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NUMERICAL HOMOGENIZATION OF WELL SINGULARITIES IN THE FLOW TRANSPORT THROUGH HETEROGENEOUS POROUS MEDIA: FULLY DISCRETE SCHEME

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Abstract. Motivated by well-driven flow transport in porous media, Chen and Yue proposed a numerical homogenization method for Green function [*Multiscale Model. Simul.* **1** (2003) 260–303]. In that paper, the authors focused on the well pore pressure, so the local error analysis in maximum norm was presented. As a continuation, we will consider a fully discrete scheme and its multiscale error analysis on the velocity field.

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1. Introduction

Upscaling or homogenization methods have become the powerful tools for the macroscopic modeling of the flow transport in the heterogenous porous media [1, 4, 6, 14, 16]. It was known that for the well-driven flow in the porous media, the standard upscaling methods did not work in the vicinity of the wells [5, 13]. In the recent paper [3], Chen and Yue developed a multiscale coarse grid algorithm for solving steady flow problem involving well singularities in heterogeneous porous medium based on the over-sampling multiscale finite element method (MsFEM) [10]. The remedy was that the well singularities (Dirac sources) of the problem were first resolved locally and then were removed; the left part could be formulated in a variational form and was solved by the multiscale finite element method.

In the previous work, focusing on the well bore pressure, Chen and Yue presented the local error analysis in maximum norm. Though the well bore pressure plays a key role in the well control [15], the flow pattern in the whole reservoir is also very important in the engineering. In this paper, we are going to consider the multiscale error analysis for the flow velocity field, *i.e.* in the energy norm.

The general idea of MsFEM is to construct finite element basis functions that capture the small scale information of the leading order differential operator. In practical implementation, this requires numerically solving a series of differential equations associated with the differential operator. It seems that the fully discrete error is not considered in the analysis on MsFEM so far (*cf.* [2, 3, 7, 11]), except for in [8, 9], a numerical homogenization method related to MsFEM was proposed and the fully discrete error analysis was presented for monotone elliptic operators and quasi-convex energies. However in these two works, only the error between the numerical

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solution and the solution of the homogenized equation was considered. In present work, we will analyze the error between the fully discrete solution of over-sampling MsFEM and the exact solution of the original problem.

Although the main singularity induced by the Dirac source is removed, what's left is only a piecewise H^2 function. In fact, the right hand side of the governing equation for the left part does not belong to L^2 (see (1.3) below). The lack of global regularity also brings some difficulties in the error analysis.

We now recall the problem and one of the main ideas in the previous study [3]. Let $\Omega \subset \mathbf{R}^2$ a bounded domain with Lipschitz boundary Γ. We consider the following problem

$$\begin{aligned} -\mathrm{div}\,(K_\varepsilon(x)\nabla u_\varepsilon) &= \delta_{x_0}, \quad \text{in } \Omega, \\ u_\varepsilon &= 0, \quad\ \ \text{on } \Gamma, \end{aligned} \tag{1.1}$$

where $0 < \varepsilon \ll 1$ is the ratio between the characteristic length scale of the micro structure and the macroscopic scale of the porous media, and $K_\varepsilon = K(x, \frac{x}{\varepsilon})$.

Let Ω_0 be a small sub-domain inside Ω such that $x_0 \in \Omega_0$, $H_0 = \mathrm{dist}\,(x_0, \partial\Omega_0)$ and there exists a constant $C > 0$ satisfying that $\mathrm{diam}(\Omega_0) \le CH_0$. Let G_ε be the local Green function associated with the domain Ω_0

$$-\mathrm{div}\,(K_\varepsilon(x)\nabla G_\varepsilon) = \delta_{x_0} \quad \text{in } \Omega_0, \qquad G_\varepsilon|_{\Sigma_0} = 0, \tag{1.2}$$

where $\Sigma_0 = \partial\Omega_0$. Let $G_\varepsilon = 0$ for $x \in \Omega\backslash\bar{\Omega}_0$ and set $\zeta_\varepsilon = u_\varepsilon - G_\varepsilon$, then $\zeta_\varepsilon \in H_0^1(\Omega)$ satisfies the following variational form

$$\int_\Omega K_\varepsilon(x)\nabla\zeta_\varepsilon\nabla v\,\mathrm{d}x = -\int_{\Sigma_0} K_\varepsilon\frac{\partial G_\varepsilon}{\partial\nu}v\,\mathrm{d}x \qquad \forall v \in C_0^\infty(\Omega). \tag{1.3}$$

The main singularity of the original solution u_ε is removed and the over-sampling MsFEM [10] can be used to discretize the above variational form on a coarse grid. The local convergence in maximum norm of the method was established in [3] for locally periodic coefficients, *i.e.* assuming $K_\varepsilon(x) = K(x, x/\varepsilon)$, where $K(x,\cdot)$ is periodic with respect to the unit square Y.

The paper is organized as follows: in Section 2 we introduce the fully discrete multiscale algorithm based on the weak formulation (1.3). In Section 3 we list some results on the homogenization of Green function and establish some new homogenization results for ζ_ε, which will be used in Section 4 to complete the multiscale error analysis for the velocity field. In the whole paper, $C > 0$ is a general constant independent of the parameters ε, H_0, the fine grid scale h and the coarse grid scale H.

2. The fully discrete scheme

In this section we are going to recall the multiscale method to solve the problem (1.3).

Let $\mathcal{M}_H$ be a regular and quasi-uniform triangulation of Ω and V_H the standard conforming linear finite element space over $\mathcal{M}_H$. For any $T \in \mathcal{M}_H$ with nodes $\{x_i^T\}_{i=1}^3$, let H_T denote the size of T, $P_1(T)$ the set of linear polynomials defined in T, and $\{\varphi_i^T\}_{i=1}^3$the basis of $P_1(T)$ satisfying $\varphi_i^T(x_j^T) = \delta_{ij}, i, j = 1, 2, 3$. For any $T \in \mathcal{M}_H$, we denote by $S = S(T)$ a macro-element which contains T and satisfies the following condition.

(H1) $H_S \le C_1 H_T$ and $\mathrm{dist}\,(\partial T, \partial S) \ge \delta_0 H_T$ for some positive constants C_1, δ_0 independent of H. The minimum angles of $S(T)$ is bounded below by some positive constant θ_0 independent of H.

Let $MS(S)$ be the multiscale finite element space spanned by $\psi_i^S, i = 1, 2, 3$, with $\psi_i^S \in H^1(S)$ being the solution of the problem

$$-\mathrm{div}\,(K_\varepsilon(x)\nabla\psi_i^S) = 0 \quad \text{in } S, \qquad \psi_i^S|_{\partial S} = \varphi_i^S. \tag{2.1}$$

Here $\{\varphi_i^S\}_{i=1}^3$ is the nodal basis of $P_1(S)$ such that $\varphi_i^S(x_j^S) = \delta_{ij}$, $i, j = 1, 2, 3$. Then we define the over-sampling finite element basis over T by

$$\bar{\psi}_i^T = c_{ij}^T \psi_i^S|_T \quad \text{ in } T, \tag{2.2}$$

with the constants c_{ij}^T so chosen that $\varphi_i^T = c_{ij}^T \varphi_j^S|_T$ in T.

