

LOCKING FREE MATCHING OF DIFFERENT THREE DIMENSIONAL MODELS IN STRUCTURAL MECHANICS

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Abstract. The present paper proposes and analyzes a general locking free mixed strategy for computing the deformation of incompressible three dimensional structures placed inside flexible membranes. The model involves as in Chapelle and Ferent [*Math. Models Methods Appl. Sci.* **13** (2003) 573–595] a bending dominated shell envelope and a quasi incompressible elastic body. The present work extends an earlier work of Arnold and Brezzi [*Math Comp.* **66** (1997) 1–14] treating the shell part and proposes a global stable finite element approximation by coupling optimal mixed finite element formulations of the different subproblems by mortar techniques. Examples of adequate finite elements are proposed. Convergence results are derived in two steps. First a global inf-sup condition is proved, deduced from the local conditions to be satisfied by the finite elements used for the external shell problem, the internal incompressible 3D problem, and the mortar coupling, respectively. Second, the analysis of Arnold and Brezzi [*Math. Comp.* **66** (1997) 1–14] is extended to the present problem and least to convergence results for the full coupled problem, with constants independent of the problem's small parameters.

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1. INTRODUCTION

Many challenging applications encountered both in industry and in biomechanics involve soft incompressible three dimensional structures placed inside flexible membranes controlling the boundary deformation. Foam structures within flexible envelopes, liver or eye surgery simulations, microcapsules used in drug delivery systems [11] are typical examples. In such situations, each component of the structure has its own model, and discretization requirements. In addition, the substructures can be very stiff along certain deformation modes: large bulk modulus for the internal quasi incompressible material, or large stiffness of membrane stress and transverse shear in the thin layered materials used for the envelope. And those critical points are precisely the locations where the local stresses must be predicted with accuracy in a fatigue, durability or stability analysis.

There are two basic computational tools to handle such situations. First, mixed finite elements treating some components of the strain tensor as independent variables provide a very systematic way of taking care of the so-called delinquent modes, that is the stiff modes which may lock in a standard finite element discretisation because of improper kinematic discretisation and oversized stiffness [1, 10]. Second, mortar methods provide an

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efficient way of weakly coupling subdomains with different scales and nonmatching finite element grids [4]. But can we match the different local locking free mixed finite elements by mortar coupling, while preserving their original stability and robustness?

The present paper answers this question in a significative example by proving the optimality of a locking free shell element weakly coupled to an incompressible 3D finite element model. The model involves as in [9] a bending dominated shell envelope and a quasi incompressible elastic body. The theoretical analysis developed herein can also be applied to fluid structure interaction problems where an incompressible viscous fluid flows inside a flexible shell. The main difference compared to earlier work such as [14] is that the convergence result should be independent of the small parameters (thickness, compressibility) which are present in the coupled problem. In addition, the present paper proposes practical examples of admissible mixed finite elements and mortar interface elements. The remaining limitation of the theory concerns the analysis of the shell part, which is restricted to the bending dominated case and for which the construction of uniformly stable finite elements satisfying an adequate inf-sup condition is still a rather open problem.

The paper is organized as follows. The coupled mechanical problem under study is introduced in Section 2. A general mixed formulation is proposed in Section 3, based on an earlier work of [1], and a stable finite element approximation is proposed in Section 4, coupling optimal mixed finite element formulations of the subproblems by mortar techniques. Convergence results are finally derived in Section 5 in two steps. First a global inf-sup condition is proved, deduced from the local conditions to be satisfied by the finite elements used for the external shell problem, the internal incompressible 3D problem, and the mortar coupling, respectively. Second, the analysis of [1] is extended to the present problem to obtain finite element convergence results for the full coupled problem, with constants independent of the problem's small parameters.

2. NOTATION AND MECHANICAL PROBLEM

We consider hereafter a soft quasi incompressible three dimensional elastic body Ω moving within a thin flexible envelope ω of thickness t . For simplicity, we will suppose that the two bodies Ω and ω are in contact on a part Γ of the boundary $\partial\Omega$ of Ω , that the three dimensional body is fixed on a part $\partial\Omega_D$ of its boundary, and that the external membrane ω is clamped to this support $\partial\Omega_D$ (Fig. 1). All length are taken to be adimensional. In other words, the physical length are divided by a reference length L associated to the overall dimension of the structure under consideration. Thus the thickness t introduced above is a small adimensional number, characterizing the ratio between the physical thickness and the physical length of the shell under consideration.

In small deformation, the fundamental description of such coupled structures under consideration involves three independent kinematic unknowns: the membrane displacement $\underline{\xi}_s$ in $H^1(\omega)$ of the surface ω , the membrane's normal rotation $\underline{\beta}$ in $H^1(\omega)$ tangent to the surface ω and describing the local infinitesimal variation $\underline{\beta} = \delta\underline{a}_3$ of the unit normal vector \underline{a}_3 to the surface ω , and the internal structure displacement $\underline{\xi}_i \in H^1(\Omega)$ describing the displacement of the internal points in the structure. The rotation $\underline{\beta}$ adds some flexural stiffness to the envelope which may be very important in zones undergoing compression or large flexions. By construction, these kinematic unknowns belong to the kinematic space

$$\begin{aligned} \mathbf{V} &= \mathbf{V}_s \times \mathbf{V}_i \\ \mathbf{V}_s &= \{ \underline{\xi} = (\underline{\xi}_s, \underline{\beta}) \in [H^1(\omega)]^3 \times [H^1(\omega)]^3, \underline{\beta} \cdot \underline{a}_3 = 0, \underline{\xi}_s = \underline{\beta} = 0 \text{ on } \partial\omega \cap \partial\Omega_D \} \\ \mathbf{V}_i &= \{ \underline{\xi}_i \in [H^1(\Omega)]^3, \underline{\xi}_i = 0 \text{ on } \partial\Omega_D \}. \end{aligned}$$

In addition, we must impose a zero displacement discontinuity

$$[[\underline{\xi}]] = \underline{\xi}_s - \frac{t}{2} \underline{\beta} - \underline{\xi}_i|_{\Gamma}$$

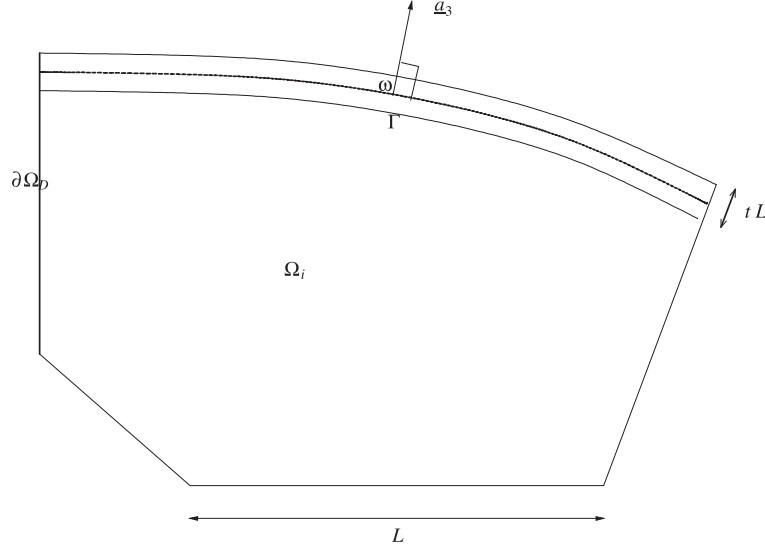


FIGURE 1. Geometry of the coupled structure with an external thin envelope of adimensional thickness t and an internal three dimensional quasi incompressible medium Ω_i . The structure is fixed on $\partial\Omega_D$.

on the interface Γ . By construction, this interface is defined from the shell mid-surface ω by

$$\Gamma = \left\{ M = m - \frac{t}{2} \underline{a}_3, m \in \omega \right\}$$

and is therefore parallel to this mid surface. To impose a zero displacement discontinuity on this interface in a weak form involves the trace space $(\text{Tr } \mathbf{V}_i)_{|\Gamma}$, and its dual \mathbf{M} in which to look for the contact reaction force \underline{g} . Endowing the space $(\text{Tr } \mathbf{V}_i)_{|\Gamma}$ with the $H^{1/2}$ like trace norm

$$\|\underline{v}\|_{tr} = \inf_{\underline{\xi}_i \in \mathbf{V}_i, \text{Tr } (\underline{\xi}_i) = \underline{v}} \|\underline{\xi}_i\|_{\mathbf{V}_i},$$

the jump $\underline{\xi} \rightarrow \llbracket \underline{\xi} \rrbracket = \underline{\xi}_s - \frac{t}{2} \underline{\beta} - \underline{\xi}_i|_{\Gamma}$ defines a continuous map from \mathbf{V} onto $(\text{Tr } \mathbf{V}_i)_{|\Gamma}$. In fact, this map is already onto from $\{0\} \times \mathbf{V}_i$ to $(\text{Tr } \mathbf{V}_i)_{|\Gamma}$ by construction of the image space.

