LARGE TIME BEHAVIOR OF SOLUTIONS IN SUPER-CRITICAL CASES TO DEGENERATE KELLER-SEGEL SYSTEMS

STEPHAN LUCKHAUS\textsuperscript{1} AND YOSHIE SUGIYAMA\textsuperscript{2}

Abstract. We consider the following reaction-diffusion equation:

\[(KS) \begin{cases} u_t = \nabla \cdot (\nabla u^m - u^{q-1} \nabla v), & x \in \mathbb{R}^N, \ 0 < t < \infty, \\ 0 = \Delta v - v + u, & x \in \mathbb{R}^N, \ 0 < t < \infty, \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \]

where $N \geq 1$, $m > 1$, $q \geq \max\{m + \frac{2}{N}, 2\}$.

In [Sugiyama, Nonlinear Anal. 63 (2005) 1051–1062; Submitted; J. Differential Equations (in press)] it was shown that in the case of $q \geq \max\{m + \frac{2}{N}, 2\}$, the above problem (KS) is solvable globally in time for “small $L^{N\frac{q-m}{2}}$ data”. Moreover, the decay of the solution $(u, v)$ in $L^p(\mathbb{R}^N)$ was proved. In this paper, we consider the case of “$q \geq \max\{m + \frac{2}{N}, 2\}$ and small $L^\ell$ data” with any fixed $\ell \geq N\frac{q-m}{2}$ and show that (i) there exists a time global solution $(u, v)$ of (KS) and it decays to 0 as $t$ tends to $\infty$ and (ii) a solution $u$ of the first equation in (KS) behaves like the Barenblatt solution asymptotically as $t$ tends to $\infty$, where the Barenblatt solution is the exact solution (with self-similarity) of the porous medium equation $u_t = \Delta u^m$ with $m > 1$.

Mathematics Subject Classification. 35B40, 35K45, 35K55, 35K65.

Received: September 22, 2005. Revised: February 27, 2006.

1. INTRODUCTION

We consider the following reaction-diffusion equation:

\[(KS) \begin{cases} u_t = \nabla \cdot (\nabla u^m - u^{q-1} \nabla v), & x \in \mathbb{R}^N, \ 0 < t < \infty, \\ 0 = \Delta v - v + u, & x \in \mathbb{R}^N, \ 0 < t < \infty, \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \]

Keywords and phrases. Degenerate parabolic system, chemotaxis, Keller-Segel model, drift term, decay property, asymptotic behavior, Fujita exponent, porous medium equation, Barenblatt solution.

\textsuperscript{1} Department of mathematics and computer science, Universit"at Leipzig, Leipzig, 04109, Germany. luckhaus@mis.mpg.de

\textsuperscript{2} Department of Mathematics and Computer Science, Tsuda College, 2-1-1, Tsuda-chou, Kodaira-shi, Tokyo, 187-8577, Japan. sugiyama@tsuda.ac.jp

© EDP Sciences, SMAI 2006
where \( N \geq 1, \ m > 1, \ q \geq \max\{m + \frac{2}{N}, 2\} \). The initial data \( u_0 \) is a non-negative function in \( L^1 \cap L^\infty(\mathbb{R}^N) \times L^1 \cap H^1 \cap W^{1,\infty}(\mathbb{R}^N) \), \( u_0^m \in H^1(\mathbb{R}^N) \). This equation is often called the Keller-Segel model describing the motion of the chemotaxis molds. (We refer to Keller-Segel [23].)

In this paper, we are interested in the large time behavior of solutions for (KS). Concerning the large time behavior of the heat equation, the following asymptotic profile is well known:

\[
\lim_{t \to \infty} t^\frac{2}{N} \|u(\cdot, t) - MG_t(\cdot)\|_{L^\infty(\mathbb{R}^N)} = 0,
\]

where \( G_t(x) \) is the heat kernel and \( M \) is the initial mass.

Also for the porous medium equation:

\[
(P) \quad u_t(x, t) = \Delta u^m(x, t)
\]

corresponding to the initial data \( u_0 \), the asymptotic profile was obtained in the following form:

\[
\lim_{t \to \infty} t^\sigma \|u(\cdot, t) - V(\cdot, t; \|u_0\|_{L^p(\mathbb{R}^N)})\|_{L^\infty(\mathbb{R}^N)} = 0 \quad \text{with} \quad \sigma = \frac{N}{N(m-1) + 2},
\]

where \( V(x, t; M) \) is the exact solution of \( P \) given by

\[
V(x, t; M) := \frac{1}{t^\nu} \left( \beta^2 M^{2\frac{(m-1)}{2mN}} - \frac{\sigma(m-1)}{2mN} \frac{|x|^2}{t^\nu} \right)_+^{m-1},
\]

with a constant \( \beta \) such that \( \int_{\mathbb{R}^N} \left( \beta^2 - \frac{\sigma(m-1)}{2mN} |y|^2 \right)_+^{m-1} \, dy = 1 \). The above \( V(x, t; M) \) holds the self-similarity and \( \int_{\mathbb{R}^N} V(x, t; M) \, dx = M \) for all \( t > 0 \). This \( V(x, t; M) \) is called the Barenblatt solution. (See Barenblatt [2].)

The asymptotic profile (1.2) was firstly proved by Kamin [20], and developed by Friedman-Kamin [10], and finally established by Vazquez [34] in the above form (1.3). (We also refer to [3, 21, 35].)

Regarding to the Keller-Segel model (KS), for the semilinear case: \( m = 1 \) of parabolic-parabolic type, Nagai-Syukuinn-Umesako [26] showed the similar asymptotic profile to (1.1). (We also refer to Biler-Cannone-Guerra-Karch [5].) Their argument is based on the representation formula. On the other hand, as for our problem (KS), there is no representation formula for solution \( u \) since \( m > 1 \). In addition, differently from the porous medium equation (P), comparison principles do not hold. Therefore, we can not employ the method by Kamin [20], Friedman-Kamin [10], Nagai-Syukuinn-Umesako [26] to our problem directly.

Our aim of this paper is to prove the following (I)-(III) without “comparison principles and representation formula”:

In the case of \( m > 1 \) and \( q \geq \max\{m + \frac{2}{N}, 2\} \):

(I) (KS) is globally solvable for the small \( L^\ell \) data with any fixed \( \ell \geq \frac{N(q-m)}{2} \);

(II) the solution \( (u, v) \) of (KS) decays to 0 in \( L^p(\mathbb{R}^N)(1 < p < \infty) \).

We also assume that \( q > m + \frac{2}{N} \). Then,

(III) the solution \( u \) to the first equation in (KS) satisfies the following asymptotic profile:

\[
\lim_{t \to \infty} \frac{1}{t^{\frac{N}{N(m-1) + 2} + 1}} \|u(\cdot, t) - V(\cdot, t; \|u_0\|_{L^p(B_t)})\|_{L^p(B_t)} = 0, \quad \varepsilon > 0 \quad \text{and} \quad 1 < p < \infty
\]

for all \( R > 0 \), where \( B_t = B_t(\varepsilon, R) := \{ x \in \mathbb{R}^N ; |x| \leq Rt^{\frac{1}{N(m-1) + 2} + \varepsilon} \} \).
In what follows, we give the definition of a weak solution \((u, v)\) for (KS).

**Definition 1.** Let \(m > 1\), \(q \geq 2\) and let \(u_0 \in L^1 \cap L^\infty(\mathbb{R}^N)\) with \(u_0^m \in H^1(\mathbb{R}^N)\) and \(u_0 \geq 0\). A pair \((u, v)\) of non-negative functions defined in \(\mathbb{R}^N \times [0, T)\) is called a weak solution of (KS) on \([0, T)\) if

(i) \(u \in L^\infty(0, T; L^1 \cap L^\infty(\mathbb{R}^N))\), \(\nabla u^m \in L^2(0, T; L^2(\mathbb{R}^N))\);

(ii) \(v \in L^\infty(0, T; H^1(\mathbb{R}^N))\);

(iii) \((u, v)\) satisfies the equations in the sense of distribution, \(i.e.,\) that

\[
\int_0^\infty \int_{\mathbb{R}^N} \left( \nabla u^m \cdot \nabla \varphi - u^{q-1} \nabla v \cdot \nabla \varphi - u \varphi_t \right) \, dx \, dt = \int_{\mathbb{R}^N} u_0(x) \varphi(x, 0) \, dx,
\]

\[
\int_{\mathbb{R}^N} \left( \nabla v \cdot \nabla \psi + v \psi_t - u \psi \right) (t) \, dx = 0 \quad \text{for a.a.} \ t \in (0, T)
\]

for all functions \(\varphi \in C_0^\infty(\mathbb{R}^N \times [0, T))\) and \(\psi \in C_0^\infty(\mathbb{R}^N)\).

In the theorem, we show the existence and decay property of a solution \((u, v)\) for (KS) with small initial data.

**Theorem 1.1** (decay property). Let \(1 \leq p < \infty\), \(N \geq 1\), \(m > 1\), \(q \geq \max\{m + \frac{q}{N}, 2\}\), \(\ell \geq \frac{N(q-m)}{2} \geq 1\). Suppose that the initial data \(u_0\) is non-negative everywhere. Then, there exist an absolute constant \(M\) and a positive number \(\varepsilon\) depending only on \(M, p, N, m, \ell\) such that if \(u_0 \in L^1 \cap L^\infty(\mathbb{R}^N)\) satisfies that

\[
\|u_0\|_{L^1(\mathbb{R}^N)} = M, \quad \|u_0\|_{L^\infty(\mathbb{R}^N)} \leq \varepsilon,
\]

then (KS) has a weak solution \((u, v)\) on \([0, \infty)\) with the following decay property: there exists a constant \(C_p\) depending only on \(p\), \(\|u_0\|_{L^1(\mathbb{R}^N)}\) together with \(N, m, q, M, \|u_0\|_{L^{N+2\ell}(\mathbb{R}^N)}\) such that

\[
\|u(t)\|_{L^p(\mathbb{R}^N)} + \|v(t)\|_{L^p(\mathbb{R}^N)} \leq C_p(1 + t)^{-d} \quad \text{for all} \ 0 < t < \infty,
\]

where

\[
d = \sigma \left(1 - \frac{1}{p}\right), \quad \sigma = \frac{N}{N(m-1) + 2}.
\]

**Remark 1.**

(i) The decay rate \(d\) depends on \(m, N\) but not on \(q\).

(ii) The above decay rate seems to be optimal. In fact, for \(m = 1\), we find that \(\sigma = \frac{N}{2}\) whose decay rate \(d\) coincides with the \(L^1\)-\(L^p\) estimate for the linear heat equation.

(iii) Concerning the following Cauchy problem

\[
(\text{PS}) \quad \begin{cases} 
  u_t = \Delta u^m + u^q, & x \in \mathbb{R}^N, \ t > 0, \\
  u(x, 0) = u_0(x), & x \in \mathbb{R}^N
\end{cases}
\]

in the case of “\(m \geq 1\) and \(q > m + \frac{2}{N}\)”, Kawanago [22] obtained the decay estimate under smallness assumption for \(\|u_0\|_{L^{N(q-m)}(\mathbb{R}^N)}\). In Remark 4.1 in [22], he mentioned that \(p_0 := \frac{N(q-m)}{2}\) is the special exponent to obtain the decay property for (PS). Regarding to (KS), we show that if \(\|u_0\|_{L^1(\mathbb{R}^N)} << 1\) for any fixed number \(\ell \geq \frac{N(q-m)}{2} \geq 1\), then the decay property is obtained, \(i.e.,\) that the exponent \(p_0\) is not special for (KS).

