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NUMERICAL ANALYSIS OF A TRANSMISSION PROBLEM WITH SIGNORINI CONTACT USING MIXED-FEM AND BEM *

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Abstract. This paper is concerned with the dual formulation of the interface problem consisting of a linear partial differential equation with variable coefficients in some bounded Lipschitz domain Ω in $\mathbb{R}^n$ $(n \geq 2)$ and the Laplace equation with some radiation condition in the unbounded exterior domain $\Omega_c := \mathbb{R}^n \backslash \bar{\Omega}$. The two problems are coupled by transmission and Signorini contact conditions on the interface $\Gamma = \partial\Omega$. The exterior part of the interface problem is rewritten using a Neumann to Dirichlet mapping (NtD) given in terms of boundary integral operators. The resulting variational formulation becomes a variational inequality with a linear operator. Then we treat the corresponding numerical scheme and discuss an approximation of the NtD mapping with an appropriate discretization of the inverse Poincaré-Steklov operator. In particular, assuming some abstract approximation properties and a discrete inf-sup condition, we show unique solvability of the discrete scheme and obtain the corresponding *a-priori* error estimate. Next, we prove that these assumptions are satisfied with Raviart-Thomas elements and piecewise constants in Ω, and continuous piecewise linear functions on Γ. We suggest a solver based on a modified Uzawa algorithm and show convergence. Finally we present some numerical results illustrating our theory.

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1. Introduction

In this work, we study a transmission problem with an inequality on the interface. This transmission problem can be seen as a scalar model problem for unilateral contact between a linear elastic unbounded medium and a deformable body. Equivalently, it can be seen as a fluid problem with a semi-permeable membrane. Such a contact can be described by Signorini boundary conditions. The particular feature of the unilateral problems is that the mathematical variational statement leads to variational inequalities set on closed convex cones. The modeling of the non-penetration condition in the discrete setting is of crucial importance.

Keywords and phrases. Raviart-Thomas space, boundary integral operator, Lagrange multiplier.

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The mathematical foundation for variational inequalities can be found in *e.g.* [8,12,16]. The numerical analysis of variational inequalities by finite elements, can be found in, *e.g.* [4,10,13,15]. An overview on more recent developments can be found in [19]. Here we take the model problem from [6] with a linear state law in the interior and combine it with the techniques from *e.g.* [11] to treat the problem in dual form, which is then reduced to a bounded domain by the use of the symmetric boundary integral representation of the Neumann to Dirichlet map and then is rewritten in a saddle point form to impose weakly the two restrictions for the interior field.

Let $\Omega \subset \mathbb{R}^n$, $n \geq 2$ be a bounded domain with Lipschitz boundary Γ. In order to describe mixed boundary conditions, we let $\Gamma = \overline{\Gamma_t \cup \Gamma_s}$ where Γ_s and Γ_t are nonempty, disjoint, and open in Γ. It is not necessary that Γ_t and Γ_s have a positive distance. Also, we let $\mathbf{n}$ denote the unit normal on Γ defined almost everywhere pointing from Ω into $\Omega_c := \mathbb{R}^n \backslash \bar{\Omega}$. Then, given $f \in L^2(\Omega)$ and a matrix-valued function $\boldsymbol{\kappa} \in [C(\bar{\Omega})]^{n\times n}$, $\boldsymbol{\kappa} = \boldsymbol{\kappa}^t$, we consider the linear partial differential equations

$$\operatorname{div}(\boldsymbol{\kappa}\nabla u) + f = 0 \quad \text{in } \Omega \quad \text{and} \quad \Delta u = 0 \quad \text{in } \Omega_c, \tag{1.1}$$

with the radiation condition as $|x| \to \infty$

$$\begin{aligned} u(x) &= o(1) && \text{if } n = 2, \\ u(x) &= \mathcal{O}(|x|^{2-n}) && \text{if } n \geq 3. \end{aligned} \tag{1.2}$$

We assume here that $\boldsymbol{\kappa}$ induces a strongly elliptic differential operator, that is there exists $\alpha > 0$ such that

$$\alpha\|\zeta\|^2 \leq (\boldsymbol{\kappa}(x)\zeta)\cdot\zeta \quad \forall x \in \Omega, \quad \forall \zeta \in \mathbb{R}^n. \tag{1.3}$$

Writing $u_1 := u$ in Ω and $u_2 := u$ in Ω_c, the tractions on Γ are given by the traces $(\boldsymbol{\kappa}\nabla u_1)\cdot\mathbf{n}$ and $-\nabla u_2 \cdot \mathbf{n}$ (note that $\mathbf{n}$ points into Ω_c). Next, given $u_0 \in H^{1/2}(\Gamma)$ and $t_0 \in H^{-1/2}(\Gamma)$, we consider transmission conditions

$$u_1 = u_2 + u_0 \quad \text{and} \quad (\boldsymbol{\kappa}\nabla u_1)\cdot\mathbf{n} = \nabla u_2\cdot\mathbf{n} + t_0 \quad \text{on } \Gamma_t, \tag{1.4}$$

and Signorini conditions

$$u_1 \leq u_2 + u_0, \quad (\boldsymbol{\kappa}\nabla u_1)\cdot\mathbf{n} = \nabla u_2\cdot\mathbf{n} + t_0 \leq 0 \quad \text{and} \quad 0 = (\boldsymbol{\kappa}\nabla u_1)\cdot\mathbf{n}\,(u_2 + u_0 - u_1) \quad \text{on } \Gamma_s \tag{1.5}$$

and assume that for $n = 2$ holds

$$\int_\Omega f(x)\,\mathrm{d}x + \int_\Gamma t_0\,\mathrm{d}x = 0. \tag{1.6}$$

In this way, we look for $u_1 \in H^1(\Omega)$ and $u_2 \in W^1(\Omega_c)$ satisfying (1.1)–(1.6) in a weak form. The Sobolev space $W^1(\Omega_c)$ is defined at the end of this section.

The purpose of this work is to study a variational formulation, based on the dual-mixed finite element method (dual-mixed FEM) and the boundary element method (BEM), for the above boundary value problem. This includes solvability of the continuous and discrete schemes, and also the associated numerical analysis yielding error estimate and rate of convergence. To this end, (1.1)–(1.6) is written as a saddle point problem given on a convex subset, where the original inequality stemming from the Signorini condition is transferred to a Lagrange multiplier.

The main advantage of using a dual-mixed method lies on the possibility of introducing further unknowns with a clear physical meaning. These unknowns are then approximated directly, which avoids any numerical postprocessing yielding additional sources of error. Also, in this case u becomes an unknown in $L^2(\Omega)$, which gives more flexibility to choose the associated finite element subspace. In particular, piecewise constant functions become a feasible choice. On the other hand, the transmission conditions of Dirichlet type, being natural in this setting, are incorporated directly into the continuous and discrete formulations, thus avoiding nonconforming Galerkin schemes.

The rest of the paper is organized as follows. In Section 2 we give the dual mixed formulation corresponding to the primal formulation on which the analysis in [6] was based and show its equivalence to the primal formulation. We apply the boundary integral equation method to rewrite the exterior problem, which leads to a dual FEM-BEM minimization problem. Further analysis leads to an equivalent dual mixed-FEM and BEM coupled formulation, which is the saddle point problem mentioned before. We show the uniqueness of its solution by proving a continuous inf-sup condition. Then, in Section 4 we deal with the numerical analysis of the discrete scheme and discuss the problems resulting from an additional approximation of the inverse Poincaré-Steklov operator, which is essential for its numerical computability. Assuming some abstract approximation properties and a discrete inf-sup condition we prove existence and uniqueness of solution for the discrete saddle point problem and show an *a-priori* estimate. Next, in Section 5 we choose Raviart-Thomas elements of order zero and piecewise constants in the domain, and continuous piecewise linear functions on the boundary, and show that the discrete inf-sup condition is satisfied for this choice of subspaces. A modified Uzawa solver and the convergence of our method is presented in Section 6. Finally, in Section 7 we present some numerical results corroborating our theory. Namely, in the Examples 7.1 and 7.2 we show that the Uzawa algorithms works well with a bounded number of iterations. Example 7.1 also demonstrates that our dual-mixed coupling method converges as predicted by Theorem 5.1, *i.e.* our numerical experiments show that the failure in obtaining optimal convergence is inherent to the scheme.

In what follows, $H^s(\mathbb{R}^n)$, $H^s(\Omega)$, $H(\mathrm{div};\Omega)$, $H^s(\Gamma)$ denote the usual Sobolev spaces (see, *e.g.* [14,17]). In particular,

$$H(\mathrm{div};\Omega) \;=\; \big\{\mathbf{q}\in[L^2(\Omega)]^n \,:\, \mathrm{div}\,\mathbf{q}\in L^2(\Omega)\big\},$$

$$H(\mathrm{div};\Omega_c) \;=\; \big\{\mathbf{q}\in[L^2(\Omega_c)]^n \,:\, \mathrm{div}\,\mathbf{q}\in L^2(\Omega_c)\big\},$$

$$H^s(\Gamma)=\big\{u|_\Gamma \,:\, u\in H^{s+1/2}(\mathbb{R}^n)\big\}\qquad (s>0),$$

$$H^0(\Gamma)\;=\;L^2(\Gamma),\quad \text{and}\quad H^s(\Gamma)\;=\;(H^{-s}(\Gamma))^*\qquad (s<0),$$

where $(H^{-s}(\Gamma))^*$ is the dual of $H^{-s}(\Gamma)$. In addition, $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$ with respect to the $L^2(\Gamma)$-inner product. Similarly, we let $\tilde{H}^{1/2}(\Gamma_s)$ be the subspace of functions in $H^{1/2}(\Gamma_s)$ whose extensions by zero to the rest of Γ belong to $H^{1/2}(\Gamma)$, and denote by $H^{-1/2}(\Gamma_s)$ its dual space. Then, $\langle\cdot,\cdot\rangle_{\Gamma_s}$ stands for the corresponding duality pairing with respect to the $L^2(\Gamma_s)$-inner product, and given $\mu\in H^{-1/2}(\Gamma_s)$, we shall write $\mu\le 0$ on Γ_s if $\langle\mu,\lambda\rangle_{\Gamma_s}\le 0$ for all $\lambda\in\tilde{H}^{1/2}(\Gamma_s)$ such that $\lambda\ge 0$. Note, that $\tilde{H}^{1/2}(\Gamma_s)$ coincides with the space $H^{1/2}_{00}(\Gamma_s)$ in [17].

The variational formulation on the unbounded domain Ω_c is based on the Beppo-Levi space $W^1(\Omega_c)$ (see [7]) and takes into account the behaviour at infinity (1.2)

$$W^1(\Omega_c)=\left\{v\;:\;\frac{v}{\sqrt{1+|x|^2}}\in L^2(\Omega_c),\nabla v\in[L^2(\Omega_c)]^n\right\},\quad n\ge 3,$$

$$W^1(\Omega_c)=\left\{v\;:\;\frac{v}{\sqrt{1+|x|^2}\log(2+|x|^2)}\in L^2(\Omega_c),\nabla v\in[L^2(\Omega_c)]^n\right\},\quad n=2.$$

The fundamental difference with the three-dimensional case is that all constant-functions belong to $W^1(\Omega_c)$, to allow the behaviour at infinity (1.2).

2. The dual mixed variational formulation

Now, we introduce the dual mixed variational formulation. We introduce the following product spaces

$$
\begin{aligned}
v \in W^1(\Omega \cup \Omega_c) &\cong H^1(\Omega) \times W^1(\Omega_c) \ni (v_1, v_2)\\
\mathbf{q} \in H(\mathrm{div}; \Omega \cup \Omega_c) &\cong H(\mathrm{div}; \Omega) \times H(\mathrm{div}; \Omega_c) \ni (\mathbf{q}_1, \mathbf{q}_2)\\
\mathbf{q} \in [L^2(\Omega \cup \Omega_c)]^n &\cong [L^2(\Omega)]^n \times [L^2(\Omega_c)]^n \ni (\mathbf{q}_1, \mathbf{q}_2).
\end{aligned}
$$

We also introduce jumps

$$[v] = v_1 - v_2, \quad [\mathbf{q}\cdot\mathbf{n}] = \mathbf{q}_1\cdot\mathbf{n} - \mathbf{q}_2\cdot\mathbf{n}$$

and two extensions of coefficients and data

$$\underline{\boldsymbol{\kappa}}(x) = \begin{cases} \boldsymbol{\kappa}(x), & x \in \Omega,\\ I, & \text{elsewhere}, \end{cases} \qquad \underline{f}(x) = \begin{cases} f(x), & x \in \Omega,\\ 0, & \text{elsewhere}. \end{cases}$$

For all $\mathbf{q} \in H(\mathrm{div}; \Omega \cup \Omega_c)$ we define the functional

$$\tilde{\Phi}(\mathbf{q}) := \frac{1}{2}\int_{\Omega\cup\Omega_c} (\underline{\boldsymbol{\kappa}}^{-1}\mathbf{q})\cdot\mathbf{q}\,\mathrm{d}x \;-\; \langle \mathbf{q}_1\cdot\mathbf{n}, u_0\rangle \tag{2.1}$$

and the subset of admissible functions

$$\tilde{C} := \Big\{ \mathbf{q} \in H(\mathrm{div}; \Omega \cup \Omega_c) \,:\, [\mathbf{q}\cdot\mathbf{n}] \;=\; t_0 \text{ on } \Gamma,\; \mathbf{q}_1\cdot\mathbf{n} \le 0 \text{ on } \Gamma_s,\; -\mathrm{div}\,\mathbf{q} \;=\; \underline{f} \text{ in } \Omega\cup\Omega_c \Big\}.$$

Note that $\tilde{\Phi}$ is convex and coercive. Then we consider the dual mixed problem (P) given by: Find $\mathbf{q}^0 \in \tilde{C}$ such that

$$\tilde{\Phi}(\mathbf{q}^0) = \min_{\mathbf{q}\in\tilde{C}} \tilde{\Phi}(\mathbf{q}). \tag{2.2}$$

Theorem 2.1. *There exists exactly one solution of* (P).

Proof. Since the set $\tilde{C}$ is convex and closed and $\tilde{\Phi}$ is convex and coercive the assertion of the theorem follows from standard arguments (*cf.* [9]). □

To the dual mixed problem (P) there corresponds the uniquely solvable (up to a constant in 2D) primal problem (*cf.* [6]): Find $u \in \mathcal{C}$ such that

$$\Phi(u) = \min_{v\in\mathcal{C}} \Phi(v) \tag{2.3}$$

where for all $v \in \mathcal{C}$

$$\Phi(v) := \frac{1}{2}\int_{\Omega\cup\Omega_c} (\underline{\boldsymbol{\kappa}}\nabla v)\cdot\nabla v\,\mathrm{d}x - \int_{\Omega\cup\Omega_c} \underline{f}v\,\mathrm{d}x - \langle t_0, v_2\rangle \tag{2.4}$$

and

$$\mathcal{C} := \Big\{ v \in W^1(\Omega\cup\Omega_c) \,:\quad [v] \;=\; u_0 \quad \text{on } \Gamma_t, \quad [v] \le u_0 \quad \text{on } \Gamma_s \Big\}.$$

The uniqueness up to a constant of (2.3) is due to the identity $\Phi(v) = \Phi(v+c)\ \forall c \in \mathbb{R}$, which follows from (1.6). We remark that in [6] the primal problem is analyzed for a non-linear differential equation, but the arguments given there are also valid for the case with variable coefficients.

