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DYNAMIC FRICTIONAL CONTACT OF A VISCOELASTIC BEAM *

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Abstract. In this paper, we study the dynamic frictional contact of a viscoelastic beam with a deformable obstacle. The beam is assumed to be situated horizontally and to move, in both horizontal and tangential directions, by the effect of applied forces. The left end of the beam is clamped and the right one is free. Its horizontal displacement is constrained because of the presence of a deformable obstacle, the so-called foundation, which is modelled by a normal compliance contact condition. The effect of the friction is included in the vertical motion of the free end, by using Tresca's law or Coulomb's law. In both cases, the variational formulation leads to a nonlinear variational equation for the horizontal displacement coupled with a nonlinear variational inequality for the vertical displacement. We recall an existence and uniqueness result. Then, by using the finite element method to approximate the spatial variable and an Euler scheme to discretize the time derivatives, a numerical scheme is proposed. Error estimates on the approximative solutions are derived. Numerical results demonstrate the application of the proposed algorithm.

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Introduction

Contact problems involving viscoplastic or viscoelastic materials appear in many industrial problems and everyday life (see, *e.g.*, the monographs[18, 19, 24] and references therein). Recently, one-dimensional contact problems for beams and rods have been studied[1, 3, 12, 17, 20, 21], including the adhesion[15], the wear[11, 22] or the damage[4]. Moreover, in[2] the adhesive contact problem of a membrane was studied.

In this work, we describe a model for the dynamic frictional contact of a viscoelastic beam with a deformable obstacle, the so-called foundation. The model was introduced in[1], where the existence of a unique weak solution was proved and its regularity was studied. That work was mainly focused on the mathematical aspects provided by the elastic case, that is, when the damping coefficient associated to the viscous effect is neglected.

Keywords and phrases. Dynamic unilateral contact, friction, viscoelastic beam, error estimates, numerical simulations.

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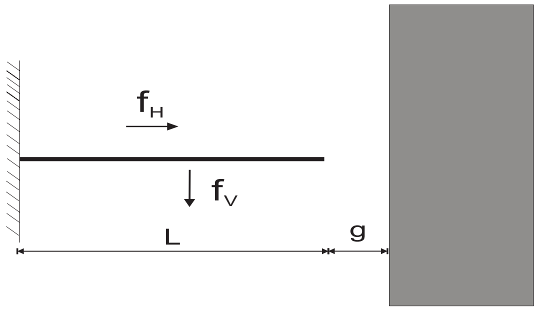

FIGURE 1. A viscoelastic beam in frictional contact.

Here, our aim is to study a numerical scheme for the viscoelastic case, based on the finite element method to approximate the spatial variable and an Euler scheme to discretize the time derivatives. Moreover, in order to show the accuracy of the proposed algorithm, some numerical simulations are presented.

1. The model and variational formulation

In this section, we follow[1] and we describe the model for the dynamic evolution of a viscoelastic beam in frictional contact with a deformable obstacle. The beam is assumed to be rigidly clamped at its left end, and its right end is free to come into frictional contact with an obstacle situated at a distance g to the right (see Fig. 1).

Let $[0, L]$, $L > 0$, be the reference configuration of the beam and let us denote by $[0, T]$, $T > 0$, the time interval of interest. The horizontal and vertical displacements of the beam are represented by $u(x, t)$ and $\tilde{u}(x, t)$, $(x, t) \in (0, L) \times (0, T)$, respectively.

The material is assumed viscoelastic and satisfying the Kelvin-Voigt viscoelastic laws[10],

$$\sigma_H(x, t) = a_1 u_x(x, t) + c_1 u_{xt}(x, t), \quad (x, t) \in (0, L) \times (0, T), \tag{1}$$

$$\sigma_V(x, t) = -(a_2 \tilde{u}_{xxx}(x, t) + c_2 \tilde{u}_{xxxt}(x, t)), \quad (x, t) \in (0, L) \times (0, T), \tag{2}$$

where, as usual, a subscript denotes a partial derivative with respect to the corresponding variable. Here, a_1, $a_2 > 0$ are elastic constants, and c_1, $c_2 > 0$ represent the viscosities. We notice that these equations are decoupled because of the assumptions on the symmetry of the cross-section of the beam and the isotropy of its material.

Since we assume that the inertia effect is not negligible, the equations of motion have the following form,

$$\rho u_{tt}(x, t) - (\sigma_H)_x(x, t) = f_H(x, t), \qquad (x, t) \in (0, L) \times (0, T), \tag{3}$$

$$\rho \tilde{u}_{tt}(x, t) - (\sigma_V)_x(x, t) = f_V(x, t), \qquad (x, t) \in (0, L) \times (0, T), \tag{4}$$

where f_H and f_V denote the horizontal and vertical applied forces, respectively. For simplicity, the density ρ is assumed to be equal to 1 and

$$f_H \in W^{1,2}(0, T; L^2(0, L)), \quad f_V \in \mathcal{C}([0, T]; L^2(0, L)). \tag{5}$$

Let u_0, v_0, $\tilde{u}_0$, $\tilde{v}_0$ be the initial conditions, that is,

$$u(x, 0) = u_0(x), \quad u_t(x, 0) = v_0(x), \quad x \in (0, L), \tag{6}$$

$$\tilde{u}(x, 0) = \tilde{u}_0(x), \quad \tilde{u}_t(x, 0) = \tilde{v}_0(x), \quad x \in (0, L). \tag{7}$$

To complete the model, we need to prescribe the boundary conditions. First, since the left end of the beam was supposed rigidly attached, we have

$$u(0,t) = 0, \quad t \in [0,T], \tag{8}$$
$$\tilde{u}(0,t) = \tilde{u}_x(0,t) = 0, \quad t \in [0,T]. \tag{9}$$

Secondly, the beam is assumed to be located at a distance g from the deformable obstacle. According to[16], the following normal compliance contact condition is considered,

$$-\sigma_H(L,t) = c_H(u(L,t) - g)_+, \quad t \in [0,T], \tag{10}$$

where $(u(L,t) - g)_+ = \max\{0, u(L,t) - g\}$ and $c_H > 0$ is a deformability coefficient (*i.e.* a hard spring or a penalty factor which is used to enforce the nonpenetration unilateral constraint).

Finally, concerning the vertical displacements, we assume that the sum of the moments on the free end is zero, *i.e.*

$$a_2\tilde{u}_{xx}(L,t) + c_2\tilde{u}_{xxt}(L,t) = 0, \quad t \in [0,T], \tag{11}$$

and also a friction condition of the form, for a.e. $t \in [0,T]$,

$$\left.\begin{array}{l} |\sigma_V(L,t)| \leq h(t), \\ \text{if } |\sigma_V(L,t)| = h(t) \Rightarrow \exists\lambda \geq 0\,;\, \tilde{u}_t(L,t) = -\lambda\sigma_V(L,t), \\ \text{if } |\sigma_V(L,t)| < h(t) \Rightarrow \tilde{u}_t(L,t) = 0. \end{array}\right\} \tag{12}$$

In (12), function $h(t)$ represents a friction bound. We will consider two different cases:

(i) $h(t) = c_V =$ constant: it corresponds to the classical Tresca's conditions.
(ii) $h(t) = c_V(u(L,t) - g)_+$, for $c_V =$ constant > 0: it leads to a particular case of Coulomb's conditions.

Remark 1.1. We notice that the case (i) results from the case (ii), when the penetration $u(L,t) - g$ is assumed to be constant. Therefore, only the case (ii) will be considered in the rest of the paper.

The dynamic frictional contact problem of a viscoelastic beam with a deformable obstacle is written as follows.

