = 0$. From (2.10), we see that $\boldsymbol{f}_0|_{\mathcal{R}_{\mathrm{RM}}} = 0$. It is also obvious that $\boldsymbol{f}_2|_{\mathcal{R}_{\mathrm{RM}}} = 0$. The bilinear forms in the Reissner–Mindlin model (2.12) and the equation (2.5) are continuous and coercive on $H_{\mathrm{RM}} = (\underset{\sim}{H}^1 \times H^1)/\mathcal{R}_{\mathrm{RM}}$. Therefore, the Reissner–Mindlin model (2.12) is well posed on H_{RM}. So is the equation (2.5). The Kirchhoff–Love model is well-posed on $H^2/\mathcal{L}$.

For the purpose to obtain rigorous estimates on the solution of (2.5), we write it as

$$(\underset{\sim}{\bar{\theta}}^{\epsilon}, \bar{w}^{\epsilon}) = (\underset{\sim}{\bar{\theta}}_0^{\epsilon}, \bar{w}_0^{\epsilon}) + \epsilon^2(\underset{\sim}{\bar{\theta}}_2^{\epsilon}, \bar{w}_2^{\epsilon}). \tag{2.15}$$

Here $(\underset{\sim}{\bar{\theta}}_0^{\epsilon}, \bar{w}_0^{\epsilon})$ solves

$$\epsilon^2 \frac{1}{3}\int_{\Omega} [\underset{\approx\approx}{C}^{*} \underset{\approx}{e}(\underset{\sim}{\bar{\theta}}_0^{\epsilon}) : \underset{\approx}{e}(\underset{\sim}{\phi})] \mathrm{d}\underset{\sim}{x} + \frac{5}{6}\mu \int_{\Omega} (\underset{\sim}{\bar{\theta}}_0^{\epsilon} - \underset{\sim}{\nabla}\bar{w}_0^{\epsilon}, \underset{\sim}{\phi} - \underset{\sim}{\nabla} z) \mathrm{d}\underset{\sim}{x} = \langle \boldsymbol{f}_0, (\underset{\sim}{\phi}, z)\rangle,\ \forall (\underset{\sim}{\phi}, z) \in \underset{\sim}{H}^1 \times H^1, \tag{2.16}$$

and $(\underset{\sim}{\bar{\theta}}_2^{\epsilon}, \bar{w}_2^{\epsilon})$ solves

$$\epsilon^2 \frac{1}{3}\int_{\Omega} [\underset{\approx\approx}{C}^{*} \underset{\approx}{e}(\underset{\sim}{\bar{\theta}}_2^{\epsilon}) : \underset{\approx}{e}(\underset{\sim}{\phi})] \mathrm{d}\underset{\sim}{x} + \frac{5}{6}\mu \int_{\Omega} (\underset{\sim}{\bar{\theta}}_2^{\epsilon} - \underset{\sim}{\nabla}\bar{w}_2^{\epsilon}, \underset{\sim}{\phi} - \underset{\sim}{\nabla} z) \mathrm{d}\underset{\sim}{x} = \langle \boldsymbol{f}_2, (\underset{\sim}{\phi}, z)\rangle,\ \forall (\underset{\sim}{\phi}, z) \in \underset{\sim}{H}^1 \times H^1. \tag{2.17}$$

They both are uniquely determined in H_{RM}.

3. Asymptotic estimates on the plate models

We derive the estimates on the solution $(\underset{\sim}{\theta}^{\epsilon}, w^{\epsilon})$ of the Reissner–Mindlin model (2.12), the solution $(\underset{\sim}{\bar{\theta}}_0^{\epsilon}, \bar{w}_0^{\epsilon})$ of (2.16), and the solution $(\underset{\sim}{\bar{\theta}}_2^{\epsilon}, \bar{w}_2^{\epsilon})$ of (2.17). Then by the relation (2.15), we obtain estimates on $(\underset{\sim}{\bar{\theta}}^{\epsilon}, \bar{w}^{\epsilon})$, the solution of (2.5). This latter estimate will lead to rigorous estimates on the constitutive residual $\underline{\underline{\varrho}}$, and provides a lower bound on the energy norm of $\underline{\bar{u}}^{\epsilon}$. Thus, by using the Prager–Synge theorem, to obtain an estimate on the relative energy norm of the error of $\underline{\bar{u}}^{\epsilon}$. These estimates will also be used to bound the difference between $(\underset{\sim}{\theta}^{\epsilon}, w^{\epsilon})$ and $(\underset{\sim}{\bar{\theta}}^{\epsilon}, \bar{w}^{\epsilon})$, thus to estimate the error of the Reissner–Mindlin model.

3.1. **An abstract analysis**

By properly defining spaces and operators, the Reissner–Mindlin model (2.12) and the equations (2.16) and (2.17) can be put in the form (3.2) below. Let H, U, and V be Hilbert spaces, and $B : H \to U$ and $S : H \to V$ be linear continuous operators. We assume that

$$\|B\boldsymbol{u}\|_U + \|S\boldsymbol{u}\|_V \simeq \|\boldsymbol{u}\|_H,\ \forall \boldsymbol{u} \in H. \tag{3.1}$$

We consider the problem: given $\boldsymbol{f} \in H^*$, the dual space of the Hilbert space H, and $\epsilon > 0$, find $\boldsymbol{u}^{\epsilon} \in H$, such that

$$\epsilon^2(B\boldsymbol{u}^{\epsilon}, B\boldsymbol{v})_U + (S\boldsymbol{u}^{\epsilon}, S\boldsymbol{v})_V = \langle \boldsymbol{f}, \boldsymbol{v}\rangle,\ \forall \boldsymbol{v} \in H. \tag{3.2}$$

This problem obviously has a unique solution in H. We assume that $W = S(H)$ is dense in V and norm W by

$$\|\boldsymbol{\zeta}\|_W := \inf_{\substack{\boldsymbol{u}\in H\\ S\boldsymbol{u}=\boldsymbol{\zeta}}} \|\boldsymbol{u}\|_H,\ \forall \boldsymbol{\zeta} \in W. \tag{3.3}$$

Then W is a dense subspace of V with continuous inclusion, so we may view V^* as a dense subspace of W^*.

The following lemmas establish the needed estimates on $\boldsymbol{u}^{\epsilon}$. Before presenting the lemmas, we recall some terminologies in Hilbert spaces. If a Hilbert space X_1 is a dense subspace of a Hilbert space X_2, then the K-functional [5] on the Hilbert couple $[X_2, X_1]$ is an ϵ-dependent norm on X_2 defined by

$$K(\epsilon, x, [X_2, X_1]) = \inf_{y\in X_1} (\|x - y\|_{X_2} + \epsilon\, \|y\|_{X_1}),\ \forall x \in X_2.$$

Since X_1 is dense in X_2, we have

$$\lim_{\epsilon\to 0} K(\epsilon, x, [X_2, X_1]) = 0, \forall x \in X_2. \tag{3.4}$$

If $x \in X_1$, then

$$K(\epsilon, x, [X_2, X_1]) \lesssim \epsilon \, \|x\|_{X_1}. \tag{3.5}$$

If x belongs to the real interpolation space $[X_2, X_1]_{\theta,p}$ for some $0 < \theta < 1$ and $1 \leq p \leq \infty$, then we have the estimate

$$K(\epsilon, x, [X_2, X_1]) \lesssim \epsilon^{\theta} \, . \tag{3.6}$$

The equation (3.2) also represents the Koiter and Naghdi shell models and it has been extensively studied [20]. It is well-known that the behavior of $\boldsymbol{u}^\epsilon$ is very different for whether $\boldsymbol{f}|_{\ker S} = 0$ or not. If $\boldsymbol{f}|_{\ker S} = 0$, it follows that there exists a unique $\boldsymbol{\xi}^* \in W^*$ such that

$$\langle \boldsymbol{\xi}^*, S\boldsymbol{v} \rangle = \langle \boldsymbol{f}, \boldsymbol{v} \rangle, \; \forall \boldsymbol{v} \in H. \tag{3.7}$$

In this case, we need to further consider whether or not $\boldsymbol{\xi}^* \in V^*$. We have the following equivalence estimates in each of the three cases. The proof can be found in [23].

Lemma 3.1. *If $\boldsymbol{f}|_{\ker S} = 0$, the equivalence*

$$\epsilon \, \|B\boldsymbol{u}^\epsilon\|_U + \|S\boldsymbol{u}^\epsilon\|_V \simeq \epsilon^{-1} \, K(\epsilon, \boldsymbol{\xi}^*, [W^*, V^*]) \tag{3.8}$$

holds.

Under the assumption $\boldsymbol{f}|_{\ker S} = 0$, if we further assume that $\boldsymbol{\xi}^* \in V^*$, we have a stronger estimate. Let $\boldsymbol{\xi} \in V$ be the Riesz representation of $\boldsymbol{\xi}^*$, then

$$(\boldsymbol{\xi}, S\boldsymbol{v})_V = \langle \boldsymbol{f}, \boldsymbol{v} \rangle, \; \forall \boldsymbol{v} \in H. \tag{3.9}$$

We have

Lemma 3.2. *If $\boldsymbol{f}|_{\ker S} = 0$ and $\boldsymbol{\xi}^* \in V^*$, the equivalence*

$$\epsilon \, \|B\boldsymbol{u}^\epsilon\|_U + \|S\boldsymbol{u}^\epsilon - \boldsymbol{\xi}\|_V \simeq K(\epsilon, \boldsymbol{\xi}, [V, W]) \tag{3.10}$$

holds.

The case of Lemma 3.2 is the S-dominated case in the sense that $\lim_{\epsilon\to 0} \frac{\epsilon^2 (B\boldsymbol{u}^\epsilon, B\boldsymbol{u}^\epsilon)_U}{(S\boldsymbol{u}^\epsilon, S\boldsymbol{u}^\epsilon)_V} = 0$, which easily follows from the property (3.4) of K-functional. If we have better estimate on the interpolation property of $\boldsymbol{\xi}$, we will obtain a rate of domination. In terms of the Reissner–Mindlin plate, this is the shear dominated case.

If $\boldsymbol{f}|_{\ker S} \neq 0$, then $\boldsymbol{u}^\epsilon$ tends to infinity in H at the rate ϵ^{-2}, and $\epsilon^2 \, \boldsymbol{u}^\epsilon$ converges to a limit. Instead of considering the convergence of $\epsilon^2 \, \boldsymbol{u}^\epsilon$, we assume $\boldsymbol{f} = \epsilon^2 \, \tilde{\boldsymbol{f}}$ with $\tilde{\boldsymbol{f}}$ independent of ϵ. Under this scaling, there is a unique nonzero element $\boldsymbol{u}^0 \in \ker S$, satisfying

$$(B\boldsymbol{u}^0, B\boldsymbol{v})_U = \langle \tilde{\boldsymbol{f}}, \boldsymbol{v} \rangle, \; \forall \boldsymbol{v} \in \ker S. \tag{3.11}$$

And we have

$$\epsilon^2 (B[\boldsymbol{u}^\epsilon - \boldsymbol{u}^0], B\boldsymbol{v})_U + (S[\boldsymbol{u}^\epsilon - \boldsymbol{u}^0], S\boldsymbol{v})_V = \epsilon^2 [\langle \tilde{\boldsymbol{f}}, \boldsymbol{v} \rangle - (B\boldsymbol{u}^0, B\boldsymbol{v})_U], \; \forall \; \boldsymbol{v} \in H. \tag{3.12}$$

In view of (3.11), the functional defined by the right hand side of this equation annihilates $\ker S$. It follows that there exists a unique $\boldsymbol{\eta}^* \in W^*$ such that

$$\langle \boldsymbol{\eta}^*, S\boldsymbol{v} \rangle = \langle \tilde{\boldsymbol{f}}, \boldsymbol{v} \rangle - (B\boldsymbol{u}^0, B\boldsymbol{v})_U, \; \forall \boldsymbol{v} \in H. \tag{3.13}$$

Applying Lemma 3.1 to the equation (3.12), we get

Lemma 3.3. *If $\boldsymbol{f}|_{\ker S} \neq 0$, we set $\boldsymbol{f} = \epsilon^2 \tilde{\boldsymbol{f}}$ with $\tilde{\boldsymbol{f}}$ being independent of ϵ. Then the equivalence*

$$\|B(\boldsymbol{u}^\epsilon - \boldsymbol{u}^0)\|_U + \epsilon^{-1}\|S\boldsymbol{u}^\epsilon\|_V \simeq K(\epsilon, \boldsymbol{\eta}^*, [W^*, V^*]) \tag{3.14}$$

holds.