Let $OMS(T) = \text{span}\{\bar{\psi}_i^T\}_{i=1}^3$ and $\Pi_T : OMS(T) \to P_1(T)$ the projection

$$\Pi_T \psi = c_i \varphi_i^T \quad \text{ if } \psi = c_i \bar{\psi}_i^T \in OMS(T). \tag{2.3}$$

Let $\bar{X}_H$ be the finite element space $\bar{X}_H = \{\psi_H : \psi_H|_T \in OMS(T),\ \forall T \in \mathcal{M}_H\}$ and define $\Pi_H : \bar{X}_H \to \Pi_{T\in\mathcal{M}_H} P_1(T)$ through the relation $\Pi_H \psi_H|_T = \Pi_T \psi_H$ for any $T \in \mathcal{M}_H$, $\psi_H \in \bar{X}_H$. The over-sampling finite element space is then defined as

$$X_H = \{\psi_H \in \bar{X}_H : \Pi_H \psi_H \in V_H \subset H^1(\Omega)\}.$$

Taking the boundary condition into account, we set $X_H^0 = \{\psi_H \in X_H : \Pi_H \psi_H = 0 \text{ on } \Gamma\}$.

The fully discrete counterparts of X_H and X_H^0 are denoted by Y_H and Y_H^0, which can be defined in the same way as above, except that the element bases $\psi_i^S, i = 1, 2, 3$ in (2.1) are replaced by their conforming piecewise linear finite element approximation $\psi_i^h, i = 1, 2, 3$ on a fine grid of size h resolving the small scale ε over the macro element S.

Note that we have now three types of finite element spaces over the same triangulation $\mathcal{M}_H$:

V_H – the conforming piecewise linear finite element space;

X_H – the common multiscale finite element space;

Y_H – the multiscale finite element space with fully discrete basis functions.

In numerical implementation, not only should the multiscale finite element basis functions be numerically constructed, but also the local Green function G_ε in Ω_0 in (1.3). We are going to approximate it by its conforming piecewise linear finite element solution G_h on a fine grid with size h resolving the scale ε over Ω_0. In order to avoid to approximate the normal derivative $\frac{\partial G_\varepsilon}{\partial \nu}$ on the subdomain boundary $\Sigma_0 = \partial\Omega_0$, we need the following equivalent form of (1.3)

$$\int_\Omega K_\varepsilon(x) \nabla \zeta_\varepsilon \nabla v \,\mathrm{d}x = \int_{\Omega_0} K_\varepsilon \nabla G_\varepsilon \nabla(\phi v) \mathrm{d}x \quad \forall v \in C_0^\infty(\Omega), \tag{2.4}$$

where ϕ is a cut-off function such that $\phi \in C^1(\Omega_0)$, $\phi \equiv 1$ in $\Omega_0 \backslash B(x_0, 3H_0/4)$, $\phi \equiv 0$ in $B(x_0, H_0/2)$. Here $B(x, r) \subset \mathbf{R}^2$ denotes a ball centered at x with a radius of r.

Now we define the fully discrete scheme: Find $\zeta_H \in Y_H^0$ such that

$$\sum_{T\in\mathcal{M}_H} \int_T K_\varepsilon(x) \nabla \zeta_H \nabla \chi_H \,\mathrm{d}x = -\int_{\Omega_0} K_\varepsilon \nabla G_h \nabla(\phi \hat{\chi}_H) \mathrm{d}x \quad \forall \chi_H \in Y_H^0, \tag{2.5}$$

where $\hat{\chi}_H = \Pi_H \chi_H \in V_H^0 = V_H \cap H_0^1(\Omega)$, the project $\Pi_H : Y_H \to V_H$ is defined in the same way as the one from X_H to V_H above.

If we make the following local periodicity assumption for the coefficient:

(H2) $K_\varepsilon(x) = K(x, x/\varepsilon)$ satisfies the uniform elliptic condition and $K \in C^1(\bar{\Omega}; C_p^1(\mathbf{R}^2))$, where $C_p^1(\mathbf{R}^2)$ stands for the collection of all $C^1(\mathbf{R}^2)$ periodic functions with respect to the unit square Y,

then for the fully discrete scheme, we have:

Theorem 2.1. *Let the assumptions* (H1)-(H2) *be fulfilled. Then there exists a constant C independent of ε, h, H and H_0 such that for $0 < h \ll \varepsilon \ll H \ll 1$, the following error estimate is valid*

$$\begin{aligned}\|\zeta_\varepsilon - \zeta_H\|_H \quad \leq \quad & C\Big(\big(H + \frac{\varepsilon}{H} + \frac{\varepsilon}{H_0} + \frac{H}{H_0} + \frac{h}{\varepsilon}\big)|\ln H_0| + \frac{\varepsilon}{H_0}|\ln \varepsilon| \\ & + \frac{\varepsilon}{H_0 H} + \sqrt{\frac{\varepsilon}{H_0}} + \frac{h}{\varepsilon H_0}|\ln H|^{1/2}\Big),\end{aligned}$$

where $\|\chi_H\|_H = \|\chi_H\|_{H,\Omega}$ *and* $\|\chi_H\|_{H,D} = \big(\sum_{T\in\mathcal{M}_H\cap D}\int_T |\nabla\chi_H|^2\,\mathrm{d}x\big)^{1/2}$.

Remark. Here we restrict ourself to a two dimensional Green function problem. In fact, in oil industry both the vertical and horizontal well should be regarded as line sources. In the discrete sense at each cross-section, we have to treat a two dimensional Dirac source. That's the reason why most of the well treatment problem in engineering is two dimensional. However most of our results and the analysis here are valid for three dimensional problem, except for the homogenization results and some *a priori* estimates related to two dimensional Green function.

3. Homogenization results

In this section we list several results of standard homogenization theory, and prove some new homogenization results for ζ_ε, which will play a basic role in the subsequent analysis.

Let $D \subseteq \Omega$ be a bounded domain with Lipschitz boundary. Given $f \in L^2(D)$, we consider the following problem with

$$\int_D a\Big(x, \frac{x}{\varepsilon}\Big)\nabla w_\varepsilon \nabla\varphi\,\mathrm{d}x = \int_D f\varphi\,\mathrm{d}x \quad \forall\varphi \in H_0^1(D). \tag{3.1}$$

Here $a(x, x/\varepsilon) = (a_{ij}(x, x/\varepsilon))$ is a symmetric matrix which satisfies the uniform ellipticity condition (in our application $a(x, x/\varepsilon) = K_\varepsilon(x)\mathbf{I}$). Furthermore, we assume that $a_{ij} \in C^1(\bar{D}; C_p^1(\mathbf{R}^2))$.