Find a horizontal displacement $u : [0,L] \times [0,T] \to \mathbb{R}$ *and a vertical displacement* $\tilde{u} : [0,L] \times [0,T] \to \mathbb{R}$ *which satisfy (1)–(12).*

In order to derive a weak formulation of the above problem, we define the following variational spaces:

$$\begin{array}{l} H = L^2(0,L), \\ E = \{w \in H^1(0,L)\,;\, w(0) = 0\}, \\ V = \{z \in H^2(0,L)\,;\, z(0) = z_x(0) = 0\}. \end{array}$$

Moreover, we denote by $(\cdot,\cdot)$ the classical inner product defined in $L^2(0,L)$ and, for a Hilbert space X, let $|\cdot|_X$ represent its norm. If X' denotes the dual space of X, then let $\langle\cdot,\cdot\rangle_{X'X}$ be the duality pairing between X' and X.

Let $j_H(u,\cdot) : E \to \mathbb{R}$ and $j_V(u,\cdot) : V \to \mathbb{R}$ be the normal compliance and friction forms defined by

$$\begin{array}{l} j_H(u,w) = c_H(u(L,t) - g)_+ w(L,t), \quad \forall w \in E, \\ j_V(u,z) = c_V(u(L,t) - g)_+ |z(L,t)|, \quad \forall z \in V. \end{array}$$

Integrating by parts the equations of motion (3)–(4) and using the previously given boundary conditions, the variational formulation is then written as follows.

Variational problem VP. *Find the horizontal displacement* $u : [0,T] \to E$ *and the vertical displacement* $\tilde{u} : [0,T] \to V$ *such that* $u(0) = u_0$, $u_t(0) = v_0$, $\tilde{u}(0) = \tilde{u}_0$, $\tilde{u}_t(0) = \tilde{v}_0$ *and for a.e.* $t \in (0,T)$,

$$\langle u_{tt}(t), w\rangle_{E'E} + a_1(u_x(t), w_x) + c_1(u_{xt}(t), w_x) + j_H(u(t), w) = (f_H(t), w), \quad \forall w \in E, \tag{13}$$

$$\langle \tilde{u}_{tt}(t), (z - \tilde{u}_t(t))\rangle_{V'V} + (a_2\tilde{u}_{xx}(t) + c_2\tilde{u}_{xxt}(t), (z - \tilde{u}_t(t))_{xx}) \\ + j_V(u(t), z) - j_V(u(t), \tilde{u}_t(t)) \geq (f_V(t), (z - \tilde{u}_t(t))), \quad \forall z \in V. \tag{14}$$

The existence of a unique solution to problem VP was proved in[1]. We recall the main result in the following.

Theorem 1.2. *Let the assumption (5) hold and assume that* $u_0 \in E$, $\tilde{u}_0 \in V$ *and* $v_0, \tilde{v}_0 \in H$. *Then, there exists a unique solution* $\{u, \tilde{u}\}$ *to problem VP with* $u \in W^{1,2}(0,T;E)$, $u_{tt} \in L^2(0,T;E')$, $\tilde{u} \in W^{1,2}(0,T;V)$ *and* $\tilde{u}_{tt} \in L^2(0,T;V')$.

2. Numerical approximation

In this section we describe a fully discrete scheme for the variational problem VP. We discretize it in two steps, both the spatial and time variables. We introduce a uniform partition of the time interval with the step size $k = T/N$ and the nodes $t_n = nk$, $n = 1, \ldots, N$. To discretize the spatial variable, we consider a uniform partition of $[0,L]$, denoted by $\{I_i\}_{i=1}^M$, in such a way that $[0,L] = \cup_{i=1}^M I_i$. We let h denote the size of the partition. Let V^h and E^h be the following finite element spaces approximating V and E,

$$E^h = \{w^h \in E \ ; \ w^h_{|_{I_i}} \in P_1(I_i), \quad 1 \leq i \leq M\},$$
$$V^h = \{z^h \in V \ ; \ z^h_{|_{I_i}} \in P_3(I_i), \quad 1 \leq i \leq M\},$$

where $P_q(I_i)$ denotes the polynomial space of degree less or equal to q restricted to I_i. We note that, since $z^h \in V \subset H^2(0,L)$, $z^h \in \mathcal{C}^1([0,L])$ and then V^h is composed of $\mathcal{C}^1$ piecewise cubic functions, while E^h is made of continuous piecewise affine functions.

We use the notation $z_n = z(t_n)$ for a continuous function $z(t)$, and for a sequence $\{z_n\}_{n=0}^N$ we denote by $\delta z_n = (z_n - z_{n-1})/k$ the divided differences. Moreover, in this section no summation is assumed over a repeated index and c denotes a generic constant which does not depend on k, h or n.

For convenience, we will consider our variational problem in terms of the velocity fields $v(t) = \dot{u}(t)$, $\tilde{v}(t) = \dot{\tilde{u}}(t)$ given by

$$u(t) = \int_0^t v(s)\mathrm{d}s + u_0, \quad \tilde{u}(t) = \int_0^t \tilde{v}(s)\mathrm{d}s + \tilde{u}_0. \tag{15}$$

Then, using an Euler scheme to discretize the time derivatives, we introduce the following fully discrete approximation of problem VP.

Fully discrete problem VP^{hk}. *Find* $v^{hk} = \{v_n^{hk}\}_{n=0}^N \subset E^h$ *and* $\tilde{v}^{hk} = \{\tilde{v}_n^{hk}\}_{n=0}^N \subset V^h$, *such that* $u_0^{hk} = u_0^h$, $v_0^{hk} = v_0^h$, $\tilde{u}_0^{hk} = \tilde{u}_0^h$, $\tilde{v}_0^{hk} = \tilde{v}_0^h$ *and, for* $n = 1, \ldots, N$,

$$(\delta v_n^{hk}, w^h) + a_1((u_n^{hk})_x, w_x^h) + c_1((v_n^{hk})_x, w_x^h) + j_H(u_{n-1}^{hk}, w^h) = ((f_H)_n, w^h), \quad \forall w^h \in E^h, \tag{16}$$

$$(\delta \tilde{v}_n^{hk}, z^h - \tilde{v}_n^{hk}) + a_2((\tilde{u}_n^{hk})_{xx}, (z^h - \tilde{v}_n^{hk})_{xx}) + c_2((\tilde{v}_n^{hk})_{xx}, (z^h - \tilde{v}_n^{hk})_{xx}) \\ + j_V(u_n^{hk}, z^h) - j_V(u_n^{hk}, \tilde{v}_n^{hk}) \geq ((f_V)_n, z^h - \tilde{v}_n^{hk}), \quad \forall z^h \in V^h, \tag{17}$$

where u_0^h, v_0^h, $\tilde{u}_0^h$ *and* $\tilde{v}_0^h$ *are appropriate approximations of the initial conditions* u_0, v_0, $\tilde{u}_0$ *and* $\tilde{v}_0$, *respectively. Moreover,* $u^{hk} = \{u_n^{hk}\}_{n=0}^N$ *and* $\tilde{u}^{hk} = \{\tilde{u}_n^{hk}\}_{n=0}^N$ *denote the displacement fields defined by*

$$u_n^{hk} = u_{n-1}^{hk} + kv_n^{hk}, \quad \tilde{u}_n^{hk} = \tilde{u}_{n-1}^{hk} + k\tilde{v}_n^{hk}, \quad n = 1, \ldots, N.$$

By induction, using classical results on variational inequalities (see[13]), we obtain that problem VP^{hk} admits a unique solution.