This is the B-dominated case in the sense that $\lim_{\epsilon\to 0} \frac{\epsilon^2 (B\boldsymbol{u}^\epsilon, B\boldsymbol{u}^\epsilon)_U}{(S\boldsymbol{u}^\epsilon, S\boldsymbol{u}^\epsilon)_V} = \infty$, which again follows from the property (3.4) of K-functional. In terms of the Reissner–Mindlin plate, this is the bending dominated case. The remaining case that corresponds to Lemma 3.1 with $\boldsymbol{\xi}^* \notin V^*$ is intermediate.

3.2. Asymptotic estimates on the model solution

To apply the above theory to the Reissner–Mindlin model, we let $H = H_{\mathrm{RM}}$. We define $U = \underset{\approx}{L}^2$ as set, but with the inner product defined by

$$(\underset{\approx}{\sigma}, \underset{\approx}{\tau})_U = \frac{1}{3}\int_\Omega [\underset{\approxeq}{C}^*(\underset{\approx}{\sigma}) : \underset{\approx}{\tau}]\mathrm{d}\underset{\sim}{x},\ \forall \underset{\approx}{\sigma}, \underset{\approx}{\tau} \in U$$

which induces a norm equivalent to the original one of $\underset{\approx}{L}^2$. We define $V = \underset{\sim}{L}^2$ as set, and define the inner product by

$$(\underset{\sim}{\boldsymbol{\xi}}, \underset{\sim}{\boldsymbol{\eta}})_V = \mu \int_\Omega (\underset{\sim}{\boldsymbol{\xi}}, \underset{\sim}{\boldsymbol{\eta}})\mathrm{d}\underset{\sim}{x},\ \forall\ \underset{\sim}{\boldsymbol{\xi}}, \underset{\sim}{\boldsymbol{\eta}} \in V.$$

So the norm of V is equivalent to the usual $\underset{\sim}{L}^2$ norm. We define the operators B and S as follows. For $\boldsymbol{v} = (\underset{\sim}{\phi}, z) \in H_{\mathrm{RM}}$,

$$B\boldsymbol{v} = \underset{\approx}{e}(\underset{\sim}{\phi}), \quad S\boldsymbol{v} = \underset{\sim}{\phi} - \underset{\sim}{\nabla} z.$$

Thus B and S are the bending strain operator and transverse shear strain operator, respectively. We see that $W = S(H)$ is dense in V. Furthermore, we have $W = \underset{\sim}{H}(\mathrm{rot})$, the space of vector valued functions with square integrable components and rotations, with equivalent norms. It is easy to see that $\ker S = H_{\mathrm{KL}}$. Therefore the condition of whether or not $\boldsymbol{f}|_{\ker S} = 0$ is whether or not $\boldsymbol{f}_{\mathrm{KL}} = 0$, or whether or not the Kirchhoff–Love solution vanishes.

To write the equations (2.16) and (2.17) in the form of (3.2), we need to define the inner product in V by $(\underset{\sim}{\boldsymbol{\xi}}, \underset{\sim}{\boldsymbol{\eta}})_V = \frac{5}{6}\mu \int_\Omega (\underset{\sim}{\boldsymbol{\xi}}, \underset{\sim}{\boldsymbol{\eta}})\mathrm{d}\underset{\sim}{x},\ \forall\ \underset{\sim}{\boldsymbol{\xi}}, \underset{\sim}{\boldsymbol{\eta}} \in V$.

To apply the above lemmas to the Reissner–Mindlin model to get estimates on its solution, we need to find an expression for $\boldsymbol{\xi}^*$ defined by (3.7) in the intermediate case and estimate the value of $K(\epsilon, \boldsymbol{\xi}^*, [W^*, V^*])$. In the shear dominated case, we need to find expression for $\boldsymbol{\xi}$ defined by (3.9) and estimate the value of $K(\epsilon, \boldsymbol{\xi}, [V, W])$. In the bending dominated case, we need to look at $\boldsymbol{\eta}^* \in W^*$ defined by (3.13). For this purpose, we reformulate the loading functional $\boldsymbol{f}_{\mathrm{RM}}$ as follows. For smooth $\underset{\sim}{\phi}$ and z, by adding to and subtracting from $-\underset{\sim}{\phi}$ the term $\underset{\sim}{\nabla} z$ in the expression (2.11) of $\boldsymbol{f}_{\mathrm{RM}}$, and performing several integration by parts, we get

$$\langle \boldsymbol{f}_{\mathrm{RM}}, (\underset{\sim}{\phi}, z)\rangle = \int_\Omega (-\underset{\sim}{p}, \underset{\sim}{\phi} - \underset{\sim}{\nabla} z)\mathrm{d}\underset{\sim}{x} + \langle q_n, \partial_n z - \phi_n\rangle_{\partial\Omega} + \langle q_s, \partial_s z - \phi_s\rangle_{\partial\Omega} + \langle \boldsymbol{f}_{\mathrm{KL}}, z\rangle.$$

Similarly, we can reformulate $\boldsymbol{f}_0$ as

$$\langle \boldsymbol{f}_0, (\underset{\sim}{\phi}, z)\rangle = \frac{5}{6}\int_\Omega (-\underset{\sim}{p}, \underset{\sim}{\phi} - \underset{\sim}{\nabla} z)\mathrm{d}\underset{\sim}{x} + \langle q_n, \partial_n z - \phi_n\rangle_{\partial\Omega} + \langle q_s, \partial_s z - \phi_s\rangle_{\partial\Omega} + \langle \boldsymbol{f}_{\mathrm{KL}}, z\rangle.$$

Recall that $\boldsymbol{f}_{\mathrm{KL}}$ is defined as the restriction of $\boldsymbol{f}_{\mathrm{RM}}$ or $\boldsymbol{f}_0$ on the subspace H_{KL} that is the kernel of the shear strain operator, *cf.* (2.13). Performing an integration by parts, we write it as

$$\langle \boldsymbol{f}_{\mathrm{KL}}, z\rangle = \int_\Omega (\operatorname{div} \underset{\sim}{p} + p_3) z \mathrm{d}\underset{\sim}{x} + \int_{\partial\Omega} (-q_n)\partial_n z \mathrm{d}s + \int_{\partial\Omega} [q_3 + \partial_s q_s - \underset{\sim}{p}\cdot\underset{\sim}{n}] z \mathrm{d}s. \tag{3.15}$$

We see that $\boldsymbol{f}_{\mathrm{RM}}|_{H_{\mathrm{KL}}} = 0$ (or equivalently $\boldsymbol{f}_0|_{H_{\mathrm{KL}}} = 0$), if and only if the following three equations are simultaneously satisfied:

$$\operatorname{div} \underset{\sim}{p} + p_3 = 0 \text{ in } \Omega, \tag{3.16}$$

$$q_n = 0 \text{ on } \partial\Omega, \tag{3.17}$$

$$q_3 + \partial_s q_s - \underset{\sim}{p}\cdot\underset{\sim}{n} = 0 \text{ on } \partial\Omega. \tag{3.18}$$

This is to say, the resultant transverse force vanishes throughout the plate, the bending moment is zero along the whole boundary of the plate, and the resultant transverse shear load defined by Kirchoff contraction is balanced by the transverse force on the edges of the lateral boundary, which, under the continuity condition (1.20), is equal to $\underset{\sim}{p}\cdot\underset{\sim}{n}$. When $\boldsymbol{f}_{\mathrm{RM}}|_{H_{\mathrm{KL}}} = 0$, we have

$$\langle \boldsymbol{f}_{\mathrm{RM}}, (\underset{\sim}{\phi}, z)\rangle = \int_\Omega (\underset{\sim}{p}, \underset{\sim}{\nabla} z - \underset{\sim}{\phi}) \mathrm{d}\underset{\sim}{x} + \langle q_s, \partial_s z - \phi_s\rangle_{\partial\Omega}, \; \forall (\underset{\sim}{\phi}, z) \in \underset{\sim}{H}^1 \times H^1, \tag{3.19}$$

and

$$\langle \boldsymbol{f}_0, (\underset{\sim}{\phi}, z)\rangle = \frac{5}{6}\int_\Omega (\underset{\sim}{p}, \underset{\sim}{\nabla} z - \underset{\sim}{\phi}) \mathrm{d}\underset{\sim}{x} + \langle q_s, \partial_s z - \phi_s\rangle_{\partial\Omega}, \; \forall (\underset{\sim}{\phi}, z) \in \underset{\sim}{H}^1 \times H^1. \tag{3.20}$$

Note that for $(\underset{\sim}{\phi}, z) \in \underset{\sim}{H}^1 \times H^1$, $\partial_s z - \phi_s$ is well-defined in $H^{-1/2}(\partial\Omega)$.

Therefore, we have the following criteria to classify the asymptotic behavior of the Reissner–Mindlin solution.

1. If any one of the three equations (3.16)–(3.18) is not satisfied ($\boldsymbol{f}_{\mathrm{RM}}|_{H_{\mathrm{KL}}} \neq 0$), the Reissner–Mindlin solution is bending dominated.
2. If the three conditions (3.16)–(3.18) are all satisfied ($\boldsymbol{f}_{\mathrm{RM}}|_{H_{\mathrm{KL}}} = 0$), then $\boldsymbol{f}_{\mathrm{RM}}$ has the expression (3.19). In this case if the twisting moment is zero ($q_s = 0$) then $\boldsymbol{\xi}^* \in V^*$ (actually $\boldsymbol{\xi} = -\frac{1}{\mu}\underset{\sim}{p}$, to be derived below). The Reissner–Mindlin solution is transverse shear dominated. Note that in this case, we also need to have $\underset{\sim}{p} \in \underset{\sim}{L}^2$ such that $\boldsymbol{\xi}^* \in V^*$, which we already assumed in Remark 2.1.
3. If the three conditions (3.16)–(3.18) are all satisfied but $q_s \not\equiv 0$, then $\boldsymbol{\xi}^* \notin V^*$. The Reissner–Mindlin solution is intermediate.

The solution of (2.16) behaves exactly in the same way.

Remark 3.1. There is a forth possibility. Namely, the Reissner–Mindlin solution could be zero. This happens when $\underset{\sim}{p} = 0$ and $d_s = 0$, in addition to the three equations (3.16)–(3.18). In terms of the original loading functions on the plate, we have $\underset{\sim}{g} = 0$, $\underline{h} = 0$, and $g_3 + f_3 = 0$. The elasticity theory yields a non-trivial higher order response to such loading [8] that can be accurately captured by $\underline{\bar{u}}^\epsilon$ with a relative error ϵ in the energy norm. But this higher order response is missed by the Reissner–Mindlin model (2.12) because the higher order term $\epsilon^2 \boldsymbol{f}_2$ was deleted from the loading functional. We will exclude such possibility in the following discussion by assuming that the Reissner–Mindlin solution is not zero.

Remark 3.2. The loading assumptions often found in the literature exclude either transverse shear dominated case or intermediate case. For example, references [11, 13, 18, 22] assume that $\underset{\sim}{g} = 0$, they, therefore, exclude transverse shear dominated behavior. References [2, 7, 9] assume free lateral boundary condition, in particular, $q_s = 0$, so they exclude intermediate behavior.

