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ON THE INF-SUP CONDITION FOR HIGHER ORDER MIXED FEM ON MESHES WITH HANGING NODES *

Vincent Heuveline[1, 2] and Friedhelm Schieweck[3]

Abstract. We consider higher order mixed finite element methods for the incompressible Stokes or Navier-Stokes equations with Q_r-elements for the velocity and discontinuous P_{r-1}-elements for the pressure where the order r can vary from element to element between 2 and a fixed bound r^*. We prove the inf-sup condition uniformly with respect to the meshwidth h on general quadrilateral and hexahedral meshes with hanging nodes.

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Introduction

The use of higher order mixed finite elements is well established for the numerical approximation of incompressible flow problems. In the last decade, many efforts have been made in that context in order to combine judiciously the h-version of the finite element method (FEM) with the p-version or spectral type methods. For properly designed meshes, the resulting hp-FEM has been shown to give exponential rates of convergence even in the presence of singularities (see *e.g.* [2, 21]). Typically the solution of incompressible Stokes or Navier-Stokes problems exhibits such singularities in the neighborhood of re-entrant corners (see *e.g.* [25] and references therein).

Due to the incompressibility constraint $\nabla \cdot u = 0$, it is well known that the pair of finite element spaces used for the approximation of velocity and pressure can not be chosen arbitrarily. A compatibility condition known as the *inf-sup condition* has to be satisfied in order to guarantee stability and uniqueness of the discrete solution (see *e.g.* [9]). Under various assumptions on the underlying meshes, the inf-sup condition has been proven for many pairs of finite element spaces both for the h-version (see *e.g.* [5, 8, 9, 13, 23]) and the p-version (see *e.g.* [3, 4, 10, 24]). However, this issue is more intricate for the case of hp-FEM which allows both local refinement by means of hanging nodes as well as various polynomial orders on the mesh elements. In the

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[1] Institute for Applied Mathematics II, University Karlsruhe (TH), Postfach 6980, 76128 Karlsruhe, Germany.
vincent.heuveline@math.uni-karlsruhe.de
[2] Scientific Supercomputing Center, University of Karlsruhe (TH), Zirkel 2, 76128 Karlsruhe, Germany.
vincent.heuveline@rz.uni-karlsruhe.de
[3] Institut für Analysis und Numerik, Otto-von-Guericke-Universität Magdeburg, Postfach 4120, 39016 Magdeburg, Germany.
friedhelm.schieweck@mathematik.uni-magdeburg.de

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two dimensional case, Schötzau *et al.* [20] have proven the inf-sup condition for the family of finite element pairs $(\mathbb{Q}_r, \mathbb{Q}_{r-2})$ on so-called anisotropically and geometrically refined meshes which describe a prototypical local refinement hierarchy with hanging nodes. The cells of the mesh are assumed to be affine equivalent to a reference unit square or a reference unit triangle. Therefore, this analysis covers only the case where the quadrilateral elements are parallelograms. Ainsworth and Coggins [1] propose a uniformly stable family of finite element pairs $(\mathbb{Q}_r, \mathbb{Q}'_{r-1})$ where $\mathbb{Q}'_{r-1}$ describes an *augmented* pressure space. While the velocity finite element functions associated with nodes at the element boundary are defined by means of the usual reference transformation, the functions corresponding to interior element nodes are based on the *Piola transformation* (see [10] for similar construction). In [1] an inf-sup condition for two dimensional meshes with hanging nodes has been proved where the inf-sup constant is uniformly bounded with respect to h and the polynomial degree r.

In this paper, our goal is to prove the inf-sup condition for the family of finite element pairs $(\mathbb{Q}_r, \mathbb{P}_{r-1}^{disc})$ on general quadrilateral and hexahedral meshes with hanging nodes where the polynomial degree r may change from element to element between 2 and a maximum value r^*. By $\mathbb{P}_{r-1}^{disc}$ we denote the space of discontinuous functions which are elementwise defined by polynomials of degree less or equal to $r-1$. Our theory covers both the case of a "mapped" and "unmapped" pressure space (see [19] and [13], respectively) where "mapped" means that the finite element functions are defined by mapping polynomials from a reference element and "unmapped" means that the finite element functions are polynomials on the original element. On the one hand, concerning implementation aspects, the mapped pressure space is more attractive since the velocity space naturally is a mapped space too. On the other hand, the mapped pressure space may have non-optimal approximation properties on general quadrilateral or hexahedral meshes [5]. However, if the mesh is created from a regular coarse mesh only by means of recursive refinement of existing elements into 2^d many son-elements (where d is the dimension of the domain) combined with the introduction of hanging nodes (see Sect. 2.2), then it has been shown that the mapped space has optimal approximation properties too [18].

For the finite element pair $(\mathbb{Q}_r, \mathbb{P}_{r-1}^{disc})$ with the same polynomial degree r on each element and a regular quadrilateral or hexahedral mesh without hanging nodes, it is already known (see [19] for the mapped and [13] for the unmapped pressure space) that the inf-sup condition holds uniformly with respect to the mesh size. In this paper, our main issue is the aspect of hanging nodes caused by non-matching mesh cells or non-matching polynomial degrees. We are interested in the question if hanging nodes affect the inf-sup condition in a negative way. The new contribution of this paper is the proof that on so-called 1-*irregular* meshes with hanging nodes the inf-sup condition holds uniformly with respect to the local and global mesh size and the number of hanging nodes.

Moreover, a main issue in our paper is that we consider general meshes consisting of non-affine equivalent elements defined by a multi-linear transformation of a reference element. Note that for such meshes in the three-dimensional case, the two-dimensional faces of the hexahedral elements can be curved in general. This, in particular, implies that the normal vector on a face is no longer constant which causes additional difficulties in the analysis. From the practical point of view, such general meshes with hanging nodes occur very naturally in applications with adaptive mesh refinement based on *a posteriori* error estimators (see *e.g.* [15,16]). Typical meshes are depicted in Figure 1.

It is well known (see [13]) that in both cases of modeling incompressible flows, by means of the Stokes equations as well as the Navier-Stokes equations, the divergence stability is guaranteed by the same condition. Therefore, in the sequel, we will consider as a model problem the Stokes equations with homogeneous Dirichlet conditions: Find a velocity field u and a pressure distribution p such that

$$\begin{aligned} -\nu\triangle u + \nabla p &= f && \text{in } \Omega, \\ \operatorname{div} u &= 0 && \text{in } \Omega, \\ u &= 0 && \text{on } \Gamma. \end{aligned} \tag{1}$$

Here, Ω denotes a bounded domain in $\mathbb{R}^d$, $d \in \{2,3\}$, $\nu > 0$ the kinematic viscosity and f a given body force per unit mass. Then, a pair (X^N, M^N) of conforming finite element spaces approximating the velocity $u \in (H_0^1(\Omega))^d$

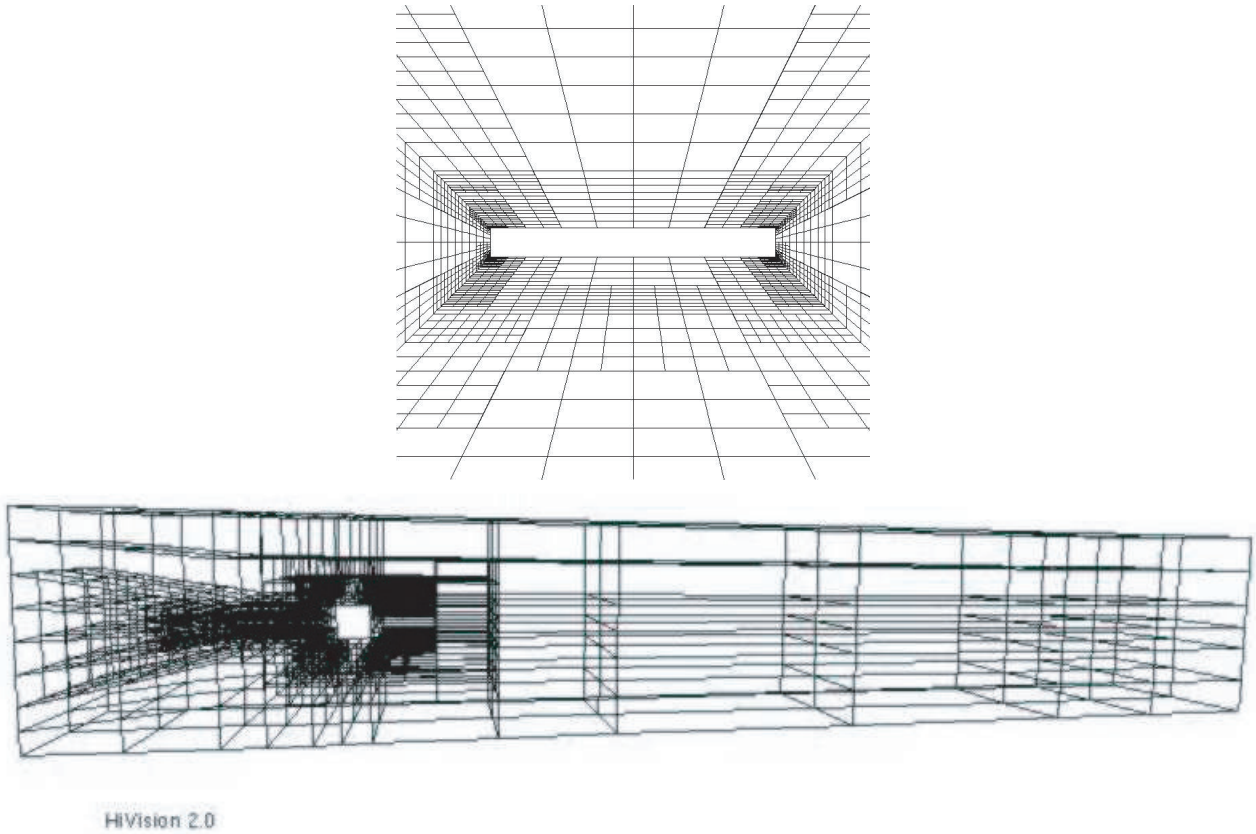

FIGURE 1. Typical local refined meshes with hanging nodes: (top) two-dimensional grid for the computation of an exterior flow problem [7]; (bottom) three-dimensional grid for a channel flow problem [14].