Our interest lies in estimating the errors $u_n - u_n^{hk}$ and $\tilde{u}_n - \tilde{u}_n^{hk}$. To that end, we make the following assumptions on the regularity of the solution,

$$u \in \mathcal{C}^1([0,T];E), \quad \ddot{u} \in \mathcal{C}([0,T];H), \tag{18}$$

$$\tilde{u} \in \mathcal{C}^1([0,T];V), \quad \ddot{\tilde{u}} \in \mathcal{C}([0,T];H). \tag{19}$$

The numerical analysis corresponding to the horizontal problem was done in [4], where the damage of the material and the thermal effects were also taken into account. Using the notation

$$I_j = \Big| \int_0^{t_j} v(s) \mathrm{d}s - \sum_{l=1}^{j} k v_l \Big|_E,$$

the following error estimate was obtained.

Theorem 2.1. *Let the assumptions of Theorem 1.2 hold. Under the regularity condition (18), the following error estimates are obtained for $n = 0, \dots, N$,*

$$|v_n - v_n^{hk}|_H^2 + \sum_{j=1}^{n} k |v_j - v_j^{hk}|_E^2 \leq c \sum_{j=1}^{n} k \left(|\dot{v}_j - \delta v_j|_H^2 + I_j^2 + \sum_{l=1}^{j-1} k |v_l - v_l^{hk}|_E^2 + |u_j - u_{j-1}|_E^2 + |v_j - w_j^h|_E^2 \right)$$
$$+ c \left(|u_1 - u_1^h|_H^2 + |u_0 - u_0^h|_E^2 + |v_1 - w_1^h|_H^2 + |v_n - w_n^h|_H^2 \right) + c k^{-1} \sum_{j=1}^{n-1} |v_j - w_j^h - \left(v_{j+1} - w_{j+1}^h \right)|_H^2,$$
$$\forall \{w_j^h\}_{j=0}^n \subset V^h. \tag{20}$$

Next, we let $z = \tilde{v}_n^{hk}$ in (14) at time $t = t_n$ to obtain, after a simple rearrangement,

$$(\dot{\tilde{v}}_n, \tilde{v}_n - \tilde{v}_n^{hk}) + c_2((\tilde{v}_n)_{xx}, (\tilde{v}_n - \tilde{v}_n^{hk})_{xx}) + a_2((\tilde{u}_n)_{xx}, (\tilde{v}_n - \tilde{v}_n^{hk})_{xx}) \leq$$
$$j_V(u_n, \tilde{v}_n^{hk}) - j_V(u_n, \tilde{v}_n) + ((f_V)_n, \tilde{v}_n - \tilde{v}_n^{hk}),$$

and the discrete variational inequality (17) is rewritten as follows,

$$- (\delta \tilde{v}_n^{hk}, \tilde{v}_n - \tilde{v}_n^{hk}) - c_2((\tilde{v}_n^{hk})_{xx}, (\tilde{v}_n - \tilde{v}_n^{hk})_{xx}) - a_2((\tilde{u}_n^{hk})_{xx}, (\tilde{v}_n - \tilde{v}_n^{hk})_{xx}) \leq$$
$$j_V(u_n^{hk}, z_n^h) - j_V(u_n^{hk}, \tilde{v}_n^{hk}) - ((f_V)_n, z_n^h - \tilde{v}_n^{hk}) - (\delta \tilde{v}_n^{hk}, \tilde{v}_n - z_n^h)$$
$$- c_2((\tilde{v}_n^{hk})_{xx}, (\tilde{v}_n - z_n^h)_{xx}) - a_2((\tilde{u}_n^{hk})_{xx}, (\tilde{v}_n - z_n^h)_{xx}),$$

for all $z_n^h \in V^h$. Adding these two inequalities, we obtain, for all $z_n^h \in V^h$,

$$(\dot{\tilde{v}}_n - \delta \tilde{v}_n^{hk}, \tilde{v}_n - \tilde{v}_n^{hk}) + c_2((\tilde{v}_n - \tilde{v}_n^{hk})_{xx}, (\tilde{v}_n - \tilde{v}_n^{hk})_{xx}) + a_2((\tilde{u}_n - \tilde{u}_n^{hk})_{xx}, (\tilde{v}_n - \tilde{v}_n^{hk})_{xx}) \leq$$
$$j_V(u_n, \tilde{v}_n^{hk}) - j_V(u_n, \tilde{v}_n) + j_V(u_n^{hk}, z_n^h) - j_V(u_n^{hk}, \tilde{v}_n^{hk}) - (\delta \tilde{v}_n^{hk}, \tilde{v}_n - z_n^h) + ((f_V)_n, \tilde{v}_n - z_n^h)$$
$$- c_2((\tilde{v}_n^{hk})_{xx}, (\tilde{v}_n - z_n^h)_{xx}) - a_2((\tilde{u}_n^{hk})_{xx}, (\tilde{v}_n - z_n^h)_{xx}).$$

Now, let us define

$$R(u_n, \tilde{u}_n, z_n^h) = (\dot{\tilde{v}}_n, z_n^h - \tilde{v}_n) + c_2((\tilde{v}_n)_{xx}, (z_n^h - \tilde{v}_n)_{xx})$$
$$+ a_2((\tilde{u}_n)_{xx}, (z_n^h - \tilde{v}_n)_{xx}) + j_V(u_n, z_n^h) - j_V(u_n, \tilde{v}_n) - ((f_V)_n, z_n^h - \tilde{v}_n).$$

Writing $\dot{\tilde{v}}_n = \delta\tilde{v}_n - (\delta\tilde{v}_n - \dot{\tilde{v}}_n)$, after easy manipulations we get

$$\begin{aligned}(\delta\tilde{v}_n - \delta\tilde{v}_n^{hk}, \tilde{v}_n - \tilde{v}_n^{hk}) &+ c_2((\tilde{v}_n - \tilde{v}_n^{hk})_{xx}, (\tilde{v}_n - \tilde{v}_n^{hk})_{xx}) \\ &\leq j_V(u_n, \tilde{v}_n^{hk}) - j_V(u_n, z_n^h) + j_V(u_n^{hk}, z_n^h) - j_V(u_n^{hk}, \tilde{v}_n^{hk}) \\ &+ c_2((\tilde{v}_n - \tilde{v}_n^{hk})_{xx}, (\tilde{v}_n - z_n^h)_{xx}) + a_2((\tilde{u}_n - \tilde{u}_n^{hk})_{xx}, (\tilde{v}_n^{hk} - z_n^h)_{xx}) \\ &+ (\dot{\tilde{v}}_n - \delta\tilde{v}_n, \tilde{v}_n^{hk} - z_n^h) + R(u_n, \tilde{u}_n, z_n^h) + (\delta\tilde{v}_n - \delta\tilde{v}_n^{hk}, \tilde{v}_n - z_n^h).\end{aligned}$$

From the definition of j_V, it follows that

$$j_V(u_n, \tilde{v}_n^{hk}) - j_V(u_n, z_n^h) + j_V(u_n^{hk}, z_n^h) - j_V(u_n^{hk}, \tilde{v}_n^{hk}) \leq c\,|u_n - u_n^{hk}|_E(|\tilde{v}_n - \tilde{v}_n^{hk}|_V + |\tilde{v}_n - z_n^h|_V).$$

