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MATHEMATICAL ANALYSIS FOR THE PERIDYNAMIC NONLOCAL CONTINUUM THEORY *

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Abstract. We develop a functional analytical framework for a linear peridynamic model of a spring network system in any space dimension. Various properties of the peridynamic operators are examined for general micromodulus functions. These properties are utilized to establish the well-posedness of both the stationary peridynamic model and the Cauchy problem of the time dependent peridynamic model. The connections to the classical elastic models are also provided.

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Introduction

The peridynamic (PD) model proposed by Silling [19] is an integral-type nonlocal continuum theory. It depends crucially upon the nonlocality of force interactions and does not explicitly involve the notion of deformation gradients. On one hand, it provides a more general framework than the classical theory for problems involving discontinuities or other singularities in the deformation; on the other hand, it can also be viewed as a continuum version of molecular dynamics.

Although a relatively recent development, the effectiveness of PD model has already been demonstrated in several sophisticated applications, including the fracture and failure of composites, crack instability, fracture of polycrystals, and nanofiber networks. Yet, from a rigorous mathematical point of view, many important and fundamental issues remain to be studied. In this work, we intend to formulate a rigorous functional analytical framework of the PD models so as to provide a better understanding of the PD model and to guide us in the development and analysis of the numerical algorithms. This in turn will help us utilize the PD theory for multiscale materials modeling. Indeed, PD can be effectively used in the multiscale modeling of materials in different ways: it can serve as a bridge between molecular dynamics (MD) and continuum elasticity (CE) to help mitigate the difficulties encountered when one attempts to couple MD and CE directly [4,5,11,15,18] and, in some situations, PD can be used as a stand-alone model to capture the behavior of materials over a wide range of spatial and temporal scales. For example, to study problems involving defects, one can use the same equations of motion over the entire body and no special treatment is needed near or at defects [6,22].

Keywords and phrases. Peridynamic model, nonlocal continuum theory, well-posedness, Navier equation.

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Such properties make PD a powerful tool for modeling problems involving cracks, interfaces or defects, we refer to [2] for a review of the recent applications of the PD framework.

Parallel to the modeling and application to practical problems, there also have been efforts to establish a sound theoretical foundation for the PD model. For instance, an abstract variational formulation is presented in [16]. It is explained in [21] how the general state-based PD material model converges to the continuum elasticity model as the ratio of the PD horizon to effective length scale decreases, assuming that the underlying deformation is sufficiently smooth. Some results on the existence and uniqueness of L^2 solutions of the PD models associated with bounded integral PD operators have been given in [8–10]. Though much of the focus of [1] is on developing homogenization theory for the PD model, some existence and uniqueness results are also provided, again for bounded integral PD operators. In [12], a nonlocal vector calculus was developed which also provided a rigorous framework for studying the boundary value problems of the nonlocal peridynamic models.

In this work, we study a linear peridynamic model for a spring network systems in $\mathbb{R}^d$. The PD models are briefly described in Section 1, then detailed analysis on the related PD operators and the associated functional spaces are given in Section 2. We consider the stationary and time-dependent PD models in Section 2.2 and Section 2.3 respectively. The properties of the models and their solutions depend crucially on the particular micromodulus functions used to specify the spring network systems. Our results are valid for more general micromodulus functions than considered in the earlier literature. Indeed, the only essential assumption on the micromodulus functions is that appropriate elastic moduli can be defined for the material model. For these more general cases, we prove the well-posedness of weak solutions to the peridynamic equation, together with studies on the solution regularity. We point out, in particular, that for some special cases of singular micromodulus functions, the solution operators still share certain smoothing properties in fractional Sobolev spaces. These mathematical results can become useful in analyzing the output of numerical simulations based on the PD models and in assessing the quality of the numerical solutions. In addition, we also examine the convergence of the solutions of peridynamic equation to that of the differential equation (classical elasticity equation) when the PD horizon parameter goes to zero. While results of this type have been presented in [10,21], our analysis is applicable to more general PD operators and only requires the minimal regularity naturally inherited by the weak solutions of the PD models. Indeed, by not assuming additional smoothness on the solutions of the PD models, such convergence results fit nicely with a distinct feature of the peridynamic modeling, that is, the formulation of physical laws with possibly non-smooth solutions.

1. A linear peridynamic model

The peridynamic model [19] of solid mechanics is a continuum theory in which the force applied on the particle at $\mathbf{x}+\mathbf{u}(t,\mathbf{x})$ by the particle at $\mathbf{x}'+\mathbf{u}(t,\mathbf{x}')$ is characterized by a force function $\mathbf{f}(\mathbf{u}(t,\mathbf{x}')-\mathbf{u}(t,\mathbf{x}),\mathbf{x}'-\mathbf{x})$, where $\mathbf{x}$ and $\mathbf{x}'$ are positions of the particles in the reference configuration, $\mathbf{u}(t,\mathbf{x})$ and $\mathbf{u}(t,\mathbf{x}')$ are displacements of particle $\mathbf{x}$ and $\mathbf{x}'$ with respect to the reference configuration. The equation of motion is

$$\rho(\mathbf{x})\ddot{\mathbf{u}}(t,\mathbf{x}) \;=\; \int_{H_x} \mathbf{f}(\mathbf{u}(t,\mathbf{x}')-\mathbf{u}(t,\mathbf{x}),\mathbf{x}'-\mathbf{x})\mathrm{d}\mathbf{x}' + \mathbf{b}(t,\mathbf{x}) \tag{1.1}$$

where ρ is the mass density, $\mathbf{u}$ the displacement, $\mathbf{f}$ the pairwise force function, $\mathbf{b}$ the external force density, and H_x the peridynamic neighborhood of $\mathbf{x}\in V$. We let V denote the reference configuration of the material and throughout the paper, we take $V=\mathbb{R}^d$ unless otherwise noted. At the same time, we take H_x to be of the form

$$H_x \;=\; \{\mathbf{x}'\in V : |\mathbf{x}'-\mathbf{x}|<\delta\} = B_\delta(\mathbf{x})$$

with $\delta>0$ being the horizon parameter and $B_\delta(\mathbf{x})$ being the d-dimensional ball centered at $\mathbf{x}$, with radius δ.

In this work, we consider only the linear peridynamic equation corresponding to an isotropic, homogeneous microelastic material with a linearized force function for small relative displacements $\mathbf{u}(t,\mathbf{x})-\mathbf{u}(t,\mathbf{x}')$:

$$\mathbf{f}(\mathbf{x}-\mathbf{x}',\mathbf{u}(t,\mathbf{x})-\mathbf{u}(t,\mathbf{x}')) = \mathbf{f}_0(\mathbf{x}-\mathbf{x}') + \mathbf{C}(\mathbf{x}-\mathbf{x}')(\mathbf{u}(t,\mathbf{x})-\mathbf{u}(t,\mathbf{x}')),$$

where $\mathbf{C}(\mathbf{x},\mathbf{x}') = \mathbf{C}(\mathbf{x}',\mathbf{x})$ is the stiffness tensor given by

$$\mathbf{C}(\mathbf{x}-\mathbf{x}') = c_\delta \zeta(|\mathbf{x}'-\mathbf{x}|)(\mathbf{x}'-\mathbf{x}) \otimes (\mathbf{x}'-\mathbf{x}) + F_0(|\mathbf{x}'-\mathbf{x}|)\mathbf{I}$$

with $\mathbf{I}$ being the identity matrix and $\zeta = \zeta(|\mathbf{x}'-\mathbf{x}|)$ being a scalar-valued function. If $F_0(|\mathbf{x}'-\mathbf{x}|) \equiv 0$, the equation models a spring network system [19] which is the case we consider here. For notational convenience, we rewrite the linear equation as:

$$\begin{cases} \mathbf{u}_{tt}(t,\mathbf{x}) = L_\delta \mathbf{u}(t,\mathbf{x}) + \mathbf{b}(t,\mathbf{x}), & \forall (t,\mathbf{x}) \in (0,T) \times \mathbb{R}^d \\ \mathbf{u}(0,\mathbf{x}) = \mathbf{g}(\mathbf{x}), & \forall \mathbf{x} \in \mathbb{R}^d \\ \mathbf{u}_t(0,\mathbf{x}) = \mathbf{h}(\mathbf{x}), & \forall \mathbf{x} \in \mathbb{R}^d \end{cases} \tag{1.2}$$

where

$$L_\delta \mathbf{u}(\mathbf{x}) = c_\delta \int_{B_\delta(\mathbf{x})} \frac{(\mathbf{x}'-\mathbf{x}) \otimes (\mathbf{x}'-\mathbf{x})}{\sigma(|\mathbf{x}'-\mathbf{x}|)} (\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x})) \mathrm{d}\mathbf{x}'. \tag{1.3}$$

Here, $c_\delta > 0$ is a positive normalization constant, and we call $\sigma = \sigma(|\mathbf{x}'-\mathbf{x}|) = 1/\zeta(|\mathbf{x}'-\mathbf{x}|)$ a *kernel function* of the peridynamic integral operator which also determines the micromodulus function.

2. Mathematical analysis of peridynamic model

To set up a suitable functional setting to discuss the well posedness and convergence properties of peridynamic model equations, we first make some assumptions on the kernel function σ:

$$\sigma(\mathbf{x}) > 0,\ \forall \mathbf{x} \in B_\delta(0), \quad \text{and} \quad \tau_\delta := c_\delta \int_{B_\delta(0)} \frac{|\mathbf{x}|^4}{\sigma(|\mathbf{x}|)} \mathrm{d}\mathbf{x} < \infty. \tag{2.1}$$

We note that, as pointed out in the literature (see for instance, eq. (2.10) in [10], and eq. (93) in [19]), the assumption on τ_δ being finite is needed in order to have a suitable definition of the elastic moduli for the corresponding material under consideration. This assumption, in fact, allows us to study the bond-based PD models with much more general kernel functions, and thus more general micromodulus functions, than those considered in the existing mathematical analysis.

As a notational convention, we use $\hat{\mathbf{u}} = \hat{\mathbf{u}}(\boldsymbol{\xi})$ to denote the Fourier transform of $\mathbf{u} = \mathbf{u}(\mathbf{x})$. Moreover, $\bar{\mathbf{u}}$ denotes the complex conjugate of $\mathbf{u}$, and $\mathbf{u}^T$ denotes the transpose of $\mathbf{u}$. By performing the Fourier transform, we can introduce an equivalent definition of our multidimensional peridynamic operator,

$$-L_\delta \mathbf{u}(\mathbf{x}) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} \mathbf{M}_\delta(\boldsymbol{\xi}) \hat{\mathbf{u}}(\boldsymbol{\xi}) \mathrm{e}^{\mathrm{i}\mathbf{x}\cdot\boldsymbol{\xi}} \mathrm{d}\boldsymbol{\xi} \tag{2.2}$$

where $\mathbf{M}_\delta(\boldsymbol{\xi})$, a real-valued and symmetric positive semi-definite $d \times d$ matrix, is the Fourier symbol of the pseudo-differential operator, see [13], $-L_\delta$:

$$\mathbf{M}_\delta(\boldsymbol{\xi}) = c_\delta \int_{B_\delta(0)} \frac{1-\cos(\boldsymbol{\xi}\cdot\mathbf{y})}{\sigma(|\mathbf{y}|)} \mathbf{y} \otimes \mathbf{y} \mathrm{d}\mathbf{y} \tag{2.3}$$

for any $\boldsymbol{\xi} \in \mathbb{R}^d$ and for any PD horizon parameter $\delta > 0$.