For any positive numbers \(\varepsilon, R\), we define \(B_t\) by

\[
B_t := \{x \in \mathbb{R}^N; \ |x| < R\varepsilon^{\frac{N(m-1)+2}{N(m-1)+2}}\}.
\]
We introduce the self-similar solution \( V(x, t; M) \) of Barenblatt \[2\]:

\[
V(x, t; M) := \frac{1}{t^\sigma} \left( \beta^2 M \frac{2(m-1)}{2mN} - \frac{|x|^2}{2mN} \right) + \frac{|x|^2}{t^\frac{1}{2N}}
\]

with a constant \( \beta \) such that

\[
\int_{\mathbb{R}^N} \left( \beta^2 - \frac{\sigma(m-1)}{2mN} |y|^2 \right) \frac{1}{t^\frac{1}{2N}} \, dy = 1.
\]

It is easily verified that

\[
\int_{\mathbb{R}^N} V(x, t; M) \, dx = M.
\]

We now give the asymptotic profile in the following theorem.

**Theorem 1.2** (asymptotic profile). Let the same assumption as that in Theorem 1.1 hold. In addition, let \( q > m + \frac{2}{N} \). Then, the weak solution \( u \) obtained in Theorem 1.1 satisfies that

\[
\lim_{t \to \infty} t^{(m-N(q-m)+\frac{1}{2}) \cdot N} \| u(\cdot, t) - V(\cdot, t; \|u_0\|_{L^1(\mathbb{R}^N)}) \|_{L^p(B_t)} = 0 \quad \text{with} \quad \varepsilon > 0, \quad 1 < p < \infty,
\]

for all \( R > 0 \), where \( B_t = B_t(\varepsilon, R) \) is the ball defined in (1.7).

**Remark 2.**

(i) It seems to be difficult to take \( \varepsilon = 0 \).

(ii) Theorem 1.2 implies that \( \Delta u^m \) is dominant to \( \nabla (u^{q-1} \nabla v) \) in the case of “\( q > m + \frac{2}{N} \) and small initial data”.

(iii) The proof of Theorem 1.2 is based on the estimate (1.6) in Theorem 1.1.

To show the asymptotic profile for (KS) with \( m > 1 \), we consider the following sequence of functions:

\[
w_k(x, t) = k^N u(kx, k^{N(m-1)+2t}) \quad \text{and} \quad z_k(x, t) = k^N v(kx, k^{N(m-1)+2t}) \quad \text{for} \quad k \geq 1.
\]

Then, (KS) can be rewritten as follows:

\[
\begin{align*}
(w_k(x, t) &= \nabla \cdot \left( \nabla (w_k)^m - k^{-N(q-m)} \cdot (w_k)^{q-1} \cdot \nabla z_k \right), \quad x \in \mathbb{R}^N, \quad 0 < t < \infty, \quad \cdots (1)_w, \\
0 &= \Delta z_k - k^2 z_k + k^2 w_k, \quad x \in \mathbb{R}^N, \quad 0 < t < \infty, \quad \cdots (2)_w, \\
w_k(x, 0) &= k^N u_0(kx), \quad x \in \mathbb{R}^N.
\end{align*}
\]

The above system \( w(\text{KS}) \) does not have any invariance under change of scaling. In addition, the second equation includes the scaling parameter \( k \). However, in the case of \( q \geq m + \frac{2}{N} \), we obtain the \( L^\infty(\mathbb{R}^N) \)-bound for \( w_k \) independently of \( k \). Next, we prove that \( (w_k)^m \) is bounded in \( H^1(\Delta, T; L^2(\mathbb{R}^N)) \cap L^\infty(\Delta, T; H^1(\mathbb{R}^N)) \) for all \( 0 < \Delta < T < \infty \). In this point, we have the difficulty such as \( \|w_k(0)\|_{L^p(\mathbb{R}^N)} = (k^{N(1-\frac{2}{p})} \|u_0\|_{L^p(\mathbb{R}^N)}) \) depends on \( k \) for all \( p \in (1, \infty) \). Under this difficulty, to obtain the boundedness in \( H^1(\Delta, T; L^2(\mathbb{R}^N)) \cap L^\infty(\Delta, T; H^1(\mathbb{R}^N)) \) independent of \( k \), we use the cut-off function which attains 0 at \( t = 0 \) and has \( C^\infty \)-regularity. As a result, we obtain the desired bounds independent of \( k \) (see Sect. 5.1 in this paper) and show that \( w_k \) converges a function \( U \). Simultaneously, we find that \( U \) satisfies (P) in a distribution sense since the power of \( k \) in the coefficient of the perturbation term is negative. (See Sect. 5.2)

Furthermore, we verify the following key fact:

\[
\text{(H)} \quad U(\cdot, t) \in L^1(\mathbb{R}^N) \quad \text{and} \quad \|U(\cdot, t)\|_{L^1(\mathbb{R}^N)} = \|u_0\|_{L^1(\mathbb{R}^N)}.
\]
To this end, we invent the crucial lemma (Lem. 2.4) in Section 2:

\[ w_k \to U \quad \text{strongly in } L^1_{\text{loc}}(\mathbb{R}^N) \]  

(1.10)

together with some additional assumption on \( w_k \), we have in fact, that

\[ \left| \int_{\mathbb{R}^N} (w_k - U) \, dx \right| \to 0 \quad \text{as } k \to \infty. \]  

(1.11)

We ensure the sufficient conditions in the above lemma (Lem. 2.4). (See Sect. 5.3.) As a result, by virtue of (1.11) and “the mass conservation law and the \( L^1 \)-scaling invariance for the initial data of \( w_k \), i.e.,

\[ \int_{\mathbb{R}^N} w_k(x, t) \, dx = \int_{\mathbb{R}^N} w_k(x, 0) \, dx = \int_{\mathbb{R}^N} u_0 \, dx \quad \text{for all } t \geq 0, \]  

(1.12)
the above (H) is verified. Once (H) is verified, we can apply Theorem 1.1 in Vazquez [34] and obtain that

\[ \| U(\cdot, t) - V(\cdot, t; \| u_0 \|_{L^1}) \|_{L^p(\mathbb{R}^N)} \to 0 \quad \text{as } t \to \infty. \]  

(1.13)

For any positive numbers \( \varepsilon, R \), we define \( B_t \) by

\[ B_t := \{ x \in \mathbb{R}^N; \ |x| < R t^{\frac{m-1}{2}+\varepsilon} \}. \]  

(1.14)

Then, taking the time variable by \( k^\varepsilon \), and combining (1.13) with the convergence of \( w_k \) to \( U \), we observe that

\[
\begin{align*}
&\| w_k(\cdot, k^\varepsilon) - V(\cdot, k^\varepsilon; \| u_0 \|_{L^1}) \|_{L^p(B_t)} \\
&\leq \| w_k(\cdot, k^\varepsilon) - U(\cdot, k^\varepsilon) \|_{L^p(B_t)} + \| U(\cdot, k^\varepsilon) - V(\cdot, k^\varepsilon; \| u_0 \|_{L^1}) \|_{L^p(\mathbb{R}^N)} \\
&\quad \to 0 \quad \text{as } k \to \infty
\end{align*}
\]  

(1.15)

for any \( p \in (1, \infty) \) and for all \( R > 0 \), where \( B_R := \{ x \in \mathbb{R}^N; \ |x| < R \} \). Moreover, taking \( k \) by \( k = t^{\frac{1}{N(m-1)+2+\varepsilon}} \) in (1.15) and using the self-similarity of the Barenblatt solution, we conclude that

\[ t^{\frac{N}{N(m-1)+2+\varepsilon}(1-\frac{1}{p})} \| u(\cdot, t) - V(\cdot, t; \| u_0 \|_{L^1}) \|_{L^p(B_t)} \to 0 \quad \text{as } t \to \infty \]  

(1.16)

for any \( \varepsilon > 0 \) and \( p \in (1, \infty) \) and for all \( R > 0 \), where \( B_t = B_t(\varepsilon, R) \) is the ball defined in (1.14). Thus, we prove Theorem 1.2. (see Section 5.4.)

In the following section, we shall prepare several lemmas which will be used in the sequel sections. In Section 3, we introduce the results obtained in [30–32] concerning the existence of a time global strong solution of the approximated problem of (KS). In Section 4, we organize the proof of the decay of a solution \((u, v)\). In Section 5, in the case of \( m > 1, q > m + \frac{2}{N} \), we prove that the solution \( u \) of (KS) behaves like the Barenblatt solution asymptotically as \( t \to \infty \) which is the exact solution of porous medium equation: \( u_t = \Delta u^m \) with \( m > 1 \).
Remark 3.

(i) In our argument, any type of comparison principles is not used.

(ii) When we substitute the second equation: \( \Delta v = v - u \) into the first equation in (KS), it holds that

\[
E \quad u_t = \Delta u^m - \nabla u^{q-1} \cdot \nabla v - u^{q-1} \Delta v = \Delta u^m + u^q - \nabla u \cdot \nabla v - u^{q-1} v.
\]

The above equation (E) includes the terms \( u_t, \Delta u^m \) and \( u^q \). Therefore, we observe that (PS) in Remark 1 is analogous to (E).

For (PS) with \( N \geq 1, m, q > 1 \), it is well known that the critical exponent \( q = m + \frac{2}{N} \) divides the situation into the global existence and the finite time blow-up of a solution. Indeed,

1. when \( q > m + \frac{2}{N} \), the problem (PS) is globally solvable for small initial data and evolves in a finite time blow-up for large initial data and
2. when \( m < q < m + \frac{2}{N} \) and \( q = m + \frac{2}{N} \), it is proved that (all) non-negative solutions of (PS) blow up in a finite time without any restriction on the size of the initial data. (See for example Galaktionov-Kurdyumov-Mikhailov-SamarskiiN [13], Galaktionov [12], Kawanago [22] and Mochizuki-Suzuki[24]. This exponent \( q = m + \frac{2}{N} \) is called the Fujita exponent [11].

For (KS) with \( N \geq 1, m > 1, q \geq 2 \), in [30–32] the Fujita’s exponent was found. Specifically, in [32] it was shown that

(i) when \( q < m + \frac{2}{N} \), the problem (KS) is globally solvable without any restriction on the size of the initial data; and
(ii) when \( m > 1 \) and \( q \geq \max\{m + \frac{2}{N}, 2\} \), the problem (KS) is globally solvable for small \( L^{\frac{N(q-m)}{2}} \) initial data. Furthermore, the decay of solution \((u, v)\) in \( L^p(\mathbb{R}^N)(1 < p < \infty) \) was proved.

In addition, in the case of \( q = 2 \) with \( 2 > m + \frac{2}{N} \):

(iii) we [33] constructed such an initial function that a solution \((u, v)\) blows up in a finite time.

In this paper, the case of (ii) above is considered.