Since

$$(\delta\tilde{v}_n - \delta\tilde{v}_n^{hk}, \tilde{v}_n - \tilde{v}_n^{hk}) \geq \frac{1}{2k}\Big[|\tilde{v}_n - \tilde{v}_n^{hk}|_H^2 - |\tilde{v}_{n-1} - \tilde{v}_{n-1}^{hk}|_H^2\Big],$$

using the Cauchy's inequality $ab \leq \epsilon a^2 + \frac{1}{4\epsilon}b^2$ for a, b, $\epsilon > 0$, the following estimate is obtained,

$$\begin{aligned}|\tilde{v}_n - \tilde{v}_n^{hk}|_H^2 + ck|\tilde{v}_n - \tilde{v}_n^{hk}|_V^2 \leq ck\Big[&|u_n - u_n^{hk}|_E^2 + |\dot{\tilde{v}}_n - \delta\tilde{v}_n|_H^2 + |\tilde{v}_n - z_n^h|_V^2 \\ &+ |\tilde{u}_n - \tilde{u}_n^{hk}|_V^2 + (\delta\tilde{v}_n - \delta\tilde{v}_n^{hk}, \tilde{v}_n - z_n^h) + |R(u_n, \tilde{u}_n, z_n^h)|\Big] + |\tilde{v}_{n-1} - \tilde{v}_{n-1}^{hk}|_H^2.\end{aligned}$$

Then, an induction argument leads to

$$\begin{aligned}|\tilde{v}_n - \tilde{v}_n^{hk}|_H^2 + k\sum_{j=1}^n |\tilde{v}_j - \tilde{v}_j^{hk}|_V^2 \leq ck\sum_{j=1}^n \Big[&|u_j - u_j^{hk}|_E^2 + |\dot{\tilde{v}}_j - \delta\tilde{v}_j|_H^2 + |\tilde{v}_j - z_j^h|_V^2 \\ &+ |\tilde{u}_j - \tilde{u}_j^{hk}|_V^2 + (\delta\tilde{v}_j - \delta\tilde{v}_j^{hk}, \tilde{v}_j - z_j^h) + |R(u_j, \tilde{u}_j, z_j^h)|\Big] + c\,|\tilde{v}_0 - \tilde{v}_0^h|_H^2.\end{aligned}$$

Keeping in mind that

$$\sum_{j=1}^n k(\delta\tilde{v}_j - \delta\tilde{v}_j^{hk}, \tilde{v}_j - z_j^h) = (\tilde{v}_n - \tilde{v}_n^{hk}, \tilde{v}_n - z_n^h) + (\tilde{v}_0^h - \tilde{v}_0, \tilde{v}_1 - z_1^h) + \sum_{j=1}^{n-1}(\tilde{v}_j - \tilde{v}_j^{hk}, \tilde{v}_j - z_j^h - (\tilde{v}_{j+1} - z_{j+1}^h)),$$

and

$$|\tilde{u}_j - \tilde{u}_j^{hk}|_V^2 \leq c\,(|\tilde{u}_0 - \tilde{u}_0^h|_V^2 + \tilde{I}_j^2 + \sum_{l=1}^j k|\tilde{v}_l - \tilde{v}_l^{hk}|_V^2), \tag{21}$$

$$|u_j - u_j^{hk}|_E^2 \leq c\,(|u_0 - u_0^h|_E^2 + I_j^2 + \sum_{l=1}^j k|v_l - v_l^{hk}|_E^2), \tag{22}$$

where $\tilde{I}_j = \Big| \int_0^{t_j} \tilde{v}(s)\mathrm{d}s - \sum_{l=1}^{j} k\tilde{v}_l \Big|_V$ is the integration error, the following estimate is obtained:

$$|\tilde{v}_n - \tilde{v}_n^{hk}|_H^2 + k\sum_{j=1}^{n} |\tilde{v}_j - \tilde{v}_j^{hk}|_V^2 \leq ck\sum_{j=1}^{n}\Bigg[\sum_{k=1}^{l} k|v_l - v_l^{hk}|_E^2 + I_j^2 + |\dot{\tilde{v}}_j - \delta\tilde{v}_j|_H^2$$
$$+|\tilde{v}_j - z_j^h|_V^2 + \tilde{I}_j^2 + \sum_{l=1}^{j} k|\tilde{v}_l - \tilde{v}_l^{hk}|_V^2 + |R\left(u_j, \tilde{u}_j, z_j^h\right)|\Bigg] + c\,|\tilde{u}_0 - \tilde{u}_0^h|_V^2 + c\,|\tilde{v}_0 - \tilde{v}_0^h|_H^2$$
$$+ c\,|\tilde{v}_1 - z_1^h|_H^2 + c\,|\tilde{v}_n - z_n^h|_H^2 + c\,|u_0 - u_0^h|_E^2 + ck^{-1}\sum_{j=1}^{n-1} |\tilde{v}_j - z_j^h - \left(\tilde{v}_{j+1} - z_{j+1}^h\right)|_H^2. \quad (23)$$

Adding now (20) and (23), we get

$$|v_n - v_n^{hk}|_H^2 + k\sum_{j=1}^{n} |v_j - v_j^{hk}|_E^2 + |\tilde{v}_n - \tilde{v}_n^{hk}|_H^2 + k\sum_{j=1}^{n} |\tilde{v}_j - \tilde{v}_j^{hk}|_V^2$$
$$\leq c\Big\{\sum_{j=1}^{n} k\Big[|\dot{\tilde{v}}_j - \delta\tilde{v}_j|_H^2 + |\dot{v}_j - \delta v_j|_H^2 + |u_j - u_{j-1}|_E^2 + |v_j - w_j^h|_E^2$$
$$+ |\tilde{v}_j - z_j^h|_V^2 + \tilde{I}_j^2 + I_j^2 + \sum_{l=1}^{j} k|\tilde{v}_l - \tilde{v}_l^{hk}|_V^2 + \sum_{l=1}^{j} k|v_l - v_l^{hk}|_E^2$$
$$+ |R(u_j, \tilde{u}_j, z_j^h)|\Big] + |v_0 - v_0^h|_H^2 + |\tilde{v}_0 - \tilde{v}_0^h|_H^2 + |u_0 - u_0^h|_E^2 + |\tilde{u}_0 - \tilde{u}_0^h|_V^2$$
$$+ |v_1 - w_1^h|_H^2 + |\tilde{v}_1 - z_1^h|_H^2 + |v_n - w_n^h|_H^2 + |\tilde{v}_n - z_n^h|_H^2$$
$$+ k^{-1}\sum_{j=1}^{n-1} |v_j - w_j^h - (v_{j+1} - w_{j+1}^h)|_H^2 + k^{-1}\sum_{j=1}^{n-1} |\tilde{v}_j - z_j^h - (\tilde{v}_{j+1} - z_{j+1}^h)|_H^2\Big\}. \quad (24)$$

Let us define the following quantities for $n = 1, \ldots, N$,

$$e_n = |v_n - v_n^{hk}|_H^2 + k\sum_{j=1}^{n} |v_j - v_j^{hk}|_E^2 + |\tilde{v}_n - \tilde{v}_n^{hk}|_H^2 + k\sum_{j=1}^{n} |\tilde{v}_j - \tilde{v}_j^{hk}|_V^2,$$

$$g_n = \sum_{j=1}^{n} k\Big[|\dot{\tilde{v}}_j - \delta\tilde{v}_j|_H^2 + |\dot{v}_j - \delta v_j|_H^2 + |u_j - u_{j-1}|_E^2 + |v_j - w_j^h|_E^2$$
$$+ |\tilde{v}_j - z_j^h|_V^2 + \tilde{I}_j^2 + I_j^2 + |R\left(u_j, \tilde{u}_j, z_j^h\right)|\Big] + |v_0 - v_0^h|_H^2 + |\tilde{v}_0 - \tilde{v}_0^h|_H^2$$
$$+ |u_0 - u_0^h|_E^2 + |\tilde{u}_0 - \tilde{u}_0^h|_V^2 + |v_1 - w_1^h|_H^2 + |\tilde{v}_1 - z_1^h|_H^2$$
$$+ |v_n - w_n^h|_H^2 + |\tilde{v}_n - z_n^h|_H^2 + k^{-1}\sum_{j=1}^{n-1} |v_j - w_j^h - \left(v_{j+1} - w_{j+1}^h\right)|_H^2$$
$$+ k^{-1}\sum_{j=1}^{n-1} |\tilde{v}_j - z_j^h - \left(\tilde{v}_{j+1} - z_{j+1}^h\right)|_H^2,$$

and

$$e_0 = g_0 = |v_0 - v_0^h|_H^2 + |\tilde{v}_0 - \tilde{v}_0^h|_H^2.$$

Therefore, estimate (24) implies that

$$e_0 \leq g_0 \quad \text{and} \quad e_n \leq c g_n + c \sum_{j=1}^{n} k_j\, e_j, \quad n = 1, \dots, N,$$

where $k_j = k$, $j = 1, \dots, N$. Then, we use the following discrete version of the Gronwall's inequality (see[14]).