By the equivalent definition of the peridynamic operator, we can define the following functional space, equipped with an associated norm:

Definition 2.1. The space $\mathcal{M}_\sigma(\mathbb{R}^d)$, which depends on the kernel function σ, consists of all the functions $\mathbf{u} \in L^2(\mathbb{R}^d)$ for which the $\mathcal{M}_\sigma(\mathbb{R}^d)$ norm

$$\|\mathbf{u}\|_{\mathcal{M}_\sigma} = \left\{ \int_{\mathbb{R}^d} \hat{\mathbf{u}}(\boldsymbol{\xi}) \cdot (\mathbf{I} + \mathbf{M}_\delta(\boldsymbol{\xi})) \hat{\mathbf{u}}(\boldsymbol{\xi}) \mathrm{d}\boldsymbol{\xi} \right\}^{\frac{1}{2}}, \tag{2.4}$$

is finite. We also define the corresponding inner product associated with the $\mathcal{M}_\sigma$ norm:

$$(\mathbf{u},\mathbf{v})_{\mathcal{M}_\sigma} = \int_{\mathbb{R}^d} \hat{\mathbf{v}}(\boldsymbol{\xi}) \cdot (\mathbf{I} + \mathbf{M}_\delta(\boldsymbol{\xi}))\hat{\mathbf{u}}(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi}, \tag{2.5}$$

for any $\mathbf{u},\mathbf{v} \in \mathcal{M}_\sigma(\mathbb{R}^d)$. In addition, we use $\mathcal{M}_\sigma^{-1}(\mathbb{R}^d)$ to denote the dual space of $\mathcal{M}_\sigma(\mathbb{R}^d)$.

Remark 2.2. The norm is well-defined since $I + \mathbf{M}_\delta(\boldsymbol{\xi})$ is real-valued symmetric positive definite matrix and it is uniformly bounded below by I.

Meanwhile, we can have the following properties:

Lemma 2.3. *The space $\mathcal{M}_\sigma(\mathbb{R}^d)$ is a Hilbert space corresponding to the inner product $(\cdot,\cdot)_{\mathcal{M}_\sigma}$.*

Proof. Let $\{\mathbf{u}_n\}$ be a Cauchy sequence in $\mathcal{M}_\sigma(\mathbb{R}^d)$. By definition, it is equivalent to say

$$\left\{(\mathbf{I} + \mathbf{M}_\delta)^{\frac{1}{2}}\hat{\mathbf{u}}_n\right\}$$

is a Cauchy sequence in $L^2(\mathbb{R}^d)$. So by the completeness of $L^2(\mathbb{R}^d)$, there exists an element $\mathbf{v} \in L^2(\mathbb{R}^d)$, such that

$$\|(\mathbf{I} + \mathbf{M}_\delta(\boldsymbol{\xi}))^{\frac{1}{2}}\hat{\mathbf{u}}_n(\boldsymbol{\xi}) - \mathbf{v}(\boldsymbol{\xi})\|_{L^2} \to 0$$

as $n \to \infty$. Then we set

$$\mathbf{u}(\mathbf{x}) = F^{-1}[(\mathbf{I} + \mathbf{M}_\delta(\boldsymbol{\xi}))^{-\frac{1}{2}}\mathbf{v}(\boldsymbol{\xi})],$$

where F^{-1} denotes the inverse Fourier transform. Then one can see that

$$\|\mathbf{u}_n - \mathbf{u}\|_{\mathcal{M}_\sigma} = \|\,(\mathbf{I} + \mathbf{M}_\delta(\boldsymbol{\xi}))^{\frac{1}{2}}\,(\hat{\mathbf{u}}_n(\boldsymbol{\xi}) - \hat{\mathbf{u}}(\boldsymbol{\xi}))\|_{L^2} \to 0.$$

So the space $\mathcal{M}_\sigma(\mathbb{R}^d)$ is complete, and it is thus a Hilbert space. □

Lemma 2.4. *The dual space of $\mathcal{M}_\sigma(\mathbb{R}^d)$ is the space of distributions:*

$$\mathcal{M}_\sigma^{-1}(\mathbb{R}^d) = \left\{\mathbf{u} : \int_{\mathbb{R}^d} \hat{\mathbf{u}}(\boldsymbol{\xi}) \cdot (\mathbf{I} + \mathbf{M}_\delta(\boldsymbol{\xi}))^{-1}\hat{\mathbf{u}}(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi} < \infty\right\},$$

equipped with the norm

$$\|u\|_{\mathcal{M}_\sigma^{-1}(\mathbb{R}^d)} = \left\{\int_{\mathbb{R}^d} \hat{\mathbf{u}}(\boldsymbol{\xi}) \cdot (\mathbf{I} + \mathbf{M}_\delta(\boldsymbol{\xi}))^{-1}\hat{\mathbf{u}}(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi}\right\}^{\frac{1}{2}}.$$

Proof. Let $l = l(\mathbf{u})$ be a bounded linear functional on $\mathcal{M}_\sigma(\mathbb{R}^d)$, then by the Riesz Representation Theorem [7] we know that there exists a unique $\mathbf{w} \in \mathcal{M}_\sigma(\mathbb{R}^d)$ such that $l(\mathbf{u}) = (\mathbf{u},\mathbf{w})_{\mathcal{M}_\sigma(\mathbb{R}^d)}$ for any $\mathbf{u} \in \mathcal{M}_\sigma(\mathbb{R}^d)$. Using the inner product given in (2.5), we have

$$l(\mathbf{u}) = \int_{\mathbb{R}^d} \hat{\mathbf{w}}(\boldsymbol{\xi}) \cdot (\mathbf{I} + \mathbf{M}_\delta(\boldsymbol{\xi}))\hat{\mathbf{u}}(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi}.$$

Let $\hat{\mathbf{v}}(\boldsymbol{\xi}) = (\mathbf{I} + \mathbf{M}_\delta(\boldsymbol{\xi}))\hat{\mathbf{w}}(\boldsymbol{\xi})$. We have $(\mathbf{I} + \mathbf{M}_\delta(\boldsymbol{\xi}))^{\frac{1}{2}}\hat{\mathbf{w}}(\boldsymbol{\xi}) \in L^2(\mathbb{R}^d)$ since $\mathbf{w} \in \mathcal{M}_\sigma(\mathbb{R}^d)$. Thus,

$$(\mathbf{I} + \mathbf{M}_\delta(\boldsymbol{\xi}))^{-\frac{1}{2}}\hat{\mathbf{v}}(\boldsymbol{\xi}) = (\mathbf{I} + \mathbf{M}_\delta(\boldsymbol{\xi}))^{\frac{1}{2}}\hat{\mathbf{w}}(\boldsymbol{\xi}) \in L^2(\mathbb{R}^d).$$

So, $\mathbf{v} \in \mathcal{M}_\sigma^{-1}(\mathbb{R}^d)$ and

$$l(\mathbf{u}) = \int_{\mathbb{R}^d} \hat{\mathbf{u}}(\boldsymbol{\xi}) \cdot \hat{\mathbf{v}}(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi}.$$

Again by the Riesz Representation Theorem, we have

$$\|l\|^2 = \|\mathbf{w}\|^2_{\mathcal{M}_\sigma} = \int_{\mathbb{R}^d} \hat{\mathbf{w}}(\boldsymbol{\xi}) \cdot (\mathbf{I} + \mathbf{M}_\delta(\boldsymbol{\xi}))\hat{\mathbf{w}}(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi} = \int_{\mathbb{R}^d} \hat{\mathbf{v}}(\boldsymbol{\xi}) \cdot (\mathbf{I} + \mathbf{M}_\delta(\boldsymbol{\xi}))^{-1}\hat{\mathbf{v}}(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi} = \|\mathbf{v}\|^2_{\mathcal{M}_\sigma^{-1}(\mathbb{R}^d)}.$$

Meanwhile, if $\mathbf{v} \in \mathcal{M}_\sigma^{-1}(\mathbb{R}^d)$, for any $\mathbf{u} \in \mathcal{M}_\sigma(\mathbb{R}^d)$,

$$\begin{aligned} |(\mathbf{u}, \mathbf{v})_{L^2}| &= \left| \int_{\mathbb{R}^d} \{\hat{\mathbf{u}}(\boldsymbol{\xi}) \cdot \hat{\mathbf{v}}(\boldsymbol{\xi})\}\mathrm{d}\boldsymbol{\xi} \right| = \left| \int_{\mathbb{R}^d} \left\{ \hat{\mathbf{u}}(\boldsymbol{\xi}) \cdot (\mathbf{I} + \mathbf{M}_\delta(\boldsymbol{\xi}))^{\frac{1}{2}} (\mathbf{I} + \mathbf{M}_\delta(\boldsymbol{\xi}))^{-\frac{1}{2}} \hat{\mathbf{v}}(\boldsymbol{\xi}) \right\} \mathrm{d}\boldsymbol{\xi} \right| \\ &\leqslant \|\mathbf{u}\|_{\mathcal{M}_\sigma(\mathbb{R}^d)} \|\mathbf{v}\|_{\mathcal{M}_\sigma^{-1}(\mathbb{R}^d)}. \end{aligned}$$

So, an element $\mathbf{v} \in \mathcal{M}_\sigma^{-1}(\mathbb{R}^d)$ corresponds a bounded linear functional on $\mathcal{M}_\sigma(\mathbb{R}^d)$. □

Lemma 2.5. *The peridynamic operator $-L_\delta$ is self-adjoint on $\mathcal{M}_\sigma(\mathbb{R}^d)$. The operator $-L_\delta + I$ is also an isometry from $\mathcal{M}_\sigma(\mathbb{R}^d)$ to $\mathcal{M}_\sigma^{-1}(\mathbb{R}^d)$, and the norm and inner product in $\mathcal{M}_\sigma(\mathbb{R}^d)$ can also be formulated as*

$$\begin{aligned} \|\mathbf{u}\|_{\mathcal{M}_\sigma} &= [(\mathbf{u}, \mathbf{u})_{\mathcal{M}_\sigma}]^{\frac{1}{2}} = [(\mathbf{u}, \mathbf{u}) + (-L_\delta \mathbf{u}, \mathbf{u})]^{\frac{1}{2}} \\ &= \left[\|\mathbf{u}\|^2_{L^2} + \frac{c_\delta}{2} \int_{\mathbb{R}^d} \int_{B_\delta(\mathbf{x})} \frac{1}{\sigma(|\mathbf{x}' - \mathbf{x}|)} [(\mathbf{u}(\mathbf{x}') - \mathbf{u}(\mathbf{x}')) \cdot (\mathbf{x}' - \mathbf{x})]^2 \mathrm{d}\mathbf{x}' \mathrm{d}\mathbf{x} \right]^{\frac{1}{2}} \end{aligned} \tag{2.6}$$

for any $\mathbf{u} \in \mathcal{M}_\sigma(\mathbb{R}^d)$.

Remark 2.6. The result of this lemma is analogous to the classical result corresponding to the differential operator $-\triangle$, that is, $-\triangle + I$ is self-adjoint and it is an isometry from $H^1(\mathbb{R}^d)$ to $H^{-1}(\mathbb{R}^d)$.