We will use the simplified notations:

1. \( \partial_t = \frac{\partial}{\partial t}, \quad \partial_i = \frac{\partial}{\partial x_i}, \quad \partial_{ij} = \partial_i \partial_j, \quad \nabla u = \left( \partial_1, \partial_2, \ldots \right), \quad \nabla^2 u = \left( \partial_{11}, \partial_{12}, \ldots \right); \)
2. \( \| \cdot \|_{L^r} = \| \cdot \|_{L^r(\mathbb{R}^N)}, (1 \leq r \leq \infty), \quad \int \cdot \, dx := \int_{\mathbb{R}^N} \cdot \, dx. \)
3. \( Q_T := \mathbb{R}^N \times (0, T), \quad B_R := \{ x \in \mathbb{R}^N; |x| < R \}. \)
4. When the weak derivatives \( \nabla u, \nabla^2 u \) and \( \partial_t u \) are in \( L^p(Q_T) \) for some \( p \geq 1 \), we say that \( u \in W^{2,1}_p(Q_T), \) i.e.,

\[
W^{2,1}_p(Q_T) := \left\{ u \in L^p(0, T; W^{2, p}(\mathbb{R}^N)) \cap W^{1, p}(0, T; L^p(\mathbb{R}^N)); \right\}
\]

\[
\|u\|_{W^{2,1}_p(Q_T)} := \|u\|_{L^p(Q_T)} + \|\nabla u\|_{L^p(Q_T)} + \|\nabla^2 u\|_{L^p(Q_T)} + \|\partial_t u\|_{L^p(Q_T)} < \infty. \]

2. Preliminary Lemmas

The following lemma gives us a version of Gagliardo-Nirenberg inequality. (See [33], Lem. 2.4. and Nakao [27])

Lemma 2.1. Let \( N \geq 1, m \geq 1, a > 2, u \in L^p(\mathbb{R}^N) \) with \( q_1 \geq 1 \) and \( u \in H^{1}(\mathbb{R}^N) \) with \( r > 0 \). Let \( q_1 \in [1, r + m - 1], q_2 \in \left[ \frac{r + m - 1}{2}, \frac{a(r + m - 1)}{2} \right] \) and

\[
\begin{cases}
1 \leq q_1 \leq q_2 \leq \infty & \text{when } N = 1, \\
1 \leq q_1 \leq q_2 < \infty & \text{when } N = 2, \\
1 \leq q_1 \leq q_2 \leq \frac{(r + m - 1)N}{N - 2} & \text{when } N \geq 3.
\end{cases}
\]
Then, it holds that
\[ \|u\|_{L^q(\mathbb{R}^N)} \leq C^{\frac{2}{r+m-1}} \|u\|_{L^{\frac{2q}{r}}(\mathbb{R}^N)} \|\nabla u\|_{L^{\frac{2q}{r+m-1}}(\mathbb{R}^N)}^{\frac{2q}{r}} \] (2.2)

with
\[ \Theta = \frac{r + m - 1}{2} \left( \frac{1}{q_1} - \frac{1}{q_2} \right) \left( \frac{1}{N} \frac{1}{2} + \frac{r + m - 1}{2q_1} \right)^{-1}, \] (2.3)

where
\[ \begin{align*}
C & \text{ depends only on } N \text{ and } a & \text{ when } q_1 \geq \frac{r + m - 1}{2}, \\
C & = c_0 \text{ with } c_0 \text{ depending only on } N \text{ and } a & \text{ when } 1 \leq q_1 < \frac{r + m - 1}{2},
\end{align*} \] (2.4)

and
\[ \beta := \frac{q_2 - \frac{r + m - 1}{2}}{q_2 - q_1} \left[ \frac{2q_1}{r + m - 1} + \left( 1 - \frac{2q_1}{r + m - 1} \right) \frac{2N}{N + 2} \right]. \] (2.5)

The following inequalities are well known. (For instance, see Duoandikoetxea [9], p. 110 and Brezis [6], IX.12.)

**Lemma 2.2.** Let \( w \in W^2, r(\mathbb{R}^N) \). Then, the following inequalities hold:
\[ \|\nabla^2 w\|_{L^r(\mathbb{R}^N)} \leq C \left( \frac{r^2}{r - 1} \right)^2 \|\Delta w\|_{L^r(\mathbb{R}^N)} \quad \text{for } 1 < r < \infty, \] (2.6)
\[ \|w\|_{L^\infty(\mathbb{R}^N)} \leq \frac{2r}{r - N} \|w\|_{W^{1, \infty}(\mathbb{R}^N)} \quad \text{for } r > N, \] (2.7)

where \( C \) is a positive constant depending only on \( N \).

We prepare a technical lemma which is used often when establishing the uniform bound of a solution \( w_k \) for \((1)_{w}\) in \((\text{KS})\).

**Lemma 2.3.** Let \( \eta = \eta(r) \) be as
\[ \eta(r) := \begin{cases} 1 & \text{for } r \geq 1, \\ \exp(1 - \frac{1}{r}) & \text{for } 0 \leq r \leq 1, \\ 0 & \text{for } r \leq 0. \end{cases} \]

We define a sequence \( \eta_\delta(t) \) of cut-off functions by \( \eta_\delta(t) := \eta(\frac{t}{\delta}) \). Then, it holds that
\[ \sup_{0 < t < \infty} t^{-p} \eta_\delta(t) \leq \frac{p^p}{\delta^p r^{p-1}} \quad \text{and} \quad \sup_{0 < t < \infty} t^{-p} \partial_t \eta_\delta(t) \leq \frac{(p + 2)p^{p+2}}{(\delta e)^{p+1}} \quad \text{for any } p \geq 1. \] (2.8)

**Proof of Lemma 2.3.** For any \( p \geq 1 \), we have
\[ \sup_{0 < t < \infty} t^{-p} \eta_\delta(t) = \sup_{0 < t < \infty} \frac{e}{\delta^p r^p} = \frac{e}{\delta^p} \sup_{0 < y < \infty} \frac{y^p}{e^y} \leq \frac{e}{\delta^p} \frac{p^p}{e^p} = \frac{p^p}{\delta^p r^{p-1}}. \] (2.9)

By the definition of \( \eta \) and \( \eta_\delta \), we see that
\[ \eta'(r) = \begin{cases} 0 & \text{for } r \geq 1, \\ \frac{1}{r^2} \exp(1 - \frac{1}{r}) & \text{for } 0 \leq r \leq 1, \end{cases} \] (2.10)
\[ \partial_t \eta_\delta(t) = \begin{cases} 
0 & \text{for } t \geq \delta, \\
\phi \cdot \exp(1 - \frac{t}{\delta}) & \text{for } 0 \leq t \leq \delta. 
\end{cases} \tag{2.11} \]

From (2.11), we observe that

\[ \sup_{0 < t < \infty} t^{-p} \partial_t \eta_\delta(t) = \sup_{0 < t \leq \delta} \frac{\delta e}{\delta^{p/2}} \cdot \frac{\gamma}{e^{y}} \leq \frac{\delta e}{\delta^{p/2}} \cdot \frac{(p+2)^{p/2}}{e^{\gamma}} = \frac{(p+2)^{p/2}}{(\delta e)^{p/2}}. \]

Thus, we complete the proof of Lemma 2.3. \qed

We present the crucial lemma which will play an important role when showing the asymptotic profile.

**Lemma 2.4.** Let \( N \geq 1 \) and \( g \) belong to \( L^1(\mathbb{R}^N) \) and assume that \( \{w_k\} \) is a sequence of non-negative \( L^1 \)-functions in \( \mathbb{R}^N \) satisfying

\[ w_k \rightharpoonup g \quad \text{strongly in } L^1_{\text{loc}}(\mathbb{R}^N). \tag{2.12} \]

We also assume that for any fixed number \( \delta > 0 \), there exist \( f_\delta \in L^1(\mathbb{R}^N) \) and \( k_0 \in \mathbb{N} \) such that

\[ \int_{\mathbb{R}^N} \left[ w_k - f_\delta \right]^+ \, dx < \frac{\delta}{6} \quad \text{for all } k > k_0, \tag{2.13} \]

where \( [s]^+ = \max(s, 0) \). Then, we have convergence:

\[ \left| \int_{\mathbb{R}^N} (w_k - g) \, dx \right| \to 0 \quad \text{as } k \to \infty. \tag{2.14} \]

**Proof of Lemma 2.4.** Let \( \delta \) be any fixed positive number and \( \Omega \) be a domain in \( \mathbb{R}^N \). Then, \( \Omega \) can be written as the union

\[ \Omega = \Omega_1 \cup \Omega_2, \]

where

\[ \Omega_1 := \{ \Omega \subset \mathbb{R}^N ; w_k(x) - f_\delta(x) \geq 0 \} \quad \text{and} \quad \Omega_2 := \{ \Omega \subset \mathbb{R}^N ; w_k(x) - f_\delta(x) < 0 \}. \]

Therefore, it holds that

\[ \left| \int_{\Omega} w_k - f_\delta \, dx \right| \leq \int_{\Omega_1} |w_k - f_\delta| \, dx + \int_{\Omega_2} |w_k - f_\delta| \, dx \]

\[ = \int_{\Omega_1} (w_k - f_\delta) \, dx + \int_{\Omega_2} (f_\delta - w_k) \, dx \]

\[ = \int_{\mathbb{R}^N} [w_k - f_\delta]^+ \, dx + \int_{\Omega} |f_\delta| \, dx. \tag{2.15} \]

On the other hand, since \( g, f_\delta \in L^1(\mathbb{R}^N) \), there exists a domain \( K_\delta \subset \mathbb{R}^N \) depending on \( \delta \) such that

\[ \int_{\mathbb{R}^N \setminus K_\delta} |g| \, dx \leq \frac{\delta}{6} \quad \text{and} \quad \int_{\mathbb{R}^N \setminus K_\delta} |f_\delta| \, dx \leq \frac{\delta}{6}. \tag{2.16} \]
Taking $\Omega$ by $\Omega = \mathbb{R}^N \backslash K_{\delta}$ in (2.15), from (2.13) and (2.16), we observe that for any fixed number $\delta > 0$, there exists $K_{\delta} \subset \mathbb{R}^N$ and $k_0 \in \mathbb{N}$ such that

$$\left| \int_{\mathbb{R}^N \backslash K_{\delta}} (w_k - f_\delta) \, dx \right| \leq \left| \int_{\mathbb{R}^N} [w_k - f_\delta]^+ \, dx + \int_{\mathbb{R}^N \backslash K_{\delta}} |f_\delta| \, dx \right| \leq \frac{\delta}{3}$$

for all $k > k_0$. \hfill (2.17)

Consequently, by (2.12), (2.16) and (2.17), we see that for any fixed number $\delta > 0$, there exists $k_0(\geq k_0) \in \mathbb{N}$ such that

$$\left| \int_{\mathbb{R}^N} (w_k(x) - g(x)) \, dx \right| \leq \left| \int_{K_{\delta}} (w_k(x) - g(x)) \, dx \right| + \left| \int_{\mathbb{R}^N \backslash K_{\delta}} (w_k(x) - g(x)) \, dx \right| \leq \| w_k - g \|_{L^1(K_{\delta})} + \left| \int_{\mathbb{R}^N \backslash K_{\delta}} (w_k(x) - f_\delta(x)) \, dx \right| + \left| \int_{\mathbb{R}^N} |f_\delta(x) + g(x)| \, dx \right| \leq \frac{\delta}{3} + \frac{\delta}{6} + \frac{\delta}{6} = \delta$$

for any $k > \tilde{k}_0$. We thus conclude (2.14) and complete the proof of Lemma 2.4.

\hfill \square

3. APPROXIMATED PROBLEM

The first equation of (KS) is a quasi-linear parabolic equation of degenerate type. Therefore, we cannot use the strong solution. We consider the space $W^{2,1}_p(Q_T)$ with $p > N/2$ and $Q_T := (0, T) \times \mathbb{R}^N$. We thus conclude (2.14) and complete the proof of Lemma 2.4.

\hfill \square

3. APPROXIMATED PROBLEM

The first equation of (KS) is a quasi-linear parabolic equation of degenerate type. Therefore, we can not expect the problem (KS) to have a classical solution at the point where the first solution $u$ vanishes. In order to justify all the formal arguments, we need to introduce the following approximated equation of (KS):

\[ \begin{cases} 
    u_{\varepsilon t}(x,t) = \nabla \cdot \left( \nabla (u_{\varepsilon} + \varepsilon)^m - (u_{\varepsilon} + \varepsilon)^{q-2}u_{\varepsilon}\nabla u_{\varepsilon} \right), & (x,t) \in \mathbb{R}^N \times (0, T), \quad \cdots (1), \\
    0(x,t) = \Delta u_{\varepsilon} - v_{\varepsilon} + u_{\varepsilon}, & (x,t) \in \mathbb{R}^N \times (0, T), \quad \cdots (2), \\
    u_{\varepsilon}(x,0) = u_{0\varepsilon}(x), & x \in \mathbb{R}^N, 
\end{cases} \]

where $q > 1$ and $\varepsilon$ is a positive parameter and $u_{0\varepsilon}$ is an approximation for the initial data $u_0$ such that

(A.1): $0 \leq u_{0\varepsilon} \in L^1 \cap W^{2,p}(\mathbb{R}^N)$ for all $\left\{ p \in \left[ \frac{N}{N+3}, \frac{N}{N+4} \right], \quad \text{for all } \varepsilon \in (0,1], \right\}$

(A.2): $\| u_{0\varepsilon} \|_{L^p} \leq \| u_0 \|_{L^p}$ for all $p \in [1, \infty]$, for all $\varepsilon \in (0,1]$, 

(A.3): $\| \nabla u_{0\varepsilon} \|_{L^2} \leq \| \nabla u_0 \|_{L^2}$ for all $\varepsilon \in (0,1]$, 

(A.4): $u_{0\varepsilon} \rightarrow u_0$ for some $p \in [1, \infty)$, as $\varepsilon \rightarrow 0$.

