

ADAPTIVE FINITE ELEMENT METHODS FOR ELLIPTIC PROBLEMS: ABSTRACT FRAMEWORK AND APPLICATIONS

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Abstract. We consider a general abstract framework of a continuous elliptic problem set on a Hilbert space V that is approximated by a family of (discrete) problems set on a finite-dimensional space of finite dimension not necessarily included into V . We give a series of realistic conditions on an error estimator that allows to conclude that the marking strategy of bulk type leads to the geometric convergence of the adaptive algorithm. These conditions are then verified for different concrete problems like convection-reaction-diffusion problems approximated by a discontinuous Galerkin method with an estimator of residual type or obtained by equilibrated fluxes. Numerical tests that confirm the geometric convergence are presented.

Mathematics Subject Classification. 65N30, 65N15, 65N50.

Received June 26, 2008.

Published online February 4, 2010.

1. INTRODUCTION

The convergence of adaptive algorithms for elliptic boundary value problems approximated by a conforming FEM started with the papers of Babuška and Vogelius [5] in 1d and of Dörfler [12] in 2d. Since this time some improvements have been proved in order to take into account the data oscillations [23–25] or to prove optimal arithmetic works [8]. On the other hand, the discontinuous Galerkin method becomes recently very popular and is a very efficient tool for the numerical approximation of reaction-convection-diffusion problems for instance. Some *a posteriori* error analysis were performed recently, let us quote [1,2,7,10,14,16,18,20,21,26] for pure diffusion (or diffusion-dominated) problems and [9,13,16] for singularly perturbed problems (*i.e.*, dominant advection or reaction); for Maxwell system see for instance [17]. In all these papers, no convergence results are proved and to our knowledge, only the recent paper of Karakashian and Pascal [19] provides a convergence result for a purely diffusion problem.

Reading carefully the papers [12,19,23–25] we can remark some similarities in the convergence proof. Hence the goal of the present paper is threefold:

- Give an abstract framework as large as possible in order to contain the setting of the previous papers. In particular since DG methods use a stability parameter $\gamma > 0$, we assume that our variational form depends on such a parameter.

Keywords and phrases. *A posteriori* estimator, adaptive FEM, discontinuous Galerkin FEM.

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- In this setting, give a series of realistic conditions on an error estimator (inspired from the above mentioned references) that allows to prove that the marking strategy of bulk type leads to the geometric convergence of the adaptive algorithm. In this way the convergence proof becomes quite easy to understand since it is not hidden between some technicalities.
- Apply this framework to some new examples, like the ones from [9,10,16], in order to deduce the convergence of the associated adaptive algorithm. According to the two points before, the convergence result is reduced to check the above mentioned conditions.

We further hope that this framework will be applied to other *a posteriori* error estimators, like the ones from [13,16] where robust methods were analyzed for problems with dominant advection or reaction.

The schedule of the paper is as follows: we state in Section 2 the abstract framework, and prove the convergence of the adaptive algorithm under some realistic conditions. In the remainder of the paper we check the conditions from this section and deduce the convergence of the associated adaptive algorithm of bulk type for different estimators built for some finite element approximations of some specific boundary value problems. In Section 3, we apply our theory to diffusion problems approximated by a discontinuous Galerkin method and an error estimator based on equilibrated fluxes; some numerical tests are also presented that confirm the theoretical results. Finally Section 4 performs the same analysis for convection-diffusion-reaction problems approximated by a discontinuous Galerkin method and an error estimator of residual type.

2. AN ABSTRACT FRAMEWORK

Let V be a real Hilbert space with norm denoted by $\|\cdot\|_V$. Let a be a bilinear continuous form on V , which is coercive in the usual sense, namely

$$\exists \alpha > 0 : a(u, u) \geq \alpha \|u\|_V^2 \quad \forall u \in V. \quad (2.1)$$

Consider the standard variational problem: given $f \in V'$, let $u \in V$ be the unique solution of

$$a(u, v) = \langle f, v \rangle, \quad \forall v \in V. \quad (2.2)$$

This problem is approximated by a discrete family of problems set on a finite dimensional space V_h , $h > 0$, which we suppose to be nested:

$$V_H \subset V_h \quad \text{if } H > h,$$

and that approach V as h goes to zero. Note that we do not assume that V_h is a subspace of V , this allows us to consider non conforming approximation like discontinuous Galerkin methods for instance.

For all $h > 0$ and a family of parameters $\gamma > 0$, we assume given a norm $\|\cdot\|_{h,\gamma}$ well defined on $V + V_h$ and a family of bilinear form $a_{h,\gamma}$ also well defined on $V + V_h$ such that $a_{h,\gamma}$ is coercive on V_h , namely we assume that there exist $\gamma_0 > 0$ and $\alpha_0 > 0$ independent of h and γ such that

$$a_{h,\gamma}(v_h, v_h) \geq \alpha_0 \|v_h\|_{h,\gamma}^2 \quad \forall v_h \in V_h, \quad \forall \gamma \geq \gamma_0. \quad (2.3)$$

With these assumptions, for $\gamma \geq \gamma_0$, we can consider $u_h \in V_h$ solution of (for shortness, we do not specify the dependence of u_h with respect to γ)

$$a_{h,\gamma}(u_h, v_h) = \langle f, v_h \rangle, \quad \forall v_h \in V_h, \quad (2.4)$$

assuming that $f \in V'_h$.

If $a_{h,\gamma}$ would be coercive on $V + V_h$, then $a_{h,\gamma}(u - u_h, u - u_h)$ would dominate the error $\|u - u_h\|_{h,\gamma}^2$, since we do not assume this coerciveness, we would lose this property. Since it plays a key rule in our analysis, we assume that it holds: there exists $\alpha'_0 > 0$ independent of h , γ , u and u_h such that

$$a_{h,\gamma}(u - u_h, u - u_h) \geq \alpha'_0 \|u - u_h\|_{h,\gamma}^2 \quad \forall \gamma \geq \gamma_0. \quad (2.5)$$

Let us emphasize on the fact that we require (2.5) only for $u \in V$ the exact solution of (2.2) and its approximation $u_h \in V_h$ solution of (2.4).

Our goal is to show that under some basic assumptions (satisfied by a quite large family of approximated problems, see below) then refinement strategy based on the bulk criterion [12,23,24] described below leads to the convergence of the algorithm.

To describe this refinement strategy in our framework, for all $h > 0$ we suppose given a finite family \mathcal{T}_h of elements, called T . Formally this family \mathcal{T}_h allows to built the space V_h by using polynomial functions on the elements T of \mathcal{T}_h for instance. Now for all $T \in \mathcal{T}_h$ we assume that we have at our disposal an estimator $\eta_T(u_h)$ (that can be computed with the help of u_h) that measures the local error on T (hence $\eta_T(u_h)$ is a nonnegative real number) and for which we have the following upper bound: there exist two positive constants $C_1, c_1 > 0$ independent of h and γ such that

$$a_{h,\gamma}(u - u_h, u - u_h) \leq C_1 \eta_h^2 + c_1 \text{osc}_h^2, \tag{2.6}$$

where the first term of this right-hand side is the global error estimator which is the sum of local contributions, while the second term osc_h^2 is the so-called oscillation term that is also the sum of local contributions

$$\eta_h^2 = \sum_{T \in \mathcal{T}_h} \eta_T(u_h)^2, \quad \text{osc}_h^2 = \sum_{T \in \mathcal{T}_h} \text{osc}_h(T)^2.$$

Now the abstract refinement strategy (of bulk type, see [12,23,24]) can be expressed as follows:

Definition 2.1 (marking strategy). Given two parameters $0 < \theta_1, \theta_2 < 1$, the new family \mathcal{T}_h (allowing to build the new space V_h) is designed with the help of a subset $\hat{\mathcal{T}}_H$ of \mathcal{T}_H constructed such that

$$\sum_{T \in \hat{\mathcal{T}}_H} \eta_H(T)^2 \geq \theta_1^2 \eta_H^2, \tag{2.7}$$

$$\sum_{T \in \hat{\mathcal{T}}_H} \text{osc}_H(T)^2 \geq \theta_2^2 \text{osc}_H^2. \tag{2.8}$$

Note that these two conditions yield

$$\sum_{T \in \hat{\mathcal{T}}_H} (\eta_H(T)^2 + \text{osc}_H(T)^2) \geq \theta^2 (\eta_H^2 + \text{osc}_H^2), \tag{2.9}$$

with $\theta = \min\{\theta_1, \theta_2\}$.

We further assume that the next error reduction holds: there exist a positive constant C_2 and a non negative constant C_3 such that for two consecutive parameters $H > h$

$$\sum_{T \in \hat{\mathcal{T}}_H} \eta_T(u_H)^2 \leq C_2 (\|u_h - u_H\|_{h,\gamma}^2 + \text{osc}_H^2) + \frac{C_3}{\gamma} \|u - u_h\|_{h,\gamma}^2 \quad \forall \gamma \geq \gamma_0. \tag{2.10}$$

Note that such estimate is usually proved locally, the local version leading to the estimate (2.10) by superposition. For the proof of the convergence of the algorithm the global version is only necessary, moreover in some applications we have in mind (see Sect. 3.2 below), only the global version is available. Hence we have restricted ourselves to the global estimate.

For the oscillation terms, we make the assumption that it reduces from one step to another one with a factor < 1 up to the consecutive error, namely we assume that there exist constants $0 < \rho_1 < 1$ and $\rho_2 > 0$ independent of h and γ such that

$$\text{osc}_h^2 \leq \rho_1 \text{osc}_H^2 + \rho_2 a_{h,\gamma}(u_h - u_H, u_h - u_H) \quad \forall \gamma \geq \gamma_0. \tag{2.11}$$

We further assume that the error between u and u_H in the norm $a_{h,\gamma}$ or in the norm $a_{H,\gamma}$ are comparable, namely we assume that

$$\exists C_4 \geq 0 : \quad a_{h,\gamma}(u - u_H, u - u_H) \leq \left(1 + \frac{C_4}{\gamma}\right) a_{H,\gamma}(u - u_H, u - u_H) \quad \forall \gamma \geq \gamma_0. \tag{2.12}$$

Finally we suppose the quasi-orthogonality relation: there exist $h_0 > 0$ and $\Lambda_h > 0$ such that

$$\Lambda_h \rightarrow 1 \quad \text{as } h \rightarrow 0,$$

and

$$a_{h,\gamma}(u - u_h, u - u_h) \leq \Lambda_h a_{h,\gamma}(u - u_H, u - u_H) - a_{h,\gamma}(u_h - u_H, u_h - u_H) \quad \forall h < h_0, \gamma \geq \gamma_0. \tag{2.13}$$

Note that this last condition is satisfied if we have a Galerkin orthogonality relation and a symmetric form $a_{h,\gamma}$ as the next lemma shows:

Lemma 2.2. *Assume that the Galerkin orthogonality relation*

$$a_{h,\gamma}(u - u_h, v_h) = 0 \quad \forall v_h \in V_h, \tag{2.14}$$

holds and that $a_{h,\gamma}$ is symmetric

$$a_{h,\gamma}(u, v) = a_{h,\gamma}(v, u), \quad \forall u, v \in V + V_h.$$

Then (2.13) holds with $\Lambda_h = 1$.

