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RESOLUTION OF THE TIME DEPENDENT $P_\mathfrak{n}$ EQUATIONS BY A GODUNOV TYPE SCHEME HAVING THE DIFFUSION LIMIT

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Abstract. We consider the $P_\mathfrak{n}$ model to approximate the time dependent transport equation in one dimension of space. In a diffusive regime, the solution of this system is solution of a diffusion equation. We are looking for a numerical scheme having the diffusion limit property: in a diffusive regime, it has to give the solution of the limiting diffusion equation on a mesh at the diffusion scale. The numerical scheme proposed is an extension of the Godunov type scheme proposed by Gosse to solve the P_1 model without absorption term. It requires the computation of the solution of the steady state $P_\mathfrak{n}$ equations. This is made by one Monte-Carlo simulation performed outside the time loop. Using formal expansions with respect to a small parameter representing the inverse of the number of mean free path in each cell, the resulting scheme is proved to have the diffusion limit. In order to avoid the CFL constraint on the time step, we give an implicit version of the scheme which preserves the positivity of the zeroth moment.

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Introduction

The $P_\mathfrak{n}$ equations ([1], p. 225 of [9]) are a good tool to approximate the neutron transport equation. This model is derived from the expansion of the neutron flux in the basis of spherical harmonics. In one dimension (1D), this model is equivalent to the $S_{\mathfrak{n}+1}$ model [9], well-known to give positive solutions; moreover its solution tends to the solution of the transport equation (see [14] for a theoretical proof) when $\mathfrak{n}$ tends to $+\infty$. In 2D, the $P_\mathfrak{n}$ model, also known as the spherical harmonics method, preserves the rotational invariance of the neutron transport equation in contrast to the $S_{\mathfrak{n}+1}$ model suffering from ray effects [18]. The $P_\mathfrak{n}$ model in 2D does not preserve the positivity of the density on the contrary of the $S_{\mathfrak{n}+1}$ model: see [22] for a detailed study of the regimes when this problem may occur. Nevertheless, in some applications, the rotational invariance of the solution is more important than its positivity. The behaviour of the $P_\mathfrak{n}$ model in the two extreme regimes of diffusion and free streaming limit is also of interest. In the diffusive regime, it recovers the solution of the diffusion equation, whatever $\mathfrak{n}$ is [21]. In the free streaming limit, it recovers the solution of the transport equation with the correct velocity in the limit of $\mathfrak{n}$ going to $+\infty$.

Recently, Brunner and McClarren [20] have proposed a finite volume approximation of the $P_\mathfrak{n}$ model: since the resulting system is hyperbolic and linear, they have considered an explicit Godunov scheme for the conservative

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part of the system, combined with an explicit centered treatment of the source term. The advantage of such a method is to take into account exactly the structure of the waves involved. The main issue is the treatment of the source terms: in the diffusive regime, the choice of a centered treatment does not give the discrete diffusion limit [15] property. Indeed, the scheme does not recover the solution of the diffusion equation with a mesh whose cell size is larger than the mean free path. In [21], they have explained this failure by the too large magnitude of the diffusive terms involved by the scheme. To overcome this defect, they have suggested to multiply these terms by an *ad hoc* factor which behaves differently, depending on the nature of the considered regime (transparent/diffusive). By this way, the scheme yields a correct discretization of the diffusion equation, but it suffers from parasite modes especially when extended to two dimensions.

At the same time, there has been a lot of work in the related field of radiative transfer to derive schemes having the discrete diffusion limit property. Among all these studies [2,4,5], we focus on the work of Gosse and Toscani [11,12] who have considered the Goldstein-Taylor model in 1D. This model is close to the P_1 model where the absorption would be neglected. They have proposed to treat the source term in such a way that the resulting scheme preserves the steady solutions of the system (well-balanced property). The Riemann problem associated with their formulation involves a stationary wave; the intermediate states are computed by solving the steady equation. A Godunov type scheme based on the resolution of this Riemann problem at each interface has thus been derived. They have shown that this scheme has also the diffusion limit.

Our main motivation is to discretize the $P_\mathfrak{n}$ equations in 1D with some scheme having the discrete diffusion limit property. The approach of Gosse seems to us more rigorous than the introduction of an *ad hoc* factor. The aim of this work is to extend this new technique to the case of the $P_\mathfrak{n}$ equations in 1D.

This paper is organized as follows. The first section is devoted to the description of the $P_\mathfrak{n}$ equations. In Section 2, we study the resolution of the P_1 equations: a Gosse type scheme is proposed. It is proved that the resulting scheme is positive and has the diffusion limit property. We describe also how to deal with a variable size mesh and non constant coefficients (σ_a, σ_t). In Section 3, we extend the discretization obtained for the P_1 equations to the $P_\mathfrak{n}$ system. Both properties, positivity and diffusion limit, are proved for the derived scheme. In Section 4, some numerical results are presented. The last section is devoted to the conclusion.

1. The $P_\mathfrak{n}$ equations

In 1D slab geometry, by taking the moments of the transport equation:

$$\frac{\partial \psi}{\partial t} + \mu \frac{\partial \psi}{\partial x} + \sigma_t \psi = (\sigma_t - \sigma_a)\tilde{\psi}, \tag{1.1}$$

we obtain the $P_\mathfrak{n}$ equations [1]:

$$\begin{cases} \dfrac{\partial \psi_0}{\partial t} + B_1 \dfrac{\partial \psi_1}{\partial x} + \sigma_a \psi_0 = 0 \\ \dfrac{\partial \psi_\ell}{\partial t} + B_{\ell+1} \dfrac{\partial \psi_{\ell+1}}{\partial x} + A_{\ell-1} \dfrac{\partial \psi_{\ell-1}}{\partial x} + \sigma_t \psi_\ell = 0 \qquad \ell = 1 ... \mathfrak{n} \\ \psi_{\mathfrak{n}+1} = 0. \end{cases} \tag{1.2}$$

The following notations are used:

- $\psi(x, \mu, t)$ is the neutron flux;
- $x \in [\mathcal{A}, \mathcal{B}]$;
- $\mu \in [-1, +1]$ is the cosinus of the angle between the neutron direction and the x axis;
- σ_a is the absorption cross section and σ_t the total cross section. These coefficients are positive;
- $\tilde{\psi} = \dfrac{1}{2} \displaystyle\int_{-1}^{+1} \psi \mathrm{d}\mu$;
- $\psi_\ell (x,t) = \displaystyle\int_{-1}^{1} \sqrt{2\ell+1}\, L_\ell(\mu)\, \psi(x,\mu,t)\, \mathrm{d}\mu$ are the moments and L_ℓ the ℓth Legendre polynomial;

- the constants A_ℓ and B_ℓ are defined by:

$$\begin{cases} A_\ell = \sqrt{\dfrac{(\ell+1)^2}{(2\ell+3)\,(2\ell+1)}} \\ B_\ell = \sqrt{\dfrac{\ell^2}{(2\ell+1)\,(2\ell-1)}}. \end{cases}$$

Let us note that the speed of the neutrons does not appear in the transport equation (1.1) because time has been adimensioned.

We must now specify the boundary conditions:

- for the transport equation (1.1), they are given by:

$$\begin{cases} \psi(\mathcal{A},\mu,t) = g_{\mathcal{A}}(\mu,t) \text{ for } \mu > 0 \\ \psi(\mathcal{B},\mu,t) = g_{\mathcal{B}}(\mu,t) \text{ for } \mu < 0 \end{cases}$$

  where $g_{\mathcal{A}}$ and $g_{\mathcal{B}}$ are some given functions;
- for the $P_{\mathfrak{n}}$ equations (1.2), the corresponding boundary conditions are:

$$\begin{cases} \displaystyle\sum_{\ell=0}^{\ell=\mathfrak{n}} \frac{\sqrt{2\ell+1}}{2}\psi_\ell\,(\mathcal{A},t)L_\ell(\mu_i) = g_{\mathcal{A}}(\mu_i,t) \;\text{ for }\; \mu_i > 0 \\ \displaystyle\sum_{\ell=0}^{\ell=\mathfrak{n}} \frac{\sqrt{2\ell+1}}{2}\psi_\ell\,(\mathcal{B},t)L_\ell(\mu_i) = g_{\mathcal{B}}(\mu_i,t) \;\text{ for }\; \mu_i < 0. \end{cases} \tag{1.3}$$

To approximate the transport equation, an alternative method is the $S_{\mathfrak{n}+1}$ method [7,8] called the discrete ordinates method. It considers an approximation of $\psi(x,\mu,t)$ at $\mathfrak{n}+1$ values of μ. Let us denote μ_i these values and u_i an approximation of $\psi(x,\mu_i,t)$. To define $\tilde{\psi}$, we choose the quadrature formula: $\tilde{\psi} \simeq \frac{1}{2}\sum_{k=1}^{k=\mathfrak{n}+1} \omega_k u_k$ which leads to the $S_{\mathfrak{n}+1}$ equations:

$$\frac{\partial u_i}{\partial t} + \mu_i \frac{\partial u_i}{\partial x} + \sigma_t u_i \quad = (\sigma_t - \sigma_a)\frac{1}{2}\sum_{k=1}^{k=\mathfrak{n}+1} \omega_k u_k \qquad i = 1 ... \mathfrak{n}+1 \tag{1.4}$$

with the boundary conditions:

$$\begin{cases} u_i(\mathcal{A},t) = g_{\mathcal{A}}(\mu_i,t) \text{ for } \mu_i > 0 \\ u_i(\mathcal{B},t) = g_{\mathcal{B}}(\mu_i,t) \text{ for } \mu_i < 0. \end{cases} \tag{1.5}$$

Let us note that this method is positive $u_i(x,t) \geq 0$, $\forall i$.

In the $S_{\mathfrak{n}+1}$ equations, we can choose $\{\mu_i,\omega_i\}_{i=1...\mathfrak{n}+1}$ as the values and the weights of the Legendre quadrature formula of order $\mathfrak{n}+1$. Thus, μ_i are the $\mathfrak{n}+1$ roots of the $(\mathfrak{n}+1)$th Legendre polynomial and the weights ω_i are defined by:

$$\omega_i = \frac{-2}{(\mathfrak{n}+2)L_{\mathfrak{n}+2}(\mu_i)L'_{\mathfrak{n}+1}(\mu_i)}.$$

It can be proved [6] that if $(u_i)_{i=1...\mathfrak{n}+1}$ is solution of the $S_{\mathfrak{n}+1}$ equations with the previous choice of $\{\mu_i, \omega_i\}_{i=1...\mathfrak{n}+1}$ and the boundary conditions (1.5), then

$$\psi_\ell\,(x,t) \;=\; \sum_{i=1}^{i=\mathfrak{n}+1} \omega_i \sqrt{2\ell+1}\, L_\ell(\mu_i)\, u_i(x,t) \tag{1.6}$$

verify the $P_{\mathfrak{n}}$ equations for $\ell = 0, ..., \mathfrak{n}$ with the boundary conditions (1.3).

Reciprocally, if $(\psi_\ell)_{\ell=0,...,\mathfrak{n}}$ are solutions of the $P_{\mathfrak{n}}$ equations with the previous boundary conditions, then

$$u_i(x,t) = \sum_{\ell=0}^{\ell=\mathfrak{n}} \frac{\sqrt{2\ell+1}}{2} \psi_\ell\,(x,t) L_\ell(\mu_i) \qquad \forall i = 1...\mathfrak{n}+1 \tag{1.7}$$

are solutions of the $S_{\mathfrak{n}+1}$ equations with the already specified $\{\mu_i, \omega_i\}_{i=1...\mathfrak{n}+1}$ and the boundary conditions (1.5).

This equivalence between both $P_{\mathfrak{n}}$ and $S_{\mathfrak{n}+1}$ models will be often used to ease calculations in the next sections.

In the following, we assume that $\mathfrak{n}$ is odd so that none value of μ_k is zero. This is not restrictive, since the $P_{\mathfrak{n}}$ approximation with $\mathfrak{n}$ even, is known to be less accurate: indeed, ψ is generally not continuous in x for $\mu = 0$ [6]. In consequence, a quadrature formula with $\mu_i = 0$ may be inappropriate.

Let us conclude this section on the behaviour of the $P_{\mathfrak{n}}$ model in the two extreme regimes of diffusion and free streaming limit.

The diffusive regime is characterized by three features: the mean free path $\dfrac{1}{\sigma_t}$ much smaller than the dimension of the domain; σ_a much smaller than σ_t; the observation time much larger than the time of collision. This regime may be obtained by the introduction of the following scaling in the transport equation [16,17,23]:

$$\frac{\partial}{\partial t} \mapsto \epsilon \frac{\partial}{\partial t}, \quad \sigma_t \mapsto \frac{\sigma_t}{\epsilon}, \quad \sigma_a \mapsto \epsilon\sigma_a$$

where the parameter ϵ represents the inverse of the number of mean free path in the domain.

This scaling can be applied to the $P_{\mathfrak{n}}$ model and it can be shown [21] that when ϵ tends to zero, ψ_0 tends to the solution of the diffusion equation:

$$\frac{\partial \psi_0}{\partial t} - \frac{\partial}{\partial x}\left(\frac{1}{3\sigma_t}\frac{\partial \psi_0}{\partial x}\right) + \sigma_a \psi_0 \quad = 0. \tag{1.8}$$

This result has first been obtained formally; since, an exact convergence result has been derived in [8].

Let us now deal with the free streaming limit characterized by $\sigma_t = \sigma_a = 0$, $\mu = \pm 1$ which corresponds to the transport of a beam in the vacuum.

Proposition 1.1. *ψ_1 and ψ_0 verify:*

$$|\psi_1(x,t)| \le \psi_0(x,t)\sqrt{3}\max_i |\mu_i|. \tag{1.9}$$

Proof. We have $\psi_1(x,t) = \displaystyle\sum_{i=1}^{i=\mathfrak{n}+1} \omega_i \sqrt{3}\mu_i u_i(x,t)$ from we deduce: $|\psi_1(x,t)| \le \left(\displaystyle\sum_{i=1}^{i=\mathfrak{n}+1} \omega_i u_i(x,t)\right)\sqrt{3}\max_i |\mu_i|$.

The inequality (1.9) follows. □

Let us note that when $\mathfrak{n}$ tends to $+\infty$, the previous inequality tends to the *flux limited property* [19,23]:

$$|\psi_1(x,t)| \leq \psi_0(x,t)\sqrt{3}.$$

This property implies the correct velocity for the propagation of the neutrons in the free streaming limit ($\psi_1(x,t) = \pm\psi_0(x,t)\sqrt{3}$ for $\mu = \pm 1$).

2. Numerical solution of the P_1 equations

The $P_{\mathfrak{n}}$ equations (1.2) in the case $\mathfrak{n} = 1$ give the P_1 model:

$$\begin{cases} \dfrac{\partial \psi_0}{\partial t} + \dfrac{1}{\sqrt{3}} \dfrac{\partial \psi_1}{\partial x} + \sigma_a\, \psi_0 = 0 \\ \dfrac{\partial \psi_1}{\partial t} + \dfrac{1}{\sqrt{3}} \dfrac{\partial \psi_0}{\partial x} + \sigma_t\, \psi_1 = 0 \end{cases} \tag{2.1}$$

with the neutron density $\psi_0(x,t) = \displaystyle\int_{-1}^{1} \psi(x,\mu,t)\mathrm{d}\mu$ and $\psi_1(x,t) = \displaystyle\int_{-1}^{1} \sqrt{3}\mu\psi(x,\mu,t)\mathrm{d}\mu$. In addition, the closure relation gives $\psi_2 = 0$. This model is equivalent to the S_2 equations:

$$\begin{cases} \dfrac{\partial u_1}{\partial t} + \dfrac{1}{\sqrt{3}} \dfrac{\partial u_1}{\partial x} + \sigma_t u_1 = \dfrac{1}{2}(\sigma_t - \sigma_a)(u_1 + u_2) \\ \dfrac{\partial u_2}{\partial t} - \dfrac{1}{\sqrt{3}} \dfrac{\partial u_2}{\partial x} + \sigma_t u_2 = \dfrac{1}{2}(\sigma_t - \sigma_a)(u_1 + u_2). \end{cases} \tag{2.2}$$

This equivalence relies on the relations (1.6) and (1.7) which rewrite: $\psi_0 = u_1 + u_2$, $\psi_1 = u_1 - u_2$ and $u_1 = \dfrac{1}{2}(\psi_0 + \psi_1)$, $u_2 = \dfrac{1}{2}(\psi_0 - \psi_1)$.