Proof. By the equivalent definition of the PD operator in (2.2) as a pseudo-differential operator with a real, nonnegative symbol, we immediately see that $-L_\delta$ is self-adjoint in $\mathcal{M}_\sigma(\mathbb{R}^d)$. The fact that $-L_\delta + I$ defines an isometry follows directly from the definitions of the norms in $\mathcal{M}_\sigma(\mathbb{R}^d)$ and $\mathcal{M}_\sigma^{-1}(\mathbb{R}^d)$. In addition, using the Parseval formula, we have

$$(-L_\delta \mathbf{u}, \mathbf{u}) + (\mathbf{u}, \mathbf{u}) = (-\widehat{L_\delta \mathbf{u}}, \hat{\mathbf{u}}) + (\hat{\mathbf{u}}, \hat{\mathbf{u}}) = \int_{\mathbb{R}^d} \hat{\mathbf{u}}(\boldsymbol{\xi}) \cdot (\mathbf{I} + \mathbf{M}_\delta(\boldsymbol{\xi}))\, \hat{\mathbf{u}}(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi},$$

which implies equation (2.6). □

For the purpose of discussing the regularity of the weak solutions, we also need to define the following space

$$\mathcal{M}_\sigma^2(\mathbb{R}^d) = \left\{ \mathbf{u} : \int_{\mathbb{R}^d} \hat{\mathbf{u}}(\boldsymbol{\xi}) \cdot (\mathbf{I} + \mathbf{M}_\delta(\boldsymbol{\xi}))^2 \hat{\mathbf{u}}(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi} < \infty \right\},$$

with the dual space

$$\mathcal{M}_\sigma^{-2}(\mathbb{R}^d) = \left\{ \mathbf{u} : \int_{\mathbb{R}^d} \hat{\mathbf{u}}(\boldsymbol{\xi}) \cdot (\mathbf{I} + \mathbf{M}_\delta(\boldsymbol{\xi}))^{-2} \hat{\mathbf{u}}(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi} < \infty \right\}$$

which share the similar properties as the ones in Lemmas 2.3, 2.4 and 2.5.

Given any $\mathbf{y} \in B_\delta(0)$, let us define a difference operator D_y by

$$D_y \mathbf{v}(\mathbf{x}) = \mathbf{v}(\mathbf{x} + \mathbf{y}) - \mathbf{v}(\mathbf{x})$$

for function $\mathbf{v}$ defined in a suitable function space. By the representation of L_δ given in (1.3), we have

$$L_\delta = c_\delta \int_{B_\delta(\mathbf{0})} \frac{\mathbf{y}\otimes\mathbf{y}}{\sigma(|\mathbf{y}|)} D_y \mathrm{d}\mathbf{y} = \frac{c_\delta}{2}\int_{B_\delta(\mathbf{0})} \frac{|\mathbf{y}|^2}{\sigma(|\mathbf{y}|)}(\mathbf{y}\otimes\mathbf{y})\frac{D_y + D_{-y}}{|\mathbf{y}|^2}\mathrm{d}\mathbf{y} \tag{2.7}$$

which gives an interesting formulation of L_δ as a linear combination difference operators (first or second order). The matrix-valued weight $|\mathbf{y}|^2/\sigma(|\mathbf{y}|)(\mathbf{y}\otimes\mathbf{y})$ is in fact in $L^1(\mathbb{R}^d)$ under the assumption (2.1). Then,

Lemma 2.7. *Let P be an scalar operator which commutes with difference operator D_y for all $\mathbf{y}\in B_\delta(0)$, then P also commutes with L_δ.*

Proof. For $\mathbf{u} = \mathbf{u}(\mathbf{x})$ with $L_\delta(\mathbf{u})$ suitably defined, we have

$$c_\delta \int_{B_\delta(\mathbf{0})} \frac{\mathbf{y}\otimes\mathbf{y}}{\sigma(|\mathbf{y}|)}(P(D_y\mathbf{u}))(\mathbf{x})\mathrm{d}\mathbf{y} = c_\delta \int_{B_\delta(\mathbf{0})} \frac{\mathbf{y}\otimes\mathbf{y}}{\sigma(|\mathbf{y}|)}(D_y(P\mathbf{u}))(\mathbf{x}))\mathrm{d}\mathbf{y}.$$

That is, $(P(L_\delta(\mathbf{u}))(\mathbf{x}) = (L_\delta(P(\mathbf{u}))(\mathbf{x})$, for any $\mathbf{x}\in\mathbb{R}^d$, so the lemma follows. □

We note that in particular, all scalar linear differential operators with constant coefficients, and their inverses (when they exist), commute with the difference operator D_y for any $\mathbf{y}$, so we have,

Corollary 2.8. *Let P be a scalar linear differential operator with constant coefficients, then $PL_\delta = L_\delta P$.*

Let us remark that the results of Lemma 2.7 and Corollary 2.8 depend crucially on the facts that in the representation of the PD operator given by (1.3), the horizon parameter δ is a uniform constant in space and that the kernel function $\sigma = \sigma(|\mathbf{x}-\mathbf{x}'|)$ is only a function of $|\mathbf{x}-\mathbf{x}'|$.

The above corollary in particular allows us to have the well-posedness of the generalized (in distribution sense) solutions to the peridynamic equation with rough data. For instance, even when the external force $\mathbf{b} = \mathbf{b}(t,\mathbf{x})$ is not in L^2, we may still get a unique solution to equation (2.28) in the appropriate weak sense by lifting $\mathbf{b}$ to $P^{-1}\mathbf{b}$ for some P which commutes with L_δ, if $P^{-1}\mathbf{b}$ can be properly defined, to get a generalized solution of (2.28) in the form $P(-L_\delta + I)^{-1}P^{-1}\mathbf{b}$.

Now we can discuss the equivalence between the defined space $\mathcal{M}_\sigma(\mathbb{R}^d)$ and standard Sobolev spaces. Similar results for the special case of scalar valued functions can be found in [3]. To treat the vector valued case, we adopt the convention that for two symmetric matrices A and B, $A < B$ means $B - A$ is positive definite, and $A \le B$ means $B - A$ is positive semi-definite. Let $\hookrightarrow$ denote the continuous embedding of function spaces.

We note that the Fourier symbol of the Navier operator of the Navier equation (2.31) with Lame coefficients $\mu = \lambda$ is

$$\mathbf{M}_0(\boldsymbol{\xi}) = \mu[|\boldsymbol{\xi}|^2\mathbf{I} + 2\boldsymbol{\xi}\otimes\boldsymbol{\xi}] \tag{2.8}$$

which corresponds to a material with Poisson ratio $\nu = 0.25$ (see related discussion on $\mathbf{M}_0(\boldsymbol{\xi})$ given in the next section). Then the following general embedding results can be shown:

Lemma 2.9. *Let the kernel function $\sigma = \sigma(|\mathbf{y}|)$ satisfy condition (2.1) and let $\mathbf{M}_0(\boldsymbol{\xi})$ be defined by (2.8) for a positive constant μ. Then*

$$0 < \mathbf{M}_\delta(\boldsymbol{\xi}) \leqslant \frac{1}{2\mu}\tau_\delta \mathbf{M}_0(\boldsymbol{\xi}), \quad \forall \boldsymbol{\xi}\in\mathbb{R}^d. \tag{2.9}$$

Consequently, we have

$$H^1(\mathbb{R}^d) \hookrightarrow \mathcal{M}_\sigma(\mathbb{R}^d), \quad H^2(\mathbb{R}^d) \hookrightarrow \mathcal{M}_\sigma^2(\mathbb{R}^d).$$

Proof. Using the inequality

$$0 \leqslant 1 - \cos(\boldsymbol{\xi}\cdot\mathbf{y}) \leqslant \frac{(\boldsymbol{\xi}\cdot\mathbf{y})^2}{2} \leqslant |\boldsymbol{\xi}|^2|\mathbf{y}|^2/2.$$

And by the form of $\mathbf{M}_0(\boldsymbol{\xi})$, we can easily see that

$$\mathbf{M}_0(\boldsymbol{\xi}) \geqslant \mu|\boldsymbol{\xi}|^2\mathbf{I}$$

with $\mathbf{I}$ being the identity matrix. So we have

$$\begin{aligned}\mathbf{M}_\delta(\boldsymbol{\xi}) &= c_\delta \int_{B_\delta(0)} \frac{1-\cos(\boldsymbol{\xi}\cdot\mathbf{y})}{\sigma(|\mathbf{y}|)}\mathbf{y}\otimes\mathbf{y}\mathrm{d}\mathbf{y} \leqslant c_\delta \left\{\int_{B_\delta(0)} \frac{1-\cos(\boldsymbol{\xi}\cdot\mathbf{y})}{\sigma(|\mathbf{y}|)}|\mathbf{y}|^2\mathrm{d}\mathbf{y}\right\}\mathbf{I} \\ &\leqslant c_\delta \left\{\int_{B_\delta(0)} \frac{|\boldsymbol{\xi}|^2}{2\sigma(|\mathbf{y}|)}|\mathbf{y}|^4\mathrm{d}\mathbf{y}\right\}\mathbf{I} = \frac{1}{2\mu}\tau_\delta\mu|\boldsymbol{\xi}|^2\mathbf{I} \leqslant \frac{1}{2\mu}\tau_\delta\mathbf{M}_0(\boldsymbol{\xi})\end{aligned}$$

which gives (2.9). The rest of the conclusions follow from the above inequality and the definitions of the relevant function spaces. □

Besides Lemma 2.9, we also show that $\mathbf{M}_0$ and $\mathbf{M}_\delta$ actually commute, a nice property that is useful to make comparison between the PD model and its local limit.

Lemma 2.10. *The matrices $\mathbf{M}_\delta$ and $\mathbf{M}_0$ defined by (2.3) and (2.8) commute, that is, $\mathbf{M}_\delta(\boldsymbol{\xi})\mathbf{M}_0(\boldsymbol{\xi}) = \mathbf{M}_0(\boldsymbol{\xi})\mathbf{M}_\delta(\boldsymbol{\xi})$.*

Proof. We only need to show that $\boldsymbol{\xi}\otimes\boldsymbol{\xi}$ commute with $\mathbf{M}_\delta(\boldsymbol{\xi})$.

Let $A = (a_{ij}) = (\boldsymbol{\xi}\otimes\boldsymbol{\xi})\mathbf{M}_\delta(\boldsymbol{\xi})$, then $A^T = \mathbf{M}_\delta(\boldsymbol{\xi})(\boldsymbol{\xi}\otimes\boldsymbol{\xi})$. By direct computation, for any $1 \le i,j \le d$, we have

$$\begin{aligned}a_{ij} - a_{ji} &= \sum_{k=1}^d c_\delta \int_{B_\delta(0)} \frac{1-\cos(\boldsymbol{\xi}\cdot\mathbf{y})}{\sigma(|\mathbf{y}|)}(\xi_i y_j - \xi_j y_i)\xi_k y_k \,\mathrm{d}\mathbf{y} = c_\delta \int_{B_\delta(0)} \frac{1-\cos(\boldsymbol{\xi}\cdot\mathbf{y})}{\sigma(|\mathbf{y}|)}\xi_i\xi_j(y_j^2 - y_i^2)\,\mathrm{d}\mathbf{y} \\ &= \frac{c_\delta}{d}\xi_i\xi_j \int_{B_\delta(0)} \frac{1-\cos(\boldsymbol{\xi}\cdot\mathbf{y})}{\sigma(|\mathbf{y}|)}(|\mathbf{y}|^2 - |\mathbf{y}|^2)\,\mathrm{d}\mathbf{y} = 0,\end{aligned}$$

where we have used the symmetry of the integrand. We then conclude that $A = A^T$. This implies that $\mathbf{M}_0$ commutes with $\boldsymbol{\xi}\otimes\boldsymbol{\xi}$ and thus commutes also with $\mathbf{M}_\delta$. □

2.1. Space equivalence for special kernel functions

We now focus on some kernel functions with special properties to establish relations between $\mathcal{M}_\sigma(\mathbb{R}^d)$ and the more conventional Sobolev spaces. To examine more singular kernel functions, we let $H^s(\mathbb{R}^d) = \{\mathbf{u} \in (L^2(\mathbb{R}^d))^d : |\boldsymbol{\xi}|^s\hat{\mathbf{u}} \in (L^2(\mathbb{R}^d))^d\}$ denote the fractional Sobolev space on $\mathbb{R}^d$ for $s \in (0,1)$.