Proof. By the symmetry property of $a_{h,\gamma}$ we see that

$$a_{h,\gamma}(u - u_H, u - u_H) - a_{h,\gamma}(u - u_h, u - u_h) - a_{h,\gamma}(u_h - u_H, u_h - u_H) = 2a_{h,\gamma}(u - u_h, u_h) - 2a_{h,\gamma}(u - u_h, u_H).$$

The conclusion follows from (2.14) because $u_H \in V_H \subset V_h$. □

Remark 2.3. For a method that does not use a parameter γ (like conforming methods for instance, see the end of this section or [9]) we can formally take $\gamma = +\infty$. In that case the results stated below remain valid and all terms of the form $\frac{C}{\gamma}$ with some positive constant C can be simply replaced by 0.

All these data allow to prove the convergence of the algorithm:

Theorem 2.4. *There exist a constant $\gamma_1 \geq \gamma_0 > 0$ sufficiently large, a mesh size h_0 sufficiently small and two constants $\kappa > 0$ and $0 < \mu < 1$ such that for all $h \leq h_0$ and $\gamma \geq \gamma_1$, one has*

$$a_{h,\gamma}(u - u_h, u - u_h) + \kappa \text{osc}_h^2 \leq \mu(a_{H,\gamma}(u - u_H, u - u_H) + \kappa \text{osc}_H^2), \tag{2.15}$$

where $h < H$ are two consecutive mesh parameters.

Proof. Using the error reduction estimate (2.10) and the coerciveness properties (2.3) and (2.5), we obtain

$$\sum_{T \in \mathcal{T}_H} \eta_T(u_H)^2 \leq C'_2(a_{h,\gamma}(u_h - u_H, u_h - u_H) + \text{osc}_H^2) + \frac{C'_3}{\gamma} a_{h,\gamma}(u - u_h, u - u_h), \tag{2.16}$$

with $C'_3 = C_3/\alpha'_0$.

Applying the upper bound (2.6) to the rough mesh parameter H , and secondly using the marking procedures (2.7) and (2.8), we have

$$\begin{aligned} \theta^2 a_{H,\gamma}(u - u_H, u - u_H) &\leq \theta^2 \left(C_1 \sum_{T \in \mathcal{T}_H} \eta_T(u_H)^2 + c_1 \text{osc}_H^2 \right) \\ &\leq C_1 \sum_{T \in \mathcal{T}_H} \eta_T(u_H)^2 + c_1 \sum_{T \in \mathcal{T}_H} \text{osc}_H(T)^2. \end{aligned}$$

Using now the estimate (2.16), we arrive at

$$\theta^2 a_{H,\gamma}(u - u_H, u - u_H) \leq C_1 C_2' a_{h,\gamma}(u_h - u_H, u_h - u_H) + C_5 \text{osc}_H^2 + \frac{C_1 C_3'}{\gamma} a_{h,\gamma}(u - u_h, u - u_h),$$

or equivalently

$$a_{h,\gamma}(u_h - u_H, u_h - u_H) \geq \frac{\theta^2}{C_1 C_2'} a_{H,\gamma}(u - u_H, u - u_H) - \frac{C_3'}{\gamma C_2'} a_{h,\gamma}(u - u_h, u - u_h) - C_6 \text{osc}_H^2. \quad (2.17)$$

Now using the quasi-orthogonality relation (2.13) and introducing a parameter $\beta \in (0, 1]$ fixed sufficiently small later on, we obtain

$$a_{h,\gamma}(u - u_h, u - u_h) \leq \Lambda_h a_{h,\gamma}(u - u_H, u - u_H) + (\beta - 1) a_{h,\gamma}(u_h - u_H, u_h - u_H) - \beta a_{h,\gamma}(u_h - u_H, u_h - u_H).$$

The last term of this right-hand side is estimated by invoking (2.17), and therefore

$$\begin{aligned} a_{h,\gamma}(u - u_h, u - u_h) &\leq \Lambda_h a_{h,\gamma}(u - u_H, u - u_H) + (\beta - 1) a_{h,\gamma}(u_h - u_H, u_h - u_H) \\ &\quad - \frac{\theta^2 \beta}{C_1 C_2'} a_{H,\gamma}(u - u_H, u - u_H) + \frac{C_3' \beta}{\gamma C_2'} a_{h,\gamma}(u - u_h, u - u_h) + \beta C_6 \text{osc}_H^2. \end{aligned}$$

Using the estimate (2.12), we arrive at

$$\begin{aligned} \left(1 - \frac{C_3' \beta}{\gamma C_2'} \right) a_{h,\gamma}(u - u_h, u - u_h) &\leq \left(\Lambda_h \left(1 + \frac{C_4}{\gamma} \right) - \frac{\theta^2 \beta}{C_1 C_2'} \right) a_{H,\gamma}(u - u_H, u - u_H) \\ &\quad + (\beta - 1) a_{h,\gamma}(u_h - u_H, u_h - u_H) + \beta C_6 \text{osc}_H^2. \end{aligned}$$

Choosing γ_1 large enough so that $1 - \frac{C_3' \beta}{\gamma C_2'} > 0$ for $\gamma \geq \gamma_1$, *i.e.*,

$$\gamma_1 \geq 1 + \frac{C_3' \beta}{C_2'}, \quad (2.18)$$

the last estimate is equivalent to

$$\begin{aligned} a_{h,\gamma}(u - u_h, u - u_h) &\leq \frac{\Lambda_h \left(1 + \frac{C_4}{\gamma} \right) - \frac{\theta^2 \beta}{C_1 C_2'}}{1 - \frac{C_3' \beta}{\gamma C_2'}} a_{H,\gamma}(u - u_H, u - u_H) \\ &\quad + \frac{\beta - 1}{1 - \frac{C_3' \beta}{\gamma C_2'}} a_{h,\gamma}(u_h - u_H, u_h - u_H) + \frac{\beta C_6}{1 - \frac{C_3' \beta}{\gamma C_2'}} \text{osc}_H^2. \end{aligned} \quad (2.19)$$

To take into account the oscillating terms, we multiply (2.11) by $\kappa := \frac{1-\beta}{\rho_2 \left(1 - \frac{C'_3\beta}{\gamma C'_2}\right)}$ and find

$$\kappa \operatorname{osc}_h^2 \leq \kappa \rho_1 \operatorname{osc}_H^2 + \frac{1-\beta}{1 - \frac{C'_3\beta}{\gamma C'_2}} a_{h,\gamma}(u_h - u_H, u_h - u_H). \tag{2.20}$$

The sum of the estimates (2.19) and (2.20) yields

$$a_{h,\gamma}(u - u_h, u - u_h) + \kappa \operatorname{osc}_h^2 \leq \frac{\Lambda_h \left(1 + \frac{C_4}{\gamma}\right) - \frac{\theta^2\beta}{C_1 C'_2}}{1 - \frac{C'_3\beta}{\gamma C'_2}} a_{H,\gamma}(u - u_H, u - u_H) + \left(\kappa \rho_1 + \frac{\beta C_6}{1 - \frac{C'_3\beta}{\gamma C'_2}}\right) \operatorname{osc}_H^2.$$

This estimate leads to the conclusion if we can chose γ_1 large enough as well as h_0 and β small enough so that there exists $0 < \mu < 1$ such that

$$\begin{aligned} \frac{\Lambda_h \left(1 + \frac{C_4}{\gamma}\right) - \frac{\theta^2\beta}{C_1 C'_2}}{1 - \frac{C'_3\beta}{\gamma C'_2}} &\leq \mu, \\ \kappa \rho_1 + \frac{\beta C_6}{1 - \frac{C'_3\beta}{\gamma C'_2}} &\leq \mu \kappa. \end{aligned}$$

These two estimates are equivalent to (using the definition of κ)

$$\Lambda_h \left(1 + \frac{C_4}{\gamma}\right) - \frac{\theta^2\beta}{C_1 C'_2} \leq \mu \left(1 - \frac{C'_3\beta}{\gamma C'_2}\right), \tag{2.21}$$

$$\rho_1 + \frac{C_6\beta\rho_2}{1-\beta} \leq \mu. \tag{2.22}$$

To guarantee the estimate (2.22), we simply chose β small enough such that

$$\rho_1 + \frac{C_6\beta\rho_2}{1-\beta} < 1,$$

which is equivalent to

$$\frac{\beta}{1-\beta} < \frac{1-\rho_1}{C_6\rho_2},$$

which is always possible since the left-hand side of this estimate tends to zero as β goes to zero.

Hence with such a choice of β , the estimate (2.22) holds with $1 > \mu \geq \rho_1 + \frac{C_6\beta\rho_2}{1-\beta}$.

Now we go on with the estimate (2.21), which is equivalent to

$$\Lambda_h \left(1 + \frac{C_4}{\gamma}\right) + \frac{C'_3\beta\mu}{\gamma C'_2} \leq \mu + \frac{\theta^2\beta}{C_1 C'_2}. \tag{2.23}$$

As $\mu < 1$ this estimate holds if

$$\Lambda_h + \frac{1}{\gamma} \left(\Lambda_h C_4 + \frac{C'_3\beta}{C'_2}\right) < \mu + \frac{\theta^2\beta}{C_1 C'_2} \tag{2.24}$$

is valid. As Λ_h tends to 1 as h goes to 0, we fix h_0 small enough such that for all $h \leq h_0$ we have

$$\Lambda_h \leq 1 + \frac{\theta^2\beta}{3C_1 C'_2}. \tag{2.25}$$

Similarly we fix γ_1 large enough in order to guarantee that

$$\frac{1}{\gamma} \left(\Lambda_h C_4 + \frac{C'_3 \beta}{C'_2} \right) \leq \frac{\theta^2 \beta}{3C_1 C'_2} \quad \forall \gamma \geq \gamma_1. \tag{2.26}$$

Indeed this estimate is equivalent to

$$\frac{\Lambda_h C_4 + \frac{C'_3 \beta}{C'_2}}{\theta^2 \beta} 3C_1 C'_2 \leq \gamma \quad \forall \gamma \geq \gamma_1,$$

and by (2.25) it holds if

$$\frac{\left(1 + \frac{\theta^2 \beta}{3C_1 C'_2} \right) C_4 + \frac{C'_3 \beta}{C'_2}}{\theta^2 \beta} 3C_1 C'_2 \leq \gamma_1. \tag{2.27}$$

This shows that (2.26) holds with γ_1 satisfying this estimate (2.27).