We call $(u_{\varepsilon}, v_{\varepsilon})$ a strong solution of (KS)$_{\varepsilon}$ if it belongs to $W^{2,1}_p \times W^{2,1}_p(Q_T)$ for some $p \geq 1$ and the equations (1), (2) in (KS)$_{\varepsilon}$ are satisfied almost everywhere.

For the strong solution, we consider the space $W(Q_T)$ defined by

$$W(Q_T) := W^1_1(Q_T) \times W^2_2(Q_T) = \left\{ \begin{array}{ll} 
    W^{2,1}_N \cap W^{2,1}_{N+3}(Q_T) \times W^{2,1}_{N+2}(Q_T), & \text{for } N \geq 2, \\
    W^{2,1}_3(Q_T) \times W^{2,1}_3(Q_T), & \text{for } N = 1. 
\end{array} \right.$$
In [30–32], the following proposition concerning the existence of the strong solution was proved:

**Proposition 3.1** (time local existence,[30–32]). Let \( N \geq 1, m > 1 \). Suppose that (A.1) is satisfied. Then, there exists a number \( T_1 = T_1(\varepsilon, \|u_0\|_{W^{2, N+2} (\mathbb{R}^N)}, m, N) > 0 \) such that \((KS)_\varepsilon\) has the unique non-negative strong solution \((u_\varepsilon, v_\varepsilon)\) belonging to \( W(Q_{T_1}) \).

**Proposition 3.2** (extension criterion,[30–32]). Let the same assumption as that in Proposition 3.1 hold and let \( T > 0 \). Suppose that \((u_\varepsilon, v_\varepsilon)\) is a strong solution of \((KS)_\varepsilon\) in the class \( W(Q_T) \). If it holds that

\[
\sup_{0 < t < T} \|u_\varepsilon(t)\|_{L^\infty(\mathbb{R}^N)} < \infty,
\]

then there is \( T' > T \) such that \((u_\varepsilon, v_\varepsilon)\) can be a strong solution of \((KS)_\varepsilon\) in \( W(Q_{T'}) \).

### 4. Proof of Theorem 1.1

For the rigorous proof, we multiply (1) in \((KS)_\varepsilon\) by \((u_\varepsilon + \varepsilon)^{-1}\). For the sake of simplicity, we multiply \( u_{\varepsilon t} = \nabla \cdot (\nabla (u_\varepsilon + \varepsilon)^m - \nabla u_\varepsilon^{q-1} \nabla v_\varepsilon) \) by \( u_\varepsilon^{-1} \), where \( r > 1 \), and integrate it over \( \mathbb{R}^N \). Then, we have

\[
\frac{d}{dt} \|u_\varepsilon\|_{L^r} \leq -mr(r-1) \int u_\varepsilon^{r+m-3} |\nabla u_\varepsilon|^2 \, dx + r(r-1) \int u_\varepsilon^q \nabla v_\varepsilon \cdot u_\varepsilon^{-2} \nabla u_\varepsilon \, dx
\]

\[
= -\frac{4mr(r-1)}{(r+m-1)^2} \|u_\varepsilon^{r+m-1}\|_{L^2}^2 + \frac{r(r-1)}{r+q-2} \int u_\varepsilon^{r+q-2} \nabla v_\varepsilon \, dx
\]

\[
\leq -\frac{4mr(r-1)}{(r+m-1)^2} \|u_\varepsilon^{r+m-1}\|_{L^2}^2 + \frac{r(r-1)}{r+q-2} \|u_\varepsilon\|_{L^{r+q-1}}^{r+q-1}
\]

for all \( r > 1 \). (4.1)

By taking \( q_1 = \frac{N(q-m)}{2} \), \( q_2 = r + q - 1 \), \( a = 2 + \frac{q}{N} \) in Lemma 2.1, we have

\[
\|u_\varepsilon\|_{L^{r+m}}^{r+m} \leq c_0 \frac{2(N+2)^{N+2}}{N} \|u_\varepsilon\|_{L^2}^{q-m} \|u_\varepsilon^{r+m-1}\|_{L^2}^2
\]

for some absolute constant \( c_0 \), where we used

\[
\frac{1}{\beta} = \frac{N+2}{N}, \quad \frac{r+m-1}{r-m+2q-1} \leq \frac{N+2}{N}
\]

by \( m \leq q - \frac{2}{N} < q \).

Combining (4.1) with (4.2), we obtain

\[
\frac{d}{dt} \|u_\varepsilon\|_{L^r} \leq \left[ \frac{r(r-1)}{r+q-2} \frac{2(N+2)^{N+2}}{N} \|u_\varepsilon\|_{L^2}^{q-m} - \frac{4mr(r-1)}{(r+m-1)^2} \|u_\varepsilon^{r+m-1}\|_{L^2}^2 \right]
\]

By the Hölder inequality, \( \|u_0\|_{L^\frac{N(q-m)}{2}} \leq \|u_0\|_{L^\frac{1}{2}} \|u_0\|_{L^\gamma}^{1-\gamma} \) for some \( \gamma = \gamma(m, q, N, \ell) \). Therefore, when we take \( \|u_0\|_{L^r} \) sufficiently small for any fixed \( \ell \geq \frac{N(q-m)}{2} \), it holds that \( \|u_0\|_{L^\frac{N(q-m)}{2}} \) is small.

On the other hand, \( \|u_\varepsilon(t)\|_{L^\frac{N(q-m)}{2}} \in C([0, T]) \). Therefore, using this continuity and (4.3) with \( r = \frac{N(q-m)}{2} \), we find that there exists a short interval \([0, t_1]\) such that \( \frac{d}{dt} \|u_\varepsilon(t)\|_{L^r} \leq 0 \) for \( t \in [0, t_1] \), and
\[ \|u(t)\|_{L^{N(1-\lambda)}} \leq \|u_0\|_{L^{N(1-\lambda)}} \text{ for } t \in [0,t]. \]

Since this implies that \( \|u(t)\|_{L^{N(1-\lambda)}} \leq \|u_0\|_{L^{N(1-\lambda)}} \), we can repeat this procedure. In consequence, we obtain

\[ \|u(t)\|_{L^{N(1-\lambda)}} \leq \|u_0\|_{L^{N(1-\lambda)}} \text{ for } t \in [0, T]. \] (4.4)

Substituting (4.4) into (4.3), we have

\[
\frac{d}{dt}\|u(t)\|_{L^r} \leq \left( \frac{r(r-1)}{r+q-2}\|u_0\|_{L^{q-1}}^{q-1} - \frac{4m r(r-1)}{(r+1)^2}\|\nabla u(t)\|_{L^2}^2 \right) \leq 0 \quad \text{for } t \in \left[ \frac{N(q-1)}{2}, \infty \right].
\] (4.5)

By applying Moser's iteration technique, we obtain

\[ \sup_{0 \leq t \leq T} \|u(t)\|_{L^{\infty}} < \infty. \] (4.6)

(see [32], Lem. 15 or [33], Sect. 5). Combining (4.6) with Propositions 3.1 and 3.2, we prove the following lemma:

**Lemma 4.1.** Let \( N \geq 1, \ m > 1, \ q \geq m + \frac{N}{2}, \ \ell \geq \frac{N(q-1)}{2}(1), \ T > 0 \) and suppose that (A.1) is satisfied. Then, there exist an absolute constant \( M \) and a positive number \( \varepsilon \) depending only on \( M, N, m, \ell \) such that if \( u_0 \in L^1 \cap L^\ell(\mathbb{R}^N) \) satisfies that

\[ \|u_0\|_{L^1(\mathbb{R}^N)} = M, \ \|u_0\|_{L^\ell(\mathbb{R}^N)} \leq \varepsilon, \] (4.7)

then (KS) has the strong solution \((u, v)\) in the class obtained in \( W(Q_T) \) with the following property:

\[ \frac{d}{dt}\|u(t)\|_{L^r} + \frac{2m r(r-1)}{(r+1)^2}\|\nabla u(t)\|_{L^2}^2 \leq 0 \quad \text{for all } t \in (0,T) \text{ and } r \in \left[ \frac{N(q-1)}{2}, \infty \right]. \]

From Lemma 4.1, we are going to show (1.6) in Theorem 1.1.

Lemma 2.1 with \( a = 3 \) and (A.2) gives that

\[ \|u(t)\|_{L^r} \leq c^{\beta_1} \|\nabla u(t)\|_{L^2}^{\beta_1} \|u_0\|_{L^1}^{\theta_1} \|\nabla u(t)\|_{L^2}^{\theta_1} \quad \text{for any } r \in [2, \infty), \] (4.8)

where \( c \) depends only on \( N \), and

\[ \beta_1 := \frac{N(r+m-2+\frac{2}{N})(r-m+1)}{N+2(r-1)(r+m-1)}, \]

\[ \theta_1 := \frac{r+m-1}{2} \cdot \left( 1 - \frac{1}{r} \right) \cdot \frac{1}{N} - \frac{1}{2} + \frac{1}{r+m-1}. \]

Here and in what follows, \( c \) denotes a general constant (not necessarily the same at different occurrences) but which depends only on \( N \).
Noting that
\[
\frac{1}{\theta_1} \leq 2, \quad \frac{1 - \theta_1}{\theta_1} (r + m - 1) \leq \frac{N + 2}{N} \quad \text{for } r \in [q, \infty),
\]
\[
\frac{1}{\beta_1} \leq \frac{2(N + 2)}{N} \quad \text{for } r \in [3(m - 1), \infty),
\]
we obtain
\[
\| u_\varepsilon \|_{L^r_x}^{\frac{r + m - 1}{\theta_1}} \leq c \| u_0 \|_{L^1}^{\frac{N + 2}{N}} \cdot \| \nabla u_\varepsilon \|_{L^2}^2 \quad \text{for any } r \in \left[ \max\{q, 3(m - 1)\}, \infty \right).
\]

By (4.9), we easily see that
\[
C_{m, r} \cdot \| u_\varepsilon \|_{L^r_x}^\lambda \leq 2mr(r - 1)(r + m - 1)^2 \| \nabla u_\varepsilon \|_{L^2}^2 \quad \text{for } m > 1 - \frac{2}{N},
\]
where
\[
\lambda := \frac{r + m - 1}{\theta_1 \cdot r} = 1 + \frac{m - 1 + \frac{2}{N}}{r - 1} > 1,
\]
\[
C_{m, r} := \frac{2mr(r - 1)}{(r + m - 1)^2} \cdot (c \| u_0 \|_{L^1})^{-\frac{N + 2}{N}}.
\]

By combining (4.10) with Lemma 4.1,
\[
\frac{d}{dt} \| u_\varepsilon(t) \|_{L^r_x} + C_{m, r} \| u_\varepsilon(t) \|_{L^r_x}^\lambda \leq 0 \quad \text{for } r \in ((N + 2)q, \infty).
\]

Let us denote \( \| u_\varepsilon(t) \|_{L^r_x} \) by \( X(t) \). Then, (4.11) gives
\[
\frac{X(t)}{X(t)^\lambda} + C_{m, r} = \frac{1}{1 - \lambda} \cdot \left( X(t)^{-\lambda+1} \right)' + C_{m, r} \leq 0.
\]