Lemma 2.2. *Assume that* $\{g_n\}_{n=0}^N$, $\{e_n\}_{n=0}^N$ *and* $\{k_n\}_{n=1}^N$ *are three sequences of nonnegative numbers satisfying*

$$\begin{aligned} & e_0 \leq g_0, \\ & e_n \leq c g_n + c \sum_{j=1}^{n} k_j\, e_j, \quad n = 1, \dots, N, \end{aligned}$$

where c *is a positive constant. Then, there exists a positive constant* $d > 0$ *such that*

$$\max_{0 \leq n \leq N} e_n \leq d \max_{0 \leq n \leq N} g_n.$$

Applying Lemma 2.2, we obtain the following result.

Theorem 2.3. *Let the assumptions of Theorem 1.2 hold true. Let us assume the additional regularity conditions* (18)–(19). *Then, the following error estimates are obtained for all* $\{w_j^h\}_{j=0}^N \subset E^h$, $\{z_j^h\}_{j=0}^N \subset V^h$,

$$\begin{aligned} & \max_{0 \leq n \leq N} \left\{ |v_n - v_n^{hk}|_H^2 + |\tilde{v}_n - \tilde{v}_n^{hk}|_H^2 \right\} + k \sum_{j=1}^{N} \left[|v_j - v_j^{hk}|_E^2 + |\tilde{v}_j - \tilde{v}_j^{hk}|_V^2 \right] \\ & \leq c \Big\{ \sum_{j=1}^{N} k \Big[|\dot{v}_j - \delta v_j|_H^2 + |\dot{\tilde{v}}_j - \delta \tilde{v}_j|_H^2 + |u_j - u_{j-1}|_E^2 + |v_j - w_j^h|_E^2 \\ & \quad + |\tilde{v}_j - z_j^h|_V^2 + \tilde{I}_j^2 + I_j^2 + |R(u_j, \tilde{u}_j, z_j^h)| \Big] + |v_0 - v_0^h|_H^2 + |\tilde{v}_0 - \tilde{v}_0^h|_H^2 \\ & \quad + k^{-1} \sum_{j=1}^{N-1} |v_j - w_j^h - (v_{j+1} - w_{j+1}^h)|_H^2 + k^{-1} \sum_{j=1}^{N-1} |\tilde{v}_j - z_j^h - (\tilde{v}_{j+1} - z_{j+1}^h)|_H^2 \\ & \quad + |u_0 - u_0^h|_E^2 + |\tilde{u}_0 - \tilde{u}_0^h|_V^2 + \max_{0 \leq n \leq N} |v_n - w_n^h|_H^2 + \max_{0 \leq n \leq N} |\tilde{v}_n - z_n^h|_H^2 \Big\}. \end{aligned} \tag{25}$$

The inequality (25) is the basis for deriving the convergence rate. Let Π^h and $\tilde{\Pi}^h$ denote the standard Lagrange and Hermite interpolation operators associated to the finite element spaces E^h or V^h (see[9]), respectively. Assume that the initial conditions u_0, v_0, $\tilde{u}_0$ and $\tilde{v}_0$ are such that

$$u_0 \in H^2(0, L), \quad v_0 \in H^1(0, L), \quad \tilde{u}_0 \in H^3(0, L), \quad \tilde{v}_0 \in H^1(0, L),$$

and define the discrete initial conditions by

$$u_0^h = \Pi^h u_0, \quad v_0^h = \Pi^h v_0, \quad \tilde{u}_0^h = \tilde{\Pi}^h \tilde{u}_0, \quad \tilde{v}_0^h = \Pi^h \tilde{v}_0.$$

Then, we have the first error estimate.

Corollary 2.4. *Let the assumptions of Theorem 1.2 hold true. Under the following additional regularity conditions*

$$u \in H^2(0,T;E) \cap \mathcal{C}^1([0,T];H^2(0,L)), \quad \dddot{u} \in L^2(0,T;H),$$
$$\tilde{u} \in H^2(0,T;V) \cap \mathcal{C}^1([0,T];H^3(0,L)), \quad \dddot{\tilde{u}} \in L^2(0,T;H),$$

we have the following error estimate,

$$\max_{0\le n\le N}\{|u_n - u_n^{hk}|_H + |\tilde{u}_n - \tilde{u}_n^{hk}|_H\} \le c(h^{1/2}+k). \tag{26}$$

Proof. We note that

$$k\sum_{j=1}^{N}\Big[|\dot{v}_j - \delta v_j^{hk}|_H^2 + |\dot{\tilde{v}}_j - \delta\tilde{v}_j^{hk}|_H^2\Big] \le ck^2\Big[|\dddot{u}|^2_{L^2(0,T;H)} + |\dddot{\tilde{u}}|^2_{L^2(0,T;H)}\Big],$$
$$k\sum_{j=1}^{N}|u_j - u_{j-1}|_E^2 \le ck^2|\dot{u}|^2_{L^2(0,T;E)},$$
$$\max_{1\le n\le N} I_n \le ck|u|_{H^2(0,T;E)}, \quad \max_{1\le n\le N}\tilde{I}_n \le ck|\tilde{u}|_{H^2(0,T;V)}.$$

We also have (see[7, 9] for details),

$$k^{-1}\sum_{j=1}^{N-1}|(v_{j+1} - w_{j+1}^h) - (v_j - w_j^h)|_H^2 \le ch^2|\dot{v}|^2_{L^2(0,T;E)},$$
$$k^{-1}\sum_{j=1}^{N-1}|(\tilde{v}_{j+1} - z_{j+1}^h) - (\tilde{v}_j - z_j^h)|_H^2 \le ch^2|\dot{\tilde{v}}|^2_{L^2(0,T;V)},$$
$$|R(u_j,\tilde{u}_j,z_j^h)| \le c|\tilde{v}_j - z_j^h|_V \le ch|\tilde{u}|_{C^1([0,T];H^3(0,L))}.$$

Using (21) and (22) it follows that

$$\max_{0\le n\le N}\{|u_n - u_n^{hk}|_E^2 + |\tilde{u}_n - \tilde{u}_n^{hk}|_V^2\} \le c\Big(|u_0 - u_0^h|_E^2 + |\tilde{u}_0 - \tilde{u}_0^h|_V^2$$
$$+ \max_{0\le n\le N}\{I_n^2 + \tilde{I}_n^2\} + \sum_{j=1}^{N}k[|v_j - v_j^{hk}|_E^2 + |\tilde{v}_j - \tilde{v}_j^{hk}|_V^2]\Big),$$

and the rest of the terms in (25) are bounded using the approximation properties of the finite element spaces E^h and V^h (see[9]).