Four fields then contribute to the strains: the membrane deformation of the envelope

$$\underline{\underline{\gamma}}(\nabla_s \underline{\xi}_s) = \frac{1}{2} ((\nabla_s \underline{\phi}_s)^t \cdot \nabla_s \underline{\xi}_s + (\nabla_s \underline{\xi}_s)^t \cdot \nabla_s \underline{\phi}_s),$$

the flexion of the envelope, measuring the variation of the shell curvature induced by the shell displacement

$$\underline{\underline{\rho}}(\nabla_s \underline{\xi}_s, \nabla_s \underline{\beta}) = \frac{1}{2} \left((\nabla_s \underline{\xi}_s)^t \cdot \nabla_s \underline{a}_3 + (\nabla_s \underline{a}_3)^t \cdot \nabla_s \underline{\xi}_s + (\nabla_s \underline{\phi}_s)^t \cdot \nabla_s \underline{\beta} + (\nabla_s \underline{\beta})^t \cdot \nabla_s \underline{\phi}_s \right),$$

the shear inside the envelope

$$\underline{\underline{\Phi}}(\nabla_s \underline{\xi}_s, \underline{\beta}) = \underline{a}_3 \cdot \nabla_s \underline{\xi}_s - \underline{\beta},$$

and the linearised strain tensor in the internal structure

$$\underline{\underline{\varepsilon}}(\nabla_s \underline{\xi}_i) = \frac{1}{2} (\nabla_s \underline{\xi}_i + \nabla^t \underline{\xi}_i).$$

internal pressure p_i , as independent variables to be looked upon in the stress space

$$\mathbf{W} = \mathbf{W}_s \times \mathbf{W}_i = \{p = (\underline{\underline{\Lambda}}, \underline{\underline{\eta}}, p_i) \in L^2(\omega, (T_\omega \times T_\omega)_{sym}) \times L^2(\omega, T_\omega) \times L^2(\Omega_i)\}.$$

Membrane stresses and shear operate on the tangent space T_ω to the shell, and the internal pressure acts inside the volume Ω_i . This stress space is to be endowed by a standard L^2 norm. Extending the ideas of [1], we then split the stiffness operator into a regular part

$$\begin{aligned} a(\underline{\underline{\xi}}, \hat{\underline{\underline{\xi}}}) &= \int_\omega E_s \underline{\underline{\rho}}(\nabla_s \underline{\underline{\xi}}_s, \nabla_s \underline{\underline{\beta}}) : \underline{\underline{\rho}}(\nabla_s \hat{\underline{\underline{\xi}}}_s, \nabla_s \hat{\underline{\underline{\beta}}}) \, d\omega + \int_\Omega 2\mu_i \underline{\underline{\varepsilon}}(\nabla \underline{\underline{\xi}}_i) : \underline{\underline{\varepsilon}}(\nabla \hat{\underline{\underline{\xi}}}_i) \, d\Omega \\ &+ c_0 \int_\omega \left(E_s \underline{\underline{\gamma}}(\nabla_s \underline{\underline{\xi}}_s) : \underline{\underline{\gamma}}(\nabla_s \hat{\underline{\underline{\xi}}}_s) + G_s \underline{\underline{\Phi}}(\nabla_s \underline{\underline{\xi}}_s, \underline{\underline{\beta}}) \cdot \underline{\underline{\Phi}}(\nabla_s \hat{\underline{\underline{\xi}}}_s, \hat{\underline{\underline{\beta}}}) \right) \, d\omega \end{aligned} \quad (4)$$

and a singular one, treated by duality

$$c(p, \hat{p}) = \frac{t^2}{1 - c_0 t^2} \int_\omega (E_s^{-1} \underline{\underline{\Lambda}} : \hat{\underline{\underline{\Lambda}}} + G_s^{-1} \underline{\underline{\eta}} \cdot \hat{\underline{\underline{\eta}}}) \, d\omega + \frac{1}{\lambda_i} \int_\Omega p_i \hat{p}_i \, d\Omega. \quad (5)$$

The form c obtained by duality defines a norm which tends to zero when the small parameters t and $1/\lambda_i$ go to zero if the associated stress remain bounded. Controlling this norm will only result in a weak control on the stress p . To emphasize the weak character of this bilinear form c , we will denote the associated norm by

$$\|\hat{p}\|_\varepsilon^2 = c(\hat{p}, \hat{p}).$$

The regular coefficient c_0 introduced in this splitting is a user defined positive constant, independent of the small parameters, which will guarantee uniform coercivity of the form a . Since the thickness is supposed to be small, the denominator $1 - c_0 t^2$ will always stay positive. The duality relating the singular term to the strain tensor is defined through the bilinear form

$$b(p, \hat{\underline{\underline{\xi}}}) = \int_\omega (\underline{\underline{\Lambda}} : \underline{\underline{\gamma}}(\nabla_s \hat{\underline{\underline{\xi}}}_s) + \underline{\underline{\eta}} \cdot \underline{\underline{\Phi}}(\nabla_s \hat{\underline{\underline{\xi}}}_s, \hat{\underline{\underline{\beta}}}_s)) \, d\omega + \int_\Omega p \operatorname{div} \hat{\underline{\underline{\xi}}}_i \, d\Omega. \quad (6)$$

We finally introduce the subspace \mathbf{V}_0 of \mathbf{V} made of those displacements in \mathbf{V} which are continuous at the interface

$$\mathbf{V}_0 = \{\underline{\underline{\xi}} \in \mathbf{V}, \int_\Gamma \llbracket \underline{\underline{\xi}} \rrbracket \hat{\underline{\underline{g}}} \, d\Gamma = 0, \quad \forall \hat{\underline{\underline{g}}} \in \mathbf{M}\}. \quad (7)$$

From the continuity of the jump $\llbracket \underline{\underline{\xi}} \rrbracket$ as a map from \mathbf{V} on $\operatorname{Tr}(\mathbf{V}_i)|_\Gamma$, the subspace \mathbf{V}_0 is a closed subspace of \mathbf{V} .

Using the obvious notation

$$L(\hat{\underline{\underline{\xi}}}) = \int_\omega (\underline{\underline{f}}_s \cdot \hat{\underline{\underline{\xi}}}_s + \underline{\underline{m}} \cdot \hat{\underline{\underline{\beta}}}) \, d\omega + \int_\Omega \underline{\underline{f}}_i \cdot \hat{\underline{\underline{\xi}}}_i \, d\Omega$$

for the imposed loading, problem (1)–(2) with weak interface continuity requirement (3) takes then the classical mixed form:

Find the generalized displacement $\underline{\underline{\xi}} = (\underline{\underline{\xi}}_s, \underline{\underline{\beta}}, \xi_i) \in \mathbf{V}_0$ and the delinquent modes $p = (\underline{\underline{\Lambda}}, \underline{\underline{\eta}}, p_i) \in \mathbf{W}$ which satisfy

$$\begin{aligned} a(\underline{\underline{\xi}}, \hat{\underline{\underline{\xi}}}) + b(p, \hat{\underline{\underline{\xi}}}) &= L(\hat{\underline{\underline{\xi}}}), \quad \forall \hat{\underline{\underline{\xi}}} \in \mathbf{V}_0, \underline{\underline{\xi}} \in \mathbf{V}_0, \\ b(\hat{p}, \underline{\underline{\xi}}) - c(p, \hat{p}) &= 0, \quad \forall \hat{p} \in \mathbf{W}, p \in \mathbf{W}. \end{aligned} \quad (8)$$

Indeed, from the second equation of (8), we first identify the delinquent modes to

$$(\underline{\underline{\Lambda}}, \underline{\eta}, p_i) = \left(\left(\frac{1}{t^2} - c_0 \right) E_s \underline{\underline{\gamma}}(\nabla_s \underline{\underline{\xi}}_s), \left(\frac{1}{t^2} - c_0 \right) G_s \underline{\underline{\Phi}}(\nabla_s \underline{\underline{\xi}}_s, \underline{\beta}), \lambda_i \operatorname{div} \underline{\underline{\xi}}_i \right).$$

Plugged in the first line of (8), this yields (1)–(2).