Let $w_0 \in H_0^1(D)$ be the unique solution of the homogenized problem

$$\int_D a^*(x)\nabla w_0 \nabla\varphi\,\mathrm{d}x = \int_D f\varphi\,\mathrm{d}x \quad \forall\varphi \in H_0^1(D), \tag{3.2}$$

where $a^*(x) = (a^*_{ij}(x))$ with

$$a^*_{ij}(x) = \frac{1}{|Y|}\int_Y a_{ik}(x,y)\Big(\delta_{kj} - \frac{\partial\chi^j}{\partial y_k}(x,y)\Big)\mathrm{d}y, \tag{3.3}$$

and $\chi^j(x,y)$ is the periodic solution of the cell problem

$$\frac{\partial}{\partial y_i}\Big(a_{ik}(x,y)\frac{\partial\chi^j}{\partial y_k}(x,y)\Big) = \frac{\partial}{\partial y_i}a_{ij}(x,y) \quad \text{in } Y, \qquad \int_Y \chi^j(x,y)\mathrm{d}y = 0. \tag{3.4}$$

Here δ_{kj} is the Kronecker delta, *i.e.* $\delta_{kj} = 1$ for $k = j$, and $\delta_{kj} = 0$ for $k \neq j$.

Set $w_1^\varepsilon(x) = w_0(x) - \varepsilon\chi^k\Big(x, \frac{x}{\varepsilon}\Big)\frac{\partial w_0}{\partial x_k}$. Let $\theta_\varepsilon = \theta_\varepsilon(w_0) \in H^1(D)$ be the boundary corrector that satisfies

$$-\mathrm{div}\,\Big(a\Big(x, \frac{x}{\varepsilon}\Big)\nabla\theta_\varepsilon\Big) = 0 \quad \text{in } D, \qquad \theta_\varepsilon|_{\partial D} = \chi^k\Big(x, \frac{x}{\varepsilon}\Big)\frac{\partial w_0}{\partial x_k}\cdot \tag{3.5}$$

The following theorem is known (*cf. e.g.* [2,12]).

Theorem 3.1. *Assume that* $w_0 \in H^2(D) \cap W^{1,\infty}(D)$. *Then there exists a constant* C *independent of* ε, *the domain* D, *and the function* f *such that*

$$\| \nabla w_\varepsilon - \nabla(w_1^\varepsilon + \varepsilon\theta_\varepsilon) \|_{0,D} \le C\varepsilon(| w_0 |_{2,D} + | w_0 |_{1,D}), \tag{3.6}$$

and for the boundary corrector θ_ε, *there exists the following estimate*

$$\| \varepsilon\nabla\theta_\varepsilon \|_{0,D} \le C\varepsilon(| w_0 |_{2,D} + | w_0 |_{1,D}) + C\sqrt{\varepsilon|\partial D|}\, | w_0 |_{1,\infty,D}, \tag{3.7}$$

where $|\partial D|$ *stands for the length of the boundary* ∂D.

Furthermore, after checking the proof of Theorem 3.1 (*cf.* [2]), we may obtain a more precise result for the boundary corrector

$$\| \varepsilon\nabla\theta_\varepsilon \|_{0,D} \le C\varepsilon(| w_0 |_{2,D_\varepsilon} + | w_0 |_{1,D_\varepsilon}) + C\sqrt{\varepsilon|\partial D|}\, | w_0 |_{1,\infty,D_\varepsilon}, \tag{3.8}$$

where $D_\varepsilon = \{x \in D : \mathrm{dist}\,(x, \partial D) \le \varepsilon\} \subset D$ is a ε-neighborhood of the boundary ∂D. This is true because one can choose the cut-off function vanishing outside the subset D_ε in the proof of (3.7).

Now we are going to establish the homogenization results for ζ_ε. As known, in standard homogenization theory, the asymptotic expansion (*e.g.* (3.6)) is valid under the assumption that the source terms be in L^2. Note that for the problem (1.3), this assumption is not true. Though the main singularity has been removed, we cannot expect that $\zeta_\varepsilon \in H^2(\Omega)$. However, benefiting from the special structure of ζ_ε, we can deduce a piecewise asymptotic expansion for it.

We first define the following homogenized problems for u_0, G_0 and ζ_0 satisfying

$$(K^*\nabla u_0, \nabla v) = v(x_0), \qquad \forall v \in H_0^1(\Omega) \cap C(\Omega), \tag{3.9}$$

$$(K^*\nabla G_0, \nabla v) = v(x_0), \qquad \forall v \in H_0^1(\Omega_0) \cap C(\Omega_0) \tag{3.10}$$

and

$$\int_\Omega K^*\nabla\zeta_0\nabla\chi\,\mathrm{d}x = -\int_{\partial\Omega_0} K^*\frac{\partial G_0}{\partial\nu}\chi, \qquad \forall\chi \in H_0^1(\Omega), \tag{3.11}$$

where K^* is defined by (3.3).

Obviously,

$$\zeta_\varepsilon = \begin{cases} u_\varepsilon, & x \in \Omega\backslash\Omega_0 \\ u_\varepsilon - G_\varepsilon, & x \in \bar\Omega_0, \end{cases} \qquad \zeta_0 = \begin{cases} u_0, & x \in \Omega\backslash\Omega_0 \\ u_0 - G_0, & x \in \bar\Omega_0. \end{cases} \tag{3.12}$$

Then we present the main results of this section.

Theorem 3.2. *Assume that* $\mathrm{diam}(\Omega_0) \le CH_0$, *then there exists a constant* C *independent of* ε *and* H_0, *such that*

$$\| \nabla(\zeta_\varepsilon - \zeta_0 + \varepsilon\chi_i\frac{\partial\zeta_0}{\partial x_i} - \varepsilon\theta_\varepsilon(\zeta_0)) \|_{0,\Omega\backslash\Omega_0} \le C\frac{\varepsilon}{H_0}|\ln\varepsilon|, \tag{3.13}$$

$$\| \nabla(\zeta_\varepsilon - \zeta_0 + \varepsilon\chi_i\frac{\partial\zeta_0}{\partial x_i} - \varepsilon\bar\theta_\varepsilon(\zeta_0)) \|_{0,\Omega_0} \le C\Big(\frac{\varepsilon}{H_0}|\ln\varepsilon| + \sqrt{\frac{\varepsilon}{H_0}}\Big), \tag{3.14}$$

where $\theta_\varepsilon(\zeta_0)$ *and* $\bar\theta_\varepsilon(\zeta_0)$ *are the boundary correctors for domains* Ω *and* Ω_0 *respectively defined as* (3.5).

Before the proof, we list some useful results on u_ε and u_0 in the previous work (*cf.* [3], Thm. 4.3, Thm. 4.5 and its proof and Lem. 4.4 respectively),

$$\|u_\varepsilon - u_0\|_{0,\Omega\backslash B(x_0,r)} \le C\varepsilon\Big(1+\ln\frac{r}{\varepsilon}\Big), \tag{3.15}$$

$$\|\,\nabla(u_\varepsilon - u_0 + \varepsilon\chi_i\frac{\partial u_0}{\partial x_i} - \varepsilon\theta_\varepsilon(u_0))\,\|_{0,\Omega\backslash B(x_0,r)} \le C\frac{\varepsilon}{r}|\ln\varepsilon|, \tag{3.16}$$

$$\begin{array}{l} |\nabla u_0(x)| \le Cr^{-1} \quad \text{for any } x\in\Omega\backslash B(x_0,r), \\ \|\,D^2 u_0\,\|_{0,\Omega\backslash B(x_0,r)} \le Cr^{-1}. \end{array} \tag{3.17}$$

Proof of Theorem 3.2. From (3.12), $\zeta_\varepsilon = u_\varepsilon$, $\zeta_0 = u_0$ in $\Omega\backslash\Omega_0$. So (3.13) is just a local asymptotic expansion for the Green function u_ε, *cf.* (3.16).