In order to establish the relation between the primal and dual problems (see Thm. 2.2 below), we need the following preliminary result.

Lemma 2.1. *Let $J_2 : W^1(\Omega \cup \Omega_c) \times [L^2(\Omega \cup \Omega_c)]^n \to \mathbb{R}$ be the functional defined by*

$$J_2(v, \mathbf{q}) := \int_{\Omega\cup\Omega_c} \mathbf{q} \cdot \nabla v \, \mathrm{d}x - \int_{\Omega\cup\Omega_c} \underline{f} v \, \mathrm{d}x - \langle t_0, v_2 \rangle$$

for all $v \in W^1(\Omega \cup \Omega_c)$, $\mathbf{q} \in [L^2(\Omega \cup \Omega_c)]^n$. Then we have

$$\inf_{v\in\mathcal{C}} J_2(v, \mathbf{q}) = \begin{cases} \langle \mathbf{q}_1 \cdot \mathbf{n}, u_0 \rangle & \text{for } \mathbf{q} \in \tilde{C}, \\ -\infty & \text{for } \mathbf{q} \notin \tilde{C}. \end{cases} \tag{2.5}$$

Proof. Since $u_0 \in H^{1/2}(\Gamma)$, the extension theorem yields the existence of $U_0 \in H^1(\Omega \cup \Omega_c)$ such that $[U_0] = u_0$ on Γ. Then we define

$$\mathcal{C}_0 := \{ v \in W^1(\Omega \cup \Omega_c) : [v] = 0 \text{ on } \Gamma_t, \ [v] \le 0 \text{ on } \Gamma_s \},$$

and observe that $\tilde{C} = U_0 + \mathcal{C}_0$. In addition, given $\mathbf{q} \in [L^2(\Omega \cup \Omega_c)]^n$, $J_2(\cdot, \mathbf{q})$ is a continuous linear functional in $W^1(\Omega \cup \Omega_c)$, and hence

$$\inf_{v\in\mathcal{C}} J_2(v, \mathbf{q}) = J_2(U_0, \mathbf{q}) + \inf_{w\in\mathcal{C}_0} J_2(w, \mathbf{q}).$$

Now, it is not difficult to see that $\mathbf{q} \in \tilde{C}$ if and only if $J_2(w, \mathbf{q}) \ge 0$ for all $w \in \mathcal{C}_0$. In fact, the first implication follows straightforward from the definition of $\tilde{C}$. Conversely, let $\mathbf{q} \in [L^2(\Omega \cup \Omega_c)]^n$ such that

$$J_2(w, \mathbf{q}) = \int_{\Omega\cup\Omega_c} \mathbf{q} \cdot \nabla w \, \mathrm{d}x - \int_{\Omega\cup\Omega_c} \underline{f} w \, \mathrm{d}x - \langle t_0, w_2 \rangle \ge 0 \quad \forall w \in \mathcal{C}_0.$$

Substituting $w = \pm(\varphi_1, \varphi_2) \in C_0^\infty(\Omega) \times C_0^\infty(\Omega_c)$ in the above inequality we obtain

$$\operatorname{div} \mathbf{q} = -\underline{f} \quad \text{in } \Omega \cup \Omega_c,$$

and hence $\mathbf{q} \in H(\operatorname{div}; \Omega \cup \Omega_c)$. Further, we have

$$\begin{aligned} J_2(w, \mathbf{q}) &= \int_{\Omega\cup\Omega_c} \mathbf{q} \cdot \nabla w \, \mathrm{d}x - \int_{\Omega\cup\Omega_c} \underline{f} w \, \mathrm{d}x - \langle t_0, w_2 \rangle \\ &= (\mathbf{q}_1, \nabla w_1)_{[L^2(\Omega)]^n} + (\operatorname{div} \mathbf{q}_1, w_1)_{L^2(\Omega)} + (\mathbf{q}_2, \nabla w_2)_{[L^2(\Omega_c)]^n} - \langle t_0, w_2 \rangle \\ &= \langle \mathbf{q}_1 \cdot \mathbf{n}, w_1 \rangle - \langle \mathbf{q}_2 \cdot \mathbf{n} + t_0, w_2 \rangle, \end{aligned} \tag{2.6}$$

and then for all $w \in \mathcal{C}_0$ there holds

$$0 \le J_2(w, \mathbf{q}) = \langle \mathbf{q}_1 \cdot \mathbf{n}, w_1 \rangle - \langle \mathbf{q}_2 \cdot \mathbf{n} + t_0, w_2 \rangle.$$

It follows that $[\mathbf{q} \cdot \mathbf{n}] = t_0$ on Γ and $\mathbf{q}_1 \cdot \mathbf{n} \le 0$ on Γ_s, and therefore $\mathbf{q} \in \tilde{C}$.

Next, given $\mathbf{q} \in \tilde{C}$ we have

$$J_2(v, \mathbf{q}) = \langle \mathbf{q}_1 \cdot \mathbf{n}, v_1 \rangle - \langle \mathbf{q}_2 \cdot \mathbf{n} + t_0, v_2 \rangle = \langle \mathbf{q}_1 \cdot \mathbf{n}, v_1 - v_2 \rangle,$$

and, since $v_1 - v_2 = u_0$ on Γ_t, we get

$$\langle \mathbf{q}_1 \cdot \mathbf{n}, v_1 - v_2 \rangle_{\Gamma_t} = \langle \mathbf{q}_1 \cdot \mathbf{n}, u_0 \rangle_{\Gamma_t}.$$

Thus, using that $\mathbf{q}_1 \cdot \mathbf{n} \le 0$ on Γ_s and $v_1 - v_2 \le u_0$ on Γ_s, we deduce that the infimum will be obtained for the upper bound of $v_1 - v_2$, that is

$$\inf_{v\in\mathcal{C}} J_2(v, \mathbf{q}) = \langle \mathbf{q}_1 \cdot \mathbf{n}, u_0 \rangle.$$

On the other hand, given $\mathbf{q} \notin \tilde{C}$, the above characterization result implies that there exists $w^0 \in \mathcal{C}_0$ such that $J_2(w^0, \mathbf{q}) < 0$. Then for for all $m > 0$ we have $m\, w^0 \in \mathcal{C}_0$ and $\lim_{m\to+\infty} J_2(m\, w^0, \mathbf{q}) = -\infty$, whence the corresponding infimum is $-\infty$. □

We are now in a position to provide the following theorem.

Theorem 2.2. *Let $u \in \mathcal{C}$ and $\mathbf{q}^0 \in \tilde{C}$ be the solutions of the primal and dual problems, respectively. Then there holds*

$$\mathbf{q}^0 = (\boldsymbol{\kappa}\nabla u_1, \nabla u_2) \quad \textit{and} \quad \tilde{\Phi}(\mathbf{q}^0) + \Phi(u) = 0. \tag{2.7}$$

Proof. Let us denote $M := [L^2(\Omega \cup \Omega_c)]^n$, and let $J : \mathcal{C} \times M \times M \to \mathbb{R}$ be the functional defined by

$$J(v, \mathbf{N}, \mathbf{q}) := \frac{1}{2}\int_{\Omega\cup\Omega_c} (\underline{\boldsymbol{\kappa}}\mathbf{N}) \cdot \mathbf{N}\, \mathrm{d}x - \int_{\Omega\cup\Omega_c} \underline{f} v\, \mathrm{d}x + \int_{\Omega\cup\Omega_c} \mathbf{q} \cdot (\nabla v - \mathbf{N})\, \mathrm{d}x - \langle t_0, v_2\rangle$$

for all $v \in \mathcal{C}$, $\mathbf{N} \in M$, $\mathbf{q} \in M$.

We easily find that

$$\sup_{\mathbf{q}_1\in[L^2(\Omega)]^n} (\mathbf{q}_1, \nabla v_1 - \mathbf{N}_1)_{[L^2(\Omega)]^n} = \begin{cases} 0 & \text{for } \mathbf{N}_1 = \nabla v_1, \\ +\infty & \text{for } \mathbf{N}_1 \neq \nabla v_1, \end{cases}$$

and

$$\sup_{\mathbf{q}_2\in[L^2(\Omega_c)]^n} (\mathbf{q}_2, \nabla v_2 - \mathbf{N}_2)_{[L^2(\Omega_c)]^n} = \begin{cases} 0 & \text{for } \mathbf{N}_2 = \nabla v_2, \\ +\infty & \text{for } \mathbf{N}_2 \neq \nabla v_2, \end{cases}$$

whence (2.4) gives

$$\inf_{v\in\mathcal{C}} \Phi(v) = \inf_{[v,\mathbf{N}]\in\mathcal{C}\times M}\ \sup_{\mathbf{q}\in M} J(v, \mathbf{N}, \mathbf{q}). \tag{2.8}$$

We now denote

$$S_0(\mathbf{q}) := \inf_{[v,\mathbf{N}]\in\mathcal{C}\times M} J(v, \mathbf{N}, \mathbf{q}) \tag{2.9}$$

and observe that

$$S_0(\mathbf{q}) \le \inf_{v\in\mathcal{C}} J(v, \nabla v, \mathbf{q}) = \inf_{v\in\mathcal{C}} \Phi(v) = \Phi(u) \quad \forall \mathbf{q} \in M, \tag{2.10}$$

which yields

$$\sup_{\mathbf{q}\in M} S_0(\mathbf{q}) \le \Phi(u). \tag{2.11}$$

Furthermore, we can split

$$S_0(\mathbf{q}) = \inf_{\mathbf{N}\in M} J_1(\mathbf{N}, \mathbf{q}) + \inf_{v\in\mathcal{C}} J_2(v, \mathbf{q}) \tag{2.12}$$

with

$$J_1(\mathbf{N}, \mathbf{q}) = \frac{1}{2}\int_{\Omega\cup\Omega_c} (\underline{\boldsymbol{\kappa}}\mathbf{N}) \cdot \mathbf{N}\, \mathrm{d}x - \int_{\Omega\cup\Omega_c} \mathbf{q} \cdot \mathbf{N}\, \mathrm{d}x$$

and J_2 as defined in Lemma 2.1. Next, it is easy to see that

$$\inf_{\mathbf{N}\in M} J_1(\mathbf{N}, \mathbf{q}) = -\frac{1}{2}\int_{\Omega\cup\Omega_c} (\underline{\boldsymbol{\kappa}}^{-1}\mathbf{q}) \cdot \mathbf{q}\, \mathrm{d}x, \tag{2.13}$$

and then, using Lemma 2.1, we can write

$$S_0(\mathbf{q}) = \begin{cases} -\dfrac{1}{2}\displaystyle\int_{\Omega\cup\Omega_c} (\underline{\boldsymbol{\kappa}}^{-1}\mathbf{q}) \cdot \mathbf{q}\, \mathrm{d}x + \langle \mathbf{q}_1 \cdot \mathbf{n}, u_0\rangle = -\tilde{\Phi}(\mathbf{q}) & \text{for } \mathbf{q} \in \tilde{C} \\ -\infty & \text{for } \mathbf{q} \notin \tilde{C}. \end{cases} \tag{2.14}$$

Therefore we have

$$\Phi(u) \geq \sup_{\mathbf{q}\in M} S_0(\mathbf{q}) = \sup_{\mathbf{q}\in\tilde{C}}[-\tilde{\Phi}(\mathbf{q})] = -\inf_{\mathbf{q}\in\tilde{C}} \tilde{\Phi}(\mathbf{q}) = -\tilde{\Phi}(\mathbf{q}^0).$$

We now show that the functional $-\tilde{\Phi}$ assumes its maximum at $\hat{\mathbf{q}} := \underline{\boldsymbol{\kappa}}\nabla u$, which, according to the uniqueness of solution for the dual problem, will imply that $\underline{\boldsymbol{\kappa}}\nabla u = \mathbf{q}^0$. Indeed, since the primal problem is equivalent to the original one, there holds

$$\operatorname{div}(\underline{\boldsymbol{\kappa}}\nabla u) = -\underline{f} \text{ in } \Omega\cup\Omega_c, \quad [(\underline{\boldsymbol{\kappa}}\nabla u)\cdot\mathbf{n}] = t_0 \text{ on } \Gamma, \quad (\boldsymbol{\kappa}\nabla u_1)\cdot\mathbf{n} \leq 0 \text{ on } \Gamma_s,$$

which is, respectively,

$$\operatorname{div}\hat{\mathbf{q}} = -\underline{f} \text{ in } \Omega\cup\Omega_c, \quad [\hat{\mathbf{q}}\cdot\mathbf{n}] = t_0 \text{ on } \Gamma, \quad \hat{\mathbf{q}}_1\cdot\mathbf{n} \leq 0 \text{ on } \Gamma_s,$$

and hence $\hat{\mathbf{q}} \in \tilde{C}$.

Now, using that $0 = (\boldsymbol{\kappa}\nabla u_1)\cdot\mathbf{n}\,(u_2+u_0-u_1)$ on Γ_s and that $u_1 = u_2+u_0$ on Γ_t, we get

$$\langle(\boldsymbol{\kappa}\nabla u_1)\cdot\mathbf{n}, u_1-u_2-u_0\rangle = 0, \tag{2.15}$$

and using $\operatorname{div}(\underline{\boldsymbol{\kappa}}\nabla u) + \underline{f} = 0$ in $\Omega\cup\Omega_c$ we obtain

$$\begin{aligned}\int_{\Omega\cup\Omega_c}(\underline{\boldsymbol{\kappa}}\nabla u)\cdot\nabla u\,\mathrm{d}x - \int_{\Omega\cup\Omega_c}\underline{f}u\,\mathrm{d}x &= \langle(\boldsymbol{\kappa}\nabla u_1)\cdot\mathbf{n}, u_1\rangle - \langle\nabla u_2\cdot\mathbf{n}, u_2\rangle\\ &= \langle(\boldsymbol{\kappa}\nabla u_1)\cdot\mathbf{n}, u_2+u_0\rangle - \langle\nabla u_2\cdot\mathbf{n}, u_2\rangle\\ &= \langle[(\boldsymbol{\kappa}\nabla u)\cdot\mathbf{n}], u_2\rangle + \langle(\boldsymbol{\kappa}\nabla u_1)\cdot\mathbf{n}, u_0\rangle\\ &= \langle t_0, u_2\rangle + \langle(\boldsymbol{\kappa}\nabla u_1)\cdot\mathbf{n}, u_0\rangle.\end{aligned}$$

It follows that

$$-\int_{\Omega\cup\Omega_c}\underline{f}u\,\mathrm{d}x - \langle t_0, u_2\rangle = -\int_{\Omega\cup\Omega_c}(\boldsymbol{\kappa}\nabla u)\cdot\nabla u\,\mathrm{d}x + \langle\hat{\mathbf{q}}_1\cdot\mathbf{n}, u_0\rangle, \tag{2.16}$$

and therefore

$$\Phi(u) = \frac{1}{2}\int_{\Omega\cup\Omega_c}(\underline{\boldsymbol{\kappa}}\nabla u)\cdot\nabla u\,\mathrm{d}x - \int_{\Omega\cup\Omega_c}\underline{f}u\,\mathrm{d}x - \langle t_0, u_2\rangle = -\tilde{\Phi}(\hat{\mathbf{q}}), \tag{2.17}$$

which ends the proof. □

3. The boundary integral formulation

Now, we collect some known properties of boundary integral operators of the Laplacian and reformulate the exterior part of the original transmission problem with the fundamental solution

$$\begin{aligned}G(x,y) &= -\frac{1}{2\pi}\log|x-y| \qquad \text{if } n=2,\\ G(x,y) &= \frac{\Gamma(\frac{n-2}{2})}{4\pi^{n/2}}\,|x-y|^{2-n} \qquad \text{if } n\geq 3,\end{aligned}$$

of the Laplacian. For $z\in\Gamma$ and $\phi\in C^\infty(\Gamma)$ we define the operators of the single layer potential V, the double layer potential K, its formal adjoint K', and the hypersingular integral operator W as

$$V\phi(z) := 2\int_\Gamma \phi(x)\,G(z,x)\,\mathrm{d}s_x, \qquad K\phi(z) := 2\int_\Gamma \phi(x)\,\frac{\partial}{\partial n_x}G(z,x)\,\mathrm{d}s_x,$$

$$K'\phi(z) := 2\int_\Gamma \phi(x)\,\frac{\partial}{\partial n_z}G(z,x)\,\mathrm{d}s_x, \quad \text{and} \quad W\phi(z) := -2\frac{\partial}{\partial n_z}\int_\Gamma \phi(x)\,\frac{\partial}{\partial n_x}G(z,x)\,\mathrm{d}s_x.$$

We define the operator R by

$$R := \frac{1}{2}(V + (I+K)W^{-1}(I+K')) \,:\, H^{-1/2}(\Gamma) \to H^{1/2}(\Gamma). \tag{3.1}$$

$-R$ is a Neumann to Dirichlet (NtD) map for the exterior domain Ω_c and the inverse Poincaré-Steklov operator. Two NtD maps can differ by a constant, but the operator R itself is well defined.