But now, as observed in [5, 8] and recalled in ([1], Lem. 1), for the proposed boundary conditions clamping ω on part of its boundary, the bilinear form

$$\begin{aligned} a_s((\underline{\underline{\xi}}_s, \underline{\beta}), (\hat{\underline{\underline{\xi}}}_s, \hat{\underline{\beta}})) &= c_0 \int_{\omega} (E_s \underline{\underline{\gamma}}(\nabla_s \underline{\underline{\xi}}_s) : \underline{\underline{\gamma}}(\nabla_s \hat{\underline{\underline{\xi}}}_s) + G_s \underline{\underline{\Phi}}(\nabla_s \underline{\underline{\xi}}_s, \underline{\beta}) \cdot \underline{\underline{\Phi}}(\nabla_s \hat{\underline{\underline{\xi}}}_s, \hat{\underline{\beta}}) \\ &\quad + E_s \underline{\underline{\rho}}(\nabla_s \underline{\underline{\xi}}_s, \nabla_s \underline{\beta}) : \underline{\underline{\rho}}(\nabla_s \hat{\underline{\underline{\xi}}}_s, \nabla_s \hat{\underline{\beta}}) \, d\omega \end{aligned}$$

is an inner product on \mathbf{V}_s , and the corresponding norm is equivalent to the $H^1(\omega)$ norm. Similarly, from the Korn's inequality and Poincaré's lemma used on the internal domain Ω , we readily prove that

$$a_i(\underline{\underline{\xi}}_i, \hat{\underline{\underline{\xi}}}_i) = \int_{\Omega} 2\mu_i \underline{\underline{\varepsilon}}(\nabla \underline{\underline{\xi}}_i) : \underline{\underline{\varepsilon}}(\nabla \hat{\underline{\underline{\xi}}}_i) \, d\Omega$$

is an inner product on \mathbf{V}_i with corresponding norm equivalent to the $H^1(\Omega)$ norm. By addition, $a(\underline{\underline{\xi}}, \hat{\underline{\underline{\xi}}})$ is an inner product on \mathbf{V} with coercivity constant c_a and continuity constant $\|a\|$, defining thus a corresponding norm equivalent to the \mathbf{V} norm. Finally, $c(\cdot, \cdot)$ defines a norm $\|\cdot\|_{\varepsilon}$ on \mathbf{W} , equivalent to the L^2 norm but with constants of equivalence depending on the values of the small parameters t and $1/\lambda_i$. We can then apply as in [1] the Lax Milgram theorem on the space $\mathbf{V}_0 \times \mathbf{W}$ endowed with the continuous coercive bilinear form

$$A((\underline{\underline{\xi}}, p), (\hat{\underline{\underline{\xi}}}, \hat{p})) = a(\underline{\underline{\xi}}, \hat{\underline{\underline{\xi}}}) + b(p, \hat{\underline{\underline{\xi}}}) - b(\hat{p}, \underline{\underline{\xi}}) + c(p, \hat{p})$$

to prove that the mixed problem (8) has a unique solution $(\underline{\underline{\xi}}, p) \in \mathbf{V}_0 \times \mathbf{W}$. Moreover, writing (8) with $\hat{\underline{\underline{\xi}}} = \underline{\underline{\xi}}$ and $\hat{p} = -p$ directly yields by addition

$$\alpha_a \|\underline{\underline{\xi}}\|_{\mathbf{V}} + \|p\|_{\varepsilon} \leq \|L\|_{\mathbf{V}'}$$

And, from the first line of (8), we obtain the last bound

$$\begin{aligned} \sup_{0 \neq \hat{\underline{\underline{\xi}}} \in \mathbf{V}_0} \frac{b(p, \hat{\underline{\underline{\xi}}})}{\|\hat{\underline{\underline{\xi}}}\|_{\mathbf{V}}} &= \sup_{0 \neq \hat{\underline{\underline{\xi}}} \in \mathbf{V}_0} \frac{-a(\underline{\underline{\xi}}, \hat{\underline{\underline{\xi}}}) + L(\hat{\underline{\underline{\xi}}})}{\|\hat{\underline{\underline{\xi}}}\|_{\mathbf{V}}} \\ &\leq \|a\| \|\underline{\underline{\xi}}\|_{\mathbf{V}} + \|L\|_{\mathbf{V}'} \\ &\leq C \|L\|_{\mathbf{V}'}, \end{aligned}$$

which finally implies

$$\|\underline{\underline{\xi}}\|_{\mathbf{V}} + \sup_{0 \neq \hat{\underline{\underline{\xi}}} \in \mathbf{V}_0} \frac{b(p, \hat{\underline{\underline{\xi}}})}{\|\hat{\underline{\underline{\xi}}}\|_{\mathbf{V}}} + \|p\|_{\varepsilon} \leq C \|L\|_{\mathbf{V}'}$$

Above, C is a constant independent of the small parameters, that is of the shell thickness t and coefficient inverse $1/\lambda_i$. Observe in contrast that the coercivity constant of the bilinear form A does depend on these small parameters when using the standard norm of $\mathbf{V} \times \mathbf{W}$, since c tend to zero when these small parameters go to zero. A specific analysis is therefore needed to get stability and convergence results which are independent of the small parameters, following the steps of [1].

Also, the above mixed formulation does not explicitly involve the contact force \underline{g} . But, it can be introduced by a direct application of the closed range theorem: by construction $a(\underline{\xi}, \hat{\underline{\xi}}) + b(p, \hat{\underline{\xi}}) - L(\hat{\underline{\xi}})$ belongs to $\mathbf{V}_0^\perp = \text{Ker}[\llbracket \cdot \rrbracket]^\perp = \text{Im}[\llbracket \cdot \rrbracket]^t$ and the jump $\llbracket \cdot \rrbracket$ is an onto map from \mathbf{V} to $\text{Tr}(\mathbf{V}_i)_{|\Gamma}$, thus there exists a contact force $\underline{g} \in \mathbf{M} = (\text{Tr}(\mathbf{V}_i)_{|\Gamma})'$ such that

$$a(\underline{\xi}, \hat{\underline{\xi}}) + b(p, \hat{\underline{\xi}}) - L(\hat{\underline{\xi}}) = \langle \underline{g}, \llbracket \hat{\underline{\xi}} \rrbracket \rangle_{\mathbf{M} \times \mathbf{M}'}, \quad \forall \hat{\underline{\xi}} \in \mathbf{V}, \quad (9)$$

and $\|\underline{g}\|_{\mathbf{M}} \leq C \|L\|_{\mathbf{V}'}$.

4. MIXED AND MORTAR FINITE ELEMENT APPROXIMATION

4.1. Finite element approximation

We have seen that the mixed formulation (8) leads to a general existence and stability result. But more important, deriving a locking free finite element approximation of such a formulation is rather straightforward when such a formulation is available for the shell part. For this purpose, we introduce finite element approximations $\mathbf{V}_{sh} \subset \mathbf{V}_s$, $\mathbf{V}_{ih} \subset \mathbf{V}_i$ of the local membrane and internal displacement spaces, $\mathbf{W}_{sh} \subset \mathbf{W}_s$, $\mathbf{W}_{ih} \subset \mathbf{W}_i$ of the stiff mode spaces, and a finite element approximation $\mathbf{M}_h \subset \mathbf{M}$ of the interface contact force space \mathbf{M} .

As proposed in [18], we endow \mathbf{M}_h with the discrete mesh dependent $H_h^{-1/2}$ norm

$$\|\underline{g}_h\|_{h,-1/2}^2 = \sum_e h_e \|\underline{g}_h\|_{0,e}^2 = \sum_e h_e \int_e |\underline{g}_h|^2 d\Gamma,$$

with dual norm

$$\|\underline{\xi}_h\|_{h,1/2}^2 = \sum_e \frac{1}{h_e} \|\underline{\xi}_h\|_{0,e}^2.$$

Here, h_e denotes the diameter of the finite element e inside the mesh used for the construction of the interface space \mathbf{M}_h . This norm is local and explicit. It does not require any additional assumption on the interface Γ and has the right scaling properties with h , while avoiding the complex and non local aspects of the $H_{00}^{1/2}(\Gamma)$ norm.

We then define the space of weakly continuous approximate displacements

$$\mathbf{V}_{0h} = \{\underline{\xi}_h = (\underline{\xi}_{sh}, \underline{\beta}_h, \underline{\xi}_{ih}) \in \mathbf{V}_{sh} \times \mathbf{V}_{ih}, \int_\Gamma \llbracket \underline{\xi}_h \rrbracket \hat{\underline{g}}_h d\Gamma = 0, \quad \forall \hat{\underline{g}}_h \in \mathbf{M}_h\}, \quad (10)$$

whose jumps are orthogonal to \mathbf{M}_h . We also need to introduce the discrete trace space

$$\mathbf{V}_{\Gamma_h} = (\text{Tr} \mathbf{V}_{ih})_{|\Gamma}$$

which the interface inherits from the finite element mesh of the internal domain, and the zero trace subspace of \mathbf{V}_{ih}

$$\hat{\mathbf{V}}_{ih} = \{\underline{\xi}_{ih} \in \mathbf{V}_{ih}, \text{Tr}(\underline{\xi}_{ih})_{|\Gamma} = 0\}.$$

The mixed finite element problem is then simply:

Find the generalized displacement $\underline{\xi}_h = (\underline{\xi}_{sh}, \underline{\beta}_h, \underline{\xi}_{ih}) \in \mathbf{V}_{0h}$ and the delinquent modes $p_h = (\underline{\Delta}_h, \underline{\eta}_h, p_{ih}) \in \mathbf{W}_h = \mathbf{W}_{sh} \times \mathbf{W}_{ih}$ which satisfy

$$\begin{aligned} a(\underline{\xi}_h, \hat{\underline{\xi}}_h) + b(p_h, \hat{\underline{\xi}}_h) &= L(\hat{\underline{\xi}}_h), \quad \forall \hat{\underline{\xi}}_h \in \mathbf{V}_{0h}, \underline{\xi}_h \in \mathbf{V}_{0h}, \\ b(\hat{p}_h, \underline{\xi}_h) - c(p_h, \hat{p}_h) &= 0, \quad \forall \hat{p}_h \in \mathbf{W}_h, p_h \in \mathbf{W}_h. \end{aligned} \quad (11)$$