For convenience, we define the following functions for $\chi > 0$ and $\alpha \ge 0$:

$$C_{\min}(\alpha,\chi) = c_\delta \int_{B_\chi(0)} \frac{1-\cos(z_1)}{|\mathbf{z}|^{2+d+2\alpha}}\min(\mathbf{z}^2)\mathrm{d}\mathbf{z}, \tag{2.10}$$

$$C_{\max}(\alpha,\chi) = c_\delta \int_{B_\chi(0)} \frac{1-\cos(z_1)}{|\mathbf{z}|^{2+d+2\alpha}}\max(\mathbf{z}^2)\mathrm{d}\mathbf{z}. \tag{2.11}$$

Here, we have $\min(\mathbf{z}^2) = \min\{z_i^2\}$ with $\{z_i\}_{i=1}^d$ being the components of $\mathbf{z}$, and similarly, $\max(\mathbf{z}^2) = \max\{z_i^2\}$.

For kernel functions σ such that $|\mathbf{x}|^2/\sigma(|\mathbf{x}|) \in L^1(B_\delta(0))$, there are already some results in [1,10] showing that $-L_\delta$ is a bounded linear operator from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$. In fact, we have:

Lemma 2.11. *Let $\sigma = \sigma(|\mathbf{y}|)$ satisfy the additional condition that*

$$\int_{B_\delta(0)} \frac{|\mathbf{y}|^2}{\sigma(|\mathbf{y}|)}\,\mathrm{d}\mathbf{y} < \infty, \tag{2.12}$$

we then have

$$\mathcal{M}_\sigma^2(\mathbb{R}^d) = \mathcal{M}_\sigma(\mathbb{R}^d) = L^2(\mathbb{R}^d),$$

and

$$\|\mathbf{u}\|_{L^2} \leqslant \|\mathbf{u}\|_{\mathcal{M}_\sigma(\mathbb{R}^d)} \leqslant \|\mathbf{u}\|_{L^2}\left(1 + c_\delta \int_{B_\delta(0)} \frac{|\mathbf{y}|^2}{\sigma(|\mathbf{y}|)}\mathrm{d}\mathbf{y}\right)^{\frac{1}{2}}, \quad \forall \mathbf{u} \in L^2(\mathbb{R}^d), \tag{2.13}$$

$$\|\mathbf{u}\|_{L^2} \leqslant \|\mathbf{u}\|_{\mathcal{M}_\sigma^2(\mathbb{R}^d)} \leqslant \|\mathbf{u}\|_{L^2}\left(1 + c_\delta \int_{B_\delta(0)} \frac{|\mathbf{y}|^2}{\sigma(|\mathbf{y}|)}\mathrm{d}\mathbf{y}\right), \quad \forall \mathbf{u} \in L^2(\mathbb{R}^d). \tag{2.14}$$

Moreover, the operators $-L_\delta$ and $(-L_\delta + I)^{-1}$ are bounded linear operators from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^d)$.

Proof. Under the condition of σ, we can see

$$0 < \mathbf{M}_\delta(\boldsymbol{\xi}) \leqslant c_\delta \left\{ \int_{B_\delta(0)} \frac{|\mathbf{y}|^2}{\sigma(|\mathbf{y}|)}\mathrm{d}\mathbf{y} \right\} \mathbf{I}.$$

By Parseval identity, we have (2.13) and (2.14) which in turn implies that $-L_\delta$ and $(-L_\delta + I)^{-1}$ are bounded operators from $L^2(\mathbb{R}^d)$ to $L^2(\mathbb{R}^2)$, and $\mathcal{M}_\sigma^{-1}(\mathbb{R}^d) = \mathcal{M}_\sigma(\mathbb{R}^d) = L^2(\mathbb{R}^d)$. □

For more general kernel functions, *i.e.* σ satisfying (2.1), the PD operator $-L_\delta$ may become unbounded in $L^2(\mathbb{R}^d)$ when the function $|\mathbf{x}|^2/\sigma(|\mathbf{x}|)$ is no longer in $L^1(B_\delta(0))$. Yet, as we demonstrate below, the basic existence and uniqueness results remain valid but with the discussion taking place in other function spaces such as $\mathcal{M}_\sigma(\mathbb{R}^d)$, as defined earlier. This is due to the fact that $-L_\delta$ becomes a bounded operator from $\mathcal{M}_\sigma(\mathbb{R}^d)$ to $\mathcal{M}_\sigma^{-1}(\mathbb{R}^d)$. To see how such spaces are related to the conventional Sobolev spaces, we first consider the space equivalence for some special kernel functions.

Lemma 2.12. *Let the kernel function $\sigma = \sigma(|\mathbf{y}|)$ satisfy the assumption (2.1) and the condition*

$$\sigma(|\mathbf{y}|) \leqslant \gamma_1 |\mathbf{y}|^{2+d+2\beta}, \quad \forall |\mathbf{y}| \leqslant \delta \tag{2.15}$$

for some exponent $\beta \in [0, 1)$ and positive constant γ_1, then we have

$$\mathcal{M}_\sigma(\mathbb{R}^d) \hookrightarrow H^\beta(\mathbb{R}^d), \quad \mathcal{M}_\sigma^2(\mathbb{R}^d) \hookrightarrow H^{2\beta}(\mathbb{R}^d),$$

$$H^{-\beta}(\mathbb{R}^d) \hookrightarrow \mathcal{M}_\sigma^{-1}(\mathbb{R}^d), \quad H^{-2\beta}(\mathbb{R}^d) \hookrightarrow \mathcal{M}_\sigma^{-2}(\mathbb{R}^d).$$

Moreover, we have

$$C_1\|\mathbf{u}\|_{H^\beta(\mathbb{R}^d)} \leqslant \|\mathbf{u}\|_{\mathcal{M}_\sigma(\mathbb{R}^d)}, \quad \forall \mathbf{u} \in \mathcal{M}_\sigma(\mathbb{R}^d), \tag{2.16}$$

$$C_1\|\mathbf{u}\|_{\mathcal{M}_\sigma^{-1}(\mathbb{R}^d)} \leqslant \|\mathbf{u}\|_{H^{-\beta}(\mathbb{R}^d)}, \quad \forall \mathbf{u} \in H^{-\beta}(\mathbb{R}^d), \tag{2.17}$$

and

$$\tilde{C}_1\|\mathbf{u}\|_{H^{2\beta}(\mathbb{R}^d)} \leqslant \|\mathbf{u}\|_{\mathcal{M}_\sigma^2(\mathbb{R}^d)}, \quad \forall \mathbf{u} \in \mathcal{M}_\sigma^2(\mathbb{R}^d), \tag{2.18}$$

$$\tilde{C}_1\|\mathbf{u}\|_{\mathcal{M}_\sigma^{-2}(\mathbb{R}^d)} \leqslant \|\mathbf{u}\|_{H^{-2\beta}(\mathbb{R}^d)}, \quad \forall \mathbf{u} \in H^{-2\beta}(\mathbb{R}^d), \tag{2.19}$$

with constants $C_1 = \min(1, (2C_{\min}(\beta,\delta)/\gamma_1)^{\frac{1}{2}})/2$ and $\tilde{C}_1 = \min(1, 2C_{\min}(\beta,\delta)/\gamma_1)/2$.

Proof. Under the condition on σ, when $|\boldsymbol{\xi}| \geqslant 1$, we do a change of variable

$$\mathbf{z} = |\boldsymbol{\xi}|\mathbf{R}\mathbf{y}$$

where $\mathbf{R}$ is an orthogonal matrix from $\mathbb{R}^d$ to $\mathbb{R}^d$ with the first row being $\boldsymbol{\xi}/|\boldsymbol{\xi}|$.

So we have

$$\begin{aligned}
\mathbf{M}_\delta(\boldsymbol{\xi}) &= c_\delta \int_{B_\delta(0)} \frac{1-\cos(\boldsymbol{\xi}\cdot\mathbf{y})}{\sigma(|\mathbf{y}|)}\mathbf{y}\otimes\mathbf{y}\mathrm{d}\mathbf{y} \geqslant \frac{c_\delta}{\gamma_1}\int_{B_\delta(0)} \frac{1-\cos(\boldsymbol{\xi}\cdot\mathbf{y})}{|\mathbf{y}|^{2+d+2\beta}}\mathbf{y}\otimes\mathbf{y}\mathrm{d}\mathbf{y} \\
&= \frac{c_\delta|\boldsymbol{\xi}|^{2+2\beta}}{\gamma_1}\int_{B_{|\boldsymbol{\xi}|\delta}(0)} \frac{1-\cos(z_1)}{|\mathbf{z}|^{2+d+2\beta}}\frac{1}{|\boldsymbol{\xi}|}\mathbf{R}^T\mathbf{z}\otimes\frac{1}{|\boldsymbol{\xi}|}\mathbf{R}^T\mathbf{z}\mathrm{d}\mathbf{z} = \frac{c_\delta|\boldsymbol{\xi}|^{2\beta}}{\gamma_1}\int_{B_{|\boldsymbol{\xi}|\delta}(0)} \frac{1-\cos(z_1)}{|\mathbf{z}|^{2+d+2\beta}}\mathbf{R}^T\mathbf{z}\otimes\mathbf{z}'\mathbf{R}\mathrm{d}\mathbf{z} \\
&= \frac{c_\delta|\boldsymbol{\xi}|^{2\beta}}{\gamma_1}\int_{B_{|\boldsymbol{\xi}|\delta}(0)} \frac{1-\cos(z_1)}{|\mathbf{z}|^{2+d+2\beta}}\mathbf{R}^T\mathrm{diag}(z_i^2)\mathbf{R}\mathrm{d}\mathbf{z} \geqslant \frac{c_\delta|\boldsymbol{\xi}|^{2\beta}}{\gamma_1}\left\{\int_{B_\delta(0)} \frac{1-\cos(z_1)}{|\mathbf{z}|^{2+d+2\beta}}\min(\mathbf{z}^2)\mathrm{d}\mathbf{z}\right\}\mathbf{I} \\
&= 2\frac{C_{\min}(\beta,\delta)}{\gamma_1}|\boldsymbol{\xi}|^{2\beta}\mathbf{I}
\end{aligned}$$

where we have used the symmetry of the integration domain. Thus the space embedding results follow. Moreover (2.16) and (2.18) are satisfied. The inequalities (2.17) and (2.19) follow by duality estimates. □

Meanwhile, we also have:

Lemma 2.13. *Let the kernel function $\sigma = \sigma(|\mathbf{y}|)$ satisfy the condition*

$$\sigma(|\mathbf{y}|) \geqslant \gamma_2|\mathbf{y}|^{2+d+2\alpha}, \quad \forall|\mathbf{y}| \leqslant \delta \tag{2.20}$$

for some exponent $\alpha \in (0,1)$ and positive constant γ_2, then we have

$$H^\alpha(\mathbb{R}^d) \hookrightarrow \mathcal{M}_\sigma(\mathbb{R}^d), \quad H^{2\alpha}(\mathbb{R}^d) \hookrightarrow \mathcal{M}_\sigma^2(\mathbb{R}^d),$$

and

$$\mathcal{M}_\sigma^{-1}(\mathbb{R}^d) \hookrightarrow H^{-\alpha}(\mathbb{R}^d), \quad \mathcal{M}_\sigma^{-2}(\mathbb{R}^d) \hookrightarrow H^{-2\alpha}(\mathbb{R}^d).$$

Moreover, we have

$$\|\mathbf{u}\|_{\mathcal{M}_\sigma(\mathbb{R}^d)} \leqslant C_2\|\mathbf{u}\|_{H^\alpha(\mathbb{R}^d)}, \quad \forall\mathbf{u} \in H^\alpha(\mathbb{R}^d), \tag{2.21}$$

$$\|\mathbf{u}\|_{H^{-\alpha}(\mathbb{R}^d)} \leqslant C_2\|\mathbf{u}\|_{\mathcal{M}_\sigma^{-1}(\mathbb{R}^d)}, \quad \forall\mathbf{u} \in \mathcal{M}_\sigma^{-1}(\mathbb{R}^d), \tag{2.22}$$

and

$$\|u\|_{\mathcal{M}_\sigma^2(\mathbb{R}^d)} \leqslant \tilde{C}_2\|\mathbf{u}\|_{H^{2\alpha}(\mathbb{R}^d)}, \quad \forall\mathbf{u} \in H^{2\alpha}(\mathbb{R}^d), \tag{2.23}$$

$$\|\mathbf{u}\|_{H^{-2\alpha}(\mathbb{R}^d)} \leqslant \tilde{C}_2\|\mathbf{u}\|_{\mathcal{M}_\sigma^{-2}(\mathbb{R}^d)}, \quad \forall\mathbf{u} \in \mathcal{M}_\sigma^{-2}(\mathbb{R}^d), \tag{2.24}$$

with constants $C_2 = \max(1, (C_{\max}(\alpha,\infty)/\gamma_2)^{\frac{1}{2}})$ and $\tilde{C}_2 = \max(1, C_{\max}(\alpha,\infty)/\gamma_2)$.