and the pressure $p \in L_0^2(\Omega)$ is said to fulfill the *discrete inf-sup condition* if the following estimate holds:

$$\exists\, \beta > 0 : \qquad \inf_{0 \neq q^N \in M^N} \sup_{0 \neq v^N \in X^N} \frac{(\operatorname{div} v^N, q^N)}{|v^N|_{1,\Omega} \|q^N\|_{0,\Omega}} \geq \beta. \tag{2}$$

It is desirable that the parameter β in (2) is independent of the mesh size and of the polynomial degree. For our family of finite element pairs $(\mathbb{Q}_r, \mathbb{P}_{r-1}^{disc})$, we prove that the inf-sup condition holds with a constant $\beta(r^*)$ which is independent of the local and global mesh size but may depend on the maximum polynomial degree r^*. However, for the situation of just one element, it has been shown in [4] that the inf-sup constant of the element pair $(\mathbb{Q}_r, \mathbb{P}_{r-1}^{disc})$ is independent of the polynomial degree r. Based on this result we will prove in a forthcoming paper that our global inf-sup constants $\beta(r^*)$ for all r^* are bounded from below by a positive constant.

The outline of this paper is as follows: In Section 1, we introduce the notations and state the assumptions to be made on the meshes and finite element spaces. In Section 2, we introduce a new interpolation operator which is uniformly H^1-stable and divergence preserving on general meshes with hanging nodes. Based on that interpolation operator, the main result is then established in Section 3 using the macro-element technique developed by Boland and Nicolaides [6] as well as the method proposed by Fortin [12].

1. Preliminaries and notation

1.1. General notation

For a measurable set $G \subset \mathbb{R}^d$, let $(\cdot,\cdot)_G$ and $\|\cdot\|_{0,G}$ denote the inner product and the norm in $L^2(G)$ or $(L^2(G))^d$, respectively, and let $L^2_0(G) := \{q \in L^2(G) : (q,1)_G = 0\}$. Furthermore, let $|\cdot|_{m,G}$ and $\|\cdot\|_{m,G}$ denote the seminorm and norm in the Sobolev space $H^m(G)$ and $(H^m(G))^d$, respectively. For a $(d-1)$-dimensional subset $E \subset \partial G$, we denote by $\langle\,\cdot,\cdot\,\rangle_E$ the inner product in $L^2(E)$, *i.e.*

$$\langle\, u, v\,\rangle_E := \int_E u(x)v(x)\,\mathrm{d}s.$$

We denote by $\mathbb{P}_m(G)$ the space of all polynomials on the domain $G \subset \mathbb{R}^d$ with total degree less or equal to m and by $\mathbb{Q}_m(G)$ the space of those polynomials where the maximum power in each coordinate is less or equal to m.

By $\mathrm{card}(J)$ we denote the number of elements of a finite set J. For a set $G \subset \mathbb{R}^d$, we denote by $\mathrm{int}(G)$ and $\overline{G}$ the interior and closure of G, respectively. Throughout this paper, C will denote a generic constant which may have different values at different places whereas special constants with fixed values are indicated by $C_1, C_2, \ldots$ All these constants occurring inside of any estimates will be independent of the local and global mesh parameter h_K and h, respectively, which will be defined below.

1.2. Description of the considered grids

Let the bounded domain $\Omega \subset \mathbb{R}^d$ be decomposed on a *mesh* $\mathcal{T}$ into elements $K \in \mathcal{T}$ which are assumed to be open quadrilaterals in the 2D-case and open hexahedrons in the 3D-case such that $\overline{\Omega} = \cup_{K\in\mathcal{T}}\overline{K}$. For an element $K \in \mathcal{T}$, we denote by h_K the diameter of the element K and by ρ_K the diameter of the largest circle inscribed into K. The *meshwidth* h of $\mathcal{T}$ is given by $h = \max_{K\in\mathcal{T}} h_K$.

We denote by $F_K : \widehat{K} \to K$ the mapping between the reference element $\widehat{K} := (-1,+1)^d$ and K. In the sequel of this paper, the mapping F_K is assumed to be multi-linear, *i.e.* $F_K \in (\mathbb{Q}_1(\widehat{K}))^d$. Therefore, the two dimensional faces of 3D hexahedral elements $K \in \mathcal{T}$ can be curved in general. In order to guarantee that the mapping F_K is bijective for general non-affine quadrilateral or hexahedral meshes, the usual shape regularity assumption $h_K/\rho_K \le C$ for all $K \in \mathcal{T}$ may not be sufficient. Therefore, we suppose the *shape regularity assumptions* given in [17, 19].

In the following, let us describe these shape regularity assumptions in more detail. By a Taylor expansion of $F_K(\hat{x})$ we get

$$F_K(\hat{x}) = b_K + B_K\hat{x} + G_K(\hat{x}), \tag{3}$$

with $b_K := F_K(0)$, $B_K := DF_K(0)$ and $G_K(\hat{x}) := F_K(\hat{x}) - F_K(0) - DF_K(0)(\hat{x})$. We denote by $\widehat{S} \subset \widehat{K}$ the d-simplex with the vertices $(0,\ldots,0)$, $(1,0,\ldots,0)$, $\ldots$, $(0,\ldots,0,1)$ and by S_K the image of $\widehat{S}$ under the affine mapping $\hat{x} \to B_K\hat{x} + b_K$. For the simplices S_K, $K \in \mathcal{T}$, we assume the usual shape regularity assumption

$$\frac{h_{S_K}}{\rho_{S_K}} \le C \qquad \forall K \in \mathcal{T}, \tag{4}$$

where $h_{S_K} := \mathrm{diam}(S_K)$ and ρ_{S_K} is the diameter of the largest ball inscribed into S_K. Note that condition (4) implies (see [11])

$$\|B_K\| \le Ch_{S_K}, \qquad \|B_K^{-1}\| \le Ch_{S_K}^{-1} \qquad \forall K \in \mathcal{T}, \tag{5}$$

where $\|B_K\|$ denotes the matrix norm induced by the Euclidean vector norm in $\mathbb{R}^d$. For each element $K \in \mathcal{T}$, we define the constant γ_K by

$$\gamma_K := \sup_{\hat{x}\in\widehat{K}} \|B_K^{-1}\, DF_K(\hat{x}) - I\|, \tag{6}$$

which is a measure of the deviation of K from a parallelogram or a parallelepiped, respectively. Note that $\gamma_K = 0$ if the mapping F_K is affine.

Definition 1. A mesh $\mathcal{T}$ of quadrilateral or hexahedral elements is called **shape regular** if the conditions (4) and

$$\gamma_K \leq \gamma_0 < 1 \qquad \forall K \in \mathcal{T}, \tag{7}$$

are satisfied.

This shape regularity assumption imposes that the distortion of the quadrilateral or hexahedral elements from a parallelogram or parallelepiped, respectively, is uniformly bounded. These conditions guarantee further that the following properties for the mapping $F_K : \widehat{K} \to K$ hold (for the proof, see [17])

$$\|DF_K(\hat{x})\| \leq Ch_K \qquad \forall \hat{x} \in \widehat{K}, \tag{8}$$

$$Ch_K^d \leq |\det(DF_K(\hat{x}))| \leq C' h_K^d \qquad \forall \hat{x} \in \widehat{K}. \tag{9}$$

In this paper, our main interest is to study the effect on the inf-sup condition of using hanging nodes. That means that the usual assumption of a *regular* grid $\mathcal{T}$ has to be weakened. In the following, we will describe the type of grids that is treated in this paper. $\mathcal{T}$ is a *multi-level grid* generated by a refinement process in the following way. We start with a partition $\mathcal{T}^0$ of the domain Ω into elements $K \in \mathcal{T}^0$ of grid level 0, *i.e.* $\Omega = \operatorname{int}\big(\bigcup_{K\in\mathcal{T}^0} \overline{K}\big)$. The grid $\mathcal{T}^0$ is assumed to be *regular* in the usual sense, *i.e.* for any two different elements $K_1, K_2 \in \mathcal{T}^0$ the intersection $\overline{K_1} \cap \overline{K_2}$ is either empty or a common $(d-m)$-dimensional face of K_1 and K_2 where $m \in \{1, \ldots, d\}$. The set of the neighboring elements of the element K is denoted by

$$\Lambda(K) := \left\{ \widetilde{K} \in \mathcal{T} : \overline{K} \cap \overline{\widetilde{K}} \neq \emptyset \right\}. \tag{10}$$

Now, starting with the elements $K \in \mathcal{T}^0$, an existing element K can be refined, *i.e.* it can be splitted into 2^d many new elements called *son-elements* and denoted by $\sigma_i(K)$, $i = 1, \ldots, 2^d$ (see Fig. 2). For a new element $K' = \sigma_i(K)$, we will say that K is the *father-element* of K' and we will write $K = \mathcal{F}(K')$. If an element K is refined then, in the partition of the domain Ω, it is replaced by the set of its son-elements $\sigma_i(K)$, $i = 1, \ldots, 2^d$. The new elements can be refined again and again and so the final partition $\mathcal{T}$ of Ω is created.

Definition 2. For an element $K \in \mathcal{T}$, generated by the refinement process from the initial grid $\mathcal{T}^0$, we define the **refinement level** $\ell(K)$ as $\ell(K) := 0$ if $K \in \mathcal{T}^0$ and $\ell(K) := m \geq 1$ if there exists a chain of m father-elements K_i, $i = 1, \ldots, m$, starting from $K_0 := K$ and defined by $K_i := \mathcal{F}(K_{i-1})$ for $i = 1, \ldots, m$, such that $K_m \in \mathcal{T}^0$.