We consider a uniform mesh of size h to discretize the spatial domain $[\mathcal{A}, \mathcal{B}]$. The cells are defined by $C_j = [x_{j-1/2}, x_{j+1/2}]$ where $x_{j+1/2} = x_j + \dfrac{h}{2}$, $j \in [1, N_x]$.

In this section, we propose to approximate the P_1 model following the ideas Gosse [11] has developed to solve the Goldstein-Taylor model. Indeed both models are very close: the Goldstein-Taylor equations may be obtained by taking $\sigma_a = 0$ and ± 1 instead of $\pm\dfrac{1}{\sqrt{3}}$ as characteristic speeds in the P_1 equations. Moreover, the numerical scheme of Gosse *et al.* has interesting properties, more particularly a good behaviour in the diffusive regime.

2.1. Derivation of a Gosse type scheme

2.1.1. *Characterization of the Riemann solver for the S_2 equations*

Following the ideas of Gosse and Toscani [12], the terms on the right hand side of system (2.2) are modified as follows:

$$\begin{cases} \dfrac{\partial u_1}{\partial t} + \dfrac{1}{\sqrt{3}} \dfrac{\partial u_1}{\partial x} = h \displaystyle\sum_j \left(\dfrac{\sigma_t}{2}(u_2 - u_1) - \dfrac{\sigma_a}{2}(u_1 + u_2) \right) \delta(x - x_{j-1/2}) \\ \dfrac{\partial u_2}{\partial t} - \dfrac{1}{\sqrt{3}} \dfrac{\partial u_2}{\partial x} = h \displaystyle\sum_j \left(\dfrac{\sigma_t}{2}(u_1 - u_2) - \dfrac{\sigma_a}{2}(u_1 + u_2) \right) \delta(x - x_{j-1/2}) \end{cases} \tag{2.3}$$

where $\delta(x - x_0)$ stands for the Dirac mass in $x = x_0$.

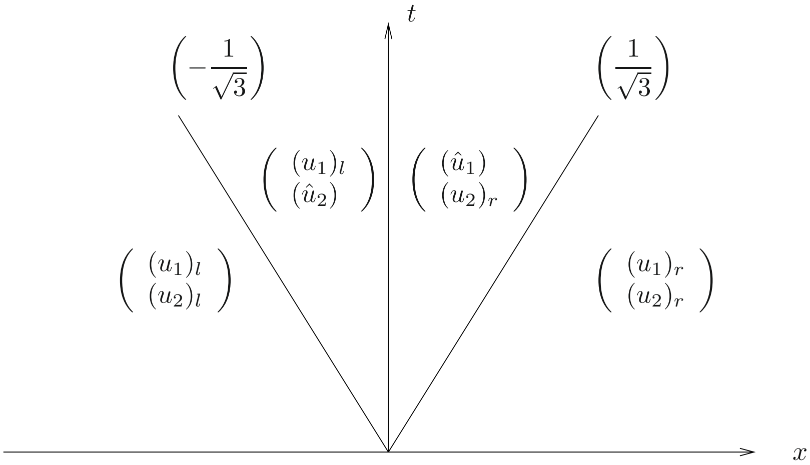

FIGURE 1. Riemann problem with the standing wave.

The Riemann problem associated with (2.3) involves a stationary contact discontinuity, which yields two unknown states $\{\hat{u_1}, \hat{u_2}\}$ (Fig. 1).

By analogy with [11,12], these states are computed by solving the steady equations:

$$\left\{\begin{array}{rcl} \dfrac{\mathrm{d}u_1^*}{\mathrm{d}\chi} & = & h\sqrt{3}\left(\dfrac{\sigma_t}{2}(u_2^*-u_1^*)-\dfrac{\sigma_a}{2}(u_1^*+u_2^*)\right) \\ -\dfrac{\mathrm{d}u_2^*}{\mathrm{d}\chi} & = & h\sqrt{3}\left(\dfrac{\sigma_t}{2}(u_1^*-u_2^*)-\dfrac{\sigma_a}{2}(u_1^*+u_2^*)\right) \end{array}\right. \quad \text{for } \chi\in[0,1] \tag{2.4}$$

with the boundary conditions:

$$\left\{\begin{array}{l} u_1^*(0)=(u_1)_l \\ u_2^*(1)=(u_2)_r. \end{array}\right. \tag{2.5}$$

In order to solve an easier system, we introduce the corresponding moments $\{\psi_0^*, \psi_1^*\}$ and we obtain:

$$\left\{\begin{array}{l} \dfrac{\mathrm{d}\psi_1^*}{\mathrm{d}\chi}=-2\beta\psi_0^* \\ \dfrac{\mathrm{d}\psi_0^*}{\mathrm{d}\chi}=-2\alpha\psi_1^* \end{array}\right. \tag{2.6}$$

where we have set $\beta = h\sqrt{3}\dfrac{\sigma_a}{2}$ and $\alpha = h\sqrt{3}\dfrac{\sigma_t}{2}$. Let us remark that α and β are positive.

The boundary conditions of the above system are given by:

$$\left\{\begin{array}{l} \psi_0^*(0)+\psi_1^*(0)=(\psi_0)_l+(\psi_1)_l \\ \psi_0^*(1)-\psi_1^*(1)=(\psi_0)_r-(\psi_1)_r. \end{array}\right.$$

As the parameter β differs from zero, ψ_1^* depends on χ. In addition, it satisfies: $\psi_1^* = -\dfrac{1}{2\alpha}\dfrac{\mathrm{d}\psi_0^*}{\mathrm{d}\chi}$. The introduction of this relation in the first equation of (2.6) leads to the following ordinary differential equation:

$$\frac{\mathrm{d}^2\psi_0^*}{\mathrm{d}^2\chi}=C^2\psi_0^*\ , \quad C=2\sqrt{\alpha\beta}$$

whose solution is:

$$\psi_0^*(\chi) = a\exp(\chi C) + b\exp(-\chi C) \;\; \{a, b\} \in \mathbb{R}.$$

This statement and the second equation of (2.6) yield $\psi_1^*(\chi) = -\frac{C}{2\alpha}[a\exp(\chi C) - b\exp(-\chi C)]$.

We are now able to compute the solutions of the system (2.4):

$$\begin{cases} u_1^*(\chi) = \frac{a}{2}\left(1 - \frac{C}{2\alpha}\right)\exp(\chi C) + \frac{b}{2}\left(1 + \frac{C}{2\alpha}\right)\exp(-\chi C) \\ u_2^*(\chi) = \frac{a}{2}\left(1 + \frac{C}{2\alpha}\right)\exp(\chi C) + \frac{b}{2}\left(1 - \frac{C}{2\alpha}\right)\exp(-\chi C). \end{cases}$$

The boundary conditions (2.5) lead to a set of 2 equations with 2 unknowns $\{a, b\}$ and we get:

$$\begin{cases} a = 4\alpha\dfrac{(2\alpha - C)\exp(-C)(u_1)_l - (2\alpha + C)(u_2)_r}{(2\alpha - C)^2\exp(-C) - (2\alpha + C)^2\exp(C)} \\ b = -4\alpha\dfrac{(2\alpha + C)\exp(C)(u_1)_l - (2\alpha - C)(u_2)_r}{(2\alpha - C)^2\exp(-C) - (2\alpha + C)^2\exp(C)}. \end{cases}$$

These identities give linear expressions of the unknown states $\{\hat{u}_1, \hat{u}_2\}$ in terms of $\{(u_1)_l, (u_2)_r\}$:

$$\begin{cases} \hat{u}_1 = u_1^*(1) = \tilde{a}(u_1)_l + \tilde{b}(u_2)_r \\ \hat{u}_2 = u_2^*(0) = \tilde{b}(u_1)_l + \tilde{a}(u_2)_r \end{cases} \tag{2.7}$$

with:

$$\begin{cases} \tilde{a} = \dfrac{2\sqrt{\alpha\beta}}{2\sqrt{\alpha\beta}\cosh(2\sqrt{\alpha\beta}) + (\alpha + \beta)\sinh(2\sqrt{\alpha\beta})} \\ \tilde{b} = \dfrac{\alpha - \beta}{\alpha + \beta + 2\sqrt{\alpha\beta}\coth(2\sqrt{\alpha\beta})}. \end{cases} \tag{2.8}$$

Lemma 2.1. *The coefficients* $\left\{\tilde{a}, \tilde{b}\right\}$ *satisfy the following properties:*

$$\begin{cases} \tilde{a} > 0 \\ \tilde{b} > 0 \\ \tilde{a} + \tilde{b} < 1. \end{cases}$$

Proof. Trivial because of the properties of the hyperbolic functions. □

This result means the unknown states $\{\hat{u}_1, \hat{u}_2\}$ are positive and satisfy $\hat{u}_{1,2} < \max((u_1)_l, (u_2)_r)$.

2.1.2. *Explicit Gosse type schemes*

If we apply the solver proposed in the last section to the cell C_j, with the notations given in Figure 2 for the different states, we can derive a Godunov type scheme to discretize the S_2 equations:

$$\begin{cases} (u_1)_j^{n+1} = (u_1)_j^n + \dfrac{\Delta t}{h\sqrt{3}}\left((\hat{u}_1)_{j-1/2}^n - (u_1)_j^n\right) \\ (u_2)_j^{n+1} = (u_2)_j^n + \dfrac{\Delta t}{h\sqrt{3}}\left((\hat{u}_2)_{j+1/2}^n - (u_2)_j^n\right) \end{cases} \tag{2.9}$$

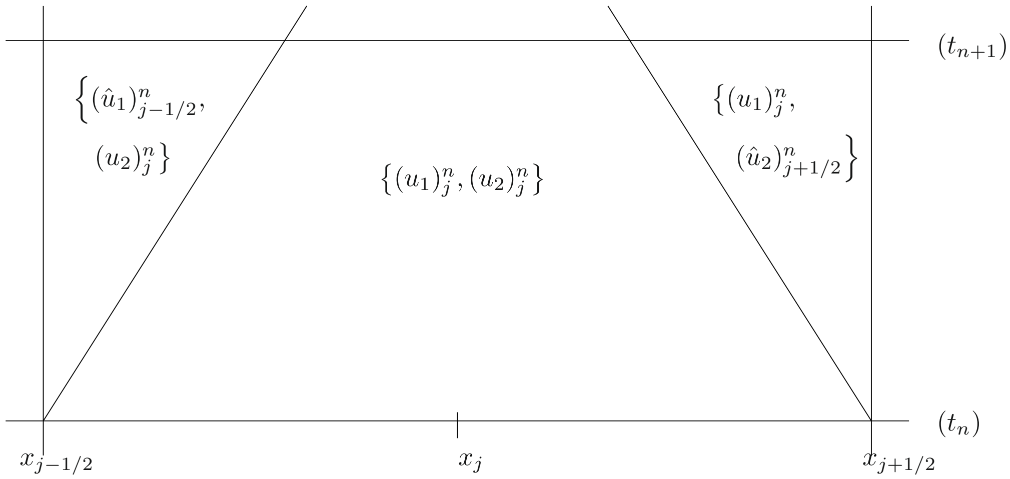

FIGURE 2. Riemann problem in the cell C_j.

with:

$$\begin{cases} (\hat{u}_1)^n_{j-1/2} = \tilde{a}(u_1)^n_{j-1} + \tilde{b}(u_2)^n_j \\ (\hat{u}_2)^n_{j+1/2} = \tilde{b}(u_1)^n_j + \tilde{a}(u_2)^n_{j+1} \end{cases} \tag{2.10}$$

because of the relations (2.7).

The numerical scheme for the P_1 equations is obtained using the last discretization (2.9) and the relations on which the equivalence between both S_2 and P_1 systems relies (see Sect. 1). So the summation and the difference of both schemes relative to the variables $\{u_1, u_2\}$ yield an explicit discretization of the P_1 model:

$$\begin{cases} (\psi_0)^{n+1}_j = (\psi_0)^n_j - \dfrac{\Delta t}{h\sqrt{3}}\tilde{a}\left((\psi_1)^n_{j+1/2} - (\psi_1)^n_{j-1/2}\right) + \dfrac{\Delta t}{h\sqrt{3}}(\tilde{a} + \tilde{b} - 1)(\psi_0)^n_j \\ (\psi_1)^{n+1}_j = (\psi_1)^n_j - \dfrac{\Delta t}{h\sqrt{3}}\tilde{a}\left((\psi_0)^n_{j+1/2} - (\psi_0)^n_{j-1/2}\right) + \dfrac{\Delta t}{h\sqrt{3}}(\tilde{a} - \tilde{b} - 1)(\psi_1)^n_j \end{cases} \tag{2.11}$$

where we have set:

$$\begin{cases} (\psi_0)^n_{j+1/2} = \dfrac{(\psi_0)^n_j + (\psi_0)^n_{j+1}}{2} + \dfrac{(\psi_1)^n_j - (\psi_1)^n_{j+1}}{2} \\ (\psi_1)^n_{j+1/2} = \dfrac{(\psi_1)^n_j + (\psi_1)^n_{j+1}}{2} + \dfrac{(\psi_0)^n_j - (\psi_0)^n_{j+1}}{2}. \end{cases} \tag{2.12}$$

Because of the construction proposed, the scheme (2.11) is equivalent to the scheme (2.9).

Remark 2.2. Let us note that Buet and Cordier have derived the expressions (2.12) in [5] with the coefficient σ_a set to zero.

Hence an explicit numerical scheme has been derived to solve both P_1 and S_2 equations. It satisfies the following properties:

Proposition 2.3. *Under the CFL condition* $\dfrac{\Delta t}{h\sqrt{3}} \leq 1$*:*

- *the explicit discretization* (2.9)–(2.11) *is* L^∞*-stable;*

- *at the discrete level, the scheme* (2.11) *satisfies the same property* (1.9) *as the* P_1 *model at the PDE level:*

$$|(\psi_1)_j^{n+1}| \le (\psi_0)_j^{n+1} \quad (\forall j, \forall n)\cdot \tag{2.13}$$

Let us remark that the second inequality (2.13) implies that for all j: $|(\psi_1)_j^{n+1}| \le \sqrt{3}(\psi_0)_j^{n+1}$, which is the *flux limited property.*

Proof. To ease the proof, we use the S_2 formulation of the discretization.

Assume both properties are true until time t_n:

$$\begin{cases} ||u_1^n||_\infty \le A,\ ||u_2^n||_\infty \le A \\ (u_1)_j^n \ge 0,\ (u_2)_j^n \ge 0 \qquad \forall j \end{cases}$$

where A is a constant independent of n and Δt.