The operator $W \,:\, H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma)$ is positive semidefinite, selfadjoint and has a one dimensional kernel (the constant functions). Therefore its range is not $H^{-1/2}(\Gamma)$ but $H_0^{-1/2}(\Gamma)$, which is the polar set of the space of constant functions. The range of $I + K'$ is also $H_0^{-1/2}(\Gamma)$, therefore the expression $W^{-1}(I + K')$ is well defined. Because the kernel of $I+K$ is composed of constant functions, the full expression $(I+K)W^{-1}(I+K')$ has a precise mathematical meaning. As a consequence the operator R is linear, selfadjoint and elliptic.

A common choice to treat the kernel of the operator W is the subspace of functions in $H^{1/2}(\Gamma)$ with integral mean zero. However, since this is not optimal from the implementational point of view, we add a least squares term to the hypersingular integral operator W. In other words, we define the functional $P : H^{1/2}(\Gamma) \to \mathbb{R}$, where $P(\phi) = \int_\Gamma \phi \, \mathrm{d}s$ for all $\phi \in H^{1/2}(\Gamma)$, and set

$$\tilde{W} := W + P'P \,:\, H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma). \tag{3.2}$$

The compact perturbation $P'P$ makes $\tilde{W}$ elliptic.

In this way, we can evaluate $u := Rt$ for $t \in H^{-1/2}(\Gamma)$ by computing $u = \frac{1}{2}(Vt + (I+K)\phi)$, where ϕ is the solution of

$$\tilde{W}\phi = (I + K')t. \tag{3.3}$$

Representing the solution ϕ of (3.3) like $\phi = \phi_0 + c_\phi$, such that $P\phi_0 = 0$ and $c_\phi \in \mathbb{R}$, and using that $\langle \tilde{W}\phi, 1\rangle = \langle (I+K')t, 1\rangle$, $W1 = 0$, and $K1 = -1$, we deduce that $\langle P1, P\phi\rangle = 0$, and consequently, $c_\phi = 0$ and $\phi = \phi_0 \in H^{1/2}(\Gamma)/\mathbb{R}$ is the unique solution of $W\phi = (I+K')t$. Therefore we can replace W by $\tilde{W}$ for the discretization without mentioning it explicitly.

Next, we give a reformulation $(\tilde{P})$ of the dual mixed problem (P) using the operator R (*cf.* (3.1)). Defining $\tilde{\Psi} : H(\mathrm{div};\Omega) \to \mathbb{R} \cup \{\infty\}$ by

$$\tilde{\Psi}(\mathbf{q}) := \frac{1}{2}\int_\Omega (\boldsymbol{\kappa}^{-1}\mathbf{q}) \cdot \mathbf{q} \, \mathrm{d}x + \frac{1}{2}\langle \mathbf{q}\cdot\mathbf{n}, R(\mathbf{q}\cdot\mathbf{n})\rangle - \langle \mathbf{q}\cdot\mathbf{n}, Rt_0 + u_0\rangle, \tag{3.4}$$

on the subset of admissible functions

$$\tilde{D} := \big\{\mathbf{q} \in H(\mathrm{div};\Omega) \,:\, \mathbf{q}\cdot\mathbf{n} \le 0 \text{ on } \Gamma_s, \quad -\mathrm{div}\,\mathbf{q} = f \quad \text{in } \Omega\big\},$$

the dual mixed problem $(\tilde{P})$ reads: Find $\mathbf{q}^D \in \tilde{D}$ such that

$$\tilde{\Psi}(\mathbf{q}^D) = \min_{\mathbf{q}\in\tilde{D}} \tilde{\Psi}(\mathbf{q}). \tag{3.5}$$

Theorem 3.1.

(a) *The dual problems (P) and $(\tilde{P})$ given by (2.2) and (3.5), respectively, are equivalent in the following sense:*

(i) *If $\mathbf{q}^D \in \tilde{D}$ is a solution of $(\tilde{P})$, and $\mathbf{q}_2^D$ is the gradient of the function u_2, which is given by the representation formula*

$$u_2(z) = \int_\Gamma v(x)\,\frac{\partial}{\partial n_x} G(z,x)\,\mathrm{d}s_x - \int_\Gamma \psi(x)\,G(z,x)\,\mathrm{d}s_x, \quad z \in \Omega_c, \tag{3.6}$$

with Cauchy data $(v,\psi) := (-R(\mathbf{q}^D\cdot\mathbf{n}-t_0), \mathbf{q}^D\cdot\mathbf{n}-t_0)$*, then* $(\mathbf{q}^D,\mathbf{q}_2^D)$ *minimizes the functional* $\tilde\Phi$ *in* (2.1).

(ii) *If* $(\mathbf{q}_1^0,\mathbf{q}_2^0)\in\tilde C$ *is the minimizer of* $\tilde\Phi$ *on* $\tilde C$*, then* $\mathbf{q}_1^0$ *is a solution of* $(\tilde P)$*.*

(b) $(\tilde P)$ *has a unique solution.*

Proof. Let $\mathbf{q}^D\in\tilde D$ be a solution of $(\tilde P)$ and define u_2 and $\mathbf{q}_2^D$ as in (i). Using the jump relations of potentials, operator identities due to the Calderon projector and the fact that $v=-R\psi$ we obtain $\nabla u_2\cdot\mathbf{n}=\psi$. Therefore u_2 is the only decaying solution of $\Delta u_2=0$ in Ω_c with $\nabla u_2\cdot\mathbf{n}=\psi$, so $u_2=-R\psi=v$ on Γ. Therefore

$$\|\mathbf{q}_2^D\|^2_{[L^2(\Omega_c)]^n} = \|\nabla u_2\|^2_{[L^2(\Omega_c)]^n} = \langle\psi, R\psi\rangle.$$

A straightforward computation shows

$$\tilde\Phi(\mathbf{q}^D,\mathbf{q}_2^D) = \tilde\Psi(\mathbf{q}^D) + \frac{1}{2}\langle t_0, Rt_0\rangle. \tag{3.7}$$

Now, given $(\mathbf{q}_1,\mathbf{q}_2)\in\tilde C$ with $\tilde\Phi(\mathbf{q}_1,\mathbf{q}_2)<\infty$, we know that $\mathbf{q}_2\in H(\mathrm{div};\Omega_c)$ and $\mathrm{div}\,\mathbf{q}_2=0$, and hence there exists $v_2\in W^1(\Omega_c)$ and $\mathbf{q}_0\in H(\mathrm{div};\Omega_c)$ such that $\mathbf{q}_2=\nabla v_2+\mathbf{q}_0$, $\nabla v_2\cdot\mathbf{n}=\mathbf{q}_1\cdot\mathbf{n}-t_0$ on Γ, $\mathrm{div}\,\mathbf{q}_0=0$ in Ω_c, and $\mathbf{q}_0\cdot\mathbf{n}=0$ on Γ. Then, we find

$$\tilde\Phi(\mathbf{q}_1,\mathbf{q}_2) = \tilde\Psi(\mathbf{q}_1) + \frac{1}{2}\langle t_0,Rt_0\rangle + \frac{1}{2}\|\mathbf{q}_0\|^2_{[L^2(\Omega_c)]^n}, \tag{3.8}$$

which implies $\tilde\Phi(\mathbf{q}_1,\mathbf{q}_2)\ge\tilde\Phi(\mathbf{q}^D,\mathbf{q}_2^D)$ for all $(\mathbf{q}_1,\mathbf{q}_2)\in\tilde C$, since (3.7) holds and $\mathbf{q}^D$ minimizes $\tilde\Psi$. Therefore $(\mathbf{q}^D,\mathbf{q}_2^D)$ minimizes $\tilde\Phi$.

Conversely, let $(\mathbf{q}_1^0,\mathbf{q}_2^0)\in\tilde C$ be a minimizer of $\tilde\Phi$ on $\tilde C$. From (3.8) we know that for all $\mathbf{q}_1\in\tilde D$ we can find a $v_2\in W^1(\Omega_c)$ such that $(\mathbf{q}_1,\nabla v_2)\in\tilde C$ and

$$\tilde\Phi(\mathbf{q}_1,\nabla v_2) = \tilde\Psi(\mathbf{q}_1) + \frac{1}{2}\langle t_0,Rt_0\rangle. \tag{3.9}$$

Using Theorem 2.2 we have for all $\mathbf{q}_1\in\tilde D$

$$\tilde\Phi(\mathbf{q}_1^0,\mathbf{q}_2^0) = \tilde\Phi(\mathbf{q}_1^0,\nabla u_2) = \tilde\Psi(\mathbf{q}_1^0) + \frac{1}{2}\langle t_0,Rt_0\rangle \le \tilde\Phi(\mathbf{q}_1,\nabla v_2) = \tilde\Psi(\mathbf{q}_1) + \frac{1}{2}\langle t_0,Rt_0\rangle,$$

which yields $\tilde\Psi(\mathbf{q}_1^0)\le\tilde\Psi(\mathbf{q}_1)$, and hence $\mathbf{q}_1^0$ minimizes $\tilde\Psi$.

The assertion (b) follows from the equivalence in (a) and the unique solvability of (P). □

Next, we introduce a saddle point formulation (M) of $(\tilde P)$. Indeed, we first define the operator $\mathcal{H}$: $H(\mathrm{div};\Omega)\times L^2(\Omega)\times\tilde H^{1/2}(\Gamma_s)\to\mathbb{R}$ as

$$\mathcal{H}(\mathbf{p},v,\mu) := \tilde\Psi(\mathbf{p}) + \int_\Omega v\,\mathrm{div}\,\mathbf{p}\,\mathrm{d}x + \int_\Omega fv\,\mathrm{d}x + \langle\mathbf{p}\cdot\mathbf{n},\mu\rangle_{\Gamma_s} \tag{3.10}$$

for all $(\mathbf{p},v,\mu)\in H(\mathrm{div};\Omega)\times L^2(\Omega)\times\tilde H^{1/2}(\Gamma_s)$, and consider the subset of admissible functions

$$\tilde H_+^{1/2}(\Gamma_s) := \{\mu\in\tilde H^{1/2}(\Gamma_s):\quad \mu\ge 0\}. \tag{3.11}$$

Then, we define problem (M) as: Find $(\hat{\mathbf{q}},\hat u,\hat\lambda)\in H(\mathrm{div};\Omega)\times L^2(\Omega)\times\tilde H_+^{1/2}(\Gamma_s)$ such that

$$\mathcal{H}(\hat{\mathbf{q}},u,\lambda)\le\mathcal{H}(\hat{\mathbf{q}},\hat u,\hat\lambda)\le\mathcal{H}(\mathbf{q},\hat u,\hat\lambda)\quad\forall(\mathbf{q},u,\lambda)\in H(\mathrm{div};\Omega)\times L^2(\Omega)\times\tilde H_+^{1/2}(\Gamma_s), \tag{3.12}$$

which is equivalent (see [9]) to finding a solution $(\hat{\mathbf{q}}, \hat{u}, \hat{\lambda}) \in H(\mathrm{div};\Omega) \times L^2(\Omega) \times \tilde{H}^{1/2}_{+}(\Gamma_s)$ of the following variational inequality:

$$a(\hat{\mathbf{q}}, \mathbf{q}) + b(\mathbf{q}, \hat{u}) + d(\mathbf{q}, \hat{\lambda}) = \langle \mathbf{q}\cdot\mathbf{n}, r\rangle \quad \forall \mathbf{q} \in H(\mathrm{div};\Omega), \tag{3.13}$$

$$b(\hat{\mathbf{q}}, u) = -\int_\Omega f u \,\mathrm{d}x \quad \forall u \in L^2(\Omega), \tag{3.14}$$

$$d(\hat{\mathbf{q}}, \lambda - \hat{\lambda}) \leq 0 \qquad \forall \lambda \in \tilde{H}^{1/2}_{+}(\Gamma_s), \tag{3.15}$$

where

$$a(\mathbf{p}, \mathbf{q}) := \int_\Omega (\boldsymbol{\kappa}^{-1}\mathbf{p})\cdot\mathbf{q}\,\mathrm{d}x + \langle \mathbf{q}\cdot\mathbf{n}, R(\mathbf{p}\cdot\mathbf{n})\rangle \qquad \forall \mathbf{p}, \mathbf{q} \in H(\mathrm{div};\Omega), \tag{3.16}$$

$$b(\mathbf{q}, u) := \int_\Omega u \,\mathrm{div}\,\mathbf{q}\,\mathrm{d}x \qquad \forall (\mathbf{q}, u) \in H(\mathrm{div};\Omega) \times L^2(\Omega), \tag{3.17}$$

$$d(\mathbf{q}, \lambda) := \langle \mathbf{q}\cdot\mathbf{n}, \lambda\rangle_{\Gamma_s} \qquad \forall (\mathbf{q}, \lambda) \in H(\mathrm{div};\Omega) \times \tilde{H}^{1/2}(\Gamma_s), \tag{3.18}$$

and $r := Rt_0 + u_0$.