The above defined *refinement level* $\ell(K)$ is equal to the number of refinement steps that is needed to generate element K from an element of the coarsest grid $\mathcal{T}^0$.

Definition 3. A grid $\mathcal{T}$, generated by the refinement process from the initial grid $\mathcal{T}^0$, is called **1-irregular** if

$$|\ell(K) - \ell(K')| \leq 1,$$

for any pair of neighbored elements $K, K' \in \mathcal{T}$ where $\partial K \cap \partial K'$ is a one- or two-dimensional manifold.

In this paper, we consider only grids $\mathcal{T}$ which are 1-*irregular*.

Finally, we need some additional notation for the analysis in the following sections. We denote by $\mathcal{E}(K)$ the set of all $(d-1)$-dimensional faces of an element K, by n^K the unit normal vector on the element boundary ∂K directed outward with respect K and by n_E^K the restriction of the normal vector n^K to the face $E \in \mathcal{E}(K)$. Let $\mathcal{E}$ be the set of all faces $E \in \mathcal{E}(K)$ of all elements $K \in \mathcal{T}$. We split $\mathcal{E}$ in the form

$$\mathcal{E} = \mathcal{E}_0 \cup \mathcal{E}(\Gamma)$$

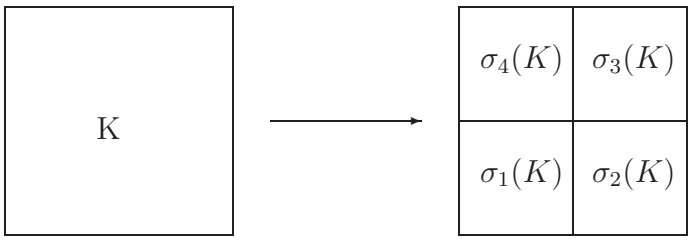

FIGURE 2. Refinement of element K into son-elements $\sigma_i(K)$, $i = 1, \ldots, 4$.

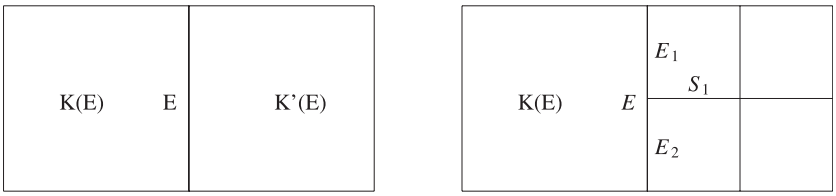

FIGURE 3. Two dimensional configuration for the case of a *regular inner face i.e.* $E \in \mathcal{E}_r$ (left) and for the case of an *irregular inner face i.e.* $E \in \mathcal{E}_i$ (right).

where $\mathcal{E}_0$ denotes the set of all inner faces of $\mathcal{E}$ and $\mathcal{E}(\Gamma)$ the set of all faces of $\mathcal{E}$ located at the boundary Γ of Ω. For any face $E \in \mathcal{E}$, we associate the set $\mathcal{T}(E)$ of adjacent elements defined by

$$\mathcal{T}(E) := \{K \in \mathcal{T} : E \in \mathcal{E}(K)\}.$$

Let $\mathcal{E}_r$ denote the set of the *regular inner faces* defined as

$$\mathcal{E}_r := \{E \in \mathcal{E}_0 : \mathrm{card}(\mathcal{T}(E)) = 2\}.$$

For each regular face $E \in \mathcal{E}_r$, there exist exactly two different elements denoted by $K(E)$ and $K'(E)$ such that E is one of their faces (see Fig. 3), *i.e.*

$$\mathcal{T}(E) = \{K(E), K'(E)\} \qquad \forall E \in \mathcal{E}_r.$$

For all other faces $E \in \mathcal{E} \setminus \mathcal{E}_r$, there is only one element denoted by $K(E)$ which has E as one of its faces, *i.e.*

$$\mathcal{T}(E) = \{K(E)\} \qquad \forall E \in \mathcal{E} \setminus \mathcal{E}_r.$$

A face $\widetilde{E} \in \mathcal{E}$ is called a *son-face* of a face $E \in \mathcal{E}$ if $\widetilde{E} \subset E$ and $|\widetilde{E}| < |E|$ where $|\widetilde{E}|$ and $|E|$ denote the $(d-1)$-dimensional measure of $\widetilde{E}$ and E, respectively. We denote by $\sigma(E)$ the set of all son-faces of E. In Figure 3 (right part) we have *e.g.* $\sigma(E) = \{E_1, E_2\}$. Note that for each regular face $E \in \mathcal{E}_r$, the set $\sigma(E)$ is empty. We denote by $\mathcal{E}_i$ the set of all *irregular inner faces* defined as

$$\mathcal{E}_i := \{E \in \mathcal{E}_0 : \sigma(E) \neq \emptyset\}.$$

Using these definitions, the set $\mathcal{E}_0$ of all inner faces can be decomposed as

$$\mathcal{E}_0 = \mathcal{E}_r \cup \mathcal{E}_i \cup \Big(\bigcup_{E \in \mathcal{E}_i} \sigma(E)\Big).$$

Let $\widetilde{E} \in \sigma(E)$ be a son-face of $E \in \mathcal{E}_i$, then the face E is called the *father-face* of $\widetilde{E}$ and we write $E = \mathcal{F}(\widetilde{E})$. We define the *set of all son-faces* by

$$\mathcal{E}_\sigma := \bigcup_{E \in \mathcal{E}_i} \sigma(E).$$

For a given element $K \in \mathcal{T}$, we define further

$$\mathcal{E}_\mu(K) := \mathcal{E}(K) \cap \mathcal{E}_\mu \qquad \text{for } \mu \in \{r, i, \sigma\}.$$

Finally, for each face $E \in \mathcal{E}$, let n_E denote the unit vector $n_E^{K(E)}$ which is normal to the face E and directed outward with respect to the element $K(E)$.

As a consequence of the shape regularity of the mesh we have

$$|E| \leq C_5 h_K^{d-1} \qquad \forall E \in \mathcal{E}(K) \tag{11}$$

where $|E|$ describes the $(d-1)$ dimensional measure of E (see [17], Lem. 5). Furthermore, the following inequality will be needed (see [17], Lem. 4)

$$C_6 h_{\widetilde{K}} \;\leq\; h_K \;\leq\; C_7 h_{\widetilde{K}} \qquad \forall \widetilde{K} \in \Lambda(K). \tag{12}$$

1.3. **Finite element spaces**

Let $\mathbf{r}$ denote the degree vector $\mathbf{r} := \{r_K : K \in \mathcal{T}\}$ which contains the polynomial degree r_K of the velocity approximation on each element K. In this paper, we assume that

$$r_K \geq 2 \qquad \forall K \in \mathcal{T}. \tag{13}$$

The finite element space of the velocity $X^N \subset (H_0^1(\Omega))^d$ associated with the mesh $\mathcal{T}$ and the degree vector $\mathbf{r}$ is defined as $X^N := (S^N)^d$ with the scalar finite element space

$$\begin{aligned} S^N &:= S(\Omega; \mathbf{r}, \mathcal{T}) \\ &:= \{\phi \in H_0^1(\Omega) : \; \phi\big|_K \circ F_K \in \mathbb{Q}_{r_K}(\widehat{K}) \quad \forall K \in \mathcal{T}\}. \end{aligned} \tag{14}$$

The finite element space $M^N \subset L_0^2(\Omega)$ for the approximation of the pressure is chosen as the following mapped space

$$\begin{aligned} M^N &:= M(\Omega; \mathbf{r}, \mathcal{T}) \\ &:= \{q \in L_0^2(\Omega) : \; q\big|_K \circ F_K \in \mathbb{P}_{r_K - 1}(\hat{K}) \quad \forall K \in \mathcal{T}\}. \end{aligned} \tag{15}$$

Then, the discrete Stokes problem reads: *Find* $(u^N, p^N) \in X^N \times M^N$ *such that*

$$\begin{array}{rcll} \nu(\nabla u^N, \nabla v^N)_\Omega \;-\; (p^N, \nabla \cdot v^N)_\Omega & = & (f, v^N)_\Omega & \forall v^N \in X^N \\ (q^N, \nabla \cdot u^N)_\Omega & = & 0 & \forall q^N \in M^N. \end{array} \tag{16}$$

Remark 4. Instead of the mapped pressure space M^N we could also choose, like in [13], Section II.3.2, the *unmapped* space $\widetilde{M}^N$ defined by

$$\widetilde{M}^N := \left\{q \in L_0^2(\Omega) : \; q\big|_K \in \mathbb{P}_{r_K - 1}(K) \quad \forall K \in \mathcal{T}\right\}. \tag{17}$$

Concerning the implementation, this space would be less attractive than the mapped space M^N defined in (15). However, in contrast to the mapped version, the unmapped pressure space guarantees optimal approximation

properties on general quadrilateral or hexahedral meshes [5]. In the following we will consider only the case of a mapped pressure space. However, the presented theory can also be applied to the case of an unmapped space (see Rem. 16 below).

For the subsequent analysis we need also the finite element spaces for the velocity with the polynomial order of 1 and 2, *i.e.* $X_h^1 := (S_h^1)^d$ and $X_h^2 := (S_h^2)^d$ based on the scalar finite element spaces

$$S_h^r := \{\phi \in H_0^1(\Omega) : \ \phi\big|_K \circ F_K \in \mathbb{Q}_r(\widehat{K}) \ \ \forall K \in \mathcal{T}\}, \ \ r \in \{1,2\}, \tag{18}$$

as well as the finite element space of the piecewise constant pressure functions

$$M_h^0 := \{q \in L_0^2(\Omega) : \ q\big|_K \in \mathbb{P}_0(K) \quad \forall K \in \mathcal{T}\}. \tag{19}$$

Some attention is required to ensure interelement continuity in (14) and (18) in the case of hanging nodes or if r_K is variable on the adjacent cells $K \in \mathcal{T}(E)$ of an inner face E. We refer to [21] for a detailed treatment of this issue.