First, we consider the transverse shear dominated case. In terms of the abstract problem, we have $\boldsymbol{\xi} \in V$ such that $\langle \boldsymbol{f}, \boldsymbol{v}\rangle = (\boldsymbol{\xi}, S\boldsymbol{v})_V$, $\forall \boldsymbol{v} \in H$. In view of the inner product of V for the Reissner–Mindlin plate, using (3.19) in which $q_s = 0$, we write

$$(\boldsymbol{\xi}, B\boldsymbol{v})_V = \mu \int_\Omega \left(-\frac{1}{\mu}\underset{\sim}{p}, \underset{\sim}{\phi} - \underset{\sim}{\nabla} z\right) \mathrm{d}\underset{\sim}{x}, \ \forall\ \boldsymbol{v} = (\underset{\sim}{\phi}, z) \in H_{\mathrm{RM}}. \tag{3.21}$$

Therefore, $\boldsymbol{\xi} = -\frac{1}{\mu}\underset{\sim}{p}$. To apply Lemma (3.2), we need to estimate $K(\epsilon, \boldsymbol{\xi}, [V, W])$. If $\underset{\sim}{p} \in \underset{\sim}{H}^1$, then $\boldsymbol{\xi} = -\frac{1}{\mu}\underset{\sim}{p}$ lies in the space W because $\boldsymbol{\xi} = S\boldsymbol{v}$ with $\boldsymbol{v} \in H_{\mathrm{RM}}$ being the equivalent class of $\left(-\frac{1}{\mu}\underset{\sim}{p}, 0\right) \in \underset{\sim}{H}^1 \times H^1$. And we have the estimate $\|\boldsymbol{\xi}\|_W \simeq \|\underset{\sim}{p}\|_{\underset{\sim}{H}(\mathrm{rot})} \lesssim \|\underset{\sim}{p}\|_{\underset{\sim}{H}^1}$. Therefore, according to (3.5),

$$K(\epsilon, \boldsymbol{\xi}, [V, W]) \lesssim \epsilon \|\boldsymbol{\xi}\|_W \lesssim \epsilon \|\underset{\sim}{p}\|_{\underset{\sim}{H}^1}. \tag{3.22}$$

By Lemma 3.2, we have

Lemma 3.4. *In the transverse shear dominated case, if* $\underset{\sim}{p} \in \underset{\sim}{H}^1$*, we have the estimate on the Reissner–Mindlin solution*

$$\epsilon \|\underset{\approx}{e}(\underset{\sim}{\theta}^\epsilon)\|_{\underset{\approx}{L}} + \|(\underset{\sim}{\theta}^\epsilon - \underset{\sim}{\nabla} w^\epsilon) + \frac{1}{\mu}\underset{\sim}{p}\|_{\underset{\sim}{L}^2} \lesssim \epsilon \|\underset{\sim}{p}\|_{\underset{\sim}{H}^1}. \tag{3.23}$$

To fit the equation (2.16) in the abstract form, we need an extra factor 5/6 in the inner product of V. But note that there is also such a factor in the expression of $\boldsymbol{\xi}$, *cf.* (3.20). For the solution of (2.16), using the same argument, we have

$$\epsilon \|\underset{\approx}{e}(\underset{\sim}{\bar{\theta}}{}_0^\epsilon)\|_{\underset{\approx}{L}} + \|(\underset{\sim}{\bar{\theta}}{}_0^\epsilon - \underset{\sim}{\nabla} \bar{w}_0^\epsilon) + \frac{1}{\mu}\underset{\sim}{p}\|_{\underset{\sim}{L}^2} \lesssim \epsilon \|\underset{\sim}{p}\|_{\underset{\sim}{H}^1} \tag{3.24}$$

hold in the shear dominated case.

If we only have $\underset{\sim}{p} \in \underset{\sim}{H}(\mathrm{div})$, as assumed earlier, then we do not have such a strong estimate. But it always holds that the left hand sides of (3.23) and (3.24) converge to zero with ϵ. This can be used to prove a convergence of the Reissner–Mindlin model to the elasticity theory, but without a convergence rate. Lower convergence rate can be obtained when $\underset{\sim}{p}$ has better regularity but less than H^1.

Remark 3.3. If there were the shear correction factor 5/6 in the Reissner–Mindlin model (2.12), we would have

$$\epsilon \|\underset{\approx}{e}(\underset{\sim}{\theta}^\epsilon)\|_{\underset{\approx}{L}} + \|\frac{5}{6}(\underset{\sim}{\theta}^\epsilon - \underset{\sim}{\nabla} w^\epsilon) + \frac{1}{\mu}\underset{\sim}{p}\|_{\underset{\sim}{L}^2} \lesssim \epsilon \|\underset{\sim}{p}\|_{\underset{\sim}{H}^1}. \tag{3.25}$$

Thus the shear strain $\underset{\sim}{\theta}^\epsilon - \underset{\sim}{\nabla} w^\epsilon$ would converge to a different limit. It is for this reason that the convergence of the Reissner–Mindlin model to the elasticity theory in the shear dominated case would be destroyed if the shear correction factor 5/6 were added.

Next, we consider the intermediate case. In this case, the loading functional is reformulated by (3.19), which gives the expression for $\boldsymbol{\xi}^* \in W^*$ in terms of the abstract problem, *i.e.*,

$$\langle \boldsymbol{f}_{\mathrm{RM}}, (\underset{\sim}{\phi}, z)\rangle = \langle \boldsymbol{\xi}^*, S\boldsymbol{v}\rangle = \int_\Omega (\underset{\sim}{p}, \underset{\sim}{\nabla} z - \underset{\sim}{\phi}) \mathrm{d}\underset{\sim}{x} + \langle q_s, \partial_s z - \phi_s\rangle_{\partial\Omega}, \ \forall \boldsymbol{v} = (\underset{\sim}{\phi}, z) \in H_{\mathrm{RM}}, \tag{3.26}$$

in which the twisting moment q_s is not identically equal to zero on $\partial\Omega$. To use Lemma 3.1, we need to estimate the K-functional value involved. We have

Lemma 3.5. *In the intermediate case, the Reissner–Mindlin loading functional can be represented by* $\boldsymbol{\xi}^* \in W^*$ *in the form of* (3.26)*. We have the estimate*

$$K(\epsilon, \boldsymbol{\xi}^*, [W^*, V^*]) \lesssim \epsilon^{1/2} \|q_s\|_{W^{1,\infty}(\partial\Omega)} + \epsilon \|\underset{\sim}{p}\|_{\underset{\sim}{L}^2}. \tag{3.27}$$

Therefore, $\boldsymbol{\xi}^* \in [W^*, V^*]_{1/2,\infty}$*, if* $q_s \in W^{1,\infty}(\partial\Omega)$ *and* $\underset{\sim}{p} \in \underset{\sim}{L}^2$.

Proof. To establish this estimate, we need to construct an approximation $\boldsymbol{\zeta}^* \in V^*$ to $\boldsymbol{\xi}^*$ such that $\|\boldsymbol{\xi}^* - \boldsymbol{\zeta}^*\|_{W^*} \lesssim \epsilon^{1/2}$ and $\|\boldsymbol{\zeta}^*\|_{V^*} \lesssim \epsilon^{-1/2}$. The estimate will follow from the definition of the K-functional. On a strip $\gamma^\epsilon \subset \Omega$ of width $O(\epsilon)$ along $\partial\Omega$, we use the usual curvilinear coordinates (ρ, s) to coordinate the point whose position vector is $\underset{\sim}{x} - \rho\underset{\sim}{n}$. Here, $\underset{\sim}{x}$ is the coordinate of a point on $\partial\Omega$ where the unit outward normal is $\underset{\sim}{n}$ and arc length parameter (with respect to a fixed reference point) is s. We extend the value of q_s by constant in the normal direction, and denote the extended by Q_s. Then we define $\boldsymbol{\zeta}^* \in W^*$ such that

$$\langle \boldsymbol{\zeta}^*, S\boldsymbol{v}\rangle = \int_\Omega (\underset{\sim}{p}, \underset{\sim}{\nabla} z - \underset{\sim}{\phi})\mathrm{d}\underset{\sim}{x} + \int_{\gamma^\epsilon} \frac{1}{\epsilon} Q_s[(\underset{\sim}{\nabla} z - \underset{\sim}{\phi}) \cdot \underset{\sim}{s}]\mathrm{d}\underset{\sim}{x},\ \forall \boldsymbol{v} = (\underset{\sim}{\phi}, z) \in H_{\mathrm{RM}}. \tag{3.28}$$

Indeed, we have $\boldsymbol{\zeta}^* \in V^*$ and we have the estimate

$$\|\boldsymbol{\zeta}^*\|_{V^*} = \sup_{\substack{\boldsymbol{v}\in H\\ S\boldsymbol{v}\neq 0}} \frac{\langle \boldsymbol{\zeta}^*, S\boldsymbol{v}\rangle}{\|S\boldsymbol{v}\|_V} \lesssim \|\underset{\sim}{p}\|_{\underset{\sim}{L}^2} + \sup_{\substack{(\underset{\sim}{\phi}, z)\in \underset{\sim}{H}^1\times H^1\\ \underset{\sim}{\nabla} z - \underset{\sim}{\phi}\neq 0}} \frac{\int_{\gamma^\epsilon} \frac{1}{\epsilon} Q_s[(\underset{\sim}{\nabla} z - \underset{\sim}{\phi}) \cdot \underset{\sim}{s}]\mathrm{d}\underset{\sim}{x}}{\|\underset{\sim}{\nabla} z - \underset{\sim}{\phi}\|_{\underset{\sim}{L}^2}}. \tag{3.29}$$

The last term is bounded by $\epsilon^{-1/2}\,\|Q_s\|_{L^\infty(\gamma^\epsilon)}$. We obtain

$$\|\boldsymbol{\zeta}^*\|_{V^*} \lesssim \|\underset{\sim}{p}\|_{\underset{\sim}{L}^2} + \epsilon^{-1/2}\,\|q_s\|_{L^\infty(\partial\Omega)}.$$

Now, we estimate $\|\boldsymbol{\xi}^* - \boldsymbol{\zeta}^*\|_{W^*}$. By the definition of W^* norm, we have

$$\|\boldsymbol{\xi}^* - \boldsymbol{\zeta}^*\|_{W^*} = \sup_{\substack{\boldsymbol{v}\in H\\ S\boldsymbol{v}\neq 0}} \frac{\langle \boldsymbol{\xi}^* - \boldsymbol{\zeta}^*, S\boldsymbol{v}\rangle}{\|S\boldsymbol{v}\|_W} = \sup_{\substack{(\underset{\sim}{\phi}, z)\in \underset{\sim}{H}^1\times H^1\\ \underset{\sim}{\nabla} z - \underset{\sim}{\phi}\neq 0}} \frac{\langle d_s, \partial_s z - \phi_s\rangle_{\partial\Omega} - \int_{\gamma^\epsilon} \frac{1}{\epsilon} Q_s[(\underset{\sim}{\nabla} z - \underset{\sim}{\phi}) \cdot \underset{\sim}{s}]\mathrm{d}\underset{\sim}{x}}{\|z\|_{H^1} + \|\underset{\sim}{\phi}\|_{\underset{\sim}{H}^1}}.$$

Performing an integration by parts with respect to the s coordinate, we write the numerator in the above fraction as

$$\left[\int_{\gamma^\epsilon} \frac{1}{\epsilon} Q_s(\underset{\sim}{\phi} \cdot \underset{\sim}{s})\mathrm{d}\underset{\sim}{x} - \int_{\partial\Omega} d_s\phi_s \mathrm{d}s\right] + \left[\int_{\gamma^\epsilon} \frac{1}{\epsilon}(\partial_s Q_s) z\mathrm{d}\underset{\sim}{x} - \int_{\partial\Omega} (\partial_s d_s) z\mathrm{d}s\right].$$

Applying the inequality

$$\left|\frac{1}{\epsilon}\int_0^\epsilon f(x)\mathrm{d}x - f(0)\right| \le \frac{2}{3}\,\epsilon^{1/2}\,|f|_{H^1(0,\epsilon)}$$

to the normal direction in the above sum of integrals, we obtain an upper bound. It is less than or equal to

$$\epsilon^{1/2}\left[\|Q_s\|_{L^\infty(\gamma^\epsilon)}\|\underset{\sim}{\phi}\|_{\underset{\sim}{H}^1} + \|\partial_s Q_s\|_{L^\infty(\gamma^\epsilon)}\|z\|_{H^1}\right].$$

Therefore,

$$\|\boldsymbol{\eta}^* - \boldsymbol{\zeta}^*\|_{W^*} \lesssim \epsilon^{1/2}\,\|q_s\|_{W^{1,\infty}(\partial\Omega)}.$$

The estimate on the K-functional then follows. $\square$

From Lemma 3.1, we have

Lemma 3.6. *In the intermediate case, we have the following estimate on the Reissner–Mindlin solution*

$$\epsilon\,\|\underset{\approx}{e}(\underset{\sim}{\theta}^\epsilon)\|_{\underset{\approx}{L}^2} + \|\underset{\sim}{\theta}^\epsilon - \underset{\sim}{\nabla} w^\epsilon\|_{\underset{\sim}{L}^2} \lesssim \epsilon^{-1/2}\,\|q_s\|_{W^{1,\infty}(\partial\Omega)} + \|\underset{\sim}{p}\|_{\underset{\sim}{L}^2}. \tag{3.30}$$

Actually, in the intermediate case, the sharper estimate $\epsilon\,\|\underset{\approx}{e}(\underset{\sim}{\theta}^\epsilon)\|_{\underset{\approx}{L}^2} + \|\underset{\sim}{\theta}^\epsilon - \underset{\sim}{\nabla} w^\epsilon\|_{\underset{\sim}{L}^2} \simeq \epsilon^{-1/2}$ holds (but we do not need such result). The same estimate holds for the solution of (2.16) in this intermediate case.