Proof. Similar to the proof of the previous lemma, under the condition on σ and for $|\boldsymbol{\xi}|>0$, we make a change of variable $\mathbf{z}=|\boldsymbol{\xi}|\mathbf{R}\mathbf{y}$ where $\mathbf{R}$ is an orthogonal matrix from $\mathbb{R}^d$ to $\mathbb{R}^d$ with the first row $\boldsymbol{\xi}/|\boldsymbol{\xi}|$. Then we have

$$\begin{aligned}
\mathbf{M}_\delta(\boldsymbol{\xi}) &= c_\delta\int_{B_\delta(0)}\frac{1-\cos(\boldsymbol{\xi}\cdot\mathbf{y})}{\sigma(|\mathbf{y}|)}\mathbf{y}\otimes\mathbf{y}d\mathbf{y}\leqslant\frac{c_\delta}{\gamma_2}\int_{B_\delta(0)}\frac{1-\cos(\boldsymbol{\xi}\cdot\mathbf{y})}{|\mathbf{y}|^{2+d+2\alpha}}\mathbf{y}\otimes\mathbf{y}d\mathbf{y}\\
&= \frac{c_\delta|\boldsymbol{\xi}|^{2+2\alpha}}{\gamma_2}\int_{B_{|\boldsymbol{\xi}|\delta}(0)}\frac{1-\cos(z_1)}{|\mathbf{z}|^{2+d+2\alpha}}\frac{1}{|\boldsymbol{\xi}|}\mathbf{R}^T\mathbf{z}\otimes\frac{1}{|\boldsymbol{\xi}|}\mathbf{R}^T\mathbf{z}d\mathbf{z}\\
&= \frac{c_\delta|\boldsymbol{\xi}|^{2\alpha}}{\gamma_2}\int_{B_{|\boldsymbol{\xi}|\delta}(0)}\frac{1-\cos(z_1)}{|\mathbf{z}|^{2+d+2\alpha}}\mathbf{R}^T\mathbf{z}\otimes\mathbf{z}'\mathbf{R}d\mathbf{z}\\
&= \frac{c_\delta|\boldsymbol{\xi}|^{2\alpha}}{\gamma_2}\int_{B_{|\boldsymbol{\xi}|\delta}(0)}\frac{1-\cos(z_1)}{|\mathbf{z}|^{2+d+2\alpha}}\mathbf{R}^T\mathrm{diag}(z_i^2)\mathbf{R}d\mathbf{z}\\
&\leqslant \left\{\frac{c_\delta|\boldsymbol{\xi}|^{2\alpha}}{\gamma_2}\int_{\mathbb{R}^d}\frac{1-\cos(z_1)}{|\mathbf{z}|^{2+d+2\alpha}}\max(\mathbf{z}^2)d\mathbf{z}\right\}\mathbf{I}=2\frac{C_{\max}(\alpha,\infty)}{\gamma_2}|\boldsymbol{\xi}|^{2\alpha}\mathbf{I}
\end{aligned}$$

where again we have used the symmetry of the integration domain. The space embedding results then follow from the respective definitions of the function spaces. Moreover, (2.21) and (2.23) are satisfied with the inequalities (2.22) and (2.24) following from duality estimates. □

Consequently, we see that under suitable conditions on the kernel function, the space $\mathcal{M}_\sigma(\mathbb{R}^d)$ is equivalent to some standard *fractional* Sobolev spaces.

Theorem 2.14. *Let the kernel function $\sigma=\sigma(|\mathbf{y}|)$ satisfy the condition*

$$\gamma_2|\mathbf{y}|^{2+d+2\alpha}\leqslant\sigma(|\mathbf{y}|)\leqslant\gamma_1|\mathbf{y}|^{2+d+2\alpha},\quad\forall|\mathbf{y}|\leqslant\delta \tag{2.25}$$

for some exponent $\alpha\in(0,1)$ and positive constants γ_1 and γ_2, then we have

$$\mathcal{M}_\sigma(\mathbb{R}^d)=H^\alpha(\mathbb{R}^d),\quad\mathcal{M}^2_\sigma(\mathbb{R}^d)=H^{2\alpha}(\mathbb{R}^d).$$

Moreover, for any $\mathbf{u}\in\mathcal{M}_\sigma(\mathbb{R}^d)$,

$$C_1\|\mathbf{u}\|_{H^\alpha(\mathbb{R}^d)}\leqslant\|\mathbf{u}\|_{\mathcal{M}_\sigma(\mathbb{R}^d)}\leqslant C_2\|\mathbf{u}\|_{H^\alpha(\mathbb{R}^d)}, \tag{2.26}$$

and for any $\mathbf{u}\in\mathcal{M}^2_\sigma(\mathbb{R}^d)$,

$$\tilde{C}_1\|\mathbf{u}\|_{H^{2\alpha}(\mathbb{R}^d)}\leqslant\|\mathbf{u}\|_{\mathcal{M}^2_\sigma(\mathbb{R}^d)}\leqslant\tilde{C}_2\|\mathbf{u}\|_{H^{2\alpha}(\mathbb{R}^d)}, \tag{2.27}$$

with the positive constants C_1, C_2, $\tilde{C}_1$ and $\tilde{C}_2$ defined in Lemmas 2.12 and 2.13.

We see from the above discussions that, under additional assumptions on the kernel function σ, we have the equivalence or continuous embedding theories between $\mathcal{M}_\sigma(\mathbb{R}^d)$ and some fractional Sobolev spaces.

2.2. Properties of stationary PD model

In this section, we give some results on the existence and uniqueness of weak solutions to the stationary (equilibrium) PD model with general kernel functions, *i.e.* L_δ may be unbounded in $L^2(\mathbb{R}^d)$:

$$-L_\delta\mathbf{u}+\mathbf{u}=\mathbf{b} \tag{2.28}$$

and some convergence properties of the solution of the stationary PD model. The term $\mathbf{u}$ is added for two purposes, one is to eliminate the need to imposing far field conditions at infinity and the other is to eliminate the nonuniqueness of solution when no boundary condition is imposed.

First, we may also establish the corresponding variational theory and some regularity properties for the stationary PD model. Then, using the properties of the PD operator provided earlier, we have:

Lemma 2.15. *Let $\sigma = \sigma(|\mathbf{y}|)$ satisfy condition (2.1), for any $\mathbf{b} \in \mathcal{M}_\sigma^{-1}(\mathbb{R}^d)$, problem (2.28) has a unique solution $\mathbf{u} \in \mathcal{M}_\sigma(\mathbb{R}^d)$ which is the minimizer of the functional:*

$$
\begin{aligned}
E(\mathbf{u}) &= \frac{1}{2}\|\mathbf{u}\|^2_{\mathcal{M}_\sigma(\mathbb{R}^d)} - (\mathbf{u},\mathbf{b})_{L^2(\mathbb{R}^d)} \\
&= \frac{1}{2}\int_{\mathbb{R}^d} \hat{\mathbf{u}}(\boldsymbol{\xi}) \cdot (\mathbf{I} + \mathbf{M}_\delta(\boldsymbol{\xi}))\hat{\mathbf{u}}(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi} - (\mathbf{u},\mathbf{b})_{L^2(\mathbb{R}^d)}
\end{aligned}
\tag{2.29}
$$

in $\mathcal{M}_\sigma(\mathbb{R}^d)$.

Proof. The conclusion follows directly from the fact that $E = E(\mathbf{u})$ is a convex quadratical functional with $-L_\delta\mathbf{u} + \mathbf{u} - \mathbf{b} \in \mathcal{M}_\sigma^{-1}(\mathbb{R}^d)$ being its variational derivative at $\mathbf{u} \in \mathcal{M}_\sigma(\mathbb{R}^d)$. □

We note that more general variational descriptions of the PD models can be found in [12,16]. As for regularity, we have for some special kernel functions that:

Lemma 2.16. *Let $\sigma = \sigma(|\mathbf{y}|)$ satisfy condition (2.25), problem (2.28) has a unique solution $\mathbf{u} \in H^{m+2\alpha}(\mathbb{R}^d)$, whenever $\mathbf{b} \in H^m(\mathbb{R}^d)$ for any $m \geqslant -2\alpha$.*

Proof. Taking the Fourier transform of equation (2.28), we get

$$
(\mathbf{M}_\delta(\boldsymbol{\xi}) + \mathbf{I})\hat{\mathbf{u}}(\boldsymbol{\xi}) = \hat{\mathbf{b}}(\boldsymbol{\xi}). \tag{2.30}
$$

Then we have

$$
((\mathbf{M}_\delta(\boldsymbol{\xi}) + \mathbf{I})\hat{\mathbf{u}}(\boldsymbol{\xi})) \cdot ((\mathbf{M}_\delta(\boldsymbol{\xi}) + \mathbf{I}))\, \hat{\mathbf{u}}(\boldsymbol{\xi})(|\boldsymbol{\xi}|^2+1)^m = |\hat{\mathbf{b}}(\boldsymbol{\xi})|^2(|\boldsymbol{\xi}|^2+1)^m \in L^1(\mathbb{R}^d).
$$

By Theorem 2.14, we have

$$
(|\boldsymbol{\xi}|^2+1)^{2\alpha+m}|\hat{\mathbf{u}}(\boldsymbol{\xi})|^2 \in L^1(\mathbb{R}^d).
$$

So the result follows. □

We now go back to the general kernel functions to consider the regularity of solution of the equilibrium equations (2.28).