The estimates (2.25) and (2.26) lead to

$$\Lambda_h + \frac{1}{\gamma} \left(\Lambda_h C_4 + \frac{C'_3 \beta}{C'_2} \right) \leq 1 + \frac{2\theta^2 \beta}{3C_1 C'_2},$$

which shows (2.24) if $\mu > 1 - \frac{\theta^2 \beta}{3C_1 C'_2}$.

The proof is complete. □

Remark 2.5. In general, there is no reason that the minimal stability parameter γ_0 satisfies (2.18) and (2.27), hence the convergence is only guaranteed for a larger threshold parameter γ_1 . Nevertheless numerical tests below reveal that the reduction factor of the error (approximated value of μ) does not vary significantly with respect to γ . Note further that from the above proof we can see that if the constants C_3 in (2.10) and C_4 in (2.12) are equal to zero then we can chose $\gamma_1 = \gamma_0$.

The remainder of this paper is to prove the convergence of some adaptive methods applied to some approximated schemes of some boundary value problems; the convergence being obtained by checking the assumptions (2.3), (2.5), (2.6), (2.10), (2.11), (2.12) and (2.13).

For instance in the framework of the paper [23], reaction-convection-diffusion problems are approximated by subspaces $V_h \subset V$ and with $a_{h,\gamma} = a$. Hence the coerciveness assumptions (2.3), (2.5) follows directly from the coerciveness of a , (2.6) is standard (see [3,28]), (2.10) is proved in Lemma 3.1 of [23], (2.11) is proved in Lemma 3.2 of [23], (2.12) is immediate and finally Lemma 2.1 of [23] is devoted to the proof of (2.13).

Remark 2.6. As said before the two main applications of our abstract framework concern DG methods where γ is the usual jump penalty parameter. We do not try to apply our framework to other methods where some penalization parameters are used (like SUPG methods for instance).

3. A POSTERIORI ERROR ESTIMATORS FOR A DISCONTINUOUS GALERKIN METHOD FOR DIFFUSION PROBLEMS

Here we revisit the results from [10] and show the convergence of an adequate adaptive algorithm at least in dimension 2 by using some recent results from [19].

More precisely we consider the two-dimensional diffusion equation in a bounded domain Ω of \mathbb{R}^2 with a polygonal boundary Γ and homogeneous mixed boundary conditions:

$$\begin{aligned} -\operatorname{div}(a \nabla u) &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \Gamma_D, \\ a \nabla u \cdot n &= 0 \quad \text{on } \Gamma_N, \end{aligned} \tag{3.1}$$

where $\Gamma = \bar{\Gamma}_D \cup \bar{\Gamma}_N$ and $\Gamma_D \cap \Gamma_N = \emptyset$.

We suppose that a is piecewise constant, namely we assume that there exists a partition \mathcal{P} of Ω into a finite set of Lipschitz polygonal domains $\Omega_1, \dots, \Omega_J$ such that, on each Ω_j , $a = a_j$, where a_j is a positive constant. For simplicity, we assume that Γ_D has a non-vanishing measure. We further assume that Ω is simply connected and that Γ is connected.

The variational formulation of (3.1) involves the bilinear form

$$a(u, v) = \int_{\Omega} a \nabla u \cdot \nabla v.$$

Given $f \in L^2(\Omega)$, the weak formulation consists in finding $u \in H_D^1(\Omega) := \{u \in H^1(\Omega) : u = 0 \text{ on } \Gamma_D\}$ such that

$$a(u, v) = (f, v) = \int_{\Omega} f v, \quad \forall v \in H_D^1(\Omega). \tag{3.2}$$

Remark 3.1. In [10] we imposed non-homogeneous boundary conditions, for the sake of simplicity we have restrict ourselves to homogeneous boundary conditions, nevertheless all the results stated below holds for non-homogeneous boundary conditions

Here to approximate problem (3.1) (or more precisely its variational formulation (3.2)), we use a discontinuous Galerkin scheme. Following [4,18,19], we consider the following discontinuous Galerkin approximation of our continuous problem: we consider a triangulation \mathcal{T}_h made of triangles T whose edges are denoted by e and assume that this triangulation is shape-regular, *i.e.*, for any element T , the ratio h_T/ρ_T is bounded by a constant $\sigma > 0$ independent of $T \in \mathcal{T}_h$ and of mesh-size $h = \max_{T \in \mathcal{T}_h} h_T$, where h_T is the diameter of T and ρ_T the diameter of its largest inscribed ball. We further assume that \mathcal{T}_h is conforming with the partition \mathcal{P} of Ω , *i.e.*, any $T \in \mathcal{T}_h$ is included in one and only one Ω_i . With each edge e of the triangulation, we associate a fixed unit normal vector n_e , and n_T stands for the outer unit normal vector of T . For boundary edges $e \subset \partial\Omega \cap \partial T$, we set $n_e = n_T$. \mathcal{E}_h represents the set of edges of the triangulation, and we assume that the Dirichlet part of the boundary Γ_D can be written as union of edges. We also need to distinguish between edges included into Ω , Γ_D or Γ_N , in other words, we set

$$\begin{aligned} \mathcal{E}_{h,\text{int}} &= \{e \in \mathcal{E}_h : e \subset \Omega\}, \\ \mathcal{E}_{h,D} &= \{e \in \mathcal{E}_h : e \subset \Gamma_D\}, \\ \mathcal{E}_{h,N} &= \{e \in \mathcal{E}_h : e \subset \Gamma_N\}. \end{aligned}$$

For shortness, we also write $\mathcal{E}_{h,ID} = \mathcal{E}_{h,\text{int}} \cup \mathcal{E}_{h,D}$. In the sequel, a_T denotes the value of the piecewise constant coefficient a restricted to the element T . Finally for $T \in \mathcal{T}_h$, ω_T denotes the patch consisting of all the triangles of \mathcal{T}_h having a nonempty intersection with T . Similarly for an edge e , ω_e denotes the patch consisting of all the triangles of \mathcal{T}_h having e as edge.

In the following, the L^2 -norm on a domain D will be denoted by $\|\cdot\|_D$; the index will be dropped if $D = \Omega$. We use $\|\cdot\|_{s,D}$ and $|\cdot|_{s,D}$ to denote the standard norm and semi-norm on $H^s(D)$ ($s \geq 0$), respectively. The energy norm is defined by $\|v\|^2 = a(v, v)$, for any $v \in H^1(\Omega)$.

Problem (3.2) is approximated in the (discontinuous) finite element space:

$$V_h = \{v_h \in L^2(\Omega) | v_h|_T \in \mathbb{P}_l(T), T \in \mathcal{T}_h\}, \tag{3.3}$$

where l is a fixed positive integer. The space V_h is equipped with the norm

$$\|q\|_{h,\gamma} := \left(\|a^{1/2} \nabla_h q\|_{\Omega}^2 + \gamma \sum_{e \in \mathcal{E}_{h,1D}} h_e^{-1} \|\llbracket q \rrbracket\|_e^2 \right)^{1/2},$$

where γ is a positive parameter fixed below and for any $q \in V_h$, we define its broken gradient $\nabla_h q$ in Ω by

$$(\nabla_h q)|_T = \nabla q|_T, \forall T \in \mathcal{T}_h.$$

As usual we need to define some jumps and means through any $e \in \mathcal{E}_h$ of the triangulation. For $e \in \mathcal{E}_h$ such that $e \subset \Omega$, we denote by T^+ and T^- the two elements of \mathcal{T}_h containing e . Let $q \in V_h$, we denote by q^\pm , the traces of q taken from T^\pm , respectively. Then we define the mean of q on e by

$$\{\{q\}\} = \frac{q^+ + q^-}{2}.$$

For $v \in [V_h]^2$, we denote similarly

$$\{\{v\}\} = \frac{v^+ + v^-}{2}.$$

The jump of q on e is now defined as follows:

$$\llbracket q \rrbracket = q^+ n_{T^+} + q^- n_{T^-}.$$

Remark that the jump $\llbracket q \rrbracket$ of q is vector-valued.

For a boundary edge e , i.e., $e \subset \partial\Omega$, there exists a unique element $T^+ \in \mathcal{T}_h$ such that $e \subset \partial T^+$. Therefore the mean and jump of q are defined by $\{\{q\}\} = q^+$ and $\llbracket q \rrbracket = q^+ n_{T^+}$.

With these notations, we define the bilinear form $a_{h,\gamma}(\cdot, \cdot)$ as follows:

$$\begin{aligned} a_{h,\gamma}(u_h, v_h) := & \sum_{T \in \mathcal{T}_h} \int_T a \nabla u_h \cdot \nabla v_h - \sum_{e \in \mathcal{E}_{h,1D}} \int_e (\{\{a \nabla_h v_h\}\} \cdot \llbracket u_h \rrbracket + \{\{a \nabla_h u_h\}\} \cdot \llbracket v_h \rrbracket) \\ & + \gamma \sum_{e \in \mathcal{E}_{h,1D}} h_e^{-1} \int_e \llbracket u_h \rrbracket \cdot \llbracket v_h \rrbracket, \quad \forall u_h, v_h \in V_h, \end{aligned}$$

where the positive parameter γ is chosen large enough to ensure coerciveness of the bilinear form $a_{h,\gamma}$ on V_h (see, e.g., Lem. 2.1 of [18]).

