

CONVERGENCE AND QUASI-OPTIMAL COMPLEXITY OF A SIMPLE ADAPTIVE FINITE ELEMENT METHOD

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Abstract. We prove convergence and quasi-optimal complexity of an adaptive finite element algorithm on triangular meshes with standard mesh refinement. Our algorithm is based on an adaptive marking strategy. In each iteration, a simple edge estimator is compared to an oscillation term and the marking of cells for refinement is done according to the dominant contribution only. In addition, we introduce an adaptive stopping criterion for iterative solution which compares an estimator for the iteration error with the estimator for the discretization error.

Mathematics Subject Classification. 65N12, 65N15, 65N30, 65N50.

Received July 27, 2008. Revised April 17, 2009.
Published online August 21, 2009.

1. INTRODUCTION

The analysis of adaptive finite element methods has made important progress in recent years. Based on classical residual-based *a posteriori* error estimators [1,16,20] it has been shown by Dörfler *et al.* [14,15,17] that an adaptive mesh refinement algorithm converges towards the solution of the Poisson equation with homogeneous Dirichlet boundary conditions and given square-integrable forces f in a two-dimensional bounded domain Ω with piecewise linear boundary $\partial\Omega$:

$$-\Delta u = f \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega. \quad (1.1)$$

In [17] the importance of controlling oscillations in data not captured on a given finite element mesh is pointed out and a special local refinement algorithm ('newest-vertex-bisection') is used in order to control a term measuring the data oscillations.

An important further result is the estimation of the dimension of the adaptively constructed discrete spaces, first achieved for wavelet discretizations by Cohen *et al.* [12]. This result is extended to a modified version of the algorithm of [17], including an additional coarsening step, by Binev *et al.* in [6]. A further significant improvement has been achieved by Stevenson [19] who shows that the additional coarsening step is not necessary in order to prove optimal complexity.

Keywords and phrases. Adaptive finite elements, *a posteriori* error analysis, convergence of adaptive algorithms, complexity estimates.

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The importance of the last-mentioned results lays in the fact that they show optimal complexity of certain adaptive algorithms: if the solution of (1.1) can be approximated by a given discretization method on a given family of meshes at a certain rate (quotient of accuracy to number of unknowns), the iteratively constructed sequence of meshes will realize this rate up to a constant factor.

The results obtained in the cited works rely on a special treatment of the before-mentioned data oscillation term and the use of the newest-vertex-bisection algorithm. To overcome the need of this special refinement algorithm, which is cumbersome for practical purposes, is a difficult task, since the classical local lower bound, which is at the heart of convergence proofs, has to be avoided. A first result in this direction has been recently proven in [18], which does however not provide any information on speed of convergence nor complexity of the algorithm. More recently, in [10] the authors provide a convergence and complexity result for an adaptive algorithm avoiding the inner node property. The idea is to replace the local lower bound by a decrease estimate of the error estimator.

Our algorithm is based on an adaptive marking strategy comparing an oscillation term with a simple estimator, following [5] where this idea has been introduced in the context of the new vertex bisection and [2] for mixed finite elements. In the present work, we avoid the inner node property, which makes it possible to use other local refinement algorithms, supposed that they satisfy a complexity estimate stated below. With respect to the cited articles, we simplify the *a posteriori* error estimator by suppressing the volume residual. This is natural, since it is known for a long time that the volume term leads to overestimation in the case of lowest-order conforming finite elements [8,9]. The resulting algorithm is attractive in practice, since the obtained refinement will in general be dominated by the edge residuals.

In addition, we introduce a new adaptive stopping criterion for the iterative solution of the discrete system in each step of the mesh refinement algorithm. The idea of this stopping criterion is to compare the iteration error ε_{it} with the estimator for the discretization error. This strategy potentially saves important computing time, since criteria with a fixed small constant, say $\varepsilon_{it} \leq 10^{-8}$, often used in practice, require close to complete solution in each step. We analyze the resulting algorithm with incomplete iterative solution and prove convergence and optimal complexity.

Throughout this paper we work with families of shape regular triangular meshes in the sense of [11]. In order to deviate as less as possible from standard notation, we denote by h an element of a family of admissible meshes \mathcal{H} , and by u_h the corresponding finite element solution. The set of cells of mesh h is denoted by \mathcal{K}_h , and the set of interior edges by \mathcal{E}_h . In addition, the set of nodes is denoted as \mathcal{N}_h . As compared to standard notation in finite element literature, h denotes a mesh in a family of meshes \mathcal{H} and *not* a global maximal cell width.

The paper is organized as follows: In Section 2 we define the adaptive algorithm. In Section 3 we prove some lemmata concerning lower/upper local/global bounds which are used later. Although the techniques used here are well established in the literature, we give complete proofs since we need to elaborate the precise dependence on iterative solution for the purpose of the subsequent analysis. In Section 4 we prove geometrical convergence of the error of the adaptive algorithm, under natural assumptions. In Section 5 we prove an asymptotic estimate for the complexity of the sequence of generated meshes. Possible concrete forms of the stopping criterion are discussed in Section 6. In Section 7 we present some numerical experiments and some conclusions are drawn in Section 8.

Throughout the paper we use the following notation. We use the standard Sobolev space $H_0^1(\Omega)$ and write for $u \in H_0^1(\Omega)$ and $\omega \subset \Omega$ $|u|_{1,\omega} := (\int_{\omega} |\nabla u|^2 dx)^{1/2}$ and $|u|_1 = |u|_{1,\Omega}$. The $L^2(\omega)$ -scalar product is denoted by $\langle \cdot, \cdot \rangle_{\omega}$ with corresponding norm $\| \cdot \|_{\omega}$, omitting the subscript in case $\omega = \Omega$.

Furthermore, P^k denotes the polynomials of maximal degree $k \geq 0$.

2. DEFINITION OF THE ADAPTIVE ALGORITHM

We define the family of admissible meshes \mathcal{H} in the following recursive way. Starting from an initial mesh h_0 , we suppose that we have a local mesh refinement algorithm $\mathcal{R}_{loc}(h, \mathcal{F})$ which refines for a given mesh $h \in \mathcal{H}$

all marked edges $\mathcal{F} \subset \mathcal{E}_h$. The resulting mesh is supposed to be conforming and the local mesh refinement algorithm is supposed to satisfy the following assumption. Let us denote by $N_h = \#\mathcal{K}_h$ the number of cells of the mesh h . Then we require that:

Assumption 2.1. *Let $h_k, k = 0, \dots, n$ be a sequence of locally refined triangulations created by the local mesh refinement algorithm, starting from the initial mesh h_0 . Let $\mathcal{F}_k \subset \mathcal{E}_{h_k}, k = 0, \dots, n - 1$ be the collection of all marked edges in step k . Then h_n is uniformly shape regular and its shape regularity only depends on that of h_0 and for all $E \in \mathcal{F}_k$ all neighboring triangles of E are at least bisected once. Furthermore, there exists a mesh-independent constant C_0 such that*

$$N_{h_n} \leq N_{h_0} + C_0 \sum_{k=0}^{n-1} \#\mathcal{F}_k. \tag{2.1}$$

Assumption 2.1 and especially the complexity estimate (2.1) are known to be true for the newest vertex bisection algorithm, see Theorem 2.4 of [6] (where the set of marked cells instead of the set of marked edges is used).