2. Two interpolation operators

In order to prove the inf-sup condition (2) for the finite element pairs (X^N, M^N) on locally refined grids with hanging nodes we will use the macro-element technique proposed by Boland and Nicolaides [6]. This technique requires a local inf-sup condition with a uniform constant for all subdomains of a nonoverlapping partitioning of Ω and a global inf-sup condition for a pair of subspaces $(\overline{X}^N, M_h^0)$ where $\overline{X}^N$ is a suitable subspace of X^N and M_h^0 is the space of discontinuous piecewise constant pressure functions. As the subdomains we choose the grid cells $K \in \mathcal{T}$. Then, the local inf-sup condition can be adopted from the literature (see Sect. 3 below) and the only thing which is still to do is to prove a global inf-sup condition for a suitable pair of spaces $(\overline{X}^N, M_h^0)$.

The goal of this section is to derive the tools that are needed to prove this condition for the low order pair (X_h^2, M_h^0). To this end, we follow the method proposed by Fortin [12] and construct an interpolation operator $\Pi_h : (H_0^1(\Omega))^d \to X_h^2$ which is H^1-stable uniformly with respect to h, *i.e.*

$$|\Pi_h v|_{1,\Omega} \le C|v|_{1,\Omega} \qquad \forall v \in (H_0^1(\Omega))^d, \tag{20}$$

and divergence preserving with respect to the test space M_h^0, *i.e.*

$$(\operatorname{div} \Pi_h v, q) = (\operatorname{div} v, q) \quad \forall q \in M_h^0. \tag{21}$$

In our approach, the interpolation operator Π_h is defined as

$$\Pi_h v := R_h v + I_h(v - R_h v) \qquad \forall v \in (H_0^1(\Omega))^d, \tag{22}$$

with a divergence preserving operator I_h presented in Section 2.1 and the Scott-Zhang type interpolation operator R_h proposed in [17].

2.1. Divergence preserving operator I_h

The operator $I_h : (H_0^1(\Omega))^d \to X_h^2$ is assumed to have the following form

$$I_h v = \sum_{E \in \mathcal{E}_r \cup \mathcal{E}_i} I_h^E v \qquad \forall v \in (H_0^1(\Omega))^d, \tag{23}$$

where the construction of the local operators I_h^E depends on the dimension d of the considered problem as well as on whether $E \in \mathcal{E}_r$ or $E \in \mathcal{E}_i$.

At first, let us describe those parts of the construction of I_h^E which can be presented in a general fashion for both the two and three-dimensional case. Let $E \in \mathcal{E}_r \cup \mathcal{E}_i$ be a given face, $K = K(E)$ the element associated with E as defined in Section 2.2 and $F_K : \widehat{K} \to K$ the corresponding mapping between the reference element $\widehat{K} = (-1,1)^d$ and K. Then, by $\widehat{E}$ we denote the face of the reference element $\widehat{K}$ that corresponds to E in the sense that $\widehat{E} = F_K^{-1}(E)$. Let $\widehat{m}_{\widehat{E}}$ be the barycenter of the reference face $\widehat{E}$ and $m_E := F_K(\widehat{m}_{\widehat{E}})$ the corresponding point at the original face E. Then, the face bubble function $\Phi_E \in X_h^2$ is defined as

$$\Phi_E(x) := \psi_E(x)\, n_E^0 \qquad \forall x \in \Omega, \tag{24}$$

where $n_E^0 := n_E(m_E)$ is the unit normal vector n_E at the point $m_E \in E$ and $\psi_E \in S_h^2$ denotes the scalar function defined by its nodal values in the following way.

Let $a_j \in \Omega$, $j \in J^2$, denote the nodal points of the quadratic finite element space S_h^2. These nodal points are the images of the reference nodes $\widehat{a}_m \in \widehat{K}$, $m = 1, \ldots, 3^d$, under the reference mappings $F_K : \widehat{K} \to K$ of all elements $K \in \mathcal{T}$. The reference nodes $\widehat{a}_m$ are the vertices of $\widehat{K}$, the barycenter of $\widehat{K}$, barycenters of the $(d-1)$-dimensional faces of $\widehat{K}$ and, in the three dimensional case, additionally the midpoints of the edges of $\widehat{K}$. For a given element $K \in \mathcal{T}$, we define by

$$J^2(K) := \{j \in J^2 : a_j = F_K(\widehat{a}_m), 1 \le m \le 3^d\}$$

the local index set of the nodal points a_j that belong to element K. Note that among the nodal points $a_j \in \Omega$, $j \in J^2$, there are also the so called *hanging nodes*. A node $j \in J^2$ is called a *hanging node* if there exists an irregular face $E \in \mathcal{E}_i$ and a son-face $\widetilde{E} \in \sigma(E)$ such that $j \in J^2(K(\widetilde{E}))$ but $j \notin J^2(K(E))$ where $K(\widetilde{E})$ and $K(E)$ denote the uniquely determined elements that are assigned to the faces $\widetilde{E}$ and E, respectively, as defined in Section 2.2. A node $j \in J^2$ is called a *regular node* if it is not a hanging node. In the following, the set of all regular nodes will be denoted by J_r^2. Now, a finite element function $\psi \in S_h^2$ is uniquely determined by its nodal values $\psi(a_j)$ for all regular nodes $j \in J_r^2$. The values $\psi(a_j)$ for hanging nodes are determined by the continuity requirements of $\psi \in S_h^2$. Note that for all regular and irregular faces E, the barycenter m_E of E corresponds to a regular node. Thus, our scalar face bubble function $\psi_E \in S_h^2$ associated with the face $E \in \mathcal{E}_r \cup \mathcal{E}_i$ is defined by

$$\psi_E(a_j) := \begin{cases} 0 & \text{for all } j \in J_r^2 \text{ with } a_j \neq m_E, \\ 1 & \text{if } a_j = m_E. \end{cases} \tag{25}$$

For the support of Φ_E, we have (see Fig. 3 for the 2D-case)

$$\operatorname{supp}(\Phi_E) = \overline{K(E) \cup K'(E)} \qquad \forall E \in \mathcal{E}_r,$$

and

$$\operatorname{supp}(\Phi_E) = \overline{K(E)} \cup \overline{\bigcup_{\widetilde{E} \in \sigma(E)} K(\widetilde{E})} \qquad \forall E \in \mathcal{E}_i.$$

From the definition of $\Phi_E \in X_h^2$ we immediately get

$$\max_{x \in \overline{\Omega}} \|\Phi_E(x)\| \le C_1 \qquad \forall E \in \mathcal{E}_r \cup \mathcal{E}_i \tag{26}$$

which implies by means of inverse inequalities the estimate

$$|\Phi_E|_{1,K} \le C_2 h_K^{d/2-1} \qquad \forall K \in \mathcal{T}. \tag{27}$$

The constants C_1 and C_2 are independent of E and K respectively.

Lemma 5. *Let $\mathcal{T}$ be a shape-regular and 1-irregular mesh. Let $E \in \mathcal{E}_r \cup \mathcal{E}_i$ and $K \in \mathcal{T}$ such that $E \in \mathcal{E}(K)$. Then, there exist positive constants C_3 and C_4 independent of E and K such that*

$$\langle \Phi_E \cdot n_E, 1 \rangle_E \geq C_3 \, h_K^{d-1}, \tag{28}$$

and

$$\|v\|_{L^2(E)} \leq C_4 \, h_K^{1/2} \, \left\{ h_K^{-1} \, \|v\|_{0,K} + |v|_{1,K} \right\} \quad \forall v \in (H^1(K))^d. \tag{29}$$

Proof. First, we prove the inequality (28).

For the two dimensional case $d = 2$, the unit normal vector n_E is constant on E, *i.e.* $n_E(x) = n_E^0$ for all points $x \in E$. By means of the Simpson's rule, we therefore have

$$\langle \Phi_E \cdot n_E, 1 \rangle_E = \langle \psi_E, 1 \rangle_E = \frac{4}{6}|E| \geq C_3 h_K,$$

where $|E|$ denotes the length of the face E and the last inequality follows from the shape regularity of the mesh.

For the three dimensional case $d = 3$, the proof is slightly more involved since $n_E(x)$ may not be constant any more for $x \in E$. We use the mapping $F_K : \widehat{K} \to K$ between the reference element $\widehat{K} = (-1, 1)^3$ and the element K in order to get a parametrization of the face $E \in \mathcal{E}(K)$. We denote by $\widehat{E} = F_K^{-1}(E)$ the face of $\widehat{K}$ that corresponds to E. First, we assume the case where $\widehat{E}$ is the face defined by $\hat{x}_1 = -1$. Then, we can represent each point $\hat{x} \in \widehat{E}$ as

$$\hat{x} = \widehat{\gamma}(t_1, t_2) := (-1, t_1, t_2)^T \qquad \text{where } (t_1, t_2) \in G := (-1, 1)^2,$$

and each point $x \in E$ as

$$x = \gamma(t_1, t_2) := F_K(\widehat{\gamma}(t_1, t_2)), \qquad \forall (t_1, t_2) \in G.$$

Then, the vector N_E defined as

$$N_E(t_1, t_2) = \left(\frac{\partial \gamma}{\partial t_1} \times \frac{\partial \gamma}{\partial t_2} \right) \Big|_{(t_1, t_2)},$$

is a normal vector of E at the point $x = \gamma(t_1, t_2)$. Furthermore, there is constant sign factor $s_E \in \{-1, +1\}$ such that the normal unit vector n_E can be uniquely expressed as

$$n_E(x) = n_E(\gamma(t_1, t_2)) = \frac{s_E}{\|N_E(t_1, t_2)\|} \, N_E(t_1, t_2) \qquad \forall (t_1, t_2) \in G,$$

under the constraint $n_E(\gamma(0, 0)) = n_E^0$. Then, we obtain

$$\langle \Phi_E \cdot n_E, 1 \rangle_E = \int_G \psi_E(\gamma(t_1, t_2)) \, \frac{N_E(0, 0) \cdot N_E(t_1, t_2)}{\|N_E(0, 0)\|} \, \mathrm{d}t_1 \mathrm{d}t_2. \tag{30}$$