Remark 2.5. We notice that (26) is just an example of an estimate which is based on the previous regularity assumptions. If we assume further regularity assumptions on the continuous solution, the linear convergence of the algorithm is achieved. For instance, if $\tilde{u} \in \mathcal{C}^1([0,T];\mathcal{C}^3([0,L]))$ (which implies that $\sigma_V \in \mathcal{C}([0,T];\mathcal{C}([0,L])))$, we have

$$|R(u_j,\tilde{u}_j,z_j^h)| \le c\,|\tilde{v}_j - z_j^h|_{H^1(0,L)} \le ch^2|\tilde{v}_j|_{H^3(0,L)},$$

and then,

$$\max_{0\le n\le N}\{|u_n - u_n^{hk}|_H + |\tilde{u}_n - \tilde{u}_n^{hk}|_H\} \le c\,(h+k).$$

3. Numerical results

In order to show the behaviour of the numerical scheme presented in the above section, some numerical experiments have been done. In this section we describe the algorithm employed to solve the fully discrete problem VP^{hk}, and we briefly present some numerical results which demonstrate the performance of the method.

3.1. Numerical resolution

First we have, for $n = 1, \ldots, N$, that the horizontal velocity is the unique solution of the following variational equation,

$$(\rho\delta v_n^{hk}, w^h) + a_1\, k((v_n^{hk})_x, w_x^h) + c_1((v_n^{hk})_x, w_x^h) + j_H(u_{n-1}^{hk}, w^h) = ((f_H)_n, w^h) - a_1((u_{n-1}^{hk})_x, w_x^h), \quad \forall w^h \in E^h. \quad (27)$$

Since problem (27) corresponds to a linear system, its resolution is done using Cholesky's method.

Secondly, the discrete vertical velocity is obtained from the solution of the following variational inequality,

$$(\rho\delta\tilde{v}_n^{hk}, z^h - \tilde{v}_n^{hk}) + a_2((\tilde{u}_n^{hk})_{xx}, (z^h - \tilde{v}_n^{hk})_{xx}) + c_2((\tilde{v}_n^{hk})_{xx}, (z^h - \tilde{v}_n^{hk})_{xx}) + j_V(u_n^{hk}, z^h) - j_V(u_n^{hk}, \tilde{v}_n^{hk}) \geq ((f_V)_n, z^h - \tilde{v}_n^{hk}), \quad \forall z^h \in V^h. \quad (28)$$

We note that problem (28) is a second kind variational inequality with the difficulty provided by the non-differentiability of the operator j_V. This can be solved using algorithms, for example, of Uzawa's type[8]. Nevertheless, we introduce here an efficient combination of a penalty-duality algorithm with a penalization over the friction condition. The penalized friction condition consists, for $0 < \mu$, in the following

$$-\sigma_V(L,t) = \Phi_\mu(\tilde{v}(L,t)), \text{ with } \quad \Phi_\mu(r) = \begin{cases} -h(t) & \text{if } \; r < -\mu, \\ \dfrac{h(t)}{\mu} r & \text{if } \; r \in [-\mu, \mu], \\ h(t) & \text{if } \; r > \mu. \end{cases} \quad (29)$$

For further smoothing techniques of the frictional problem the reader may see [23, 25].

Using (29) instead of (12), another second kind variational inequality is derived for the vertical velocity,

$$(\rho\delta\tilde{v}_\mu^{hk}, z^h - \tilde{v}_\mu^{hk}) + a_2((\tilde{u}_\mu^{hk})_{xx}, (z^h - \tilde{v}_\mu^{hk})_{xx}) + c_2((\tilde{v}_\mu^{hk})_{xx}, (z^h - \tilde{v}_\mu^{hk})_{xx}) + j_\mu(u_n^{hk}, z^h) - j_\mu(u_n^{hk}, \tilde{v}_\mu^{hk}) \geq ((f_V)_n, z^h - \tilde{v}_\mu^{hk}), \quad \forall z^h \in V^h, \quad (30)$$

where $j_\mu(u_n^{hk}, \cdot) : V \to \mathbb{R}$ is a differentiable functional defined by

$$j_\mu(u_n^{hk}, v) = \begin{cases} -h(u_n^{hk})v - h(u_n^{hk})\dfrac{\mu}{2} & \text{if } \; v < -\mu, \\ \dfrac{h(u_n^{hk})}{2\mu} v^2 & \text{if } \; v \in [-\mu, \mu], \\ h(u_n^{hk})v - h(u_n^{hk})\dfrac{\mu}{2} & \text{if } \; v > \mu, \end{cases}$$

where either $h(u_n^{hk}) = \mathit{constant}$ (Tresca's law) or $h(u_n^{hk}) = c_V(u_n^{hk} - g)_+$ (Coulomb's law).

Problem (30) is solved using the penalty-duality algorithm introduced in[5]. Moreover, in[6] it was proved that

$$|\tilde{v}_n^{hk} - \tilde{v}_\mu^{hk}|_V \leq c\mu(h + k + |\tilde{v}|_{C([0,T];V)}).$$

TABLE 1. Example 1: numerical errors for some n and k (stick case).

$n \downarrow k \rightarrow$	0.01	0.005	0.002	0.001	0.0005	0.0001
100	5.042e-3	5.042e-3	5.042e-3	5.042e-3	5.042e-3	5.042e-3
200	2.856e-3	2.856e-3	2.856e-3	2.856e-3	2.856e-3	2.856e-3
400	1.699e-3	1.699e-3	1.699e-3	1.699e-3	1.699e-3	1.699e-3
800	1.008e-3	1.008e-3	1.008e-3	1.008e-3	1.008e-3	1.008e-3
1600	5.639e-4	5.637e-4	5.637e-4	5.636e-4	5.636e-4	5.636e-4
3200	2.028e-4	2.015e-4	2.008e-4	2.007e-4	2.005e-4	2.003e-4

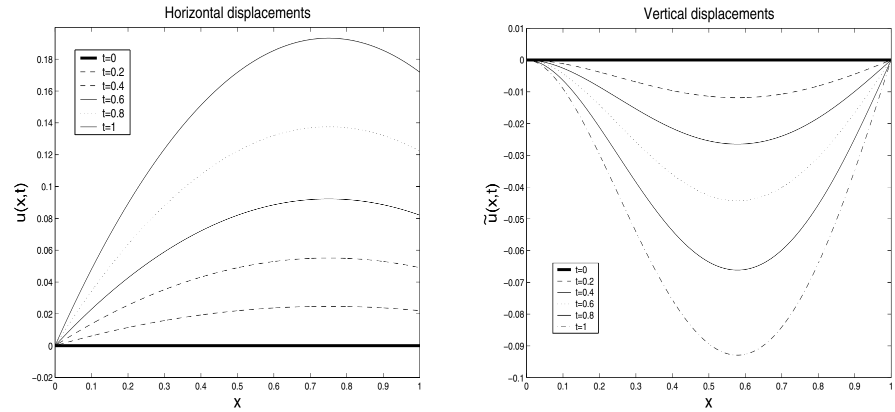

FIGURE 2. Example 1: horizontal and vertical displacements at different instants.

3.2. **Numerical simulations**

3.2.1. *First example: stick case*

As a first example, we have considered a beam of length $L = 1$ m with its left end rigidly attached, and no gap is assumed between the right end and the obstacle ($g = 0$ m). Coulomb's contact conditions were employed.