The scheme (2.9) can be rewritten as:

$$\begin{cases} (u_1)_j^{n+1} = \left(1 - \dfrac{\Delta t}{h\sqrt{3}}\right)(u_1)_j^n + \dfrac{\Delta t}{h\sqrt{3}}\tilde{a}(u_1)_{j-1}^n + \dfrac{\Delta t}{h\sqrt{3}}\tilde{b}(u_2)_j^n \\ (u_2)_j^{n+1} = \left(1 - \dfrac{\Delta t}{h\sqrt{3}}\right)(u_2)_j^n + \dfrac{\Delta t}{h\sqrt{3}}\tilde{a}(u_2)_{j+1}^n + \dfrac{\Delta t}{h\sqrt{3}}\tilde{b}(u_1)_j^n. \end{cases}$$

From Lemma 2.1 and the CFL condition $\dfrac{\Delta t}{h\sqrt{3}} \le 1$, we deduce that $(u_1)_j^{n+1}$ is a linear combination of $\{(u_1)_j^n, (u_1)_{j-1}^n, (u_2)_j^n\}$ with positive coefficients; the same results holds for $(u_2)_j^{n+1}$ in terms of $\{(u_2)_j^n, (u_2)_{j+1}^n, (u_1)_j^n\}$. Hence, the scheme is positive: $(u_1)_j^{n+1} \ge 0,\ (u_2)_j^{n+1} \ge 0,\ \forall j$. The inequality (2.13) follows.

Moreover, the following inequalities may be written:

$$\begin{cases} (u_1)_j^{n+1} \le \left[1 - \dfrac{\Delta t}{h\sqrt{3}}(1 - \tilde{a} - \tilde{b})\right] A \le A \qquad \forall j \\ (u_2)_j^{n+1} \le \left[1 - \dfrac{\Delta t}{h\sqrt{3}}(1 - \tilde{a} - \tilde{b})\right] A \le A \qquad \forall j \end{cases}$$

which proves the L^∞-stability because of Lemma 2.1. □

2.1.3. *Implicit Gosse type schemes*

This section is devoted to present an implicit version of the above scheme, which will be used later because of its unconditional stability.

We first consider the implicit Godunov type scheme:

$$\begin{cases} (u_1)_j^{n+1} = (u_1)_j^n + \dfrac{\Delta t}{h\sqrt{3}}\left((\hat{u}_1)_{j-1/2}^{n+1} - (u_1)_j^{n+1}\right) \\ (u_2)_j^{n+1} = (u_2)_j^n + \dfrac{\Delta t}{h\sqrt{3}}\left((\hat{u}_2)_{j+1/2}^{n+1} - (u_2)_j^{n+1}\right) \end{cases} \tag{2.14}$$

corresponding to the S_2 equations. The definition of the quantities $\{(\hat{u}_1)^{n+1}_{j-1/2}, (\hat{u}_2)^{n+1}_{j+1/2}\}$ is the canonical extension of the relations (2.10):

$$\left\{\begin{array}{l} (\hat{u}_1)^{n+1}_{j-1/2} = \tilde{a}(u_1)^{n+1}_{j-1} + \tilde{b}(u_2)^{n+1}_j \\ \\ (\hat{u}_2)^{n+1}_{j+1/2} = \tilde{b}(u_1)^{n+1}_j + \tilde{a}(u_2)^{n+1}_{j+1}. \end{array}\right. \tag{2.15}$$

Remark 2.4. Let us note that all the terms are treated implicitly in the scheme (2.14), not only the stiff convection terms as in the scheme prescribed in [11] for the Goldstein-Taylor model.

As in the explicit case, we can derive the following implicit scheme for the P_1 equations:

$$\left\{\begin{array}{l} (\psi_0)^{n+1}_j = (\psi_0)^n_j - \dfrac{\Delta t}{h\sqrt{3}}\tilde{a}\left((\psi_1)^{n+1}_{j+1/2} - (\psi_1)^{n+1}_{j-1/2}\right) + \dfrac{\Delta t}{h\sqrt{3}}(\tilde{a}+\tilde{b}-1)(\psi_0)^{n+1}_j \\ \\ (\psi_1)^{n+1}_j = (\psi_1)^n_j - \dfrac{\Delta t}{h\sqrt{3}}\tilde{a}\left((\psi_0)^{n+1}_{j+1/2} - (\psi_0)^{n+1}_{j-1/2}\right) + \dfrac{\Delta t}{h\sqrt{3}}(\tilde{a}-\tilde{b}-1)(\psi_1)^{n+1}_j \end{array}\right. \tag{2.16}$$

where $\left\{(\psi_0)^{n+1}_{j+1/2}, (\psi_1)^{n+1}_{j+1/2}\right\}$ are defined by the extension of the explicit identities (2.12):

$$\left\{\begin{array}{l} (\psi_0)^{n+1}_{j+1/2} = \dfrac{(\psi_0)^{n+1}_j + (\psi_0)^{n+1}_{j+1}}{2} + \dfrac{(\psi_1)^{n+1}_j - (\psi_1)^{n+1}_{j+1}}{2} \\ (\psi_1)^{n+1}_{j+1/2} = \dfrac{(\psi_1)^{n+1}_j + (\psi_1)^{n+1}_{j+1}}{2} + \dfrac{(\psi_0)^{n+1}_j - (\psi_0)^{n+1}_{j+1}}{2}. \end{array}\right.$$

Proposition 2.5.

- *The implicit discretization (2.14)–(2.16) is unconditionally L^∞-stable.*
- *At the discrete level, the scheme (2.16) satisfies the same property (1.9) as the P_1 model at the PDE level:*

$$|(\psi_1)^{n+1}_j| \le (\psi_0)^{n+1}_j \quad (\forall j, \forall n). \tag{2.17}$$

Let us remark that the second inequality (2.17) implies that for all j: $|(\psi_1)^{n+1}_j| \le \sqrt{3}(\psi_0)^{n+1}_j$, which is the *flux limited property.*

Proof. Let us establish some properties on the discretization (2.14) of the S_2 model. It can be rewritten as:

$$\left\{\begin{array}{l} -\dfrac{\Delta t}{h\sqrt{3}}\tilde{a}(u_1)^{n+1}_{j-1} + \left(1+\dfrac{\Delta t}{h\sqrt{3}}\right)(u_1)^{n+1}_j - \dfrac{\Delta t}{h\sqrt{3}}\tilde{b}(u_2)^{n+1}_j = (u_1)^n_j \\ -\dfrac{\Delta t}{h\sqrt{3}}\tilde{b}(u_1)^{n+1}_j + \left(1+\dfrac{\Delta t}{h\sqrt{3}}\right)(u_2)^{n+1}_j - \dfrac{\Delta t}{h\sqrt{3}}\tilde{a}(u_2)^{n+1}_{j+1} = (u_2)^n_j. \end{array}\right. \tag{2.18}$$

This form leads to the resolution of a linear system to achieve the computation of $\left\{u_1^{n+1}, u_2^{n+1}\right\}$. According to the properties the coefficients $\{\tilde{a}, \tilde{b}\}$ satisfy (see Lem. 2.1), the matrix of this linear system is a M-matrix. This property implies that the scheme (2.14) is positive which gives the inequality (2.17).

To show the L^∞-stability, we assume the property is true until time t_n:

$$||u_1^n||_\infty \le A,\ ||u_2^n||_\infty \le A.$$

Let us denote j_0 the cell where

$$(u_l)^{n+1}_{j_0} = \max_j\left((u_1)^{n+1}_j, (u_2)^{n+1}_j\right). \tag{2.19}$$

Assume $l = 1$. Then, we have:

$$(u_1)_{j_0}^{n+1} = \frac{\dfrac{\Delta t}{h\sqrt{3}}\tilde{a}(u_1)_{j_0-1}^{n+1} + (u_1)_{j_0}^{n} + \dfrac{\Delta t}{h\sqrt{3}}\tilde{b}(u_2)_{j_0}^{n+1}}{1 + \dfrac{\Delta t}{h\sqrt{3}}}.$$

Because of the definition (2.19) and Lemma 2.1, the following inequality may be derived:

$$(u_1)_{j_0}^{n+1} \leq \frac{\dfrac{\Delta t}{h\sqrt{3}}}{1 + \dfrac{\Delta t}{h\sqrt{3}}}(u_1)_{j_0}^{n+1} + \frac{1}{1 + \dfrac{\Delta t}{h\sqrt{3}}}(u_1)_{j_0}^{n}$$

from which we deduce:

$$(u_1)_{j_0}^{n+1} < (u_1)_{j_0}^{n}$$

and:

$$\begin{aligned} ||u_1^{n+1}||_\infty &\leq (u_1)_{j_0}^{n+1} \leq ||u_1^n||_\infty \leq A \\ ||u_2^{n+1}||_\infty &\leq (u_1)_{j_0}^{n+1} \leq ||u_1^n||_\infty \leq A. \end{aligned}$$

If (2.19) is realized by $l = 2$, in the same way as in the previous case, we can show:

$$\begin{aligned} ||u_1^{n+1}||_\infty &\leq (u_2)_{j_0}^{n+1} \leq ||u_2^n||_\infty \leq A \\ ||u_2^{n+1}||_\infty &\leq (u_2)_{j_0}^{n+1} \leq ||u_2^n||_\infty \leq A \end{aligned}$$

which ends the proof. □

Let us conclude this section by some remarks on the matrix of the linear system (2.18) which may be useful to solve this system:

- it is non symmetric;
- if $\{(u_1)_j, (u_2)_j\}$ are stored by pairs, it is block tridiagonal;
- because $0 < \tilde{b} < 1$, the diagonal blocks are not singular. They are given by:

$$\begin{pmatrix} 1 + \dfrac{\Delta t}{h\sqrt{3}} & -\dfrac{\Delta t}{h\sqrt{3}}\tilde{b} \\ -\dfrac{\Delta t}{h\sqrt{3}}\tilde{b} & 1 + \dfrac{\Delta t}{h\sqrt{3}} \end{pmatrix}.$$

Hence, a convenient way to solve the linear system is to use the block Gauss Seidel iteration method: its convergence is ensured by the above properties of the matrix.

2.2. **Well-balanced property**

According to [13], we recall the meaning of this property:

Definition 2.6. A numerical scheme is said well-balanced (WB) if it preserves at the discrete level the steady states of the partial differential equations it discretizes.

Proposition 2.7. *The explicit scheme* (2.9)–(2.11) *is well-balanced.*

Proof. Let us introduce $\{u_1^{ex}, u_2^{ex}\}$ the exact solutions of the steady equations:

$$\begin{cases} \dfrac{1}{\sqrt{3}}\dfrac{\mathrm{d}u_1^{ex}}{\mathrm{d}x} + \sigma_t u_1^{ex} = \dfrac{1}{2}(\sigma_t - \sigma_a)(u_1^{ex} + u_2^{ex}) \\ -\dfrac{1}{\sqrt{3}}\dfrac{\mathrm{d}u_2^{ex}}{\mathrm{d}x} + \sigma_t u_2^{ex} = \dfrac{1}{2}(\sigma_t - \sigma_a)(u_1^{ex} + u_2^{ex}) \end{cases}$$

with the boundary conditions:

$$\begin{cases} u_1^{ex}(\mathcal{A}) = g_{\mathcal{A}}(\mu_1) \\ u_2^{ex}(\mathcal{B}) = g_{\mathcal{B}}(\mu_2). \end{cases}$$

Assume that the initial states $\{(u_1)_j^0, (u_2)_j^0\}$ in the cell C_j are a discretization of $\{u_1^{ex}, u_2^{ex}\}$ at the center of the cell. Assume now the WB property is satisfied until the discrete time level t_n. Let us show it is true at the time t_{n+1}.

Since $\{u_1^{ex}, u_2^{ex}\}$ are the exact solutions of the steady equations, they verify:

$$\begin{cases} u_1^{ex}(x_j) = \tilde{a}u_1^{ex}(x_{j-1}) + \tilde{b}u_2^{ex}(x_j) \\ u_2^{ex}(x_j) = \tilde{b}u_1^{ex}(x_j) + \tilde{a}u_2^{ex}(x_{j+1}) \end{cases} \tag{2.20}$$

because of the relations (2.7) applied on both intervals $[x_{j-1}, x_j]$ and $[x_j, x_{j+1}]$ respectively for both values $\{u_1^{ex}, u_2^{ex}\}$ respectively.

As the WB property is satisfied until the discrete time level t_n, we also have: $u_1^{ex}(x_j) = (u_1)_j^n$ and $u_2^{ex}(x_j) = (u_2)_j^n$, $\forall j$. Hence, we deduce:

$$\begin{cases} (u_1)_j^n = \tilde{a}(u_1)_{j-1}^n + \tilde{b}(u_2)_j^n \\ (u_2)_j^n = \tilde{b}(u_1)_j^n + \tilde{a}(u_2)_{j+1}^n \end{cases}$$

and:

$$\begin{cases} (\hat{u}_1)_{j-1/2}^n = (u_1)_j^n \\ (\hat{u}_2)_{j+1/2}^n = (u_2)_j^n \end{cases}$$

because of the identities (2.10). The scheme (2.9) is involved to obtain the desired property:

$$\begin{cases} (u_1)_j^{n+1} = (u_1)_j^n \\ (u_2)_j^{n+1} = (u_2)_j^n. \end{cases}$$

□

Proposition 2.8. *The implicit scheme* (2.14)–(2.16) *is well-balanced.*

Proof. We make the same assumptions as in the explicit case: the initial states $\{(u_1)_j^0, (u_2)_j^0\}$ are a discretization of $\{u_1^{ex}, u_2^{ex}\}$ at the center of the cell C_j; the WB property is satisfied until the discrete time level t_n. Let us show it is true at the time t_{n+1}.

The linear system involved by the implicit scheme has an unique solution since the corresponding matrix of the linear system is a M-matrix (see previous section). So we just have to show that $\{u_1^{ex}(x_j), u_2^{ex}(x_j)\}$ is this unique solution. Let us introduce them in the scheme (2.14) and the relations (2.15). The recurrence assumption leads to:

$$\begin{cases} u_1^{ex}(x_j) = \tilde{a}u_1^{ex}(x_{j-1}) + \tilde{b}u_2^{ex}(x_j) \\ u_2^{ex}(x_j) = \tilde{b}u_1^{ex}(x_j) + \tilde{a}u_2^{ex}(x_{j+1}) \end{cases}$$

which ends the proof because of the definition of $\{u_1^{ex}, u_2^{ex}\}$ (see relation (2.20)). □

2.3. Asymptotic preserving property

This section is devoted to the behaviour of the numerical schemes, derived in the last sections, in the diffusive regime. To study this point, we introduce the following definition:

Definition 2.9. A numerical scheme is said to be asymptotic preserving (AP) or to have the diffusion limit if it satisfies a consistent discretization of the limiting diffusion equation (1.8), when ϵ tends to zero.

The analysis of this property requires the introduction of the "discrete diffusive scaling":

$$\left\{\Delta t \mapsto \frac{\Delta t}{\epsilon},\ \sigma_t \mapsto \frac{\sigma_t}{\epsilon},\ \sigma_a \mapsto \epsilon\sigma_a\right\}$$

into the numerical schemes. The parameter ϵ represents a characteristic value of the inverse of the number of mean free path in all cells.

In this section, among the different schemes proposed, we only consider the implicit discretization (2.14)–(2.16). Indeed, when the explicit scheme (2.9)–(2.11) is used to compute the diffusive regime, the CFL condition becomes $\dfrac{\Delta t}{\epsilon h\sqrt{3}} \leq 1$, which is much too restrictive.