Let $h \in \mathcal{H}$. We use the Courant finite element space

$$V_h := \{v_h \in H_0^1(\Omega) : v_h|_K \in P^1 \text{ for all } K \in \mathcal{K}_h\}.$$

The Ritz projection $u_h \in V_h$ is defined by

$$\langle \nabla u_h, \nabla v_h \rangle = \langle f, v_h \rangle \quad \forall v_h \in V_h. \tag{2.2}$$

In order to approximately solve the discrete system in (2.2), we use a multigrid algorithm, which is based on the sequence of nested spaces $V_1 \subset \dots \subset V_l \subset \dots \subset V_L$ with $V_L = V_h$ resulting from the hierarchical mesh refinement. The multigrid algorithm employs mesh transfer operators and a smoothing operator only acting on the locally refined meshes, see [7,21] for detailed definitions and proofs of the mesh-independent convergence of the resulting algorithms.

We denote by u_h^m the discrete function obtained from m iterations of the multigrid algorithm starting with the solution from the previous step of the mesh-refinement algorithm. In order to estimate the iteration error, we use an *a posteriori* error estimator for the definition of the stopping criterion of the multigrid iteration. Let $\zeta_h(u_h^m)$ be an estimator satisfying the upper bound

$$\|u_h - u_h^m\|_1^2 \leq C_{mg} \zeta_h^2(u_h^m). \tag{2.3}$$

Several estimators satisfying (2.3) are possible, see Section 6 for a detailed discussion. Our results on convergence and complexity of the adaptive algorithm do not depend on the concrete form of the estimator but only (2.3).

Let $h \in \mathcal{H}$ and let ω_z be the set of cells joining a node $z \in \mathcal{N}_h$ and denote by π_ω the mean-value operator ($\pi_\omega(f) := \int_\omega f \, dx / |\omega|$). We define for given $z \in \mathcal{N}_h$ and $\mathcal{P} \subset \mathcal{N}_h$ an oscillation term

$$\text{osc}_z := |\omega_z|^{1/2} \|f - \pi_{\omega_z} f\|_{\omega_z}, \quad \text{osc}_h(\mathcal{P}) := \left(\sum_{z \in \mathcal{P}} \text{osc}_z^2 \right)^{1/2}. \tag{2.4}$$

Next we define for $E \in \mathcal{E}_h$ and any given subset $\mathcal{F} \subseteq \mathcal{E}_h$ the standard edge residuals for a given function $v_h \in V_h$:

$$J_E(v_h) := |E|^{1/2} \left\| \left[\frac{\partial v_h}{\partial n} \right] \right\|_E, \quad J_h(v_h, \mathcal{F}) := \left(\sum_{E \in \mathcal{F}} J_E^2(v_h) \right)^{1/2}, \tag{2.5}$$

where $[\cdot]_E$ denotes as usual the jump of a cell-wise polynomial function. We set for brevity $\text{osc}_h := \text{osc}_h(\mathcal{N}_h)$ and $J_h(v_h) := J_h(v_h, \mathcal{E}_h)$.

The purpose of this article is to analyze the following adaptive finite element algorithm:

Algorithm 1 \mathcal{AFEM}

- (0) Choose parameters $0 < \theta, \sigma < 1, \gamma > 0, \alpha > 0$ and an initial mesh h_0 , and set $k = 0$.
- (1) Do m iterations of the iterative solution algorithm applied to the discrete system (2.2) with h replaced by h_k to obtain the finite element solution u_k^m ; m is determined by the condition to be the smallest integer verifying:

$$\zeta_{h_k}^2(u_k^m) \leq \alpha J_{h_k}^2(u_k^m). \tag{2.6}$$

- (2) Compute the *a posteriori* error estimator J_{h_k} and oscillation term osc_{h_k} .
- (3) – If $\text{osc}_{h_k}^2 \leq \gamma J_{h_k}^2(u_k^m)$ then mark a set $\mathcal{F} \subset \mathcal{E}_{h_k}$ with minimal cardinality such that

$$J_{h_k}^2(\mathcal{F}) \geq \theta J_{h_k}^2 \tag{2.7}$$

- else find a set $\mathcal{P} \subset \mathcal{N}_{h_k}$ with minimal cardinality such that

$$\text{osc}_{h_k}^2(\mathcal{P}) \geq \sigma \text{osc}_{h_k}^2 \tag{2.8}$$

and define \mathcal{F} to be the set of edges containing at least one node in \mathcal{P} .

- (4) Adapt the mesh: $h_{k+1} := \mathcal{R}_{\text{loc}}(h_k, \mathcal{F})$.
 - (5) Set $k := k + 1$ and go to step (1).
-

Remark 2.2. The refinement is only governed by the oscillation term, if it is big compared to the estimator, following the idea of [5]. Therefore, in most practical cases, the edge residuals alone dominate the error estimation, such as suggested in the work of Carstensen and Verfürth [9].

Remark 2.3. The choice of parameters can be guided by our theoretical results. The stopping criterion α has to be small enough relative to θ in order to yield convergence; the other parameters θ, σ , and γ are arbitrary. The fact that γ is arbitrary for our convergence result indicates that the edge residuals play the dominant role in the overall refinement.

In order to achieve optimal complexity, in addition to the condition on α , the marking parameter θ has to be small enough and γ has to satisfy a condition, whereas σ is free.

Such a condition on θ is known from other complexity estimates [19]. In [6], $\theta = 1$ is possible, but an additional coarsening step is used.

3. SOME TECHNICAL LEMMATA

For the purpose of the later convergence and complexity proofs, we collect here some lemmata concerning upper and lower bounds of the estimators. Although the techniques used here are well established in the literature, see [8–10,17,19], we give complete proofs since the subsequent analysis requires to elaborate the precise dependence on iterative solution.

Lemma 3.1 (upper bounds). *Let $h \in \mathcal{H}$. There exists a constant $C_1 > 0$ depending only on the minimum angle of h_0 such that for $u_h \in V_h$ the solution of (2.2) and arbitrary $w_h \in V_h$*

$$|u - w_h|_1^2 \leq C_1 (J_h^2(w_h) + \text{osc}_h^2) + 2 |u_h - w_h|_1^2. \tag{3.1}$$

Suppose in addition that $H \in \mathcal{H}$ and $\mathcal{F} \subset \mathcal{E}_H$ are such that $h = \mathcal{R}_{\text{loc}}(H, \mathcal{F})$. Letting $\mathcal{P} \subset \mathcal{N}_H$ the set of nodes included in \mathcal{F} and $u_H \in V_H$ the discrete solution, we have

$$|u_h - w_H|_1^2 \leq C_1 (J_H^2(w_H, \mathcal{F}) + \text{osc}_H^2(\mathcal{P}) + |u_H - w_H|_1^2) \quad \forall w_H \in V_H, \tag{3.2}$$

and

$$\#\mathcal{F} \leq C_3 (N_h - N_H). \tag{3.3}$$

Proof. The global upper bound (3.1) with $w_h = u_h$ has been proven by Carstensen in [8] introducing a weighted Clément-type quasi-interpolation operator, denoted by C_h . The generalization to $w_h \neq u_h$ follows from the triangle inequality.