The discontinuous Galerkin approximation of problem (3.2) reads now: find $u_h \in V_h$, such that

$$a_{h,\gamma}(u_h, v_h) = \int_{\Omega} f v_h, \forall v_h \in V_h. \tag{3.4}$$

3.1. The *a posteriori* error analysis based on Raviart-Thomas finite elements

Error estimators can be constructed in many different ways as, for example, using residual type error estimators which measure locally the jump of the discrete flux [18,19]. Here, introducing the flux $j = a \nabla u$ as auxiliary variable, we locally define an error estimator based on a $H(\text{div})$ -conforming approximation of this variable. Hence the discrete flux approximation j_h will be searched in a $H(\text{div})$ -conforming space RT_h based

on the Raviart-Thomas finite elements. This means that our error estimate of the conforming part of the error is based on the error between $a\nabla_h u_h$ and an approximating flux j_h of j that we search in the Raviart-Thomas finite element space

$$RT_h = \{v_h \in H(\operatorname{div}, \Omega) | v_h|_T \in RT_{l-1}(T), T \in \mathcal{T}_h\}.$$

For a similar approach where the fluxes are computed using local Neumann problems, see for instance [6,22].

On a triangle T , an element p of $RT_{l-1}(T)$ is characterized by the degrees of freedom given by

- $\int_e p \cdot n q, \quad \forall q \in \mathbb{P}_{l-1}(e), \forall e \subset \partial T,$
- $\int_T p \cdot q, \quad \forall q \in [\mathbb{P}_{l-2}(T)]^2.$

Therefore we fix the discrete flux j_h by setting

$$\int_e j_h \cdot n_T q = \int_e g_{T,e} q, \quad \forall q \in \mathbb{P}_{l-1}(e), \forall e \subset \partial T, \quad (3.5)$$

$$\int_T j_h \cdot q = \int_T a\nabla_h u_h \cdot q - a_T l_{\partial T}(q), \quad \forall q \in [\mathbb{P}_{l-2}(T)]^2, \quad (3.6)$$

where for all $e \subset \partial T$, $g_{T,e}$ is defined by

$$\begin{aligned} g_{T,e} &= (\{a\nabla_h u_h\} - \gamma h_e^{-1} \llbracket u_h \rrbracket) \cdot n_T & \text{if } e \in \mathcal{E}_{h,\text{int}}, \\ g_{T,e} &= a\nabla_h u_h \cdot n_T - \gamma h_e^{-1} u_h & \text{if } e \in \mathcal{E}_{h,D}, \\ g_{T,e} &= 0 & \text{if } e \in \mathcal{E}_{h,N}, \end{aligned}$$

and the linear form $l_{\partial T}$ is given by

$$l_{\partial T}(q) = \frac{1}{2} \sum_{e \subset \partial T \setminus \Gamma} \int_e \llbracket u_h \rrbracket \cdot q + \sum_{e \subset \partial T \cap \Gamma_D} \int_e u_h q \cdot n_T.$$

Denote by Π_{l-1} the L^2 -projection on $W_h = \{w_h \in L^2(\Omega) | w_h|_T \in \mathbb{P}_{l-1}(T), T \in \mathcal{T}_h\}$. Then, we have the following main property (see Lem. 3.1 of [10])

$$\operatorname{div} j_h = -\Pi_{l-1} f. \quad (3.7)$$

We now recall the estimator introduced in [10]: it consists in three parts: a conforming part that only involves the difference between the discrete flux approximation j_h and $a\nabla_h u_h$:

$$\eta_{CF,T}(u_h) = \|a^{-1/2} (a\nabla_h u_h - j_h)\|_T. \quad (3.8)$$

The nonconforming part is built by using the Oswald interpolation operator of u_h , namely the unique element $w_h \in V_h \cap H_D^1(\Omega)$ defined in the following natural way (see Thm. 2.2 of [18]): to each node n of the mesh in $\Omega \cup \Gamma_N$, corresponding to Lagrangian-type degree of freedom of $V_h \cap H_D^1(\Omega)$, the value of w_h is the average of the values of u_h at this node n , i.e., $w_h(n) = \frac{\sum_{n \in \mathcal{T}} |T| u_h|_T(n)}{\sum_{n \in \mathcal{T}} |T|}$. Then the non conforming estimator is simply

$$\eta_{NC,T}(u_h) = \|a^{1/2} \nabla(w_h - u_h)\|_T. \quad (3.9)$$

Finally we introduce the estimator corresponding to jumps of u_h :

$$\eta_{J,T}(u_h)^2 = \frac{1}{2} \sum_{e \in \mathcal{E}_{h,\text{int}} \cap T} \eta_{J,e}(u_h)^2 + \sum_{e \in \mathcal{E}_{h,D} \cap T} \eta_{J,e}(u_h)^2, \quad \eta_{J,e}(u_h)^2 = \frac{\gamma}{h_e} \|\llbracket u_h \rrbracket\|_e^2.$$

The estimator on T is then defined by

$$\eta_T(u_h)^2 = \eta_{CF,T}(u_h)^2 + \eta_{NC,T}(u_h)^2 + \eta_{J,T}^2(u_h)^2.$$

The oscillating terms depending on the datum f is defined as usual by

$$\text{osc}_h^2 = \sum_{T \in \mathcal{T}_h} h_T^2 a_T^{-1} \|f - \Pi_{l-1} f\|_T^2.$$

Now using the results from Section 2, we describe a convergent algorithm for the above estimators.

3.2. Checking the assumptions from Section 2

Since $V_h \not\subset V = H_D^1(\Omega)$ and $a_{h,\gamma} \neq a$, the coerciveness estimates (2.3) and (2.5) are not direct but they are respectively proved in Lemma 3.1 and Proposition 4.2 of [19]. The estimate (2.12) is proved as follows: first Proposition 4.1 of [19] shows that there exists $C > 0$ independent of h and γ such that

$$a_{h,\gamma}(u - u_H, u - u_H) \leq a_{H,\gamma}(u - u_H, u - u_H) + C\gamma \sum_{e \in \mathcal{E}_{h,TD}} h_e^{-1} \|\llbracket u_H \rrbracket\|_e^2.$$

But Theorem 3.2 of [19] guarantees that

$$\gamma \sum_{e \in \mathcal{E}_{h,TD}} h_e^{-1} \|\llbracket u_H \rrbracket\|_e^2 \leq \frac{C}{\gamma} \|\nabla_H(u - u_H)\|^2.$$

As Proposition 4.2 of [19] implies that

$$\|\nabla_H(u - u_H)\|^2 \leq 2a_{H,\gamma}(u - u_H, u - u_H),$$

for $\gamma \geq \gamma_1$ and γ_1 large enough (depending on the order l and on the shape regularity constant σ), the three above estimates lead to (2.12).

Since $a_{h,\gamma}$ is symmetric and (2.14) holds (see the identity (3.2) of [19]), the quasi-orthogonality estimate (2.13) holds with $\Lambda_h = 1$ due to Lemma 2.2.

The estimate (2.6) was proved in Theorem 3.4 of [10] since (2.5) holds.

The oscillation reduction estimate (2.11) follows from Lemma 3.2 of [23] if the marking strategy from Definition 2.1 is used and if the successive meshes are constructed *via* the procedure REFINE of Morin, Nochetto and Siebert [23–25].

It remains the error reduction estimate (2.10): first in the proof of Theorem 3.6 of [10] we see that

$$\eta_{CF,T}(u_H)^2 \leq C(a) \left(\sum_{e \subset T} h_T \|J_{e,n}(u_H)\|_e^2 + \frac{1}{\gamma} \eta_{J,T}^2(u_H) \right),$$

where $C(a)$ is a positive constant depending on a and the jump terms are the usual ones defined by

$$J_{e,n}(u_H) = \begin{cases} \llbracket a \nabla u_H \cdot n_e \rrbracket & \text{for interior edges of } \mathcal{T}_H, \\ 0 & \text{for Dirichlet boundary edges of } \mathcal{T}_H, \\ \nabla u_H \cdot n_e & \text{for Neumann boundary edges of } \mathcal{T}_H. \end{cases}$$

On the other hand using Theorem 2.2 of [18], there exists $C > 0$ independent of γ and H such that

$$\eta_{NC,T}(u_H)^2 \leq \frac{C}{\gamma} \eta_{J,T}^2(u_H).$$

These two estimates imply that

$$\sum_{T \in \mathcal{T}_H} \eta_T(u_H)^2 \leq C(a) \left(\sum_{e \in \mathcal{E}_H} h_T \|J_{e,n}(u_H)\|_e^2 + \sum_{e \in \mathcal{E}_{H,1D}} h_e^{-1} \|\llbracket u_H \rrbracket\|_e^2 \right),$$

for some $C(a) > 0$ depending only on a and shape regularity constant σ . The first term of this right hand side is a part of the estimator from [19] and by the estimate (4.31) of [19] we have

$$\sum_{e \in \mathcal{E}_H} h_T \|J_{e,n}(u_H)\|_e^2 \leq C \left(\|\nabla_h(u_h - u_H)\|^2 + \sum_{e \in \mathcal{E}_{h,1D}} h_e^{-1} \|\llbracket u_h \rrbracket\|_e^2 \right),$$

and consequently

$$\begin{aligned} \sum_{T \in \mathcal{T}_H} \eta_T(u_H)^2 &\leq C(a) \left(\|\nabla_h(u_h - u_H)\|^2 + \sum_{e \in \mathcal{E}_{H,1D}} h_e^{-1} \|\llbracket u_H \rrbracket\|_e^2 + \sum_{e \in \mathcal{E}_{h,1D}} h_e^{-1} \|\llbracket u_h \rrbracket\|_e^2 \right) \\ &\leq 2C(a) \left(\|\nabla_h(u_h - u_H)\|^2 + \sum_{e \in \mathcal{E}_{H,1D}} h_e^{-1} \|\llbracket u_H - u_h \rrbracket\|_e^2 + \sum_{e \in \mathcal{E}_{h,1D}} h_e^{-1} \|\llbracket u_h \rrbracket\|_e^2 \right). \end{aligned}$$

For the two first terms of this right-hand side using the definition of the norm $\|\cdot\|_{h,\gamma}$, we have

$$\|\nabla_h(u_h - u_H)\|^2 + \sum_{e \in \mathcal{E}_{H,1D}} h_e^{-1} \|\llbracket u_H - u_h \rrbracket\|_e^2 \leq C \|u_h - u_H\|_{h,\gamma}^2.$$

For the third term we notice that

$$\sum_{e \in \mathcal{E}_{h,1D}} h_e^{-1} \|\llbracket u_h \rrbracket\|_e^2 = \sum_{e \in \mathcal{E}_{h,1D}} h_e^{-1} \|\llbracket u - u_h \rrbracket\|_e^2 \leq \frac{1}{\gamma} \|u - u_h\|_{h,\gamma}^2.$$

The three above estimates lead to the estimate (2.10).