Next, we introduce the bilinear form

$$B(\mathbf{q}, (u, \lambda)) = b(\mathbf{q}, u) + d(\mathbf{q}, \lambda) \quad \forall (\mathbf{q}, u, \lambda) \in H(\mathrm{div};\Omega) \times L^2(\Omega) \times \tilde{H}^{1/2}(\Gamma_s), \tag{3.19}$$

and rewrite the variational inequality (3.13)–(3.15) as

$$a(\hat{\mathbf{q}}, \mathbf{q}) + B(\mathbf{q}, (\hat{u}, \hat{\lambda})) = \langle \mathbf{q}\cdot\mathbf{n}, r\rangle \qquad \forall \mathbf{q} \in H(\mathrm{div};\Omega) \tag{3.20}$$

$$B(\hat{\mathbf{q}}, (u - \hat{u}, \lambda - \hat{\lambda})) \leq -\int_\Omega f(u - \hat{u})\,\mathrm{d}x \qquad \forall (u, \lambda) \in L^2(\Omega) \times \tilde{H}^{1/2}_{+}(\Gamma_s). \tag{3.21}$$

Problems $(\tilde{P})$ and (M) are connected as follows.

Theorem 3.2. *$(\tilde{P})$ and (M) are equivalent in the following sense:*

(i) *If $(\hat{\mathbf{q}}, \hat{u}, \hat{\lambda})$ solves (M), then $\hat{\mathbf{q}} \in \tilde{D}$ is the solution of problem $(\tilde{P})$. Additionally, there holds $\hat{\mathbf{q}} = \boldsymbol{\kappa}\nabla\hat{u}$, $\hat{u} = -R(\hat{\mathbf{q}}\cdot\mathbf{n} - t_0) + u_0$ on Γ_t and $\hat{\lambda} = -R(\hat{\mathbf{q}}\cdot\mathbf{n} - t_0) + u_0 - \hat{u}$ on Γ_s.*

(ii) *Let $\mathbf{q}^D \in \tilde{D}$ be the solution of $(\tilde{P})$. Then there is a unique solution $\hat{u} \in H^1(\Omega)$ of the Neumann problem*

$$-\,\mathrm{div}(\boldsymbol{\kappa}\nabla\hat{u}) = f \ \text{ in } \Omega, \quad (\boldsymbol{\kappa}\nabla\hat{u})\cdot\mathbf{n} = \mathbf{q}^D\cdot\mathbf{n} \ \text{ on } \Gamma$$

satisfying the additional condition

$$\langle \mu, \hat{u} + R(\mathbf{q}^D\cdot\mathbf{n} - t_0) - u_0\rangle \geq 0 \quad \forall \mu \in H^{-1/2}(\Gamma) \text{ with } \mu \leq -\mathbf{q}^D\cdot\mathbf{n} \text{ on } \Gamma_s.$$

Defining $\hat{\lambda} := -R(\mathbf{q}^D\cdot\mathbf{n} - t_0) + u_0 - \hat{u}$ on Γ_s, it follows that $(\mathbf{q}^D, \hat{u}, \hat{\lambda})$ is a solution of (M).

Proof.

(i) If $(\hat{\mathbf{q}}, \hat{u}, \hat{\lambda})$ is a solution to (M), then (3.14) implies that $\mathrm{div}\,\hat{\mathbf{q}} = -f$. Then, taking $\lambda = 0$ and $\lambda = 2\hat{\lambda}$, we find that (3.15) is equivalent to

$$d(\hat{\mathbf{q}}, \hat{\lambda}) = 0 \quad \text{and} \quad d(\hat{\mathbf{q}}, \lambda) \leq 0 \qquad \forall \lambda \in \tilde{H}^{1/2}_{+}(\Gamma_s). \tag{3.22}$$

Therefore $\hat{\mathbf{q}} \in \tilde{D}$ and $\tilde{\Psi}(\hat{\mathbf{q}}) = \mathcal{H}(\hat{\mathbf{q}}, \hat{u}, \hat{\lambda})$. Since $\mathcal{H}(\mathbf{q}, \hat{u}, \hat{\lambda}) \le \tilde{\Psi}(\mathbf{q})$ for all $\mathbf{q} \in \tilde{D}$, then $\hat{\mathbf{q}}$ solves $(\tilde{P})$. Finally, elementary arguments show that (3.13) is equivalent to $\hat{\mathbf{q}} = \boldsymbol{\kappa}\nabla\hat{u}$ and to

$$\langle \mathbf{q}\cdot\mathbf{n}, \hat{u} + R(\mathbf{q}\cdot\mathbf{n}) - r\rangle + \langle \mathbf{q}\cdot\mathbf{n}, \hat{\lambda}\rangle_{\Gamma_s} = 0 \quad \forall \mathbf{q} \in H(\mathrm{div};\Omega), \tag{3.23}$$

which is equivalent to the remaining assertions.

(ii) Let $\mathbf{q}^D$ be the solution of $(\tilde{P})$. Because $\mathbf{q}^D \in \tilde{D}$, we know that (3.14) is satisfied and

$$d(\mathbf{q}^D, \lambda) \le 0 \qquad \forall \lambda \in \tilde{H}_+^{1/2}(\Gamma_s). \tag{3.24}$$

Theorem 2.2 implies that the problem

$$-\operatorname{div}(\boldsymbol{\kappa}\nabla\hat{u}) = f \quad \text{in } \Omega, \qquad (\boldsymbol{\kappa}\nabla\hat{u})\cdot\mathbf{n} = \mathbf{q}^D\cdot\mathbf{n} \quad \text{on } \Gamma, \tag{3.25}$$

has a solution, unique up to an additive constant. Let us fix the constant by demanding that

$$\langle R(\mathbf{q}^D\cdot\mathbf{n}) + \hat{u} - r, 1\rangle = 0. \tag{3.26}$$

Using Theorem 2.2 again it is clear that $\mathbf{q}^D = \boldsymbol{\kappa}\nabla\hat{u}$. Let us finally define $\hat{\lambda} = r - \hat{u} - R(\mathbf{q}^D\cdot\mathbf{n})$. The minimization problem $(\tilde{P})$ is equivalent to the variational equation

$$a(\mathbf{q}^D, \mathbf{q} - \mathbf{q}^D) - \langle r, (\mathbf{q} - \mathbf{q}^D)\cdot\mathbf{n}\rangle \ge 0 \quad \forall \mathbf{q} \in \tilde{D}. \tag{3.27}$$

Applying that $\mathbf{q}^D = \boldsymbol{\kappa}\nabla\hat{u}$, the definition of $\hat{\lambda}$ and the fact that $\langle\hat{\lambda}, 1\rangle = 0$, the above variational inequality implies that

$$\langle \hat{\lambda}, c + (\mathbf{q}^D - \mathbf{q})\cdot\mathbf{n}\rangle \ge 0 \qquad \forall \mathbf{q} \in \tilde{D} \quad \forall c \in \mathbb{R}. \tag{3.28}$$

The following claim is the fact that the set

$$\Big\{\mu \in H^{-1/2}(\Gamma): \quad \mu = c + (\mathbf{q}^D - \mathbf{q})\cdot\mathbf{n}, \quad \mathbf{q} \in \tilde{D}, \quad c \in \mathbb{R}\Big\} \tag{3.29}$$

contains the following elements: (a) any $\mu \ge 0$ on Γ; (b) any $\mu \in H^{-1/2}(\Gamma)$ such that $\mu = 0$ in Γ_s, and (c) the two particular elements given by

$$\mu = \begin{cases} \pm\mathbf{q}^D\cdot\mathbf{n}, & \text{in } \Gamma_s \\ 0 & \text{elsewhere.} \end{cases} \tag{3.30}$$

With (a) we prove from (3.28) that $\hat{\lambda} \ge 0$, with (b) that $\hat{\lambda} = 0$ on Γ_t and with (c) that $d(\mathbf{q}^D, \hat{\lambda}) = 0$. Therefore, (3.15) is satisfied and $\hat{\lambda} \in \tilde{H}_+^{1/2}(\Gamma_s)$. Finally, we have to study

$$\mathrm{Res}(\mathbf{q}) := a(\mathbf{q}^D, \mathbf{q}) + b(\mathbf{q}, \hat{u}) + d(\mathbf{q}, \hat{\lambda}) - \langle \mathbf{q}\cdot\mathbf{n}, r\rangle. \tag{3.31}$$

Using again that $\mathbf{q}^D = \boldsymbol{\kappa}\nabla\hat{u}$ we prove that

$$\mathrm{Res}(\mathbf{q}) = \langle \mathbf{q}\cdot\mathbf{n}, \hat{u} + R(\mathbf{q}^D\cdot\mathbf{n}) - r\rangle + \langle \mathbf{q}\cdot\mathbf{n}, \hat{\lambda}\rangle_{\Gamma_s} = 0, \tag{3.32}$$

so (3.13) is also satisfied and the result is proved.

□

An *a-priori* estimate for the solution of problem (M) is established next.

Theorem 3.3. *There exists a constant C, independent of r $(:= Rt_0 + u_0)$ and f, such that*

$$\|\hat{\mathbf{q}}\|_{H(\mathrm{div};\Omega)} + \|\hat{u}\|_{L^2(\Omega)} + \|\hat{\lambda}\|_{\tilde{H}^{1/2}(\Gamma_s)} \leq C\,\left\{\|r\|_{H^{1/2}(\Gamma)} + \|f\|_{L^2(\Omega)}\right\}. \tag{3.33}$$

Proof. First, we note from the proof of Theorem 2.1 in [2] that there exists a constant $\beta > 0$ such that for all $(u, \lambda) \in L^2(\Omega) \times \tilde{H}^{1/2}(\Gamma_s)$ there holds

$$\beta\,\|(u,\lambda)\|_{L^2(\Omega)\times\tilde{H}^{1/2}(\Gamma_s)} \leq \sup_{\substack{\mathbf{q}\in H(\mathrm{div};\Omega)\\ \mathbf{q}\neq 0}} \frac{B(\mathbf{q},(u,\lambda))}{\|\mathbf{q}\|_{H(\mathrm{div};\Omega)}}. \tag{3.34}$$

On the other hand, (3.20) leads to

$$\begin{aligned}\beta\,\|(\hat{u},\hat{\lambda})\| \;\leq\; & \sup_{\substack{\mathbf{q}\in H(\mathrm{div};\Omega)\\ \mathbf{q}\neq 0}} \frac{B(\mathbf{q},(\hat{u},\hat{\lambda}))}{\|\mathbf{q}\|_{H(\mathrm{div};\Omega)}} = \sup_{\substack{\mathbf{q}\in H(\mathrm{div};\Omega)\\ \mathbf{q}\neq 0}} \frac{\langle \mathbf{q}\cdot\mathbf{n}, r\rangle - a(\hat{\mathbf{q}},\mathbf{q})}{\|\mathbf{q}\|_{H(\mathrm{div};\Omega)}}\\ & \leq C\,\left\{\|r\|_{H^{1/2}(\Gamma)} + \|\hat{\mathbf{q}}\|_{H(\mathrm{div};\Omega)}\right\}.\end{aligned} \tag{3.35}$$

Now, with (3.13)—(3.15) we obtain

$$\begin{aligned} a(\hat{\mathbf{q}},\hat{\mathbf{q}}) &= \langle \hat{\mathbf{q}}\cdot\mathbf{n}, r\rangle - b(\hat{\mathbf{q}},\hat{u}) - d(\hat{\mathbf{q}},\hat{\lambda}) = \langle \hat{\mathbf{q}}\cdot\mathbf{n}, r\rangle + (f,\hat{u})_{L^2(\Omega)}\\ &\leq \|r\|_{H^{1/2}(\Gamma)}\|\hat{\mathbf{q}}\|_{H(\mathrm{div};\Omega)} + \|f\|_{L^2(\Omega)}\|\hat{u}\|_{L^2(\Omega)}.\end{aligned}$$

Using that $\mathrm{div}\,\hat{\mathbf{q}} = -f$ and (3.35), this yields

$$\begin{aligned}\|\hat{\mathbf{q}}\|^2_{H(\mathrm{div};\Omega)} &\leq c'\cdot a(\hat{\mathbf{q}},\hat{\mathbf{q}}) + \|\,\mathrm{div}\,\hat{\mathbf{q}}\|^2_{L^2(\Omega)}\\ &\leq c'\left\{\|r\|_{H^{1/2}(\Gamma)}\|\hat{\mathbf{q}}\|_{H(\mathrm{div};\Omega)} + \|f\|_{L^2(\Omega)}\|\hat{u}\|_{L^2(\Omega)}\right\} + \|f\|^2_{L^2(\Omega)}\\ &\leq C\left\{\|\hat{\mathbf{q}}\|_{H(\mathrm{div};\Omega)}\left(\|r\|_{H^{1/2}(\Gamma)} + \|f\|_{L^2(\Omega)}\right) + \|r\|^2_{H^{1/2}(\Gamma)} + \|f\|^2_{L^2(\Omega)}\right\},\end{aligned}$$

and therefore

$$\|\hat{\mathbf{q}}\|^2_{H(\mathrm{div};\Omega)} \leq C\left\{\|r\|^2_{H^{1/2}(\Gamma)} + \|f\|^2_{L^2(\Omega)}\right\}.$$

□

4. General numerical approximation

In this section we treat the numerical approximation for problem (M) by using mixed finite elements in Ω and boundary elements on Γ. For simplicity we assume that Γ_t and Γ_s are polygonal hypersurfaces.

Let $(\mathcal{T}_h)_h$ be a family of regular triangulations [3] of the domain $\bar{\Omega}$ by triangles/tetrahedrons T of diameter h_T such that $h := \max\{h_T : T \in \mathcal{T}_h\}$. We denote by ρ_T the diameter of the inscribed circle/sphere in T, and assume that there exists a constant $\kappa > 0$ such that, for any h and for any T in $\mathcal{T}_h$, the inequality $\dfrac{h_T}{\rho_T} \leq \kappa$ holds. Moreover we assume that there exists a constant $C > 0$ such that for any h and for any triangle/tetrahedron T in $\mathcal{T}_h$ such that $T \cap \partial\Omega$ is a whole edge/face of T, there holds $|\,T \cap \partial\Omega\,| \geq C\,h^{n-1}$, where $|T \cap \partial\Omega|$ denotes the length/area of $T \cap \partial\Omega$. This means that the family of triangulations is uniformly regular near the boundary.

We also assume that all the points/curves in $\bar{\Gamma}_t \cap \bar{\Gamma}_s$ become vertices/edges of $\mathcal{T}_h$ for all $h > 0$. Then we denote by $\mathbf{E}_h$ the set of all edges/faces e of $\mathcal{T}_h$ and put $\mathcal{G}_h := \{e \in \mathbf{E}_h : e \subset \Gamma\}$. Further, we let $(\tau_{\tilde{h}})_{\tilde{h}}$ be a family of independent regular triangulations of the boundary part Γ_s by line segments/triangles Δ of diameter $\tilde{h}_\Delta$ such that $\tilde{h} := \max\{\tilde{h}_\Delta : \Delta \in \tau_{\tilde{h}}\}$.