Finally, we treat the bending dominated case in which $\boldsymbol{f}_{\mathrm{RM}}|_{H_{\mathrm{KL}}} \neq 0$. From Lemma 3.3 we see that $(\underset{\sim}{\theta}^\epsilon, w^\epsilon)$ tends to infinity in H_{RM} at the rate ϵ^{-2}. We assume $\boldsymbol{f}_{\mathrm{RM}} = \epsilon^2\,\tilde{\boldsymbol{f}}_{\mathrm{RM}}$ with $\tilde{\boldsymbol{f}}_{\mathrm{RM}}$ being independent of ϵ. This is to say that we impose a scaling on all the loading functions by assuming $\underset{\sim}{g} = \epsilon^2\,\tilde{\underset{\sim}{g}}$, etc. So we also have $\boldsymbol{f}_0 = \epsilon^2\,\tilde{\boldsymbol{f}}_0$, $\boldsymbol{f}_2 = \epsilon^2\,\tilde{\boldsymbol{f}}_2$, and $\boldsymbol{f}_{\mathrm{KL}} = \epsilon^2\,\tilde{\boldsymbol{f}}_{\mathrm{KL}}$. In place of the limiting problem (3.11), we have

$$\frac{1}{3}\int_\Omega [\underset{\approx}{C}^*(\underset{\approx}{e}\underset{\sim}{\nabla} w^0) : \underset{\approx}{e}\underset{\sim}{\nabla} z]\mathrm{d}\underset{\sim}{x} = \langle \tilde{\boldsymbol{f}}_{\mathrm{KL}}, z\rangle,\ \forall z \in H^2. \tag{3.31}$$

This is actually the classical Kirchhoff–Love plate bending model that uniquely determines $w^0 \in H^2/\mathcal{L}$. Then under this loading scaling, the Reissner–Mindlin solution $(\underset{\sim}{\theta}^\epsilon, w^\epsilon)$ converges to the fixed limit $(\underset{\sim}{\nabla} w^0, w^0)$ in H_{RM}. In order to apply Lemma 3.3, we need to find expression for $\boldsymbol{\eta}^*$ as defined in (3.13) and estimate the K-functional in (3.14).

Referring to (3.11) and (3.15), Performing integration by parts, we write the Kirchhoff–Love model (3.31) as a biharmonic equation with two boundary conditions:

$$\begin{aligned}
\frac{1}{3}(2\mu + \lambda^*)\Delta^2 w^0 &= \operatorname{div}\tilde{\underset{\sim}{p}} + \tilde{p}_3 \ \text{ in } \Omega,\\
\frac{1}{3}[\underset{\approx}{C}^*(\underset{\approx}{e}\underset{\sim}{\nabla} w^0)\underset{\sim}{n}]\cdot\underset{\sim}{n} = \frac{1}{3}(2\mu\partial^2_{nn}w^0 + \lambda^*\Delta w^0) &= -\tilde{q}_n \ \text{ on } \partial\Omega,\\
\frac{1}{3}(2\mu + \lambda^*)\partial_n\Delta w^0 + \frac{2\mu}{3}\partial_s(\partial^2_{sn}w^0 - \kappa\partial_s w^0) &= -(\tilde{q}_3 + \partial_s\tilde{q}_s - \hat{\tilde{h}}_3) \ \text{ on } \partial\Omega.
\end{aligned} \tag{3.32}$$

Here, $\kappa = \partial_s \underset{\sim}{n}\cdot\underset{\sim}{s}$ is the curvature of $\partial\Omega$. According to the classical regularity theory of elliptic equations, the solution of (3.32) is sufficiently regular if $\partial\Omega$ and the resultant loading functions involved in the equation are sufficiently regular.

For any $\boldsymbol{v} = (\underset{\sim}{\phi}, z)$ with $\underset{\sim}{\phi}$ and z smooth, we have

$$\begin{aligned}
\langle \tilde{\boldsymbol{f}}_{\mathrm{RM}}, \boldsymbol{v}\rangle - (B\boldsymbol{u}^0, B\boldsymbol{v})_U = \int_\Omega (\tilde{\underset{\sim}{p}}, \underset{\sim}{\nabla} z - \underset{\sim}{\phi})\mathrm{d}\underset{\sim}{x} + \langle \tilde{d}_n, \partial_n z - \phi_n\rangle_{\partial\Omega} + \langle \tilde{d}_s, \partial_s z - \phi_s\rangle_{\partial\Omega} + \langle \tilde{\boldsymbol{f}}_{\mathrm{KL}}, z\rangle\\
- \frac{1}{3}\int_\Omega [\underset{\approx}{C}^*(\underset{\approx}{e}\underset{\sim}{\nabla} w^0) : \underset{\approx}{e}(\underset{\sim}{\phi} - \underset{\sim}{\nabla} z)]\mathrm{d}\underset{\sim}{x} - \frac{1}{3}\int_\Omega [\underset{\approx}{C}^*(\underset{\approx}{e}\underset{\sim}{\nabla} w^0) : \underset{\approx}{e}\underset{\sim}{\nabla} z]\mathrm{d}\underset{\sim}{x}.
\end{aligned}$$

In view of the Kirchhoff–Love model (3.31), the above expression simplifies to

$$\int_\Omega (\tilde{\underset{\sim}{p}}, \underset{\sim}{\nabla} z - \underset{\sim}{\phi})\mathrm{d}\underset{\sim}{x} + \int_{\partial\Omega}[\tilde{q}_n(\partial_n z - \phi_n) + \tilde{q}_s(\partial_s z - \phi_s)]\mathrm{d}s + \frac{1}{3}\int_\Omega [\underset{\approx}{C}^*(\underset{\approx}{e}\underset{\sim}{\nabla} w^0) : \underset{\approx}{e}(\underset{\sim}{\nabla} z - \underset{\sim}{\phi})]\mathrm{d}\underset{\sim}{x}.$$

Performing integration by parts on the last term, and using the first boundary condition in (3.32), we obtain the expression for $\boldsymbol{\eta}^* \in W^*$: for any $\boldsymbol{v} = (\underset{\sim}{\phi}, z)$ sufficiently smooth, we have

$$\langle \boldsymbol{\eta}^*, S\boldsymbol{v}\rangle = \int_\Omega (\tilde{\underset{\sim}{p}} - \frac{1}{3}\underset{\sim}{\operatorname{div}}\underset{\approx}{C}^*\underset{\approx}{e}\underset{\sim}{\nabla} w^0, \underset{\sim}{\nabla} z - \underset{\sim}{\phi})\mathrm{d}\underset{\sim}{x} + \langle \tilde{q}_s + \frac{2\mu}{3}(\partial^2_{sn}w^0 - \kappa\partial_s w^0), \partial_s z - \phi_s\rangle_{\partial\Omega}. \tag{3.33}$$

This is certainly true for any $\boldsymbol{v} = (\underset{\sim}{\phi}, z) \in \underset{\sim}{H}^1 \times H^1$. This expression is in the same form as that of $\boldsymbol{\xi}^*$, *cf.* (3.26). We see that $\boldsymbol{\eta}^* \in V^*$ if and only if $\tilde{q}_s + \frac{2\mu}{3}(\partial^2_{sn}w^0 - \kappa\partial_s w^0) = 0$ on $\partial\Omega$. Generally, this can not hold because the solution of the Kirchhoff–Love model does not satisfy such a boundary condition. It follows from

Lemma 3.5 that

$$K(\epsilon, \boldsymbol{\eta}^*, [W^*, V^*]) \lesssim \epsilon^{1/2}[\|\tilde{q}_s\|_{W^{1,\infty}(\partial\Omega)} + \|\partial^2_{sn} w^0 - \kappa \partial_s w^0\|_{W^{1,\infty}(\partial\Omega)}] + \epsilon[\|\tilde{\underset{\sim}{p}}\|_{\underset{\sim}{L}^2} + |w^0|_{H^3}]. \tag{3.34}$$

Therefore, $\boldsymbol{\eta}^* \in [W^*, V^*]_{1/2,\infty}$, if the data of the plate problem is sufficiently regular such that the norms appeared in the above right hand side are finite.

The following estimate on the Reissner–Mindlin solution then follows from Lemma 3.3.

Lemma 3.7. *In the bending dominated case, under the loading scaling* $\boldsymbol{f}_{\rm RM} = \epsilon^2 \tilde{\boldsymbol{f}}_{\rm RM}$ *with* $\tilde{\boldsymbol{f}}_{\rm RM}$ *being independent of* ϵ*, we have the estimate on the Reissner–Mindlin solution that*

$$\|\underset{\approx}{e}(\underset{\sim}{\theta}^\epsilon) - \underset{\approx}{e}(\underset{\sim}{\nabla} w^0)\|_{\underset{\approx}{L}^2} + \epsilon^{-1}\,\|\underset{\sim}{\nabla} w^\epsilon - \underset{\sim}{\theta}^\epsilon\|_{\underset{\sim}{L}^2} \lesssim \epsilon^{1/2}\,. \tag{3.35}$$

Here $w^0 \neq 0$ *is independent of* ϵ*. It is the solution of the Kirchhoff–Love model* (3.31).

For the very same reason, we have the following estimate on the solution $(\underset{\sim}{\bar{\theta}}{}^\epsilon_0, \bar{w}^\epsilon_0)$ of the equation (2.16) (with the loading functions scaled by ϵ^2):

$$\|\underset{\approx}{e}(\underset{\sim}{\bar{\theta}}{}^\epsilon_0) - \underset{\approx}{e}(\underset{\sim}{\nabla} w^0)\|_{\underset{\approx}{L}^2} + \epsilon^{-1}\,\|\underset{\sim}{\nabla} \bar{w}^\epsilon_0 - \underset{\sim}{\bar{\theta}}{}^\epsilon_0\|_{\underset{\sim}{L}^2} \lesssim \epsilon^{1/2}\,. \tag{3.36}$$

Without assuming the required regularity on the plate data, we have that the left hand sides of (3.35) and (3.36) converge to zero with ϵ. We can use such convergence to prove that the Reissner–Mindlin model converges to the elasticity theory in the bending dominated case. The result of Lemma 3.7 is not new. It was proved, for example, in [2] for free boundary conditions by asymptotic expansion, where more estimates in various norms can be found. And that method can be adapted (with substantial work) to non-zero stress boundary conditions. It seems that our method is simpler, and we require less regularity on the plate boundary and loading forces.