3.3. Some numerical tests

In order to illustrate the performance of our estimator η_h and the convergence of the adaptive algorithm, for two benchmark examples we show the meshes obtained after some iterations, as well as the experimental convergence orders of the error

$$EOC_e = 2 \frac{\lg \frac{\|u - u_H\|_{H,\gamma}}{\|u - u_h\|_{h,\gamma}}}{\lg \frac{DOF_h}{DOF_H}},$$

the effectivity indices

$$Eff = \eta_h / \|u - u_h\|_{h,\gamma}$$

and the reduction factors of the error (approximated value of the constant $\sqrt{\mu}$ appearing in Thm. 2.4)

$$RFE = \|u - u_h\|_{h,\gamma} / \|u - u_H\|_{H,\gamma},$$

calculated during the different iterations. We use the iterative algorithm of bulk type described in Definition 2.1 with $\theta_1 = \theta_2 = \theta = 0.75, 0.8$ or 0.9 and refine the triangles of \mathcal{T}_H by a standard refinement procedure with a limitation on the minimal angle.

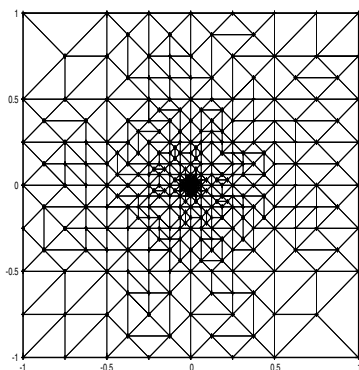


FIGURE 1. Adaptive mesh after 10 iterations for the checkerboard ($a_1 = 5$, $\theta = 0.8$).

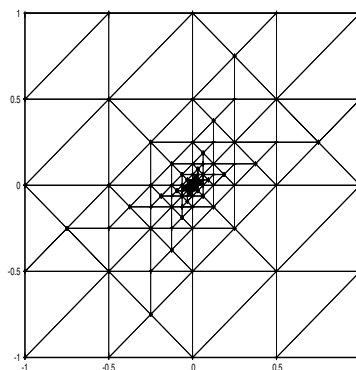


FIGURE 2. Adaptive mesh after 10 iterations for the checkerboard ($a_1 = 100$, $\theta = 0.9$).

For the first example we consider the checkerboard example, namely we take $\Omega = (-1, 1)^2$, $\Gamma_D = \Gamma$ and a discontinuous coefficient a . Namely we decompose Ω into 4 sub-domains Ω_i , $i = 1, \dots, 4$ with $\Omega_1 = (0, 1) \times (0, 1)$, $\Omega_2 = (-1, 0) \times (0, 1)$, $\Omega_3 = (-1, 0) \times (-1, 0)$ and $\Omega_4 = (0, 1) \times (-1, 0)$ and take $a = a_i$ on Ω_i , with $a_2 = a_4 = 1$ and $a_1 = a_3 = 5$ or 100. The discretization will be made with piecewise polynomials of order less than 1 (*i.e.* we choose $l = 1$) and with $\gamma = 25$ for $a_1 = 5$ and $\gamma = 500$ for 100, the experiments have shown that this choice of γ is the optimal one.

Using polar coordinates centered at $(0, 0)$, we take as exact solution,

$$S(x, y) = r^\alpha \phi(\theta),$$

where $\alpha \in (0, 1)$ and ϕ are chosen such that S is harmonic on each sub-domain Ω_i , $i = 1, \dots, 4$ and satisfies the jump conditions:

$$[[S]] = 0 \text{ and } [[a \nabla S \cdot n]] = 0$$

on the interfaces (*i.e.* the segments $\bar{\Omega}_i \cap \bar{\Omega}_{i+1} \pmod{4}$, $i = 1, \dots, 4$). We fix non-homogeneous Dirichlet boundary conditions on Γ accordingly.

It is easy to see (see for instance [11]) that α is the root of the transcendental equation

$$\tan \frac{\alpha\pi}{4} = \sqrt{a_1}.$$

This solution has a singular behavior around the point $(0, 0)$ (because $\alpha < 1$). Therefore a refinement of the mesh near this point can be expected. This can be seen in Figures 1 and 2 on the meshes obtained for $a_1 = 5$ and $a_1 = 100$ respectively and for which $\alpha \approx 0.53544094560$ and $\alpha \approx 0.1269020697$. The approximated convergence rates of the error are presented in Tables 1 and 2 and show a convergence rate approximatively equal to 1 (the case $a_1 = 100$ is less accurate due to the high singular behavior of the solution). There we see that the different effectivity indices are approximatively equal to 1.5 and 2.8 respectively and confirm the efficiency of the estimator. As the reduction factors of the error are around 0.7 and 0.9, the convergence of the adaptive algorithms is confirmed.

In order to see the effect of the parameter γ on the convergence of our method, we have computed the approximated solutions obtained by the adaptive algorithm described above for different values of γ and a fixed θ . In Tables 3 and 4 we give the effectivity index, the reduction factor of the error and the convergence rate for the checkerboard with $a_1 = 5$ and $a_1 = 100$ after 10 iterations. Even if an effect does exist between these parameters and γ , it is quite mild since they do not vary significantly with respect to γ . In particular

TABLE 1. Effectivity indices, reduction factors of the error and convergence rates for the checkerboard with $a_1 = 5$, $\gamma = 25$, $\theta = 0.8$.

it	DOF	η	$\ u - u_h\ _{h,\gamma}$	$Eff.$	RFE	EOC_e
1	96	0.7679	0.4278	1.7948		
2	174	0.6023	0.3332	1.8077	0.7788	0.8408
4	390	0.3748	0.2166	1.7306	0.7974	1.0961
6	1077	0.2082	0.1292	1.6118	0.7848	1.0176
8	3816	0.1100	0.0702	1.5679	0.7413	0.9515
10	13 731	0.0573	0.0373	1.5357	0.7274	0.9707
12	49 194	0.0297	0.0196	1.5146	0.7195	1.0171
14	182 247	0.0152	0.0102	1.5017	0.7204	1.0215

TABLE 2. Effectivity indices, reduction factors of the error and convergence rates for the checkerboard with $a_1 = 100$, $\gamma = 500$, $\theta = 0.9$.

it	DOF	η	$\ u - u_h\ _{h,\gamma}$	$Eff.$	RFE	EOC_e
1	96	1.4435	0.3656	3.9481		
2	174	1.3912	0.3436	4.0494	0.9397	0.2092
4	342	1.3120	0.3251	4.0361	0.9762	0.1710
6	510	1.2093	0.3040	3.9783	0.9646	0.4007
8	678	1.0983	0.2801	3.9207	0.9588	0.6358
10	846	0.9879	0.2567	3.8484	0.9570	0.8403
14	1758	0.7589	0.2021	3.7553	0.9471	1.0056
18	2982	0.5420	0.1566	3.4615	0.9593	2.9061
22	5556	0.3877	0.1254	3.0918	0.9514	0.6318
26	9258	0.2748	0.0938	2.9284	0.9273	1.3345
30	17 742	0.1940	0.0692	2.8015	0.9268	0.9876

TABLE 3. Effectivity indices, reduction factors of the error and convergence rates for the checkerboard with $a_1 = 5$, $\theta = 0.8$ and different γ .

γ	$Eff.$	RFE	EOC_e
5	1.5238	0.7422	0.9193
10	1.5423	0.7231	0.9890
15	1.5358	0.7275	0.9707
20	1.5373	0.7250	0.9865
25	1.5373	0.7250	0.9865
50	1.5373	0.7250	0.9865

the reduction factor of the error is mainly independent of γ and therefore the convergence of the algorithm is relatively independent on the variation of γ .

As second example, we take again the L -shape domain $\Omega = (-1, 1)^2 \setminus (-1, 0) \times (0, 1)$, $a = 1$, $\Gamma_D = \Gamma$ and as exact solution

$$S = r^{2/3} \sin(2\theta/3).$$

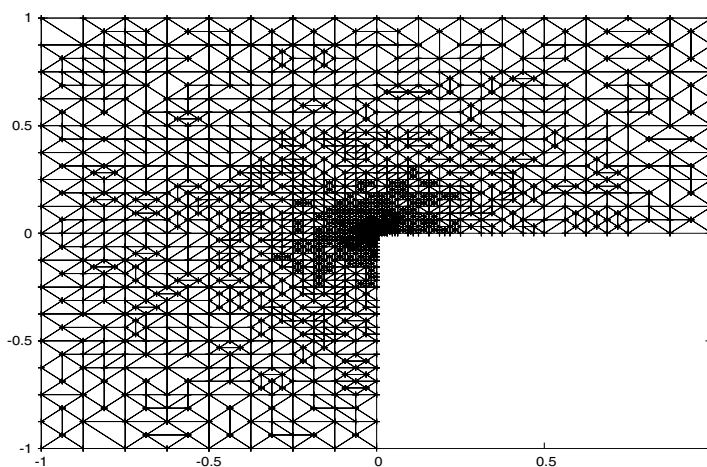
The discretization is still performed with piecewise polynomials of order less than 1 and with $\gamma = 10$. The singular behavior at $(0, 0)$ of the solution induces refinement of the meshes near this point, which can be seen in Figure 3. As before an approximated convergence rate 1 of the error and effectivity indices of order 1.3

TABLE 4. Effectivity indices, reduction factors of the error and convergence rates for the checkerboard with $a_1 = 100$, $\theta = 0.9$ and different γ .

γ	$Eff.$	RFE	EOC_e
25	3.8048	0.9572	0.9232
50	3.8048	0.9572	0.9232
100	3.7681	0.9562	0.9469
250	3.8048	0.9572	0.9232
500	3.8048	0.9572	0.9232
1000	3.8048	0.9572	0.9232

TABLE 5. Effectivity indices, reduction factors of the error and convergence rates for the L -shape with $\theta = 0.75$.

it	DOF	η	$\ u - u_h\ _{h,\gamma}$	$Eff.$	RFE	EOC_e
1	72	0.2746	0.2726	1.0074		
2	117	0.2351	0.2112	1.1127	0.7748	1.0508
4	420	0.1417	0.1144	1.2387	0.7464	1.1187
6	1869	0.07330	0.05565	1.3172	0.7049	1.0209
8	7860	0.03636	0.02664	1.3649	0.6990	1.0185
10	29 691	0.01872	0.01339	1.3986	0.7127	1.0491

FIGURE 3. Adaptive mesh after 12 iterations for the L -shape with $\theta = 0.75$.

are noticed in Table 5. Since the reduction factors of the error are approximately equal to 0.7, the adaptive algorithm is convergent.