Next, we consider $(X_{h,\tilde{h}})_{h,\tilde{h}} = (L_h \times H_h \times H_h^{-1/2} \times H_h^{1/2} \times H_{s,\tilde{h}}^{1/2})_{h,\tilde{h}}$ be a family of finite-dimensional subspaces of $X = L^2(\Omega)\times H(\mathrm{div};\Omega)\times H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)/\mathbb{R}\times \tilde{H}^{1/2}(\Gamma_s)$, subordinated to the corresponding triangulations such that the following approximation property holds

$$\lim_{\substack{h\to 0\\ \tilde{h}\to 0}} \left\{ \inf_{(u_h,\mathbf{q}_h,\psi_h,\phi_h,\lambda_{\tilde{h}})\in X_{h,\tilde{h}}} \|(u,\mathbf{q},\psi,\phi,\lambda) - (u_h,\mathbf{q}_h,\psi_h,\phi_h,\lambda_{\tilde{h}})\|_X \right\} = 0 \tag{4.1}$$

for all $(u,\mathbf{q},\psi,\phi,\lambda)\in X$. In addition, we assume that

$$\left\{ \mathrm{div}\,\mathbf{q}_h \,:\, \mathbf{q}_h \in H_h \right\} \subseteq L_h \tag{4.2}$$

and

$$H_h^{-1/2} = \{\mathbf{q}_h|_\Gamma \cdot \mathbf{n} \,:\, q_h \in H_h\} \tag{4.3}$$

i.e. $H_h^{-1/2}$ is the space of normal traces on Γ of the functions in H_h. Also, the subspaces $L_h \times H_{s,\tilde{h}}^{1/2}$ and H_h are supposed to satisfy the usual discrete Babuška-Brezzi condition, which means that there exists $\beta^* > 0$ such that

$$\inf_{\substack{(u_h,\lambda_{\tilde{h}})\in L_h\times H_{s,\tilde{h}}^{1/2}\\ (u_h,\lambda_{\tilde{h}})\neq 0}} \sup_{\substack{\mathbf{q}_h\in H_h\\ \mathbf{q}_h\neq 0}} \frac{B(\mathbf{q}_h,(u_h,\lambda_{\tilde{h}}))}{\|\mathbf{q}_h\|_{H(\mathrm{div};\Omega)}\|(u_h,\lambda_{\tilde{h}})\|_{L^2(\Omega)\times\tilde{H}^{1/2}(\Gamma_s)}} \geq \beta^*. \tag{4.4}$$

Now, we let $i_h : L_h \hookrightarrow L^2(\Omega)$, $j_h : H_h \hookrightarrow H(\mathrm{div};\Omega)$, $k_h : H_h^{-1/2} \hookrightarrow H^{-1/2}(\Gamma)$, $l_h : H_h^{1/2} \hookrightarrow H^{1/2}(\Gamma)/\mathbb{R}$ and $m_{\tilde{h}} : H_{s,\tilde{h}}^{1/2} \hookrightarrow \tilde{H}^{1/2}(\Gamma_s)$ denote the canonical imbeddings with their corresponding adjoints i_h^*, j_h^*, k_h^*, l_h^* and $m_{\tilde{h}}^*$. Let $\gamma : H(\mathrm{div};\Omega) \to H^{-1/2}(\Gamma)$ be the trace operator giving the normal component of functions in $H(\mathrm{div};\Omega)$. Then we have $k_h H_h^{-1/2} = \gamma j_h H_h$.

In order to approximate R we define the discrete operators

$$R_h := k_h^* R k_h \quad \text{and} \quad \tilde{R}_h := \frac{1}{2}\left(k_h^* V k_h + k_h^*(I+K)l_h(l_h^* W l_h)^{-1} l_h^*(I+K')k_h\right).$$

We remark that the computation of $\tilde{R}_h$ requires the numerical solution of a linear system with a symmetric positive definite matrix $W_h := l_h^* W l_h$. In general, there holds $\tilde{R}_h \neq R_h$ because $\tilde{R}_h$ is a Schur complement of a discretized matrix while R_h is a discretized Schur complement of an operator.

Then, in order to approximate the solution of problem (M), we consider the following Galerkin scheme (M_h) with an approximated bilinear form $a_h(\cdot,\cdot)$: Find $(\hat{\mathbf{q}}_h, \hat{u}_h, \hat{\lambda}_{\tilde{h}}) \in H_h \times L_h \times H_{s,+,\tilde{h}}^{1/2}$ such that

$$a_h(\hat{\mathbf{q}}_h,\mathbf{q}_h) + b(\mathbf{q}_h,\hat{u}_h) + d(\mathbf{q}_h,\hat{\lambda}_{\tilde{h}}) = \langle \mathbf{q}_h\cdot\mathbf{n}, r_h\rangle \qquad \forall \mathbf{q}_h \in H_h, \tag{4.5}$$

$$b(\hat{\mathbf{q}}_h,u_h) = -\int_\Omega f u_h \,\mathrm{d}x \qquad \forall u_h \in L_h, \tag{4.6}$$

$$d(\hat{\mathbf{q}}_h,\lambda_{\tilde{h}} - \hat{\lambda}_{\tilde{h}}) \leq 0 \qquad \forall \lambda_{\tilde{h}} \in H_{s,+,\tilde{h}}^{1/2}, \tag{4.7}$$

where

$$H_{s,+,\tilde{h}}^{1/2} := \left\{\mu \in H_{s,\tilde{h}}^{1/2} : \quad \mu \geq 0\right\}, \tag{4.8}$$

$$a_h(\mathbf{p},\mathbf{q}) = \int_\Omega (\boldsymbol{\kappa}^{-1}\mathbf{p})\cdot\mathbf{q}\,\mathrm{d}x + \left\langle \mathbf{q}\cdot\mathbf{n}, \tilde{R}_h(\mathbf{p}\cdot\mathbf{n})\right\rangle \quad \forall \mathbf{p},\mathbf{q}\in H_h, \tag{4.9}$$

and

$$r_h := k_h^* \left(\frac{1}{2}(V + (I+K)l_h(l_h^* W l_h)^{-1} l_h^*(I+K'))t_0 + u_0 \right).$$

Note that the nonconformity of problem (M_h) arises from the bilinear form $a_h(\cdot,\cdot)$ approximating $a(\cdot,\cdot)$.

The following lemma provides bounds for the approximation error introduced by using a discrete Schur complement.

Lemma 4.1. *Let the symmetric operator $\delta_{r,h}$ be defined for all $t \in H^{-1/2}(\Gamma)$ by*

$$\delta_{r,h} t := \frac{1}{2}(I+K)\left(l_h(l_h^* W l_h)^{-1} l_h^* - W^{-1}\right)(I+K')t \in H^{1/2}(\Gamma). \tag{4.10}$$

Then there exists $c_0 > 0$, independent of h, such that for all $t \in H^{-1/2}(\Gamma)$ there holds

$$\|\delta_{r,h} t\|_{H^{1/2}(\Gamma)} \le c_0 \inf_{\phi_h \in H_h^{1/2}} \|W^{-1}(I+K')t - \phi_h\|_{H^{1/2}(\Gamma)}, \tag{4.11}$$

$$\langle t, \delta_{r,h} t\rangle \ge 0. \tag{4.12}$$

Proof. Since $W : H^{1/2}(\Gamma)/\mathbb{R} \to H^{-1/2}(\Gamma)$ is positive definite and we have $\langle (I+K')t, 1\rangle = \langle t, (I+K)1\rangle = 0$ there exists a unique solution $z \in H^{1/2}(\Gamma)/\mathbb{R}$ of $Wz = (I+K')t$ and $z_h \in H_h^{1/2}$ of $(l_h^* W l_h) z_h = l_h^*(I+K')t$, *i.e.* z_h is the Galerkin approximation of z. Using the boundedness of $(I+K)$ and the quasi-optimal error estimate we have

$$\|\delta_{r,h} t\|_{H^{1/2}(\Gamma)} \le \frac{1}{2}\|(I+K)\| \cdot \|z - l_h z_h\|_{H^{1/2}(\Gamma)} \le c_0 \operatorname{dist}_{H^{1/2}(\Gamma)}\left(W^{-1}(I+K')t, H_h^{1/2}\right).$$

Therefore, we obtain the assertion (4.11), *cf.* [5], Lemma 9.

Additionally, we can write for the error in the energy norm

$$0 \le \|z - z_h\|_W^2 = \|l_h z_h\|_W^2 - \|z\|_W^2 = t^*(I+K)(l_h(l_h^* W l_h)^{-1} l_h^* - W^{-1})(I+K')t = \langle t, \delta_{r,h} t\rangle.$$

This proves assertion (4.12). □

The following lemma provides an upper bound for the difference between the discrete and the continuous solutions above.

Lemma 4.2. *Let $(\hat{\mathbf{q}}, \hat{u}, \hat{\lambda})$ and $(\hat{\mathbf{q}}_h, \hat{u}_h, \hat{\lambda}_{\tilde{h}})$ be the solutions of problems (M) and (M_h), respectively. Then there exists $c > 0$, independent of h and $\tilde{h}$, such that*

$$\begin{aligned}\|\hat{u} - \hat{u}_h\|_{L^2(\Omega)}^2 + \|\hat{\lambda} - \hat{\lambda}_{\tilde{h}}\|_{\tilde{H}^{1/2}(\Gamma_s)}^2 \le c \Big\{ &\|\hat{\mathbf{q}} - \hat{\mathbf{q}}_h\|_{H(\mathrm{div};\Omega)}^2 + \|\hat{u} - u_h\|_{L^2(\Omega)}^2 + \|\hat{\lambda} - \lambda_{\tilde{h}}\|_{\tilde{H}^{1/2}(\Gamma_s)}^2 \\ &+ \|\delta_{r,h}(t_0 - \hat{\mathbf{q}}_h \cdot \mathbf{n})\|_{H^{1/2}(\Gamma)}^2 \Big\}\end{aligned} \tag{4.13}$$

and

$$\begin{aligned}\|\hat{\mathbf{q}} - \hat{\mathbf{q}}_h\|_{H(\mathrm{div};\Omega)}^2 \le c \Big\{ &\|\hat{\mathbf{q}} - \mathbf{q}_h\|_{H(\mathrm{div};\Omega)}^2 + \|\hat{u} - u_h\|_{L^2(\Omega)}^2 + \|\hat{\lambda} - \lambda_{\tilde{h}}\|_{\tilde{H}^{1/2}(\Gamma_s)}^2 \\ &- \langle (\mathbf{q}_h - \hat{\mathbf{q}}_h)\cdot \mathbf{n}, \delta_{r,h}(t_0 - \hat{\mathbf{q}}_h \cdot \mathbf{n})\rangle - d(\hat{\mathbf{q}} - \hat{\mathbf{q}}_h, \lambda_{\tilde{h}} - \hat{\lambda}_{\tilde{h}}) \Big\}\end{aligned} \tag{4.14}$$

for all $(\mathbf{q}_h, u_h, \lambda_{\tilde{h}}) \in H_h \times L_h \times H_{s,+,\tilde{h}}^{1/2}$.

Proof. Given $(\mathbf{q}_h, u_h, \lambda_{\tilde{h}}) \in H_h \times L_h \times H^{1/2}_{s,+,\tilde{h}}$, we observe from (4.5) and (3.13) that

$$a(\hat{\mathbf{q}} - \hat{\mathbf{q}}_h, \mathbf{q}_h) + \langle \mathbf{q}_h \cdot \mathbf{n}, (R - \tilde{R}_h)(\hat{\mathbf{q}}_h \cdot \mathbf{n})\rangle + b(\mathbf{q}_h, \hat{u} - \hat{u}_h) + d(\mathbf{q}_h, \hat{\lambda} - \hat{\lambda}_{\tilde{h}}) = \langle \mathbf{q}_h \cdot \mathbf{n}, r - r_h\rangle, \tag{4.15}$$

whence

$$\begin{aligned} b(\mathbf{q}_h, -\hat{u}_h) + d(\mathbf{q}_h, -\hat{\lambda}_{\tilde{h}}) &= b(\mathbf{q}_h, -\hat{u}) + d(\mathbf{q}_h, -\hat{\lambda}) \\ &\quad - a(\hat{\mathbf{q}} - \hat{\mathbf{q}}_h, \mathbf{q}_h) + \langle \mathbf{q}_h \cdot \mathbf{n}, r - r_h - (R - \tilde{R}_h)(\hat{\mathbf{q}}_h \cdot \mathbf{n})\rangle. \end{aligned} \tag{4.16}$$

Note that we have $\langle \mathbf{q}_h \cdot \mathbf{n}, r - r_h - (R - \tilde{R}_h)(\hat{\mathbf{q}}_h \cdot \mathbf{n})\rangle = -\langle \mathbf{q}_h \cdot \mathbf{n}, \delta_{r,h}(t_0 - \hat{\mathbf{q}}_h \cdot \mathbf{n})\rangle$. Similarly, (4.6) and (3.14) give

$$b(\hat{\mathbf{q}} - \hat{\mathbf{q}}_h, u_h) = 0. \tag{4.17}$$

Next, applying the discrete inf-sup condition (4.4) and making use of (4.16), we obtain

$$\begin{aligned} \beta^* \|(u_h - \hat{u}_h, \lambda_{\tilde{h}} - \hat{\lambda}_{\tilde{h}})\|_{L^2(\Omega)\times \tilde{H}^{1/2}(\Gamma_s)} &\le \sup_{\substack{\mathbf{q}_h \in H_h \\ \mathbf{q}_h \neq 0}} \frac{b(\mathbf{q}_h, u_h - \hat{u}_h) + d(\mathbf{q}_h, \lambda_{\tilde{h}} - \hat{\lambda}_{\tilde{h}})}{\|\mathbf{q}_h\|_{H(\mathrm{div};\Omega)}} \\ = \sup_{\substack{\mathbf{q}_h \in H_h \\ \mathbf{q}_h \neq 0}} & \frac{b(\mathbf{q}_h, u_h - \hat{u}) + d(\mathbf{q}_h, \lambda_{\tilde{h}} - \hat{\lambda}) - a(\hat{\mathbf{q}} - \hat{\mathbf{q}}_h, \mathbf{q}_h) - \langle \mathbf{q}_h \cdot \mathbf{n}, \delta_{r,h}(t_0 - \hat{\mathbf{q}}_h \cdot \mathbf{n})\rangle}{\|\mathbf{q}_h\|_{H(\mathrm{div};\Omega)}} \\ \le C & \left\{ \|\hat{u} - u_h\|_{L^2(\Omega)} + \|\hat{\lambda} - \lambda_{\tilde{h}}\|_{\tilde{H}^{1/2}(\Gamma_s)} + \|\hat{\mathbf{q}} - \hat{\mathbf{q}}_h\|_{H(\mathrm{div};\Omega)} + \|\delta_{r,h}(t_0 - \hat{\mathbf{q}}_h \cdot \mathbf{n})\|_{H^{1/2}(\Gamma)} \right\}, \end{aligned}$$

which, combined with the triangle inequality applied to $(\hat{u} - \hat{u}_h, \hat{\lambda} - \hat{\lambda}_{\tilde{h}}) = (\hat{u} - u_h, \hat{\lambda} - \lambda_{\tilde{h}}) + (u_h - \hat{u}_h, \lambda_{\tilde{h}} - \hat{\lambda}_{\tilde{h}})$, yield (4.13) (see also [13], Thm. 5.1' and Rem. 5.6).