4.2. Admissibility conditions on the finite element spaces

In this construction, the local finite element spaces on the membrane and inside the body can be chosen independently one from another. Nevertheless, we expect them to satisfy three local inf-sup compatibility conditions:

$$\inf_{(\hat{\underline{\Lambda}}_h, \hat{\underline{\eta}}_h) \in \mathbf{W}_{sh}} \sup_{(\hat{\underline{\xi}}_{sh}, \hat{\underline{\beta}}_h) \in \mathbf{V}_{sh}} \frac{\int_{\omega} \hat{\underline{\Lambda}}_h : \underline{\underline{\gamma}}(\nabla_s \hat{\underline{\xi}}_{sh}) + \hat{\underline{\eta}}_h \cdot \underline{\underline{\Phi}}(\nabla_s \hat{\underline{\xi}}_{sh}, \hat{\underline{\beta}}_h) d\omega}{\|(\hat{\underline{\Lambda}}_h, \hat{\underline{\eta}}_h)\|_{\mathbf{W}_s} \|(\hat{\underline{\xi}}_{sh}, \hat{\underline{\beta}}_h)\|_{\mathbf{V}_s}} \geq \gamma, \quad (12)$$

$$\inf_{\hat{p}_{ih} \in \mathbf{W}_{ih}} \sup_{\hat{\underline{\xi}}_{ih} \in \mathbf{V}_{ih}} \frac{\int_{\Omega} \hat{p}_{ih} \operatorname{div} \hat{\underline{\xi}}_{ih} d\Omega}{\|\hat{p}_{ih}\|_{\mathbf{W}_i} \|\hat{\underline{\xi}}_{ih}\|_{\mathbf{V}_i}} \geq \gamma, \quad (13)$$

$$\inf_{\hat{g}_h \in \mathbf{M}_h} \sup_{\hat{\underline{\xi}}_{ih} \in \mathbf{V}_{\Gamma_h}} \frac{\int_{\Gamma} \hat{g}_h \cdot \hat{\underline{\xi}}_{ih} d\Gamma}{\|\hat{g}_h\|_{h, -1/2} \|\hat{\underline{\xi}}_{ih}\|_{h, 1/2}} \geq \gamma. \quad (14)$$

The first condition is the classical condition which would guarantee a uniformly valid mixed approximation of a shell problem [1], the second condition corresponds to the classical compatibility condition used for the mixed finite element approximation of the incompressible Stokes problem [6], and the third one is the condition used by [18] in her analysis of mortar coupling techniques. Above, the triple norm on \mathbf{W}_s is the dual norm introduced in [1] which writes here

$$\|(\hat{\underline{\Lambda}}, \hat{\underline{\eta}})\|_{\mathbf{W}_s} = \sup_{(\hat{\underline{\xi}}_s, \hat{\underline{\beta}}) \in \mathbf{V}_s} \frac{\int_{\omega} \hat{\underline{\Lambda}} : \underline{\underline{\gamma}}(\nabla_s \hat{\underline{\xi}}_s) + \hat{\underline{\eta}} \cdot \underline{\underline{\Phi}}(\nabla_s \hat{\underline{\xi}}_s, \hat{\underline{\beta}}) d\omega}{\|(\hat{\underline{\xi}}_s, \hat{\underline{\beta}})\|_{\mathbf{V}_s}},$$

and whose extension to \mathbf{W} is defined by

$$\begin{aligned} \|\hat{p}\|_{\mathbf{W}} &= \sup_{\hat{\underline{\xi}} \in \mathbf{V}} \frac{b(\hat{p}, \hat{\underline{\xi}})}{\|\hat{\underline{\xi}}\|_{\mathbf{V}}} \\ &\approx \|(\hat{\underline{\Lambda}}, \hat{\underline{\eta}})\|_{\mathbf{W}_s} + \|\hat{p}\|_{L^2(\Omega)}. \end{aligned}$$

Moreover, in the mortar coupling condition (14), we have chosen the internal finite element space \mathbf{V}_{ih} as the non mortar side in control of the interface tractions $\hat{g}_h \in \mathbf{M}_h$. This choice is important for the subsequent derivation of a global inf-sup condition.

4.3. Examples of finite elements

Finding shell elements which exactly satisfy the inf-sup condition (12) is difficult [2]. A possible choice, which is proved to work in specific situations is the one advocated by [1], using second order elements with central bubbles for the displacement \mathbf{V}_{sh} , and piecewise constants for the membrane and shear stresses (Fig. 2). We can also use the choice proposed in [19] for cylindrical shells, using continuous piecewise $P2$ interpolation for the displacements, and continuous piecewise $P1$ interpolation for membrane and shear stresses.

For the quasi incompressible internal body, the first classical elements satisfying the inf-sup condition (13) are the celebrated Taylor Hood hexaedral or tetraedral elements, using continuous second order elements for the displacement and first order continuous elements for the pressure [6]. But for complex constitutive laws, continuous pressure elements are not very convenient. Discontinuous linear pressure fields $\mathbf{W}_{ih} = \mathbf{P}_1(K)_{disc}$ are then preferred. In order to satisfy the inf-sup condition (13), the displacement finite element \mathbf{V}_{ih} should be either the full Q2 hexaedra (27 nodes) when using hexaedral elements, or the enriched ($P2 \oplus b_{ij} \oplus b_i$) 15 nodes second order tetraedral element [6]. In the latter case, as described in Figure 3, in addition to the standard nodal and mid edge degrees of freedom used to build a second order polynomial interpolation, bubble functions

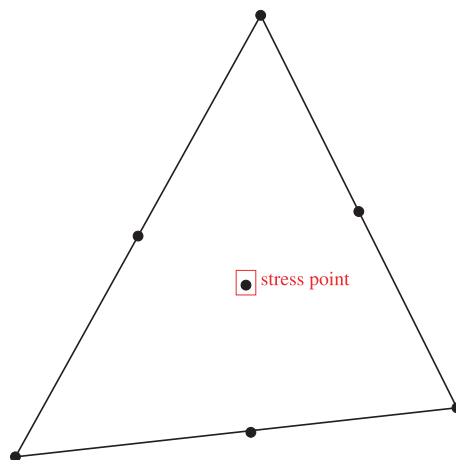


FIGURE 2. The mixed shell element proposed in [1]. The displacements $\underline{\xi}_s$ and rotations $\underline{\beta}$ are interpolated at the vertices, midpoint and center of the triangle (black dots), while the membrane and shear stresses $(\underline{\Lambda}, \underline{\eta})$ are constant on the element.

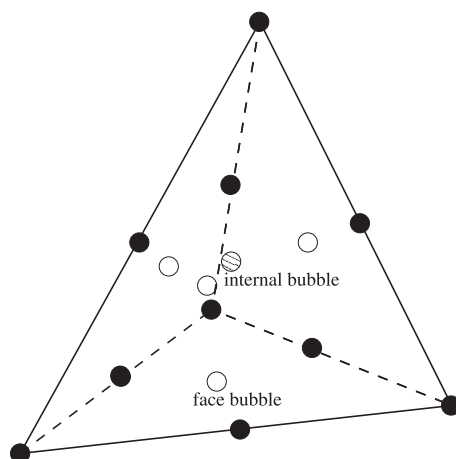


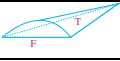
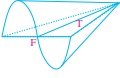
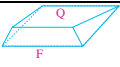
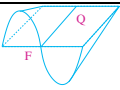
FIGURE 3. The enriched 15 nodes elements with additional bubbles b_{ij} associated to nodes at the center of each face (in white) and b_i associated to the center of the element (in grey pattern). Displacements are interpolated at each of these 15 nodes while pressure is linear per element and discontinuous at interelement boundaries.

are added at the center of the element and at the center of each face. The inf sup condition (13) is then proved by a direct application of the macroelement technique of [17].

For mortars, an efficient choice uses discontinuous elements for \mathbf{M}_h . This then requires to locally enrich the internal space \mathbf{V}_{ih} by face bubbles as described in [7, 13] and reviewed below.

In the tetraedral case, when using linear elements $\mathbf{P}_1(K)$ inside the volume, one interface bubble function \mathbf{b}_F must be added on each triangular interface, defined in parametric coordinates by $\mathbf{b}_F(M) = \lambda_1 \lambda_2 \lambda_3 (1 - \lambda_4)$, where λ_4 corresponds to the internal vertex of T which is not on the interface. When using quadratic elements $\mathbf{P}_2(K)$, one must add three interface bubbles \mathbf{b}_F^i (one for each vertex) locally defined in K by $\mathbf{b}_F^i(M) = (\lambda_i - 1/2) \lambda_1 \lambda_2 \lambda_3 (1 - \lambda_4)$, $i = 1, 3$. The situation is similar for the hexaedral case. When using quadratic

TABLE 1. Examples of admissible pairs of compatible finite elements in the case of piecewise discontinuous mortar elements.