As before the effect of the parameter γ on the convergence of our method is presented in Table 6 where we give the effectivity index, the reduction factor of the error and the convergence rate for the L -shape after 10 iterations. Clearly this dependence is quite mild since for $\gamma \geq 10$, the parameters take the same values (up to 5 digits in fact).

TABLE 6. Effectivity indices, reduction factors of the error and convergence rates the L -shape with $\theta = 0.75$ and different γ .

γ	$Eff.$	RFE	EOC_e
5	1.4104	0.7148	1.0481
10	1.3986	0.7127	1.0491
25	1.3986	0.7127	1.0491
50	1.3986	0.7127	1.0491
100	1.3986	0.7127	1.0491
500	1.3986	0.7127	1.0491

4. *A POSTERIORI* ERROR ESTIMATORS FOR A DISCONTINUOUS GALERKIN METHOD FOR CONVECTION-DIFFUSION-REACTION PROBLEMS

The discontinuous Galerkin method is an efficient method for solving convection-diffusion-reaction problems. Hence in this section we approximate such problems by a method proposed in [16] and show the convergence of the adaptive algorithm for an estimator of residual type.

In this section our main goal is to perform a convergence analysis but not to obtain its robustness with respect to large Péclet and/or Damkohler numbers. Hence these numbers are supposed to be fixed and we do not give the dependence of the obtained constants with respect to these numbers. For large Péclet and/or Damkohler numbers, another DG-method should be used with another penalization strategy (like the one described in [16] for instance).

In a bounded domain Ω of \mathbb{R}^2 , for $f \in L^2(\Omega)$, we consider the problem

$$\begin{cases} Au := -\operatorname{div}(K\nabla u) + \beta \cdot \nabla u + bu = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma, \end{cases} \quad (4.1)$$

where the diffusion tensor K , the velocity field β and the reaction function b satisfy the following assumptions:

$$\begin{aligned} \beta &\in W^{1,\infty}(\Omega)^2, \quad b \in L^\infty(\Omega), \\ \exists \mu_0 > 0 : b - \frac{1}{2} \operatorname{div} \beta &\geq \mu_0, \\ K \in \mathbb{R}^{2 \times 2} &\text{ is symmetric,} \\ \exists \alpha_0 > 0 : A\xi \cdot \xi &\geq \alpha_0, \quad \forall \xi \in \mathbb{R}^2. \end{aligned}$$

Obviously if $K = a\mathbb{I}$, $b = 0$ and $\beta = 0$, we recover the problem considered in the previous section. The variational formulation of this problem is quite standard and uses the bilinear form

$$a(u, v) = \int_{\Omega} (K\nabla u \cdot \nabla v + \beta \cdot \nabla uv + buv).$$

Due to the above assumptions, a is coercive on $H_0^1(\Omega)$ equipped with the norm

$$\|u\|^2 = \int_{\Omega} \left(K\nabla u \cdot \nabla u + \left(b - \frac{1}{2} \operatorname{div} \beta \right) u^2 \right).$$

Given $f \in L^2(\Omega)$, the weak formulation consists in finding $u \in H_0^1(\Omega)$ solution of (3.2) (with $H_D^1(\Omega) = H_0^1(\Omega)$).

We approximate problem (4.1) (or more precisely its variational formulation (3.2)) by a discontinuous Galerkin scheme introduced in [16] (see also [15,27]). As before we consider a family of regular triangulations \mathcal{T}_h made of triangles T satisfying the same assumptions than before and use the notations from the previous section.

Problem (3.2) is approximated in the (discontinuous) finite element space defined by (3.3), here equipped with the norm

$$\|q\|_{h,\gamma} := \left(\|K^{1/2}\nabla_h q\|_{\Omega}^2 + \left\| \left(b - \frac{1}{2}\operatorname{div} \beta \right) q \right\|_{\Omega}^2 + \gamma \sum_{e \in \mathcal{E}_h} h_e^{-1} \|\llbracket q \rrbracket\|_e^2 \right)^{1/2},$$

where γ is a positive parameter fixed below.

The interior penalty DG method uses the bilinear form $a_{h,\gamma}(\cdot, \cdot)$ defined as follows:

$$\begin{aligned} a_{h,\gamma}(u_h, v_h) := & \sum_{T \in \mathcal{T}_h} \int_T (K \nabla u_h \cdot \nabla v_h + (b - \operatorname{div} \beta) u_h v_h + u_h \beta \cdot \nabla v_h) \\ & - \sum_{e \in \mathcal{E}_h} \int_e (\{ \{ K \nabla_h v_h \} \} \cdot \llbracket u_h \rrbracket + \{ \{ K \nabla_h u_h \} \} \cdot \llbracket v_h \rrbracket) \\ & + \sum_{e \in \mathcal{E}_h} \left(\gamma h_e^{-1} \int_e \llbracket u_h \rrbracket \cdot \llbracket v_h \rrbracket + \int_e \{ \{ u_h \} \} \beta \cdot \llbracket v_h \rrbracket \right), \quad \forall u_h, v_h \in V_h. \end{aligned}$$

This form is coercive if the positive parameter γ is chosen large enough because by element-wise integration by parts we have that

$$\begin{aligned} a_{h,\gamma}(u_h, u_h) = & \sum_{T \in \mathcal{T}_h} \int_T \left(K \nabla u_h \cdot \nabla u_h + \left(b - \frac{1}{2} \operatorname{div} \beta \right) u_h^2 \right) \\ & - 2 \sum_{e \in \mathcal{E}_h} \int_e \{ \{ K \nabla_h u_h \} \} \cdot \llbracket u_h \rrbracket \\ & + \sum_{e \in \mathcal{E}_h} \gamma h_e^{-1} \int_e |\llbracket u_h \rrbracket|^2, \end{aligned}$$

and the coerciveness (2.3) follows as in Lemma 2.1 of [18] for instance.

The discontinuous Galerkin approximation of problem (3.2) is to find $u_h \in V_h$ solution of (3.4).

Note that the form $a_{h,\gamma}$ is consistent in the sense that the solution $u \in H_0^1(\Omega)$ of (3.2) satisfies

$$a_{h,\gamma}(u, v_h) = \int_{\Omega} f v_h, \quad \forall v_h \in V_h, \tag{4.2}$$

and therefore the orthogonality relation (2.14) holds. Unfortunately we cannot invoke Lemma 2.2 to obtain the quasi-orthogonality relations (2.13) because $a_{h,\gamma}$ is not symmetric. Nevertheless (2.13) is valid as we will see later on.

Remark that the consistency property (4.2) and since the solution $u \in H_0^1(\Omega)$ of (3.2) is at least in $H^s(\Omega)$ for some $s \in (3/2, 2]$, the convergence of u_h to u is guaranteed and we have the *a priori* error estimate (see for instance Sect. 5.1 of [4])

$$\|u - u_h\|_{h,\gamma} \leq C h^{s-1} \|f\|, \tag{4.3}$$

where C is a positive constant that depends on the data K, β, b , on the domain Ω and on the (fixed) parameter γ .

Similarly (4.2) and the fact that the adjoint form of $a_{h,\gamma}$ is also consistent, the so-called Aubin-Nitsche trick holds:

Lemma 4.1. *There exists a positive constant C that depends on the data K, β, b , on the domain Ω and on the (fixed) parameter γ such that*

$$\|u - u_h\| \leq Ch^{s-1} \|u - u_h\|_{h,\gamma}.$$

4.1. A posteriori error analysis of residual type

Following [18,19] we introduce residual type error estimators which essentially measure locally the jump of the discrete flux.

For all $T \in \mathcal{T}_h$, we introduce the local estimator on T defined by

$$\eta_T(u_h)^2 = h_T^2 \|f - Au_h\|_T^2 + \sum_{e \in \mathcal{E}_h \cap T} h_e \| [[K \nabla u_h \cdot n_e]] \|_e^2.$$

Now using the results from Section 2 and extending some results from [19] to our setting, we describe a convergent algorithm for the above estimators.

From now on for the sake of simplicity we suppose that f is piecewise \mathbb{P}_{l-1} , hence there is no oscillation terms and (2.11) directly holds. If f is not piecewise \mathbb{P}_{l-1} , then a standard oscillation term has to be added and the oscillation reduction estimate (2.11) follows from Lemma 3.2 of [23] if the marking strategy from Definition 2.1 is used and if the successive meshes are constructed *via* the procedure REFINE of Morin, Nochetto and Siebert [23–25].

4.2. Convergence of an adaptive algorithm

In this section we state some results that are similar to the ones stated in Sections 3 and 4 of [19] that we extend to our setting. These results and Section 2 lead to the convergence of the adaptive algorithm described in Definition 2.1.

For shortness we denote by $e_h = u - u_h$ and by

$$\begin{aligned} \| \|e_h\| \|_h^2 &= \sum_{T \in \mathcal{T}_h} \| \|e_h\| \|_T^2, \\ \| \|e_h\| \|_T^2 &= \int_T \left(K \nabla e_h \cdot \nabla e_h + \left(b - \frac{1}{2} \operatorname{div} \beta \right) e_h^2 \right). \end{aligned}$$

We first start by showing the efficiency of the estimator and a very important estimate between the L^2 -norm of the jumps and the estimator (see Thm. 3.2 of [19]):

Theorem 4.2. *There exists a positive constant c independent of the mesh-size and of γ such that for all $T \in \mathcal{T}_h$, the next estimates hold:*

$$h_T^2 \|f - Au_h\|_T^2 \leq c \| \|e_h\| \|_T^2, \tag{4.4}$$

$$h_e \| [[K \nabla u_h \cdot n_e]] \|_e^2 \leq c \sum_{T' \subset \omega_e} \| \|e_h\| \|_{T'}^2, \quad \forall e \in \mathcal{E}_h \cap T, \tag{4.5}$$

$$\gamma^2 \sum_{e \in \mathcal{E}_h} h_e^{-1} \int_e | [[u_h]] |^2 \leq c \eta_h^2. \tag{4.6}$$

Proof. The proof of the estimates (4.4) and (4.5) are standard and are obtained by using element and edge bubble functions respectively. Let us concentrate on the proof of (4.6) that is adapted from Theorem 3.2 of [19].