The above arguments can be applied to the equation (2.17), which is the same as (2.16) except for the loading functional, to get estimates on the higher order term $\epsilon^2(\underset{\sim}{\bar{\theta}}{}^\epsilon_2, \bar{w}^\epsilon_2)$. Then by superposition, we obtain the following estimates on the solution $(\underset{\sim}{\bar{\theta}}{}^\epsilon, \bar{w}^\epsilon)$ of the equation (2.5). If $\boldsymbol{f}_{\rm RM}|_{H_{\rm KL}} = 0$ and $q_s \equiv 0$, we have

$$\epsilon\,\|\underset{\approx}{e}(\underset{\sim}{\bar{\theta}}{}^\epsilon)\|_{\underset{\approx}{L}} + \|(\underset{\sim}{\bar{\theta}}{}^\epsilon - \underset{\sim}{\nabla} \bar{w}^\epsilon) + \frac{1}{\mu}\underset{\sim}{p}\|_{\underset{\sim}{L}^2} \lesssim \epsilon\,. \tag{3.37}$$

If $\boldsymbol{f}_{\rm RM}|_{H_{\rm KL}} \neq 0$, we let $\boldsymbol{f}_{\rm RM} = \epsilon^2 \tilde{\boldsymbol{f}}_{\rm RM}$ with $\tilde{\boldsymbol{f}}_{\rm RM}$ being independent of ϵ, then we have

$$\|\underset{\approx}{e}(\underset{\sim}{\bar{\theta}}{}^\epsilon) - \underset{\approx}{e}(\underset{\sim}{\nabla} w^0)\|_{\underset{\approx}{L}^2} + \epsilon^{-1}\,\|\underset{\sim}{\nabla} \bar{w}^\epsilon - \underset{\sim}{\bar{\theta}}{}^\epsilon\|_{\underset{\sim}{L}^2} \lesssim \epsilon^{1/2}\,. \tag{3.38}$$

Here $w^0 \in H^2/\mathcal{L}$ is the nonzero solution of the biharmonic model (3.31)

4. On the accuracy of the Reissner–Mindlin model

Based on the solution $(\underset{\sim}{\bar{\theta}}{}^\epsilon, \bar{w}^\epsilon)$ of (2.5), we define a three-dimensional displacement field $\underline{\bar{u}}^\epsilon$ on the plate domain P^ϵ by the formula (2.4) in which w_2 is defined by (2.7). The stress field defined by (2.2) and (2.8), as we have emphasized, is statically admissible. So we shall use the Prager–Synge theorem to bound the error of $\underline{\bar{u}}^\epsilon$.

Theorem 4.1 (the Prager–Synge theorem)**.** *Suppose that* $\underline{\underline{\sigma}} \in \underline{\underline{H}}(\underline{\rm div}, P^\epsilon)$ *is statically admissible, i.e.*

$$\underline{\rm div}\underline{\underline{\sigma}} + \underline{f}^\epsilon = 0 \quad in \quad P^\epsilon, \quad \underline{\underline{\sigma}}\underline{n} = \underline{g}^\epsilon_\pm \quad on \quad \partial P^\epsilon_\pm, \quad \underline{\underline{\sigma}}\underline{n} = \underline{h}^\epsilon \quad on \quad \partial P^\epsilon_L,$$

where $\underline{n}$ is the unit outer normal to the surface, and suppose $\underline{u} \in \underline{H}^1(P^\epsilon)$ is kinematically admissible (when the plate is subject to stress boundary conditions, this latter requirement is trivial). Then

$$\|\underline{u} - \underline{u}^\epsilon_{3\mathrm{D}}\|^2_{E^\epsilon} + \int_{P^\epsilon} [\underline{\underline{\underline{A}}}(\underline{\underline{\sigma}} - \underline{\underline{\sigma}}^\epsilon_{3\mathrm{D}}) : (\underline{\underline{\sigma}} - \underline{\underline{\sigma}}^\epsilon_{3\mathrm{D}})]\mathrm{d}\underline{x} = \int_{P^\epsilon} [\underline{\underline{\underline{C}}}\underline{\underline{\varrho}} : \underline{\underline{\varrho}}]\mathrm{d}\underline{x}.$$

Here $\underline{\underline{\varrho}} = \underline{\underline{\underline{A}}}\underline{\underline{\sigma}} - \underline{\underline{e}}(\underline{u})$ is the residual of the constitutive equation.

Note that the above integrals are equivalent to square of the $\underline{L}^2(P^\epsilon)$ norms of $\underline{\underline{\sigma}} - \underline{\underline{\sigma}}^\epsilon_{3\mathrm{D}}$ and $\underline{\underline{\varrho}}$, respectively. By using this theorem, we prove that in the shear dominated case, $\underline{\bar{u}}^\epsilon$ converges to $\underline{u}^\epsilon_{3\mathrm{D}}$ in the relative energy norm at the rate ϵ. We also prove that the convergence holds in the bending dominated case, but at the rate $\epsilon^{1/2}$. The needed admissible stress field is defined by (2.2).

Based on the Reissner–Mindlin solution, we define a three-dimensional displacement field $\underline{u}^\epsilon_{\mathrm{RM}}$ by

$$\underline{u}^\epsilon_{\mathrm{RM}} = \begin{pmatrix} -x_3 \underset{\sim}{\theta}^\epsilon(\underset{\sim}{x}) \\ w^\epsilon(\underset{\sim}{x}) + \epsilon^2 r(x_3) w^\epsilon_2(\underset{\sim}{x}) \end{pmatrix}. \tag{4.1}$$

Here

$$w^\epsilon_2 = \frac{\lambda}{2(2\mu+\lambda)}[\operatorname{div} \underset{\sim}{\theta}^\epsilon - g_3] - \frac{10\mu(2\mu+3\lambda)+3\lambda^2}{70\mu(2\mu+\lambda)(2\mu+3\lambda)}(g_3 + f_3 + \operatorname{div} \underset{\sim}{g}). \tag{4.2}$$

The infinitesimal strain tensor engendered by $\underline{u}^\epsilon_{\mathrm{RM}}$ is

$$\underline{\underline{e}}(\underline{u}^\epsilon_{\mathrm{RM}}) = \begin{pmatrix} -x_3 \underset{\approx}{e}(\underset{\sim}{\theta}^\epsilon), & \frac{1}{2}(\underset{\sim}{\nabla} w^\epsilon - \underset{\sim}{\theta}^\epsilon) + \epsilon^2 \frac{1}{2} r(x_3) \underset{\sim}{\nabla} w^\epsilon_2 \\ \text{symmetric}, & \epsilon^2 r'(x_3) w^\epsilon_2 \end{pmatrix}. \tag{4.3}$$

The expression for $\underline{\underline{e}}(\underline{\bar{u}}^\epsilon)$ is similar. We thus have the formula

$$\begin{aligned} &\underline{\underline{e}}(\underline{u}^\epsilon_{\mathrm{RM}}) - \underline{\underline{e}}(\underline{\bar{u}}^\epsilon) \\ &= \begin{pmatrix} -x_3[\underset{\approx}{e}(\underset{\sim}{\theta}^\epsilon) - \underset{\approx}{e}(\underset{\sim}{\bar{\theta}}^\epsilon)], & \frac{1}{2}[(\underset{\sim}{\nabla} w^\epsilon - \underset{\sim}{\theta}^\epsilon) - (\underset{\sim}{\nabla} \bar{w}^\epsilon - \underset{\sim}{\bar{\theta}}^\epsilon)] + \epsilon^2 r(x_3) \frac{\lambda}{4(2\mu+\lambda)} \underset{\sim}{\nabla} \operatorname{div}(\underset{\sim}{\theta}^\epsilon - \underset{\sim}{\bar{\theta}}^\epsilon) \\ \text{symmetric}, & \epsilon^2 r'(x_3) \frac{\lambda}{2(2\mu+\lambda)} \operatorname{div}(\underset{\sim}{\theta}^\epsilon - \underset{\sim}{\bar{\theta}}^\epsilon) \end{pmatrix}. \end{aligned} \tag{4.4}$$

We shall show that $\underline{u}^\epsilon_{\mathrm{RM}}$ is close to $\underline{\bar{u}}^\epsilon$ in the relative energy norm, which is bounded by ϵ in the shear dominated case, and by $\epsilon^{1/2}$ in the bending dominated case, thus give a justification for the Reissner–Mindlin model in these two cases. Finally, we show that neither the Reissner–Mindlin solution or the solution of (2.5) converges to the elasticity solution in the intermediate case.

4.1. **Shear dominated case**

This is the case in which $\boldsymbol{f}_{\mathrm{KL}} = 0$ and the bending moment $q_s \equiv 0$. The Kirchhoff–Love solution is zero, and so the model is useless. In this case, we prove that the Reissner–Mindlin model is an accurate approximation to the elasticity theory. From the Prager–Synge theorem, we see $\|\underline{u}^\epsilon_{3\mathrm{D}} - \underline{\bar{u}}^\epsilon\|_{E^\epsilon} \lesssim \|\underline{\underline{\varrho}}\|_{\underline{\underline{L}}^2(P^\epsilon)}$. We need to bound the L^2 norm of $\underline{\underline{\varrho}}$. We will need the following estimate on the correction $\bar{w}^\epsilon_2$ defined by (2.7).

Lemma 4.2. *Under the condition that the Reissner–Mindlin solution is in the shear dominated case, the estimate*

$$\|\underset{\sim}{\nabla} \bar{w}^\epsilon_2\|_{\underset{\sim}{L}^2} \lesssim \epsilon^{-1} \tag{4.5}$$

holds on the correction function defined by (2.7).

Proof. From the expression (2.7), we see

$$\epsilon^2 \|\underset{\sim}{\nabla} \bar{w}_2^\epsilon\|_{\underset{\sim}{L}^2} \lesssim \epsilon^2(\|\operatorname{div} \underset{\sim}{\bar{\theta}}^\epsilon\|_{H^1} + 1).$$

To get a bound on $\epsilon^2 \|\operatorname{div} \underset{\sim}{\bar{\theta}}^\epsilon\|_{H^1}$, we invoke the equation (2.5). This equation, under the shear domination assumption (that the equations (3.16)–(3.18) are satisfied by the loading functions and $q_s = 0$), says that $\underset{\sim}{\bar{\theta}}^\epsilon$ satisfies the equation

$$\epsilon^2 \frac{1}{3} \int_\Omega (\underset{\approx}{C}^* \underset{\approx}{e}(\underset{\sim}{\bar{\theta}}^\epsilon) : \underset{\approx}{e}(\underset{\sim}{\phi}))\mathrm{d}\underset{\sim}{x} = -\frac{5}{6}\mu \int_\Omega (\underset{\sim}{\bar{\theta}}^\epsilon - \underset{\sim}{\nabla} \bar{w}^\epsilon + \frac{1}{\mu}\underset{\sim}{g}, \underset{\sim}{\phi})\mathrm{d}\underset{\sim}{x} - \epsilon^2 \frac{\lambda}{3(2\mu+\lambda)} \int_\Omega g_3 \operatorname{div} \underset{\sim}{\phi} \mathrm{d}\underset{\sim}{x},\ \forall \underset{\sim}{\phi} \in \underset{\sim}{H}^1.$$

This is the same as the equation of plane elasticity. By the classical regularity results of plane elasticity, we get

$$\epsilon^2 \|\underset{\sim}{\bar{\theta}}^\epsilon\|_{\underset{\sim}{H}^2/\mathcal{R}_{2D}} \lesssim \|\underset{\sim}{\bar{\theta}}^\epsilon - \underset{\sim}{\nabla} \bar{w}^\epsilon + \frac{1}{\mu}\underset{\sim}{g}\|_{\underset{\sim}{L}^2} + \epsilon^2 \|g_3\|_{H^1} + \epsilon^2 \|g_3\|_{H^{1/2}(\partial\Omega)}.$$

Here $\mathcal{R}_{2D} = \{(a + cx_2, b - cx_1) | (a, b, c) \in \mathbb{R}^3\}$ is the space of two-dimensional infinitesimal rigid body motions. Using the estimate (3.37), we get $\epsilon^2 \|\underset{\sim}{\bar{\theta}}^\epsilon\|_{\underset{\sim}{H}^2/\mathcal{R}_{2D}} \lesssim \epsilon$. Therefore

$$\epsilon^2 \|\operatorname{div} \underset{\sim}{\bar{\theta}}^\epsilon\|_{H^1} \lesssim \epsilon^2 \|\underset{\sim}{\bar{\theta}}^\epsilon\|_{\underset{\sim}{H}^2/\mathcal{R}_{2D}} \lesssim \epsilon\,. \tag{4.6}$$

Therefore

$$\epsilon^2 \|\underset{\sim}{\nabla} \bar{w}_2^\epsilon\|_{\underset{\sim}{L}^2} \lesssim \epsilon^2(\|\operatorname{div} \underset{\sim}{\bar{\theta}}^\epsilon\|_{H^1} + 1) \lesssim \epsilon\,.$$

□

This estimate will also be needed to get a lower bound on $\|\underline{\bar{u}}^\epsilon\|_{E^\epsilon}$.