We now let $Y := H(\mathrm{div};\Omega) \times L^2(\Omega) \times \tilde{H}^{1/2}(\Gamma_s)$ and define the bounded bilinear form $\mathcal{A} : Y \times Y \to \mathbb{R}$ by

$$\mathcal{A}((\hat{\mathbf{q}}, \hat{u}, \hat{\lambda}), (\mathbf{q}, u, \lambda)) := a(\hat{\mathbf{q}}, \mathbf{q}) + b(\mathbf{q}, \hat{u}) + d(\mathbf{q}, \hat{\lambda}) - b(\hat{\mathbf{q}}, u) - d(\hat{\mathbf{q}}, \lambda)$$

for all $(\hat{\mathbf{q}}, \hat{u}, \hat{\lambda}), (\mathbf{q}, u, \lambda) \in Y$. It follows that

$$\begin{aligned} a(\hat{\mathbf{q}} - \hat{\mathbf{q}}_h, \hat{\mathbf{q}} - \hat{\mathbf{q}}_h) &= \mathcal{A}((\hat{\mathbf{q}} - \hat{\mathbf{q}}_h, \hat{u} - \hat{u}_h, \hat{\lambda} - \hat{\lambda}_{\tilde{h}}), (\hat{\mathbf{q}} - \hat{\mathbf{q}}_h, \hat{u} - \hat{u}_h, \hat{\lambda} - \hat{\lambda}_{\tilde{h}})) \\ &= \mathcal{A}((\hat{\mathbf{q}} - \hat{\mathbf{q}}_h, \hat{u} - \hat{u}_h, \hat{\lambda} - \hat{\lambda}_{\tilde{h}}), (\hat{\mathbf{q}} - \mathbf{q}_h, \hat{u} - u_h, \hat{\lambda} - \lambda_{\tilde{h}})) \\ &\quad + \mathcal{A}((\hat{\mathbf{q}} - \hat{\mathbf{q}}_h, \hat{u} - \hat{u}_h, \hat{\lambda} - \hat{\lambda}_{\tilde{h}}), (\mathbf{q}_h - \hat{\mathbf{q}}_h, u_h - \hat{u}_h, \lambda_{\tilde{h}} - \hat{\lambda}_{\tilde{h}})) \\ &\le \|(\hat{\mathbf{q}} - \hat{\mathbf{q}}_h, \hat{u} - \hat{u}_h, \hat{\lambda} - \hat{\lambda}_{\tilde{h}})\|_Y \; \|(\hat{\mathbf{q}} - \mathbf{q}_h, \hat{u} - u_h, \hat{\lambda} - \lambda_{\tilde{h}})\|_Y \\ &\quad + \mathcal{A}((\hat{\mathbf{q}} - \hat{\mathbf{q}}_h, \hat{u} - \hat{u}_h, \hat{\lambda} - \hat{\lambda}_{\tilde{h}}), (\mathbf{q}_h - \hat{\mathbf{q}}_h, u_h - \hat{u}_h, \lambda_{\tilde{h}} - \hat{\lambda}_{\tilde{h}})). \end{aligned} \tag{4.18}$$

Then, using (4.15) and (4.17), we obtain

$$\begin{aligned} &\mathcal{A}\left(\left(\hat{\mathbf{q}} - \hat{\mathbf{q}}_h, \hat{u} - \hat{u}_h, \hat{\lambda} - \hat{\lambda}_{\tilde{h}}\right), \left(\mathbf{q}_h - \hat{\mathbf{q}}_h, u_h - \hat{u}_h, \lambda_{\tilde{h}} - \hat{\lambda}_{\tilde{h}}\right)\right) \\ &\quad = -d(\hat{\mathbf{q}} - \hat{\mathbf{q}}_h, \lambda_{\tilde{h}} - \hat{\lambda}_{\tilde{h}}) - \langle (\mathbf{q}_h - \hat{\mathbf{q}}_h) \cdot \mathbf{n}, \delta_{r,h}(t_0 - \hat{\mathbf{q}}_h \cdot \mathbf{n})\rangle. \end{aligned} \tag{4.19}$$

Since $\mathrm{div}(\mathbf{q}_h - \hat{\mathbf{q}}_h) \in L_h$ and $(\mathrm{div}(\hat{\mathbf{q}} - \hat{\mathbf{q}}_h), u_h)_{L^2(\Omega)} = 0$ for all $u_h \in L_h$, we deduce that

$$\|\,\mathrm{div}(\hat{\mathbf{q}} - \hat{\mathbf{q}}_h)\|_{L^2(\Omega)} \le \|\,\mathrm{div}(\hat{\mathbf{q}} - \mathbf{q}_h)\|_{L^2(\Omega)} \le \|\hat{\mathbf{q}} - \mathbf{q}_h\|_{H(\mathrm{div};\Omega)} \quad \forall \mathbf{q}_h \in H_h. \tag{4.20}$$

Finally, combining (4.18), (4.19), (4.20) and (4.13), and applying the generalized Cauchy-Schwarz inequality, we arrive at (4.14). □

The main result of this section is established as follows.

Theorem 4.1. *Let* $(\hat{\mathbf{q}}, \hat{u}, \hat{\lambda})$ *and* $(\hat{\mathbf{q}}_h, \hat{u}_h, \hat{\lambda}_{\tilde{h}})$ *be the solutions of problems* (M) *and* (M_h)*, respectively. Define* $\hat{\phi} := W^{-1}(I + K')(t_0 - \hat{\mathbf{q}} \cdot \mathbf{n})$*. Then there exists* $c > 0$*, independent of* h *and* $\tilde{h}$*, such that the following Céa type estimate holds*

$$\begin{aligned}
&\|\hat{\mathbf{q}} - \hat{\mathbf{q}}_h\|_{H(\mathrm{div};\Omega)} + \|\hat{u} - \hat{u}_h\|_{L^2(\Omega)} + \|\hat{\lambda} - \hat{\lambda}_{\tilde{h}}\|_{\tilde{H}^{1/2}(\Gamma_s)} \\
&\leq \quad c \left\{ \inf_{\mathbf{q}_h \in H_h} \|\hat{\mathbf{q}} - \mathbf{q}_h\|_{H(\mathrm{div};\Omega)} + \inf_{u_h \in L_h} \|\hat{u} - u_h\|_{L^2(\Omega)} + \inf_{\lambda_{\tilde{h}} \in H^{1/2}_{s,+,\tilde{h}}} \|\hat{\lambda} - \lambda_{\tilde{h}}\|^{1/2}_{\tilde{H}^{1/2}(\Gamma_s)} \right. \\
&\qquad \left. + \inf_{\phi_h \in H^{1/2}_h} \|\hat{\phi} - \phi_h\|_{H^{1/2}(\Gamma)} \right\}. \qquad (4.21)
\end{aligned}$$

Proof. We apply the estimates provided in Lemmas 4.2 and 4.1. Due to (4.7) and (3.15), the term $-d(\hat{\mathbf{q}} - \hat{\mathbf{q}}_h, \lambda_{\tilde{h}} - \hat{\lambda}_{\tilde{h}})$ given in Lemma 4.2 is estimated as follows

$$\begin{aligned}
-\, d(\hat{\mathbf{q}} - \hat{\mathbf{q}}_h, \lambda_{\tilde{h}} - \hat{\lambda}_{\tilde{h}}) \quad &= \quad d(\hat{\mathbf{q}}, \hat{\lambda}_{\tilde{h}} - \lambda_{\tilde{h}}) + d(\hat{\mathbf{q}}_h, \lambda_{\tilde{h}} - \hat{\lambda}_{\tilde{h}}) \\
\leq\ d(\hat{\mathbf{q}}, \hat{\lambda}_{\tilde{h}} - \lambda_{\tilde{h}}) \quad &= \quad d(\hat{\mathbf{q}}, \hat{\lambda}_{\tilde{h}} - \hat{\lambda}) \ +\ d(\hat{\mathbf{q}}, \hat{\lambda} - \lambda_{\tilde{h}}) \ \leq\ d(\hat{\mathbf{q}}, \hat{\lambda} - \lambda_{\tilde{h}}).
\end{aligned}$$

Then we estimate the term $d(\hat{\mathbf{q}}, \hat{\lambda} - \lambda_{\tilde{h}})$ by $\|\hat{\mathbf{q}}\|_{H(\mathrm{div};\Omega)} \|\hat{\lambda} - \lambda_{\tilde{h}}\|_{\tilde{H}^{1/2}(\Gamma_s)}$ and note that $\|\hat{\mathbf{q}}\|_{H(\mathrm{div};\Omega)}$ is bounded due to Theorem 3.3. Due to Lemma 4.1 the term $\|\delta_{r,h}(t_0 - \hat{\mathbf{q}}_h \cdot \mathbf{n})\|^2_{H^{1/2}(\Gamma)}$ is bounded by $c_0 \Big\{ \inf_{\phi_h \in H^{1/2}_h} \|W^{-1}(I + K')$ $(t_0 - \hat{\mathbf{q}} \cdot \mathbf{n}) - \phi_h\|_{H^{1/2}(\Gamma)} + \|\hat{\mathbf{q}} \cdot \mathbf{n} - \hat{\mathbf{q}}_h \cdot \mathbf{n}\|_{H^{-1/2}(\Gamma)} \Big\}$. Then, we can estimate

$$\begin{aligned}
&-\langle (\mathbf{q}_h - \hat{\mathbf{q}}_h) \cdot \mathbf{n}, \delta_{r,h}(t_0 - \hat{\mathbf{q}}_h \cdot \mathbf{n}) \rangle \\
&\quad = \quad -\langle \mathbf{q}_h \cdot \mathbf{n} - \hat{\mathbf{q}} \cdot \mathbf{n} + \hat{\mathbf{q}} \cdot \mathbf{n} - \hat{\mathbf{q}}_h \cdot \mathbf{n}, \delta_{r,h}(t_0 - \hat{\mathbf{q}} \cdot \mathbf{n} + \hat{\mathbf{q}} \cdot \mathbf{n} - \hat{\mathbf{q}}_h \cdot \mathbf{n}) \rangle \\
&\quad = \quad -\langle (\mathbf{q}_h - \hat{\mathbf{q}}) \cdot \mathbf{n}, \delta_{r,h}(t_0 - \hat{\mathbf{q}} \cdot \mathbf{n}) \rangle - \langle (\mathbf{q}_h - \hat{\mathbf{q}}) \cdot \mathbf{n}, \delta_{r,h}(\hat{\mathbf{q}} \cdot \mathbf{n} - \hat{\mathbf{q}}_h \cdot \mathbf{n}) \rangle \\
&\qquad\quad -\langle (\hat{\mathbf{q}} - \hat{\mathbf{q}}_h) \cdot \mathbf{n}, \delta_{r,h}(t_0 - \hat{\mathbf{q}} \cdot \mathbf{n}) \rangle - \langle (\hat{\mathbf{q}} - \hat{\mathbf{q}}_h) \cdot \mathbf{n}, \delta_{r,h}(\hat{\mathbf{q}} \cdot \mathbf{n} - \hat{\mathbf{q}}_h \cdot \mathbf{n}) \rangle \\
&\quad \leq \quad \frac{1}{2} \|\hat{\mathbf{q}} - \mathbf{q}_h\|^2_{H(\mathrm{div};\Omega)} + \frac{1}{2} \|\delta_{r,h}(t_0 - \hat{\mathbf{q}} \cdot \mathbf{n})\|^2_{H^{1/2}(\Gamma)} \\
&\qquad\quad + \frac{1}{2\epsilon} \|\hat{\mathbf{q}} - \mathbf{q}_h\|^2_{H(\mathrm{div};\Omega)} + \frac{\epsilon}{2} \|\delta_{r,h}(\hat{\mathbf{q}} \cdot \mathbf{n} - \hat{\mathbf{q}}_h \cdot \mathbf{n})\|^2_{H^{1/2}(\Gamma)} \\
&\qquad\quad + \frac{\epsilon}{2} \|\hat{\mathbf{q}} - \hat{\mathbf{q}}_h\|^2_{H(\mathrm{div};\Omega)} + \frac{1}{2\epsilon} \|\delta_{r,h}(t_0 - \hat{\mathbf{q}} \cdot \mathbf{n})\|^2_{H^{1/2}(\Gamma)}.
\end{aligned}$$

Note, due to Lemma 4.1 we have $-\langle (\hat{\mathbf{q}} - \hat{\mathbf{q}}_h) \cdot \mathbf{n}, \delta_{r,h}(\hat{\mathbf{q}} \cdot \mathbf{n} - \hat{\mathbf{q}}_h \cdot \mathbf{n}) \rangle \leq 0$. Finally, after choosing a suitable $\epsilon > 0$ we take the infimum. □

5. Numerical approximation

In this section we apply the general approximation theory of the previous section to a particular choice of finite dimensional subspaces. In order to define them, we first consider an element T of a regular triangulation $\mathcal{T}_h$ of the domain Ω, and for each integer $k \geq 0$, we denote by $P^k(T)$ the space of polynomial functions of degree $\leq k$ on T. A similar definition holds for $P^k(e)$ where e is an edge of $\mathcal{T}_h$. We define the lowest-order Raviart-Thomas space

$$RT_0(\mathcal{T}_h) := \{ \mathbf{q} \in H(\mathrm{div};\Omega) \,:\, \mathbf{q}|_T \in [P^0(T)]^n + \mathbf{x}\, P^0(T) \quad \forall T \in \mathcal{T}_h \}$$

where $\mathbf{x} = (x_1, \ldots, x_n)^{\mathbf{T}}$. Then, we define the subspaces

$$\begin{aligned}
H_h &:= RT_0(\mathcal{T}_h),\\
L_h &:= \{v_h \in L^2(\Omega): \quad v_h|_{\bar{T}} \in P^0(T) \quad \forall T \in \mathcal{T}_h\},\\
H^{1/2}_{s,\tilde{h}} &:= \{\lambda_{\tilde{h}} \in C(\Gamma_s): \quad \lambda_{\tilde{h}}|_e \in P^1(e) \quad \forall e \in \tau_{\tilde{h}}\},\\
H^{-1/2}_h &:= \{\psi_h \in L^2(\Gamma): \quad \psi_h|_e \in P^0(e) \quad \forall e \in \mathcal{G}_h\},\\
H^{-1/2}_{s,h} &:= \{\psi_h \in L^2(\Gamma_s): \quad \psi_h|_e \in P^0(e) \quad \forall e \in \mathcal{G}_h\}.
\end{aligned}$$

Note, $H^{-1/2}_h$ is the space of normal traces on Γ of the functions in H_h, and $H^{-1/2}_{s,h}$ is the space of normal traces on Γ_s of the functions in H_h.

In order to obtain the following inf-sup condition for the bilinear form $B(\cdot,(\cdot,\cdot))$ we must impose some additional restrictions on the discrete spaces which are the direct 3d analogues of the assumptions for the 2d case treated in [2].