For the proof of the localized upper bound (3.2) we make use of the operator C_h which allows for the local interpolation estimates, known from the Clément-operator. First, there exists a mesh-independent constant C_{int} such that

$$\max(|v - C_h v|_K, |K|^{1/4} |v - C_h v|_{\partial K}) \leq C_{\text{int}} |K|^{1/2} |v|_{1, \omega_K},$$

where ω_K is the union of the neighboring elements. In addition, C_h is constructed in such a way that, with another mesh independent constant C_{orth} ,

$$\langle f, v - C_h v \rangle \leq C \sum_{z \in \mathcal{N}_H} |\omega_z| |f - \pi_{\omega_z} f|_{\omega_z} |v|_{1, \omega_z} \leq C_{\text{orth}} \text{osc}_H |v|_1.$$

Let now $v_H \in V_H$ denote the quasi-interpolant of $v_h = u_h - w_H \in V_h$. Then the support of $v_h - v_H$ is located to the locally refined region augmented by one layer of elements. Therefore we obtain the bound (3.3) with \mathcal{F} to be defined as the set of edges belonging to the support of $v_h - v_H$:

$$\begin{aligned} |u_h - w_H|_1^2 &= \langle \nabla(u_h - w_H), \nabla(v_h - v_H) \rangle + \langle \nabla(u_H - w_H), \nabla v_H \rangle \\ &\leq \langle f, v_h - v_H \rangle - \langle \nabla w_H, \nabla(v_h - v_H) \rangle + |u_H - w_H|_1 |v_H|_1 \\ &\leq (C_{\text{orth}} \text{osc}_H(\mathcal{P}) + C_{\text{int}} J_H(w_H, \mathcal{F})) |v_h|_1 + 2C_{\text{stab}} |u_H - w_H|_1 |v_h|_1, \end{aligned}$$

where C_{stab} is the H_0^1 -stability constant of the weighted Clément operator. We conclude by setting $C_1 = \max(C_{\text{orth}}, C_{\text{int}}, 2C_{\text{stab}})$. \square

The next lemma concerns lower bounds of the error. Whereas the global bound has a standard form, the local bound is different from the one commonly used for convergence proofs, since the inner node property is not assumed here.

Lemma 3.2 (lower bounds). *There exists a constant $C_2 > 0$ depending only on the minimum angle of h_0 such that for all $v_H \in V_H$*

$$J_H^2(v_H) \leq C_2 (|u - v_H|_1^2 + \text{osc}_H^2). \tag{3.4}$$

There exists a constant $C_4 > 0$ depending only on the minimum angle of h_0 such that for $\mathcal{F} \subset \mathcal{E}_H$, $h = \mathcal{R}_{\text{loc}}(H, \mathcal{F})$ and arbitrary $\delta > 0$

$$J_h^2(v_h) \leq (1 + \delta) J_H^2(v_H) - \frac{1 + \delta}{2} J_H^2(v_H, \mathcal{F}) + C_4 (1 + 1/\delta) |v_h - v_H|_1^2 \quad \forall v_h \in V_h, v_H \in V_H. \tag{3.5}$$

Proof. The global local bound (3.4) is obtained using the standard techniques of [20] as follows. First we recall that for a given triangle $K \in \mathcal{K}_H$ and given edge $E \in \mathcal{E}_H$ there exist a cubic bubble function $b_K \in H_0^1(K)$ and a quadratic bubble function $b_E \in H_0^1(\omega_E)$, where ω_E denotes the union of the neighboring cells of E . Taking the test function $w_E := [\frac{\partial v_H}{\partial n}] b_E$ we have with positive constants $C, c > 0$ and $f_K = \pi_K f$ that

$$\begin{aligned} c \left\| \left[\frac{\partial v_H}{\partial n} \right] \right\|_E^2 &\leq \int_E \left[\frac{\partial v_H}{\partial n} \right] w_E \, ds = \int_{\omega_E} \nabla v_H \nabla w_E \, dx = \int_{\omega_E} \nabla(v_H - u) \nabla w_E \, dx + \int_{\omega_E} f w_E \, dx \\ &= \int_{\omega_E} \nabla(v_H - u) \nabla w_E \, dx + \sum_{K \subset \omega_E} \int_{\omega_E} f_K w_E \, dx + \int_{\omega_E} (f - f_K) w_E \, dx \\ &\leq C \left(|E|^{-1/2} |u - v_H|_{1, \omega_E} + |E|^{1/2} \sum_{K \subset \omega_E} (\|f_K\|_K + \|f - f_K\|_K) \right) \left\| \left[\frac{\partial v_H}{\partial n} \right] \right\|_E, \end{aligned}$$

which yields

$$|E| \left\| \left[\frac{\partial v_H}{\partial n} \right] \right\|_E^2 \leq C \left(|u - v_H|_{1,\omega_E}^2 + \sum_{K \subset \omega_E} |K| \|f - f_K\|_K^2 + \sum_{K \subset \omega_E} |K| \|f_K\|_K^2 \right). \tag{3.6}$$

In order to estimate the last term in (3.6), we choose the test function $w_K := f_K b_K$, yielding

$$\int_K \nabla(u - v_H) \cdot \nabla w_K \, dx = \int_K f w_K \, dx = \int_K f_K w_K \, dx + \int_K (f - f_K) w_K \, dx,$$

from which we deduce that

$$c \|f_K\|_K^2 \leq C \left(|K|^{-1/2} |u - v_H|_{1,K} + \|f - f_K\|_K \right) \|f_K\|_K,$$

and therefore

$$|K| \|f_K\|_K^2 \leq C \left(|u - v_H|_{1,K}^2 + |K| \|f - f_K\|_K^2 \right).$$

Combining the last inequality with (3.6) and making use of the estimate

$$\sum_{K \in \mathcal{K}_H} |K| \|f - f_K\|_K^2 \leq C \operatorname{osc}_H^2 \tag{3.7}$$

gives (3.1). The bound (3.7) follows from the fact that for a node $z \in \mathcal{N}_H$, it holds $\sum_{K \subset \omega_z} \|f - f_K\|_K^2 \leq \sum_{K \subset \omega_z} \|f - \pi_z f\|_K^2$ and that the number of cells sharing a node is bounded for the family of meshes.

Next we prove (3.5), following [10]. We divide the edges created by the local refinement into three sets. The first set consists of newly created edges in the interior of a coarse cell. For such an edge E we have that

$$|E| \int_E \left[\frac{\partial v_h}{\partial n} \right]^2 \, ds = |E| \int_E \left[\frac{\partial(v_h - v_H)}{\partial n} \right]^2 \, ds \leq C |v_h - v_H|_{1,\omega_E}^2.$$

The second set consists of edges obtained from dividing a coarse edge E into two new edges, $E = E_1 \cup E_2$. For a coarse edge from this set, we have by Young’s inequality

$$\begin{aligned} & |E_1| \int_{E_1} \left[\frac{\partial v_h}{\partial n} \right]^2 \, ds + |E_2| \int_{E_2} \left[\frac{\partial v_h}{\partial n} \right]^2 \, ds \\ & \leq (1 + \delta) \frac{|E|}{2} \int_E \left[\frac{\partial v_H}{\partial n} \right]^2 \, ds + (1 + 1/\delta) \frac{|E|}{2} \int_E \left[\frac{\partial(v_h - v_H)}{\partial n} \right]^2 \, ds \\ & \leq \left(1 + \delta - \frac{1 + \delta}{2} \right) |E| \int_E \left[\frac{\partial v_H}{\partial n} \right]^2 \, ds + (1 + 1/\delta) \frac{|E|}{2} \int_E \left[\frac{\partial(v_h - v_H)}{\partial n} \right]^2 \, ds, \end{aligned}$$

yielding (3.5) in this case.

The last set is the set of unrefined edges. Here we have

$$\begin{aligned} |E| \int_E \left[\frac{\partial v_h}{\partial n} \right]^2 \, ds & \leq (1 + \delta) |E| \int_E \left[\frac{\partial v_H}{\partial n} \right]^2 \, ds + (1 + 1/\delta) |E| \int_E \left[\frac{\partial(v_h - v_H)}{\partial n} \right]^2 \, ds \\ & \leq (1 + \delta) |E| \int_E \left[\frac{\partial v_H}{\partial n} \right]^2 \, ds + C(1 + 1/\delta) |v_h - v_H|_{1,\omega_E}^2. \end{aligned}$$

This completes the proof. □

4. CONVERGENCE PROOF

We prove error reduction with respect to the following error measure:

$$e(h, m) := |u - u_h^m|_1^2 + \beta_1 \text{osc}_h^2 + \beta_2 J_h^2(u_h^m) \tag{4.1}$$

for some constants $\beta_1 > 0$ and $\beta_2 > 0$.