Lemma 4.3. *If the Reissner–Mindlin solution falls in the shear dominated case, then the displacement field* $\underline{\bar{u}}^\epsilon$ *defined in terms of the solution of* (2.5) *converges to the elasticity solution, and the estimate*

$$\frac{\|\underline{u}_{3\mathrm{D}}^\epsilon - \underline{\bar{u}}^\epsilon\|_{E^\epsilon}}{\|\underline{u}_{3\mathrm{D}}^\epsilon\|_{E^\epsilon}} \lesssim \epsilon \tag{4.7}$$

holds.

Proof. By the Prager–Synge theorem, $\|\underline{u}_{3\mathrm{D}}^\epsilon - \underline{\bar{u}}^\epsilon\|_{E^\epsilon} \lesssim \|\underline{\underline{\varrho}}\|_{\underline{\underline{L}}^2(P^\epsilon)}$. From the expression of σ (the third equation in (2.8)), we see $\|\underset{\approx}{\varrho}\|_{\underset{\approx}{L}^2(P^\epsilon)} \lesssim \epsilon^{3/2}$ and $\|\varrho_{33}\|_{L^2(P^\epsilon)} \lesssim \epsilon^{3/2}$. Using the estimate (3.37) and (4.5), we get $\|\underset{\sim}{\varrho}\|_{\underset{\sim}{L}^2(P^\epsilon)} \lesssim \epsilon^{3/2}$. Thus we obtain the upper bound

$$\|\underline{u}_{3\mathrm{D}}^\epsilon - \underline{\bar{u}}^\epsilon\|_{E^\epsilon} \lesssim \|\underline{\underline{\varrho}}\|_{\underline{\underline{L}}^2(P^\epsilon)} \lesssim \epsilon^{3/2}\,.$$

From the expression (4.3), we see that the transverse shear part of the strain tensor $\underline{\underline{e}}(\underline{\bar{u}}^\epsilon)$ is

$$\underset{\sim}{e}(\underline{\bar{u}}^\epsilon) = \frac{1}{2}(\underset{\sim}{\nabla} \bar{w}^\epsilon - \underset{\sim}{\bar{\theta}}^\epsilon) + \epsilon^2 \frac{1}{2} r(x_3) \underset{\sim}{\nabla} \bar{w}_2^\epsilon.$$

From the estimate (3.37), we see that $\|\frac{1}{2}(\underset{\sim}{\nabla} \bar{w}^\epsilon - \underset{\sim}{\bar{\theta}}^\epsilon)\|_{\underset{\sim}{L}^2(P^\epsilon)} \simeq \|\underset{\sim}{g}\|_{\underset{\sim}{L}^2(P^\epsilon)} \simeq \epsilon^{1/2}$, while (4.5) says that

$$\|\, \epsilon^2 \frac{1}{2} r(x_3) \underset{\sim}{\nabla} \bar{w}_2^\epsilon\|_{\underset{\sim}{L}^2(P^\epsilon)} \lesssim \epsilon^{3/2}\,.$$

Therefore

$$\|\underline{\bar{u}}^\epsilon\|_{E^\epsilon} \simeq \|\underline{\underline{e}}(\underline{\bar{u}}^\epsilon)\|_{\underline{\underline{L}}^2(P^\epsilon)} \gtrsim \epsilon^{1/2}\,.$$

□

Now we estimate the difference between $\underline{\bar{u}}^\epsilon$ and the displacement field $\underline{u}^\epsilon_{\mathrm{RM}}$ that is reconstructed from the Reissner–Mindlin solution. We give upper bound for every term in the difference of their strain tensor, *cf.* (4.4). From (3.23) and (3.37) we see that both $\|\underset{\approx}{e}(\underset{\sim}{\bar{\theta}}^\epsilon)\|_{\underset{\approx}{L}}$ and $\|\underset{\approx}{e}(\underset{\sim}{\theta}^\epsilon)\|_{\underset{\approx}{L}}$ are bounded by constant. We also see that $\|(\underset{\sim}{\nabla} w^\epsilon - \underset{\sim}{\theta}^\epsilon) - (\underset{\sim}{\nabla}\bar{w}^\epsilon - \underset{\sim}{\bar{\theta}}^\epsilon)\|_{\underset{\sim}{L}^2} \lesssim \epsilon$. An estimate of the form of (4.6) can be established for $\underset{\sim}{\theta}^\epsilon$, and an estimate of the form (4.5) can be established to w_2^ϵ in exactly the same way. Then we have $\epsilon^2\,\|\operatorname{div}(\underset{\sim}{\theta}^\epsilon - \underset{\sim}{\bar{\theta}}^\epsilon)\|_{L^2} \lesssim \epsilon$ and $\epsilon^2\,\|\underset{\sim}{\nabla}\operatorname{div}(\underset{\sim}{\theta}^\epsilon - \underset{\sim}{\bar{\theta}}^\epsilon)\|_{\underset{\sim}{L}^2} \lesssim \epsilon$. All these allow us to obtain

$$\|\underline{\underline{e}}(\underline{u}^\epsilon_{\mathrm{RM}}) - \underline{\underline{e}}(\underline{\bar{u}}^\epsilon)\|_{\underline{\underline{L}}^2(P^\epsilon)} \lesssim \epsilon^{3/2}\,.$$

This estimate together with Lemma 4.3 leads to the following theorem.

Theorem 4.4. *If the plate P^ϵ is loaded in such a way that the solution of the Reissner–Mindlin model* (2.12) *is transverse shear dominated, i.e., the corresponding Kirchhoff–love solution w^0 is zero and the twisting moment q_s is also zero, the Reissner–Mindlin model* (2.12) *yields a non-trivial solution $(\underset{\sim}{\theta}^\epsilon, w^\epsilon)$. Based on this solution, we define a three-dimensional displacement field $\underline{u}^\epsilon_{\mathrm{RM}}$ on the plate domain P^ϵ by the formula* (4.1) *with a higher order correction on the transverse deflection defined by* (4.2). *Then $\underline{u}^\epsilon_{\mathrm{RM}}$ converges to $\underline{u}^\epsilon_{\mathrm{3D}}$ in the relative energy norm. If the plate boundary and loading forces are sufficiently smooth, we have the error estimate in the relative energy norm that*

$$\frac{\|\underline{u}^\epsilon_{\mathrm{3D}} - \underline{u}^\epsilon_{\mathrm{RM}}\|_{E^\epsilon}}{\|\underline{u}^\epsilon_{\mathrm{3D}}\|_{E^\epsilon}} \lesssim \epsilon\,. \tag{4.8}$$

Remark 4.1. If the shear correction factor 5/6 were added in the Reissner–Mindlin model (2.12), then we would have the estimate (3.25). By using this estimate in place of (3.23), we see the convergence

$$\lim_{\epsilon\to 0} \frac{\|\underline{u}^\epsilon_{\mathrm{3D}} - \underline{u}^\epsilon_{\mathrm{RM}}\|_{E^\epsilon}}{\|\underline{u}^\epsilon_{\mathrm{3D}}\|_{E^\epsilon}} = 1/6.$$

We thus lost the validity of the Reissner–Mindlin model in the shear dominated case. Any shear correction factor different from 1 would do a similar damage.

4.2. **Bending dominated case**

In this case, the Kirchhoff–Love model has a non-zero solution $w^0 \in H^2/\mathcal{L}$. Based on this, we define a displacement field

$$\underline{u}^\epsilon_{\mathrm{KL}} = \begin{pmatrix} -x_3 \underset{\sim}{\nabla} w^0(\underset{\sim}{x}) \\ w^0(\underset{\sim}{x}) + \epsilon^2\, \tilde{r}(x_3) w_2^0(\underset{\sim}{x}) \end{pmatrix}. \tag{4.9}$$

Here

$$w_2^0 = \frac{\lambda}{2(2\mu+\lambda)}[\Delta w^0 - g_3] - \frac{10\mu(2\mu+3\lambda)+3\lambda^2}{70\mu(2\mu+\lambda)(2\mu+3\lambda)}(g_3 + f_3 + \operatorname{div} \underset{\sim}{g}). \tag{4.10}$$

Both this displacement field and $\underline{u}^\epsilon_{\mathrm{RM}}$ defined in the previous subsection converge to $\underline{u}^\epsilon_{\mathrm{3D}}$ in the relative energy norm. We have

Theorem 4.5. *If the plate P^ϵ is loaded in such a way that the solution of the Reissner–Mindlin model* (2.12) *is bending dominated, i.e., the corresponding Kirchhoff–love solution w^0 is not zero, then the displacement field $\underline{u}^\epsilon_{\mathrm{RM}}$ constructed from the Reissner–Mindlin solution by the formula* (4.1) *with a higher order correction on the transverse deflection defined by* (4.2) *converges to the elasticity solution $\underline{u}^\epsilon_{\mathrm{3D}}$ in the relative energy norm.*

So is the displacement field $\underline{u}^\epsilon_{\mathrm{KL}}$. *If the plate boundary and the loading forces are sufficiently smooth, we have the error estimate in the relative energy norm:*

$$\frac{\|\underline{u}^\epsilon_{3\mathrm{D}} - \underline{u}^\epsilon_{\mathrm{RM}}\|_{E^\epsilon}}{\|\underline{u}^\epsilon_{3\mathrm{D}}\|_{E^\epsilon}} \lesssim \epsilon^{1/2}, \quad \frac{\|\underline{u}^\epsilon_{3\mathrm{D}} - \underline{u}^\epsilon_{\mathrm{KL}}\|_{E^\epsilon}}{\|\underline{u}^\epsilon_{3\mathrm{D}}\|_{E^\epsilon}} \lesssim \epsilon^{1/2}. \tag{4.11}$$

Proof. We first prove that

$$\frac{\|\underline{u}^\epsilon_{3\mathrm{D}} - \underline{\bar{u}}^\epsilon\|_{E^\epsilon}}{\|\underline{\bar{u}}^\epsilon\|_{E^\epsilon}} \lesssim \epsilon^{1/2}. \tag{4.12}$$

Recall that $\underline{\bar{u}}^\epsilon$ is defined in terms of the solution of (2.5). For this we use the Prager–Synge theorem. Then we bound $\|\underline{\bar{u}}^\epsilon - \underline{u}^\epsilon_{\mathrm{RM}}\|_{E^\epsilon}$ and $\|\underline{\bar{u}}^\epsilon - \underline{u}^\epsilon_{\mathrm{KL}}\|_{E^\epsilon}$.