Proposition 2.10. *The scheme* (2.16) *is AP.* ψ_0 *at the order* 0 *satisfies the following consistent discretization of the diffusion equation* (1.8)*:*

$$(\psi_0)_j^{n+1,(0)} = (\psi_0)_j^{n,(0)} + \frac{\Delta t}{3\sigma_t h^2}\left((\psi_0)_{j-1}^{n+1,(0)} - 2(\psi_0)_j^{n+1,(0)} + (\psi_0)_{j+1}^{n+1,(0)}\right) - \Delta t\sigma_a(\psi_0)_j^{n+1,(0)} + O(h^2). \quad (2.21)$$

Proof. To study the diffusion limit, two main calculations have to be performed: the introduction of the "discrete diffusive scaling" into the numerical scheme and the expansion of the numerical approximation of $\{\psi_0, \psi_1\}$. Let us note:

$$(\psi_0)_k^p = \sum_{l=0}^{l=+\infty}(\psi_0)_k^{p,(l)}\epsilon^l; \qquad (\psi_1)_k^p = \sum_{l=0}^{l=+\infty}(\psi_1)_k^{p,(l)}\epsilon^l$$

where k stands for the space discretization and p the time discretization.

The "discrete diffusive scaling" is applied to the scheme (2.16):

$$\left\{\begin{array}{l}(\psi_0)_j^{n+1} = (\psi_0)_j^n - \dfrac{\Delta t}{\epsilon h\sqrt{3}}\tilde{a}_\epsilon\left((\psi_1)_{j+1/2}^{n+1} - (\psi_1)_{j-1/2}^{n+1}\right) + \dfrac{\Delta t}{\epsilon h\sqrt{3}}(\tilde{a}_\epsilon + \tilde{b}_\epsilon - 1)(\psi_0)_j^{n+1}\\ (\psi_1)_j^{n+1} = (\psi_1)_j^n - \dfrac{\Delta t}{\epsilon h\sqrt{3}}\tilde{a}_\epsilon\left((\psi_0)_{j+1/2}^{n+1} - (\psi_0)_{j-1/2}^{n+1}\right) + \dfrac{\Delta t}{\epsilon h\sqrt{3}}(\tilde{a}_\epsilon - \tilde{b}_\epsilon - 1)(\psi_1)_j^{n+1}\end{array}\right. \quad (2.22)$$

where $\{\tilde{a}_\epsilon, \tilde{b}_\epsilon\}$ denote the natural extension of $\{\tilde{a}, \tilde{b}\}$ defined by the expressions (2.8):

$$\left\{\begin{array}{l}\tilde{a}_\epsilon = \dfrac{2\sqrt{\alpha_\epsilon\beta_\epsilon}}{2\sqrt{\alpha_\epsilon\beta_\epsilon}\cosh(2\sqrt{\alpha_\epsilon\beta_\epsilon}) + (\alpha_\epsilon + \beta_\epsilon)\sinh(2\sqrt{\alpha_\epsilon\beta_\epsilon})}\\ \tilde{b}_\epsilon = \dfrac{\alpha_\epsilon - \beta_\epsilon}{\alpha_\epsilon + \beta_\epsilon + 2\sqrt{\alpha_\epsilon\beta_\epsilon}\coth(2\sqrt{\alpha_\epsilon\beta_\epsilon})}\end{array}\right.$$

with $\alpha_\epsilon = \dfrac{\alpha}{\epsilon}$ and $\beta_\epsilon = \beta\epsilon$. Let us note that $\alpha_\epsilon\beta_\epsilon$ remains constant, as the constant C, in spite of the scaling.

These relations lead to:

$$\left\{\begin{array}{lcl}\dfrac{\tilde{a}_\epsilon}{\epsilon} & = & \dfrac{-8\alpha C}{4\alpha^2[\exp(-C)-\exp(C)]-4\alpha C[\exp(-C)+\exp(C)]\epsilon+C^2[\exp(-C)-\exp(C)]\epsilon^2} \\ \dfrac{\tilde{a}_\epsilon+\tilde{b}_\epsilon-1}{\epsilon} & = & \dfrac{4\alpha C[\exp(-C)+\exp(C)-2]+2C^2[\exp(C)-\exp(-C)]\epsilon}{4\alpha^2[\exp(-C)-\exp(C)]-4\alpha C[\exp(-C)+\exp(C)]\epsilon+C^2[\exp(-C)-\exp(C)]\epsilon^2} \\ \dfrac{\tilde{a}_\epsilon-\tilde{b}_\epsilon-1}{\epsilon} & = & -\dfrac{2\alpha[\exp(C)-\exp(-C)]+C[\exp(C)+\exp(-C)-2]\epsilon}{\alpha[\exp(C)-\exp(-C)]\epsilon+C[\exp(C)+\exp(-C)]\epsilon^2+\dfrac{C^2}{4\alpha}[\exp(C)-\exp(-C)]\epsilon^3}.\end{array}\right. \tag{2.23}$$

In the second equation of (2.22), the term $\dfrac{\tilde{a}_\epsilon}{\epsilon}$ being of order 0 in ϵ, there is only one term function of $\dfrac{1}{\epsilon}$: $\dfrac{-2}{\epsilon h\sqrt{3}}$ coming from $\dfrac{\tilde{a}_\epsilon-\tilde{b}_\epsilon-1}{\epsilon h\sqrt{3}}$. This remark yields:

$$(\psi_1)_j^{n+1,(0)}=0 \qquad (\forall j) \quad\Longrightarrow\quad (\psi_1)_{j+1/2}^{n+1,(0)}=\frac{(\psi_0)_j^{n+1,(0)}-(\psi_0)_{j+1}^{n+1,(0)}}{2} \qquad (\forall j).$$

Because of the last statement and the identities (2.23), the scheme (2.22) on $(\psi_0)_j^{n+1}$ rewrites as follows:

$$\begin{array}{rcl}(\psi_0)_j^{n+1,(0)} & = & (\psi_0)_j^{n,(0)}+\dfrac{\Delta t}{h\sqrt{3}}\left[\dfrac{2C}{\alpha(\exp(-C)-\exp(C))}\right]\left[\dfrac{-(\psi_0)_{j+1}^{n+1,(0)}+2(\psi_0)_j^{n+1,(0)}-(\psi_0)_{j-1}^{n+1,(0)}}{2}\right] \\ & & +\dfrac{\Delta t}{h\sqrt{3}}\left[\dfrac{C(\exp(-C)+\exp(C)-2)}{\alpha(\exp(-C)-\exp(C))}\right](\psi_0)_j^{n+1,(0)}\end{array} \tag{2.24}$$

at the order 0 in ϵ.

Remember that α and C depend on h: $\alpha=h\sqrt{3}\dfrac{\sigma_t}{2}$ and $C=h\sqrt{3\sigma_t\sigma_a}$. The introduction of these definitions in the relation satisfied by $(\psi_0)_j^{n+1,(0)}$ leads to an expression only depending on C. To obtain a result free of this constant, let us make a Taylor expansion about h:

$$\begin{array}{l}(\psi_0)_j^{n+1,(0)}=(\psi_0)_j^{n,(0)}+\dfrac{\Delta t}{3\sigma_t}\left(\dfrac{(\psi_0)_{j+1}^{n+1,(0)}-2(\psi_0)_j^{n+1,(0)}+(\psi_0)_{j-1}^{n+1,(0)}}{h^2}\right)\left(1-\dfrac{1}{2}\sigma_t\sigma_a h^2+O(h^4)\right) \\ \qquad\qquad -\Delta t\sigma_a\left(1-\dfrac{1}{4}\sigma_t\sigma_a h^2+O(h^4)\right)(\psi_0)_j^{n+1,(0)}\end{array}$$

which gives the relation (2.21) because $\dfrac{(\psi_0)_{j+1}^{n+1,(0)}-2(\psi_0)_j^{n+1,(0)}+(\psi_0)_{j-1}^{n+1,(0)}}{h^2}=O(1)$. □

2.4. Extension to a non uniform mesh and non constant $(\boldsymbol{\sigma_a},\boldsymbol{\sigma_t})$

If σ_a and σ_t are not constant and the mesh is not uniform, one must replace the system (2.4) by the following set of ordinary differential equations:

$$\left\{\begin{array}{rcl}\dfrac{\mathrm{d}u_1^*}{\mathrm{d}\chi} & = & \dfrac{h_{j-1}+h_j}{2}\sqrt{3}\left(\dfrac{\sigma_t(\chi)}{2}(u_2^*-u_1^*)-\dfrac{\sigma_a(\chi)}{2}(u_1^*+u_2^*)\right) \\ -\dfrac{\mathrm{d}u_2^*}{\mathrm{d}\chi} & = & \dfrac{h_{j-1}+h_j}{2}\sqrt{3}\left(\dfrac{\sigma_t(\chi)}{2}(u_1^*-u_2^*)-\dfrac{\sigma_a(\chi)}{2}(u_1^*+u_2^*)\right)\end{array}\right. \tag{2.25}$$

to derive a numerical WB scheme. The mesh size h_j satisfies $x_{j+1/2} = x_j + \frac{h_j}{2}$ and the coefficients $\sigma_{t,a}(\chi)$ are defined by:

$$\sigma_{t,a}(\chi) = \begin{cases} (\sigma_{t,a})_{j-1} & \text{for} \quad \chi < \dfrac{x_{j-1/2} - x_{j-1}}{x_j - x_{j-1}} \\ (\sigma_{t,a})_j & \text{for} \quad \chi > \dfrac{x_{j-1/2} - x_{j-1}}{x_j - x_{j-1}}. \end{cases}$$

The resolution of (2.25) can be made by introducing the variables (ψ_0^*, ψ_1^*) once again. In this case, we obtain a diffusion equation on ψ_0^* with variable coefficients:

$$\frac{\mathrm{d}}{\mathrm{d}\chi}\left(\frac{-1}{2\alpha}\frac{\mathrm{d}\psi_0^*}{\mathrm{d}\chi}\right) + 2\beta\psi_0^* = 0$$

which can easily be solved by classical techniques.

3. A numerical scheme for the $P_{\mathfrak{n}}$ equations

For a sake of simplicity, let us consider the same uniform mesh as before.

Here, we extend the ideas developed in the last section to discretize the $P_{\mathfrak{n}}$ system (1.2).

Let us note we sort the μ_k in the following way: $k = 1$ to $k = \frac{\mathfrak{n}+1}{2}$ refer to positive and increasing μ_k while $k = \frac{\mathfrak{n}+3}{2}$ to $k = \mathfrak{n}+1$ refer to negative and decreasing μ_k.

Moreover, remark that the μ_k are symmetric: $\mu_k = -\mu_{k+\frac{\mathfrak{n}+1}{2}}$ for $k \in \left\{1, ..., \frac{\mathfrak{n}+1}{2}\right\}$ because the $L_{\mathfrak{n}+1}$ Legendre polynomial is even if $\mathfrak{n}$ is odd.

3.1. Derivation of a Gosse type scheme

3.1.1. *Characterization of the Riemann solver for the $S_{\mathfrak{n}+1}$ equations*

According to the scheme derived for the P_1 equations, the system $S_{\mathfrak{n}+1}$ (1.4) is replaced by the following one:

$$\frac{\partial u_i}{\partial t} + \mu_i \frac{\partial u_i}{\partial x} = h \sum_j \left((\sigma_t - \sigma_a)\frac{1}{2}\sum_{k=1}^{k=\mathfrak{n}+1} \omega_k u_k - \sigma_t u_i \right) \delta(x - x_{j-1/2}) \qquad i = 1...\mathfrak{n}+1. \tag{3.1}$$

As for the S_2 equations, the Riemann problem associated with (3.1) involves a stationary contact discontinuity, which yields $\frac{\mathfrak{n}+1}{2}$ unknown states $(\hat{u}_i)_{i/\mu_i>0}$ at the right of this wave and $\frac{\mathfrak{n}+1}{2}$ unknown states $(\hat{u}_i)_{i/\mu_i<0}$ on its left: see Figure 3 for the example of the S_4 equations.

These states are computed by solving the steady equations:

$$\mu_i \frac{\mathrm{d}u_i^*}{\mathrm{d}\chi} = h\left((\sigma_t - \sigma_a)\frac{1}{2}\sum_{k=1}^{k=\mathfrak{n}+1} \omega_k u_k^* - \sigma_t u_i^* \right) \qquad \text{for } \chi \in [0,1] \qquad i = 1...\mathfrak{n}+1 \tag{3.2}$$

with the boundary conditions:

$$\begin{cases} u_i^*(0) = (u_i)_l \text{ for } i \in \left\{1...\dfrac{\mathfrak{n}+1}{2}\right\} \\ u_i^*(1) = (u_i)_r \text{ for } i \in \left\{\dfrac{\mathfrak{n}+3}{2}...\mathfrak{n}+1\right\}. \end{cases} \tag{3.3}$$

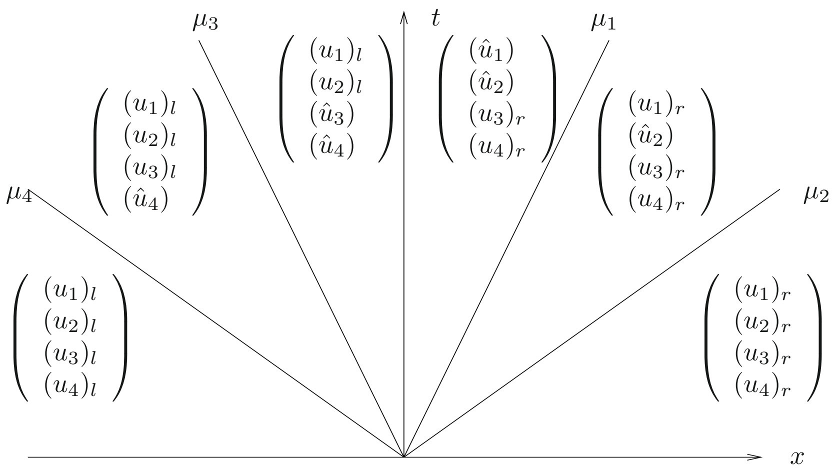

FIGURE 3. Riemann problem with the standing wave for the S_4 equations.

The unknown states are defined by:

$$\hat{u}_i = \begin{cases} u_i^*(1) \text{ for } i \in \left\{1 ... \frac{\mathfrak{n}+1}{2}\right\} \\ \\ u_i^*(0) \text{ for } i \in \left\{\frac{\mathfrak{n}+3}{2} ... \mathfrak{n}+1\right\}. \end{cases} \tag{3.4}$$

On the contrary of the S_2 equations case, we are not able to give an analytical expression for these unknown states, since the resolution of (3.2) is no longer trivial. Indeed, it would be equivalent to solve exactly the stationary $S_{\mathfrak{n}+1}$ equations which has been done only in very restrictive cases to the best of our knowledge [6].

Since the system (3.2) is linear, we can give a linear formulation of the unknown states in terms of the initial conditions of the Riemann problem:

$$\hat{u}_i = \begin{cases} \sum_{i'=1}^{i'=\frac{\mathfrak{n}+1}{2}} a_{i,i'}(u_{i'})_l + \sum_{i'=\frac{\mathfrak{n}+3}{2}}^{i'=\mathfrak{n}+1} b_{i,i'}(u_{i'})_r \quad \text{for } i \in \left\{1 ... \frac{\mathfrak{n}+1}{2}\right\} \\ \sum_{i'=1}^{i'=\frac{\mathfrak{n}+1}{2}} c_{i,i'}(u_{i'})_l + \sum_{i'=\frac{\mathfrak{n}+3}{2}}^{i'=\mathfrak{n}+1} d_{i,i'}(u_{i'})_r \quad \text{for } i \in \left\{\frac{\mathfrak{n}+3}{2} ... \mathfrak{n}+1\right\}. \end{cases} \tag{3.5}$$

Let us note that a Monte-Carlo algorithm is designed to compute the coefficients $a_{i,i'}$, $b_{i,i'}$, $c_{i,i'}$ and $d_{i,i'}$ in the Appendix A.