Theorem 4.1. *Assume that ζ satisfies (2.3). Let $\{h_k\}_{k \geq 0}$ be a sequence of meshes generated by algorithm \mathcal{AFEM} and let $\{u_{h_k}^{m_k}\}_{k \geq 0}$ be the corresponding sequence of finite element solutions. Suppose that*

$$0 \leq \alpha < \frac{\theta^2}{64C_4C_{mg}}, \tag{4.2}$$

then there exist constants $\beta_1 > 0$, $\beta_2 > 0$, and $\rho < 1$ such that for all $k = 1, 2, \dots$

$$e(h_{k+1}, m_{k+1}) \leq \rho e(h_k, m_k). \tag{4.3}$$

Remark 4.2. For the convergence result of Theorem 4.1, the parameters γ , $\theta < 1$, and $\sigma < 1$ can be chosen arbitrarily.

Proof. We set $\alpha' := \alpha C_{mg}$.

First we set $u_h := u_{h_{k+1}}$, $u_H := u_{h_k}$, $u_h^m = u_{h_{k+1}}^{m_{k+1}}$, $u_H^l = u_{h_k}^{m_k}$, $J_h = J_h(u_h^m)$, $J_H = J_H(u_H^l)$, $J_H(\mathcal{F}) := J_H(u_H^l, \mathcal{F})$ and similarly for the other involved quantities. We consider the two cases of the algorithm separately.

Let us start with the first case. We make use of the Galerkin property of the finite element discretization. Since for arbitrary $\xi > 0$ we have

$$\begin{aligned} \langle \nabla(u - u_h^m), \nabla(u_h^m - u_H^l) \rangle &= \langle \nabla(u_h - u_h^m), \nabla(u_h^m - u_H^l) \rangle \\ &\leq (1/2\xi)|u_h - u_h^m|_1^2 + \xi/2|u_h^m - u_H^l|_1^2, \end{aligned}$$

we find by means of the stopping criterion that

$$\begin{aligned} |u - u_h^m|_1^2 &= |u - u_H^l|_1^2 - |u_h^m - u_H^l|_1^2 - 2\langle \nabla(u - u_h^m), \nabla(u_h^m - u_H^l) \rangle \\ &\leq |u - u_H^l|_1^2 - (1 - \xi)|u_h^m - u_H^l|_1^2 + \frac{\alpha'}{\xi} J_h^2. \end{aligned}$$

Combining the last inequality with (3.5) and introducing $\beta'_2 := \beta_2 + \alpha'/\xi$ we obtain

$$\begin{aligned} |u - u_h^m|_1^2 + \beta_2 J_h^2 &\leq |u - u_H^l|_1^2 - (1 - \xi)|u_h^m - u_H^l|_1^2 + \beta'_2 J_h^2 \\ &\leq |u - u_H^l|_1^2 + (\beta'_2 C_4(1 + 1/\delta) - 1 + \xi) |u_h^m - u_H^l|_1^2 \\ &\quad + \beta'_2 \left((1 + \delta) J_H^2 - \frac{1 + \delta}{2} J_H^2(\mathcal{F}) \right), \end{aligned}$$

which leads to

$$\begin{aligned} e(h, m) &\leq |u - u_H^l|_1^2 + (\beta'_2 C_4(1 + 1/\delta) - 1 + \xi) |u_h^m - u_H^l|_1^2 + \beta_1 \text{osc}_h^2 \\ &\quad + \beta'_2 \left((1 + \delta) J_H^2 - \frac{1 + \delta}{2} J_H^2(\mathcal{F}) \right). \end{aligned} \tag{4.4}$$

Imposing the condition

$$\left(\beta_2 + \frac{\alpha'}{\xi} \right) C_4(1 + 1/\delta) - 1 + \xi \leq 0, \tag{4.5}$$

and using the refinement criterion, we have

$$e(h, m) \leq |u - u_H^l|_1^2 + \beta_1 \operatorname{osc}_h^2 + \beta_2' \left((1 + \delta) J_H^2 - \frac{\theta(1 + \delta)}{2} J_H^2 \right).$$

We next introduce $\theta' = (1 + \delta)\theta/2 - \delta$ and parameters $a, b > 0$ to be chosen later.

$$\begin{aligned} e(h, m) &\leq |u - u_H^l|_1^2 + \beta_1 \operatorname{osc}_h^2 + \beta_2'(1 - \theta') J_H^2 \\ &\leq |u - u_H^l|_1^2 - a\theta' \beta_2' J_H^2 + \beta_1 \operatorname{osc}_h^2 - (1 - a - b)\theta' \beta_2' J_H^2 + \beta_2'(1 - b\theta') J_H^2. \end{aligned}$$

Using the upper bound we have $|u - u_H^l|_1^2 \leq (C_1 + 2\alpha')(J_H^2 + \operatorname{osc}_H^2)$. Taking into account $\operatorname{osc}_H^2 \leq \gamma J_H^2$ we find

$$\begin{aligned} e(h, m) &\leq \left(1 - \frac{a\beta_2'\theta'}{C_1 + 2\alpha'} \right) |u - u_H^l|_1^2 + (\beta_1 + a\theta' \beta_2') \operatorname{osc}_H^2 \\ &\quad - (1 - a - b)\theta' \beta_2' J_H^2 + \beta_2'(1 - b\theta') J_H^2 \\ &\leq \left(1 - \frac{a\beta_2'\theta'}{C_1 + 2\alpha'} \right) |u - u_H^l|_1^2 + \beta_1 \left(1 + \theta' \frac{\beta_2'}{\beta_1} \left(a - \frac{1 - a - b}{\gamma} \right) \right) \operatorname{osc}_H^2 \\ &\quad + \beta_2'(1 - b\theta') J_H^2 \\ &\leq \rho_1 |u - u_H^l|_1^2 + \rho_2 \beta_1 \operatorname{osc}_H^2 + \rho_3 \beta_2 J_H^2 \\ &\leq \rho^1 e(H, l), \end{aligned}$$

with $\rho^1 = \max(\rho_1, \rho_2, \rho_3)$ and

$$\rho_1 = 1 - a\beta_2'\theta'(C_1 + 2\alpha')^{-1}, \quad \rho_2 = 1 + \theta' \frac{\beta_2'}{\beta_1} \left(a - \frac{1 - a - b}{\gamma} \right), \quad \rho_3 = (\beta_2'/\beta_2)(1 - \theta'b).$$

We now show that under the condition (4.2) we have $0 < \rho_i < 1$ for $i = 1, 2, 3$.