The following data were used in the simulations:

$$
\begin{aligned}
&T = 1\,\mathrm{s}, \quad a_1 = a_2 = 1000\,\mathrm{N}, \quad c_1 = c_2 = 1\,\mathrm{N}\cdot\mathrm{s}, \\
&\rho = 10^{-4}\,\mathrm{kg/m^3}, \quad c_H = 10^3, \quad c_V = 5 \times 10^6, \quad \mu = 10^{-10}, \\
&f_H(x,t) = 400(e^t - 1)\,\mathrm{N/m}, \quad f_V(x,t) = -10\,000(e^t - 1)\,\mathrm{N/m} \text{ in } [0,1], \\
&u_0(x) = \tilde{u}_0(x) = 0\,\mathrm{m}, \quad v_0(x) = \tilde{v}_0(x) = 0\,\mathrm{m/s} \quad \text{in } [0,1].
\end{aligned}
$$

In order to see the behaviour of the algorithm, a sequence of numerical solutions, based on uniform partitions of both the time interval and the spatial domain, have been computed. Here, the spatial domain $[0,1]$ is divided into n equal parts ($h = 1/n$). We start with $n = 100$, which is successively halved, and $k = 0.5$, taking as "exact" solution that obtained with $n = 12\,800$ and $k = 10^{-5}$. In Table 1, the numerical errors

$$
\max_{o \le n \le N} \{ |u_n - u_n^{hk}|_H + |\tilde{u}_n - \tilde{u}_n^{hk}|_H \}, \tag{31}
$$

obtained with different discretization parameters, are shown. As we can see, the convergence of the numerical scheme is deduced.

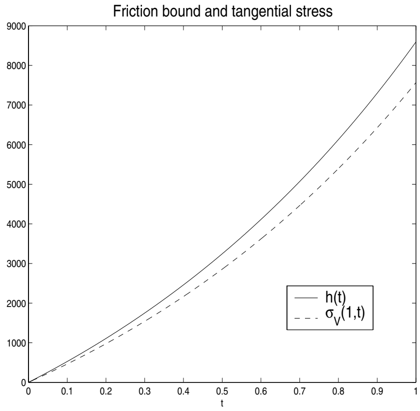

FIGURE 3. Example 1: evolution in time of the tangential stress $\sigma_V(1,t)$ and the friction bound $h(t) = c_V u(1,t)$.

TABLE 2. Example 2: numerical errors for some n and k (Slip case).

$n \downarrow k \rightarrow$	0.01	0.005	0.002	0.001	0.0005	0.0001
100	1.522e-2	1.060e-2	7.923e-3	7.056e-3	6.598e-3	5.944e-3
200	1.512e-2	1.044e-2	7.627e-3	6.669e-3	6.156e-3	5.417e-3
400	1.512e-2	1.043e-2	7.610e-3	6.649e-3	6.133e-3	5.382e-3
800	1.514e-2	1.045e-2	7.634e-3	6.674e-3	6.159e-3	5.409e-3
1600	1.516e-2	1.049e-2	7.670e-3	6.718e-3	6.200e-3	5.453e-3
3200	1.498e-2	1.044e-2	7.702e-3	6.755e-3	6.281e-3	5.549e-3

Using $h = k = 10^{-3}$ as discretization parameters, in Figure 2 (left-hand side), the horizontal displacements are shown at different times, while the vertical displacements are depicted at the same times on the right-hand side. We can observe that the right end of the beam keeps clamped, because the tangential stress does not achieve the friction bound, and the beam remains sticked to the obstacle (see Fig. 3).

3.2.2. *Second example: slip case*

As a second example, we consider the same setting that in the above example. However, we assume now that the horizontal force is given by $f_H(x,t) = 100(e^t - 1)\, N/m$ for $x \in [0,1]$, the vertical one by $f_V(x,t) = -1000(e^t - 1)\, N/m$ for $x \in [0,1]$ and the friction coefficient $c_V = 10^4$. We notice that with these new data, there is always movement of the contact node since the tangential stress equals the friction bound.

We have computed a sequence of numerical solutions following the previous example. In Table 2, the numerical errors (31) are depicted for different discretization parameters. Again, the convergence of the algorithm is shown.

Using $h = k = 10^{-3}$, the vertical displacements at different times are shown on the left-hand side of Figure 4. On the right-hand side, the evolution in time of the vertical displacement of the contact node is plotted. As we can observe, the right end of the beam is always moving.

3.2.3. *Third example: a test with Tresca's conditions*

As a third example, we consider a viscoelastic beam of length $L = 1$ m which is rigidly attached at its left end. Tresca's conditions are employed for the modelling of the contact. Since the problem decouples and the horizontal displacement was considered in previous studies (see, *e.g.*,[4]), our interest concerns only the vertical displacement.

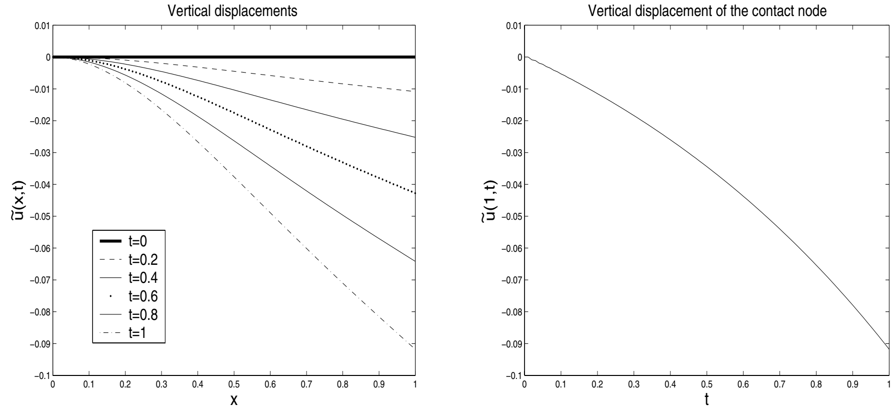

FIGURE 4. Example 2: vertical displacements at different times and evolution in time of the vertical displacement of the contact node.

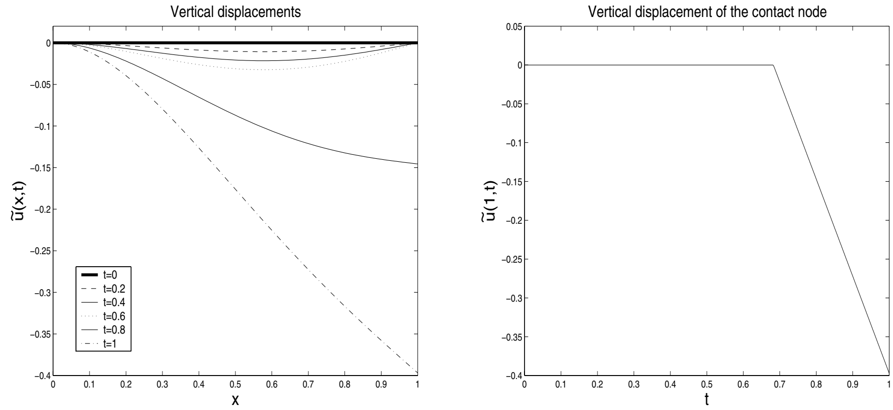

FIGURE 5. Example 3: vertical displacements at several times and evolution in time of the vertical displacement of the contact node.

The following data were used in the simulations:

$$\begin{aligned}&T = 1\,\mathrm{s}, \quad a_2 = 1000\,\mathrm{N}, \quad c_2 = 1\,\mathrm{N}\cdot\mathrm{s}, \quad c_V = 3000, \quad \mu = 10^{-10},\\ &\rho = 10^{-4}\,\mathrm{kg/m}^3, \quad f_V(x,t) = -10\,000\,\mathrm{t\,N/m} \text{ in } [0,1],\\ &\tilde{u}_0(x) = 0\,\mathrm{m}, \quad \tilde{v}_0(x) = 0\,\mathrm{m/s} \quad \text{in } [0,1].\end{aligned}$$

Taking $h = k = 10^{-3}$, in Figures 5 and 6 we show the numerical results obtained. In Figure 5, the vertical displacements at several times (left) and the evolution in time of the vertical displacement of the contact node (right) are plotted. Moreover, the tangential stress is depicted in Figure 6. As we can observe, the contact node is clamped until time $t = 0.68$ s, when the tangential stress $\sigma_V(1,t)$ reaches the friction bound.