Lemma 3.1. *The coefficients $a_{i,i'}$, $b_{i,i'}$, $c_{i,i'}$ and $d_{i,i'}$ are non negative. Moreover, they verify:*

$$\sum_{i'=1}^{i'=\frac{\mathfrak{n}+1}{2}} a_{i,i'} + \sum_{i'=\frac{\mathfrak{n}+3}{2}}^{i'=\mathfrak{n}+1} b_{i,i'} \leq 1 \quad \textit{for } i \in \left\{1 ... \frac{\mathfrak{n}+1}{2}\right\} \tag{3.6}$$

and

$$\sum_{i'=1}^{i'=\frac{\mathfrak{n}+1}{2}} c_{i,i'} + \sum_{i'=\frac{\mathfrak{n}+3}{2}}^{i'=\mathfrak{n}+1} d_{i,i'} \leq 1 \qquad \textit{for } i \in \left\{\frac{\mathfrak{n}+3}{2}...\mathfrak{n}+1\right\}. \tag{3.7}$$

Proof. For example, to prove that a_{i_0,i'_0} is non negative, we take $(u_{i'})_l = \delta_{i'}^{i'_0}$ and $(u_{i'})_r = 0$ for all i', thus we obtain $u^*_{i_0}(1) = a_{i_0,i'_0}$. We deduce that a_{i_0,i'_0} is non negative, because the solution of (3.2) with positive boundary conditions is non negative.

Recall that $u^*_i(\chi)$ is the solution of the steady $S_{\mathfrak{n}+1}$ equations (3.2) with the *ad hoc* boundary conditions. Because of the maximum principle, we have:

$$u^*_i(\chi) \leq \max\left((u_{i/i\in\{1...\frac{\mathfrak{n}+1}{2}\}})_l, (u_{i/i\in\{\frac{\mathfrak{n}+3}{2}...\mathfrak{n}+1\}})_r\right) \qquad \forall\chi\in[0,1].$$

This inequality taken at $\chi = 0$ and $\chi = 1$ combined with the identities (3.5) leads to:

$$\begin{cases} \displaystyle\sum_{i'=1}^{i'=\frac{\mathfrak{n}+1}{2}} a_{i,i'}(u_{i'})_l + \sum_{i'=\frac{\mathfrak{n}+3}{2}}^{i'=\mathfrak{n}+1} b_{i,i'}(u_{i'})_r \leq \max\left((u_{i/i\in\{1...\frac{\mathfrak{n}+1}{2}\}})_l, (u_{i/i\in\{\frac{\mathfrak{n}+3}{2}...\mathfrak{n}+1\}})_r\right) & \text{for } i\in\left\{1...\dfrac{\mathfrak{n}+1}{2}\right\} \\ \displaystyle\sum_{i'=1}^{i'=\frac{\mathfrak{n}+1}{2}} c_{i,i'}(u_{i'})_l + \sum_{i'=\frac{\mathfrak{n}+3}{2}}^{i'=\mathfrak{n}+1} d_{i,i'}(u_{i'})_r \leq \max\left((u_{i/i\in\{1...\frac{\mathfrak{n}+1}{2}\}})_l, (u_{i/i\in\{\frac{\mathfrak{n}+3}{2}...\mathfrak{n}+1\}})_r\right) & \text{for } i\in\left\{\dfrac{\mathfrak{n}+3}{2}...\mathfrak{n}+1\right\} \end{cases}$$

whatever the boundary conditions $(u_{i/i\in\{1...\frac{\mathfrak{n}+1}{2}\}})_l$ and $(u_{i/i\in\{\frac{\mathfrak{n}+3}{2}...\mathfrak{n}+1\}})_r$ are. So, we can choose a particular value for each one: the choice $(u_{i/i\in\{1...\frac{\mathfrak{n}+1}{2}\}})_l = (u_{i/i\in\{\frac{\mathfrak{n}+3}{2}...\mathfrak{n}+1\}})_r = 1$ gives the result. □

3.1.2. *Explicit Gosse type schemes*

If we apply the above solver in the cell C_j, we obtain the following explicit Godunov type scheme to discretize the $S_{\mathfrak{n}+1}$ equations:

$$\begin{cases} (u_i)^{n+1}_j = (u_i)^n_j + \dfrac{\mu_i\Delta t}{h}\left((\hat{u}_i)^n_{j-1/2} - (u_i)^n_j\right) & \text{for } i\in\left\{1...\dfrac{\mathfrak{n}+1}{2}\right\} \\ (u_i)^{n+1}_j = (u_i)^n_j - \dfrac{\mu_i\Delta t}{h}\left((\hat{u}_i)^n_{j+1/2} - (u_i)^n_j\right) & \text{for } i\in\left\{\dfrac{\mathfrak{n}+3}{2}...\mathfrak{n}+1\right\} \end{cases} \tag{3.8}$$

with:

$$\begin{cases} (\hat{u}_i)^n_{j-1/2} = \displaystyle\sum_{i'=1}^{i'=\frac{\mathfrak{n}+1}{2}} a_{i,i'}(u_{i'})^n_{j-1} + \sum_{i'=\frac{\mathfrak{n}+3}{2}}^{i'=\mathfrak{n}+1} b_{i,i'}(u_{i'})^n_j & \text{for } i\in\left\{1...\dfrac{\mathfrak{n}+1}{2}\right\} \\ (\hat{u}_i)^n_{j+1/2} = \displaystyle\sum_{i'=1}^{i'=\frac{\mathfrak{n}+1}{2}} c_{i,i'}(u_{i'})^n_j + \sum_{i'=\frac{\mathfrak{n}+3}{2}}^{i'=\mathfrak{n}+1} d_{i,i'}(u_{i'})^n_{j+1} & \text{for } i\in\left\{\dfrac{\mathfrak{n}+3}{2}...\mathfrak{n}+1\right\} \end{cases}$$

because of the relations (3.5).

An explicit scheme for the $P_{\mathfrak{n}}$ equations may be derived from the last scheme. The summation of the identities (3.8) after they have been multiplied by $\omega_i\sqrt{2\ell+1}\,L_\ell(\mu_i)$ leads to:

$$(\psi_l)^{n+1}_j = (\psi_l)^n_j + \frac{\Delta t}{h}\left(\sum_{i=1}^{i=\frac{\mathfrak{n}+1}{2}} \Gamma_{li}\left((\hat{u}_i)^n_{j-1/2} - (u_i)^n_j\right) - \sum_{i=\frac{\mathfrak{n}+3}{2}}^{i=\mathfrak{n}+1} \Gamma_{li}\left((\hat{u}_i)^n_{j+1/2} - (u_i)^n_j\right)\right) \quad \text{for } l\in\{0...\mathfrak{n}\} \tag{3.9}$$

because of the relation:

$$(\psi_\ell)_j = \sum_{i=1}^{i=\mathfrak{n}+1} \omega_i \sqrt{2\ell+1}\, L_\ell(\mu_i)\,(u_i)_j$$

with the notation $\Gamma_{li} = \omega_i \sqrt{2\ell+1}\, L_\ell(\mu_i)\,\mu_i$.

Now, we can replace $(u_i)_j$ and $(\hat{u}_i)_{j\pm 1/2}$ by the moments ψ_l thanks to the identity

$$(u_i)_j = \sum_{\ell=0}^{\ell=\mathfrak{n}} M_{il}(\psi_\ell)_j,$$

where $M_{il} = \dfrac{\sqrt{2\ell+1}}{2} L_\ell(\mu_i)$. Hence, the explicit scheme relative to the $P_\mathfrak{n}$ equations is obtained:

$$(\psi_l)_j^{n+1} = (\psi_l)_j^n + \frac{\Delta t}{h} \sum_{k=0}^{k=\mathfrak{n}} [A_{lk}(\psi_k)_{j-1}^n + (B_{lk} - C_{lk})(\psi_k)_j^n - D_{lk}(\psi_k)_{j+1}^n] \quad \text{for } l \in \{0 ... \mathfrak{n}\} \tag{3.10}$$

with the notations $A_{lk} = \displaystyle\sum_{i=1}^{i=\frac{\mathfrak{n}+1}{2}} \sum_{i'=1}^{i'=\frac{\mathfrak{n}+1}{2}} \Gamma_{li} a_{i,i'} M_{i'k}$, $B_{lk} = \displaystyle\sum_{i=1}^{i=\frac{\mathfrak{n}+1}{2}} \Gamma_{li} \left(\sum_{i'=\frac{\mathfrak{n}+3}{2}}^{i'=\mathfrak{n}+1} b_{i,i'} M_{i'k} - M_{ik} \right)$,

$C_{lk} = \displaystyle\sum_{i=\frac{\mathfrak{n}+3}{2}}^{i=\mathfrak{n}+1} \Gamma_{li} \left(\sum_{i'=1}^{i'=\frac{\mathfrak{n}+1}{2}} c_{i,i'} M_{i'k} - M_{ik} \right)$ and $D_{lk} = \displaystyle\sum_{i=\frac{\mathfrak{n}+3}{2}}^{i=\mathfrak{n}+1} \sum_{i'=\frac{\mathfrak{n}+3}{2}}^{i'=\mathfrak{n}+1} \Gamma_{li} d_{i,i'} M_{i'k}$.

Because of the construction proposed, the scheme (3.10) is equivalent to the scheme (3.8).

Proposition 3.2. *Under the CFL condition* $\dfrac{\Delta t}{h} \max_k |\mu_k| \leq 1$*:*

- *the explicit discretization* (3.8)–(3.10) *is* L^∞*-stable;*
- *at the discrete level, the scheme* (3.8) *is positive as the* $S_{\mathfrak{n}+1}$ *model at the PDE level.*

Let us remark that the positivity of the scheme (3.8) implies that for all j: $|(\psi_1)_j^{n+1}| \leq \sqrt{3} \max_i |\mu_i| (\psi_0)_j^{n+1}$, which implies the *flux limited property*.

Proof. It is the same as the S_2-P_1 case. □

3.1.3. *Implicit Gosse type schemes*

In this section, we give an implicit version of the scheme (3.8) corresponding to the discretization of the $S_{\mathfrak{n}+1}$ equations:

$$\begin{cases} \left(1 + \dfrac{\mu_i \Delta t}{h}\right) (u_i)_j^{n+1} - \dfrac{\mu_i \Delta t}{h} \left(\displaystyle\sum_{i'=1}^{i'=\frac{\mathfrak{n}+1}{2}} a_{i,i'} (u_{i'})_{j-1}^{n+1} + \sum_{i'=\frac{\mathfrak{n}+3}{2}}^{i'=\mathfrak{n}+1} b_{i,i'} (u_{i'})_j^{n+1} \right) = (u_i)_j^n & \text{for } i \in \left\{1 ... \dfrac{\mathfrak{n}+1}{2}\right\} \\[2ex] \left(1 - \dfrac{\mu_i \Delta t}{h}\right) (u_i)_j^{n+1} + \dfrac{\mu_i \Delta t}{h} \left(\displaystyle\sum_{i'=1}^{i'=\frac{\mathfrak{n}+1}{2}} c_{i,i'} (u_{i'})_j^{n+1} + \sum_{i'=\frac{\mathfrak{n}+3}{2}}^{i'=\mathfrak{n}+1} d_{i,i'} (u_{i'})_{j+1}^{n+1} \right) = (u_i)_j^n & \text{for } i \in \left\{\dfrac{\mathfrak{n}+3}{2} ... \mathfrak{n}+1\right\}. \end{cases} \tag{3.11}$$

The implicit version of the scheme (3.10) is given by:

$$(\psi_l)_j^{n+1} = (\psi_l)_j^n + \frac{\Delta t}{h} \sum_{k=0}^{k=\mathfrak{n}} [A_{lk}(\psi_k)_{j-1}^{n+1} + (B_{lk} - C_{lk})(\psi_k)_j^{n+1} - D_{lk}(\psi_k)_{j+1}^{n+1}] \quad \text{for } l \in \{0 ... \mathfrak{n}\}. \tag{3.12}$$

Proposition 3.3.

- *The implicit discretization (3.11)–(3.12) is unconditionally L^∞-stable.*
- *At the discrete level, the scheme (3.11) is positive as the $S_{\mathfrak{n}+1}$ model at the PDE level.*

Proof. The implicit scheme (3.11) leads to the resolution of a linear system to achieve the computation of $(u_i)_j^{n+1}$. Because of the inequalities (3.6) and (3.7) and the non negativity of the coefficients $a_{i,i'}$, $b_{i,i'}$, $c_{i,i'}$ and $d_{i,i'}$, the matrix of the linear system is a M-matrix. This property is used to prove the L^∞-stability and the positivity of the scheme (3.11) in the same way as in the S_2 case. The unconditional stability of the scheme (3.12) is deduced from the equivalence between both schemes (3.12) and (3.11). □

By construction, this scheme is equivalent to the scheme (3.11). Let us remark that the positivity of the scheme (3.11) implies that for all j: $|(\psi_1)_j^{n+1}| \le \sqrt{3}\max_i |\mu_i|(\psi_0)_j^{n+1}$, which implies the *flux limited property.*

3.2. Well-balanced property

Proposition 3.4. *The explicit and implicit discretizations proposed in Section 3.1 are well-balanced.*

Proof. The well-balanced property of the explicit and implicit schemes (3.8)–(3.11) can be proved as for the S_2 case, except that equations (2.20) must be replaced by:

$$\begin{cases} u_i^{ex}(x_j) = \displaystyle\sum_{i'=1}^{i'=\frac{\mathfrak{n}+1}{2}} a_{i,i'}(u_{i'}^{ex})(x_{j-1}) + \sum_{i'=\frac{\mathfrak{n}+3}{2}}^{i'=\mathfrak{n}+1} b_{i,i'}(u_{i'}^{ex})(x_j) & \text{for } i \in \left\{1 \ldots \dfrac{\mathfrak{n}+1}{2}\right\} \\ u_i^{ex}(x_j) = \displaystyle\sum_{i'=1}^{i'=\frac{\mathfrak{n}+1}{2}} c_{i,i'}(u_{i'}^{ex})(x_j) + \sum_{i'=\frac{\mathfrak{n}+3}{2}}^{i'=\mathfrak{n}+1} d_{i,i'}(u_{i'}^{ex})(x_{j+1}) & \text{for } i \in \left\{\dfrac{\mathfrak{n}+3}{2} \ldots \mathfrak{n}+1\right\}. \end{cases}$$ □

3.3. Asymptotic preserving property

For the same reasons as in the P_1 case, we only consider the implicit scheme derived before.