From (4.12), we obtain
\[
X(t) \leq \frac{1}{\left( (\lambda - 1)C_{m, r} \cdot t + X(0)^{-\lambda+1} \right)^{\frac{1}{1 - \lambda}}} \cdot \frac{1}{\min \left\{ (\lambda - 1)C_{m, r}, \| u_0 \|_{L^r_x}^{-(\lambda+1)} \right\}^\frac{1}{1 - \lambda}} \cdot (1 + t)^{-\frac{1}{1 - \lambda}}.
\]

This means that
\[
\| u_\varepsilon(t) \|_{L^r_x} \leq \max \left\{ \left( (\lambda - 1)C_{m, r} \right)^{-\frac{1}{1 - \lambda}}, \| u_0 \|_{L^r_x} \right\} \cdot (1 + t)^{-\frac{1}{1 - \lambda}}.
\]

(4.13)
where
\[
\tilde{C}_{0,r} := \max \left\{ \left\{ \frac{(r + m - 1)^2}{r} \cdot \frac{1}{2m(m - 1 + \frac{1}{N})} \cdot (c\|u_0\|_{L^1})^{\frac{N-1}{2m(m - 1 + \frac{1}{N})}} \right\} \left[ \frac{N}{m-1+N+2} \right]^{(1 - \frac{1}{p})}, \|u_0\|_{L^r} \right\}.
\]

We thus establish the decay estimate for \( r \in [(N + 2)q, \infty) \). On the other hand, by the Hölder inequality and the mass conservation law,
\[
\|u_\varepsilon\|_{L^p} \leq \|u_0\|_{L^1}^{1 - \frac{1}{p}} \|u_\varepsilon\|_{L^r}^{\frac{1}{r}} \quad \text{for } p \in [1, r].
\]

Therefore, we have the \( L^p \)-decay estimates for all \( p \in [1, \infty) \) as follows:
\[
\|u_\varepsilon(t)\|_{L^p} \leq C_{0,p} \cdot (1 + t)^{-\frac{N}{2m(m - 1 + \frac{1}{N})}} \quad \text{for } p \in [1, \infty),
\]
where
\[
C_{0,p} := \|u_0\|_{L^1}^{1 - \frac{1}{p}} \|u_\varepsilon\|_{L^r}^{\frac{1}{r}} \cdot \tilde{C}_{0,r}^{\frac{1}{p}}.
\]

In addition, a solution \( u_\varepsilon \) of the second equation in (KS) can be expressed by the Bessel potential. Therefore, we obtain the same decay estimate as (4.16) for \( u_\varepsilon \).

Furthermore, by the similar argument to that in Section 5 in [32], we can prove that there exists a subsequence \( \{u_{\varepsilon_n}\} \) such that
\[
u_n \rightarrow u \quad \text{strongly in } C((0, \infty); L^p_{loc}(\mathbb{R}^N)),
\]
\[
\nabla u_n \rightarrow \nabla u \quad \text{weakly star in } L^\infty(0, \infty; L^2(\mathbb{R}^N)),
\]
\[
v_n \rightarrow v \quad \text{weakly star in } L^\infty(0, \infty; L^p(\mathbb{R}^N)),
\]
\[
\nabla v_n \rightarrow \nabla v \quad \text{weakly star in } L^\infty(0, \infty; L^p(\mathbb{R}^N)),
\]
\[
\Delta v_n \rightarrow \Delta v \quad \text{weakly star in } L^\infty(0, \infty; L^p(\mathbb{R}^N)),
\]
for any \( p \in [1, \infty) \) and any \( s \in [1, \infty] \). Hence, by the standard convergence argument, we prove the existence of a weak solution \((u, v)\) for (KS). Moreover, by the lower semi-continuity of the norm for \( p \in (1, \infty) \) and Fatou lemma for \( p = 1 \), we obtain the decay estimate (1.6) in Theorem 1.1. Thus, we complete the proof of Theorem 1.1.

5. Proof of Theorem 1.2

We set \((w_k, v_k)\) by
\[
w_k(x, t) := k^N u(kx, k^{N(m-1)+2t}) \quad \text{and} \quad z_k(x, t) := k^N v(kx, k^{N(m-1)+2t}) \quad \text{for } k > 1.
\]

Then, the above \((w_k, v_k)\) becomes a non-negative weak solution of the following problem:
\[
\begin{align*}
\begin{cases}
\begin{align*}
\text{w(KS)} & \quad \text{in } \mathbb{R}^N \times (0, \infty), \quad \cdots (1)_w
\end{align*}
\end{cases}
\end{align*}
\]

where \( N \geq 1, \ m > 1, \ q \geq \max\{m + \frac{2}{N}, 2\} \).
5.1. A priori estimate for \( w_k \)

By (4.15)–(4.16) in the proof of Theorem 1.1, we see that there exists a constant \( C_{0,p} = C_{0,p}(m, q, N, \|u_0\|_{L^1}, \|u_0\|_{L^{(N+2)q}}) \) independently of \( k \) such that

\[
\|w_k(t)\|_{L^p} = k^{N(1 - \frac{1}{p})}\|u(k^{N(m-1)+2}t)\|_{L^p} \leq C_{0,p} t^{-\frac{N}{m(m-1)+2}} (1 + \frac{t}{\delta}) \quad \text{for } t > 0 \text{ and } p \in [1, \infty).
\]

Moreover, we obtain the \( L^\infty(\delta, T; L^\infty(\mathbb{R}^N)) \)-estimate (for any \( \delta > 0 \)) for \( w_k \) by Moser’s iteration technique. To this end, we prepare the \( L^\infty(\delta, T; L^p(\mathbb{R}^N)) \)-estimate for \( w_k \) using (5.2).

**Lemma 5.1.** Let \( \delta > 0, p \geq 1 \) and let \( N \geq 1, m > 1, q \geq \max \{m + \frac{1}{q}, 2\} \). Suppose that \( (w_k, z_k) \) is a weak solution of \( w(\text{KS}) \). We assume that \( w_k \) satisfies (5.2). Then, there exist positive numbers \( R_{\delta,p}, Q_{\delta,p} \) depending on \( \delta, p, m, q, N, \|u_0\|_{L^1(\mathbb{R}^N)} \) but not on \( k \) such that

\[
\sup_{0 < t < \infty} \left( \|w_k\|_{L^p(\mathbb{R}^N)}^p \right) \leq R_{\delta,p} \quad \text{and} \quad \sup_{0 < t < \infty} \left( \|w_k\|_{L^p(\mathbb{R}^N)} \right) \leq Q_{\delta,p},
\]

where \( \eta(t) \) is a sequence of cut-off functions defined by \( \eta(t) : = \eta(\frac{t}{\delta}) \) with \( \eta \) introduced in Lemma 2.3.

**Proof of Lemma 5.1.** From (5.2) and Lemma 2.3, we see that

\[
\sup_{0 < t < \infty} \left( \|w_k\|_{L^p(\mathbb{R}^N)}^p \right) \leq \sup_{0 < t < \delta} \left( \|w_k\|_{L^p} \right) + \sup_{\delta < t < \infty} \left( \|w_k\|_{L^p} \right) \leq (C_{0,p})^p \sup_{0 < t < \delta} \left( t^{-\frac{N}{m(m-1)+2}} \|\eta(t)\|_{L^1} \right) + (C_{0,p})^p \sup_{\delta < t < \infty} \left( t^{-\frac{N}{m(m-1)+2}} \right) \leq (C_{0,p})^p \frac{\mu^\mu}{\delta^{\mu \cdot p^\mu - 1}} \quad \text{for } \mu := \frac{N}{N(m-1)+2} \cdot (p - 1).
\]

From (2) in \( w(\text{KS}) \) and (5.2), we see by the standard argument that

\[
\sup_{\delta < t < \infty} \|\Delta z_k(t)\|_{L^p} \leq 2k^2 C_{0,p} \delta^{-\frac{p}{2}}, \quad \sup_{\delta < t < \infty} \|A z_k(t)\|_{L^p} \leq k(p + N^2 - 1) \delta \leq 2k^2 C_{0,p} \delta^{-\frac{p}{2}} \quad \text{for any } \delta > 0 \text{ and } p \in [2, \infty),
\]

where \( C_{0,p} \) is the constant in (5.2) and \( \mu := \frac{N}{N(m-1)+2} \cdot (p - 1) \). Moreover, by (2.6) and (2.7) in Lemma 2.2, we see that

\[
\|\nabla z_k(t)\|_{L^\infty} \leq \frac{2p}{p - N} \left( \|\nabla z_k(t)\|_{L^p} + \|\nabla^2 z_k(t)\|_{L^p} \right) \leq \frac{2p}{p - N} \cdot c \left( \frac{p^2}{p - 1} \right)^2 \left( \|\nabla z_k(t)\|_{L^p} + \|\Delta z_k(t)\|_{L^p} \right) \leq \frac{2p}{p - N} \cdot c \left( \frac{p^2}{p - 1} \right)^2 \cdot (p + N^2 + 1) C_{0,p} \delta^{-\frac{p}{2}} \cdot k^2 \quad \text{for } p \in (N, \infty).
\]
Therefore, taking $p = N + 1$, we have
\[ \sup_{\delta < t < \infty} \| \nabla z_k(t) \|_{L^\infty} \leq c \cdot C_{0,N+1} \delta^{-\frac{m}{m-1}} \cdot k^2 =: C \nabla z \cdot k^2 \quad \text{for any } \delta > 0. \quad (5.6) \]

Using (5.5), (5.6) and Lemma 5.1, we establish the $L^\infty$-estimate for $w_k$ in the following lemma.

**Lemma 5.2.** Let the same assumption as that in Lemma 5.1 hold and let $T > 0$. Then, there exists a positive number $W_{\delta,T}$ depending on $\delta, m, q, N, \| u_0 \|_{L^{1,\gamma(L,\gamma-2)_N}(\mathbb{R}^N)}$, $T$ but not on $k$ such that
\[ \sup_{\delta < t < T} \| w_k(t) \|_{L^\infty(\mathbb{R}^N)} \leq W_{\delta,T} \quad \text{for all } \delta > 0. \quad (5.7) \]

**Proof of Lemma 5.2.** We follow the argument employed in [33], Proof of Lemma 10. (For the sake of simplicity, we perform only the formal calculation.)