To prove the second result (3.14), we first rewrite by (3.12),

$$-\mathrm{div}\,(K_\varepsilon\nabla\zeta_\varepsilon) = 0, \ \text{ in } \Omega_0, \quad \zeta_\varepsilon|_{\partial\Omega_0} = u_\varepsilon, \tag{3.18}$$

$$-\mathrm{div}\,(K^*\nabla\zeta_0) = 0, \ \text{ in } \Omega_0, \quad \zeta_0|_{\partial\Omega_0} = u_0. \tag{3.19}$$

Define $\bar\zeta_\varepsilon$ by

$$-\mathrm{div}\,(K_\varepsilon\nabla\bar\zeta_\varepsilon) = 0, \ \text{ in } \Omega_0, \quad \bar\zeta_\varepsilon|_{\partial\Omega_0} = u_\varepsilon - u_0, \tag{3.20}$$

then ζ_0 is the homogenization of $(\zeta_\varepsilon - \bar\zeta_\varepsilon)$. Therefore, by standard homogenization theory (3.1),

$$\begin{aligned} \|\,\nabla\big((\zeta_\varepsilon-\bar\zeta_\varepsilon) - \zeta_0 + \varepsilon\chi_i\frac{\partial\zeta_0}{\partial x_i} - \varepsilon\bar\theta_\varepsilon(\zeta_0)\big)\,\|_{0,\Omega_0} &\le C\varepsilon(\|\,\zeta_0\,\|_{2,\Omega_0} + \|\,\zeta_0\,\|_{1,\Omega_0}) \\ &\le C\frac{\varepsilon}{H_0}|\ln H_0|, \end{aligned} \tag{3.21}$$

where we have used the following *a priori* estimates

$$\|\,\zeta_0\,\|_{1,\Omega_0} \le C|\ln H_0|, \quad \|\,\zeta_0\,\|_{2,\Omega_0} \le C|\ln H_0|/H_0, \tag{3.22}$$

which can be obtained from (3.19) and from the local *a priori* estimates for the Green function u_0 (*cf.* (3.17)) by checking the governing equation for the new variable $w = \zeta_0 - \phi u_0$ with the cut-off function ϕ introduced in (2.4).

What's left is to bound the term $\|\,\nabla\bar\zeta_\varepsilon\,\|_{0,\Omega_0}$. We split $\bar\zeta_\varepsilon$ into two parts $\bar\zeta_\varepsilon = \hat\zeta_\varepsilon + \tilde\zeta_\varepsilon$ by

$$\begin{aligned} &-\mathrm{div}\,(K_\varepsilon\nabla\hat\zeta_\varepsilon) = 0 \ \text{ in } \Omega_0, \\ &\hat\zeta_\varepsilon|_{\partial\Omega_0} = u_\varepsilon - u_0 + \varepsilon\chi_i\frac{\partial u_0}{\partial x_i} - \varepsilon\theta_\varepsilon(u_0), \end{aligned} \tag{3.23}$$

and

$$\begin{aligned} &-\mathrm{div}\,(K_\varepsilon\nabla\tilde\zeta_\varepsilon) = 0 \ \text{ in } \Omega_0, \\ &\tilde\zeta_\varepsilon|_{\partial\Omega_0} = -\varepsilon\chi_i\frac{\partial u_0}{\partial x_i} + \varepsilon\theta_\varepsilon(u_0). \end{aligned} \tag{3.24}$$

For the first part $\hat{\zeta}_\varepsilon$, recalling the cut-off function $\phi \in C^1(\Omega_0)$ introduced in (2.4), $\phi \equiv 0$ in $B(x_0, H_0/2)$, $\phi \equiv 1$ in $\Omega_0 \backslash B(x_0, 3H_0/4)$ such that $|\nabla \phi| \le \frac{C}{H_0}$, we have

$$\begin{aligned} \| \nabla \hat{\zeta}_\varepsilon \|_{0,\Omega_0} &\le C \| \nabla \big(\phi (u_\varepsilon - u_0 + \varepsilon \chi_i \frac{\partial u_0}{\partial x_i} - \varepsilon \theta_\varepsilon(u_0)) \big) \|_{0,\Omega_0} \\ &\le C \| \nabla \big(u_\varepsilon - u_0 + \varepsilon \chi_i \frac{\partial u_0}{\partial x_i} - \varepsilon \theta_\varepsilon(u_0) \big) \|_{0,\Omega_0 \backslash B(x_0,H_0/2)} \\ &+ C \| \nabla \phi \big(u_\varepsilon - u_0 + \varepsilon \chi_i \frac{\partial u_0}{\partial x_i} - \varepsilon \theta_\varepsilon(u_0) \big) \|_{0,\Omega_0 \backslash B(x_0,H_0/2)} \le C \frac{\varepsilon}{H_0} |\ln \varepsilon|, \end{aligned} \tag{3.25}$$

where we have used the previous results (3.16), (3.15) and (3.17).

The second part $\tilde{\zeta}_\varepsilon$ can be regarded as a boundary corrector, so similarly to bound the corrector $\theta_\varepsilon(w_0)$ in Theorem 3.1 (see, *e.g.* [2]), we first introduce a cut-off function $\psi_\varepsilon \in C^2(\Omega_0), 0 \le \psi_\varepsilon \le 1$ in Ω_0, $\psi_\varepsilon = 1$ on the boundary $\partial\Omega_0$, $\psi_\varepsilon \equiv 0$ outside the ε-neighborhood $\Omega_0^\varepsilon \subset \Omega_0$ of the boundary $\partial\Omega_0$, and $|\nabla \psi_\varepsilon| \le C/\varepsilon$ in Ω_0 with C independent of ε and Ω_0. It is clear that $|\Omega_0^\varepsilon| \le \varepsilon |\partial\Omega_0| \le C\varepsilon H_0$. Then thanks to (3.17),

$$\begin{aligned} \| \nabla \tilde{\zeta}_\varepsilon \|_{0,\Omega_0} &\le C \| \varepsilon \nabla \big(\psi_\varepsilon (-\chi_i \frac{\partial u_0}{\partial x_i} + \theta_\varepsilon(u_0)) \big) \|_{0,\Omega_0} \\ &\le C \sqrt{|\Omega_0^\varepsilon|} \, \| \theta_\varepsilon(u_0) - \chi_i \frac{\partial u_0}{\partial x_i} \|_{0,\infty,\Omega_0^\varepsilon} + \| \varepsilon \nabla (\theta_\varepsilon(u_0) - \chi_i \frac{\partial u_0}{\partial x_i}) \|_{0,\Omega_0^\varepsilon} \\ &\le C \sqrt{\varepsilon H_0} \| u_0 \|_{1,\infty,\Omega \backslash B(x_0,H_0/2)} + C\varepsilon \| u_0 \|_{2,\Omega \backslash B(x_0,H_0/2)} \\ &\le C \big(\sqrt{\frac{\varepsilon}{H_0}} + \frac{\varepsilon}{H_0} \big). \end{aligned} \tag{3.26}$$

Combining (3.21), (3.25) and (3.26), we obtain (3.14), and the proof is completed. □

4. Error estimates

We first derive the error estimate for the numerical approximation of the multiscale finite element basis functions.