Consider the Galerkin approximation of u in $V_h^c := V_h \cap H_0^1(\Omega)$, namely let $u_h^G \in V_h^c$ be the unique solution of

$$a(u_h^G, v_h) = \int_{\Omega} f v_h \quad \forall v_h \in V_h^c.$$

By integration by parts, we see that

$$a(v_h, w_h) = a_{h,\gamma}(v_h, w_h) \quad \forall v_h, w_h \in V_h^c,$$

this leads to the orthogonality relation

$$a_{h,\gamma}(u - u_h^G, v_h) = a(u - u_h^G, v_h) = 0 \quad \forall v_h \in V_h^c.$$

This property and the other orthogonality relation (2.14) allow to write

$$a_{h,\gamma}(u_h - u_h^G, u_h - u_h^G) = a_{h,\gamma}(u - u_h^G, u_h - u_h^G) = a_{h,\gamma}(u - u_h^G, u_h - u_h^G - \chi),$$

for any $\chi \in V_h^c$. Using the definition of $a_{h,\gamma}$ and then using the splitting $u = e_h + u_h$, we get

$$\begin{aligned} a_{h,\gamma}(u_h - u_h^G, u_h - u_h^G) &= \sum_{T \in \mathcal{T}_h} \int_T \left(K \nabla(u - u_h^G) \cdot \nabla(u_h - u_h^G - \chi) \right. \\ &\quad \left. + (b - \operatorname{div} \beta)(u - u_h^G)(u_h - u_h^G - \chi) + (u - u_h^G) \beta \cdot \nabla(u_h - u_h^G - \chi) \right) \\ &\quad - \sum_{e \in \mathcal{E}_h} \int_e \left\{ \{ K \nabla_h(u - u_h^G) \} \right\} \cdot \llbracket u_h \rrbracket + \sum_{e \in \mathcal{E}_h} \int_e \left\{ \{ u - u_h^G \} \right\} \beta \cdot \llbracket u_h \rrbracket \\ &= \sum_{T \in \mathcal{T}_h} \int_T \left(K \nabla e_h \cdot \nabla(u_h - u_h^G - \chi) + (b - \operatorname{div} \beta) e_h (u_h - u_h^G - \chi) \right. \\ &\quad \left. + e_h \beta \cdot \nabla(u_h - u_h^G - \chi) \right) + \sum_{T \in \mathcal{T}_h} \int_T \left(K \nabla(u_h - u_h^G) \cdot \nabla(u_h - u_h^G - \chi) \right. \\ &\quad \left. + (b - \operatorname{div} \beta)(u_h - u_h^G)(u_h - u_h^G - \chi) + (u_h - u_h^G) \beta \cdot \nabla(u_h - u_h^G - \chi) \right) \\ &\quad - \sum_{e \in \mathcal{E}_h} \int_e \left\{ \{ K \nabla_h(u - u_h^G) \} \right\} \cdot \llbracket u_h \rrbracket + \sum_{e \in \mathcal{E}_h} \int_e \left\{ \{ u - u_h^G \} \right\} \beta \cdot \llbracket u_h \rrbracket. \end{aligned}$$

Integrating by parts the terms $\int_T K \nabla e_h \cdot \nabla(u_h - u_h^G - \chi)$ and $\int_T e_h \beta \cdot \nabla(u_h - u_h^G - \chi)$, we arrive at

$$\begin{aligned} a_{h,\gamma}(u_h - u_h^G, u_h - u_h^G) &= \sum_{T \in \mathcal{T}_h} \int_T (f - Au_h)(u_h - u_h^G - \chi) + \sum_{T \in \mathcal{T}_h} \int_T \left(K \nabla(u_h - u_h^G) \cdot \nabla(u_h - u_h^G - \chi) \right. \\ &\quad \left. + (b - \operatorname{div} \beta)(u_h - u_h^G)(u_h - u_h^G - \chi) + (u_h - u_h^G) \beta \cdot \nabla(u_h - u_h^G - \chi) \right) \\ &\quad - \sum_{e \in \mathcal{E}_h} \int_e \left(\left\{ \{ K \nabla_h(u_h - u_h^G) \} \right\} \cdot \llbracket u_h \rrbracket + \llbracket K \nabla_h u_h \rrbracket \cdot \left\{ \{ u_h - u_h^G - \chi \} \right\} \right) \\ &\quad + \sum_{e \in \mathcal{E}_h} \int_e \left(\left\{ \{ u_h - u_h^G \} \right\} \beta \cdot \llbracket u_h \rrbracket + \left\{ \{ u_h - u_h^G - \chi \} \right\} \beta \cdot \llbracket u_h \rrbracket \right). \end{aligned}$$

In comparison with the proof of Theorem 3.2 of [19], some terms related to the vector field β and to b appear. Nevertheless using χ as in Lemma 3.1 of [19] and some trace and inverse inequality we arrive at

$$a_{h,\gamma}(u_h - u_h^G, u_h - u_h^G) \leq C_1 \epsilon \| \|u_h - u_h^G \| \|_h^2 + C_2 \epsilon^{-1} \gamma^{-1} \eta_h^2 + (C_3 \epsilon \gamma + C_4(\epsilon)) \sum_{e \in \mathcal{E}_h} h_e^{-1} \int_e | \llbracket u_h \rrbracket |^2,$$

if $\gamma \geq 1$, where C_1, C_2 and C_3 are positive constants independent of γ, ϵ and the meshsize, while $C_4(\epsilon)$ is positive constant independent of γ and of the mesh size but that may depend on ϵ . Using the coerciveness of $a_{h,\gamma}$, we get for $\gamma \geq \max\{1, \gamma_0\}$,

$$\begin{aligned} \alpha_0 \left(\| \|u_h - u_h^G \| \|_h^2 + \gamma \sum_{e \in \mathcal{E}_h} h_e^{-1} \int_e | \llbracket u_h \rrbracket |^2 \right) &\leq C_1 \epsilon \| \|u_h - u_h^G \| \|_h^2 + C_2 \epsilon^{-1} \gamma^{-1} \eta_h^2 \\ &+ (C_3 \epsilon \gamma + C_4(\epsilon)) \sum_{e \in \mathcal{E}_h} h_e^{-1} \int_e | \llbracket u_h \rrbracket |^2. \end{aligned}$$

This leads to (4.6) by choosing (and fixing) ϵ small enough such that $2C_1 \epsilon \leq \alpha_0$ as well as $2C_3 \epsilon \leq \alpha_0$, and then γ_0 large enough such that $\alpha_0 \gamma_0 - C_4(\epsilon) > 0$. □

This result leads to the reliability of the estimator, namely:

Corollary 4.3. *The upper bound (2.6) holds.*

Proof. By standard arguments based on element-wise integration by parts, interpolation error estimates and Young’s inequality, we have (see for instance Thm. 3.1 of [19])

$$a_{h,\gamma}(e_h, e_h) \leq C \left(\eta_h^2 + \gamma^2 \sum_{e \in \mathcal{E}_h} h_e^{-1} \int_e | \llbracket u_h \rrbracket |^2 \right),$$

for some $C > 0$ independent of the meshsize and of γ . We conclude by the estimate (4.6). □

We go on by proving (2.5) (compare with Prop. 4.2 of [19]).

Lemma 4.4. *There exists $\gamma_1 > 0$ large enough and $C_1, C_2 > 0$ independent of γ such that for $\gamma \geq \gamma_1$, we have*

$$a_{h,\gamma}(e_h, e_h) \geq \frac{1}{2} \| \|e_h \| \|_h + C_1 \gamma^2 \sum_{e \in \mathcal{E}_h} h_e^{-1} \| \llbracket u_h \rrbracket \|_e^2, \tag{4.7}$$

$$a_{h,\gamma}(e_h, e_h) \leq 2 \| \|e_h \| \|_h + C_2 \gamma \sum_{e \in \mathcal{E}_h} h_e^{-1} \| \llbracket u_h \rrbracket \|_e^2. \tag{4.8}$$

Proof. As before we notice that

$$a_{h,\gamma}(e_h, e_h) = \sum_{T \in \mathcal{T}_h} \int_T \left(K \nabla e_h \cdot \nabla e_h + \left(b - \frac{1}{2} \operatorname{div} \beta \right) e_h^2 \right) - 2 \sum_{e \in \mathcal{E}_h} \int_e (\{ \{ K \nabla_h e_h \} \} \cdot \llbracket e_h \rrbracket) + \sum_{e \in \mathcal{E}_h} \gamma h_e^{-1} \int_e | \llbracket e_h \rrbracket |^2.$$

Secondly by using (4.2), we get

$$a_{h,\gamma}(e_h, \chi - u_h) = 0 \quad \forall \chi \in V_h \cap H_0^1(\Omega),$$

which yields

$$\begin{aligned} \sum_{e \in \mathcal{E}_h} \int_e \{ \{ K \nabla_h e_h \} \} \cdot \llbracket e_h \rrbracket &= \sum_{T \in \mathcal{T}_h} \int_T \left(K \nabla e_h \cdot \nabla (\chi - u_h) + (b - \operatorname{div} \beta) e_h (\chi - u_h) - e_h \beta \cdot \nabla (\chi - u_h) \right) \\ &\quad - \sum_{e \in \mathcal{E}_h} \int_e \{ \{ K \nabla_h (\chi - u_h) \} \} \cdot \llbracket e_h \rrbracket \\ &\quad + \sum_{e \in \mathcal{E}_h} \left(\gamma h_e^{-1} \int_e \llbracket e_h \rrbracket \cdot \llbracket \chi - u_h \rrbracket + \int_e \{ \{ e_h \} \} \beta \cdot \llbracket \chi - u_h \rrbracket \right). \end{aligned}$$