In the bending dominated case, we need to scale the loading functions on the plate by ϵ^2 so that the model solution converges to a fixed limit. We let $f_3 = \epsilon^2 \tilde{f}_3$, $\underline{g} = \epsilon^2 \underline{\tilde{g}}$, and $\underline{h} = \epsilon^2 \underline{\tilde{h}}$. Then all the resultant functionals becomes $\boldsymbol{f}_{\mathrm{RM}} = \epsilon^2 \tilde{\boldsymbol{f}}_{\mathrm{RM}}$, etc. All the quantities with tilde are independent of ϵ. By doing so, in the expression (2.9) of constitutive residual, σ becomes $\frac{\epsilon^3}{2}(\tilde{g}_3 + \tilde{f}_3 + \operatorname{div} \underset{\sim}{\tilde{g}})$, *cf.* the third equation in (2.8). Thus

$$\|\underset{\approx}{\varrho}\|_{\underset{\approx}{L}^2(P^\epsilon)} \lesssim \epsilon^{7/2}, \quad \|\varrho_{33}\|_{L^2(P^\epsilon)} \lesssim \epsilon^{7/2}.$$

The main task is to bound $\|\underset{\sim}{\varrho}\|_{\underset{\sim}{L}^2(P^\epsilon)}$. From (2.9), we see that

$$\|\underset{\sim}{\varrho}\|_{\underset{\sim}{L}^2(P^\epsilon)} \lesssim \|\underset{\sim}{\nabla} \bar{w}^\epsilon - \underset{\sim}{\bar{\theta}}^\epsilon\|_{\underset{\sim}{L}^2(P^\epsilon)} + \epsilon^2 \|\underset{\sim}{\tilde{g}}\|_{\underset{\sim}{L}^2(P^\epsilon)} + \epsilon^2 \|\underset{\sim}{\nabla} \bar{w}^\epsilon_2\|_{\underset{\sim}{L}^2(P^\epsilon)}. \tag{4.13}$$

From (3.38), we see that $\|\underset{\sim}{\nabla} \bar{w}^\epsilon - \underset{\sim}{\bar{\theta}}^\epsilon\|_{\underset{\sim}{L}^2} \lesssim \epsilon^{3/2}$. Therefore,

$$\|\underset{\sim}{\nabla} \bar{w}^\epsilon - \underset{\sim}{\bar{\theta}}^\epsilon\|_{\underset{\sim}{L}^2(P^\epsilon)} \lesssim \epsilon^2.$$

From the expression (2.7), we see

$$\epsilon^2 \|\underset{\sim}{\nabla} \bar{w}^\epsilon_2\|_{\underset{\sim}{L}^2} \lesssim \epsilon^2 (\|\operatorname{div} \underset{\sim}{\bar{\theta}}^\epsilon\|_{H^1} + \epsilon^2).$$

To get a bound on $\epsilon^2 \|\operatorname{div} \underset{\sim}{\bar{\theta}}^\epsilon\|_{H^1}$, we again invoke the equation (2.5) to obtain a plane elasticity equation of the form

$$\epsilon^2 \frac{1}{3} \int_\Omega (\underset{\approx}{C}^* \underset{\approx}{e}(\underset{\sim}{\theta}) : \underset{\approx}{e}(\underset{\sim}{\phi})) \mathrm{d}\underset{\sim}{x} = -\frac{5}{6}\mu \int_\Omega (\underset{\sim}{\theta} - \underset{\sim}{\nabla} w, \underset{\sim}{\phi}) \mathrm{d}\underset{\sim}{x} + \epsilon^2 \frac{5}{6} \int_\Omega (\underset{\sim}{\tilde{g}}, -\underset{\sim}{\phi}) \mathrm{d}\underset{\sim}{x} + \epsilon^2 \int_{\partial\Omega} (-\underset{\sim}{\tilde{g}}, \underset{\sim}{\phi}) \mathrm{d}s$$
$$- \epsilon^4 \frac{\lambda}{3(2\mu + \lambda)} \int_\Omega [\tilde{g}_3 + \frac{1}{5}(\tilde{g}_3 + \tilde{f}_3 + \operatorname{div} \underset{\sim}{\tilde{g}})] \operatorname{div} \underset{\sim}{\phi} \, \mathrm{d}\underset{\sim}{x}.$$

Using the regularity of plane elasticity, we see

$$\epsilon^2 \|\underset{\sim}{\nabla} \bar{w}^\epsilon_2\|_{\underset{\sim}{L}^2} \lesssim \epsilon^2 \|\operatorname{div} \underset{\sim}{\bar{\theta}}^\epsilon\|_{H^1} + \epsilon^2 \lesssim \epsilon^2 \|\underset{\sim}{\bar{\theta}}^\epsilon\|_{\underset{\sim}{H}^2/\mathcal{R}_{2\mathrm{D}}} + \epsilon^2 \lesssim \|\underset{\sim}{\bar{\theta}}^\epsilon - \underset{\sim}{\nabla} \bar{w}^\epsilon\|_{\underset{\sim}{L}^2} + \epsilon^2 \lesssim \epsilon^{3/2}. \tag{4.14}$$

Going back to (4.13), we see $\|\underset{\sim}{\varrho}\|_{\underset{\sim}{L}^2(P^\epsilon)} \lesssim \epsilon^2$. Thus, by the Prager–Synge theorem, $\|\underline{u}^\epsilon_{3\mathrm{D}} - \underline{\bar{u}}^\epsilon\|_{E^\epsilon} \lesssim \epsilon^2$. For a lower bound on $\|\underline{\bar{u}}^\epsilon\|_{E^\epsilon}$, we see from the expression (4.3) that $\|\underline{\bar{u}}^\epsilon\|_{E^\epsilon} \gtrsim \epsilon \|\underset{\approx}{e}(\underset{\sim}{\bar{\theta}}^\epsilon)\|_{\underset{\approx}{L}^2(P^\epsilon)}$. The estimate (3.38) shows that

$$\|\underset{\approx}{e}(\underset{\sim}{\bar{\theta}}^\epsilon)\|_{\underset{\approx}{L}^2} \simeq \|\underset{\approx}{e}(\underset{\sim}{\nabla} w^0)\|_{\underset{\approx}{L}^2} \gtrsim \|w^0\|_{H^2/\mathcal{L}} \neq 0.$$

Therefore, $\|\underline{\bar{u}}^\epsilon\|_{E^\epsilon} \gtrsim \epsilon^{3/2}$. This proves (4.12).

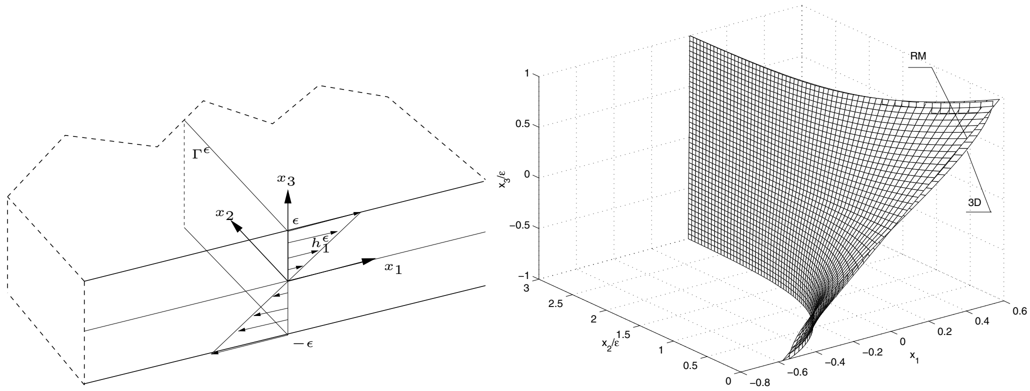

FIGURE 4.1. A semi-infinite plate loaded by a twisting moment (left) and the deformations of the section Γ^ϵ determined by Reissner–Mindlin (RM) and elasticity (3D) theories.

For the estimate on $\|\underline{\bar{u}}^\epsilon - \underline{u}^\epsilon_{\mathrm{RM}}\|_{E^\epsilon}$, we use the expression (4.4). The estimate (4.14) can be proved for w_2^ϵ and $\underset{\sim}{\theta}^\epsilon$ in exactly the same way. Using such estimate and the estimates (3.35) and (3.38), we show that $\|\underline{\bar{u}}^\epsilon - \underline{u}^\epsilon_{\mathrm{RM}}\|_{E^\epsilon} \lesssim \epsilon^2$. Similarly, we have $\|\underline{\bar{u}}^\epsilon - \underline{u}^\epsilon_{\mathrm{KL}}\|_{E^\epsilon} \lesssim \epsilon^2$. The proof is complete. □

4.3. **Intermediate case**

In this case the Kirchhoff–Love solution is zero. But there is a nonzero twisting moment applied on the plate lateral boundary so the elasticity solution and the Reissner–Mindlin solution are not trivial. The Kirchhoff–Love model is useless. The question is whether or not the Reissner–Mindlin model gives an accurate approximation to the elasticity theory. Unfortunately, the above method is not applicable to this case: no convergence estimate can be achieved in the relative energy norm. Such convergence actually does not exist. We give a simple example to show that the Reissner–Mindlin model is not asymptotically consistent with the elasticity theory in the intermediate case.

We consider a semi-infinite plate whose mid-section is $\Omega = (-\infty,\infty)\times(0,\infty)$. The boundary $\partial\Omega$ is the x_1 axis. The plate is only loaded by a twisting moment on the lateral face $\partial P_L^\epsilon = \{(x_1,x_2,x_3)|x_1 \in (-\infty,\infty), x_2 = 0, x_3 \in (-\epsilon,\epsilon)\}$ such that $\underline{h}^\epsilon(\underline{x}) = (x_3\frac{3}{\epsilon^2}q_s, 0, 0)$. Here q_s is a constant independent of x_1. See the left figure in Figure 4.1. In the resultant loading functions in the Reissner–Mindlin model, *cf.* (1.10), only the twisting moment q_s is not zero. It is easy to see that the loading functional in the Kirchhoff–Love model is zero because $\partial_s q_s = \partial_1 q_s = 0$, *cf.* (2.13). We are thus in the intermediate case.

For this problem, the Reissner–Mindlin model (2.12), written in differential form, determines that $w^\epsilon = 0$, $\theta_2^\epsilon = 0$ and θ_1^ϵ is independent of x_1. The dependence of θ_1^ϵ on x_2 is governed by the boundary value problem of ordinary differential equation

$$-\frac{1}{3}\epsilon^2\mu\partial_{22}\theta_1^\epsilon + \mu\theta_1^\epsilon = 0,\ \forall x_2 \in (0,\infty), \quad \frac{1}{3}\epsilon^2\mu\partial_2\theta_1^\epsilon = q_s \ \text{ at } x_2 = 0 \ \text{ and } \theta_1^\epsilon < \infty \ \text{ at } x_2 = \infty. \tag{4.15}$$

From this we see that

$$\theta_1^\epsilon = -\frac{\sqrt{3}}{\mu\,\epsilon}d_s \mathrm{e}^{-\frac{\sqrt{3}}{\epsilon}x_2}. \tag{4.16}$$

The displacement field $\underline{u}^\epsilon_{\mathrm{RM}}$ constructed from the Reissner–Mindlin solution by the formula (4.1) is

$$u^\epsilon_{RM,1} = x_3\frac{\sqrt{3}}{\mu\,\epsilon}q_s\mathrm{e}^{-\frac{\sqrt{3}}{\epsilon}x_2}, \quad u^\epsilon_{RM,2} = u^\epsilon_{RM,3} = 0. \tag{4.17}$$

The elasticity equation (1.1) is also explicitly solvable. It yields $\underline{u}^\epsilon_{\rm 3D} = (u^\epsilon_{\rm 3D,1}, 0, 0)$, in which $u^\epsilon_{\rm 3D,1}$ is independent of x_1. The restriction of $u^\epsilon_{\rm 3D,1}$ on the x_2x_3-plane, still denoted by $u^\epsilon_{\rm 3D,1}$, is determined by the following boundary value problem on the the semi-infinite strip Γ^ϵ that is the intersection of P^ϵ with the x_2x_3-plane. The boundary of Γ^ϵ is composed of $\partial\Gamma^\epsilon_L$ and $\partial\Gamma^\epsilon_\pm$, corresponding to $x_2 = 0$ and $x_3 = \pm\,\epsilon$.

$$\begin{gathered}\partial^2_{22}u^\epsilon_{3D,1} + \partial^2_{33}u^\epsilon_{3D,1} = 0 \ \text{ in } \Gamma^\epsilon,\\ \partial_3 u^\epsilon_{3D,1} = 0 \ \text{ on } \partial\Gamma^\epsilon_\pm, \quad \partial_2 u^\epsilon_{3D,1} = -\frac{3}{\mu\,\epsilon^2}q_s x_3 \ \text{ on } \partial\Gamma^\epsilon_L.\end{gathered} \tag{4.18}$$