Proposition 3.5. *The implicit scheme (3.12) is AP. ψ_0 at the order 0 verifies the same consistent discretization (2.21) of the diffusion equation (1.8) as in the P_1 case.*

Proof. To study the diffusive regime, the discrete diffusive scaling is introduced in the implicit version of the scheme (3.9) which gives for $l = 0$:

$$(\psi_0)_j^{n+1} = (\psi_0)_j^n + \frac{\Delta t}{\epsilon h}\left(-\sum_{i=1}^{i=\frac{\mathfrak{n}+1}{2}} \mu_i\omega_i(u_i)_j^{n+1} - \sum_{i=\frac{\mathfrak{n}+3}{2}}^{i=\mathfrak{n}+1} \mu_i\omega_i(\hat{u}_i)_{j+1/2}^{n+1} + \sum_{i=\frac{\mathfrak{n}+3}{2}}^{i=\mathfrak{n}+1} \mu_i\omega_i(u_i)_j^{n+1} + \sum_{i=1}^{i=\frac{\mathfrak{n}+1}{2}} \mu_i\omega_i(\hat{u}_i)_{j-1/2}^{n+1} \right). \tag{3.13}$$

Let us denote $(\psi_l)_{j\pm1/2}(\chi) = \sum_{i=1}^{i=\mathfrak{n}+1} \omega_i\sqrt{2l+1}L_l(\mu_i)u_i^*(\chi)$ where $u_i^*(\chi)$ stands for the solution of the steady equations (3.2) at the interface $x_{j\pm1/2}$ with the boundary conditions (3.3). Introducing the definition of the unknown states (3.4), we obtain:

$$\begin{cases} (\psi_1)_{j+1/2}^{n+1}(0) = \displaystyle\sum_{i=1}^{i=\frac{\mathfrak{n}+1}{2}} \sqrt{3}\mu_i\omega_i(u_i)_j^{n+1} + \sum_{i=\frac{\mathfrak{n}+3}{2}}^{i=\mathfrak{n}+1} \sqrt{3}\mu_i\omega_i(\hat{u}_i)_{j+1/2}^{n+1} \\ (\psi_1)_{j-1/2}^{n+1}(1) = \displaystyle\sum_{i=\frac{\mathfrak{n}+3}{2}}^{i=\mathfrak{n}+1} \sqrt{3}\mu_i\omega_i(u_i)_j^{n+1} + \sum_{i=1}^{i=\frac{\mathfrak{n}+1}{2}} \sqrt{3}\mu_i\omega_i(\hat{u}_i)_{j-1/2}^{n+1}. \end{cases}$$

Thus, the scheme (3.13) can be rewritten as:

$$(\psi_0)_j^{n+1} = (\psi_0)_j^n + \frac{\Delta t}{\epsilon h\sqrt{3}}\Big(-(\psi_1)_{j+1/2}^{n+1}(0) + (\psi_1)_{j-1/2}^{n+1}(1)\Big). \tag{3.14}$$

Then, by definition, $(\psi_l)_{j\pm1/2}^{n+1}(\chi)$ is solution of the steady $P_{\mathfrak{n}}$ equations:

$$\begin{cases} B_1\,\dfrac{\mathrm{d}(\psi_1)_{j\pm1/2}^{n+1}(\chi)}{\mathrm{d}\chi} = -\epsilon h\sigma_a(\psi_0)_{j\pm1/2}^{n+1}(\chi) \\ B_{\ell+1}\,\dfrac{\mathrm{d}(\psi_{\ell+1})_{j\pm1/2}^{n+1}(\chi)}{\mathrm{d}\chi} + A_{\ell-1}\,\dfrac{\mathrm{d}(\psi_{\ell-1})_{j\pm1/2}^{n+1}(\chi)}{\mathrm{d}\chi} = -h\dfrac{\sigma_t}{\epsilon}(\psi_\ell)_{j\pm1/2}^{n+1}(\chi) \qquad \ell = 1 \ldots \mathfrak{n}. \\ (\psi_{\mathfrak{n}+1})_{j\pm1/2}^{n+1}(\chi) = 0. \end{cases} \tag{3.15}$$

By taking the zeroth order terms in ϵ in the relation (3.14), we obtain the identity:

$$(\psi_0)_j^{n+1,(0)} = (\psi_0)_j^{n,(0)} + \frac{\Delta t}{h\sqrt{3}}\Big(-(\psi_1)_{j+1/2}^{n+1,(1)}(0) + (\psi_1)_{j-1/2}^{n+1,(1)}(1)\Big). \tag{3.16}$$

The introduction of the ψ_ℓ expansion in the equations (3.15) and the identification of the ϵ^{-1} terms lead to $(\psi_\ell)_{j\pm1/2}^{n+1,(0)}(\chi) = 0$ for $l > 0$. Gathering terms of order ϵ^0 gives:

$$\begin{cases} B_1\,\dfrac{\mathrm{d}(\psi_1)_{j\pm1/2}^{n+1,(0)}(\chi)}{\mathrm{d}\chi} = 0 \\ B_{\ell+1}\,\dfrac{\mathrm{d}(\psi_{\ell+1})_{j\pm1/2}^{n+1,(0)}(\chi)}{\mathrm{d}\chi} + A_{\ell-1}\,\dfrac{\mathrm{d}(\psi_{\ell-1})_{j\pm1/2}^{n+1,(0)}(\chi)}{\mathrm{d}\chi} = -h\sigma_t(\psi_\ell)_{j\pm1/2}^{n+1,(1)}(\chi) \qquad \ell = 1 \ldots \mathfrak{n}. \\ (\psi_{\mathfrak{n}+1})_{j\pm1/2}^{n+1,(0)}(\chi) = 0. \end{cases} \tag{3.17}$$

Since $(\psi_\ell)_{j\pm1/2}^{n+1,(0)}(\chi) = 0$ for $l > 0$, the second equation of (3.17) leads to $(\psi_\ell)_{j\pm1/2}^{n+1,(1)}(\chi) = 0$ for $l > 1$. Hence, these both identities for $l = 2$ yield $(\psi_2)_{j\pm1/2}^{n+1}(\chi) = O(\epsilon^2)$ so that $\Big((\psi_0)_{j\pm1/2}^{n+1}(\chi), (\psi_1)_{j\pm1/2}^{n+1}(\chi)\Big)$ verify the system:

$$\begin{cases} \dfrac{1}{\epsilon}\dfrac{\mathrm{d}(\psi_1)_{j\pm1/2}^{n+1}(\chi)}{\mathrm{d}\chi} = -2\beta(\psi_0)_{j\pm1/2}^{n+1}(\chi) \\ \dfrac{1}{\epsilon}\dfrac{\mathrm{d}(\psi_0)_{j\pm1/2}^{n+1}(\chi)}{\mathrm{d}\chi} = -2\dfrac{\alpha}{\epsilon^2}(\psi_1)_{j\pm1/2}^{n+1}(\chi) + O(\epsilon). \end{cases} \tag{3.18}$$

If we consider the ϵ^{-1} term in the second equation of (3.18) and the zeroth order term in the first equation of this system, we obtain:

$$\begin{cases} (\psi_1)_{j\pm1/2}^{n+1,(1)}(\chi) = -\dfrac{1}{2\alpha}\dfrac{\mathrm{d}(\psi_0)_{j\pm1/2}^{n+1,(0)}(\chi)}{\mathrm{d}\chi} \\ \dfrac{\mathrm{d}^2(\psi_0)_{j\pm1/2}^{n+1,(0)}(\chi)}{\mathrm{d}^2\chi} = C^2(\psi_0)_{j\pm1/2}^{n+1,(0)}(\chi). \end{cases} \tag{3.19}$$

Let us now establish the boundary conditions for the second equation of the above system (3.19) at $x_{j+1/2}$.

The expansion of u_i is introduced in the implicit scheme relative to the $S_{\mathfrak{n}+1}$ formulation:

$$\begin{cases} (u_i)_j^{n+1} = (u_i)_j^n + \dfrac{\mu_i \Delta t}{\epsilon h}\left((\hat{u}_i)_{j-1/2}^{n+1} - (u_i)_j^{n+1}\right) & \text{for } i \in \left\{1 ... \dfrac{\mathfrak{n}+1}{2}\right\} \\ (u_i)_j^{n+1} = (u_i)_j^n - \dfrac{\mu_i \Delta t}{\epsilon h}\left((\hat{u}_i)_{j+1/2}^{n+1} - (u_i)_j^{n+1}\right) & \text{for } i \in \left\{\dfrac{\mathfrak{n}+3}{2} ... \mathfrak{n}+1\right\}, \end{cases}$$

the ϵ^{-1} terms yield ($\forall j$):

$$\begin{cases} (\hat{u}_i)_{j-1/2}^{n+1,(0)} = (u_i)_j^{n+1,(0)} & \text{for } i \in \left\{1 ... \dfrac{\mathfrak{n}+1}{2}\right\} \\ (\hat{u}_i)_{j+1/2}^{n+1,(0)} = (u_i)_j^{n+1,(0)} & \text{for } i \in \left\{\dfrac{\mathfrak{n}+3}{2} ... \mathfrak{n}+1\right\} \end{cases}$$

which implies:

$$\begin{cases} (\psi_0)_{j+1/2}^{n+1,(0)}(0) = \displaystyle\sum_{i=1}^{i=\frac{\mathfrak{n}+1}{2}} (u_i)_j^{n+1,(0)} + \sum_{i=\frac{\mathfrak{n}+3}{2}}^{i=\mathfrak{n}+1} (\hat{u}_i)_{j+1/2}^{n+1,(0)} = \sum_{i=1}^{i=\frac{\mathfrak{n}+1}{2}} (u_i)_j^{n+1,(0)} + \sum_{i=\frac{\mathfrak{n}+3}{2}}^{i=\mathfrak{n}+1} (u_i)_j^{n+1,(0)} = (\psi_0)_j^{n+1,(0)} \\ (\psi_0)_{j+1/2}^{n+1,(0)}(1) = \displaystyle\sum_{i=1}^{i=\frac{\mathfrak{n}+1}{2}} (\hat{u}_i)_{j+1/2}^{n+1,(0)} + \sum_{i=\frac{\mathfrak{n}+3}{2}}^{i=\mathfrak{n}+1} (u_i)_{j+1}^{n+1,(0)} = \sum_{i=1}^{i=\frac{\mathfrak{n}+1}{2}} (u_i)_{j+1}^{n+1,(0)} + \sum_{i=\frac{\mathfrak{n}+3}{2}}^{i=\mathfrak{n}+1} (u_i)_{j+1}^{n+1,(0)} = (\psi_0)_{j+1}^{n+1,(0)}. \end{cases}$$

Thus the boundary conditions for the second equation of (3.19) has been established in $x_{j+1/2}$. The same calculations can be performed at $x_{j-1/2}$.

Let us now prove that the identity (3.16) combined with (3.19) and the above boundary conditions give the relation (2.24) already obtained in the P_1 case.

The first equation of (3.18) at zeroth order in ϵ leads to:

$$\frac{1}{2}(\psi_1)_{j\pm1/2}^{n+1,(1)}(1) = \frac{1}{2}(\psi_1)_{j\pm1/2}^{n+1,(1)}(0) - \int_0^1 \mathrm{d}\chi \beta (\psi_0)_{j\pm1/2}^{n+1,(0)}(\chi). \tag{3.20}$$

Moreover the identity (3.16) may be rewritten as:

$$(\psi_0)_j^{n+1,(0)} = (\psi_0)_j^{n,(0)} + \frac{\Delta t}{h\sqrt{3}}\left(-\frac{1}{2}(\psi_1)_{j+1/2}^{n+1,(1)}(0) - \frac{1}{2}(\psi_1)_{j+1/2}^{n+1,(1)}(0) + \frac{1}{2}(\psi_1)_{j-1/2}^{n+1,(1)}(1) + \frac{1}{2}(\psi_1)_{j-1/2}^{n+1,(1)}(1)\right).$$

This relation combined with the statement (3.20) gives:

$$(\psi_0)_j^{n+1,(0)} = (\psi_0)_j^{n,(0)} + \frac{\Delta t}{h\sqrt{3}}\left(-(\hat{\psi}_1)_{j+1/2}^{n+1,(1)} + (\hat{\psi}_1)_{j-1/2}^{n+1,(1)} - \int_0^1 \mathrm{d}\chi \beta (\psi_0)_{j-1/2}^{n+1,(0)}(\chi) - \int_0^1 \mathrm{d}\chi \beta (\psi_0)_{j+1/2}^{n+1,(0)}(\chi)\right)$$

where $(\hat{\psi}_1)_{j\pm1/2}^{n+1,(1)} = \dfrac{1}{2}\left((\psi_1)_{j\pm1/2}^{n+1,(1)}(0) + (\psi_1)_{j\pm1/2}^{n+1,(1)}(1)\right)$.

The expression of $(\psi_0)^{n+1,(0)}_{j+1/2}$ is obtained by solving the second order differential equation (3.19) at $x_{j+1/2}$ with the corresponding boundary conditions:

$$(\psi_0)^{n+1,(0)}_{j+1/2}(\chi) = \frac{(\psi_0)^{n+1,(0)}_{j+1}[\exp(\chi C) - \exp(-\chi C)] + (\psi_0)^{n+1,(0)}_{j}[\exp((1-\chi)C) - \exp(-(1-\chi)C)]}{\exp(C) - \exp(-C)}. \tag{3.21}$$

Moreover, the derivative of the last relation yields:

$$\frac{1}{2}\left[\frac{\mathrm{d}(\psi_0)^{n+1,(0)}_{j+1/2}}{\mathrm{d}\chi}(0) + \frac{\mathrm{d}(\psi_0)^{n+1,(0)}_{j+1/2}}{\mathrm{d}\chi}(1)\right] = \frac{2C + C\exp(C) + C\exp(-C)}{2(\exp(C) - \exp(-C))}\left((\psi_0)^{n+1,(0)}_{j+1} - (\psi_0)^{n+1,(0)}_{j}\right).$$

This last statement combined with the first equation of (3.19) gives:

$$(\hat{\psi}_1)^{n+1,(1)}_{j+1/2} = \frac{1}{2\alpha}\left(\frac{2C + C\exp(C) + C\exp(-C)}{2(\exp(C) - \exp(-C))}\right)\left((\psi_0)^{n+1,(0)}_{j} - (\psi_0)^{n+1,(0)}_{j+1}\right).$$

The expression of $(\hat{\psi}_1)^{n+1,(1)}_{j-1/2}$ can be obtained in the same way as $(\hat{\psi}_1)^{n+1,(1)}_{j+1/2}$.

Hence, we get the following identity:

$$-(\hat{\psi}_1)^{n+1,(1)}_{j+1/2} + (\hat{\psi}_1)^{n+1,(1)}_{j-1/2} = \frac{1}{2\alpha}\left(\frac{2C + C\exp(C) + C\exp(-C)}{2(\exp(C) - \exp(-C))}\right)\left((\psi_0)^{n+1,(0)}_{j-1} - 2(\psi_0)^{n+1,(0)}_{j} + (\psi_0)^{n+1,(0)}_{j+1}\right).$$

From the relation (3.21), we have also:

$$\int_0^1 \mathrm{d}\chi(\psi_0)^{n+1,(0)}_{j-1/2}(\chi) + \int_0^1 \mathrm{d}\chi(\psi_0)^{n+1,(0)}_{j+1/2}(\chi) = \frac{-2 + \exp(C) + \exp(-C)}{C(\exp(C) - \exp(-C))}\left((\psi_0)^{n+1,(0)}_{j-1} + 2(\psi_0)^{n+1,(0)}_{j} + (\psi_0)^{n+1,(0)}_{j+1}\right).$$

Finally, after some easy calculation, we obtain:

$$\begin{aligned}(\psi_0)^{n+1,(0)}_{j} = {} & (\psi_0)^{n,(0)}_{j} + \frac{\Delta t}{h\sqrt{3}}\left(\frac{1}{2\alpha}\right)\left(\frac{2C + C\exp(C) + C\exp(-C)}{2(\exp(C) - \exp(-C))}\right)\left((\psi_0)^{n+1,(0)}_{j-1} - 2(\psi_0)^{n+1,(0)}_{j} + (\psi_0)^{n+1,(0)}_{j+1}\right)\\ & -\beta\frac{\Delta t}{h\sqrt{3}}\left(\frac{-2 + \exp(C) + \exp(-C)}{C(\exp(C) - \exp(-C))}\right)\left((\psi_0)^{n+1,(0)}_{j-1} + 2(\psi_0)^{n+1,(0)}_{j} + (\psi_0)^{n+1,(0)}_{j+1}\right)\end{aligned}$$

which gives the statement (2.24) already obtained to prove the asymptotic preserving property in the P_1 case. Thus, the end of this proof is exactly the same as the one established in the P_1 case. □

4. Numerical results

4.1. **Diffusive case**

We solve the P_1 equations obtained with the diffusive scaling:

$$\begin{cases} \dfrac{\partial \psi_0}{\partial t} + \dfrac{1}{\epsilon\sqrt{3}}\dfrac{\partial \psi_1}{\partial x} + \sigma_a \psi_0 = 0 \\ \dfrac{\partial \psi_1}{\partial t} + \dfrac{1}{\epsilon\sqrt{3}}\dfrac{\partial \psi_0}{\partial x} + \dfrac{\sigma_t}{\epsilon^2}\psi_1 = 0 \end{cases} \tag{4.1}$$

with $\sigma_a = 1$, $\sigma_t = 2$, on the interval $[\mathcal{A} = 0, \mathcal{B} = 1]$.