A simple calculation shows that the integrand in (30) is a polynomial in the space $\mathbb{Q}_3(G)$, such that Simpson's rule yields the exact value of the integral, *i.e.*

$$\langle \Phi_E \cdot n_E, 1 \rangle_E = \frac{16}{9} \, \psi_E(\gamma(0, 0)) \cdot \|N_E(0, 0)\| = \frac{16}{9} \, \|N_E(0, 0)\|. \tag{31}$$

Now, we have

$$\begin{aligned} \det(DF_K(\widehat{\gamma}(0, 0))) &= \frac{\partial F_K}{\partial \hat{x}_1} \cdot \left(\frac{\partial F_K}{\partial \hat{x}_2} \times \frac{\partial F_K}{\partial \hat{x}_3} \right) \Big|_{\hat{x} = \widehat{\gamma}(0,0)} \\ &= \frac{\partial F_K}{\partial \hat{x}_1} \Big|_{\hat{x} = \widehat{\gamma}(0,0)} \cdot N_E(0, 0), \end{aligned}$$

which implies the estimate

$$|\det(DF_K(\widehat{\gamma}(0,0)))| \leq \left\| \frac{\partial F_K}{\partial \hat{x}_1}(\widehat{\gamma}(0,0)) \right\| \cdot \|N_E(0,0)\|.$$

Using the estimates (9) and (8), this leads to

$$\|N_E(0,0)\| \geq Ch_K^2.$$

Together with (31) this completes the proof of (28) in the case that $\widehat{E}$ corresponds to $\hat{x}_1 = -1$. The proof where $\widehat{E}$ corresponds to $\hat{x}_i = \pm 1$ is completely analogous.

Now, we prove the estimate (29). At first, we need to prove the estimate

$$\|v\|_{L^2(E)} \leq C\, h_K^{(d-1)/2}\, \|\hat{v}\|_{L^2(\widehat{E})}, \tag{32}$$

where the function $\hat{v} \in (H^1(\widehat{K}))^d$ is defined by $\hat{v} := v(F_K(\hat{x}))$ for all $\hat{x} \in \widehat{K}$. We present the proof of (32) only for the three dimensional case $d = 3$. The case $d = 2$ follows easily by simple transformation of the integrals corresponding to both sides of (32).

Again we consider only the special case where $\widehat{E} = F_K^{-1}(E)$ corresponds to $\hat{x}_1 = -1$. For the face E, we get

$$\|v\|_{L^2(E)}^2 = \int_G v(\gamma(t_1,t_2))^2\, \|N_E(t_1,t_2)\|\, \mathrm{d}t_1 \mathrm{d}t_2,$$

and by means of (8)

$$\|N_E(t_1,t_2)\| \leq \left\| \frac{\partial F_K}{\partial \hat{x}_2}(\widehat{\gamma}(t_1,t_2)) \right\| \cdot \left\| \frac{\partial F_K}{\partial \hat{x}_3}(\widehat{\gamma}(t_1,t_2)) \right\| \leq Ch_K^2,$$

which implies

$$\|v\|_{L^2(E)}^2 \leq Ch_K^2 \int_G v(\gamma(t_1,t_2))^2\, \mathrm{d}t_1 \mathrm{d}t_2. \tag{33}$$

The parametrization of $\widehat{E}$ is given by $\hat{x} = \widehat{\gamma}(t_1,t_2) = (-1,t_1,t_2)^T$ for $(t_1,t_2) \in G$ and the corresponding normal vector is

$$\widehat{N}_{\widehat{E}}(t_1,t_2) := \left(\frac{\partial \widehat{\gamma}}{\partial t_1} \times \frac{\partial \widehat{\gamma}}{\partial t_2} \right)\Big|_{(t_1,t_2)} = (1,0,0)^T.$$

Therefore, we get

$$\begin{aligned} \|\hat{v}\|_{L^2(\widehat{E})}^2 &= \int_G \hat{v}(\widehat{\gamma}(t_1,t_2))^2\, \|\widehat{N}_{\widehat{E}}(t_1,t_2)\| \mathrm{d}t_1 \mathrm{d}t_2 \\ &= \int_G v(\gamma(t_1,t_2))^2 \mathrm{d}t_1 \mathrm{d}t_2. \end{aligned}$$

Together with (33), this proves (32) for the case $d = 3$. Now, we apply the trace theorem on the reference element $\widehat{K}$ and well-known estimates between the norms of $\hat{v}$ and $\widehat{K}$ and the norms of v on K (see *e.g.* [11]) and get

$$\|\hat{v}\|_{L^2(\widehat{E})} \leq C\|\hat{v}\|_{0,\widehat{K}} + C|\hat{v}|_{1,\widehat{K}} \leq Ch_K^{-d/2}\|v\|_{0,K} + Ch_K^{1-d/2}|v|_{1,K}.$$

Together with (32), this proves the estimate (29). □

In case of a regular face $E \in \mathcal{E}_r$, we impose for an arbitrary function $v \in (H_0^1(\Omega))^d$

$$I_h^E v = c_E(v)\Phi_E \tag{34}$$

where $c_E(v) \in \mathbb{R}$ is a constant which is chosen such that

$$c_E(v) \left\langle \Phi_E \cdot n_E, 1 \right\rangle_E = \left\langle v \cdot n_E, 1 \right\rangle_E . \tag{35}$$

Due to (28) the term $\left\langle \Phi_E \cdot n_E^K, 1 \right\rangle_E$ does not vanish. By definition of Φ_E it holds $\Phi_E = 0$ on $\partial K \setminus E$ for all $K \in \mathcal{T}(E)$. Thus, the operator I_h^E has the property that for each function $v \in (H_0^1(\Omega))^d$ it holds

$$\left\langle I_h^E v \cdot n^K, 1 \right\rangle_{\partial K} = \left\langle v \cdot n_E^K, 1 \right\rangle_E , \tag{36}$$

for all $E \in \mathcal{E}_r$ and for all $K \in \mathcal{T}(E) = \{K(E), K'(E)\}$.

In the following, we will describe those parts of the construction of I_h^E which are different in the two and three-dimensional case.

2.1.1. *Irregular faces in the two dimensional case*

In case of $E \in \mathcal{E}_i$, we denote by E_1 and E_2 the son-faces of E and by $K_m := K(E_m)$, $m \in \{1, 2\}$, the element associated with the face E_m as depicted in Figure 3 (right part). Moreover, we denote by $S_1 := \overline{K_1} \cap \overline{K_2}$ the common face of the elements K_1 and K_2. Then, we impose

$$I_h^E v := c_E(v)\Phi_E + c_{E,S_1}(v)\Phi_{S_1}, \tag{37}$$

where $c_E(v)$ and $c_{E,S_1}(v)$ are suitable constants. The constant $c_E(v)$ again is chosen such that equation (35) is satisfied which is always possible due to the fact that $\left\langle \Phi_E \cdot n_E, 1 \right\rangle_E \neq 0$. Since the face-bubble function Φ_{S_1} is zero on the face E and $\Phi_E = 0$ on $\partial K(E) \setminus E$, the operator I_h^E has the following property for all functions $v \in (H_0^1(\Omega))^d$

$$\left\langle I_h^E v \cdot n^{K(E)}, 1 \right\rangle_{\partial K(E)} = \left\langle v \cdot n_E^{K(E)}, 1 \right\rangle_E , \tag{38}$$

for all $E \in \mathcal{E}_i$. The constant $c_{E,S_1}(v)$ will be taken such that the operator I_h^E satisfies the following condition

$$\left\langle I_h^E v \cdot n^{K(\widetilde{E})}, 1 \right\rangle_{\partial K(\widetilde{E})} = \left\langle v \cdot n_{\widetilde{E}}^{K(\widetilde{E})}, 1 \right\rangle_{\widetilde{E}} , \tag{39}$$

for all $E \in \mathcal{E}_i$ and for all $\widetilde{E} \in \sigma(E)$. A simple calculation starting from equation (39) with $\widetilde{E} = E_i$ leads to the expression

$$c_{E,S_1}(v) = \left\langle \Phi_{S_1} \cdot n_{S_1}^{K_i}, 1 \right\rangle_{S_1}^{-1} \left\{ \left\langle v \cdot n_{E_i}^{K_i}, 1 \right\rangle_{E_i} - c_E(v) \left(\left\langle \Phi_E \cdot n_{E_i}^{K_i}, 1 \right\rangle_{E_i} + \left\langle \Phi_E \cdot n_{S_1}^{K_i}, 1 \right\rangle_{S_1} \right) \right\} . \tag{40}$$

It is easy to show that the value on the right hand side is the same for $i \in \{1, 2\}$. The expression for $c_{E,S_1}(v)$ in (40) is well defined since by (28) it holds $|\left\langle \Phi_{S_1} \cdot n_{S_1}^{K_1}, 1 \right\rangle_{S_1}| \geq C h_{K_1}$. Using this estimate, (26) and the estimate (12) between h_{K_1} and h_K with $K := K(E)$, we obtain from (40) with $i = 1$

$$\begin{aligned} |c_{E,S_1}(v)| &\leq C h_{K_1}^{-1} \left\{ |E_1|^{1/2} \|v\|_{L^2(E_1)} + |c_E(v)| \max_{x \in \overline{\Omega}} \|\Phi_E(x)\| \left(|E_1| + |S_1| \right) \right\} \\ &\leq C h_K^{-1/2} \|v\|_{L^2(E)} + C|c_E(v)|. \end{aligned} \tag{41}$$

2.1.2. *Irregular faces in the three dimensional case*

In case of $E \in \mathcal{E}_i$, we will denote by E_m, $m = 1, \ldots, 4$, the son-faces of E and by $K_m := K(E_m)$ the element associated with the face E_m as depicted in Figure 4. Furthermore, we define $S_m := \overline{K_m} \cap \overline{K_{\widetilde{m}}}$ as the common