Lemma 2.17. *Let the kernel function $\sigma = \sigma(|\mathbf{y}|)$ satisfy (2.1), the problem (2.28) has a unique solution $\mathbf{u} \in \mathcal{M}_\sigma(\mathbb{R}^d)$ for $\mathbf{b} \in \mathcal{M}_\sigma^{-1}(\mathbb{R}^d)$. Moreover, if $\mathbf{b} \in L^2(\mathbb{R}^d)$, the solution of the equilibrium equation (2.28) satisfies*

$$
\mathbf{u} \in \mathcal{M}_\sigma^2(\mathbb{R}^d).
$$

Proof. The first part follows from the isometry property given in Lemma 2.5. The proof of regularity is similar to that of Lemma 2.16. □

By Lemma 2.7 and Corollary 2.8, we also have the following regularity:

Lemma 2.18. *Let P be a linear scalar operator with constant coefficients, if $\mathbf{u}$ is a solution of the equation (2.28) with a given function $\mathbf{b}$, then $P\mathbf{u}$ is the solutions of (2.28) with the right hand side being $P\mathbf{b}$.*

Proof. From Lemma 2.7 and Corollary 2.8, we know $-L_\delta P = -PL_\delta$. So

$$-L_\delta(P\mathbf{u}) = P(-L_\delta \mathbf{u}) = P\mathbf{b}.$$

This leads to the result of the lemma. □

Considering the convergence of the PD model to differential equation, we first denote by $\mathbf{u}_o = \mathbf{u}_o(\mathbf{x}) \in H^2(\mathbb{R}^d)$ the solution of

$$-\mu\triangle \mathbf{u}_o(\mathbf{x}) - (\mu+\lambda)\nabla \mathrm{div}\, \mathbf{u}_o(\mathbf{x}) + \mathbf{u}_o(\mathbf{x}) = \mathbf{b}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d \tag{2.31}$$

for $\mathbf{b} \in L^2(\mathbb{R}^d)$, where μ and λ are Lamé coefficients with $\mu = \lambda$. We note that, in this case, $\mathbf{I} + \mathbf{M}_0(\xi)$ defined in (2.8) is the Fourier symbol of equation (2.31).

To indicate the dependence of the solutions of the PD model on the parameter δ, we set $\mathbf{u}_\delta = \mathbf{u}_\delta(\mathbf{x})$ as the solution of (2.28). Then we have the following theorem.

Theorem 2.19. *Let the kernel function $\sigma = \sigma(|\mathbf{y}|)$ satisfy (2.1). Then, as $\delta \to 0$, the solution of (2.28) converges to the solution of (2.31) as measured by the norm $\|\cdot\|_{\mathcal{M}^2_\sigma(\mathbb{R}^d)}$ (and consequently in $L^2(\mathbb{R}^d)$), provided that $\mathbf{b} \in L^2(\mathbb{R}^d)$ and*

$$\tau_\delta \to 2d(d+2)\mu, \tag{2.32}$$

for the constant τ_δ defined in (2.1).

Proof. We consider the problem in Fourier space, we can see, the Fourier coefficients of the solutions for equation (2.28) and (2.31) are

$$\hat{\mathbf{u}}_\delta(\boldsymbol{\xi}) = (\mathbf{I} + \mathbf{M}_\delta(\boldsymbol{\xi}))^{-1}\hat{\mathbf{b}}(\boldsymbol{\xi}), \quad \hat{\mathbf{u}}_0(\boldsymbol{\xi}) = (\mathbf{I} + \mathbf{M}_0(\boldsymbol{\xi}))^{-1}\hat{\mathbf{b}}(\boldsymbol{\xi}).$$

By simple calculation, we can get

$$\|\mathbf{u}_\delta - \mathbf{u}_0\|^2_{\mathcal{M}^2_\sigma(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \hat{\mathbf{b}}(\boldsymbol{\xi}) \cdot (\mathbf{I} + \mathbf{M}_0(\boldsymbol{\xi}))^{-1}(\mathbf{M}_0(\boldsymbol{\xi}) - \mathbf{M}_\delta(\boldsymbol{\xi}))^2(\mathbf{I} + \mathbf{M}_0(\boldsymbol{\xi}))^{-1}\hat{\mathbf{b}}(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi}.$$

Meanwhile, by Lemma 2.9, we have

$$(\mathbf{I} + \mathbf{M}_0(\boldsymbol{\xi}))^{-1}(\mathbf{M}_0(\boldsymbol{\xi}) - \mathbf{M}_\delta(\boldsymbol{\xi}))^2(\mathbf{I} + \mathbf{M}_0(\boldsymbol{\xi}))^{-1} \leqslant (1 + \frac{\tau_\delta}{2\mu})^2\mathbf{I}.$$

This implies that $\|\mathbf{u}_\delta - \mathbf{u}_0\|^2_{\mathcal{M}^2_\sigma(\mathbb{R}^d)}$ is uniformly bounded. We note also the fact that

$$\int_{B_\delta(0)} f(|\mathbf{x}|)x_i^4 \mathrm{d}\mathbf{x} = 3\int_{B_\delta(0)} f(|\mathbf{x}|)x_k^2 x_l^2 \mathrm{d}\mathbf{x}, \quad k \neq l,\ i,k,l = 1,2,\ldots,d \tag{2.33}$$

where $f = f(x)$ is any function such that the integral is well defined.

Then by Taylor expansion, condition (2.32) and the identities given in (2.33), we can easily see that for any given $\boldsymbol{\xi}$,

$$
\begin{aligned}
\mathbf{M}_\delta(\boldsymbol{\xi}) &= \frac{c_\delta}{2}\int_{B_\delta(0)} \frac{(\boldsymbol{\xi}\cdot\mathbf{y})^2}{\sigma(|\mathbf{y}|)}\mathbf{y}\otimes\mathbf{y}\mathrm{d}\mathbf{y} + \alpha_\delta(\boldsymbol{\xi}) \\
&= \frac{c_\delta}{2}\int_{B_\delta(0)} \frac{\sum_{i=1}^d y_i^2\xi_i^2}{\sigma(|\mathbf{y}|)}\mathbf{y}\otimes\mathbf{y}\mathrm{d}\mathbf{y} + \frac{c_\delta}{2}\int_{B_\delta(0)} \frac{\sum_{i<j} y_iy_j\xi_i\xi_j}{\sigma(|\mathbf{y}|)}\mathbf{y}\otimes\mathbf{y}\mathrm{d}\mathbf{y} + \alpha_\delta(\boldsymbol{\xi}) \\
&= \frac{c_\delta}{2}\int_{B_\delta(0)} \frac{1}{\sigma(|\mathbf{y}|)}\sum_{i=1}^d y_i^2\xi_i^2 \mathrm{diag}(y_i^2)\mathrm{d}\mathbf{y} \\
&\quad + \frac{c_\delta}{2}\int_{B_\delta(0)} \frac{1}{\sigma(|\mathbf{y}|)}[(\mathbf{y}^2\boldsymbol{\xi})\otimes(\mathbf{y}^2\boldsymbol{\xi}) - \mathrm{diag}(y_i^4\xi_i^2)]\mathrm{d}\mathbf{y} + \alpha_\delta(\boldsymbol{\xi}) \\
&= \left[\frac{c_\delta}{2}\int_{B_\delta(0)} \frac{y_1^2y_d^2}{\sigma(|\mathbf{y}|)}|\boldsymbol{\xi}|^2\mathrm{d}\mathbf{y}\right]\mathbf{I} + \frac{c_\delta}{2}\int_{B_\delta(0)} \frac{y_1^2y_d^2}{\sigma(|\mathbf{y}|)}\boldsymbol{\xi}\otimes\boldsymbol{\xi}\mathrm{d}\mathbf{y} + \alpha_\delta(\boldsymbol{\xi}) \\
&= \frac{c_\delta}{2}\int_{B_\delta(0)} \frac{y_1^2y_d^2}{\sigma(|\mathbf{y}|)}\mathrm{d}\mathbf{y}[|\boldsymbol{\xi}|^2\mathbf{I} + 2\boldsymbol{\xi}\otimes\boldsymbol{\xi}] + \alpha_\delta(\boldsymbol{\xi})
\end{aligned}
$$

where $\mathbf{y}^2\boldsymbol{\xi}$ means component-wise multiplication which remains a vector and

$$\alpha_\delta(\boldsymbol{\xi}) = -\frac{c_\delta}{24}\int_{B_\delta(0)} \frac{(\boldsymbol{\xi}\cdot\mathbf{y})^4\cos(\theta)}{\sigma(|\mathbf{y}|)}\mathbf{y}\otimes\mathbf{y}\mathrm{d}\mathbf{y}$$

for some θ. We can see that for any $\boldsymbol{\xi}$,

$$|\alpha_\delta(\boldsymbol{\xi})| \leqslant \frac{\tau_\delta}{24}\delta^2|\boldsymbol{\xi}|^4\mathbf{I}.$$

Hence, by the assumption on τ_δ and the identity (2.33), we have

$$\mathbf{M}_\delta(\boldsymbol{\xi}) \to \mu[|\boldsymbol{\xi}|^2\mathbf{I} + 2\boldsymbol{\xi}\otimes\boldsymbol{\xi}] = \mathbf{M}_0(\boldsymbol{\xi}), \quad \forall\boldsymbol{\xi}\in\mathbb{R}^d$$

as $\delta\to 0$ where

$$\mu = \lim_{\delta\to 0}\frac{c_\delta}{2}\int_{B_\delta(0)} \frac{y_1^2y_d^2}{\sigma(|\mathbf{y}|)}\mathrm{d}\mathbf{y} = \lim_{\delta\to 0}\frac{\tau_\delta}{2d(d+2)}\cdot \tag{2.34}$$

We may then use the dominant convergence theorem to get $\|\mathbf{u}_\delta - \mathbf{u}_0\|^2_{\mathcal{M}^2_\sigma(\mathbb{R}^d)} \to 0$ as $\delta\to 0$. □

Remark 2.20. For any $\boldsymbol{\xi}\in\mathbb{R}^d$, we use $\lambda_{\delta,1}$ and $\lambda_{0,1}$ to denote the largest eigenvalues of $\mathbf{M}_\delta(\boldsymbol{\xi})$ and $\mathbf{M}_0(\boldsymbol{\xi})$ respectively. We also let $\{\lambda_{\delta,i}\}_{i=2}^d$ and $\{\lambda_{0,i}\}_{i=2}^d$ be the remaining eigenvalues of $\mathbf{M}_\delta(\boldsymbol{\xi})$ and $\mathbf{M}_0(\boldsymbol{\xi})$. By Lemma 2.10, $\mathbf{M}_\delta$ and $\mathbf{M}_0$ has the same set of eigenvectors. Direct computation shows that $\lambda_{0,1} = 3\mu|\boldsymbol{\xi}|^2$ and $\lambda_{0,i} = \mu|\boldsymbol{\xi}|^2$ for $i\geq 2$. From the proof of the above theorem, we can see that $\lambda_{\delta,i}\to\lambda_{0,i}$ as $\delta\to 0$ for all i. Moreover, by the Lemmas 2.9 and 2.10, we can get $0<\lambda_{\delta,1}\leq 3\tau_\delta|\boldsymbol{\xi}|^2/2$ while $0<\lambda_{\delta,i}\leq\tau_\delta|\boldsymbol{\xi}|^2/2$ for $2\leq i\leq d$.

It is easy to see that the convergence stated in the above theorem holds in $L^2(\mathbb{R}^d)$, based on the uniform bounds in δ on the respective norms established earlier. Moreover, the condition (2.32) is a natural assumption that leads to a consistent definition of the elastic constant of the material under consideration [10,20]. In addition, we note that while the Navier equation corresponding to PD model for the spring network system under consideration here only takes on a Poisson ratio of $\nu = 0.25$ (since $\mu = \lambda$), it is possible to recover limiting equations with other Poisson ratios from the more general PD models with the peridynamic states [20].

Unlike discussions in the literature [1,10], the above convergence theorem does not require extra assumptions on the regularities of the solutions to the PD model beyond those already established in this work and remains valid for weak solutions. Moreover, using the properties stated in the Corollary 2.8, one may establish a general convergence in distribution sense.