Lemma 5.1. *Let $(\mathcal{T}_h)_h$ be uniformly regular near Γ_s, which means that there exists $C > 0$, independent of h, such that $\min\{h_T : T \in \mathcal{T}_h,\ \bar{T} \cap \Gamma_s \neq \phi\} \geq C\,h$. Furthermore, let the partition $(\tau_{\tilde{h}})$ of Γ_s be uniformly regular, that is there exists $C > 0$ such that $\min\{h_\Delta : \Delta \in \tau_{\tilde{h}}\} \geq C\tilde{h}$. Then there exist constants $C_0, \beta > 0$, independent of h and $\tilde{h}$, such that for all $h \leq C_0\tilde{h}$*

$$\inf_{(u_h,\lambda_{\tilde{h}}) \in L_h \times H^{1/2}_{s,\tilde{h}} \setminus \{0\}} \ \sup_{\mathbf{q}_h \in H_h \setminus \{0\}} \frac{B(\mathbf{q}_h,(u_h,\lambda_{\tilde{h}}))}{\|\mathbf{q}_h\|_{H(\mathrm{div};\Omega)} \|(u_h,\lambda_{\tilde{h}})\|_{L^2(\Omega)\times \tilde{H}^{1/2}(\Gamma_s)}} \geq \beta > 0. \tag{5.1}$$

Proof. The assertion is shown by generalizing the results in [2], covering the 2d situation, to the 3d situation given here. We omit the details for brevity. □

This section is completed with a result on the rate of convergence of the nonconforming Galerkin scheme (M_h). For this, we need the following approximation properties of the subspaces L_h, H_h, $H^{-1/2}_h$, $H^{1/2}_h$, and $H^{1/2}_{s,\tilde{h}}$, respectively (see, *e.g.* [1,3,18]):

For all $v \in H^1(\Omega)$ there exists $v_h \in L_h$ such that

$$\|v - v_h\|_{L^2(\Omega)} \leq C\,h\,\|v\|_{H^1(\Omega)}. \tag{5.2}$$

For all $\mathbf{q} \in [H^1(\Omega)]^n$ with $\operatorname{div}\mathbf{q} \in H^1(\Omega)$, there exists $\mathbf{q}_h \in H_h$ such that

$$\|\mathbf{q} - \mathbf{q}_h\|_{H(\mathrm{div};\Omega)} \leq C\,h\,\left\{\|\mathbf{q}\|_{[H^1(\Omega)]^n} + \|\operatorname{div}\mathbf{q}\|_{H^1(\Omega)}\right\}. \tag{5.3}$$

For all $\psi \in H^{1/2}(\Gamma)$, there exists $\psi_h \in H^{-1/2}_h$ such that

$$\|\psi - \psi_h\|_{H^{-1/2}(\Gamma)} \leq C\,h\,\|\psi\|_{H^{1/2}(\Gamma)}. \tag{5.4}$$

For all $\phi \in H^{3/2}(\Gamma)$, there exists $\phi_h \in H^{1/2}_h$ such that

$$\|\phi - \phi_h\|_{H^{1/2}(\Gamma)} \leq C\,h\,\|\phi\|_{H^{3/2}(\Gamma)}. \tag{5.5}$$

For all $\lambda \in \tilde{H}^{3/2}(\Gamma_s)$, there exists $\lambda_{\tilde{h}} \in H^{1/2}_{s,\tilde{h}}$ such that

$$\|\lambda - \lambda_{\tilde{h}}\|_{\tilde{H}^{1/2}(\Gamma_s)} \leq C\,\tilde{h}\,\|\lambda\|_{\tilde{H}^{3/2}(\Gamma_s)}. \tag{5.6}$$

Theorem 5.1. *Let* $(\hat{\mathbf{q}}, \hat{u}, \hat{\lambda})$ *and* $(\hat{\mathbf{q}}_h, \hat{u}_h, \hat{\lambda}_{\tilde{h}})$ *be the solutions of problems* (M) *and* (M_h)*, respectively. Assume that* $\hat{\mathbf{q}} \in [H^1(\Omega)]^n$, $\operatorname{div}\hat{\mathbf{q}} \in H^1(\Omega)$, $\hat{u} \in L^2(\Omega)$, $\hat{\lambda} \in \tilde{H}^{3/2}(\Gamma_s)$, $\hat{\phi} := W^{-1}(I + K')(t_0 - \hat{\mathbf{q}} \cdot \mathbf{n}) \in H^{3/2}(\Gamma)$*. Then the following* a priori *error estimate holds*

$$\begin{aligned} &\|\hat{\mathbf{q}} - \hat{\mathbf{q}}_h\|_{H(\operatorname{div};\Omega)} + \|\hat{u} - \hat{u}_h\|_{L^2(\Omega)} + \|\hat{\lambda} - \hat{\lambda}_{\tilde{h}}\|_{\tilde{H}^{1/2}(\Gamma_s)} \\ &\leq C\,h \left\{ \|\hat{\mathbf{q}}\|_{[H^1(\Omega)]^n} + \|\operatorname{div}\hat{\mathbf{q}}\|_{H^1(\Omega)} + \|\hat{u}\|_{H^1(\Omega)} + \|\hat{\phi}\|_{H^{3/2}(\Gamma)} \right\} + C\,\tilde{h}^{1/2}\,\|\hat{\lambda}\|^{1/2}_{\tilde{H}^{3/2}(\Gamma_s)} \end{aligned}$$

with a constant C *independent of* h *and* $\tilde{h}$*, with* $h \leq C_0 \tilde{h}$*.*

Proof. The estimates for $\hat{\mathbf{q}}$, $\hat{u}$, and $\hat{\lambda}$ follow directly from Theorem 4.1, the assumed regularity of the solution of (M) and the approximation properties (5.2), (5.3), (5.5) and (5.6). □

6. Solvers for FEM-BEM dual-mixed coupling problems

As solver for the discretized saddle point problem (M_h) we take a modified Uzawa algorithm that utilizes the equation for the Lagrange multiplier u. First, we introduce a projector $P_{\tilde{h}} : H^{1/2}_{s,\tilde{h}} \to H^{1/2}_{s,+,\tilde{h}}$ and a linear mapping $\Phi : H(\operatorname{div};\Omega) \to H^{1/2}_{s,\tilde{h}} \subset \tilde{H}^{1/2}(\Gamma_s)$, which satisfy for given $\lambda \in H^{1/2}_{s,\tilde{h}}$ and $\mathbf{q} \in H(\operatorname{div};\Omega)$

$$(P_{\tilde{h}}\lambda - \lambda, \lambda_{\tilde{h}} - P_{\tilde{h}}\lambda)_{\tilde{H}^{1/2}(\Gamma_s)} \geq 0 \qquad \forall \lambda_{\tilde{h}} \in H^{1/2}_{s,+,\tilde{h}}, \tag{6.1}$$

and

$$d(\mathbf{q}, \lambda_{\tilde{h}}) = (\lambda_{\tilde{h}}, \Phi(\mathbf{q}))_{\tilde{H}^{1/2}(\Gamma_s)} \qquad \forall \lambda_{\tilde{h}} \in H^{1/2}_{s,\tilde{h}}. \tag{6.2}$$

Therefore

$$\|\Phi(\mathbf{q})\|^2_{\tilde{H}^{1/2}(\Gamma_s)} = d(\mathbf{q}, \Phi(\mathbf{q})) = \langle \mathbf{q}\cdot\mathbf{n}, \Phi(\mathbf{q})\rangle_{\Gamma_s} \leq \|\mathbf{q}\cdot\mathbf{n}\|_{H^{-1/2}(\Gamma_s)}\,\|\Phi(\mathbf{q})\|_{\tilde{H}^{1/2}(\Gamma_s)},$$

yielding

$$\|\Phi(\mathbf{q})\|_{\tilde{H}^{1/2}(\Gamma_s)} \leq \|\mathbf{q}\cdot\mathbf{n}\|_{H^{-1/2}(\Gamma_s)} \leq \|\mathbf{q}\|_{H(\operatorname{div};\Omega)} \qquad \forall \mathbf{q} \in H(\operatorname{div};\Omega), \tag{6.3}$$

which proves the continuity of the linear mapping Φ.

Algorithm 6.1 (modified Uzawa)**.**

1. *Choose an initial* $\lambda^{(0)} \in H^{1/2}_{s,+,\tilde{h}}$.
2. *Given* $\lambda^{(n)} \in H^{1/2}_{s,+,\tilde{h}}$*, find* $(\mathbf{q}^{(n)}, u^{(n)}) \in H_h \times L_h$ *such that*

$$\begin{aligned} a_h(\mathbf{q}^{(n)}, \mathbf{q}_h) + b(\mathbf{q}_h, u^{(n)}) &= \langle r_h, \mathbf{q}_h\cdot\mathbf{n}\rangle - d(\mathbf{q}_h, \lambda^{(n)}) \quad \forall \mathbf{q}_h \in H_h, &(6.4)\\ b(\mathbf{q}^{(n)}, u_h) &= -(f, u_h)_{L^2(\Omega)} \quad \forall u_h \in L_h. &(6.5) \end{aligned}$$

3. *Compute* $\lambda^{(n+1)}$ *by*

$$\lambda^{(n+1)} = P_{\tilde{h}}(\lambda^{(n)} + \varrho\,\Phi(\mathbf{q}^{(n)})). \tag{6.6}$$

4. *Check for some stopping criterion and if it is not fulfilled, then go to step* 2.

The convergence of this algorithm is established in the following theorem.

Theorem 6.1. *With* $0 < \varrho < 2$ *the modified Uzawa algorithm converges towards the solution of the discrete problem* (M_h) *for arbitrary starting value* $\lambda_0 \in H^{1/2}_{s,+,\tilde{h}}$.

Proof. Let $(\hat{\mathbf{q}}_h, \hat{u}_h, \hat{\lambda}_{\tilde{h}}) \in H_h \times L_h \times H^{1/2}_{s,+,\tilde{h}}$ solve problem (M_h). Then (4.7) and (6.2) give

$$(\lambda_{\tilde{h}} - \hat{\lambda}_{\tilde{h}}, \hat{\lambda}_{\tilde{h}} - (\hat{\lambda}_{\tilde{h}} + \varrho\Phi(\hat{\mathbf{q}}_h)))_{\tilde{H}^{1/2}(\Gamma_s)} \geq 0 \quad \forall \lambda_{\tilde{h}} \in H^{1/2}_{s,+,\tilde{h}},$$

which means that

$$\hat{\lambda}_{\tilde{h}} = P_{\tilde{h}}(\hat{\lambda}_{\tilde{h}} + \varrho\,\Phi(\hat{\mathbf{q}}_h)). \tag{6.7}$$

Therefore, it follows that

$$\|\lambda^{(n+1)} - \hat{\lambda}_{\tilde{h}}\|^2_{\tilde{H}^{1/2}(\Gamma_s)} = \|P_{\tilde{h}}(\lambda^{(n)} + \varrho\,\Phi(\mathbf{q}^{(n)})) - P_{\tilde{h}}(\hat{\lambda}_{\tilde{h}} + \varrho\,\Phi(\hat{\mathbf{q}}_h))\|^2_{\tilde{H}^{1/2}(\Gamma_s)}$$
$$\leq \|\lambda^{(n)} - \hat{\lambda}_{\tilde{h}}\|^2_{\tilde{H}^{1/2}(\Gamma_s)} + 2\,\varrho\, d(\mathbf{q}^{(n)} - \hat{\mathbf{q}}_h, \lambda^{(n)} - \hat{\lambda}_{\tilde{h}}) + \varrho^2\,\|\Phi(\mathbf{q}^{(n)}) - \Phi(\hat{\mathbf{q}}_h)\|^2_{\tilde{H}^{1/2}(\Gamma_s)}.$$

Then, using (4.5) and (6.4), we obtain

$$d(\mathbf{q}^{(n)} - \hat{\mathbf{q}}_h, \lambda^{(n)} - \hat{\lambda}_{\tilde{h}}) = -\,a_h(\mathbf{q}^{(n)} - \hat{\mathbf{q}}_h, \mathbf{q}^{(n)} - \hat{\mathbf{q}}_h),$$

and due to (4.6) and (6.5) we have for $\mathbf{q}^{(n)} - \hat{\mathbf{q}}_h \in H_h$ that

$$(\mathrm{div}(\mathbf{q}^{(n)} - \hat{\mathbf{q}}_h), u_h)_{L^2(\Omega)} = 0 \quad \forall u_h \in L_h,$$

which gives $\mathrm{div}(\mathbf{q}^{(n)} - \hat{\mathbf{q}}_h) = 0$. Thus, there holds

$$d(\mathbf{q}^{(n)} - \hat{\mathbf{q}}_h, \lambda^{(n)} - \hat{\lambda}_{\tilde{h}}) = -\,a_h(\mathbf{q}^{(n)} - \hat{\mathbf{q}}_h, \mathbf{q}^{(n)} - \hat{\mathbf{q}}_h) \leq -\,\|\mathbf{q}^{(n)} - \hat{\mathbf{q}}_h\|^2_{L^2(\Omega)} = -\,\|\mathbf{q}^{(n)} - \hat{\mathbf{q}}_h\|^2_{H(\mathrm{div};\Omega)}.$$

Together with (6.3) this yields

$$\|\lambda^{(n+1)} - \hat{\lambda}_{\tilde{h}}\|^2_{\tilde{H}^{1/2}(\Gamma_s)} \leq \|\lambda^{(n)} - \hat{\lambda}_{\tilde{h}}\|^2_{\tilde{H}^{1/2}(\Gamma_s)} - \varrho\,(2-\varrho)\,\|\mathbf{q}^{(n)} - \hat{\mathbf{q}}_h\|^2_{H(\mathrm{div};\Omega)}. \tag{6.8}$$

Thus $\|\lambda^{(n)} - \hat{\lambda}_{\tilde{h}}\|^2_{\tilde{H}^{1/2}(\Gamma_s)}$ is monotonically decreasing and bounded from below, and therefore convergent. Hence, we get

$$0 \leq \varrho\,(2-\varrho)\,\|\mathbf{q}^{(n)} - \hat{\mathbf{q}}_h\|^2_{H(\mathrm{div};\Omega)} \leq \|\lambda^{(n)} - \hat{\lambda}_{\tilde{h}}\|^2_{\tilde{H}^{1/2}(\Gamma_s)} - \|\lambda^{(n+1)} - \hat{\lambda}_{\tilde{h}}\|^2_{\tilde{H}^{1/2}(\Gamma_s)},$$

which goes to 0 as $n \to \infty$, and therefore

$$\lim_{n\to\infty} \|\mathbf{q}^{(n)} - \hat{\mathbf{q}}_h\|^2_{H(\mathrm{div};\Omega)} = 0. \tag{6.9}$$

Now, from (4.5) and (6.4) we have

$$B(\mathbf{q}_h, (u^{(n)} - \hat{u}_h, \lambda^{(n)} - \hat{\lambda}_{\tilde{h}})) = a_h(\hat{\mathbf{q}}_h - \mathbf{q}^{(n)}, \mathbf{q}_h) \quad \forall \mathbf{q}_h \in H_h.$$

Applying the discrete inf-sup condition (5.1) for B we obtain

$$\beta\,\|(u^{(n)}, \lambda^{(n)}) - (\hat{u}_h, \hat{\lambda}_{\tilde{h}})\|_{L^2(\Omega)\times\tilde{H}^{1/2}(\Gamma_s)} \leq \sup_{\mathbf{q}_h \in H_h\setminus\{0\}} \frac{B(\mathbf{q}_h, (u^{(n)} - \hat{u}_h, \lambda^{(n)} - \hat{\lambda}_{\tilde{h}}))}{\|\mathbf{q}_h\|_{H(\mathrm{div};\Omega)}}$$

$$= \sup_{\mathbf{q}_h \in H_h\setminus\{0\}} \frac{a_h(\hat{\mathbf{q}}_h - \mathbf{q}^{(n)}, \mathbf{q}_h)}{\|\mathbf{q}_h\|_{H(\mathrm{div};\Omega)}} \leq C\,\|\hat{\mathbf{q}}_h - \mathbf{q}^{(n)}\|_{H(\mathrm{div};\Omega)},$$

which shows with (6.9) the claimed convergence of $(u^{(n)}, \lambda^{(n)})$. □

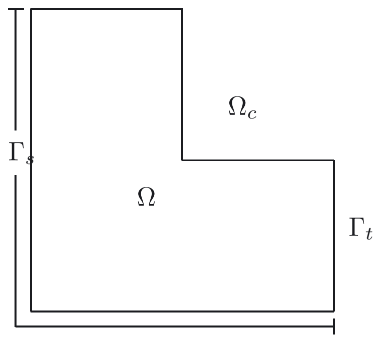

FIGURE 1. The model problem, geometry and interface conditions.