3D shape	internal space $\mathbf{V}_{ih}(K)$	mortar $\mathbf{M}_h(F)$	Bubbles \mathbf{b}_F
Tetraedron	$\mathbf{P}_1(K) \oplus \mathbf{b}_F$	$\mathbf{P}_0(F)_{disc}$	
Tetraedron	$\mathbf{P}_2(K) \oplus \text{span}\{\mathbf{b}_F^i, i = 1, \dots, 3\}$	$\mathbf{P}_1(F)_{disc}$	
Hexaedron	$\mathbf{Q}_1(K) \oplus \mathbf{b}_F$	$\mathbf{P}_0(F)_{disc}$	
Hexaedron	$\mathbf{Q}_2(K) \oplus \text{span}\{\mathbf{b}_F^i, i = 1, \dots, 2\}$	$\mathbf{P}_1(F)_{disc}$	

elements $\mathbf{Q}_2(K)$, one must add two interface bubbles \mathbf{b}_F^i (one for each local coordinate) locally defined in K by $\mathbf{b}_F^i(M) = \hat{x}_i(1 - \hat{x}_1^2)(1 - \hat{x}_2^2)(1 - \hat{x}_3^2)$, $i = 1, 2$.

Remark 1. The weak continuity constraint $\int_{\Gamma} \hat{\underline{g}}_h \cdot \hat{\underline{\xi}}_{ih} d\Gamma = 0$ cannot be included in the definition of the bilinear form b because this constraint is not continuous in natural norm on the product space $\mathbf{M} \times \mathbf{V}$. We will see later that continuity of this constraint for the $H_h^{-1/2} \times H_h^{1/2}$ norm can only be achieved in the subspace \mathbf{V}_{0h} .

5. CONVERGENCE ANALYSIS

5.1. Review of fundamental lemma

The technical results which are recalled below are introduced in [18], with proofs detailed in [13]. The first lemma introduces a continuous projection on the interface space \mathbf{V}_{Γ_h} , orthogonally to the mortar space \mathbf{M}_h .

Lemma 1. *Under the inf-sup condition (14), there exists a projection P_m from $L^2(\Gamma)$ onto \mathbf{V}_{Γ_h} orthogonally to \mathbf{M}_h which satisfies*

$$\int_{\Gamma} (P_m \hat{\underline{\xi}} - \hat{\underline{\xi}}) \hat{\underline{g}}_h = 0, \quad \forall \hat{\underline{g}}_h \in \mathbf{M}_h, P_m \hat{\underline{\xi}} \in \mathbf{V}_{\Gamma_h}, \quad (15)$$

$$\|P_m \hat{\underline{\xi}}\|_{h,1/2} \leq C \|\hat{\underline{\xi}}\|_{h,1/2}, \quad \forall \hat{\underline{\xi}} \in L^2(\Gamma). \quad (16)$$

Proof. From the inf-sup condition (14), the injection \mathcal{I} has a continuous inverse \mathcal{I}^{-1} from \mathbf{M}'_h to \mathbf{V}_{Γ_h} , when endowing both spaces with the discrete $H_h^{1/2}$ norm. We then define P_m by $P_m = \mathcal{I}^{-1} \circ \mathcal{R}$, with \mathcal{R} the duality mapping from $L^2(\Gamma)$ to \mathbf{M}'_h , defined by

$$\langle \mathcal{R} \underline{v}, \hat{\underline{g}}_h \rangle = \int_{\Gamma} \underline{v} \cdot \hat{\underline{g}}_h, \quad \forall \hat{\underline{g}}_h \in \mathbf{M}_h.$$

□

The second lemma proves the continuity of the map $\llbracket \cdot \rrbracket$ from \mathbf{V}_{0h} endowed with the H^1 norm $\|\cdot\|_{\mathbf{V}}$ to $L^2(\Gamma)$ endowed with the $H_h^{1/2}$ norm.

Lemma 2. *The jump satisfies*

$$\|\llbracket \hat{\underline{\xi}}_h \rrbracket\|_{h,1/2} \leq C \|\hat{\underline{\xi}}_h\|_{\mathbf{V}}, \quad \forall \hat{\underline{\xi}}_h \in \mathbf{V}_{0h}.$$

Proof. The proof can also be found in detail in ([13], (4.46)) for a more complex situation. It only holds for test functions $\hat{\underline{\xi}}_h$ which satisfy the weak continuity requirement at the interface. The key point is to observe that since by assumption $\hat{\underline{\xi}}_h$ belongs to \mathbf{V}_{0h} , we have $P_m(\llbracket \hat{\underline{\xi}}_h \rrbracket) = 0$. Introducing then the finite element interpolation $I_h(\underline{v})$ from $C^0(\Gamma)$ onto \mathbf{V}_{Γ_h} , we have then

$$\begin{aligned} \llbracket \hat{\underline{\xi}}_h \rrbracket &= (I - P_m)(\llbracket \hat{\underline{\xi}}_h \rrbracket - I_h(\llbracket \hat{\underline{\xi}}_h \rrbracket)) \\ &= (I - P_m)(\text{Tr } {}_s \hat{\underline{\xi}}_{sh} - I_h(\text{Tr } {}_s \hat{\underline{\xi}}_{sh})) - (I - P_m)(\hat{\underline{\xi}}_{ih} - I_h(\hat{\underline{\xi}}_{ih})). \end{aligned}$$

Here, $\text{Tr } {}_s \hat{\underline{\xi}}_{sh}$ denotes the trace $\hat{\underline{\xi}}_{sh} - \frac{t}{2} \underline{\beta}_h$ of the shell displacement on the interface Γ . From the previous lemma, the operator $I - P_m$ is bounded for the $H_h^{1/2}$ norm. On the other hand, by construction, $\hat{\underline{\xi}}_{ih} - I_h(\hat{\underline{\xi}}_{ih}) = 0$, and from standard interpolation estimates, we have

$$\|\hat{\underline{\xi}}_{sh} - I_h(\hat{\underline{\xi}}_{sh})\|_{h,1/2}^2 = \sum_e h_e^{-1} \|\hat{\underline{\xi}}_{sh} - I_h(\hat{\underline{\xi}}_{sh})\|_{0,e}^2 \leq C \|\hat{\underline{\xi}}_{sh}\|_{H^1(\Gamma)}^2 \leq C \|\hat{\underline{\xi}}_{sh}\|_{\mathbf{V}_s}^2.$$

□

The third lemma proves that the truncation error on \mathbf{V}_{0h} is bounded by the interpolation error on $\mathbf{V}_{sh} \times \mathbf{V}_{ih}$.

Lemma 3. *For any $\hat{\underline{\xi}} \in \mathbf{V}_0$, we have*

$$\inf_{\tilde{\underline{\xi}}_h \in \mathbf{V}_{0h}} \|\hat{\underline{\xi}} - \tilde{\underline{\xi}}_h\|_{\mathbf{V}} \leq C \|\hat{\underline{\xi}} - I_h \hat{\underline{\xi}}\|_{\mathbf{V}},$$

where $I_h(\hat{\underline{\xi}})$ denotes the finite element interpolate of $\hat{\underline{\xi}}$ on the product space $\mathbf{V}_{sh} \times \mathbf{V}_{ih}$.

Proof. We build $\tilde{\underline{\xi}}_h \in \mathbf{V}_{0h}$ as in [14] by correcting the interface jump $\llbracket \hat{\underline{\xi}}_h \rrbracket$ of the interpolate $\hat{\underline{\xi}}_h = (I_{hs}(\hat{\underline{\xi}}_s), I_{hi}(\hat{\underline{\xi}}_i))$ of $\hat{\underline{\xi}}$ in $\mathbf{V}_{sh} \times \mathbf{V}_{ih}$ on the slave side, that is in the internal domain. For this purpose, we set

$$\tilde{\underline{\xi}}_h = \left(\hat{\underline{\xi}}_{sh}, \hat{\underline{\xi}}_{ih} - \text{Ext}_h(P_m(\llbracket \hat{\underline{\xi}}_h \rrbracket)) \right).$$

Here $\text{Ext}_h \underline{v}_h$ denotes the finite element function of \mathbf{V}_{ih} whose nodal values are equal to those of \underline{v}_h on Γ and zero elsewhere. We therefore need to prove that the correction $\text{Ext}_h(P_m(\llbracket \hat{\underline{\xi}}_h \rrbracket))$ is bounded in \mathbf{V}_i by $C \|\hat{\underline{\xi}}_h - \hat{\underline{\xi}}\|_{\mathbf{V}}$. We already know from basic estimate [13, 18] and Lemma 1 that we have

$$\|\text{Ext}_h(P_m(\llbracket \hat{\underline{\xi}}_h \rrbracket))\|_{\mathbf{V}_i} \leq C \|P_m(\llbracket \hat{\underline{\xi}}_h \rrbracket)\|_{h,1/2} \leq \|\llbracket \hat{\underline{\xi}}_h \rrbracket\|_{h,1/2}.$$

On the other hand, by assumption, $\hat{\underline{\xi}}$ is continuous on the interface, and therefore, we have

$$\llbracket \hat{\underline{\xi}}_h \rrbracket = \llbracket \hat{\underline{\xi}}_h - \hat{\underline{\xi}} \rrbracket = I_{hs}(\hat{\underline{\xi}}_s) - \hat{\underline{\xi}}_s - I_{hi}(\hat{\underline{\xi}}_i) + \hat{\underline{\xi}}_i.$$