We multiply (1) in $w(KS)$ by $(w_k)^{p-1} \eta_k(t)$ and integrate it over $\mathbb{R}^N$, where $p > 1$ and $\eta_k$ is the function defined in Lemma 5.1. Then, by $q \geq m + \frac{2}{N}$, (5.6) and Young inequality, we have
\[ \frac{1}{p} \cdot \frac{d}{dt} \left( \| w_k \|_{L^p(q)} \eta_k(t) \right) \leq -\frac{4m(p-1)}{(p+m-1)^2} \| \nabla w_k \|_{L^q}^\frac{p+m-1}{2} \eta_k(t) \]
\[ + \frac{1}{p} \cdot \| w_k \|_{L^p(q)} \partial_t \eta_k(t) \quad + \quad k^{-N(1-m)}(p-1) \int (w_k)^{q-1} \nabla z_k \cdot (w_k)^{p-2} \nabla w_k \cdot \eta_k(t) \ dx \]
\[ \leq -\frac{2m(p-1)}{(p+m-1)^2} \| \nabla w_k \|_{L^q}^\frac{p+m-1}{2} \eta_k(t) \]
\[ + \frac{1}{pt^2} \| w_k \|_{L^p(q)}^p \eta_k(t) \]
\[ + \frac{1}{pt^2} \| w_k \|_{L^p(q)}^p \eta_k(t) \quad - \quad \frac{1}{2m} \cdot k^{-2N(q-m)+2} (C \nabla z)^2 \| w_k \|^p_{L^{p+2(q-2)+2(N-1)}} \eta_k(t) \]
\[ \leq -\frac{2m(p-1)}{(p+m-1)^2} \| \nabla w_k \|_{L^q}^\frac{p+m-1}{2} \eta_k(t) \]
\[ + \frac{1}{pt^2} \| w_k \|_{L^p(q)}^p \eta_k(t) \quad + \quad \frac{1}{2m} (C \nabla z)^2 \| w_k(0) \|_{L^{p+2(q-2)+2(N-1)}} \eta_k(t) \]
\[ \leq \quad \frac{p^2 \| w_k \|_{L^{p+2(q-2)+2(N-1)}}} \quad \text{for all } p \in [2m - 2q + 3, \infty). \]

Using Lemma 2.1, we see that there exists a positive number $p_0$ depending only on $m, q, N$ such that
\[ p^2 \| w_k \|_{L^{p+2(q-2)+2(N-1)}} \leq \frac{m(q+1)}{(q+m-1)^2} \| \nabla (w_k)^{\frac{p+m-1}{2}} \|_{L^2}^2 + \frac{(p+m-1)^2}{2} \| w_k \|_{L^q}^\frac{p+m-1}{2} \eta_k(t) \]
\[ \quad \text{for all } p \in [p_0, \infty). \quad (5.9) \]

and
\[ \frac{1}{p} \cdot \frac{d}{dt} \left( \| w_k \|_{L^p(q)}^p \right) \leq \frac{m(p-1)}{(p+m-1)^2} \| \nabla w_k \|_{L^q}^\frac{p+m-1}{2} \eta_k(t) \]
\[ + \frac{1}{p} \cdot \left( \frac{\eta_k(t)}{\eta_k(t)} \right) \eta_k(t) \eta_k(t) \eta_k(t) \]
\[ \leq \frac{m(p-1)}{(p+m-1)^2} \| \nabla (w_k)^{\frac{p+m-1}{2}} \|_{L^2}^2 + \frac{(p+m-1)^2}{2} \| w_k \|_{L^q}^\frac{p+m-1}{2} \eta_k(t) \]
\[ \quad \text{for all } p \in [p_0, \infty). \quad (5.10) \]
Substituting (5.9) and (5.10) into (5.8) and integrating it from 0 to \( t \), we have

\[
\sup_{0 < t < T} \| w_k \eta_\delta(t) \|_{L^p}^\delta \leq \left( \frac{1}{2m} (C_{\mathcal{V}_\gamma})^2 \| u_0 \|_{L^p} \right)^{\frac{p+1}{2m-1}} T + 2pT + pT(p + m - 1) \sup_{0 < t < T} \| w_k \eta_\delta(t) \|_{L^p}^\delta \tag{5.11}
\]

for all \( p \in [p_0, \infty) \). Applying the Moser iteration technique to (5.11), we have

\[
\sup_{0 < t < T} \| w_k \eta_\delta(t) \|_{L^p}^\delta \leq 4^{C_1p_0} 4^{2-p_0} C_1^{2-p_0} \max \left\{ \left( \frac{1}{2m} (C_{\mathcal{V}_\gamma})^2 \right)^{\frac{1}{p-1}}, 1, R_{\delta,4p_0} \right\} \tag{5.12}
\]

where \( C_1 \) is a constant depending only on \( \delta, m, q, N, \| u_0 \|_{L^1}, \| u_0 \|_{L^{N+1}} \). Consequently, by letting \( p \) tend to \( \infty \), we see that \( w_k \in L^\infty(\delta, \infty; L^\infty(\mathbb{R}^N)) \) and

\[
\| w_k(t) \|_{L^\infty(\mathbb{R}^N)} \leq 4^{C_1(p_0)q} 4^{2-q_0} C_1^{2-q_0} \max \left\{ \left( \frac{1}{2m} (C_{\mathcal{V}_\gamma})^2 \right)^{\frac{1}{q-1}}, 1, R_{\delta,4q_0} \right\} =: W_{\delta,T} \tag{5.13}
\]

for any fixed number \( \delta > 0 \). Thus, the \( L^\infty(\mathbb{R}^N) \)-bound for \( w_k \) is obtained, which complete the proof of Lemma 5.2.

\[\square\]

**Lemma 5.3.** Let the same assumption as that in Lemma 5.1 hold and let \( T > 0 \). Then, there exists a positive number \( N_{\delta,T} \) depending on \( \delta, m, q, N, \| u_0 \|_{L^1}, \| u_0 \|_{L^{N+1}} \), \( T \) but not on \( k \) such that

\[
\int_{\delta}^{T} \| \partial_t (w_k)^m \|_{L^2(\mathbb{R}^N)}^2 \ dt + \sup_{\delta < t < T} \| \nabla (w_k)^m(t) \|_{L^2(\mathbb{R}^N)}^2 \leq N_{\delta,T}. \tag{5.14}
\]

**Proof of Lemma 5.3.** For the sake of simplicity, we perform only the formal calculation. We multiply (1) in \( \mathcal{W}(K) \) by \((w_k)^{p-1} \eta_\delta(t)\) and integrate it over \( \mathbb{R}^N \), where \( p > 1 \) and \( \eta_\delta \) is the function defined in Lemma 5.1. Then, similarly to (4.1), we have

\[
\frac{d}{dt} \left( \| w_k \|_{L^p}^p \right) + \frac{4mp(p-1)}{(p+m-1)^2} \| \nabla w_k \|_{L^2}^2 \| \eta_\delta \|_{L^2} \leq \| w_k \|_{L^p}^p \partial_t \eta_\delta(t)
\]

\[
+ k^{-N(q-m)+2} \cdot \frac{p(p-1)}{p+q-2} \| w_k \|_{L^{p+q-1}} \eta_\delta(t) \quad \text{for all } p \in (1, \infty). \tag{5.15}
\]

Integrating (5.15) from 0 to \( t \), from Lemma 5.1 and \( q \geq m + \frac{2}{N} \), we have

\[
\| w_k \|_{L^p}^p \eta_\delta(t) + \frac{4mp(p-1)}{(p+m-1)^2} \int_0^t \| \nabla w_k \|_{L^2}^2 \eta_\delta(s) \ ds \leq \int_0^t \| w_k \|_{L^p}^p \partial_t \eta_\delta(s) \ ds
\]

\[
+ k^{-N(q-m)+2} \cdot \frac{p(p-1)}{p+q-2} \int_0^t \| w_k \|_{L^{p+q-1}} \eta_\delta(s) \ ds \leq Q_{\delta,p} T + \frac{p(p-1)}{p+q-2} \cdot R_{\delta, p+q-1} T \tag{5.16}
\]

for all \( p \in (1, \infty) \) and \( t \in (0, T) \).
Thus, we obtain
\[
\int_0^T \| \nabla w_k^{\frac{m+1}{4}} \|^2_{L^2} \eta_k(s) \, ds \\
\leq \frac{(p + m - 1)^2}{4mp(p - 1)} \left( Q_{\delta,p} + \frac{p(p - 1)}{p + q - 2} \right) R_{\delta,p+q-1} T =: L_{\delta,p} T \quad \text{for all } p \in (1, \infty).
\] (5.17)

By the similar argument, we find that there exists a positive number \( \tilde{L}_{\delta,p} \) depending on \( \delta, p, m, q, N, \) \( \| u_0 \|_{L^1}, \| u_0 \|_{L^p} \) but not \( k \) such that
\[
\int_0^T \| \nabla w_k^{\frac{p+m-1}{2}} \|^2_{L^2} \eta_k(t) \frac{1}{t^2} \, dt \leq \tilde{L}_{\delta,p} T \quad \text{for all } p \in (1, \infty).
\] (5.18)

Next, we multiply (1) in \( u \) by \( \partial_t (w_k)^m \cdot \eta_k(t) \) and integrate it over \( \mathbb{R}^N \). Noting that \( \Delta z_k = k^2(z_k - w_k) \), from (5.4) and (5.6), we have
\[
\frac{4m}{(m + 1)^2} \left( \| \partial_t (w_k)^{m+1} \|^2_{L^2} \eta_k(t) + \frac{1}{2} \frac{d}{dt} \| (w_k)^m \|^2_{L^2} \eta_k(t) \right) \\
= \frac{1}{2} \| \nabla (w_k)^m \|^2_{L^2} \frac{\eta_k(t)}{\delta} - k^{-N(q-m)} \frac{2m}{m + 1} \int \left( w_k^{\frac{m+1}{2}} \nabla (w_k)^q \right) \partial_t (w_k)^{m+1} \eta_k(t) \, dx \\
\leq \frac{1}{2} \| \nabla (w_k)^m \|^2_{L^2} \frac{\eta_k(t)}{\delta} + \frac{m}{(m + 1)^2} \left( \frac{\delta}{\eta_k(t)} \right)^2 + k^{-N(q-m)} m \int \left( w_k^{m+1} \nabla (w_k)^q \right)^2 \eta_k(t) \, dx \\
\leq \frac{1}{2} \| \nabla (w_k)^m \|^2_{L^2} \frac{\eta_k(t)}{\delta} + \frac{m}{(m + 1)^2} \left( \frac{\delta}{\eta_k(t)} \right)^2 + k^{-N(q-m)} m \left( \| w_k \|_{L^4}^{m+2q-1} + \| w_k \|_{L^{4q-1}}^{m+2q-3} + \| z_k \|^{4q}_{L^4} \right) \eta_k(t). 
\] (5.19)

Integrating (5.19) from 0 to \( t \) and using (5.3), (5.4), (5.17), (5.18) and \( q \geq \max \{ m + \frac{3}{2}, 2 \} \), we have
\[
\frac{3m}{(m + 1)^2} \int_0^t \left( \frac{\delta}{\eta_k(t)} \right)^2 \frac{d}{dt} \| \nabla (w_k)^m \|^2_{L^2} \eta_k(t) \\
\leq \frac{\delta}{2} \int_0^t \| \nabla (w_k)^m \|^2_{L^2} \eta_k(t) \frac{1}{t^2} \\
+ k^{-2N(q-m)+4} \cdot 2m \left( \frac{q - 1}{q + \frac{m-3}{2}} \right)^2 \left( C \nabla z \right)^2 \int_0^t \| \nabla (w_k)^q \|^2_{L^2} \eta_k(t) \, dt \\
+ k^{-2N(q-m)+4} \cdot 2m \left( \| w_k \|_{L^{m+2q-1}}^{m+2q-1} + \| w_k \|_{L^{4q-1}}^{m+2q-3} + \| z_k \|^{4q}_{L^4} \right) \eta_k(t) \, dt \\
\leq \frac{\delta}{2} \tilde{L}_{\delta,m+1} T + 2m \left( \frac{q - 1}{q + \frac{m-3}{2}} \right)^2 \left( C \nabla z \right)^2 \cdot L_{\delta,2q-2} T \\
+ 2m \left( 2R_{\delta,m+2q-1} + R_{\delta,2(m+2q-3)} + (C_0) \delta^{\frac{4q}{m+3}} \right) T
\]
for \( t \in (0, T) \) and \( \mu = \frac{N}{N(m-1)+2} \cdot (p-1) \). Therefore, we find that
\[
\sup_{\delta < t < T} \| \nabla (w_k)^m(t) \|_{L^2}^2 \leq \delta \tilde{L}_{\delta,m+1} T + \frac{(m+1)^2}{6m} \int_\delta \| \partial_t (w_k)^{m+1} \|_{L^2}^2 \, dt
\]
\[
+ 2m \left( 2 R_{\delta,m+2q-1} + R_{\delta,2(m+2q-3)} + (C_{0,4})^4 \delta^{-\frac{4q}{p}} \right) T
\]
\[
=: N_{\delta}^{(1)} T.
\]
Moreover, it holds that
\[
\int_\delta^T \| \partial_t (w_k)^m \|_{L^2}^2 \, dt \leq \frac{(m+1)^2}{6m} \int_\delta^T \| \partial_t (w_k)^{m+1} \|_{L^2}^2 \, dt
\]
\[
+ 2m \left( 2 R_{\delta,m+2q-1} + R_{\delta,2(m+2q-3)} + (C_{0,4})^4 \delta^{-\frac{4q}{p}} \right) T
\]
\[
=: N_{\delta}^{(2)} T.
\]
From (5.20) and Lemma 5.2, we find that
\[
\int_\delta^T \| \partial_t (w_k)^m \|_{L^2}^2 \, dt \leq \left( \frac{2m}{m+1} \right)^2 (W_{\delta,T})^{m-1} \int_\delta^T \| \partial_t (w_k)^{m+1} \|_{L^2}^2 \, dt
\]
\[
\leq \left( \frac{2m}{m+1} \right)^2 (W_{\delta,T})^{m-1} N_{\delta,T}^{(2)} =: N_{\delta,T}^{(3)} T.
\]
Thus, we establish
\[
\int_\delta^T \| \partial_t (w_k)^m \|_{L^2}^2 \, dt + \sup_{\delta < t < T} \| \nabla (w_k)^m(t) \|_{L^2}^2 \leq (N_{\delta}^{(1)} + N_{\delta,T}^{(3)}) T =: N_{\delta,T} T
\]
for any fixed number \( \delta > 0 \), which completes the proof of Lemma 5.3. \( \square \)