Now, if additional assumptions on the kernel functions are made, then one may get convergence in conventional Sobolev space as well using the space equivalence established earlier.

Proposition 2.21. *If the kernel function $\sigma = \sigma(|\mathbf{y}|)$ satisfies (2.20) with τ_δ satisfying (2.32), then for $\mathbf{b} \in H^{-\alpha}(\mathbb{R}^d)$, we have*

$$\|\mathbf{u}_0 - \mathbf{u}_\delta\|_{H^\alpha(\mathbb{R}^d)} \leqslant C_1 \|\mathbf{u}_0 - \mathbf{u}_\delta\|_{\mathcal{M}_\sigma(\mathbb{R}^d)} \to 0,$$

for some constant C_1, independent of δ and $\mathbf{b}$.

Proof. We only mention that when the conditions in the theorem are satisfied,

$$\|\mathbf{b}\|_{\mathcal{M}_\sigma^{-1}(\mathbb{R}^d)} \leqslant C_0 \|\mathbf{b}\|_{H^{-\alpha}(\mathbb{R}^d)}$$

where C_0 is independent of δ. The other part of the proof is similar to the previous theorem. □

To conclude the discussion here, let us note that if additional regularity on the function $\mathbf{b}$ can be assumed together with the order of convergence of τ_δ to $d(d+2)\mu$, then one may also establish the order of convergence of $\mathbf{u}_0 - \mathbf{u}_\delta$ in suitable norms using techniques similar to that in the proof of the Theorem 2.19. One can also examine further the higher order expansions of the PD operator in δ, see for instance the formal derivations given in [23] for smoothly defined solutions.

2.3. The time-dependent PD model

With the suitable function spaces for the PD operator and the stationary peridynamic model given earlier, we now proceed to discuss the existence and uniqueness of the solutions of the time-dependent PD model (1.2) in these spaces, again for more general kernel functions $\sigma = \sigma(|\mathbf{y}|)$.

Using the Fourier transform, we first rewrite the PD equation (1.2) as

$$\begin{cases} \hat{\mathbf{u}}_{tt}(t,\boldsymbol{\xi}) + \mathbf{M}_\delta(\boldsymbol{\xi})\hat{\mathbf{u}}(t,\boldsymbol{\xi}) = \hat{\mathbf{b}}(t,\boldsymbol{\xi}), \\ \hat{\mathbf{u}}(0,\boldsymbol{\xi}) = \hat{\mathbf{g}}(\boldsymbol{\xi}), \\ \hat{\mathbf{u}}_t(0,\boldsymbol{\xi}) = \hat{\mathbf{h}}(\boldsymbol{\xi}). \end{cases} \tag{2.35}$$

By Duhamel's principle, we formally have

$$\hat{\mathbf{u}}(t,\boldsymbol{\xi}) = \cos\big(\sqrt{\mathbf{M}_\delta(\boldsymbol{\xi})}t\big)\hat{\mathbf{g}}(\boldsymbol{\xi}) + \frac{\sin\big(\sqrt{\mathbf{M}_\delta(\boldsymbol{\xi})}t\big)}{\sqrt{\mathbf{M}_\delta(\boldsymbol{\xi})}}\hat{\mathbf{h}}(\boldsymbol{\xi}) + \frac{1}{\sqrt{\mathbf{M}_\delta(\boldsymbol{\xi})}}\int_0^t \sin\big(\sqrt{\mathbf{M}_\delta(\boldsymbol{\xi})}s\big)\hat{\mathbf{b}}(t-s,\boldsymbol{\xi})\mathrm{d}s. \tag{2.36}$$

Then by taking the inverse Fourier transform, we can get

$$\mathbf{u}(t,\mathbf{x}) = \int_{\mathbb{R}^d} \frac{\mathrm{d}}{\mathrm{d}t}G(t,\mathbf{y})\mathbf{g}(\mathbf{x}-\mathbf{y})\mathrm{d}\mathbf{y} + \int_{\mathbb{R}^d} G(t,\mathbf{y})\mathbf{h}(\mathbf{x}-\mathbf{y})\mathrm{d}y + \int_0^t\int_{\mathbb{R}^d} G(t,\mathbf{y})\mathbf{b}(t-s,\mathbf{x}-\mathbf{y})\mathrm{d}y\mathrm{d}s, \tag{2.37}$$

where $G(t,\mathbf{y}) = F^{-1}\left(\frac{\sin\big(\sqrt{\mathbf{M}_\delta(\boldsymbol{\xi})}t\big)}{\sqrt{\mathbf{M}_\delta(\boldsymbol{\xi})}}\right)$, see also [23].

From equation (2.36), we can see:

Theorem 2.22. *If the kernel function $\sigma = \sigma(|\mathbf{y}|)$ satisfies (2.1), and*

$$\mathbf{g} \in \mathcal{M}_\sigma(\mathbb{R}^d), \quad \mathbf{h} \in L^2(\mathbb{R}^d), \quad \mathbf{b} \in L^2(0,T;L^2(\mathbb{R}^d)), \tag{2.38}$$

for some $T > 0$, then the PD equation (1.2) *has a unique solution $\mathbf{u} = \mathbf{u}(t, \mathbf{x})$ given by* (2.37)*. Moreover,*

$$\mathbf{u} \in C([0,T]; \mathcal{M}_\sigma(\mathbb{R}^d)), \quad \mathbf{u}_t \in L^2(0,T; L^2(\mathbb{R}^d)). \tag{2.39}$$

Proof. From (2.36) and (2.37), we can see the solution $\mathbf{u}$ can be expressed by the given quantities $\mathbf{b}$, $\mathbf{g}$ and $\mathbf{h}$, *i.e.* the solution of the PD equation (1.2) uniquely exists, so it is suffice to give the proper space that the solution belongs to.

First, we note that since $\mathbf{M}_\delta(\boldsymbol{\xi})$ is real symmetric and positive definite, it can be diagonalized by an orthogonal matrix $\mathbf{Q} = \mathbf{Q}(\boldsymbol{\xi})$, *i.e.* $\mathbf{M}_\delta(\boldsymbol{\xi}) = \mathbf{Q}^T \mathrm{diag}(\lambda_{\delta,i})\mathbf{Q}$. And we denote the three terms on the right hand side of (2.36) as $\hat{\mathbf{u}}_1(t,\boldsymbol{\xi})$, $\hat{\mathbf{u}}_2(t,\boldsymbol{\xi})$ and $\hat{\mathbf{u}}_3(t,\boldsymbol{\xi})$ respectively so that

$$\hat{\mathbf{u}}(t,\boldsymbol{\xi}) = \hat{\mathbf{u}}_1(t,\boldsymbol{\xi}) + \hat{\mathbf{u}}_2(t,\boldsymbol{\xi}) + \hat{\mathbf{u}}_3(t,\boldsymbol{\xi}).$$

By the condition (2.38), we readily have

$$\begin{aligned}
\|\mathbf{u}_1(t,\boldsymbol{x})\|^2_{\mathcal{M}_\sigma} &= \int_{\mathbb{R}^d} \hat{\mathbf{g}}(\boldsymbol{\xi}) \cdot \cos(\sqrt{\mathbf{M}_\delta(\boldsymbol{\xi})}t)(\mathbf{I} + \mathbf{M}_\delta(\boldsymbol{\xi}))\cos(\sqrt{\mathbf{M}_\delta(\boldsymbol{\xi})}t)\hat{\mathbf{g}}(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi} \\
&= \int_{\mathbb{R}^d} \hat{\mathbf{g}}(\boldsymbol{\xi}) \cdot \mathbf{Q}^T \mathrm{diag}(\cos(\sqrt{\lambda_{\delta,i}}t)\mathbf{Q})(\mathbf{I} + \mathbf{M}_\delta(\boldsymbol{\xi}))\mathbf{Q}^T \mathrm{diag}(\cos(\sqrt{\lambda_{\delta,i}}t))\mathbf{Q}\hat{\mathbf{g}}(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi} \\
&= \int_{\mathbb{R}^d} \hat{\mathbf{g}}(\boldsymbol{\xi}) \cdot \mathbf{Q}^T \mathrm{diag}(\cos(\sqrt{\lambda_{\delta,i}}t)\mathrm{diag})(1 + \lambda_{\delta,i})\mathrm{diag}(\cos(\sqrt{\lambda_{\delta,i}}t))\mathbf{Q}\hat{\mathbf{g}}(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi} \\
&= \int_{\mathbb{R}^d} \hat{\mathbf{g}}(\boldsymbol{\xi}) \cdot \mathbf{Q}^T \mathrm{diag}((1 + \lambda_{\delta,i})\cos^2(\sqrt{\lambda_{\delta,i}}t))\mathbf{Q}\hat{\mathbf{g}}(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi} \\
&\leqslant \int_{\mathbb{R}^d} \hat{\mathbf{g}}(\boldsymbol{\xi}) \cdot \mathbf{Q}^T \mathrm{diag}(1 + \lambda_{\delta,i})\mathbf{Q}\hat{\mathbf{g}}(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi} = \int_{\mathbb{R}^d} \hat{\mathbf{g}}(\boldsymbol{\xi}) \cdot (\mathbf{I} + \mathbf{M}_\delta(\boldsymbol{\xi}))\hat{\mathbf{g}}(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi} = \|\mathbf{g}\|^2_{\mathcal{M}_\sigma}.
\end{aligned}$$

So we can see that $\mathbf{u}_1 = \mathbf{u}_1(t,\mathbf{x})$ is uniformly bounded in $C([0,T]; \mathcal{M}_\sigma(\mathbb{R}^d))$. Similarly, we can also deduce that

$$\|\mathbf{u}_2(t,\mathbf{x})\|^2_{\mathcal{M}_\sigma} \le (1+T^2)\|\mathbf{h}\|^2_{L^2(\mathbb{R}^d)} \quad \text{and} \quad \|\mathbf{u}_3(t,\mathbf{x})\|^2_{\mathcal{M}_\sigma} \le (1+T^2)T\|\mathbf{b}\|^2_{L^2(0,T;L^2(\mathbb{R}^d))}$$

uniformly in $[0,T]$. Therefore, it implies that $\mathbf{u} = \mathbf{u}(t,\mathbf{x})$ is bounded uniformly in $\mathcal{M}_\sigma(\mathbb{R}^d)$ for any $t \in [0,T]$, *i.e.* $\mathbf{u} \in C([0,T]; \mathcal{M}_\sigma(\mathbb{R}^d))$.

Differentiating (2.36) with respect to t, we get

$$\hat{\mathbf{u}}_t(t,\boldsymbol{\xi}) = -\sqrt{\mathbf{M}_\delta(\boldsymbol{\xi})}\sin(\sqrt{\mathbf{M}_\delta(\boldsymbol{\xi})}t)\hat{\mathbf{g}}(\boldsymbol{\xi}) + \cos(\sqrt{\mathbf{M}_\delta(\boldsymbol{\xi})}t)\hat{\mathbf{h}}(\boldsymbol{\xi}) + \int_0^t \cos(\sqrt{\mathbf{M}_\delta(\boldsymbol{\xi})}(t-s))\hat{\mathbf{b}}(s,\boldsymbol{\xi})\mathrm{d}s. \tag{2.40}$$

Then through a similar calculation as in the above, we can get that $\mathbf{u}_t$ is uniformly bounded in $L^2(0,T; L^2(\mathbb{R}^d))$.

These *a priori* estimates, together with standard PDE theory [14,17], lead to the existence and uniqueness of the solution $\mathbf{u}$ of (1.2) in $C([0,T]; \mathcal{M}_\sigma(\mathbb{R}^d)) \cap H^1(0,T; L^2(\mathbb{R}^d))$. □

Note that for the linear time-dependent equation, we can easily get the following lemma.