Lemma 4.1. *For any $\chi \in X_H$, if $\chi|_T = \sum_{i=1}^3 c_i \bar{\psi}_i^T, T \in \mathcal{M}_H$, then its fully discrete counterpart $\chi_h \in Y_H$ is defined as $\chi_h|_T = \sum_{i=1}^3 c_i \bar{\psi}_i^{T,h}$, where $\bar{\psi}_i^{T,h}, i = 1,2,3$, are the numerically constructed bases, and its counterpart in V_H has been defined in* (2.3) *as $\hat{\chi} = \Pi_H \chi$. The following error estimate is valid*

$$\|\chi - \chi_h\|_H \le C \frac{h}{\varepsilon} \|\nabla \hat{\chi}\|_{0,\Omega}.$$

Proof. Thanks to (2.1) and (2.2), for each macro element $S \supset T \in \mathcal{M}_H$,

$$\begin{cases} -\nabla \cdot (K_\varepsilon \nabla \chi) = 0 & \text{in } S \\ \chi = \hat{\chi} & \text{on } \partial S. \end{cases} \tag{4.1}$$

χ_h is actually the piecewise linear finite element solution of the above problem on fine grid of size h. Noting that $\hat{\chi} \in P_1(S)$, it is direct to deduce that

$$\|\nabla(\chi - \chi_h)\|_{0,S} \le Ch|\chi|_{2,S} \le C \frac{h}{\varepsilon} \|\nabla \hat{\chi}\|_{0,S} \le C \frac{h}{\varepsilon} \|\nabla \hat{\chi}\|_{0,T}, \tag{4.2}$$

where we have used the assumption (H1) and the *a priori* estimate $|\chi|_{2,S} \le (C/\varepsilon)\|\nabla\hat{\chi}\|_{0,S}$, which can be seen easily if we write down the governing equation for the auxiliary variable $w = \chi - \hat{\chi}$

$$\begin{cases} -\Delta w = \frac{1}{K_\varepsilon}\nabla K_\varepsilon \cdot \nabla \chi & \text{in } S \\ \qquad\quad w = 0 & \text{on } \partial S. \end{cases}$$

The standard elliptic regularity yields that $|w|_{2,S} \le (C/\varepsilon)\|\nabla\chi\|_{0,S} \le (C/\varepsilon)\|\nabla\hat{\chi}\|_{0,S}$.

Finally, we have

$$\|\chi - \chi_h\|_H^2 = \sum_{T\in\mathcal{M}_H}\int_T |\nabla(\chi-\chi_h)|^2\,\mathrm{d}x \le \sum_{T\in\mathcal{M}_H}\int_S |\nabla(\chi-\chi_h)|^2 \le C\left(\frac{h}{\varepsilon}\right)^2\|\nabla\hat{\chi}\|_{0,\Omega}^2. \qquad \square$$

We are now going to prove Theorem 2.1. Before that, we list some previous results. The following lemma can be found in [3], Section 5.

Lemma 4.2. *Under the assumptions* (H1)-(H2) *there exist positive constants* C *independent of* H, ε *such that for sufficiently small* $H > 0$*, the following estimates are valid*

$$\|\nabla\hat{\chi}_H\|_{0,T} \le C\|\nabla\chi_H\|_{0,T}, \qquad \|\nabla\chi_H\|_{0,T} \le C\|\nabla\hat{\chi}_H\|_{0,T} \tag{4.3}$$

$$\|\nabla(\chi_H - \hat{\chi}_H + \varepsilon\chi_i\frac{\partial\hat{\chi}_H}{\partial x_i} - \varepsilon\tilde{\theta}_\varepsilon(\hat{\chi}_H))\|_{0,T} \le C(\frac{\varepsilon}{H} + H)\|\nabla\hat{\chi}_H\|_{0,T} \tag{4.4}$$

for any $\chi_H \in X_H, \hat{\chi}_H = \Pi_H\chi_H \in V_H$*, where* $\tilde{\theta}_\varepsilon(\hat{\chi}_H)$ *is the associated boundary corrector that satisfies, for any* $T \in \mathcal{M}_H$*,* $S = S(T)$ *the over-sampling element,*

$$-\mathrm{div}\,(K_\varepsilon(x)\nabla\tilde{\theta}_\varepsilon(\hat{\chi}_H)) = 0 \quad \text{in } S, \qquad \tilde{\theta}_\varepsilon(\hat{\chi}_H)|_{\partial S} = \chi^k\Big(x, \frac{x}{\varepsilon}\Big)\frac{\partial\hat{\chi}_H}{\partial x_k} \quad \text{on } \partial S,$$

and $\tilde{\theta}_\varepsilon(\hat{\chi}_H)$ *is bounded in* [7] *by*

$$\|\nabla\tilde{\theta}_\varepsilon(\hat{\chi}_H)\|_{0,T} \le C\frac{\varepsilon}{H}\|\nabla\hat{\chi}_H\|_{0,T}. \tag{4.5}$$

The following result is well known (*cf.* [7]).

Lemma 4.3. *Let* $N(x,y)$ *be periodic in* y *with respect to the unit square* Y *in* $\mathbf{R}^2$ *such that* $\int_Y N(x,y)\mathrm{d}y = 0$ *for any* $x \in D$*. Moreover, assume that* $|N(x,y)| + |\nabla_x N(x,y)| \le C$ *for any* $x \in D, y \in \mathbf{R}^2$*. Then, for any* $\xi \in H^1(D) \cap L^\infty(D)$*, we have*

$$\Big|\int_D \xi(x)N\left(x, \frac{x}{\varepsilon}\right)\mathrm{d}x\Big| \le C\varepsilon|D|^{1/2}\|\xi\|_{1,D} + C\varepsilon|\partial D|\|\xi\|_{0,\infty,D}.$$

The next lemma, which considered the local error between the local Green function G_ε and its fine scale piecewise linear finite element approximation G_h, can be found in [3], Theorem 6.2.