### 5.2. Convergence

From (5.2), we find that \( \| w_k(t) \|_{L^p} \) is bounded on \( [\delta, \infty) \) for any \( \delta > 0 \). Therefore, we can extract a subsequence \( \{w_{k_n}\} \) which converges in \( L^p (1 < p < \infty) \) such that
\[
w_{k_n} \rightharpoonup U \quad \text{weakly in } L^p(\delta, T; L^p(\mathbb{R}^N))
\]
for any \( T > 0 \) and \( \delta \in (0, T) \). Moreover, we see that there exists a subsequence, still denoted by \( \{w_{k_n}\} \) such that
\[
w_{k_n} \rightarrow U \quad \text{strongly in } C((0, T); L^p_{\text{loc}}(\mathbb{R}^N)),
\]
\[
\nabla (w_{k_n})^m \rightarrow \nabla U^m \quad \text{weakly in } L^2(\delta, T; L^2(\mathbb{R}^N))
\]
for any \( p \) with \( 1 < p < \infty \) and for any \( T > 0 \). The above (5.22) and (5.23) are shown as follows:

From Lemma 5.3, we see that \( (w_k)^m \) is bounded in \( H^1(\delta, T; L^2(\mathbb{R}^N)) \cap L^\infty(\delta, T; H^1(\mathbb{R}^N)) \) for any \( T > 0 \) and \( \delta \in (0, T) \). Therefore, we can extract a subsequence such that
\[
(w_{k_n})^m \rightarrow \xi \quad \text{strongly in } C((\delta, T); L^2_{\text{loc}}(\mathbb{R}^N))
\]
(5.24)
for any \( T > 0 \) and \( \delta \in (0, T) \). This gives
\[
(w_{k_n})^m(x, t) \rightarrow \xi(x, t) \quad \text{a.a. } x \in \mathbb{R}^N, \ t \in (0, T).
\]

A function \( g(w) = w^{\frac{m}{m-1}} \) is continuous with respect to \( u \).

Thus, we see that
\[
w_{k_n}(x, t) \rightarrow \xi^m(x, t) \quad \text{a.a. } x \in \mathbb{R}^N, \ t \in (0, T).
\]

On the other hand, by Lemma 5.2, we see that there exists a constant \( W_{\delta,T} \) independently of \( k \) such that
\[
\sup_{\delta < t < T} \| w_{k_n}(t) \|_{L^\infty} \leq W_{\delta,T}.
\]

Therefore, Lebesgue dominated convergence theorem and (5.21) gives that
\[
w_{k_n} \rightarrow \xi^m = U \quad \text{strongly in } L^p((\delta, T); L^p_{\text{loc}}(\mathbb{R}^N)) \quad (5.26)
\]
for any \( p \in (1, \infty) \). From (5.26), we observe that
\[
w_{k_n}(x, t) \rightarrow \xi^m(x, t) = U(x, t) \quad \text{a.a. } x \in \mathbb{R}^N, \ t \in (0, T).
\]

From (5.24) and (5.27),
\[
w_{k_n}^m \rightarrow U^m \quad \text{strongly in } C((\delta, T); L^2_{\text{loc}}(\mathbb{R}^N)) \quad (5.28)
\]
In addition, since \(|b-a|^m \leq |b^m - a^m| \) for \( 0 \leq a \leq b \) and \( m \geq 1 \), from (5.22) we see that
\[
w_{k_n} \rightarrow U \quad \text{strongly in } C((\delta, T); L^{2m}_{\text{loc}}(\mathbb{R}^N)).
\]

By Hölder inequality and (5.29), in all cases of \( 1 < p < \infty \), it holds that
\[
w_{k_n} \rightarrow U \quad \text{strongly in } C((0, T); L^p_{\text{loc}}(\mathbb{R}^N)) \quad (5.30)
\]
for any \( T > 0 \) and \( \delta \in (0, T) \). Now let \( \delta \rightarrow 0 \). Employing a diagonal process,
\[
w_{k_n} \rightarrow U \quad \text{strongly in } C((0, T); L^p_{\text{loc}}(\mathbb{R}^N)),
\]
which prove (5.22). From (5.14) in Lemma 5.3 and (5.30), we obtain (5.23).

Using (5.22)-(5.23), we find by \( q > m + \frac{2}{N} \) that \( U(x,t) \) satisfies the porous medium equation in a distribution sense, i.e., that
\[
\int_{\tau_1}^{\tau_2} \int U \varphi_t + U^m \Delta \varphi \, dxdt = \int U(x, \tau_2) \varphi(x, \tau_2) \, dx - \int U(x, \tau_1) \varphi(x, \tau_1) \, dx
\]
for all \( C^2 \) functions \( \varphi(x,t) \) with compact support in \( \mathbb{R}^N \times (0, T] \), and all \( 0 < \tau_1 < \tau_2 < T \). We now remark that the critical case of \( q = m + \frac{2}{N} \) should be excluded from Theorem 1.2.
5.3. Key lemma

We are now in a position to prove the following key lemma:

**Lemma 5.4** (key lemma). Let $N \geq 1$, $m > 1$ and $q \geq \max\{m + \frac{2}{N}, 2\}$. Let $(w_k, z_k)$ be a weak solution of $(KS)$ and let $U$ satisfy (5.31). We assume that $w_k$ converges to $U$ in $C((0, T); L_{loc}^p(\mathbb{R}^N))$ with $1 \leq p < \infty$. Then, it holds that

$$\left| \int_{\mathbb{R}^N} (w_k(x, t) - U(x, t)) \, dx \right| \to 0 \quad \text{a.a. } t \in (0, T) \quad \text{as } k \to \infty. \quad (5.32)$$

**Proof of Lemma 5.4.** It is easy to verify that $w_k \to U$ strongly in $L_{loc}^1(\mathbb{R}^N)$. Indeed, it holds that

$$\|w_{k_n}(t) - U(t)\|_{L^1(K)} \leq \|w_{k_n}(t) - U(t)\|_{L^p(K)} \frac{2}{m-1} \to 0 \quad \text{all } t \in (0, T) \quad (5.33)$$

as $k_n \to \infty$ for any compact set $K \subset \subset \mathbb{R}^N$ and for any $T > 0$.

Next, we fix a time $t$ in $(0, \infty)$ and prove that for any $\lambda > 0$, there exist $f_\lambda(\cdot, t) \in L^1(\mathbb{R}^N)$ and $k_0 \in \mathbb{N}$ such that

$$\int [w_k(x, t) - f_\lambda(x, t)]^+ \, dx < \frac{\lambda}{6} \quad \text{for any } k > k_0. \quad (5.34)$$

To this end, we prepare the following lemma:

**Lemma 5.5.** Let $N \geq 1$, $m > 1$ and $q \geq \max\{m + \frac{2}{N}, 2\}$. Let $(u, v)$ be the weak solution of $(KS)$ obtained in Theorem 1.1 and let $U$ satisfy (5.31). Then, there exists a function $g \in L^1(0, \infty)$ such that

$$\int_{\mathbb{R}^N} [u(t_2) - U(t_2)]^+ \, dx dt - \int_{\mathbb{R}^N} [u(t_1) - U(t_1)]^+ \, dx dt \leq \int_{t_1}^{t_2} g(t) \, dt \quad \text{for all } t_1, t_2 \text{ with } 0 < t_1 < t_2 < \infty. \quad (5.35)$$

**Proof of Lemma 5.5.** We give the formal calculation. $\eta_n(r)$ is a sequence of cut-off functions defined by $\eta_n(r) := \eta(n t)$ with $\eta$ defined in Lemma 2.3. By multiplying the first equation in $(KS)$ by $\eta_n(u^m - U^m)$ and integrating it over $\mathbb{R}^N$, we get

$$\int_{\mathbb{R}^N} (u - U)_t \cdot \eta_n(u^m - U^m) \, dx = - \int_{\mathbb{R}^N} |\nabla (u^m - U^m)|^2 \cdot \eta_n'(u^m - U^m) \, dx$$

$$- \int_{\mathbb{R}^N} \left( \nabla u^{q-1} \cdot \nabla v + u^{q-1} \cdot \Delta v \right) \cdot \eta_n(u^m - U^m) \, dx$$

$$\leq - \int_{\mathbb{R}^N} (\nabla u^{q-1} \cdot \nabla v \cdot \eta_n(u^m - U^m) + u^q \cdot \eta_n(u^m - U^m)) \, dx$$

$$\leq \int_{\mathbb{R}^N} (|q - 1| u^{q-2} |\nabla u| |\nabla v| + u^q) \, dx$$

$$\leq \int_{\mathbb{R}^N} u^{q-2} |\nabla u|^2 \, dx + c_q \int_{\mathbb{R}^N} (u^q + |\nabla v|^q) \, dx$$

$$\leq \int_{\mathbb{R}^N} \frac{u^{q-2} |\nabla u|^2}{c_q^q} \, dx + c_q \int_{\mathbb{R}^N} u^q \, dx =: I + J, \quad (5.36)$$
where
\[ I(t) := \int_{\mathbb{R}^N} u^{q-2}|\nabla u|^2(t) \, dx, \quad J(t) := \int_{\mathbb{R}^N} u^q(t) \, dx \]
and \( c_q \) and \( c_q' \) are constants depending only on \( q \).

We are now going to show that \( I, J \in L^1(0, \infty) \). Similarly to (4.1), we have
\[
\|u(t)\|_{L^r} + mr(r-1) \int_0^t \int_{\mathbb{R}^N} u^{r+m-3}|\nabla u|^2 \, dz \, ds \leq \|u_0\|_{L^r}^{(r-1)} + \frac{r(r-1)}{r+q-2} \int_0^t \|u(s)\|_{L^{r+m}}^{r+q-1} \, ds \quad \text{for all } r \in (1, \infty). \tag{5.37}
\]
Taking \( r = q - m + 1 \) \((> 1 + \frac{2}{N})\) in (5.37), from Theorem 1.1 and \( q \geq m + \frac{2}{N} > (m + \frac{1}{N}) \), we have
\[
\int_0^\infty I(t) \, dt \leq c_{m,q} \|u_0\|_{L^{N(m-1)+2}}^{\frac{q-m+1}{m}} + c_{m,q} \int_0^\infty \|u(s)\|_{L^{N(m-1)+2}}^{2q-m} \, ds \leq c_{m,q} \|u_0\|_{L^{N(m-1)+2}}^{\frac{q-m+1}{m}} + C_2 \int_0^\infty (1 + t)^{-\frac{N}{m-1} + q-1} \, dt \leq c_{m,q} \|u_0\|_{L^{N(m-1)+2}}^{\frac{q-m+1}{m}} + C_3, \tag{5.38}
\]
where \( c_{m,q} \) is a constant depending only on \( m, q \) and \( C_2 \) and \( C_3 \) are constants depending only on \( m, q, N, \|u_0\|_{L^1}, \) and \( \|u_0\|_{L^{2q-m}} \).