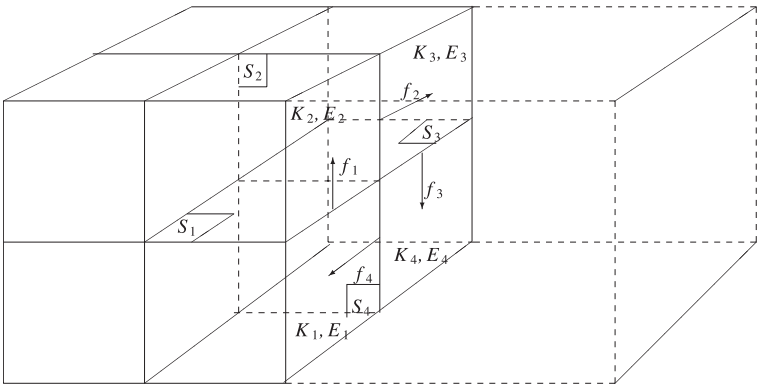

FIGURE 4. Three dimensional configuration for the case of an *irregular inner face i.e.* $E \in \mathcal{E}_i$.

face of the elements K_m and $K_{\widetilde{m}}$ where $\widetilde{m} := m+1$ if $m \in \{1,2,3\}$ and $\widetilde{m} := 1$ if $m = 4$. Now, we impose

$$I_h^E v := c_E(v)\Phi_E + \sum_{m=1}^{4} c_{E,S_m}(v)\Phi_{S_m} \tag{42}$$

where $c_E(v)$ and $\{c_{E,S_m}(v)\}_{1\le m\le 4}$ are suitable constants. For a given function $v \in (H_0^1(\Omega))^d$, the constant $c_E(v)$ is chosen such that equation (35) is satisfied which is always possible due to the fact that $\langle\, \Phi_E \cdot n_E, 1\,\rangle_E \neq 0$. Since the face-bubble functions $\{\Phi_{S_m}\}_{1\le m\le 4}$ are zero on the face E and $\Phi_E = 0$ on $\partial K(E) \setminus E$, the operator I_h^E has the same property as in the 2D-case that for all functions $v \in (H_0^1(\Omega))^d$

$$\left\langle\, I_h^E v \cdot n^{K(E)}, 1\,\right\rangle_{\partial K(E)} = \left\langle\, v \cdot n_E^{K(E)}, 1\,\right\rangle_E, \tag{43}$$

for all $E \in \mathcal{E}_i$.

Lemma 6. *Let $\mathcal{T}$ be a shape-regular and 1-irregular mesh. For a face $E \in \mathcal{E}_i$, let $K = K(E)$ and $I_h^E v$ be the local interpolation defined in (42). Then, for any given constant $c_E(v)$ there exist constants $\{c_{E,S_m}(v)\}_{1\le m\le 4}$ such that*

$$\left\langle\, I_h^E v \cdot n^{K(\widetilde{E})}, 1\,\right\rangle_{\partial K(\widetilde{E})} = \left\langle\, v \cdot n_{\widetilde{E}}^{K(\widetilde{E})}, 1\,\right\rangle_{\widetilde{E}} \qquad \forall\, \widetilde{E} \in \sigma(E) \tag{44}$$

and

$$|\, c_{E,S_m}(v)| \le C_8 h_K^{-1} \|v\|_{L^2(E)} + C_9 |\, c_E(v)| \qquad \forall\, m = 1,\dots,4, \tag{45}$$

where the constants C_8 and C_9 do not depend on E, h_K and the function $v \in (H_0^1(\Omega))^3$.

Proof. In the following, we will consider the notations given in Figure 4. The set of son-faces of E is supposed to be $\sigma(E) = \{E_1, E_2, E_3, E_4\}$. For a given index $m \in \{1,2,3,4\}$ we define $K_m := K(E_m)$ and the index m' as

$$m' := m-1 \qquad \forall\, m \in \{2,3,4\} \qquad \text{and} \qquad m' := 4 \ \text{ for } \ m = 1.$$

Then, the boundary of element K_m contains the faces E_m, S_m and $S_{m'}$. Considering the definition (42) of $I_h^E v$, taking into account the local support of the bubble functions $\{\Phi_{S_m}\}_{1\le m\le 4}$, and using that $\Phi_E(x) = 0$ for all $x \in \partial K_m \setminus (E_m \cup S_m \cup S_{m'})$, we obtain by a simple calculation the following equivalent form to the condition (44) for $\widetilde{E} = E_m$

$$\begin{aligned} &c_{E,S_m}(v)\,\langle\, \phi_{S_m} \cdot n^{K_m}, 1\,\rangle_{\partial K_m} + c_{E,S_{m'}}(v)\,\langle\, \phi_{S_{m'}} \cdot n^{K_m}, 1\,\rangle_{\partial K_m} \\ &= \left\langle\, v \cdot n_{E_m}^{K_m}, 1\,\right\rangle_{E_m} - c_E(v)\,\langle\, \Phi_E \cdot n^{K_m}, 1\,\rangle_{E_m \cup S_m \cup S_{m'}}, \end{aligned}$$

for all $m \in \{1,2,3,4\}$. In matrix form these equations lead to

$$\begin{pmatrix} f_1 & 0 & 0 & -f_4 \\ -f_1 & f_2 & 0 & 0 \\ 0 & -f_2 & f_3 & 0 \\ 0 & 0 & -f_3 & f_4 \end{pmatrix} \begin{pmatrix} c_{E,S_1}(v) \\ c_{E,S_2}(v) \\ c_{E,S_3}(v) \\ c_{E,S_4}(v) \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix}, \tag{46}$$

where f_m denotes the normal flux

$$f_m := \left\langle \Phi_{S_m} \cdot n^{K_m}, 1 \right\rangle_{S_m} \qquad \forall\, m \in \{1,2,3,4\},$$

and

$$b_m := \left\langle v \cdot n_{E_m}^{K_m}, 1 \right\rangle_{E_m} - c_E(v) \left\langle \Phi_E \cdot n^{K_m}, 1 \right\rangle_{E_m \cup S_m \cup S_{m'}},$$

for all $m \in \{1,2,3,4\}$. By the property (28) of the face bubble functions $\{\Phi_{S_m}\}_{1 \le m \le 4}$ in the case $d = 3$ and the estimate (12) between h_{K_m} and h_K we have

$$|f_m| \ge C h_{K_m}^2 \ge C h_K^2 \qquad \forall\, m \in \{1,2,3,4\}. \tag{47}$$

One can easily check that

$$\begin{aligned} c_{E,S_m}(v) &= \frac{1}{f_m}\Big(\sum_{i=1}^{m} b_i\Big), \qquad \forall\, m \in \{1,2,3\}, \\ c_{E,S_4}(v) &= 0, \end{aligned} \tag{48}$$

is an explicit solution of the linear system (46). For each $m \in \{1,2,3,4\}$, we can estimate b_m using (26) and the estimate (12) between h_{K_m} and h_K

$$\begin{aligned} |b_m| &\le |E_m|^{1/2} \|v\|_{L^2(E_m)} + |c_E(v)| \max_{x \in \overline{\Omega}} \|\Phi_E(x)\| \left\{|E_m| + |S_m| + |S_{m'}|\right\} \\ &\le C h_{K_m} \|v\|_{L^2(E)} + C h_{K_m}^2 |c_E(v)| \\ &\le C h_K \|v\|_{L^2(E)} + C h_K^2 |c_E(v)|. \end{aligned}$$

Together with (47) this proves the estimate (45). □

2.1.3. *Divergence preserving property of I_h*

Lemma 7. *Let $\mathcal{T}$ be a shape-regular and 1-irregular mesh. The operator I_h that has been defined in (23) by means of the local operators I_h^E, developed in Section 2.1.1 for the two dimensional case and in Section 2.1.2 for the three dimensional case, is divergence preserving with respect to the test space M_h^0, i.e. for $v \in (H_0^1(\Omega))^d$ it holds*

$$(\operatorname{div} I_h v, q) = (\operatorname{div} v, q) \quad \forall\, q \in M_h^0. \tag{49}$$

Proof. We denote by q_K the value of $q \in M_h^0$ on the element $K \in \mathcal{T}$. Then, using partial integration and the properties (36), (38), (39) for the two dimensional case (resp. (36), (43), (44) for the three dimensional case) of

the operator I_h^E we have

$$
\begin{aligned}
(\operatorname{div} v, q) &= \sum_{K\in\mathcal{T}} \sum_{E\in\mathcal{E}(K)} q_K \left\langle v\cdot n_E^K, 1\right\rangle_E = \sum_{E\in\mathcal{E}_0} \sum_{K\in\mathcal{T}(E)} q_K \left\langle v\cdot n_E^K, 1\right\rangle_E \\
&= \sum_{E\in\mathcal{E}_r} \left\{ q_{K(E)} \left\langle v\cdot n_E^{K(E)}, 1\right\rangle_E + q_{K'(E)} \left\langle v\cdot n_E^{K'(E)}, 1\right\rangle_E \right\} \\
&\quad + \sum_{E\in\mathcal{E}_i} \left\{ q_{K(E)} \left\langle v\cdot n_E^{K(E)}, 1\right\rangle_E + \sum_{\widetilde{E}\in\sigma(E)} q_{K(\widetilde{E})} \left\langle v\cdot n_{\widetilde{E}}^{K(\widetilde{E})}, 1\right\rangle_{\widetilde{E}} \right\} \\
&= \sum_{E\in\mathcal{E}_r} \left\{ q_{K(E)} \left\langle I_h^E v\cdot n^{K(E)}, 1\right\rangle_{\partial K(E)} + q_{K'(E)} \left\langle I_h^E v\cdot n^{K'(E)}, 1\right\rangle_{\partial K'(E)} \right\} \\
&\quad + \sum_{E\in\mathcal{E}_i} \left\{ q_{K(E)} \left\langle I_h^E v\cdot n^{K(E)}, 1\right\rangle_{\partial K(E)} + \sum_{\widetilde{E}\in\sigma(E)} q_{K(\widetilde{E})} \left\langle I_h^E v\cdot n^{K(\widetilde{E})}, 1\right\rangle_{\partial K(\widetilde{E})} \right\} \\
&= \sum_{E\in\mathcal{E}_r} \left\{ (\operatorname{div} I_h^E v, q)_{K(E)\cup K'(E)} \right\} + \sum_{E\in\mathcal{E}_i} \left\{ (\operatorname{div} I_h^E v, q)_{K(E)} + \sum_{\widetilde{E}\in\sigma(E)} (\operatorname{div} I_h^E v, q)_{K(\widetilde{E})} \right\}.
\end{aligned}
$$