Lemma 4.4. *There exists a constant* C *independent of* h *and* H_0 *such that*

$$\|\nabla(G_\varepsilon - G_h)\|_{0,\Omega_0\setminus\bar{B}(x_0,H_0/2)} \le C\frac{h}{\varepsilon H_0}.$$

Proof of Theorem 2.1. First by Strang's second lemma, we have from (1.3) and (2.5) that

$$\begin{aligned}\|\zeta_\varepsilon-\zeta_H\|_H &\le C\inf_{\chi_H\in Y_H^0}\|\zeta_\varepsilon-\chi_H\|_H\\ &+ C\sup_{\chi_H\in Y_H^0}\frac{|\big(K_\varepsilon\nabla G_h,\nabla(\phi\hat\chi_H)\big)_{\Omega_0}-\sum_{T\in\mathcal{M}_H}\big(K_\varepsilon\nabla\zeta_\varepsilon,\nabla\chi_H\big)_T|}{\|\chi_H\|_H}.\end{aligned}\tag{4.6}$$

In (4.6) the error is divided into two parts "conforming error" and "nonconforming error". The "conforming error" is dominated by the interpolation error. For ζ_0, we may define its different node interpolations: $I_X\zeta_0\in X_H^0$, $I_Y\zeta_0\in Y_H^0$ and $I_V\zeta_0\in V_H^0$ as for each $T\in\mathcal{M}_H$, $I_X\zeta_0|_T=\sum_{i=1}^3\zeta_0(x_i)\bar\psi_i^T$, where $x_i, i=1,2,3$, are the vertices of T, and $\bar\psi_i^T, i=1,2,3$, are the corresponding over-sampling finite element bases (*cf.* (2.2)); $I_Y\zeta_0$ and $I_V\zeta_0$ are defined in the same way.

Split the interpolation error into two parts

$$\|\zeta_\varepsilon-I_Y\zeta_0\|_H\le\|\zeta_\varepsilon-I_X\zeta_0\|_H+\|I_X\zeta_0-I_Y\zeta_0\|_H.\tag{4.7}$$

The first part is also the interpolation error for the multiscale finite element space X_H, and the second part is the error due to the numerical approximation of the finite element basis functions. Thanks to Lemma 4.1, (3.17) and (3.22), we have

$$\|I_X\zeta_0-I_Y\zeta_0\|_H\le C\frac{h}{\varepsilon}\|\nabla(\Pi_H I_X\zeta_0)\|_{0,\Omega}=C\frac{h}{\varepsilon}\|\nabla I_V\zeta_0\|_{0,\Omega}\le C\frac{h}{\varepsilon}|\ln H_0|.\tag{4.8}$$

The interpolation error in the right hand side of (4.7) can be treated under the framework of [7], Section 3.1, though some more attention should be paid on the lack of global regularity and on the precise dependence on the different length scales ε, H, H_0.

First we begin with

$$\|\zeta_\varepsilon-I_X\zeta_0\|_H^2=\|\zeta_\varepsilon-I_X\zeta_0\|_{H,\Omega\backslash\Omega_0}^2+\|\zeta_\varepsilon-I_X\zeta_0\|_{H,\Omega_0}^2.\tag{4.9}$$

Denoting by $v_H=I_X\zeta_0\in X_H^0$ and by $\hat v_H=\Pi_H(I_X\zeta_0)=I_V\zeta_0\in V_H^0$, we have for any $T\in\mathcal{M}_H\cap(\Omega\backslash\Omega_0)$

$$\begin{aligned}\|\zeta_\varepsilon-I_X\zeta_0\|_{0,T}&\le\|\,\nabla(\zeta_\varepsilon-\zeta_0-\varepsilon\chi_i\frac{\partial\zeta_0}{\partial x_i}+\varepsilon\theta_\varepsilon(\zeta_0))\,\|_{0,T}\\ &+\|\,\nabla(v_H-\hat v_H-\varepsilon\chi_i\frac{\partial\hat v_H}{\partial x_i}+\varepsilon\tilde\theta_\varepsilon(\hat v_H))\,\|_{0,T}+\|\,\nabla(\zeta_0-\hat v_H)\,\|_{0,T}\\ &+\|\,\nabla(\varepsilon\chi_i\frac{\partial}{\partial x_i}(\zeta_0-\hat v_H))\,\|_{0,T}+\|\,\varepsilon\nabla\theta_\varepsilon(\zeta_0)\,\|_{0,T}+\|\,\varepsilon\nabla\tilde\theta_\varepsilon(\hat v_H)\,\|_{0,T}.\end{aligned}\tag{4.10}$$

Then we have

$$\begin{aligned}\|\zeta_\varepsilon-I_X\zeta_0\|_{H,\Omega\backslash\Omega_0}&\le\|\,\nabla(\zeta_\varepsilon-\zeta_0-\varepsilon\chi_i\frac{\partial\zeta_0}{\partial x_i}+\varepsilon\theta_\varepsilon(\zeta_0))\,\|_{0,\Omega\backslash\Omega_0}\\ &+\Big(\sum_{T\in\Omega\backslash\Omega_0}\|\,\nabla(v_H-\hat v_H-\varepsilon\chi_i\frac{\partial\hat v_H}{\partial x_i}+\varepsilon\tilde\theta_\varepsilon(\hat v_H))\,\|_{0,T}^2\Big)^{1/2}+\|\,\nabla(\zeta_0-\hat v_H)\,\|_{0,\Omega\backslash\Omega_0}\\ &+\|\,\varepsilon\nabla\theta_\varepsilon(\zeta_0)\,\|_{0,\Omega\backslash\Omega_0}+\Big(\sum_{T\in\Omega\backslash\Omega_0}\|\,\nabla(\varepsilon\chi_i\frac{\partial}{\partial x_i}(\zeta_0-\hat v_H))\,\|_{0,T}^2\Big)^{1/2}\\ &+\Big(\sum_{T\in\Omega\backslash\Omega_0}\|\,\varepsilon\nabla\tilde\theta_\varepsilon(\hat v_H)\,\|_{0,T}^2\Big)^{1/2}\equiv I_1+I_2+\cdots+I_6.\end{aligned}\tag{4.11}$$

The first two terms have been bounded in Theorem 3.2 and Lemma 4.2 respectively. The third term is the standard interpolation error. Due to (3.8) and (3.17), we have

$$I_4 \leq \| \varepsilon \nabla \theta_\varepsilon(\zeta_0) \|_{0,\Omega} \leq C(\varepsilon + \sqrt{\varepsilon}). \tag{4.12}$$

The fifth term can be estimated from (3.17) as

$$I_5 \leq C(\varepsilon + H) \| \zeta_0 \|_{2,\Omega\setminus\Omega_0} \leq C(\varepsilon + H)/H_0.$$

Thanks to (4.5) and (3.17),

$$I_6 \leq C\frac{\varepsilon}{H} \| \nabla \hat{v}_H \|_{0,\Omega\setminus\Omega_0} \leq C\frac{\varepsilon}{H} \| \nabla \zeta_0 \|_{0,\Omega\setminus\Omega_0} \leq C\frac{\varepsilon}{H} |\ln H_0|.$$

Combining all the six terms together, we have

$$\|\zeta_\varepsilon - I_X\zeta_0\|_{H,\Omega\setminus\Omega_0} \quad \leq \quad C\Big(\frac{\varepsilon}{H_0}|\ln \varepsilon| + (\frac{\varepsilon}{H} + H)|\ln H_0| + \frac{H}{H_0} + \sqrt{\varepsilon}\Big). \tag{4.13}$$