By Theorem 1.1 and \( q \geq m + \frac{2}{N} \), similarly to (5.38), we have
\[
\int_0^\infty J(t) \, dt \leq c_q \int_0^\infty (1 + t)^{-\frac{N}{m-1} + q-1} \, dt < C_4, \tag{5.39}
\]
where \( C_4 \) depends only on \( m, q, N, \|u_0\|_{L^1}, \) and \( \|u_0\|_{L^\infty} \).

We set \( [s]^+ = \max(s, 0) \) and
\[
\left\{ \begin{array}{ll}
sign_0^+(s) = 1 & \text{for } s > 0, \\
sign_0^+ (s) = 0 & \text{for } s \leq 0.
\end{array} \right.
\]
Noting that \( \eta_n \) converge to the sign function \( \text{sign}_0^+ \) as \( n \to \infty \) and \( \partial_t [u - U]^+ = \partial_t (u - U) \cdot \text{sign}_0^+ (u - U) \). (See Gilbarg-Trudinger [14].)

Taking the limit in (5.36) as \( n \to \infty \) and integrating it from \( t_1 \) to \( t_2 \), we have
\[
\int [u(t_2) - U(t_2)]^+ \, dx - \int [u(t_1) - U(t_1)]^+ \, dx \leq \int_{t_1}^{t_2} I(s) + J(s) \, ds < \infty \tag{5.40}
\]
for any \( t_1, t_2 \) with \( 0 < t_1 \leq t_2 < \infty \). Thus, we observe that there exists \( g = (I + J) \in L^1(0, \infty) \) satisfying (5.35). Thus, we complete the proof of Lemma 5.5. \( \square \)

Let us define \( \beta \) and \( V(x, t; M) \) by
\[
\int_{\mathbb{R}^N} \left( \beta^2 - \frac{m-1}{2m(N(m-1)+2)} \right) |y|^2 \, dy = 1 \quad \text{and}
V(x, t; M) := \frac{1}{t^{N(N(m-1)+2)}} \left( \beta^2 M^{\frac{2(m-1)}{N(N(m-1)+2)}} - \frac{m-1}{2m(N(m-1)+2)} \right) t^{\frac{m-1}{N(N(m-1)+2)}} |x|^2^{\frac{1}{m-1}}.
\]
Then, we easily see that \( \int_{\mathbb{R}^N} V(x, t; M) \, dx = M \). It is known that the above function \( V(x, t, M) \) is the exact solution of \( u_t = \Delta u^{m} \) which is the so-called Barenblatt solution. (we refer to Barenblatt [2] for instance.) Therefore, we can take \( U(x, t) \) by \( U(x, t) = V(x, t; M) \) in Lemma 5.5. Consequently, we see that for any \( \lambda > 0 \), there exists \( T_\lambda > 0 \) sufficiently large such that

\[
\sup_{T_\lambda < t < \infty} \left( \int_{\mathbb{R}^N} [u(x, t) - V(x, t; M)]^+ \, dx - \int_{\mathbb{R}^N} [u(x, T_\lambda) - V(x, T_\lambda; M)]^+ \, dx \right) < \frac{\lambda}{12}. \tag{5.41}
\]

Moreover, for any fixed \( \lambda > 0 \), we can find a constant \( M(\lambda) > 0 \) such that

\[
\int_{\mathbb{R}^N} [u(x, T_\lambda) - V(x, T_\lambda; M(\lambda))]^+ \, dx < \frac{\lambda}{12}. \tag{5.42}
\]

Indeed, for any \( \lambda > 0 \), there exists a compact set \( K_\lambda \subset \subset \mathbb{R}^N \) such that

\[
\int_{\mathbb{R}^N} [u(x, T_\lambda) - V(x, T_\lambda; M)]^+ \, dx \leq \int_{\mathbb{R}^N \setminus K_\lambda} [u(x, T_\lambda) - V(x, T_\lambda; M)]^+ \, dx + \int_{K_\lambda} [u(x, T_\lambda) - V(x, T_\lambda; M)]^+ \, dx
\]

\[
\leq \frac{\lambda}{12} + \int_{K_\lambda} [u(x, T_\lambda) - V(x, T_\lambda; M)]^+ \, dx. \tag{5.43}
\]

On the other hand,

\[
\text{supp } V(\cdot, T_\lambda; M) = \left\{ x \in \mathbb{R}^N; |x| \leq M \frac{2m(N(m-1)+2)}{m-1} \beta \left( \frac{2m(N(m-1)+2)}{m-1} \right)^{\frac{1}{2}} T_\lambda^{\frac{m-1}{2}} \right\}
\]

and

\[
V(x, T_\lambda; M)^{m-1} = M^{\frac{2(m-1)}{N(m-1)+2}} T_\lambda^{\frac{N(m-1)}{N(m-1)+2}} \beta^2 - \frac{m-1}{2m(N(m-1)+2)} T_\lambda^{-1} |x|^2 \text{ in supp } V(\cdot, T_\lambda; M).
\]

By Theorem 1.1, the weak solution \( u \) belongs to \( L^\infty(0, \infty; L^1 \cap L^\infty(\mathbb{R}^N)) \). Therefore, taking \( M = M(\lambda, K_\lambda, T_\lambda) \) sufficiently large, we verify

\[
\int_{K_\lambda} [u(x, T_\lambda) - V(x, T_\lambda; M)]^+ \, dx = 0. \tag{5.44}
\]

Combining (5.43) with (5.44), we obtain (5.42).

In consequence, from (5.41) and (5.42), we observe that there exist \( T_\lambda > 0 \) and \( M_\lambda > 0 \) such that

\[
\sup_{T_\lambda < t < \infty} \int_{\mathbb{R}^N} [u(x, t) - V(x, t; M_\lambda)]^+ \, dx \leq \frac{\lambda}{6}. \tag{5.45}
\]

On the other hand, it is easy to see that \( V(x, t; M_\lambda) = k^N V(kx, k^{N(m-1)+2}t; M_\lambda) \) all \( x \in \mathbb{R}^N \) and \( t \in (0, \infty) \). Therefore, the above (5.45) is equivalent to

\[
\sup_{T_\lambda < k^{N(m-1)+2} < \infty} \int_{\mathbb{R}^N} [w_k(x, t) - V(x, t; M_\lambda)]^+ \, dx \leq \frac{\lambda}{6}. \tag{5.46}
\]
From (5.46), we see that for any fixed time $t \in (0, \infty)$ and $\lambda > 0$, there exist $V(\cdot, t; M_\lambda) \in L^1(\mathbb{R}^N)$ and $k_0 = \left( \frac{T_\lambda}{T} \right)^{\frac{2\lambda}{N+4\lambda}}$ such that

$$\int_{\mathbb{R}^N} [w_k(x, t) - V(x, t; M_\lambda)]^+ \, dx < \frac{\lambda}{6} \quad \text{for any } k > k_0. \quad (5.47)$$

We thus conclude by Lemma 2.4 that

$$\left| \int_{\mathbb{R}^N} (w_k(x, t) - U(x, t)) \, dx \right| \to 0 \quad \text{as } k \to \infty$$

for all $t \in (0, \infty)$.

### 5.4. Proof of Theorem 1.2

From (5.48), we see that

$$\int U(1) \, dx = \int u_0(x) \, dx. \quad (5.48)$$

Indeed, by the mass conservation law ($\|w_k(t)\|_{L^1} = \|w_k(0)\|_{L^1}$ for all $t \geq 0$) and (5.48), it holds that

$$\int u_0(x) \, dx = \int w_k(x, 0) \, dx = \int w_k(x, t) \, dx \to \int U(x, t) \, dx \quad \text{for a.a. } t \in (0, \infty), \quad (5.49)$$

which yields (5.48). On the other hand, by virtue of (5.22), (5.23), (5.31), $U$ satisfies that

$$U(\cdot, 1) \in L^1_{\text{loc}}(\mathbb{R}^N) \quad \text{and} \quad U^m, \nabla U^m \in L^1(1, T; L^1_{\text{loc}}(\mathbb{R}^N)), \quad (5.50)$$

and

$$\int_1^\infty \int (U \varphi_t + U^m \nabla \varphi) \, dx \, dt = \int U(x, 1) \varphi(x, 1) \, dx \quad (5.51)$$

for all smooth functions $\varphi(x, t)$ with compact support in $\mathbb{R}^N \times [1, \infty)$. Therefore, by Theorem 1.1 in Vazquez [34] we obtain that

$$t^{\frac{N}{(m-1)\max\{\gamma, 2\}}} \|U(\cdot, t) - V(\cdot, t; \|u_0\|_{L^1(\mathbb{R}^N)})\|_{L^p(\mathbb{R}^N)} \to 0 \quad \text{as } t \to \infty \quad (5.52)$$

for any $p \in [1, \infty]$.

For any positive numbers $\varepsilon, R$, we define $B_t$ by

$$B_t := \{x \in \mathbb{R}^N; \ |x| < R t^{\frac{1}{N(m-1)+2\gamma}} \}. \quad (5.53)$$

From (5.27), $U(x, t)$ is defined almost all $(x, t) \in \mathbb{R}^N \times (0, \infty)$. From (5.22), $w_k(x, t)$ converges to $U$ in $L^p_{\text{loc}}(\mathbb{R}^N)(1 < p < \infty)$ for all $t \in [1, \infty)$. Therefore, for any fixed positive number $\varepsilon$, taking $t = k^\varepsilon \geq 1$ and noting that all subsequence $\{w_{k_n}\}$ of $\{w_k\}$ have the subsequence $\{w_{k_n}\}$ of $\{w_k\}$ which converges to the Barenblatt solution, we see that

$$\|w_k(\cdot, k^\varepsilon) - V(\cdot, k^\varepsilon; \|u_0\|_{L^1})\|_{L^p(B_R)} \leq \|w_k(\cdot, k^\varepsilon) - U(\cdot, k^\varepsilon)\|_{L^p(B_R)} + \|U(\cdot, k^\varepsilon) - V(\cdot, k^\varepsilon; \|u_0\|_{L^1})\|_{L^p(\mathbb{R}^N)}$$

$$\to 0 \quad \text{as } k(\in \mathbb{R}) \to \infty \quad (5.54)$$

for all $p \in (1, \infty)$.
Since $V(x, t; M)$ is a self-similar solution, from (5.54), we obtain
\[
k^{N(1 - \frac{1}{p})} \|u(\cdot, k^N(M^{-1} + 2 + \varepsilon); \|u_0\|_{L^1})\|_{L^p(B_R)} \to 0 \text{ as } k \to \infty
\] (5.55)
for any $\varepsilon > 0$ and $p \in (1, \infty)$.

Taking $k$ by $k = t^{-\frac{N}{N - (m - 1) + 2 + \varepsilon}}$ in (5.55), we thus obtain
\[
t^{-\frac{N}{N - (m - 1) + 2 + \varepsilon}} \|u(\cdot, t) - V(\cdot, t; \|u_0\|_{L^1})\|_{L^p(B_1)} \to 0 \text{ as } t \to \infty,
\] (5.56)
for any $\varepsilon > 0$ and $p \in (1, \infty)$ and for all $R > 0$, where $B_t = B_t(\varepsilon, R)$ is the ball defined in (5.53).

Thus, we complete the proof of Theorem 1.2 for all cases of $p \in (1, \infty)$.

Acknowledgements. This article was written while the second author stayed at Max Planck Institute for Mathematics in Leipzig. The second author would like to express her sincere gratitude to Professor A. Stevens and all other members at MPI for their cordial hospitality. The second author also wishes to express her sincere gratitude to Professors T. Nagai and H. Kozono for many stimulating conversations and helpful advice.

References


To access this journal online:

www.edpsciences.org