Using the fact that the support of $I_h^E v$ is $\overline{K(E)} \cup \overline{K'(E)}$ in the case $E \in \mathcal{E}_r$ and $\overline{K(E)} \cup (\bigcup_{\widetilde{E}\in\sigma(E)} \overline{K(\widetilde{E})})$ in the case $E \in \mathcal{E}_i$ and applying the definition (23) of the operator I_h we get

$$
(\operatorname{div} v, q) = \sum_{E\in\mathcal{E}_r} (\operatorname{div} I_h^E v, q) + \sum_{E\in\mathcal{E}_i} (\operatorname{div} I_h^E v, q) \;=\; (\operatorname{div} I_h v, q). \qquad \square
$$

2.1.4. *Stability properties of I_h*

In order to establish the needed stability properties of the interpolation operator I_h we need first its representation on a given element $K \in \mathcal{T}$. Due to its construction, the interpolation $I_h^E v$ of $v \in (H_0^1(\Omega))^d$ has a local support, *i.e.*

$$
\operatorname{supp}(I_h^E v) \;\subseteq\; \begin{cases} \overline{K(E)} \cup \overline{K'(E)}, & \text{if } E \in \mathcal{E}_r \\ \overline{K(E)} \cup \displaystyle\bigcup_{\widetilde{E}\in\sigma(E)} \overline{K(\widetilde{E})}, & \text{if } E \in \mathcal{E}_i. \end{cases}
$$

This leads to the following local representation of $I_h v$

$$
I_h v|_K = \sum_{E\in\mathcal{E}_r(K)\cup\mathcal{E}_i(K)} I_h^E v|_K + \sum_{\widetilde{E}\in\mathcal{E}_\sigma(K)} I_h^{\mathcal{F}(\widetilde{E})} v|_K. \tag{50}
$$

Lemma 8. *Let $\mathcal{T}$ be a shape-regular and 1-irregular mesh. Let $K \in \mathcal{T}$ and I_h the interpolation operator defined in* (23)*. Then, for $v \in (H_0^1(\Omega))^d$ it holds*

$$
|I_h v|_{1,K}^2 \le C \sum_{\widetilde{K}\in\Lambda(K)} \left\{ h_{\widetilde{K}}^{-2} \|v\|_{0,\widetilde{K}}^2 \;+\; |v|_{1,\widetilde{K}}^2 \right\}, \tag{51}
$$

with $\Lambda(K)$ from (10) *and*

$$
|I_h v|_{1,\Omega}^2 \le C \sum_{K\in\mathcal{T}} \left\{ h_K^{-2} \|v\|_{0,K}^2 \;+\; |v|_{1,K}^2 \right\}. \tag{52}
$$

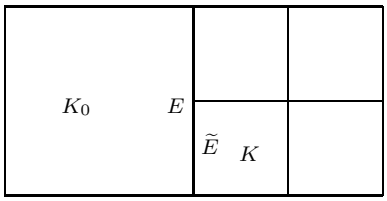

FIGURE 5. Two dimensional configuration for the case of $\widetilde{E} \in \mathcal{E}_\sigma(K)$, $E := \mathcal{F}(\widetilde{E})$ and $K_0 := K(E)$.

Proof. First we prove the estimate (51). Let $E \in \mathcal{E}_r(K) \cup \mathcal{E}_i(K)$ for a given $K \in \mathcal{T}$. Due to (34), (37) and (42) we have

$$I_h^E v|_K = c_E(v)\phi_E|_K,$$

where we have used the fact that for $E \in \mathcal{E}_i(K)$ we have $\phi_{S_m}|_K = 0$ for all $m = 1, \dots, 4^{d-2}$. Due to (35) and (28), we get for $c_E(v)$

$$|c_E(v)| = \frac{\langle v \cdot n_E, 1 \rangle_E}{\langle \Phi_E \cdot n_E, 1 \rangle_E} \leq \frac{|E|^{1/2}\|v\|_{0,E}}{C_3 h_K^{d-1}}, \tag{53}$$

$$\leq \frac{C_5^{1/2}\, C_4}{C_3}\, h_K^{-d/2+1} \left\{ h_K^{-1}\, \|v\|_{0,K} + |v|_{1,K} \right\}, \tag{54}$$

where the last estimate follows from (11) and (29). Together with (27) we obtain

$$|I_h^E v|_{1,K} \leq C_{10}\, \left\{ h_K^{-1}\, \|v\|_{0,K} + |v|_{1,K} \right\} \qquad \forall E \in \mathcal{E}_r(K) \cup \mathcal{E}_i(K), \tag{55}$$

with $C_{10} := C_5^{1/2} C_4 C_2 / C_3$.

Now, we consider the configuration corresponding to the Figure 5 *i.e.* $\widetilde{E} \in \mathcal{E}_\sigma(K)$, $E := \mathcal{F}(\widetilde{E})$ and $K_0 := K(E)$. Based on the representation (50), we obtain

$$I_h^E v|_K = c_E(v)\Phi_E|_K + \sum_{m=1}^{4^{d-2}} c_{E,S_m}(v)\Phi_{S_m}|_K. \tag{56}$$

Since $E \in \mathcal{E}(K_0)$, we get similarly to (53)

$$|c_E(v)| \leq \frac{C_5^{1/2}}{C_3}\, h_{K_0}^{-(d-1)/2}\, \|v\|_{0,E}. \tag{57}$$

Using (27), (12) and (29) we get

$$|c_E(v)\Phi_E|_{1,K} \leq \frac{C_5^{1/2} C_2 C_7^{\frac{d}{2}-1}}{C_3}\, h_{K_0}^{-1/2}\, \|v\|_{0,E} \tag{58}$$

$$\leq \frac{C_5^{1/2} C_2 C_7^{\frac{d}{2}-1} C_4}{C_3} \left\{ h_{K_0}^{-1}\, \|v\|_{0,K_0} + |v|_{1,K_0} \right\}. \tag{59}$$

Based on the estimates (41) and (45) we deduce similarly the following estimates

$$|c_{E,S_m}(v)| \leq C_8\, h_{K_0}^{-(d-1)/2}\, \|v\|_{L^2(E)} + C_9 |c_E(v)|,$$

$$\leq C h_{K_0}^{1-\frac{d}{2}} \left\{ h_{K_0}^{-1}\, \|v\|_{0,K_0} + |v|_{1,K_0} \right\}. \tag{60}$$

Together with (27) we then obtain

$$|c_{E,S_m}(v)\Phi_{S_m}|_{1,K} \leq C \left\{ h_{K_0}^{-1} \|v\|_{0,K_0} + |v|_{1,K_0} \right\}, \tag{61}$$

for all $m \in \left\{1, \ldots, 4^{d-2}\right\}$. Finally from (56), (59) and (61) we get

$$|I_h^E v|_{1,K} \leq C \left\{ h_{K_0}^{-1} \|v\|_{0,K_0} + |v|_{1,K_0} \right\} \quad \text{where } K_0 = K(E). \tag{62}$$

Using the representation formula (50) of $I_h v|_K$ we obtain

$$\begin{aligned} |I_h v|_{1,K}^2 &\leq C \left\{ h_K^{-2} \|v\|_{0,K}^2 + |v|_{1,K}^2 \right\} + C \sum_{\widetilde{E} \in \mathcal{E}_\sigma(K)} \left\{ h_{K(\mathcal{F}(\widetilde{E}))}^{-2} \|v\|_{0,K(\mathcal{F}(\widetilde{E}))}^2 + |v|_{1,K(\mathcal{F}(\widetilde{E}))}^2 \right\} \\ &\leq C \sum_{\widetilde{K} \in \Lambda(K)} \left\{ h_{\widetilde{K}}^{-2} \|v\|_{0,\widetilde{K}}^2 + |v|_{1,\widetilde{K}}^2 \right\} \end{aligned}$$

which concludes the proof of (51). The estimate (52) is a direct consequence of (51). □

2.2. H^1-stable interpolation operator R_h

The definition of the interpolation operator Π_h in (22) also requires the construction of an interpolation operator $R_h : (H_0^1(\Omega))^d \to X_h^1$ which aims at approximating possibly non-smooth functions in H^1 by means of continuous piecewise polynomials of low order. Such an operator for H^1 functions has been proposed by Scott and Zhang [22] and its construction is based on averaging over $(d-1)$-dimensional faces of the d-dimensional elements. However, the restriction of this approach is that it assumes affine equivalent elements and a regular mesh which excludes hanging nodes. In [17] an extension of the Scott-Zhang operator for the case of a shape regular and 1-irregular mesh is proposed and the following properties have been proven.