For the second term on the right hand side of (4.9), we have

$$\begin{aligned} \|\zeta_\varepsilon - I_X\zeta_0\|_{H,\Omega_0} &\leq \| \nabla(\zeta_\varepsilon - \zeta_0 - \varepsilon\chi_i\frac{\partial \zeta_0}{\partial x_i} + \varepsilon\bar{\theta}_\varepsilon(\zeta_0)) \|_{0,\Omega_0} \\ &+ \Big(\sum_{T\in\Omega_0} \| \nabla(v_H - \hat{v}_H - \varepsilon\chi_i\frac{\partial \hat{v}_H}{\partial x_i} + \varepsilon\tilde{\theta}_\varepsilon(\hat{v}_H)) \|_{0,T}^2\Big)^{1/2} + \| \nabla(\zeta_0 - \hat{v}_H) \|_{0,\Omega_0} \\ &+ \| \varepsilon\nabla\bar{\theta}_\varepsilon(\zeta_0) \|_{0,\Omega_0} + \Big(\sum_{T\in\Omega_0} \| \nabla(\varepsilon\chi_i\frac{\partial}{\partial x_i}(\zeta_0 - \hat{v}_H)) \|_{0,T}^2\Big)^{1/2} \\ &+ \Big(\sum_{T\in\Omega_0} \| \varepsilon\nabla\tilde{\theta}_\varepsilon(\hat{v}_H)) \|_{0,T}^2\Big)^{1/2} \;\equiv\; \hat{I}_1 + \hat{I}_2 + \cdots + \hat{I}_6. \end{aligned} \tag{4.14}$$

The first term has been bounded in Theorem 3.2. Thanks to Lemma 4.2 and (3.22), we have

$$\hat{I}_2 \leq C\big(\frac{\varepsilon}{H} + H\big)\| \nabla\hat{v}_H \|_{0,\Omega_0} \leq C\big(\frac{\varepsilon}{H} + H\big)\| \nabla\zeta_0 \|_{0,\Omega_0} \leq C\big(\frac{\varepsilon}{H} + H\big)|\ln H_0|.$$

For the third and fifth terms, we have by (3.22)

$$\hat{I}_3 + \hat{I}_5 \leq C(H + \varepsilon)\| \zeta_0 \|_{2,\Omega_0} \leq C(H + \varepsilon)|\ln H_0|/H_0.$$

The fourth term can be estimated in the same way as in (4.11). Thanks to (3.8) and (3.22),

$$\begin{aligned} \hat{I}_4 \quad &\leq \quad \| \varepsilon\nabla\bar{\theta}_\varepsilon(\zeta_0) \|_{0,\Omega_0} \leq C(\varepsilon|\zeta_0|_{2,\Omega_0^\varepsilon} + \sqrt{\varepsilon H_0}\ |\zeta_0|_{1,\infty,\Omega_0^\varepsilon}) \\ &\leq \quad C\big(\frac{\varepsilon}{H_0}|\ln H_0| + \sqrt{\frac{\varepsilon}{H_0}}\big), \end{aligned} \tag{4.15}$$

where $\Omega_0^\varepsilon \subset \Omega_0$ is the ε-neighborhood of $\partial\Omega_0$ and (3.17) has been used to obtain $|\zeta_0|_{1,\infty,\Omega_0^\varepsilon} = |u_0 - G_0|_{1,\infty,\Omega_0^\varepsilon} \leq C/H_0$.

Due to (3.22), the sixth term can be bounded by

$$\hat{I}_6 \leq C\frac{\varepsilon}{H}\| \nabla\hat{v}_H \|_{0,\Omega_0} \leq C\frac{\varepsilon}{H}\| \nabla\zeta_0 \|_{0,\Omega_0} \leq C\frac{\varepsilon}{H}|\ln H_0|.$$

Then we obtain

$$\|\zeta_\varepsilon - I_X\zeta_0\|_{H,\Omega_0} \leq C\big(\frac{\varepsilon}{H_0}|\ln\varepsilon| + \sqrt{\frac{\varepsilon}{H_0}} + (\frac{\varepsilon}{H} + \frac{\varepsilon+H}{H_0})|\ln H_0|\big). \tag{4.16}$$

Hence, combining (4.8), (4.9), (4.13) and (4.16), the "conforming error" can be bounded by

$$\inf_{\chi_H\in Y_H^0}\|\zeta_\varepsilon - \chi_H\|_H \leq C\Big(\frac{\varepsilon}{H_0}|\ln\varepsilon| + \sqrt{\frac{\varepsilon}{H_0}} + (\frac{\varepsilon}{H} + \frac{\varepsilon+H}{H_0} + \frac{h}{\varepsilon})|\ln H_0|\Big). \tag{4.17}$$

We now turn to bound the "nonconforming error" in (4.6). Due to (2.4), for $\chi_H \in Y_H^0$,

$$\begin{aligned}\big(K_\varepsilon\nabla G_h, \nabla(\phi\hat{\chi}_H)\big)_{\Omega_0} - \sum_{T\in\mathcal{M}_H}\big(K_\varepsilon\nabla\zeta_\varepsilon, \nabla\chi_H\big)_T &= \big(K_\varepsilon\nabla G_h - K_\varepsilon\nabla G_\varepsilon, \nabla(\phi\hat{\chi}_H)\big)_{\Omega_0}\\ + \sum_{T\in\mathcal{M}_H}\big(K_\varepsilon\nabla\zeta_\varepsilon, \nabla(\bar{\chi}_H - \chi_H)\big)_T &+ \sum_{T\in\mathcal{M}_H}\big(K_\varepsilon\nabla\zeta_\varepsilon, \nabla(\hat{\chi}_H - \bar{\chi}_H)\big)_T,\end{aligned} \tag{4.18}$$

where $\bar{\chi}_H \in X_H^0$ is the counterpart of $\chi_H \in Y_H^0$. To bound the first term on the right hand side, recalling the definition of the cut-off function ϕ in (2.4), we have $|\nabla\phi| \leq CH_0^{-1}$, so for any $w_H \in V_H$,

$$\begin{aligned}\|\nabla(\phi w_H)\|_{0,\Omega_0} &\leq \frac{C}{H_0}|B(x_0,3H_0/4)\backslash B(x_0,H_0/2)|\,\|w_H\|_{0,\infty,\Omega_0} + \|\nabla w_H\|_{0,\Omega_0}\\ &\leq C|\ln H|^{1/2}\|\nabla w_H\|_{0,\Omega},\end{aligned}$$

where we have used the well known 2-d discrete interpolate inequality

$$\|w_H\|_{0,\infty,\Omega} \leq C|\ln H|^{1/2}\|\nabla w_H\|_{0,\Omega} \quad \text{for } w_H \in V_H.$$

Due to Lemma 4.4, we obtain

$$\big(K_\varepsilon\nabla G_h - K_\varepsilon\nabla G_\varepsilon, \nabla(\phi\hat{\chi}_H)\big)_{\Omega_0} \leq C\frac{h}{\varepsilon H_0}|\ln H|^{1/2}\|\nabla\hat{\chi}_H\|_{0,\Omega}. \tag{4.19}$$

The second term on the right hand side of (4.18) is the error due to the numerical approximation of the multiscale finite element basis. Thanks to Lemma 4.1,

$$\sum_{T\in\mathcal{M}_H}\big(K_\varepsilon\nabla\zeta_\varepsilon, \nabla(\bar{\chi}_H - \chi_H)\big)_T \leq C\|\nabla\zeta_\varepsilon\|_{0,\Omega}\ \|\nabla(\bar{\chi}_H - \chi_H)\|_H \leq C\frac{h}{\varepsilon}|\ln H_0|\|\nabla\hat{\chi}_H\|_{0,\Omega}, \tag{4.20}$$

where we have used the *a priori* estimate $\|\nabla\zeta_\varepsilon\|_{0,\Omega} \leq C|\ln H_0|$, which can be obtain by choosing $v = \zeta_\varepsilon$ in (2.4).