First we notice that (4.2) implies, since $1 - x < \sqrt{1 - x} \leq 1 - x/2 - x^2/8$ for $0 < x < 1$ (which we apply with $x = \theta/2$),

$$\alpha' < \frac{\theta^2}{64C_4} \leq \frac{1 - \theta/4 - \sqrt{1 - \theta/2}}{2C_4} < \frac{1 - \theta/4 - (1 - \theta/2)}{2C_4} = \frac{\theta}{8C_4}. \tag{4.6}$$

Next we consequently choose the parameters δ, b , and β_2 . First consider the quadratic equation

$$\delta^2 + A\delta + B = 0, \quad A = \frac{4\alpha'C_4 - \theta/2}{1 - \theta/2}, \quad B = \frac{4\alpha'C_4}{1 - \theta/2}. \tag{4.7}$$

By (4.6) we have $A < 0$ and $A^2/4 - B > 0$. Therefore (4.7) has a positive root and this implies existence of $\delta^* > 0$ such that

$$(1 - \theta/2)\delta^{*2} + (4\alpha'C_4 - \theta/2)\delta^* < -4\alpha'C_4. \tag{4.8}$$

We choose $\delta < \min(\delta^*, \frac{\theta/2}{1 - \theta/2})$. (4.8) implies

$$\frac{\alpha'C_4(1 + 1/\delta)}{\theta'} = \frac{\alpha'C_4(1 + 1/\delta)}{\theta/2 - \delta(1 - \theta/2)} < \frac{1}{4}. \tag{4.9}$$

Due to (4.9) the quadratic equation

$$x - x^2 = \frac{\alpha'C_4(1 + 1/\delta)}{\theta'} \tag{4.10}$$

has two roots satisfying $0 < \xi_1 < \xi_2 < 1$. Choose $\xi_1 < \xi < \xi_2$, such that

$$\xi - \xi^2 > \frac{\alpha' C_4(1 + 1/\delta)}{\theta'}. \tag{4.11}$$

Direct manipulation leads to

$$\frac{1 - \theta' \alpha'}{\theta' \xi} < \frac{1 - \xi}{C_4(1 + 1/\delta)} - \frac{\alpha'}{\xi}. \tag{4.12}$$

We can therefore choose $\beta_2 > 0$ such that

$$\frac{1 - \theta' \alpha'}{\theta' \xi} < \beta_2 < \frac{1 - \xi}{C_4(1 + 1/\delta)} - \frac{\alpha'}{\xi}. \tag{4.13}$$

The left inequality in (4.13) leads to

$$\theta' > \frac{\alpha'/\xi}{\beta_2 + \alpha'/\xi} = 1 - \frac{\beta_2}{\beta_2 + \alpha'/\xi} = 1 - \frac{\beta_2}{\beta_2'}. \tag{4.14}$$

This allows us to choose b such that

$$1 > b > \left(1 - \frac{\beta_2}{\beta_2'}\right) / \theta', \tag{4.15}$$

which implies

$$\rho_3 = \frac{\beta_2'}{\beta_2}(1 - b\theta') < 1. \tag{4.16}$$

The right inequality in (4.13) leads to

$$\beta_2' C_4(1 + 1/\delta) + \xi \leq 1,$$

which is (4.5).

It remains to choose $a < \frac{1-b}{1+\gamma}$ to obtain $\rho_2 < 1$. Since for the indicated choices, it holds $\rho_1 < 1$, we have completed the proof in the first case. Notice that $\beta_1 > 0$ is arbitrary up to now.

Now we consider the second case of the algorithm. We have the following property concerning the oscillation term involving a constant $0 < \mu < 1$:

$$\text{osc}_H^2 - \text{osc}_h^2 \geq \mu \text{osc}_H^2(\mathcal{P}), \tag{4.17}$$

which follows from the facts that the number of triangles containing a given node $z \in \mathcal{N}_h$ is bounded and that the marked cells are at least bisected once, see [18] for more details.

The inequality (4.17) implies $\text{osc}_h^2 \leq (1 - \mu\sigma) \text{osc}_H^2$. With the same choice of β_2 , ξ , and δ as before, it follows from (4.8) under the condition (4.5) that:

$$\begin{aligned} e(h, m) &\leq |u - u_H^l|_1^2 + (\beta_2' C_4(1 + 1/\delta) - 1 + \xi) |u_h^m - u_H^l|_1^2 + \beta_1 \text{osc}_h^2 \\ &\quad + \beta_2' \left((1 + \delta) J_H^2 - \frac{1 + \delta}{2} J_H^2(\mathcal{F}) \right) \\ &\leq |u - u_H^l|_1^2 + \beta_1 \text{osc}_h^2 + \beta_2'(1 + \delta) J_H^2 \\ &\leq |u - u_H^l|_1^2 - a\beta_1\mu\sigma \text{osc}_H^2 + \beta_1(\text{osc}_H^2 - (1 - a - b)\mu\sigma \text{osc}_H^2) \\ &\quad + \beta_2'(1 + \delta) J_H^2 - b\beta_1\mu\sigma \text{osc}_H^2 \\ &\leq \left(1 - a \frac{\beta_1\mu\sigma\gamma}{C_1(1 + \gamma) + 2\alpha'} \right) |u - u_H^l|_1^2 + \beta_1(1 - (1 - a - b)\mu\sigma) \text{osc}_H^2 \\ &\quad + \beta_2 \left(\left(1 + \frac{\alpha'}{\beta_2\xi} \right) (1 + \delta) - b \frac{\beta_1}{\beta_2} \gamma \mu \sigma \right) J_H^2 \\ &\leq \rho^{(2)} e(H, l). \end{aligned}$$

We choose a, b such that $1 - a - b > 0$ and β_1 such that

$$\beta_1 > \frac{\beta'_2(1 + \delta)}{b\gamma\mu\sigma}. \tag{4.18}$$

It follows that $0 < \rho^{(2)} < 1$. We conclude the proof by setting $\rho = \max(\rho^1, \rho^{(2)})$. □

5. COMPLEXITY ESTIMATE

In order to express the optimal complexity, we introduce some notation from nonlinear approximation theory, developed in [6,13]. Let \mathcal{H}_N be the set of all triangulations h which satisfy $N_h \leq N$.

Next we define the approximation class

$$\mathcal{W}^s := \left\{ (u, f) \in (H_0^1(\Omega), L^2(\Omega)) : \|(u, f)\|_{\mathcal{W}^s} < +\infty \right\} \tag{5.1}$$

with

$$\|(u, f)\|_{\mathcal{W}^s} := \sup_{N \geq N_0} N^s \inf_{h \in \mathcal{H}_N} \left(|u - u_h|_1^2 + \text{osc}_h^2 \right).$$

We say that an adaptive finite element method realizes optimal convergence rates if, whenever $(u, f) \in \mathcal{W}^s$, it produces a triangulation h_k with dimension N_k and corresponding approximation u_k such that

$$|u - u_k|_1^2 \leq CN_k^{-s}. \tag{5.2}$$

Theorem 5.1. *Suppose $(f, u) \in \mathcal{W}^s$. Let $\{h_k\}_{k \geq 0}$ be a sequence of meshes generated by algorithm \mathcal{AFEM} and let $\{V_k\}_{k \geq 0}$ and $\{u_k\}_{k \geq 0}$ be the corresponding sequences of finite element spaces and solutions. Let $\varepsilon_k := |u - u_k^{m_k}|_1^2 + \text{osc}_k^2$ and $N_k = \dim(V_k)$. Assuming the parameters γ and θ to satisfy*

$$0 < \gamma < 1/(C_2(1 + C_1)), \quad 0 < \theta < \frac{1 - \gamma C_2(1 + C_1)}{C_2(C_1 + C_{mg}\alpha)}, \tag{5.3}$$

we have the following estimate on the complexity of the algorithm: there exists a constant C such that

$$N_k \leq C \varepsilon_k^{-1/s}. \tag{5.4}$$

Remark 5.2. In case of exact solution, $\alpha = 0$, and zero data oscillation $\text{osc}_h = 0$ (and consequently $\gamma = 0$), we get the condition $\theta < 1/(C_1 C_2)$. This condition states that the percentage of marked cells should not be larger than the inverse of the efficiency index of the estimator $C_1 C_2$. Such a condition seems natural, since $\theta = 1$ corresponds to uniform refinement.

Proof. We use the same notation as in the convergence proof. As before, we consider the two cases separately.