The boundary conditions are $\psi_0(0,t) + \psi_1(0,t) = 1$ and $\psi_0(1,t) - \psi_1(1,t) = 1$.

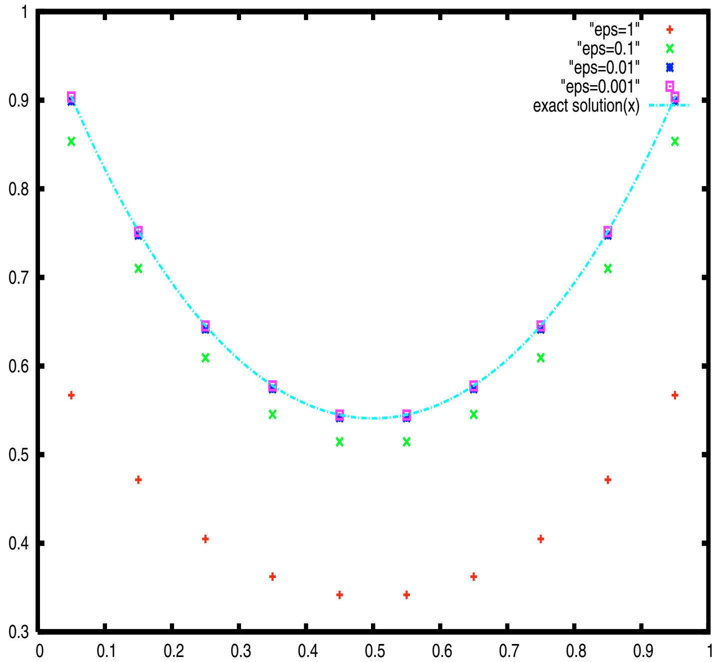

FIGURE 4. Diffusive case – Gosse type scheme for the P_1 equations: convergence to the steady solution of the diffusion equation.

The initial conditions are given by: $\psi_0(x,0) = f(x) + \sin(\pi x)$ and $\psi_1(x,0) = 0$. The function f is defined by: $f(x) = \left(\dfrac{1-\exp(-c)}{\exp(c)-\exp(-c)}\right)\exp(xc) + \left(\dfrac{\exp(c)-1}{\exp(c)-\exp(-c)}\right)\exp(-xc)$.

When ϵ tends to zero, the density ψ_0 is solution of the diffusion equation (1.8) with the following boundary and initial conditions:

$$\begin{cases} \psi_0(0,t) = \psi_0(1,t) = 1 \\ \psi_0(x,0) = f(x) + \sin(\pi x). \end{cases}$$

In this case, ψ_0 is given by:

$$\psi_0(x,t) = f(x) + \sin(\pi x)\exp[-(\pi^2 D + \sigma_a)t] \tag{4.2}$$

with $c = \sqrt{3\sigma_a\sigma_t}$ and $D = \dfrac{1}{3\sigma_t}$. The steady state solution is $f(x)$.

In the case of the $P_{\mathfrak{n}}$ equations, when ϵ tends to zero, the density ψ_0 tends to the same density (4.2). When ϵ is not small enough, the solutions depend on $\mathfrak{n}$.

4.1.1. *Steady state case*

Due to the well-balanced property of the scheme, we expect the convergence of the discrete solution to the exact solution $f(x)$ whatever the mesh is.

On Figure 4, we observe the convergence to the steady solution of the diffusion equation when ϵ tends to 0: for the time of observation $T = 10$, the solution ψ_0 of the scaled P_1 equations (4.1) discretized with the implicit scheme (2.16) is compared to the analytical one on a crude mesh ($N_x = 10$), for various values of ϵ. The time of observation has been set to get the stationary solution.

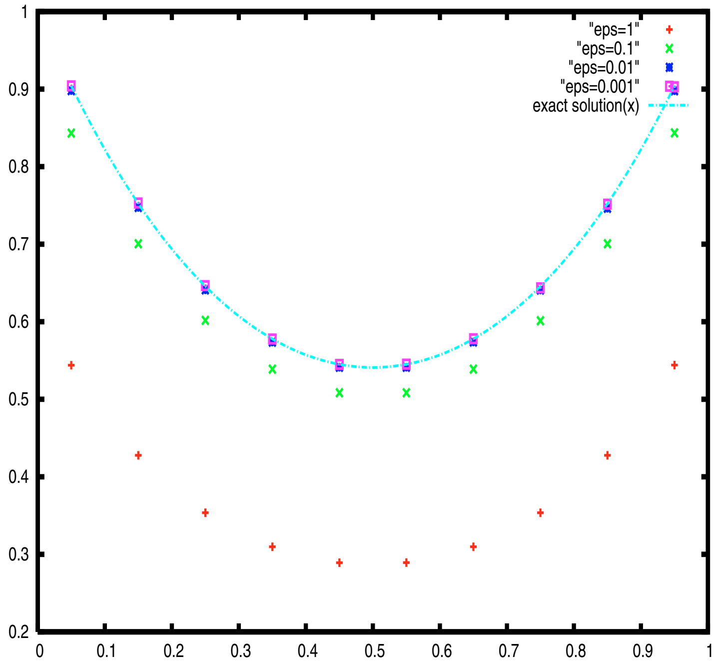

FIGURE 5. Diffusive case – Gosse type scheme for the P_3 equations: convergence to the steady solution of the diffusion equation.

On Figure 5, we observe the convergence to the steady solution of the diffusion equation, for the time of observation $T = 10$: the solution ψ_0 of the scaled P_3 equations discretized with the implicit scheme (3.11) is compared to the exact solution on a crude mesh ($N_x = 10$), for various values of ϵ.

Let us note that the solutions of the P_1 and P_3 models for ϵ not small (see for example $\epsilon = 1$) are different because the medium is not diffusive enough for these values of ϵ but their limits are the same for ϵ tending to zero.

4.1.2. *Unsteady case*

Now, the discrete solution ψ_0 of the P_1 equations discretized with the implicit scheme (2.16) is compared to the exact solution (4.2) of the unsteady diffusion equation at various times $T = 0.01$, $T = 0.1$ and $T = 1$. ϵ is set to 0.001 and N_x to 100 so that $h\dfrac{\sigma_t}{\epsilon}$ is equal to 20: this value is large enough to ensure that the asymptotic analysis is valid. On Figure 6, we observe a very good agreement between the calculated solution and the exact diffusion solution which confirms the results of the asymptotic analysis. The difference between both solutions would be larger on a cruder mesh since the asymptotic solution of the P_1 equations is solution of the discretized diffusion equation and not its exact solution.

4.2. **Plane source**

This problem involves a purely scattering problem: $\sigma_a = 0$, $\sigma_t = 1$. The boundary conditions are: $g_{\mathcal{A}}(\mu, t) = 0$ and $g_{\mathcal{B}}(\mu < 0, t) = 0$ with $\mathcal{A} = -10$, $\mathcal{B} = 10$.

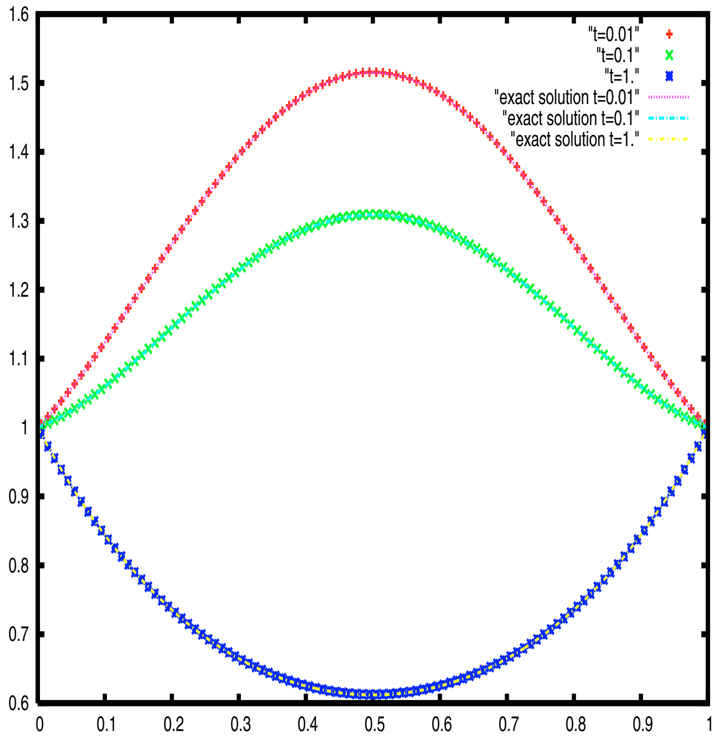

FIGURE 6. Diffusive case – Gosse type scheme for the P_1 equations: comparison of the solution of the unsteady diffusion equation with the solution obtained by the implicit scheme.

4.2.1. P_1 *equations*

The initial condition is a pulse of particles located at $x = 0$:

$$\begin{cases} \psi_0(x,0) = \delta(x) \\ \psi_1(x,0) = 0. \end{cases}$$

In this case, the system of P_1 equations (2.1) has an exact solution [3] (see Fig. 7):

$$\begin{aligned} \psi_0(x,t) \quad = \quad & \frac{\sqrt{3}}{4}\sigma_t \exp\left(-\frac{\sigma_t t}{2}\right) I_0\left(\frac{\sigma_t}{2}\sqrt{t^2-3x^2}\right) H(t-\sqrt{3}|x|) \\ & + \frac{\sqrt{3}}{4}\sigma_t \exp\left(-\frac{\sigma_t t}{2}\right) \frac{t}{\sqrt{t^2-3x^2}} I_1\left(\frac{\sigma_t}{2}\sqrt{t^2-3x^2}\right) H(t-\sqrt{3}|x|) \\ & + \frac{\sqrt{3}}{2}\exp\left(-\frac{\sigma_t t}{2}\right) \delta(t-\sqrt{3}|x|) \end{aligned}$$

where H is the Heaviside function and I_0, I_1 are Bessel functions of the first and second kind.

We compare the discrete solution obtained with the explicit scheme and the exact solution. The mesh is defined by $N_x = 5001$, the CFL constant is equal to 0.99, so that $\Delta t = 0.006854$. We observe a good agreement between both solutions. The speed of the two opposite waves $-\dfrac{1}{\sqrt{3}}$ and $\dfrac{1}{\sqrt{3}}$ is well reproduced by the scheme (Fig. 8).

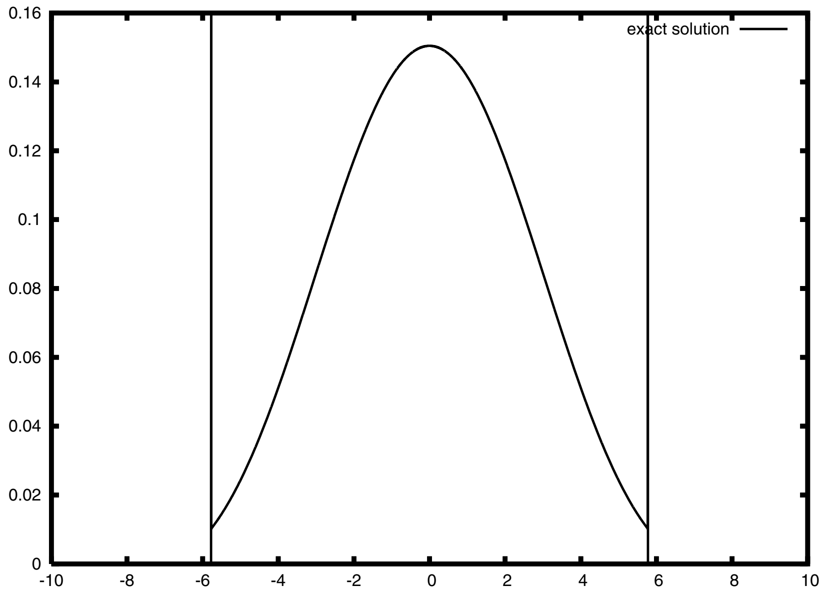

Figure 7. Plane source – exact solution of the P_1 system at time $T = 10$.

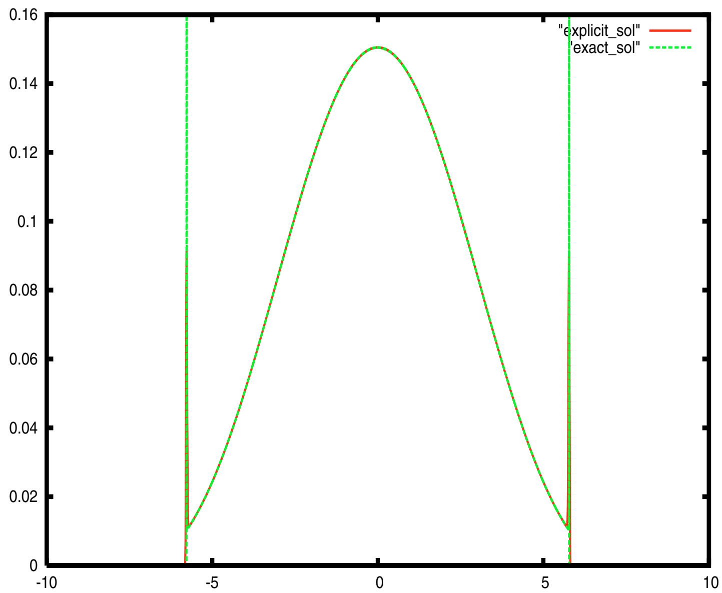

Figure 8. Plane source with P_1 equations – comparison of the exact density and the one obtained with the explicit scheme (2.11).

The use of the implicit scheme with $\Delta t = 0.01$ smears the solution at the position of the wave fronts, while the accuracy of the solution is preserved (Fig. 9).

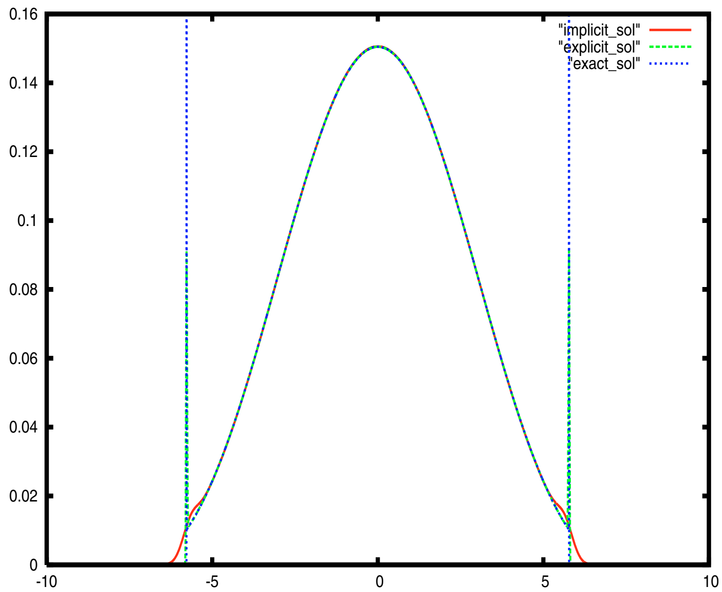

FIGURE 9. Plane source with P_1 equations – comparison of the exact solution and the ones obtained with the explicit and implicit schemes (2.11)–(2.16).

4.2.2. $P_{\mathfrak{n}}$ *equations*

The boundary and initial conditions are the same as previously.