Lemma 9. *Let $\mathcal{T}$ be a shape-regular and 1-irregular mesh. There exists an interpolation operator R_h such that for $K \in \mathcal{T}$ it holds*

$$|R_h v|_{1,K} \leq C |v|_{1,\omega(K)} \qquad \forall v \in (H^1(\Omega))^d \tag{63}$$

$$\|v - R_h v\|_{0,K} \leq C h_K |v|_{1,\omega(K)} \qquad \forall v \in (H^1(\Omega))^d \tag{64}$$

where

$$\delta(K) := \bigcup_{\widetilde{K} \in \Lambda(K)} \overline{\widetilde{K}}, \qquad \omega(K) := \bigcup_{\widetilde{K} \in \delta(K)} \delta(\widetilde{K}).$$

Proof. See [17], Theorem 8. □

2.3. Properties of Π_h

Now, for the operator $\Pi_h : (H_0^1(\Omega))^d \to X_h^2$ defined in (22) by means of the operators $I_h : (H_0^1(\Omega))^d \to X_h^2$ and $R_h : (H_0^1(\Omega))^d \to X_h^1$ with $X_h^1 \subset X_h^2$, we are able to prove the needed properties (20) and (21).

Lemma 10. *Let $\mathcal{T}$ be a shape-regular and 1-irregular mesh. Then, the operator Π_h defined in (22) is H^1-stable uniformly with respect to h in the sense of (20) and divergence preserving in the sense of (21).*

Proof. From the definition (22) of Π_h, the stability of R_h in (63) and of I_h in (52) we obtain

$$\begin{aligned} |\Pi_h v|_{1,\Omega}^2 &\leq 2|R_h v|_{1,\Omega}^2 + 2|I_h(v - R_h v)|_{1,\Omega}^2, \\ &\leq C \left\{ |v|_{1,\Omega}^2 + \sum_{K \in \mathcal{T}} \left\{ h_K^{-2} \|v - R_h v\|_{0,K}^2 + |v - R_h v|_{1,K}^2 \right\} \right\}, \\ &\leq C |v|_{1,\Omega}^2, \end{aligned}$$

where the last estimate relies on the interpolation property (64) of R_h. This shows that Π_h is H^1-stable in the sense of (20). The property (21) that Π_h is divergence preserving is an easy consequence of the fact that the operator I_h is divergence preserving. □

3. Proving the inf-sup condition

Our approach to prove the inf-sup condition is based on the macro-element technique proposed by Boland and Nicolaides [6] (see also [13], Sect. II.1.4). In its general setup, this technique relies on a partitioning of the domain Ω by means of R nonoverlapping open subdomains Ω_r, $r = 1, \dots, R$, with Lipschitz continuous boundary. In our case, these subdomains Ω_r are built by the mesh elements $K \in \mathcal{T}$. Based on the finite element spaces $X^N = (S^N)^d$ and M^N defined in (14) and (15), respectively, we define the following local spaces

$$X^N(K) := \left\{ v \in (H_0^1(K))^d : \; v \circ F_K \in (\mathbb{Q}_{r_K}(\widehat{K}))^d \right\}, \tag{65}$$

$$M^N(K) := \left\{ q \in L_0^2(K) : \; q \circ F_K \in \mathbb{P}_{r_K - 1}(\widehat{K}) \right\}. \tag{66}$$

We say that the finite element spaces X^N and M^N fulfill the *local inf-sup condition uniformly with respect to* $K \in \mathcal{T}$ if there exist constants $\lambda(r_K) > 0$, such that for all $K \in \mathcal{T}$ it holds

$$\inf_{q^N \in M^N(K) \setminus \{0\}} \; \sup_{v^N \in X^N(K)} \frac{(\nabla \cdot v^N, q^N)_K}{|v^N|_{1,K} \|q^N\|_{0,K}} \geq \lambda(r_K), \tag{67}$$

where $\lambda(r_K)$ is independent of the mesh cell K and the mesh parameter h but may depend on the polynomial degree r_K of the local spaces $X^N(K)$ and $M^N(K)$.

Now, in order to prove the global inf-sup condition for our finite element spaces $X^N = (S^N)^d$ and M^N, defined in (14) and (15), respectively, we only have to verify the local inf-sup condition (67) and the global inf-sup condition for a suitable pair of low order subspaces $(\overline{X}^N, M_h^0)$. This is justified by the following lemma.

Lemma 11. *Let the local inf-sup condition* (67) *be fulfilled uniformly with respect to $K \in \mathcal{T}$ with positive constants $\lambda(r_K)$ independent of K and the mesh parameter h. Furthermore, we assume that there exists a subspace $\overline{X}^N \subset X^N$ such that the pair of finite element spaces $(\overline{X}^N, M_h^0)$ fulfills the global inf-sup condition*

$$\inf_{q_h \in M_h^0 \setminus \{0\}} \; \sup_{v_h \in \overline{X}^N} \frac{(\nabla \cdot v_h, q_h)_\Omega}{|v_h|_{1,\Omega} \|q_h\|_{0,\Omega}} \geq \bar{\beta} > 0, \tag{68}$$

with a constant $\overline{\beta} > 0$ independent of h. Then, the pair (X^N, M^N) satisfies the global inf-sup condition (2) *with a constant $\beta > 0$ which depends on $\bar{\beta}$ and on*

$$\lambda^* := \min_{r=2,\dots,r^*} \lambda(r), \qquad \textit{where} \quad r^* := \max_{K \in \mathcal{T}} r_K, \tag{69}$$

but which is independent of h.

Proof. Due to the definition of λ^* in (69), the local inf-sup condition (67) is satisfied with the fixed positive constant λ^* for each element $K \in \mathcal{T}$. Therefore, the statement of Lemma 11 directly follows from Theorem II.1.12 in [13]. □

A nice feature in the analysis of our pair of higher order finite element spaces (X^N, M^N) on a 1-irregular mesh with hanging nodes is that the local spaces $X^N(K_0)$ and $M^N(K_0)$ for a fixed element $K_0 \in \mathcal{T}$ with the polynomial degree $r_0 = r_{K_0}$ are completely the same as in the case of a higher order pair (X_h, M_h) based on the uniform polynomial degree r_0 and a mesh $\mathcal{T}_h$ without hanging nodes that contains K_0. Therefore, the proof of the local inf-sup condition (67) for our local spaces defined in (65) and (66) is the same as in [19].

Lemma 12. *Let $\mathcal{T}$ be a shape-regular mesh. Then, for each element $K \in \mathcal{T}$, the pair $(X^N(K), M^N(K))$ of the mapped local finite element spaces defined in* (65) *and* (66)*, satisfies the local inf-sup condition* (67) *with a constant $\lambda(r_K) > 0$ which is independent of K and the mesh parameter h.*

Proof. See the proof of Theorem 8 in [19]. □

The global inf-sup condition (68) of the pair of spaces $(\overline{X}^N, M_h^0)$ can be proved for the second order subspace $\overline{X}^N = X_h^2 \subset X^N$.

Lemma 13. *Let $\mathcal{T}$ be a locally refined mesh, which is assumed to be* 1*-irregular and shape-regular. Then, the pair of finite element spaces $(\overline{X}^N, M_h^0)$ with $\overline{X}^N := X_h^2$ satisfies the global inf-sup condition* (68) *with a constant $\overline{\beta} > 0$ independent of h.*

Proof. In Section 2.3, we have constructed an interpolation operator $\Pi_h : (H_0^1(\Omega))^d \to X_h^2$ which is H^1-stable uniformly with respect to h (see (20)) and divergence preserving (see (21)). From these properties, due to Fortin [12] (see also [13], Lem. II.1.1), we deduce that the inf-sup condition (68) holds. □

Now, we can state our main result.

Theorem 14. *Let $\mathcal{T}$ be locally refined mesh which is shape-regular and* 1*-irregular in the sense of Definitions* 1 *and* 3*, respectively. For the pair of finite element spaces (X^N, M^N), defined in* (14) *and* (15) *with $X^N = (S^N)^d$, it holds the global inf-sup condition* (2) *with an h-independent constant $\beta > 0$ which depends on the constants $\bar{\beta}$ from Lemma* 13 *and λ^* defined in* (69) *with the constants $\lambda(r)$ from Lemma* 12*.*

Proof. The proof of Theorem 14 is a direct consequence of Lemmas 11, 12 and 13. □

Remark 15. Theorem 14 shows that, even for a locally refined, 1-irregular and shape regular mesh $\mathcal{T}$, the inf-sup constant β in (2) is independent of the local and global mesh parameters h_K and h, respectively. However, the dependency of β on the polynomial degree vector $\mathbf{r} := \{r_K : K \in \mathcal{T}\}$ is not investigated. For the situation of just one cell $\mathcal{T} = \{K_0\}$ with $K_0 = (-1,1)^d$, it has been shown in [4] that the inf-sup constant β is independent of the polynomial degree $r = r_{K_0}$ of the spaces $X^N(K_0)$ and $M^N(K_0)$ defined in (65) and (66). The proof that the local inf-sup-constants $\lambda(r_K)$ in (67) are independent of the polynomial degree r_K on grids described in Section 1.2 with a sufficiently small mesh size $h \le h_0$ will be the object of a forthcoming paper.

Remark 16. For the pair of finite element spaces X^N and $\widetilde{M}^N$ with the so-called *unmapped* pressure space defined in [13], Section II.3.2, we can derive an analogous result as in Theorem 14. To this end, we need the local inf-sup condition (67) with the local unmapped pressure space

$$\widetilde{M}^N(K) := \left\{ q \in L_0^2(K) : \; q \in \mathbb{P}_{r_K - 1}(K) \right\},$$

instead of the local mapped space $M^N(K)$ defined in (66). Again the inf-sup constants $\lambda(r_K)$ have to be independent of K and the mesh parameter h. This local inf-sup condition has been proven in [13] Theorem II.3.2, for the 2D-case and the proof can be generalized in a straightforward way to the 3D-case. The global inf-sup condition (68) for the low order subspaces $(\overline{X}^N, M_h^0) \subset (X^N, \widetilde{M}^N)$ in case of the unmapped pressure space is exactly the same as in the case of the mapped pressure space M^N. Therefore, this condition follows from Lemma 13. Thus, applying Lemma 11, Theorem II.3.2 in [13] and its generalization to the 3D-case, we obtain the analogous result as in Theorem 14 for the pair of finite element spaces $(X^N, \widetilde{M}^N)$.

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