First let $\text{osc}_H^2 \leq \gamma J_H^2$. From our regularity assumption we have existence of a mesh $h^* \in \mathcal{H}$ such that for $\lambda > 0$ to be chosen below

$$|u - u_{h^*}|_1^2 + \text{osc}_{h^*}^2 \leq \lambda (|u - u_H^l|_1^2 + \text{osc}_H^2) \tag{5.5}$$

and

$$N_{h^*} \leq C (|u - u_H^l|_1^2 + \text{osc}_H^2)^{-1/s}. \tag{5.6}$$

Following the proof of Stevenson [19] (proof of Lem. 5.2), we can suppose that h^* is a refinement of H , if we replace (5.6) by:

$$N_{h^*} - N_H \leq (|u - u_H^l|_1^2 + \text{osc}_H^2)^{-1/s}. \tag{5.7}$$

Let $\mathcal{F}^* \subset \mathcal{E}_H$ be the set of refined edges and let \mathcal{P}^* be the set of corresponding nodes. We now prove that

$$J_H^2(\mathcal{F}^*) \geq \theta J_H^2. \tag{5.8}$$

From (3.2) and the stopping criterion we have, since $|u - u_{h^*}|_1^2 = |u - u_H^l|_1^2 - |u_H^l - u_{h^*}|_1^2$,

$$\begin{aligned} (C_1 + \alpha')J_H^2(\mathcal{F}^*) &\geq |u_{h^*} - u_H^l|_1^2 - C_1 \text{osc}_H^2(\mathcal{P}^*) = |u - u_H^l|_1^2 - |u - u_{h^*}|_1^2 - C_1 \text{osc}_H^2(\mathcal{P}^*) \\ &\geq |u - u_H^l|_1^2 - \lambda (|u - u_H^l|_1^2 + \text{osc}_H^2) + \text{osc}_{h^*}^2 - C_1 \text{osc}_H^2(\mathcal{P}^*) \\ &\geq (1 - \lambda) (|u - u_H^l|_1^2 + \text{osc}_H^2) - \text{osc}_H^2 - C_1 \text{osc}_H^2(\mathcal{P}^*) \\ &\geq \frac{1 - \lambda}{C_2} J_H^2 - (1 + C_1) \text{osc}_H^2 \\ &\geq \left(\frac{1 - \lambda}{C_2} - \gamma(1 + C_1) \right) J_H^2. \end{aligned}$$

Choosing $\lambda = 1 - C_2(C_1 + \alpha')\theta - \gamma C_2(1 + C_1)$, which is positive by assumption, we obtain the desired bound (5.8).

Since \mathcal{F} is chosen to be the set with minimal cardinality satisfying the bound (5.8), we find that

$$\#\mathcal{F}_k \leq \#\mathcal{F}^* \leq N_{h^*} - N_k \leq \varepsilon_k^{-1/s}. \tag{5.9}$$

Now we consider the second case. Similarly as before, there exists a mesh h^* such that with λ to be chosen below

$$|u - u_{h^*}|_1^2 + \text{osc}_{h^*}^2 \leq \lambda (|u - u_H^l|_1^2 + \text{osc}_H^2) \tag{5.10}$$

and

$$N_{h^*} - N_h \leq C (|u - u_H^l|_1^2 + \text{osc}_H^2)^{-1/s}. \tag{5.11}$$

It follows from (5.10) and (3.1) that

$$\begin{aligned} \text{osc}_H^2 - \text{osc}_H^2(\mathcal{P}^*) &\leq \text{osc}_{h^*}^2 \leq \lambda (|u - u_H^l|_1^2 + \text{osc}_H^2) \\ &\leq \lambda ((C_1 + \alpha')(J_H^2 + \text{osc}_H^2) + \text{osc}_H^2) \\ &\leq \lambda \left(1 + (C_1 + \alpha') \left(1 + \frac{1}{\gamma} \right) \right) \text{osc}_H^2. \end{aligned}$$

From this we obtain

$$\text{osc}_H^2(\mathcal{P}^*) \geq \left(1 - \lambda \left(1 + (C_1 + \alpha') \left(1 + \frac{1}{\gamma} \right) \right) \right) \text{osc}_H^2.$$

We choose $\lambda = \frac{1 - \sigma}{1 + (C_1 + \alpha')(1 + \frac{1}{\gamma})}$ to obtain

$$\text{osc}_H^2(\mathcal{P}^*) \geq \sigma \text{osc}_H^2. \tag{5.12}$$

With the same argument as before we get (5.9).

Let now $e_l := |u - u_{h_l}^{m_l}|_1^2 + \beta_1 \text{osc}_{h_l}^2 + \beta_2 J_{h_l}^2(u_{h_l}^{m_l})$. From Theorem 4.1 we know that with a constant $0 < \rho < 1$ there holds

$$e_k \leq \rho^{k-l} e_l, \quad k \geq l \geq 0.$$

Due to the definitions and the first lower bound (3.4) of Lemma 3.2 e_k and ε_k are equivalent. Then we also have with a constant C depending on β_1, β_2 that

$$\varepsilon_k \leq C \rho^{k-l} \varepsilon_l, \quad k \geq l \geq 0. \tag{5.13}$$

The estimate (5.13) together with Assumption 2.1 imply

$$\begin{aligned} N_{k+1} - N_0 &\leq C \sum_{l=0}^k \#\mathcal{F}_k \leq C \sum_{l=0}^k \varepsilon_l^{-1/s} \\ &\leq C \varepsilon_k^{-1/s} \sum_{l=0}^k (\varepsilon_k/\varepsilon_l)^{1/s} \leq C \varepsilon_k^{-1/s} \left(\sum_{l=0}^k \rho^{(k-l)/s} \right) \leq \frac{C}{1 - \rho^{1/s}} \varepsilon_k^{-1/s}. \end{aligned}$$

This concludes the proof. □

By Theorem 5.1, the number of unknowns is asymptotically optimal. The convergence analysis of multigrid on the type of meshes considered here, see [7,21], can be used to check that the stopping criterion of the multigrid iteration can be fulfilled with linear work count (supposing we use the nested algorithm). We note that it also has already been discussed in the literature, see [14], how the logarithmic factor required in the complete sorting of the estimator can be avoided.

Corollary 5.3. *The algorithm AFEM can be implemented with optimal work count in the sense that for a given accuracy $\varepsilon > 0$, the algorithm provides a discrete solution u_h satisfying $|u - u_h|_1 \leq \varepsilon$ with a number of operations proportional to $\varepsilon^{-1/2s}$.*

6. STOPPING CRITERION

One way to obtain an *a posteriori* error estimator satisfying (2.3) is to follow the idea of [4].

The hierarchical refinement leads to a sequence of nested spaces $V_1 \subset \dots \subset V_l \subset \dots \subset V_L$ with $V_L = V_h$. Let the discrete operators $A_l : V_l \rightarrow V_l$ be defined by $\langle A_l u_l, v_l \rangle = \langle \nabla u_l, \nabla v_l \rangle$ for all $v_l \in V_l$ and $Q_l : L^2(\Omega) \rightarrow V_l$ be the $L^2(\Omega)$ -projection.