We compare the solution obtained with the explicit scheme for the P_5 equations and the one obtained with a $S_{\mathfrak{n}}$ diamond difference scheme [8,9] and $\mathfrak{n} = 6$ which can be considered as the reference one. Both solutions have been computed on the same mesh $N_x = 5001$, at the same time $T = 1$. We observe a good agreement between both solutions (Fig. 10). The P_5 solution is composed of six Dirac peaks plus a smooth solution. These peaks correspond to particles moving with the velocities μ_i (the μ_i are the six discrete ordinates) and having not suffered a collision. The smooth part of the solution represents particles having suffered at least one collision.

At a later time, $T = 10$, the Dirac peaks have completely disappeared from the P_5 calculated solution (Fig. 11). We observe a good agreement between this solution and the transport reference one obtained on a converged mesh ($N_x = 10\,001$) with a $S_{\mathfrak{n}}$ diamond difference scheme and $\mathfrak{n} = 32$. The P_5 approximation is thus sufficient to capture the transport solution at this time. This is not the case of the P_1 equations whose exact solution is far from the transport reference solution. We notice that the transport solution is free from singular Dirac peaks.

5. Conclusion

In this paper, we have extended the method of Gosse, initially designed for the Goldstein-Taylor model, to the $P_{\mathfrak{n}}$ equations with absorption in 1D. The resulting Godunov scheme preserves the steady state solution (well-balanced property). Moreover, it gives the solution of the diffusion equation in the diffusive case, on a mesh resolving the diffusion scale much larger than the transport scale (diffusion limit property). This last result was proved by making formal expansions of the solution with respect to a small parameter representing the inverse of the number of mean free path in each cell. In the transparent scale, the scheme maintains the finite speed of propagation of the hyperbolic system. To avoid the CFL constraint on the time step which can be prohibitive in the diffusive case, the scheme has been made implicit. We have proved that the matrix of the resulting linear system is a M-matrix which ensures the positivity of the solution. The coefficients of the matrix

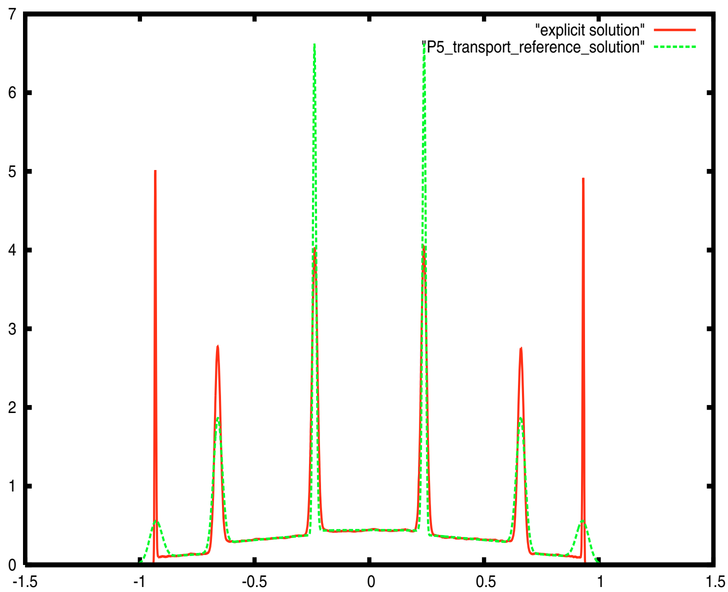

FIGURE 10. Plane source at $T = 1$ – comparison of the S_6 diamond difference scheme solution with the solution using P_5 model solved by the explicit scheme (3.10).

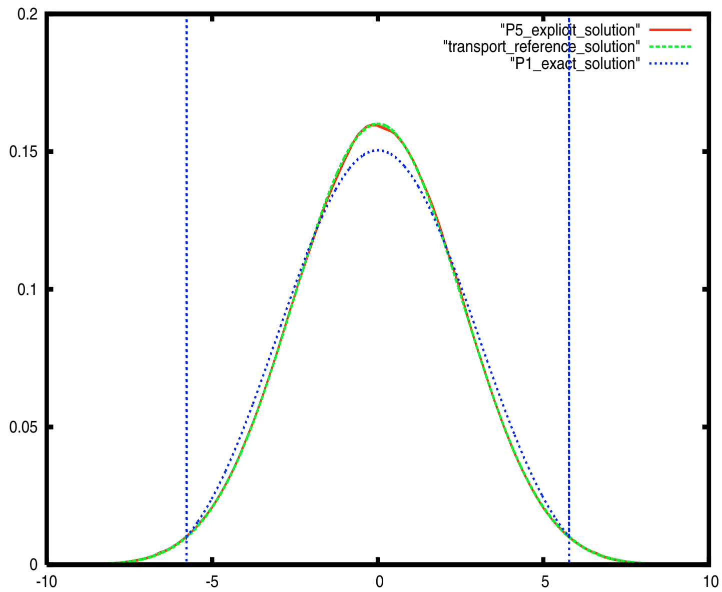

FIGURE 11. Plane source at $T = 10$ – transport reference solution, P_1 exact solution and solution using P_5 model solved by the explicit scheme (3.10).

have been precomputed by performing one Monte-Carlo steady state calculation at each interface of the mesh. We have verified numerically that the asymptotic preserving property is satisfied.

To improve this work, we could propose an extension of the proposed Gosse type scheme to second order, at least in space: the first order is too restrictive to compute sharp solutions, like in the plane source test for example. We are also interested in solving bidimensional problems. In 2D, there is no equivalence between the discrete ordinates equations and the $P_{\mathfrak{n}}$ equations, whatever the choice of the angular directions. So, as the $SP_{\mathfrak{n}}$ equations by direction are the one dimensional $P_{\mathfrak{n}}$ equations (see Appendix B), a second extension of this work could be the discretization of the $SP_{\mathfrak{n}}$ equations on Cartesian geometries. Indeed, we can propose to solve this system by a splitting technique and the scheme (3.10). Finally, it would be interesting to study this approach on unstructured meshes.

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A. Resolution of the Riemann problem involved in the $P_{\mathfrak{n}}$ scheme

The use of the explicit and implicit Gosse type schemes (3.8)–(3.11) requires the knowledge of the coefficients $a_{i,i'}$, $b_{i,i'}$, $c_{i,i'}$ and $d_{i,i'}$. As we are unable to compute their analytical expression, we have to find another way to calculate them.

Here, we are interested in the case where σ_a, σ_t do not change in time. So we propose to make a Monte-Carlo simulation to evaluate them. Let us note that a good accuracy can be achieved with such a simulation. Indeed, as it is made outside the time loop, we can use a large amount of particles.

To give a physical insight of this Monte-Carlo simulation, it is easier to introduce the currents $\tilde{u}_i = \omega_i|\mu_i|u_i$, $\forall i \in \{1...\mathfrak{n}+1\}$. Let us express the identities (3.5) for these new variables:

$$\begin{cases} \tilde{u}_i^*(1) = \displaystyle\sum_{i'=1}^{i'=\frac{\mathfrak{n}+1}{2}} \omega_i|\mu_i|a_{i,i'}\frac{(\tilde{u}_{i'})_l}{\omega_{i'}|\mu_{i'}|} + \sum_{i'=\frac{\mathfrak{n}+3}{2}}^{i'=\mathfrak{n}+1} \omega_i|\mu_i|b_{i,i'}\frac{(\tilde{u}_{i'})_r}{\omega_{i'}|\mu_{i'}|} & \text{for } i \in \left\{1...\dfrac{\mathfrak{n}+1}{2}\right\} \\ \tilde{u}_i^*(0) = \displaystyle\sum_{i'=1}^{i'=\frac{\mathfrak{n}+1}{2}} \omega_i|\mu_i|c_{i,i'}\frac{(\tilde{u}_{i'})_l}{\omega_{i'}|\mu_{i'}|} + \sum_{i'=\frac{\mathfrak{n}+3}{2}}^{i'=\mathfrak{n}+1} \omega_i|\mu_i|d_{i,i'}\frac{(\tilde{u}_{i'})_r}{\omega_{i'}|\mu_{i'}|} & \text{for } i \in \left\{\dfrac{\mathfrak{n}+3}{2}...\mathfrak{n}+1\right\}. \end{cases}$$

For example, to compute a_{i,i'_0} for all $i \in \left\{1...\frac{\mathfrak{n}+1}{2}\right\}$ and c_{i,i'_0} for all $i \in \left\{\frac{\mathfrak{n}+3}{2}...\mathfrak{n}+1\right\}$, we set $(\tilde{u}_{i''})_l = \delta_{i''}^{i'_0}$ and $(\tilde{u}_{i''})_r = 0$, $\forall i''$. Then, we have:

$$\begin{cases} \tilde{u}_i^*(1) = \dfrac{\omega_i|\mu_i|a_{i,i'_0}}{\omega_{i'_0}|\mu_{i'_0}|} & \text{for } i \in \left\{1...\dfrac{\mathfrak{n}+1}{2}\right\} \\ \tilde{u}_i^*(0) = \dfrac{\omega_i|\mu_i|c_{i,i'_0}}{\omega_{i'_0}|\mu_{i'_0}|} & \text{for } i \in \left\{\dfrac{\mathfrak{n}+3}{2}...\mathfrak{n}+1\right\}. \end{cases}$$

To simulate the $S_{\mathfrak{n}+1}$ equations with the previous boundary conditions, a Monte-Carlo algorithm can be prescribed:

- Step 1: sampling of the boundary condition.
  One generates at the left of the interval $[0,1]$ a particle with the direction $\mu_i = \mu_{i'_0}$.
- Step 2: sampling of the distance of collision.
  We distinguish two events.
  - The particle escapes with the probability P_{esc}.
    Denote ξ_i the counter to estimate the particle leakage.

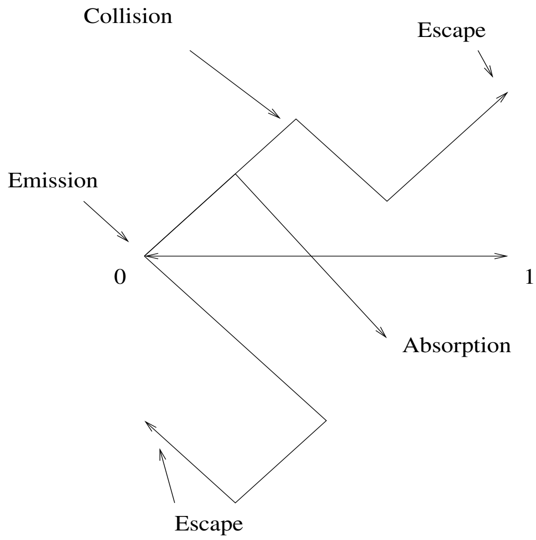

FIGURE 12. Examples of tracks.

If $\mu_i < 0$, the particle escapes at the left of the domain, one scores $\xi_i + 1 \mapsto \xi_i$ and the particle is killed. The mean of ξ_i is an estimator for $\tilde{u}_i(0)$.
If $\mu_i > 0$, the particle escapes at the right of the domain, one scores $\xi_i + 1 \mapsto \xi_i$ and the particle is killed. The mean of ξ_i is an estimator for $\tilde{u}_i(1)$.
Let us note that the probability for the particle of not having a collision between χ and 0 is defined by $P_{\text{esc}} = \exp\left(-\frac{\sigma_t h\chi}{|\mu_i|}\right)$ if $\mu_i < 0$ and $P_{\text{esc}} = \exp\left(-\frac{\sigma_t h(1-\chi)}{|\mu_i|}\right)$ if $\mu_i > 0$.

– The particle suffers a collision with the probability $1 - P_{\text{esc}}$.
Then, one computes the distance l of collision by sampling the probability $P_{\text{coll}}(l)$ on $[0, 1-\chi]$ or $[0, \chi]$ according to the sign of μ_i. This probability satisfies: $P_{\text{coll}}(l) = \frac{1}{1-P_{\text{esc}}}\frac{\sigma_t h}{|\mu_i|}\exp\left(-\frac{\sigma_t hl}{|\mu_i|}\right) dl$.
The particle is moved to $\chi = \chi + \text{sign}(\mu)l$ for its next collision. The particle is killed with the probability $\frac{\sigma_a}{\sigma_t}$ while it survives with the probability $1 - \frac{\sigma_a}{\sigma_t}$. The new direction of the particle is then sampled using a discrete probability: the probability for the particle to get the direction $\mu_{i'_1}$ is $\frac{\omega_{i'_1}}{2}$. We come back to Step 2 with the direction $\mu_i = \mu_{i'_1}$.

From the mean of ξ_i for all i, we deduce the coefficients a_{i,i'_0} and c_{i,i'_0} for all i. An example of three tracks starting from the same direction i'_0 is given in Figure 12.

Remark A.1. This procedure can be extended in a straightforward manner to a non uniform mesh and non constant coefficients σ_a, σ_t.

Remark A.2. If (σ_a, σ_t) evolve in time, because of the cost of an accurate Monte-Carlo simulation, we would prefer a deterministic method to approximate the solution of the stationary $S_{\mathfrak{n}+1}$ equations, at each interface $x_{j+1/2}$ of the mesh.

B. $SP_{\mathfrak{n}}$ EQUATIONS

On the contrary of the bidimensional $P_{\mathfrak{n}}$ equations, the $SP_{\mathfrak{n}}$ equations by direction are the one dimensional $P_{\mathfrak{n}}$ equations. These equations were first obtained [10] by considering, in 1D $P_{\mathfrak{n}}$ equations, the odd moments as vectors, the even moments remaining scalars. Then, the partial derivative $\frac{\partial}{\partial x}$ is replaced by the divergence operator in the equations for the even moments and by the gradient operator in the equations for the odd moments.

For a sake of simplicity, we take the example of the SP_3 equations:

$$\begin{cases} \dfrac{\partial \psi_0}{\partial t} + B_1 \vec{\nabla} \cdot \vec{\psi}_1 + \sigma_a \psi_0 = 0 \\ \dfrac{\partial \vec{\psi}_1}{\partial t} + B_2 \vec{\nabla} \psi_2 + A_0 \vec{\nabla} \psi_0 + \sigma_t \vec{\psi}_1 = \vec{0} \\ \dfrac{\partial \psi_2}{\partial t} + B_3 \vec{\nabla} \cdot \vec{\psi}_3 + A_1 \vec{\nabla} \cdot \vec{\psi}_1 + \sigma_t \psi_2 = 0 \\ \dfrac{\partial \vec{\psi}_3}{\partial t} + A_2 \vec{\nabla} \psi_2 + \sigma_t \vec{\psi}_3 = \vec{0} \end{cases}$$

with $\vec{\psi}_1 = (\psi_1^x, \psi_1^y)$ and $\vec{\psi}_3 = (\psi_3^x, \psi_3^y)$.

For example, for the x direction, we obtain:

$$\begin{cases} \dfrac{\partial \psi_0}{\partial t} + B_1 \dfrac{\partial \psi_1^x}{\partial x} + \sigma_a \psi_0 = 0 \\ \dfrac{\partial \psi_1^x}{\partial t} + B_2 \dfrac{\partial \psi_2}{\partial x} + A_0 \dfrac{\partial \psi_0}{\partial x} + \sigma_t \psi_1^x = 0 \\ \dfrac{\partial \psi_2}{\partial t} + B_3 \dfrac{\partial \psi_3^x}{\partial x} + A_1 \dfrac{\partial \psi_1^x}{\partial x} + \sigma_t \psi_2 = 0 \\ \dfrac{\partial \psi_3^x}{\partial t} + A_2 \dfrac{\partial \psi_2}{\partial x} + \sigma_t \psi_3^x = 0. \end{cases}$$

We can see that we recover the one dimensional P_3 equations. It is also true for the y direction.

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