The solution, found by Fourier series method, is

$$u^\epsilon_{3D,1} = -\frac{48q_s}{\mu}\sum_{k=0}^{\infty}\frac{1}{[(2k+1)\pi]^3}\cos\frac{(2k+1)\pi(x_3+\epsilon)}{2\,\epsilon}\mathrm{e}^{-\frac{(2k+1)\pi}{2\,\epsilon}x_2}. \tag{4.19}$$

The right figure in Figure 4.1 depicts the deformed shapes of Γ^ϵ determined by the Reissner–Mindlin and elasticity models, in which we have taken $q_s = \mu$. Note that both $\underline{u}^\epsilon_{\rm RM}$ and $\underline{u}^\epsilon_{\rm 3D}$ are constant in x_1. The surface with straight line cross-section with x_2 = constant represents the Reissner–Mindlin solution. It appears that $\underline{u}^\epsilon_{\rm RM}$ and $\underline{u}^\epsilon_{\rm 3D}$ are close to each other. A closer look shows that they are different. The key point here is that the relative error between $\underline{u}^\epsilon_{\rm RM}$ and $\underline{u}^\epsilon_{\rm 3D}$ does not change with respect to ϵ in any norm. More specifically,

$$\frac{\|\underline{u}^\epsilon_{\rm RM} - \underline{u}^\epsilon_{\rm 3D}\|_{E^\epsilon}}{\|\underline{u}^\epsilon_{\rm 3D}\|_{E^\epsilon}} \simeq 1. \tag{4.20}$$

Here the norm should be understood as the energy norm on, say, $(0,1)\times\Gamma^\epsilon$. Therefore, there is no convergence of the Reissner–Mindlin solution to the elasticity solution when $\epsilon \to 0$ in the intermediate case (however, the right figure in Figure 4.1 shows that the relative error of the Reissner–Mindlin model is significantly less than 1, but the relative error of Kirchhoff–Love is equal to 1. In this sense, we may say that the Reissner–Mindlin is more accurate). This is in sharp contrast with the shear dominated and bending dominated cases, *cf.* (4.8) and (4.11). We note that for this example both $\underline{u}^\epsilon_{\rm RM}$ and $\underline{u}^\epsilon_{\rm 3D}$ are just boundary layers, and their difference is negligible away from the plate boundary. As the elasticity solution shows, a normal fiber near the boundary of the semi-infinite plate is deformed to a curve that is not of polynomial profile. We know that there is a whole hierarchy of plate models of increasing order, see [21] or [19], for example. But each one in this hierarchy yields a deformation of polynomial profile, and it has a relative error not converging to zero. Therefore, any model in the hierarchy is not asymptotically correct for this example. However, this is not at odds with the possibility that when the order of plate model in the hierarchy tends to infinity, we would have a convergence. See [3]. This argument can be used to more general intermediate problems.

For a general intermediate problem, we split the load on the plate to two sets of forces:

$$\{\underline{f}^\epsilon, \underline{g}^\epsilon, \underline{h}^\epsilon\} = \{\underline{f}^\epsilon, \underline{g}^\epsilon, [(\underset{\sim}{h}^\epsilon\cdot\underset{\sim}{n})\underset{\sim}{n}, \hat{h}_3]\} \oplus \{0, 0, [(\underset{\sim}{h}^\epsilon\cdot\underset{\sim}{s})\underset{\sim}{s}, \frac{3}{2}\left(1-\frac{x_3^2}{\epsilon^2}\right)(q_3-\hat{h}_3)]\}. \tag{4.21}$$

It is easy to verify that both the two sets of forces are compatible and satisfy the continuity condition (1.20). Therefore there exist $\underline{u}^\epsilon_{I,\rm 3D}$ and $\underline{u}^\epsilon_{II,\rm 3D}$ solving the elasticity equation with the corresponding loads. Furthermore, the Kirchhoff–Love model yields a trivial solution for both the two problems. We have

$$\underline{u}^\epsilon_{\rm 3D} = \underline{u}^\epsilon_{I,\rm 3D} + \underline{u}^\epsilon_{II,\rm 3D}.$$

Corresponding to this loading splitting, the resultant loading functional in the Reissner–Mindlin model, *cf.* (3.19), is split to

$$\int_\Omega(\underset{\sim}{p}, \underset{\sim}{\nabla} z - \underset{\sim}{\phi})\mathrm{d}\underset{\sim}{x} \oplus \langle q_s, \partial_s z - \phi_s\rangle_{\partial\Omega},\ \forall(\underset{\sim}{\phi}, z)\in H_{\rm RM},$$

And the Reissner–Mindlin solution is split to

$$(\underset{\sim}{\theta}^\epsilon, w^\epsilon) = (\underset{\sim}{\theta}_I^\epsilon, w_I^\epsilon) + (\underset{\sim}{\theta}_{II}^\epsilon, w_{II}^\epsilon).$$

We see that $(\underset{\sim}{\theta}_I^\epsilon, w_I^\epsilon)$ is shear dominated (if $\underset{\sim}{p} \neq 0$). Therefore, according to Theorem 4.4, we have

$$\frac{\|\underline{u}_{I,3\mathrm{D}}^\epsilon - \underline{u}_{I,\mathrm{RM}}^\epsilon\|_{E^\epsilon}}{\|\underline{u}_{I,3\mathrm{D}}^\epsilon\|_{E^\epsilon}} \lesssim \epsilon\,.$$

Here $\underline{u}_{I,\mathrm{RM}}^\epsilon$ is a displacement field constructed from $(\underset{\sim}{\theta}_I^\epsilon, w_I^\epsilon)$ in the same way as above. The strain energy due to $(\underset{\sim}{\theta}_I^\epsilon, w_I^\epsilon)$ is mainly of transverse shear, and the total strain energy is

$$E(\underset{\sim}{\theta}_I^\epsilon, w_I^\epsilon) := \epsilon^2 \frac{1}{3}\int_\Omega [\underset{\approx}{C}^* \underset{\approx}{e}(\underset{\sim}{\theta}_I^\epsilon) : \underset{\approx}{e}(\underset{\sim}{\theta}_I^\epsilon)]\mathrm{d}\underset{\sim}{x} + \mu\int_\Omega (\underset{\sim}{\theta}_I^\epsilon - \underset{\sim}{\nabla} w_I^\epsilon, \underset{\sim}{\theta}_I^\epsilon - \underset{\sim}{\nabla} w_I^\epsilon)\mathrm{d}\underset{\sim}{x} \simeq 1.$$

However, the displacement field $\underline{u}_{II,\mathrm{RM}}^\epsilon$ constructed on the remaining part $(\underset{\sim}{\theta}_{II}^\epsilon, w_{II}^\epsilon)$ does not converge to $\underline{u}_{II,3\mathrm{D}}^\epsilon$ in the relative energy norm. Namely, we have

$$\frac{\|\underline{u}_{II,3\mathrm{D}}^\epsilon - \underline{u}_{II,\mathrm{RM}}^\epsilon\|_{E^\epsilon}}{\|\underline{u}_{II,3\mathrm{D}}^\epsilon\|_{E^\epsilon}} \simeq 1.$$

Furthermore, the difference between $\underline{u}_{II,3\mathrm{D}}^\epsilon$ and $\underline{u}_{II,\mathrm{RM}}^\epsilon$ is not confined in a boundary layer. If $\partial\Omega$ and q_s are sufficiently smooth, by using the method of [2] (in which only free boundary condition can be found), we obtain

$$(\underset{\sim}{\theta}_{II}^\epsilon, w_{II}^\epsilon) = \frac{1}{\epsilon}(\underset{\sim}{\nabla} w_1, w_1) + \left(-\frac{\sqrt{3}}{\mu\,\epsilon} q_s \mathrm{e}^{-\frac{\sqrt{3}}{\epsilon}\rho} \boldsymbol{\chi} \underset{\sim}{s}, 0\right) + O(\epsilon). \tag{4.22}$$

Here $w_1 \in H^2/\mathcal{L}$ is determined by

$$\begin{aligned} \Delta^2 w_1 = 0 \quad \text{in } \Omega,& \\ 2\mu\partial_{nn}^2 w_1 + \lambda^*\Delta w_1 = -\sqrt{3}\partial_s q_s \quad \text{on} \quad \partial\Omega,& \\ (2\mu+\lambda^*)\partial_n\Delta w_1 + 2\mu\partial_s(\partial_{sn}^2 w_1 - \kappa\partial_s w_1) = -\sqrt{3}\partial_s(\kappa q_s) \quad \text{on} \quad \partial\Omega.& \end{aligned} \tag{4.23}$$

Recall that (ρ, s) is the curvilinear coordinates defined on a strip γ^ϵ along $\partial\Omega$, and $\boldsymbol{\chi}$ is a cut-off function that is identically equal to zero away from $\partial\Omega$ by a distance $O(\epsilon)$ and equal to 1 on the boundary strip. Thus $(\underset{\sim}{\theta}_{II}^\epsilon, w_{II}^\epsilon)$ has a regular part and a boundary layer part. The regular part, given by $\frac{1}{\epsilon}(\underset{\sim}{\nabla} w_1, w_1)$, although has a $1/\epsilon$ magnitude, has a finite bending strain and has no transverse shear strain, and its total strain energy $E(\frac{1}{\epsilon}\underset{\sim}{\nabla} w_1, \frac{1}{\epsilon}w_1) \simeq 1$. The boundary layer part only has a non-zero rotational part that is tangential to the plate boundary, but it bears both bending and transverse shear strain energies that are equal in magnitude and its total strain energy is proportional to ϵ^{-1}. Therefore, the boundary layer part bears a dominant portion of the strain energy. From the equation (4.23), we see that $(\underset{\sim}{\theta}_{II}^\epsilon, w_{II}^\epsilon)$ is a boundary layer only in two cases. One is that $\partial\Omega$ is straight and q_s is constant, which is the example above. Another is that $\partial\Omega$ is a circular and d_s is constant.

For the elasticity solution $\underline{u}_{II,3\mathrm{D}}^\epsilon$, we have the following asymptotic expansion. In this case, using the method of [9], [8], and [6] we can prove that

$$\underline{u}_{B3\mathrm{D}}^\epsilon = \frac{1}{\epsilon}(-x_3\underset{\sim}{\nabla} W_1, W_1) + \underline{v}\left(\frac{\rho}{\epsilon}, s, \frac{x_3}{\epsilon}\right)\boldsymbol{\chi} + O(\epsilon). \tag{4.24}$$

Here W_1 is a biharmonic function on Ω, but satisfying a boundary condition different from that of w_1 in (4.23). The function $\underline{v}$ is a boundary layer, similar to that in (4.22), but with a different profile on $\partial\Omega$. Therefore, the difference between $\underline{u}^{\epsilon}_{II,3\mathrm{D}}$ and $\underline{u}^{\epsilon}_{II,\mathrm{RM}}$ is not only in a boundary layer, but throughout Ω. But note that in $\underline{u}^{\epsilon}_{II,\mathrm{RM}}$ there is not interior transverse shear. And in $\underline{u}^{\epsilon}_{II,3\mathrm{D}}$ the transverse shear is a higher order term. The transverse shear effect went to $\underline{u}^{\epsilon}_{I,\mathrm{RM}}$ and $\underline{u}^{\epsilon}_{I,3\mathrm{D}}$. The conclusion is that in the intermediate case the Reissner–Mindlin model is not asymptotically correct. It fails to capture the interior bending strain and both bending and shear strains in the boundary layer, but it accurately captures the shear strain in the interior of the domain.

There is a possibility to enhance the interior accuracy of the Reissner–Mindlin model. The function W_1 that dictates the interior bending in the elasticity solution is a biharmonic function. It satisfies boundary conditions that are explicitly expressible in terms of q_s [9] (which is different from that in (4.23)). So in principle we can compute W_1 thus to capture in the interior bending. Together with $(\underset{\sim}{\theta}^{\epsilon}_{I}, w^{\epsilon}_{I})$ that captures the interior shear deformation, we would obtain an interior accurate approximation to $\underline{u}^{\epsilon}_{3\mathrm{D}}$ in the intermediate case.

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