Let u_h^m be the approximate solution of the multigrid iteration after m iterations. We define with $h_l \approx 2^{-l}$

$$\zeta_h(u_h^m) := \sum_{l=1}^L h_l \| (Q_l - Q_{l-1})R^m \| + \| Q_0 R^m \|, \quad R^m := Q_h f - A_h u_h^m. \tag{6.1}$$

Using the relation $Q_l A_h = A_l P_l$ where P_l is the Ritz-projection on V_l , it can be shown that the residual terms appearing in the definition of (6.1) can be related to the residuals of the multigrid iteration, see [4]. As compared to the estimator proposed there, (6.1) seems to be more appropriate on locally refined meshes. In order to prove that (2.3) is satisfied, let $w := u_h - u_h^m$. We then have

$$\begin{aligned} |u_h - u_h^m|_1^2 &= \langle \nabla(u_h - u_h^m), \nabla w \rangle = \langle Q_h f - A_h u_h^m, w \rangle \\ &= \sum_{l=1}^L \langle R^m, (Q_l - Q_{l-1})w \rangle + \langle R^m, Q_0 w \rangle \\ &= \sum_{l=1}^L \langle (Q_l - Q_{l-1})R^m, w \rangle + \langle Q_0 R^m, w \rangle \\ &\leq C \left(\sum_{l=1}^L h_l \| (Q_l - Q_{l-1})R^m \| + \| Q_0 R^m \| \right) |w|_1 \end{aligned}$$

by the use of the Clément interpolation and the Poincaré inequality.

Another way to derive such an *a posteriori* error estimator, which might also be valid for other iterative methods, is given next. Suppose that an *a priori* estimate of the form

$$|u - u_h^{m+1}|_1 \leq \rho_{mg} |u - u_h^m|_1, \tag{6.2}$$

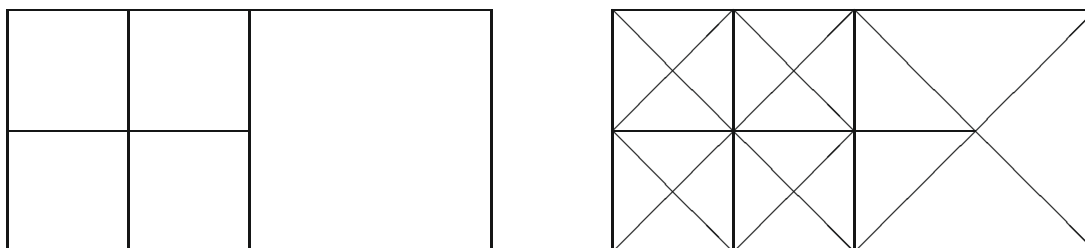


FIGURE 1. Refinement creating a hanging node and a hanging nodes and corresponding conforming refinement for triangles.

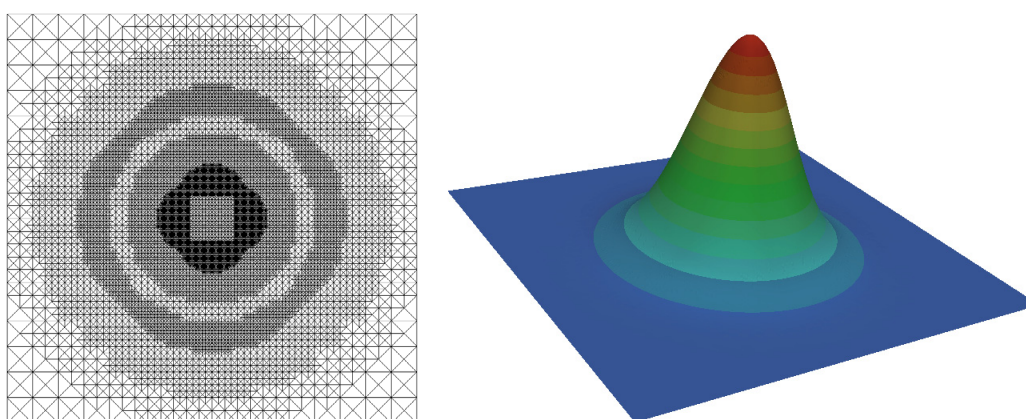


FIGURE 2. Typical mesh and plot of exact solution for Example 1.

with $\rho_{mg} < 1$ holds. Then we define a simple estimator by

$$\zeta_h(u_h^m) := \frac{\rho_{mg}}{1 - \rho_{mg}} |u_h^{m+1} - u_h^m|_1. \tag{6.3}$$

Notice that (6.3) is easy to implement, since it is at hand in standard implementations of the multigrid algorithm and other iterative solvers.

7. NUMERICAL EXPERIMENTS

We present two numerical experiments. The first one has a smooth solution and is used to examine the behavior of the algorithm with respect to the different involved parameters. It is also used to compare with other algorithms known from the literature. The second example employs a solution which is not in $H^2(\Omega)$. It illustrate the complexity estimate by showing that the optimal error decrease can be recovered by our adaptive algorithm.

For both examples we use $\Omega =]-1, 1[\times]-1, 1[$. The local mesh refinement uses quadrilateral meshes with hanging nodes which are cut into triangles as indicates in Figure 1. A proof of Assumption 1 for this algorithm as well as further details can be found in [3].

In the first case, we use $u(x) = \exp(-10|x|^2)$ and $f(x) = -\Delta u(x)$. We employ inhomogeneous Dirichlet data, the generalization of the estimator is immediate. This example has been treated before in [17]. A typical mesh is shown in Figure 2.

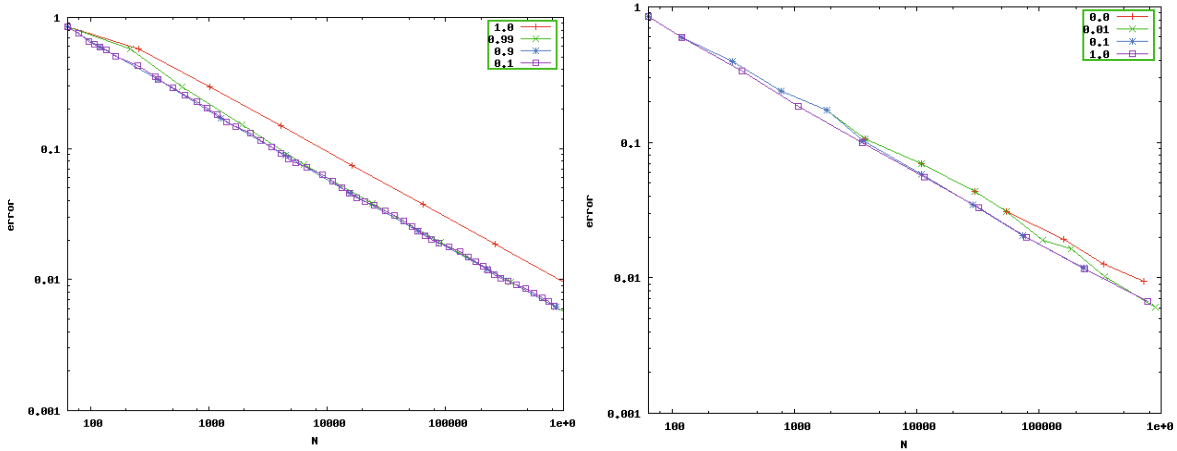


FIGURE 3. Example 1: Error *versus* N for different values of θ (left, $\gamma = 0.1$) and γ (right, $\theta = 0.8$).

We examine the behaviour of the algorithm with respect to the implied parameters. The important parameters are θ and γ . For simplicity, we set throughout $\sigma = \theta$ and $\alpha = 0$. The ratio between the true error $e = \|\nabla(u - u_h)\|$ and the estimator varied between 0.5 and 0.288 in all cases, and between 0.292 and 0.288 for all computations with $N \geq 500$.

The quality of the generated meshes for different values of θ are shown in the left picture of Figure 3. Since $u \in H^2(\Omega)$, all plots have the optimal slope $s = 1/2$. We have only represented $\theta = 0.9$ and the extreme values $\theta = 0.99$ and $\theta = 0.1$. It can be noticed that the behaviour is hardly influenced by the value of θ , as long as $\theta < 1.0$. In this case (adaptive refinement), there is a gain in the number of unknowns of a factor 2.5–3 as compared to uniform refinement ($\theta = 1.0$), which is explained by a better constant in $e(N) \sim N^{-s}$.