3.2.4. *Fourth example: oscillating forces*

As a final test, we have considered a beam of length $L = 1\,\mathrm{m}$ with its left end rigidly attached, and no gap is assumed between the right end and the obstacle ($g = 0\,\mathrm{m}$). Coulomb's contact conditions were employed with

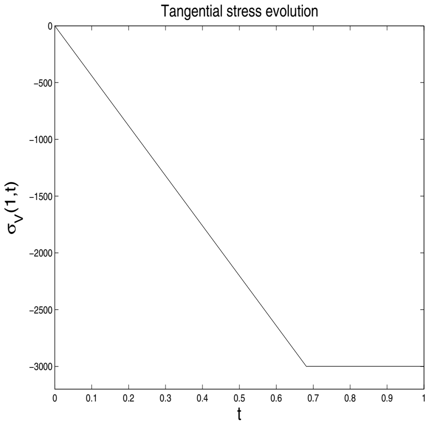

FIGURE 6. Example 3: evolution in time of the tangential stress.

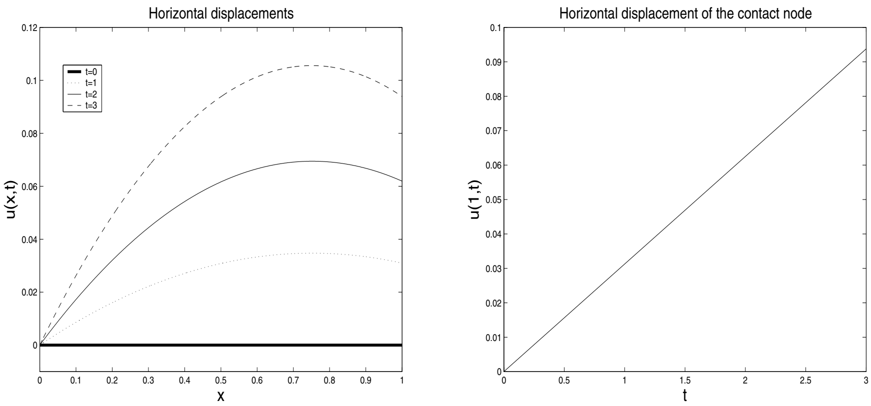

FIGURE 7. Example 4: horizontal displacements at different times and evolution in time of the horizontal displacement of the contact node.

the following data:

$$
\begin{aligned}
&T = 3\,\mathrm{s}, \quad a_1 = a_2 = 1000\,\mathrm{N}, \quad c_1 = c_2 = 1\,\mathrm{N}\cdot\mathrm{s},\\
&\rho = 10^{-4}\,\mathrm{kg/m}^3, \quad c_H = 10^3, \quad c_V = 5\times 10^4, \quad \mu = 10^{-10},\\
&f_H(x,t) = 125\ \mathrm{t\,N/m}, \quad f_V(x,t) = 10\,000 \sin\frac{\pi t}{2}\,\mathrm{N/m} \text{ in } [0,1],\\
&u_0(x) = \tilde{u}_0(x) = 0\,\mathrm{m}, \quad v_0(x) = \tilde{v}_0(x) = 0\,\mathrm{m/s} \quad \text{in } [0,1].
\end{aligned}
$$

Discretization parameters $h = k = 0.001$ were used for solving the corresponding discrete problem VP^{hk}. The horizontal displacements at several times and the evolution in time of the horizontal displacement of the contact node are shown in Figure 7. We notice that there is always penetration of the beam into the obstacle.

The vertical displacements at several times and the evolution in time of the vertical displacement of the contact node are plotted in Figure 8. Moreover, in Figure 9 it is shown the evolution in time of the friction bound $h(t) = c_V u(1,t)$ (left) and the tangential stress (right). Finally, the sum of L^2-norms of u and $\tilde{u}$ is plotted in Figure 10 depending on the deformability and friction coefficients c_H and c_V. The stability of the solution is clearly observed.

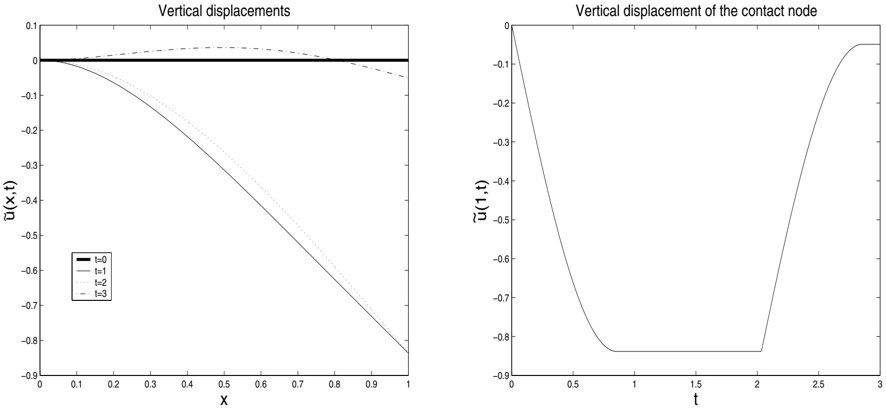

FIGURE 8. Example 4: vertical displacements at different times and evolution in time of the vertical displacement of the contact node.

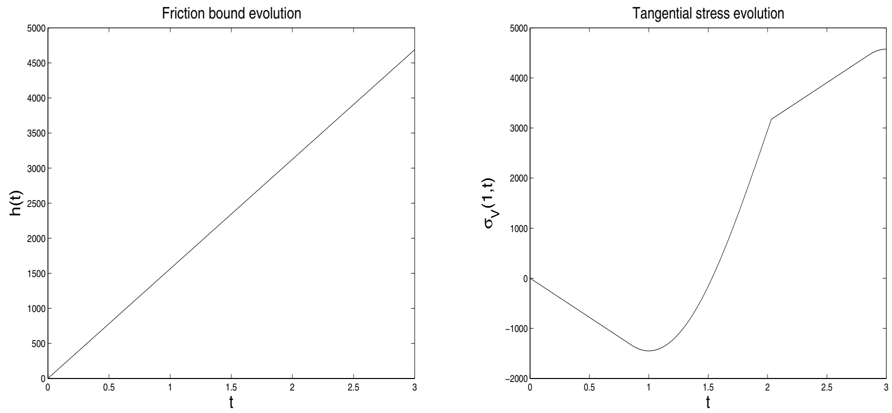

FIGURE 9. Example 4: evolution in time of the friction bound and the tangential stress.

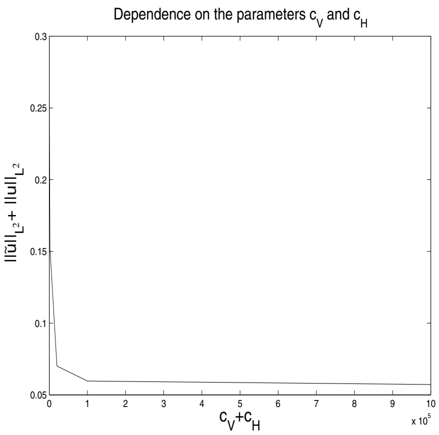

FIGURE 10. Example 4: sum of the L^2-norms depending on c_H and c_V.

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