The first term's $H_h^{1/2}$ norm is bounded by

$$\begin{aligned} \|I_{hs}(\hat{\underline{\xi}}_s) - \hat{\underline{\xi}}_s\|_{h,1/2}^2 &= \sum_e h_e^{-1} \|I_{hs}(\hat{\underline{\xi}}_s) - \hat{\underline{\xi}}_s\|_{0,e}^2 \\ &\leq C \|I_{hs}(\hat{\underline{\xi}}_s) - \hat{\underline{\xi}}_s\|_{H^1(\Gamma)}^2 \\ &\leq C \|I_{hs}(\hat{\underline{\xi}}_s) - \hat{\underline{\xi}}_s\|_{\mathbf{V}_s}^2. \end{aligned}$$

To bound the second term, one needs to use the trace theorem on the volume elements V_e of Ω associated to the faces e of Γ and standard interpolation estimates to get

$$\begin{aligned} \|I_{hi}(\hat{\underline{\xi}}_i) - \hat{\underline{\xi}}_i\|_{h,1/2}^2 &= \sum_e h_e^{-1} \|I_{hi}(\hat{\underline{\xi}}_i) - \hat{\underline{\xi}}_i\|_{0,e}^2 \\ &\leq C \sum_{V_e} h_e^{-2} \|I_{hi}(\hat{\underline{\xi}}_i) - \hat{\underline{\xi}}_i\|_{0,V_e}^2 + |I_{hi}(\hat{\underline{\xi}}_i) - \hat{\underline{\xi}}_i|_{1,V_e}^2 \\ &\leq C \sum_{V_e} |I_{hi}(\hat{\underline{\xi}}_i) - \hat{\underline{\xi}}_i|_{1,V_e}^2 \\ &\leq C \|I_{hi}(\hat{\underline{\xi}}_i) - \hat{\underline{\xi}}_i\|_{\mathbf{V}_i}^2, \end{aligned}$$

which completes the proof. \square

5.2. Global inf-sup condition

The key result in our coupling problem proves that when the three local inf-sup conditions (12)–(14) are satisfied on the membrane, on the incompressible internal structure and on the mortar coupling, then a global inf-sup condition holds for the global duality form b .

Theorem 1. *The three local inf-sup conditions (12)–(14) imply the global inf-sup condition*

$$\inf_{\hat{p}_h \in \mathbf{W}_h} \sup_{\hat{\underline{\xi}}_h \in \mathbf{V}_{0h}} \frac{b(\hat{p}_h, \hat{\underline{\xi}}_h)}{\|\hat{p}_h\| \|\mathbf{W}\| \|\hat{\underline{\xi}}_h\| \|\mathbf{V}\|} \geq \gamma. \quad (17)$$

Proof. The proof uses successively the displacements fields appearing in each individual inf-sup condition. Let therefore be $\hat{p}_h = (\hat{\underline{\underline{\Lambda}}}_h, \hat{\underline{\eta}}_h, \hat{p}_{ih})$ be a given stress field in \mathbf{W}_h . Since (12) holds, there exists a field $(\hat{\underline{\xi}}_{sh}, \hat{\underline{\beta}}_h) \in \mathbf{V}_{sh}$ with unit norm such that

$$\gamma \|\hat{\underline{\underline{\Lambda}}}_h, \hat{\underline{\eta}}_h\| \|\mathbf{W}_s\| \leq \int_{\omega} \hat{\underline{\underline{\Lambda}}}_h : \underline{\underline{\gamma}} (\nabla_s \hat{\underline{\xi}}_{sh}) + \hat{\underline{\eta}}_h \cdot \underline{\underline{\Phi}} (\nabla_s \hat{\underline{\xi}}_{sh}, \hat{\underline{\beta}}_h) \, d\omega. \quad (18)$$

Let us then construct the \mathbf{V}_{Γ_h} Scott and Zhang finite element interpolation $I_{Z\Gamma_h}(\hat{\underline{\xi}}_{sh} - \frac{t}{2}\hat{\underline{\beta}}_h)$ of its interface trace onto the trace space \mathbf{V}_{Γ_h} . We also introduce the \mathbf{V}_{ih} Scott and Zhang approximation [16] $I_{Z\Omega_h}$ of the harmonic extension Ext_H inside Ω of this interpolate

$$\underline{\xi}_{ih}^1 = I_{Z\Omega_h} \circ \text{Ext}_H \circ I_{Z\Gamma_h} \left(\hat{\underline{\xi}}_{sh} - \frac{t}{2} \hat{\underline{\beta}}_h \right).$$

Here, by definition, the harmonic extension $\text{Ext}_H v$ of a given function v defined on Γ is the function of \mathbf{V}_i which is harmonic on Ω and whose trace on Γ is equal to v . By construction, it satisfies

$$\|\text{Ext}_H v\|_{\mathbf{V}_i} \leq C \|v\|_{H^{1/2}(\Gamma)} \leq C \|v\|_{H^1(\Gamma)}.$$

The Scott and Zhang interpolation being stable in H^1 , we have that $\underline{\xi}_{ih}^1$ is of bounded norm in \mathbf{V}_i

$$\begin{aligned} \|\underline{\xi}_{ih}^1\|_{\mathbf{V}_i} &\leq C \|\text{Ext}_H \circ I_{Z\Gamma_h}(\hat{\underline{\xi}}_{sh} - \frac{t}{2}\hat{\underline{\beta}}_h)\|_{\mathbf{V}_i} \\ &\leq C \|I_{Z\Gamma_h}(\hat{\underline{\xi}}_{sh} - \frac{t}{2}\hat{\underline{\beta}}_h)\|_{H^1(\omega)} \\ &\leq C \|(\hat{\underline{\xi}}_{sh}, \hat{\underline{\beta}}_h)\|_{H^1(\omega)} \\ &\leq C. \end{aligned}$$

In the above inequalities, C denote various constants independent of the mesh size h and of any small parameter. We also introduce as in Lemma 3 the internal displacement field

$$\underline{\xi}_{ih}^2 = \text{Ext}_h \circ P_m \circ \left(\hat{\underline{\xi}}_{sh} - \frac{t}{2} \hat{\underline{\beta}}_h - I_{Z\Gamma_h} \left(\hat{\underline{\xi}}_{sh} - \frac{t}{2} \hat{\underline{\beta}}_h \right) \right).$$

From the boundedness of Ext_h as a map from $H_h^{1/2}(\Gamma)$ to \mathbf{V}_i , and from Lemma 1, we have

$$\begin{aligned} \|\underline{\xi}_{ih}^2\|_{\mathbf{V}_i}^2 &\leq C \left\| \hat{\underline{\xi}}_{sh} - \frac{t}{2} \hat{\underline{\beta}}_h - I_{Z\Gamma_h} \left(\hat{\underline{\xi}}_{sh} - \frac{t}{2} \hat{\underline{\beta}}_h \right) \right\|_{h,1/2}^2 \\ &\leq \sum_e h_e^{-1} \left\| \hat{\underline{\xi}}_{sh} - \frac{t}{2} \hat{\underline{\beta}}_h - I_{Z\Gamma_h} \left(\hat{\underline{\xi}}_{sh} - \frac{t}{2} \hat{\underline{\beta}}_h \right) \right\|_{0,e}^2. \end{aligned}$$

From the convergence properties of the Scott and Zhang interpolation $\|v - I_{Z\Gamma_h} v\|_{L^2(\omega)} \leq Ch\|v\|_{H^1(\omega)}$, this yields

$$\begin{aligned} \|\underline{\xi}_{ih}^2\|_{\mathbf{V}_i}^2 &\leq \left\| \hat{\underline{\xi}}_{sh} - \frac{t}{2} \hat{\underline{\beta}}_h \right\|_{1,\Gamma}^2 \\ &\leq C. \end{aligned}$$

We finally introduce the solution $(\hat{\underline{\xi}}_{ih}, \tilde{p}_{ih}) \in \hat{\mathbf{V}}_{ih} \times \mathbf{W}_{ih}$ of the Stokes problem

$$\int_{\Omega} 2\mu_i \underline{\varepsilon}(\nabla(\hat{\underline{\xi}}_{ih} + \underline{\xi}_{ih}^1 + \underline{\xi}_{ih}^2)) : \underline{\varepsilon}(\nabla \hat{\underline{\xi}}_{ih}) + \tilde{p}_{ih} \text{div} \hat{\underline{\xi}}_{ih} \, d\Omega = 0, \quad \forall \hat{\underline{\xi}}_i \in \hat{\mathbf{V}}_{ih}, \quad (19)$$

$$\int_{\Omega} \hat{q}_h \text{div} (\hat{\underline{\xi}}_{ih} + \underline{\xi}_{ih}^1 + \underline{\xi}_{ih}^2) \, d\Omega = \int_{\Omega} \frac{\hat{p}_{ih}}{\|\hat{p}_{ih}\|_{L^2(\Omega)}} \hat{q}_h \, d\Omega, \quad \forall \hat{q}_h \in \mathbf{W}_{ih}. \quad (20)$$

From the inf-sup condition (13), the above problem has a unique solution $(\hat{\underline{\xi}}_{ih}, \tilde{p}_{ih}) \in \hat{\mathbf{V}}_{ih} \times \mathbf{W}_{ih}$ and this solution is uniformly bounded

$$\|\hat{\underline{\xi}}_{ih}\|_{\mathbf{V}_i} + \|\tilde{p}_{ih}\|_{\mathbf{W}_i} \leq \|\underline{\xi}_{ih}^1 + \underline{\xi}_{ih}^2\|_{\mathbf{V}_i} + \left\| \frac{\hat{p}_{ih}}{\|\hat{p}_{ih}\|_{L^2(\Omega)}} \right\|_{L^2(\Omega)} \leq C.$$