Inserting this expression in the previous identity we arrive at

$$\begin{aligned} a_{h,\gamma}(e_h, e_h) &= \|\| e_h \|\|_h^2 - 2 \sum_{T \in \mathcal{T}_h} \int_T \left(K \nabla e_h \cdot \nabla (\chi - u_h) + (b - \operatorname{div} \beta) e_h (\chi - u_h) - e_h \beta \cdot \nabla (\chi - u_h) \right) \\ &\quad - 2 \sum_{e \in \mathcal{E}_h} \int_e \{ \{ K \nabla_h (\chi - u_h) \} \} \cdot \llbracket e_h \rrbracket \\ &\quad + 2 \sum_{e \in \mathcal{E}_h} \left(\gamma h_e^{-1} \int_e \llbracket e_h \rrbracket \cdot \llbracket \chi - u_h \rrbracket + 2 \int_e \{ \{ e_h \} \} \beta \cdot \llbracket \chi - u_h \rrbracket \right) + \sum_{e \in \mathcal{E}_h} \gamma h_e^{-1} \int_e \|\llbracket e_h \rrbracket\|^2. \end{aligned}$$

Now using χ as in Theorem 2.1 of [19], trace and inverse inequalities and Young’s inequality we conclude as in Proposition 4.2 of [19] that

$$\begin{aligned} a_{h,\gamma}(e_h, e_h) &\geq (1 - C_3 \epsilon) \|\| e_h \|\|_h^2 - C_4 (\gamma + \epsilon^{-1}) \sum_{e \in \mathcal{E}_h} h_e^{-1} \|\llbracket u_h \rrbracket\|_e^2, \\ a_{h,\gamma}(e_h, e_h) &\leq (1 + C_3 \epsilon) \|\| e_h \|\|_h^2 + C_4 (\gamma + \epsilon^{-1}) \sum_{e \in \mathcal{E}_h} h_e^{-1} \|\llbracket u_h \rrbracket\|_e^2, \end{aligned}$$

for any $\epsilon > 0$. The second estimate directly leads to (4.8) by choosing $\epsilon = \gamma^{-1}$. The first estimate (4.7) follows from the first above estimate and Theorem 4.2. □

The next lemma shows that (2.13) holds (compare with Lem. 2.1 of [23]).

Lemma 4.5. *There exists h_0 small enough (depending on the shape regularity constant of the meshes \mathcal{T}_h and on Ω) such that for all $h \leq h_0$, the quasi-orthogonality relations (2.13) holds.*

Proof. By element-wise integration by parts we see that

$$a_{h,\gamma}(u - u_H, u - u_H) - a_{h,\gamma}(u - u_h, u - u_h) - a_{h,\gamma}(u_h - u_H, u_h - u_H) = \int_{\Omega} \operatorname{div} \beta (u_H - u_h)(u - u_h).$$

Hence using Young’s inequality we find that

$$\begin{aligned} a_{h,\gamma}(u - u_H, u - u_H) - a_{h,\gamma}(u - u_h, u - u_h) - a_{h,\gamma}(u_h - u_H, u_h - u_H) \\ \geq - \left(\frac{\epsilon}{2} \|u - u_h\|^2 + \frac{\|\operatorname{div} \beta\|_{\infty, \Omega}^2}{2\epsilon} \|u_H - u_h\|^2 \right), \end{aligned}$$

for any $\epsilon > 0$. The Aubin-Nitsche trick from Lemma 4.1 and the coerciveness of the form $a_{h,\gamma}$ then yield

$$a_{h,\gamma}(u - u_H, u - u_H) - a_{h,\gamma}(u - u_h, u - u_h) - a_{h,\gamma}(u_h - u_H, u_h - u_H) \geq - \left(C_1 h^{2s} \epsilon \|u - u_h\|_{h,\gamma}^2 + \frac{C_2}{\epsilon} a_{h,\gamma}(u_H - u_h, u_H - u_h) \right),$$

for some $C_1, C_2 > 0$ depending on the data K, β, b and on Ω . Since we have shown that (2.5) holds, we deduce that

$$a_{h,\gamma}(u - u_H, u - u_H) - a_{h,\gamma}(u - u_h, u - u_h) - a_{h,\gamma}(u_h - u_H, u_h - u_H) \geq - \left(C_3 h^{2s} \epsilon a_{h,\gamma}(u - u_h, u - u_h) + \frac{C_2}{\epsilon} a_{h,\gamma}(u_H - u_h, u_H - u_h) \right),$$

with $C_3 = C_1 \alpha'_0$. The conclusion follows by taking $\epsilon^2 = \frac{C_2}{C_3} h^{-2s}$ and by choosing h_0 small enough such that $C_3 C_2 h_0^{2s} < 1$. \square

It remains the error reduction:

Lemma 4.6. *If \mathcal{T}_h is obtained from $\hat{\mathcal{T}}_H$ by the procedure REFINE from [23] (see also [19]). Then the estimate (2.10) is valid.*

Proof. Taking an arbitrary $v \in V_h^c$, by element-wise integration by parts we notice that

$$a_{h,\gamma}(u - u_H, v) = \sum_{T \in \mathcal{T}_h} \int_T (f - Au_H)v + \sum_{e \in \mathcal{E}_h} \int_e (\beta \llbracket u_H \rrbracket - \llbracket K \nabla u_H \rrbracket)v.$$

Since the orthogonality relation (2.14) holds, we deduce that

$$a_{h,\gamma}(u_h - u_H, v) = \sum_{T \in \mathcal{T}_h} \int_T (f - Au_H)v + \sum_{e \in \mathcal{E}_h} \int_e (\beta \llbracket u_H \rrbracket - \llbracket K \nabla u_H \rrbracket)v. \tag{4.9}$$

First for $T \in \hat{\mathcal{T}}_H$, take $v = (f - Au_H)b_T$, where b_T is the unique element in V_h^c with $l = 1$ (i.e. b_T is piecewise \mathbb{P}_1) such that $b_T(x_T) = 1$, where x_T is an interior node of T generated by the procedure REFINE (that is a node of \mathcal{T}_h) and $b_T(x) = 0$ for all other nodes of \mathcal{T}_h . For such a choice of v in (4.9), we get

$$\|f - Au_H\|_T^2 \leq C \left(\int_T (f - Au_H)v + a_{h,\gamma}(u_h - u_H, v) \right).$$

By Cauchy-Schwarz's and inverse inequalities, we arrive at

$$h_K^2 \|f - Au_H\|_T^2 \leq C \left(\sum_{T' \in \mathcal{T}_h, T' \subset T} \|u_h - u_H\|_{T'}^2 + \sum_{e \in \mathcal{E}_h, e \subset T} h_e^{-1} \|\llbracket u_h \rrbracket\|_e^2 \right). \tag{4.10}$$

Similarly if $E \in \mathcal{E}_H$ is an interior edge of $T \in \hat{\mathcal{T}}_H$, we fix an interior node x_E of E created by REFINE and that is a node of \mathcal{T}_h and take $v = b_E E_h(j_E)$, where b_E is the unique element in V_h^c with $l = 1$ such that $b_T(x_E) = 1$ and $b_T(x) = 0$ for all other nodes x of \mathcal{T}_h ; $j_E = \llbracket \nabla u_H \rrbracket$ and $E_h(j_E)$ is an extension of j_E inside ω_E

obtained in the usual way (see for instance p. 1813 of [23]). With such a choice and using (4.9), we have

$$\begin{aligned} \|[K\nabla u_H]\|_E^2 &\leq C \int_E \|[K\nabla u_H]\] \cdot v \\ &\leq C \left(a_{h,\gamma}(u_h - u_H, v) + \sum_{T \in \mathcal{T}_h} \int_T (f - Au_H)v + \sum_{e \in \mathcal{E}_h} \int_e \beta \cdot \llbracket u_H \rrbracket v \right). \end{aligned}$$

Using Cauchy-Schwarz's and inverse inequalities we obtain

$$h_E^{\frac{1}{2}} \|[K\nabla u_H]\|_E \leq C \left(h_T \|f - Au_H\|_T + \|u_h - u_H\|_T + \sum_{e \in \mathcal{E}_h, e \subset T} (h_e^{-1} \|[u_h - u_H]\|_e + h_e \llbracket u_H \rrbracket) \right).$$

By the triangular inequality we obtain

$$h_E^{\frac{1}{2}} \|[K\nabla u_H]\|_E \leq C \left(h_T \|f - Au_H\|_T + \|u_h - u_H\|_T + \sum_{e \in \mathcal{E}_h, e \subset T} (h_e^{-1} \|[u_h - u_H]\|_e + h_e \llbracket u_h \rrbracket) \right).$$

Using the estimate (4.10), for any $T \in \hat{\mathcal{T}}_H$ we arrive at

$$\eta_T(u_H)^2 \leq C \left(\sum_{T' \in \mathcal{T}_h, T' \subset T} \|u_h - u_H\|_{T'}^2 + \sum_{e \in \mathcal{E}_h, e \subset T} h_e^{-1} (\|[u_h]\|_e^2 + \|[u_h - u_H]\|_e^2) \right).$$

Summing this estimate on $T \in \hat{\mathcal{T}}_H$, we arrive at (2.10) by using the definition of the norm $\|\cdot\|_{h,\gamma}$ because we recall that for $e \in \mathcal{E}_h$, $\llbracket u_h \rrbracket = \llbracket u - u_h \rrbracket$. □

We finally show that in our DG context, (2.12) holds:

Lemma 4.7. *If \mathcal{T}_h is obtained from $\hat{\mathcal{T}}_H$ by the procedure REFINE from [23] (see also [19]). Then the estimate (2.12) is valid.*

Proof. First we remark that the refinement procedure yields (see Prop. 4.1 of [19])

$$a_{h,\gamma}(u - u_H, u - u_H) \leq a_{H,\gamma}(u - u_H, u - u_H) + c\gamma \sum_{e \in \mathcal{E}_H} h_e^{-1} \|[u_H]\|_e^2. \tag{4.11}$$

Now applying Theorem 4.2 to the level H , we deduce that

$$\gamma \sum_{e \in \mathcal{E}_H} h_e^{-1} \|[u_H]\|_e^2 \leq \frac{C}{\gamma} \|u - u_H\|^2.$$

The coerciveness of $a_{H,\gamma}$ then yields

$$\gamma \sum_{e \in \mathcal{E}_H} h_e^{-1} \|[u_H]\|_e^2 \leq \frac{C}{\alpha_0 \gamma} a_{H,\gamma}(u - u_H, u - u_H).$$

This estimate in (4.11) leads to the conclusion. □

Remark 4.8. In view of the results from [16], especially Theorems 6.7 and 7.2, and using our above results and Lemma 3.1 of [23], the adaptive algorithm from Definition 2.1 based on the estimator of flux type introduced in [16] (see Thm. 6.7) is also convergent.

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