The third term on the right hand side of (4.18) can be estimated by the similar way as in [7], Section 3.1. Noting that $\hat{\chi}_H$ is piecewise linear, for any $T \in \mathcal{M}_H$, we have

$$\begin{aligned}\int_T K_\varepsilon\nabla\zeta_\varepsilon\nabla(\bar{\chi}_H - \hat{\chi}_H) &= \int_T K_\varepsilon\nabla\zeta_\varepsilon\nabla\Big(\bar{\chi}_H - \hat{\chi}_H + \varepsilon\chi_i(x,\frac{x}{\varepsilon})\frac{\partial\hat{\chi}_H}{\partial x_i}\Big)\mathrm{d}x\\ &- \int_T K_\varepsilon\frac{\partial\zeta_\varepsilon}{\partial x_j}\varepsilon\frac{\partial}{\partial x_j}\chi_i(x,y)\frac{\partial\hat{\chi}_H}{\partial x_i}\,\mathrm{d}x - \int_T K_\varepsilon\frac{\partial\zeta_\varepsilon}{\partial x_j}\frac{\partial}{\partial y_j}\chi_i(x,y)\frac{\partial\hat{\chi}_H}{\partial x_i}\,\mathrm{d}x.\\ &\equiv T_1 + T_2 + T_3.\end{aligned} \tag{4.21}$$

We have from Lemma 4.2,

$$\begin{aligned}\sum_{T\in\mathcal{M}_H}|T_1|+|T_2| &\le \sum_{T\in\mathcal{M}_H} C\big(\varepsilon+\frac{\varepsilon}{H}+H\big)\|\nabla\zeta_\varepsilon\|_{0,T}\|\,\nabla\hat\chi_H\,\|_{0,T}\\ &\le C\big(\varepsilon+\frac{\varepsilon}{H}+H\big)\|\nabla\zeta_\varepsilon\|_{0,\Omega}\|\,\nabla\hat\chi_H\,\|_{0,\Omega}. \end{aligned}\tag{4.22}$$

Denoting by $\zeta_1^\varepsilon=\zeta_0-\varepsilon\chi_j\frac{\partial\zeta_0}{\partial x_j}$, for any $T\in\mathcal{M}_H\cap\Omega\backslash\Omega_0$,

$$\begin{aligned}T_3=&-\int_T K_\varepsilon\frac{\partial}{\partial x_i}\Big(\zeta_\varepsilon-\zeta_1^\varepsilon-\varepsilon\theta_\varepsilon(\zeta_0)\Big)\frac{\partial}{\partial y_i}\chi_p(x,y)\frac{\partial\hat\chi_H}{\partial x_p}\,\mathrm{d}x\\ &-\int_T\Big(K_\varepsilon\frac{\partial\zeta_1^\varepsilon}{\partial x_i}-k_{ij}^*\frac{\partial\zeta_0}{\partial x_j}\Big)\frac{\partial}{\partial y_i}\chi_p(x,y)\frac{\partial\hat\chi_H}{\partial x_p}\,\mathrm{d}x-\int_T k_{ij}^*\frac{\partial\zeta_0}{\partial x_j}\frac{\partial}{\partial y_i}\chi_p(x,y)\frac{\partial\hat\chi_H}{\partial x_p}\,\mathrm{d}x\\ &-\int_T\varepsilon K_\varepsilon\frac{\partial\theta_\varepsilon(\zeta_0)}{\partial x_i}\frac{\partial}{\partial y_i}\chi_p(x,y)\frac{\partial\hat\chi_H}{\partial x_p}\,\mathrm{d}x\ \equiv\ T_{31}+T_{32}+T_{33}+T_{34}.\end{aligned}\tag{4.23}$$

For $T\in\mathcal{M}_H\cap\Omega_0$, $\theta_\varepsilon(\zeta_0)$ should be replaced by $\bar\theta_\varepsilon(\zeta_0)$ in the above formula. The terms T_{31} and T_{34} can be bounded by using Theorems 3.2 and 3.1. The term T_{33} can be estimated by using Lemma 4.3. Term T_{32} can be treated by the standard argument (*cf.* [12], Sect. 1.3, (1.48)). Hence we have

$$|T_{32}|+|T_{33}|\ \le\ C\varepsilon\big(|\zeta_0|_{2,T}+|\zeta_0|_{1,\infty,T}\big)\|\,\nabla\hat\chi_H\,\|_{0,T},$$

and by using (4.12) and (4.15),

$$\begin{aligned}\Big|\sum_{T\in\mathcal{M}_H}T_3\Big| &= \Big|\sum_{T\in\Omega\backslash\Omega_0}T_3+\sum_{T\in\Omega_0}T_3\Big|\le C\Big(\frac{\varepsilon}{H_0}|\ln\varepsilon|+\sqrt{\varepsilon}+\frac{\varepsilon}{H_0}+\frac{\varepsilon}{H_0H}\Big)\|\,\nabla\hat\chi_H\,\|_{0,\Omega\backslash\Omega_0}\\ &\quad+C\Big(\frac{\varepsilon}{H_0}|\ln\varepsilon|+\sqrt{\frac{\varepsilon}{H_0}}+\frac{\varepsilon}{H_0}|\ln H_0|+\frac{\varepsilon}{H}|\ln H_0|\Big)\|\,\nabla\hat\chi_H\,\|_{0,\Omega_0}.\end{aligned}\tag{4.24}$$

Thanks to Lemma 4.1 and and the stability result (4.3) of Lemma 4.2, we can deduce that for $h\ll\varepsilon\ll H\ll 1$,

$$\|\nabla\hat\chi_H\|_{0,\Omega}\le C\|\chi_H\|_H,\ \forall\chi_H\in Y_H^0.$$

So the "nonconforming error" can be bounded by

$$\begin{aligned}\sup_{\chi_H\in Y_H^0}\frac{\big|\big(K_\varepsilon\nabla G_h,\nabla(\phi\hat\chi_H)\big)_{\Omega_0}-\sum_{T\in\mathcal{M}_H}\big(K_\varepsilon\nabla\zeta_\varepsilon,\nabla\chi_H\big)_T\big|}{\|\chi_H\|_H}&\le C\Big(\frac{\varepsilon}{H_0}|\ln\varepsilon|+\sqrt{\frac{\varepsilon}{H_0}}\\ &+\frac{\varepsilon}{H_0H}+\Big(\varepsilon+H+\frac{\varepsilon}{H}+\frac{\varepsilon}{H_0}+\frac{h}{\varepsilon}\Big)|\ln H_0|\Big).\end{aligned}\tag{4.25}$$

Finally combining the conforming error (4.17) and nonconforming error (4.25) together, we complete the proof of Theorem 2.1. □

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