Lemma 2.23. *Let P be a time-independent linear operator which commutes with L_δ, then it commutes with the solution operator of system* (1.2)*.*

Similar to the stationary case, this again allows us to establish the well-posedness of even more generalized solutions to (1.2).

Theorem 2.24. *Let the kernel function $\sigma = \sigma(|\mathbf{y}|)$ satisfy (2.1), and P be a time-independent linear operator which commutes with L_δ. Then for the initial conditions and the forcing term satisfying*

$$P\mathbf{g} \in \mathcal{M}_\sigma(\mathbb{R}^d), \quad P\mathbf{h} \in L^2(\mathbb{R}^d), \quad P\mathbf{b} \in L^2(0,T;L^2(\mathbb{R}^d)),$$

the PD equation (1.2) has a unique solution $\mathbf{u} = \mathbf{u}(t,\mathbf{x})$ with

$$P\mathbf{u} \in C([0,T];\mathcal{M}_\sigma(\mathbb{R}^d)) \cap H^1(0,T;L^2(\mathbb{R}^d)).$$

In particular, we can take $P = (-L_\delta + I)^{-1/2}$, then we get the existence and uniqueness of weak solution $\mathbf{u} = \mathbf{u}(t,\mathbf{x})$ to (1.2) with

$$\mathbf{u} \in C([0,T];L^2(\mathbb{R}^d)) \cap H^1(0,T;\mathcal{M}_\sigma^{-1}(\mathbb{R}^d))$$

for $\mathbf{g} \in L^2(\mathbb{R}^d)$, $\mathbf{h} \in \mathcal{M}_\sigma^{-1}(\mathbb{R}^d)$ and $\mathbf{b} \in L^2(0,T;\mathcal{M}_\sigma^{-1}(\mathbb{R}^d))$.

The proof is straightforward by verifying that $P\mathbf{u}$ is also the solution of the PD equation with the transformed data. Note that the theorem also implies the well-posedness of the Cauchy problem for the time-dependent PD equation even when the initial displacement is given as a distribution.

Similar to the discussion given in the previous section, we may establish the convergence property of the solution of PD equation (1.2) to the one of the Navier equation in the limit when $\delta \to 0$:

$$\begin{cases} \mathbf{u}_{tt}(t,\mathbf{x}) = \mu\triangle\mathbf{u}(t,\mathbf{x}) + (\mu+\lambda)\nabla\mathrm{div}\,\mathbf{u}(t,\mathbf{x}) + \mathbf{b}(t,\mathbf{x}), \\ \mathbf{u}(0,\mathbf{x}) = \mathbf{g}(\mathbf{x}), \quad \text{in } \mathbb{R}^d, \\ \mathbf{u}_t(0,\mathbf{x}) = \mathbf{h}(\mathbf{x}), \quad \text{in } \mathbb{R}^d, \end{cases} \tag{2.41}$$

where $\mu = \lambda$, as defined in (2.34).

Theorem 2.25. *Let the kernel function $\sigma = \sigma(|\mathbf{y}|)$ satisfy (2.1), then the solution of (1.2) converges, in the conventional norms of $L^2(0,T;\mathcal{M}_\sigma(\mathbb{R}^d)) \cap H^1(0,T;L^2(\mathbb{R}^d))$, to the solution of (2.41) as $\delta \to 0$, provided that (2.32) is satisfied and*

$$\mathbf{g} \in H^1(\mathbb{R}^d), \quad \mathbf{h} \in L^2(\mathbb{R}^d), \quad \mathbf{b} \in L^2(0,T;L^2(\mathbb{R}^d)). \tag{2.42}$$

Proof. By Duhamel's principle, the solution of (2.41) can be expressed as

$$\hat{\mathbf{u}}_0(t,\boldsymbol{\xi}) = \cos(\sqrt{\mathbf{M}_0(\boldsymbol{\xi})}t)\hat{\mathbf{g}}(\boldsymbol{\xi}) + \frac{\sin(\sqrt{\mathbf{M}_0(\boldsymbol{\xi})}t)}{\sqrt{\mathbf{M}_0(\boldsymbol{\xi})}}\hat{\mathbf{h}}(\boldsymbol{\xi}) + \frac{1}{\sqrt{\mathbf{M}_0(\boldsymbol{\xi})}}\int_0^t \sin(\sqrt{\mathbf{M}_0(\boldsymbol{\xi})}s)\hat{\mathbf{b}}(t-s,\boldsymbol{\xi})\mathrm{d}s.$$

Using similar expression of $\mathbf{u}_\delta$, we get

$$\begin{aligned} \hat{\mathbf{u}}_\delta(t,\boldsymbol{\xi}) - \hat{\mathbf{u}}_0(t,\boldsymbol{\xi}) &= (\cos(\sqrt{\mathbf{M}_\delta(\boldsymbol{\xi})}t) - \cos(\sqrt{\mathbf{M}_0(\boldsymbol{\xi})}t))\hat{\mathbf{g}}(\boldsymbol{\xi}) + \left(\frac{\sin(\sqrt{\mathbf{M}_\delta(\boldsymbol{\xi})}t)}{\sqrt{\mathbf{M}_\delta(\boldsymbol{\xi})}} - \frac{\sin(\sqrt{\mathbf{M}_0(\boldsymbol{\xi})}t)}{\sqrt{\mathbf{M}_0(\boldsymbol{\xi})}}\right)\hat{\mathbf{h}}(\boldsymbol{\xi}) \\ &+ \int_0^t \left(\frac{\sin(\sqrt{\mathbf{M}_\delta(\boldsymbol{\xi})}s)}{\sqrt{\mathbf{M}_\delta(\boldsymbol{\xi})}} - \frac{\sin(\sqrt{\mathbf{M}_0(\boldsymbol{\xi})}s)}{\sqrt{\mathbf{M}_0(\boldsymbol{\xi})}}\right)\hat{\mathbf{b}}(t-s,\boldsymbol{\xi})\mathrm{d}s = \hat{\mathbf{u}}_{c1} + \hat{\mathbf{u}}_{c2} + \hat{\mathbf{u}}_{c3}. \end{aligned}$$

For any $\boldsymbol{\xi} \in \mathbb{R}^d$, let $\mathbf{Q} = \mathbf{Q}(\boldsymbol{\xi})$ denote an orthogonal matrix that diagonalizes the matrix $\mathbf{M}_0(\boldsymbol{\xi})$ such that

$$\mathbf{M}_0(\boldsymbol{\xi}) = \mathbf{Q}^T\mathrm{diag}(\lambda_{0,i})\mathbf{Q}.$$

By Lemma 2.10, $\mathbf{Q}$ also diagonalizes $\mathbf{M}_\delta(\boldsymbol{\xi})$. Then, we have

$$\hat{\mathbf{g}}(\boldsymbol{\xi}) \cdot (\cos(\sqrt{\mathbf{M}_\delta(\boldsymbol{\xi})}t) - \cos(\sqrt{\mathbf{M}_0(\boldsymbol{\xi})}t))(I + \mathbf{M}_\delta)(\cos(\sqrt{\mathbf{M}_\delta(\boldsymbol{\xi})}t) - \cos(\sqrt{\mathbf{M}_0(\boldsymbol{\xi})}t))\hat{\mathbf{g}}(\boldsymbol{\xi})$$
$$= (\mathbf{Q}\hat{\mathbf{g}}(\boldsymbol{\xi})) \cdot \mathrm{diag}((\cos(\sqrt{\lambda_{\delta,i}}t) - \cos(\sqrt{\lambda_{0,i}}t))^2(1 + \lambda_{\delta,i}))\,\mathbf{Q}\hat{\mathbf{g}}(\boldsymbol{\xi}). \quad (2.43)$$

Since $1 \le 1 + \lambda_{\delta,i} \le \max(1, \tau_\delta/(2\mu))(1 + \lambda_{0,i})$, we get

$$\|\mathbf{u}_{c1}\|^2_{\mathcal{M}_\sigma} \le \max(4, 2\tau_\delta/\mu)\|\mathbf{g}\|_1^2$$

uniformly for any $t \in [0, T]$. This implies in particular that,

$$\int_0^T \|\mathbf{u}_{c1}\|^2_{\mathcal{M}_\sigma} \mathrm{d}t \le \max(4T, 2T\tau_\delta/\mu)\|\mathbf{g}\|_1^2.$$

Similarly, under the assumptions on $\mathbf{h}$ and $\mathbf{b}$, one can prove the uniform upper bounds for $\mathbf{u}_{c2}$, $\mathbf{u}_{c3}$ and thus $\mathbf{u}_\delta - \mathbf{u}_0$ in $L^2(0, T; \mathcal{M}_\sigma(\mathbb{R}^d))$. Based on the remark after Theorem 2.19, we can further see for any $\boldsymbol{\xi} \in \mathbb{R}^d$ and $t \in (0, T)$ that, $\sqrt{\lambda_{\delta,i}}t \to \sqrt{\lambda_{0,i}}t$, which implies that the left side of (2.43) converges to zero in $(0, T) \times \mathbb{R}^d$. By the dominant convergence theorem, we get as $\delta \to 0$,

$$\int_0^T \|\mathbf{u}_{c1}\|^2_{\mathcal{M}_\sigma} \mathrm{d}t \to 0.$$

By applying similar arguments to the terms $\mathbf{u}_{c2}$ and $\mathbf{u}_{c3}$, we get

$$\int_0^T \|\mathbf{u}_\delta - \mathbf{u}_0\|^2_{\mathcal{M}_\sigma} \mathrm{d}t \to 0.$$

We thus have $\mathbf{u}_\delta \to \mathbf{u}_0$ in $L^2(0, T; \mathcal{M}_\sigma(\mathbb{R}^d))$. Following the same idea, using expression (2.40), one can prove $\mathbf{u}_{\delta,t} \to \mathbf{u}_{0,t}$ in $L^2(0, T; L^2(\mathbb{R}^d))$. Together, we get the theorem. □

Remark 2.26. Again, the results given here are generalizations of the results in [10]. Moreover, similar to the time-independent case, for kernel functions with additional properties, we can say more about the convergence. In fact, if the kernel function $\sigma = \sigma(|\mathbf{y}|)$ satisfies (2.15) and τ_δ satisfies (2.32), then for $\mathbf{g} \in H^1$, $\mathbf{h} \in L^2$ and $\mathbf{b} \in L^2(0, T; L^2(\mathbb{R}^d))$, we have that, as $\delta \to 0$, $\|\mathbf{u} - \mathbf{u}_\delta\|_{L^2(0,T;H^\alpha(\mathbb{R}^d)) \cap H^1(0,T;L^2(\mathbb{R}^d))} \to 0$.

3. Conclusion

In this work, a general functional analytical framework is provided for the mathematical and numerical analysis of the linear peridynamic models. For illustration, we focus on the case of a linear constitutive relation corresponding to the spring system in multi-dimensional space. Various analytical issues are established here under the unified framework, extending some of the results given in the literature.

The techniques developed here can be extended to study more general nonlocal models using the peridynamic state [20]. Moreover, we may also consider the corresponding functional spaces for nonlocal boundary value problems defined on a bounded domain [12]. We note that the analytical frameworks and the studies of the solution regularity properties associated with the PD models can also be useful in establishing basic convergence and error estimates of their numerical approximations such as the Galerkin finite element approximation [24]. While the Fourier based techniques similar to that developed here can still be used in the analysis of certain special nonlocal boundary value problems for the linear bond based PD models defined on box-like domains [24], other techniques need to be further developed in the future to treat more generic boundary conditions, arbitrary geometry and nonlinear models.

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