We now introduce the field $\hat{\underline{\xi}}_h = (\hat{\underline{\xi}}_{sh}, \hat{\underline{\beta}}_h, \hat{\underline{\xi}}_{ih} + \underline{\xi}_{ih}^1 + \underline{\xi}_{ih}^2) \in \mathbf{V}_h$ whose \mathbf{V} norm is bounded by C_{sup} by construction of $(\hat{\underline{\xi}}_{sh}, \hat{\underline{\beta}}_h)$, $\hat{\underline{\xi}}_{ih}$, $\underline{\xi}_{ih}^1$ and $\underline{\xi}_{ih}^2$. By construction of these functions, we also have that

$$\begin{aligned} \int_{\Gamma} \hat{\underline{g}}_h \cdot \llbracket \hat{\underline{\xi}}_h \rrbracket &= \int_{\Gamma} \hat{\underline{g}}_h \cdot \left(\hat{\underline{\xi}}_{sh} - \frac{t}{2} \hat{\underline{\beta}}_h - I_{Z\Gamma_h} \left(\hat{\underline{\xi}}_{sh} - \frac{t}{2} \hat{\underline{\beta}}_h \right) - P_m \left(\hat{\underline{\xi}}_{sh} - \frac{t}{2} \hat{\underline{\beta}}_h - I_{Z\Gamma_h} \left(\hat{\underline{\xi}}_{sh} - \frac{t}{2} \hat{\underline{\beta}}_h \right) \right) \right) \\ &= 0, \quad \forall \hat{\underline{g}}_h \in \mathbf{M}_h. \end{aligned}$$

Thus $\hat{\underline{\xi}}_h$ is in \mathbf{V}_{0h} . From (18) and from (20) written with $\hat{q}_h = \hat{p}_{ih}$, we finally get

$$\begin{aligned} b(\hat{p}, \hat{\underline{\xi}}_h) &= \int_{\omega} \hat{\underline{\Lambda}}_h : \underline{\gamma}(\nabla_s \hat{\underline{\xi}}_{sh}) + \hat{\underline{\eta}}_h \cdot \underline{\Phi}(\nabla_s \hat{\underline{\xi}}_{sh}, \hat{\underline{\beta}}_h) \, d\omega + \int_{\Omega} \hat{p}_{ih} \text{div} \hat{\underline{\xi}}_{ih} \, d\Omega \\ &\geq \gamma \|\hat{\underline{\Lambda}}_h, \hat{\underline{\eta}}_h\|_{\mathbf{W}_s} + \|\hat{p}_{ih}\|_{L^2(\Omega)} \\ &\geq (\gamma \|\hat{\underline{\Lambda}}_h, \hat{\underline{\eta}}_h\|_{\mathbf{W}_s} + \|\hat{p}_{ih}\|_{L^2(\Omega)}) \frac{\|\hat{\underline{\xi}}_h\|_{\mathbf{V}}}{C_{\text{sup}}} \end{aligned}$$

which concludes our proof. \square

5.3. Convergence and stability result

With the above lemmas, we can now extend the discrete stability and convergence result of [1] to our global coupled problem. In other words, combining stable locking free finite element approximations of the external shell problem on one hand, of the internal incompressible structure on the other hand, and a proper weak mortar coupling between the two structures yields a stable locking free approximation of the full coupled problem:

Theorem 2. *If the contact force \underline{g} is in $L^2(\Gamma)$, there exists a constant C depending on the continuity and coercivity constants $\|a\|$ and c_a of the bilinear form a on \mathbf{V} and on the inf-sup constant γ , but not on the small parameters t and $1/\lambda_i$, which bounds the error between the solutions $(\underline{\xi}, p)$ and $(\underline{\xi}_h, p_h)$ of the continuous and discrete problems (8) and (11) by*

$$\begin{aligned} & \|\underline{\xi} - \underline{\xi}_h\|_{\mathbf{V}} + \frac{1}{\sqrt{c_a}} \|p - p_h\|_{\varepsilon} + \|p - p_h\|_{\mathbf{W}} + \leq C \|\underline{\xi} - I_h \underline{\xi}\|_{\mathbf{V}} \\ & + C \inf_{\tilde{p}_h \in \mathbf{W}_h} (\|p - \tilde{p}_h\|_{\mathbf{W}} + \|p - \tilde{p}_h\|_{\varepsilon}) + C \inf_{\tilde{g}_h \in \mathbf{M}_h} \|\underline{g} - \tilde{g}_h\|_{h, -1/2}. \end{aligned} \quad (21)$$

Proof. The proof detailed below is quite classical. It follows exactly the steps of [1], but with the additional complexity arising from the weak interface continuity constraint, and from the presence of two small parameters instead of one.

By writing the continuous problem (8) with the Lagrange multiplier \underline{g} in $L^2(\Gamma)$, and the discrete problem (11), we first get

$$\begin{aligned} a(\underline{\xi}, \hat{\underline{\xi}}_h) + b(p, \hat{\underline{\xi}}_h) - \int_{\Gamma} \underline{g} \cdot \llbracket \hat{\underline{\xi}}_h \rrbracket d\Gamma &= L(\hat{\underline{\xi}}_h), \quad \forall \hat{\underline{\xi}}_h \in \mathbf{V}_{0h}, \\ b(\hat{p}_h, \underline{\xi}) - c(p, \hat{p}_h) &= 0, \quad \forall \hat{p}_h \in \mathbf{W}_h, \end{aligned}$$

$$\begin{aligned} a(\underline{\xi}_h, \hat{\underline{\xi}}_h) + b(p_h, \hat{\underline{\xi}}_h) &= L(\hat{\underline{\xi}}_h), \quad \forall \hat{\underline{\xi}}_h \in \mathbf{V}_{0h}, \\ b(\hat{p}_h, \underline{\xi}_h) - c(p_h, \hat{p}_h) &= 0, \quad \forall \hat{p}_h \in \mathbf{W}_h. \end{aligned}$$

By subtraction, and since $\hat{\underline{\xi}}_h \in \mathbf{V}_{0h}$ by construction, which implies that we have

$$\int_{\Gamma} \tilde{g}_h \cdot \llbracket \hat{\underline{\xi}}_h \rrbracket d\Gamma = 0, \quad \forall \tilde{g}_h \in \mathbf{M}_h,$$

we get

$$a(\underline{\xi}_h - \underline{\xi}, \hat{\underline{\xi}}_h) + b(p_h - p, \hat{\underline{\xi}}_h) + \int_{\Gamma} (\underline{g} - \tilde{g}_h) \cdot \llbracket \hat{\underline{\xi}}_h \rrbracket d\Gamma = 0, \quad \forall \hat{\underline{\xi}}_h \in \mathbf{V}_{0h}, \quad (22)$$

$$b(\hat{p}_h, \underline{\xi}_h - \underline{\xi}) - c(p_h - p, \hat{p}_h) = 0, \quad \forall \hat{p}_h \in \mathbf{W}_h. \quad (23)$$

Subtracting the second line from the first line, and using as test functions $(\hat{\underline{\xi}}_h, \hat{p}_h) = (\underline{\xi}_h - \tilde{\underline{\xi}}_h, p_h - \tilde{p}_h)$ yields:

$$\begin{aligned} a(\hat{\underline{\xi}}_h, \hat{\underline{\xi}}_h) + c(\hat{p}_h, \hat{p}_h) &= - \int_{\Gamma} (\underline{g} - \tilde{g}_h) \cdot \llbracket \hat{\underline{\xi}}_h \rrbracket d\Gamma \\ &+ a(\underline{\xi} - \tilde{\underline{\xi}}_h, \hat{\underline{\xi}}_h) + b(p - \tilde{p}_h, \hat{\underline{\xi}}_h) - b(\hat{p}_h, \underline{\xi} - \tilde{\underline{\xi}}_h) + c(p - \tilde{p}_h, \hat{p}_h), \\ \forall (\tilde{\underline{\xi}}_h, \tilde{p}_h, \tilde{g}_h) &\in \mathbf{V}_{0h} \times \mathbf{W}_h \times \mathbf{M}_h. \end{aligned}$$

Using the coercivity of a , the definition of the norm $\|\cdot\|_\varepsilon$ and the triple norm $\|\|\cdot\|\|$ whose definition yields

$$-b(\hat{p}_h, \underline{\xi} - \tilde{\xi}_h) \leq \|\|\hat{p}_h\|\| \|\underline{\xi} - \tilde{\xi}_h\|_{\mathbf{V}},$$