In Table 1 we make a comparison of our algorithm with the MNS algorithm [17]. We have chosen $\gamma = 0.1$ and $\theta = 0.7$. We denote by M_η and M_{osc} the number of marked cells due to the estimator and the oscillation term, respectively. As can be seen from Table 1, in contrast to the MNS algorithm, the algorithm presented in this article leads to refinement according to the oscillation term only during the first steps. In the following iterations, the refinement is only done according to the jump estimator, which reflects the fact that the oscillations are of higher-order accuracy. Notice that for practical purposes, one would use a larger value of θ and an even larger value for σ . The choice of $\theta = 0.9$ and $\sigma = 0.95$ lead to a reduction of the number of adaptive iterations from 13 to for the same final accuracy in our tests.

Our second numerical example is intended to examine the complexity estimate. We compute the Poisson equation in the L-shaped domain $\Omega =]-1, 1[\times]-1, 1[\setminus [0, 1] \times [0, 1]$ and with right-hand side $f = \text{div } g$. The vector $g = (g_1, g_2)$ is given by $g_1 = -1$ if $x > -0.5$ and $g_1 = 1$ otherwise, and $g_2 = -1$ if $y > -0.5$ and $g_2 = 1$ otherwise. A plot of the solution is shown in Figure 4.

Although the presented theory cannot be applied directly, since $f \notin L^2(\Omega)$, we consider this as an interesting example to see, if the optimal slope $s = -1/2$ can be reached by the adaptive algorithm. The problem presents a singularity along the lines of discontinuity of g , as well as the corner singularity. Both are resolved to a certain extent by the adaptive mesh refinement, as can be seen from Figure 4.

Finally, we show the decrease of the estimator for different values of θ in the right plot of Figure 5. It turns out that slope $s = -1/2$ can be recovered for $\theta = 0.9$. For $\theta = 0.1$ the complexity is not improved, but the number of iterations required to achieve $\eta_h < 0.1$ is increased from 12 to 74. The choice of $\theta = 0.99$ leads to less favorable behavior, but the optimal slope seems to be recovered when the mesh is fine enough.

The number of nodes is approximately doubled during the adaptive refinement for $\theta \approx 0.9$, which could be expected from isotropic refinement in two dimensions of a one-dimensional singularity. A much better behavior could be expected for this example from anisotropic refinement, but this is out of the present scope.

TABLE 1. Comparison between MNS algorithm (columns 2–4, from [17]) and the present one (columns 5–7, $\gamma = 0.1$).

Iter	N	M_η	M_{osc}	N	M_η	M_{osc}
0	4	8	0	4	0	4
1	64	16	16	16	0	12
2	704	56	8	54	0	12
3	2256	80	0	94	0	8
4	4208	96	8	120	0	24
5	6624	112	24	202	0	80
6	8752	344	0	492	0	208
7	17512	432	0	1224	644	0
8	28368	608	0	3588	1796	0
9	42896	768	16	9396	3304	0
10	60216	2192	0	19466	11476	0
11	113040	2296	24	54388	28356	0
12	160592	3816	24	141144	48360	0

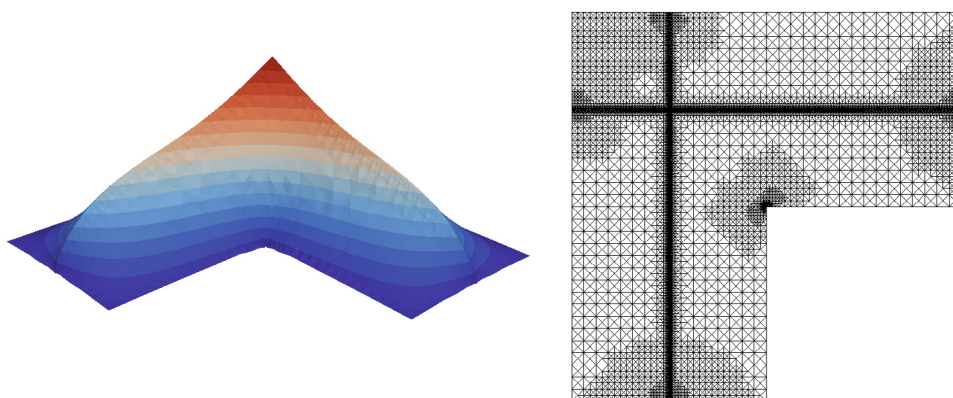


FIGURE 4. Typical mesh and plot of exact solution for Example 2.

8. CONCLUSION

We have proposed a new adaptive algorithm for conforming finite elements based on an adaptive marking strategy and an adaptive stopping criterion for iterative solution.

In this article, we have carried out the proofs of geometric convergence of the error and asymptotic complexity of the resulting meshes in the case of lowest-order finite elements in two space dimensions. The generalization to three dimensions seems to be possible.

Numerical examples have been presented in order to compare the adaptive marking strategy with the collective marking known from the literature and in order to illustrate the complexity of the sequence of meshes generated by the adaptive algorithm for different values of the involved parameters. The second example illustrates the gain in adaptivity, but it also suggests the possible improvement by anisotropic refinement. However, the proof of convergence and complexity results for anisotropic adaptive finite element algorithms requires further consideration.

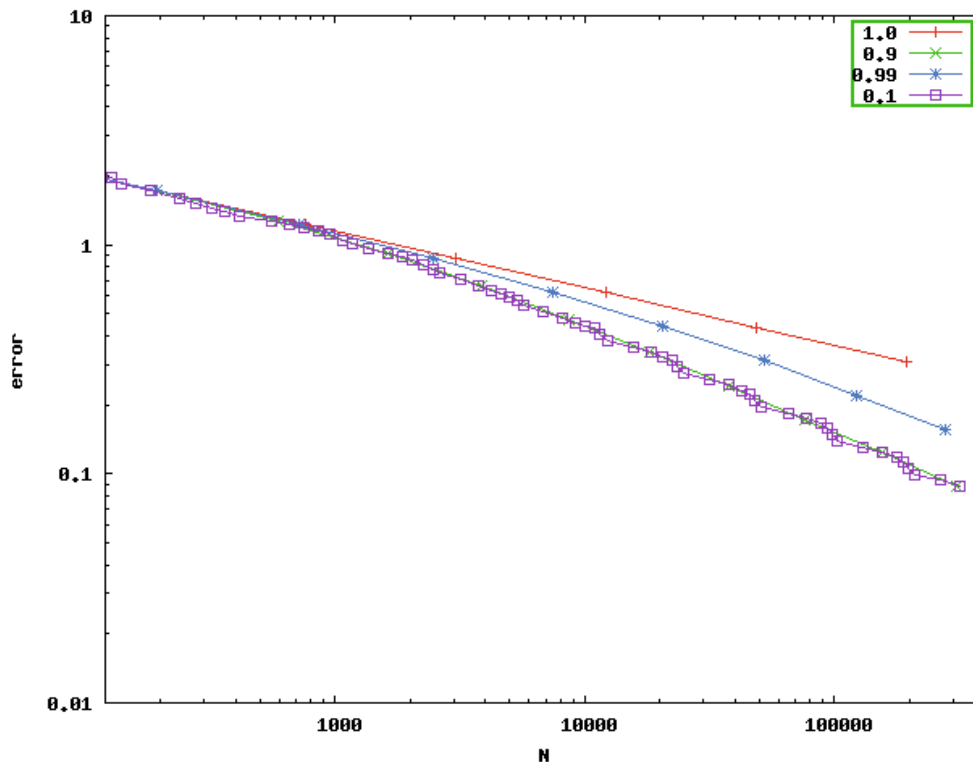


FIGURE 5. Decrease of the estimator for Example 2.

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