Lemma 2 and Cauchy Schwarz imply

$$\begin{aligned} c_a \|\hat{\underline{\xi}}_h\|_{\mathbf{V}}^2 + \|\hat{p}_h\|_\varepsilon^2 &\leq \|\underline{g} - \tilde{g}_h\|_{h,-1/2} \|\hat{\underline{\xi}}_h\|_{\mathbf{V}} + \|a\| \|\underline{\xi} - \tilde{\xi}_h\|_{\mathbf{V}} \|\hat{\underline{\xi}}_h\|_{\mathbf{V}} \\ &\quad + \|\|p - \tilde{p}_h\|\|_{\mathbf{W}} \|\hat{\underline{\xi}}_h\|_{\mathbf{V}} + \|\|\hat{p}_h\|\|_{\mathbf{W}} \|\tilde{\xi}_h - \underline{\xi}\|_{\mathbf{V}} + \|\|p - \tilde{p}_h\|\|_\varepsilon \|\hat{p}_h\|_\varepsilon. \end{aligned} \quad (24)$$

On the other hand, from the global inf-sup condition (17), (22) and Lemma 2, we have

$$\begin{aligned} \|\|\hat{p}_h\|\|_{\mathbf{W}} &\leq \|\|p - \tilde{p}_h\|\|_{\mathbf{W}} + \|\|p_h - p\|\|_{\mathbf{W}} \quad (25) \\ &\leq \|\|p - \tilde{p}_h\|\|_{\mathbf{W}} + \frac{1}{\gamma} \sup_{\tilde{\xi}_h \in \mathbf{V}_{0h}} \frac{b(p - p_h, \tilde{\xi}_h)}{\|\tilde{\xi}_h\|_{\mathbf{V}}} \\ &\leq \|\|p - \tilde{p}_h\|\|_{\mathbf{W}} + \frac{1}{\gamma} \sup_{\tilde{\xi}_h \in \mathbf{V}_{0h}} \frac{a(\underline{\xi}_h - \underline{\xi}, \tilde{\xi}_h) + \int_\Gamma (\underline{g} - \tilde{g}_h) \cdot \llbracket \tilde{\xi}_h \rrbracket d\Gamma}{\|\tilde{\xi}_h\|_{\mathbf{V}}} \\ &\leq \|\|p - \tilde{p}_h\|\|_{\mathbf{W}} + \frac{1}{\gamma} (\|a\| \|\underline{\xi}_h - \underline{\xi}\|_{\mathbf{V}} + \|\underline{g} - \tilde{g}_h\|_{h,-1/2}) \\ &\leq \|\|p - \tilde{p}_h\|\|_{\mathbf{W}} + \frac{1}{\gamma} (\|a\| \|\hat{\underline{\xi}}_h\|_{\mathbf{V}} + \|a\| \|\tilde{\xi}_h - \underline{\xi}\|_{\mathbf{V}} + \|\underline{g} - \tilde{g}_h\|_{h,-1/2}). \end{aligned} \quad (26)$$

Plugged in (24), and dividing by the coercivity constant c_a of the bilinear form a on \mathbf{V} , the above inequality then implies

$$\begin{aligned} \|\hat{\underline{\xi}}_h\|_{\mathbf{V}}^2 + \frac{1}{c_a} \|\hat{p}_h\|_\varepsilon^2 &\leq \frac{1}{c_a} \|\underline{g} - \tilde{g}_h\|_{h,-1/2} \|\hat{\underline{\xi}}_h\|_{\mathbf{V}} + \frac{1}{c_a} (1 + \frac{1}{\gamma}) \|a\| \|\underline{\xi} - \tilde{\xi}_h\|_{\mathbf{V}} \|\hat{\underline{\xi}}_h\|_{\mathbf{V}} \\ &\quad + \frac{1}{c_a} \|\|p - \tilde{p}_h\|\|_{\mathbf{W}} \|\hat{\underline{\xi}}_h\|_{\mathbf{V}} + \frac{1}{c_a} \|p - \tilde{p}_h\|_\varepsilon \|\hat{p}_h\|_\varepsilon \\ &\quad + \frac{1}{c_a} \|\tilde{\xi}_h - \underline{\xi}\|_{\mathbf{V}} \left[(\|\|p - \tilde{p}_h\|\|_{\mathbf{W}} + \frac{1}{\gamma} (\|a\| \|\tilde{\xi}_h - \underline{\xi}\|_{\mathbf{V}} + \|\underline{g} - \tilde{g}_h\|_{h,-1/2})) \right]. \end{aligned}$$

Writing this inequality as $x^2 - 2Kx - K^2 \leq 0$ with $x = \|\hat{\underline{\xi}}_h\|_{\mathbf{V}} + \frac{1}{\sqrt{c_a}} \|\hat{p}_h\|_\varepsilon$, this implies that x is bounded by $(1 + \sqrt{2})K$, which writes

$$\|\hat{\underline{\xi}}_h\|_{\mathbf{V}} + \frac{1}{\sqrt{c_a}} \|\hat{p}_h\|_\varepsilon \leq C \left(\|\underline{g} - \tilde{g}_h\|_{h,-1/2} + \|\underline{\xi} - \tilde{\xi}_h\|_{\mathbf{V}} + \|\|p - \tilde{p}_h\|\|_{\mathbf{W}} + \|p - \tilde{p}_h\|_\varepsilon \right).$$

Adding (26) finally yields

$$\begin{aligned} \|\hat{\underline{\xi}}_h\|_{\mathbf{V}} + \frac{1}{\sqrt{c_a}} \|\hat{p}_h\|_\varepsilon + \|\|\hat{p}_h\|\|_{\mathbf{W}} &\leq C \left(\|\underline{g} - \tilde{g}_h\|_{h,-1/2} + \|\underline{\xi} - \tilde{\xi}_h\|_{\mathbf{V}} \right. \\ &\quad \left. + \|\|p - \tilde{p}_h\|\|_{\mathbf{W}} + \|p - \tilde{p}_h\|_\varepsilon \right), \quad \forall (\tilde{\xi}_h, \tilde{p}_h, \tilde{g}_h) \in \mathbf{V}_{0h} \times \mathbf{W}_h \times \mathbf{M}_h. \end{aligned} \quad (27)$$

This exactly the desired result, because, from Lemma 3, we have

$$\inf_{\tilde{\xi}_h \in \mathbf{V}_{0h}} \|\underline{\xi} - \tilde{\xi}_h\|_{\mathbf{V}} \leq C \|\underline{\xi} - I_h \underline{\xi}\|_{\mathbf{V}}. \quad \square$$

Remark 2. Estimates on the approximation error $\|p - \tilde{p}_h\|_{\mathbf{W}} + \|p - \tilde{p}_h\|_{\varepsilon}$ can be directly derived from the estimates given in [1] on $\|(\underline{\underline{\Lambda}} - \tilde{\underline{\underline{\Lambda}}}, \underline{\underline{\eta}} - \tilde{\underline{\underline{\eta}}})\|_{\mathbf{W}_s} + t\|(\underline{\underline{\Lambda}} - \tilde{\underline{\underline{\Lambda}}}, \underline{\underline{\eta}} - \tilde{\underline{\underline{\eta}}})\|_{L^2}$. And for the contact forces, direct local interpolation yields

$$\inf_{\tilde{\underline{\underline{g}}}_h \in \mathbf{M}_h} \|\underline{\underline{g}} - \tilde{\underline{\underline{g}}}_h\|_{h, -1/2} \leq Ch^q \|\underline{\underline{g}}\|_{H^{q-1/2}},$$

when \mathbf{M}_h uses finite elements of order q .

6. CONCLUSIONS

We have indicated herein how to efficiently couple locking free mixed elements in the case of an elastic shell interacting with an elastic incompressible solid, while preserving the accuracy and the stability of the original elements. More important, if one uses locking free elements on the shell, the constants of stability and convergence remain independent of the shell thickness even after coupling.

This technique, treating interface continuity with mortar techniques, allows to use completely different finite element discretisations on the internal side and on the shell. It can be applied to a wide variety of structure structure or of fluid structure interaction problems and yield accurate results in situations where they are needed such as the onset of instabilities for microcapsules or blood flows in arteries. It can also handle geometric incompatibilities. Indeed, in most practical situations, the internal finite element velocity or displacement field is defined on an internal domain with piecewise linear boundaries and the shell displacement field is defined on a curved surface. After discretisation, the outer boundary of the fluid is therefore geometrically different from the shell internal surface. In such situations, imposing a nodal continuity at vertices is not sufficiently accurate, and neglects all the coupling effects associated to the rotation of the unit normal. Imposing instead a strict pointwise continuity on the whole interface requires the introduction of curved finite elements inside the domain, which may be very complex. As explained in [12, 15], mortar techniques such as those introduced herein handle this lack of geometric conformity by using the original shape functions within the fluid, and by imposing a zero weighted average between the shell displacement field and the internal displacement field on the real interface.

Unfortunately, on the shell side, the theory presented herein is mainly restricted to the bending dominated case and is subjected to the same limitations as in the shell theory. Checking the discrete inf-sup condition (12) in a very general framework is still out of reach. When using finite element spaces \mathbf{W}_{sh} and \mathbf{V}_{sh} as proposed in [1], it can only be proved for very specific shell geometries whose fundamental form are piecewise constant [19]. To our knowledge also, the theory has not yet been backed by numerical simulations, focusing in particular on the numerical verification of the discrete inf-sup conditions as in [3]. Moreover, the numerical efficiency of the proposed approach remains to be assessed in situations where the shell has no inextensional modes, but has extensional modes of reasonably low energy. Typical situations concern capsules enclosing soft matter.

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