We remark that the operators $P_{\tilde{h}}$ and Φ are defined with respect to the scalar product of $\tilde{H}^{1/2}(\Gamma_s)$, which is not practical from the computational point of view. Fortunately, inspection of the proof of Theorem 6.1 shows that it suffices that the norm induced by the scalar product used in the algorithm be equivalent to the $\tilde{H}^{1/2}(\Gamma_s)$-norm. Therefore we can use the bilinear form $\langle W\cdot,\cdot\rangle$ instead of the scalar product $(\cdot,\cdot)_{\tilde{H}^{1/2}(\Gamma_s)}$. Then, we have to solve: Find $P_{\tilde{h}}\lambda \in H^{1/2}_{s,+,\tilde{h}}$ such that

$$\langle WP_{\tilde{h}}\lambda, \lambda_{\tilde{h}} - P_{\tilde{h}}\lambda\rangle \;\geq\; \langle W\lambda, \lambda_{\tilde{h}} - P_{\tilde{h}}\lambda\rangle \qquad \forall \lambda_{\tilde{h}} \,\in\, H^{1/2}_{s,+,\tilde{h}}, \tag{6.10}$$

and find $\Phi(\mathbf{q}) \in H^{1/2}_{s,\tilde{h}}$ such that

$$\langle W\Phi(\mathbf{q}), \lambda_{\tilde{h}}\rangle \;=\; d(\mathbf{q}, \lambda_{\tilde{h}}) \qquad \forall \lambda_{\tilde{h}} \,\in\, H^{1/2}_{s,\tilde{h}}. \tag{6.11}$$

Both systems are small compared with the total size of the problem, because they are only defined on the Signorini part Γ_s of the interface Γ. Applying (6.11) to (6.10) we obtain for $\lambda \,=\, \lambda^{(n)} \,+\, \varrho\,\Phi(\mathbf{q}^{(n)})$,

$$\langle W\lambda, \lambda_{\tilde{h}} - P_{\tilde{h}}\lambda\rangle \quad=\quad \langle W\lambda^{(n)}, \lambda_{\tilde{h}} \,-\, P_{\tilde{h}}\lambda\rangle \,+\, \varrho\, d(\mathbf{q}^{(n)}, \lambda_{\tilde{h}} - P_{\tilde{h}}\lambda),$$

and hence, the explicit solution of (6.11) is avoided.

7. NUMERICAL EXAMPLES

In the following we present a numerical example for the interface problem with Signorini interface conditions. The geometry is the L-Shaped domain shown in Figure 1 with corners $(0,0), (0,0.25), (-0.25,0.25), (-0.25,-0.25), (0.25,-0.25), (0.25,0)$. We take $\boldsymbol{\kappa} \equiv I_{2\times 2}$ and vanishing body forces, *i.e.* $f = 0$ and Signorini conditions on both large edges of the L-Shape.

Here we are using rectangular elements and it was necessary to use $\tilde{h} = 2h$ for $H^{1/2}_{s,\tilde{h}}$ to fulfill the requirement $h \leq C_0\tilde{h}$ of the discrete Babuška-Brezzi condition in Lemma 5.1. The numerical solutions of the discrete problem (M_h) are computed with the modification (6.10) of the Uzawa algorithm 6.1. Then, Tables 1 and 2 give the numbers of outer iterations $It_{\rm Uz}$ for the Uzawa algorithm with $\varrho = 2.5$. We notice that the numbers of outer iterations are nearly independent of $\dim H^{1/2}_{s,\tilde{h}}$. Note also, that $\dim H^{1/2}_{s,\tilde{h}}$ is very small compared with the total size of the problem (M_h). For solving the contact problem on $H^{1/2}_{s,\tilde{h}}$ we use the Polyak algorithm, iterations are given by $It_{\rm Cont}$. The inner linear systems are solved with the MINRES algorithm. The iteration numbers are given by $It_{\rm Int}$.

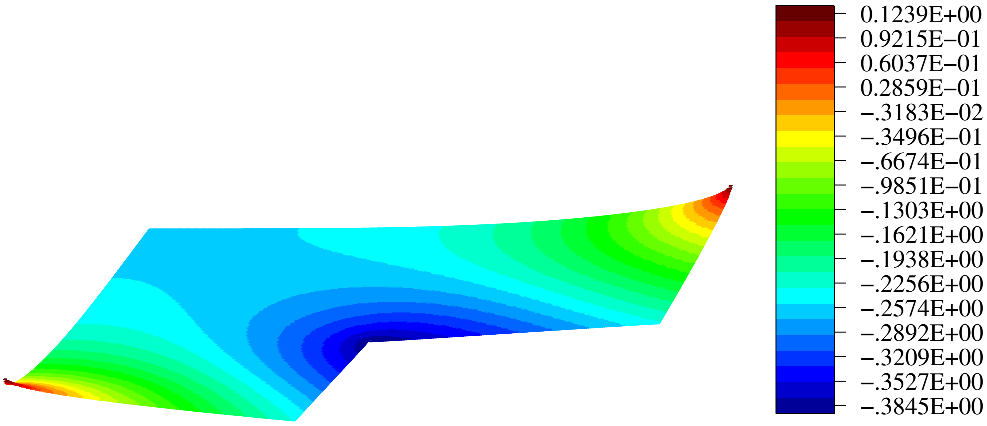

FIGURE 2. Displacement u (Ex. 7.1). (Figure in color available online at www.esaim-m2an.org.)

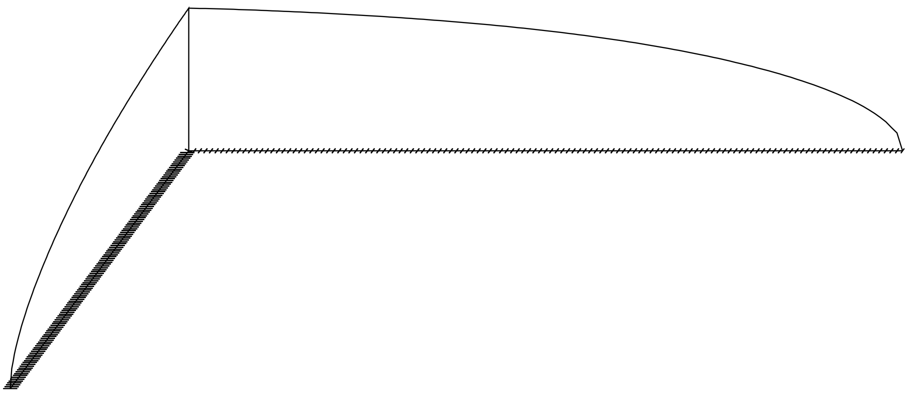

FIGURE 3. Gap variable λ (Ex. 7.1).

TABLE 1. Iteration numbers and convergence rates for the Uzawa algorithm (Ex. 7.1).

dim H_h	dim L_h	dim $H^{1/2}_{s,\tilde{h}}$	It_{Uz}	It_{Int}	It_{Cont}	$\tilde{\psi}(\mathbf{q}_h)$	$\delta\psi$	α	Time (s)
32	12	3	23	29	3	−0.594833896	0.3044368		0.02800
112	48	7	22	90	5	−0.570673630	0.2106211	−0.291	0.04800
416	192	15	21	219	9	−0.559802953	0.1503992	−0.256	0.20001
1600	768	31	21	556	14	−0.554313982	0.1078979	−0.246	1.66410
6272	3072	63	20	1502	20	−0.551465713	0.0771066	−0.246	18.5132
24 832	12 288	127	20	4157	30	−0.549998703	0.0548763	−0.247	234.755
98 816	49 152	255	19	11 919	44	−0.549255639	0.0390548	−0.246	3040.01

We use the values of the quadratic functional $\tilde{\Psi}(\mathbf{q}_h)$ to compute a convergence rate for our example (the minimizer of $\tilde{\Psi}$ is the solution of the problem). We determine an approximation for the minimum of $\tilde{\Psi}$ by extrapolating the values for $\tilde{\Psi}(\mathbf{q}_h)$. We measure the error by $\delta\psi := \sqrt{2|\tilde{\psi}(\mathbf{q}_{ex}) - \tilde{\psi}(\mathbf{q}_h)|}$.

TABLE 2. Iteration numbers and convergence rates for the Uzawa algorithm (Ex. 7.2).

dim H_h	dim L_h	dim $H^{1/2}_{s,\tilde{h}}$	It_{Uz}	It_{Int}	It_{Cont}	$\tilde{\psi}(\mathbf{q}_h)$	$\delta\psi$	α	Time (s)
32	12	3	23	29	2	–0.171814118	0.1763513		0.03200
112	48	7	21	90	7	–0.181108322	0.1118542	–0.359	0.04000
416	192	15	20	218	21	–0.185091190	0.0674212	–0.385	0.14801
1600	768	31	19	552	33	–0.186498230	0.0416118	–0.358	1.16807
6272	3072	63	18	1497	51	–0.187083558	0.0236830	–0.412	14.1249
24 832	12 288	127	18	4174	127	–0.187270910	0.0136448	–0.401	182.475
98 816	49 152	255	18	11 760	183	–0.187333101	0.0078612	–0.399	2476.43

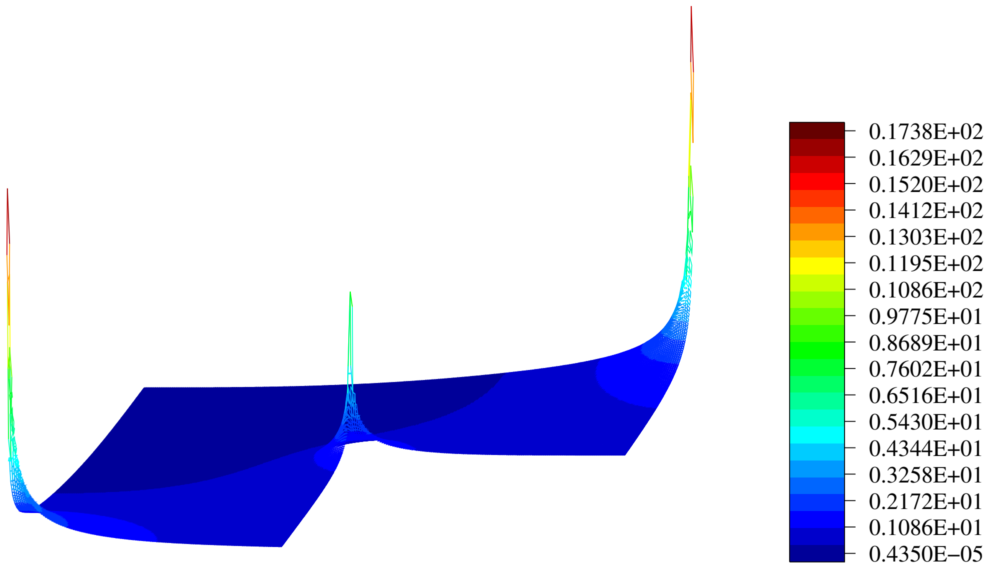

FIGURE 4. Stress q (Ex. 7.1). (Figure in color available online at `www.esaim-m2an.org`.)

Example 7.1. In this first example we choose $u_0 = r^{2/3} \sin \frac{2}{3}(\varphi - \frac{\pi}{2})$ and $t_0 = \frac{\partial}{\partial \mathbf{n}} u_0$. Then there holds $\Delta u_0 = 0$, and hence

$$0 \;=\; (\Delta u_0, 1)_{L^2(\Omega)} \;=\; -\,(\nabla u_0, \nabla 1)_{[L^2(\Omega)]^n} \;+\; \left\langle 1, \frac{\partial}{\partial \mathbf{n}}\, u_0 \right\rangle,$$

that is $\langle 1, t_0 \rangle = 0$, whence the corresponding uniqueness requirement (1.6) is fulfilled. The extrapolated value for $\tilde{\Psi}$ is $\tilde{\Psi}(\mathbf{q}_{ex}) = -0.548493$.

Furthermore, Table 1 gives the experimental convergence rates $\alpha \sim -1/4$ which corresponds to $h^{1/2}$ (the theoretical result of Thm. 5.1) due to $N \sim h^{-2}$. We notice also that the number of Uzawa iteration is bounded (see Thm. 6.1). Figure 3 shows that the gap variable λ is non-zero, *i.e.* in this example there is no actual unilateral contact. The approximated displacements and stresses are shown in Figures 2 and 4, respectively.

Example 7.2. In this second example we choose $u_0 = -2r^{2/3} \sin \frac{2}{3}(\varphi - \frac{\pi}{2})$ and $t_0 = \frac{\partial}{\partial \mathbf{n}} u_0$. Then there holds again $\Delta u_0 = 0$, and hence

$$0 \;=\; (\Delta u_0, 1)_{L^2(\Omega)} \;=\; -\,(\nabla u_0, \nabla 1)_{[L^2(\Omega)]^n} \;+\; \left\langle 1, \frac{\partial}{\partial \mathbf{n}}\, u_0 \right\rangle,$$

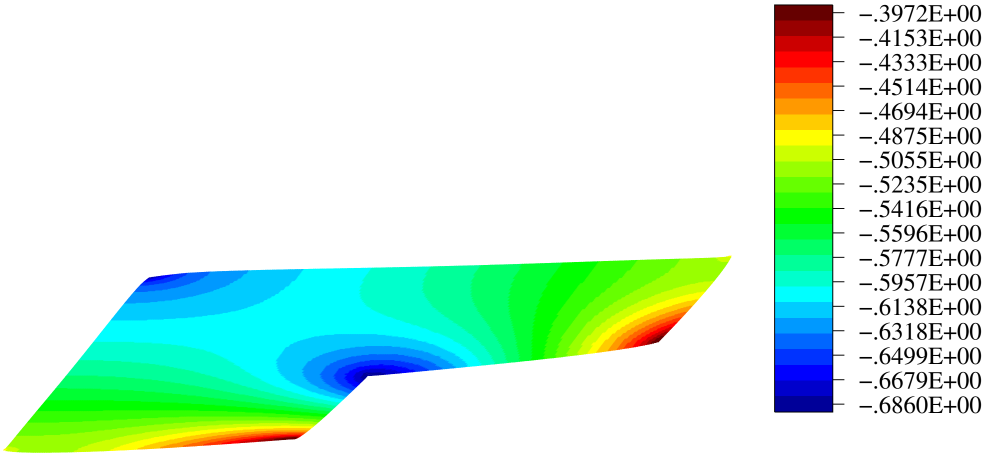

FIGURE 5. Displacement u (Ex. 7.2). (Figure in color available online at www.esaim-m2an.org.)

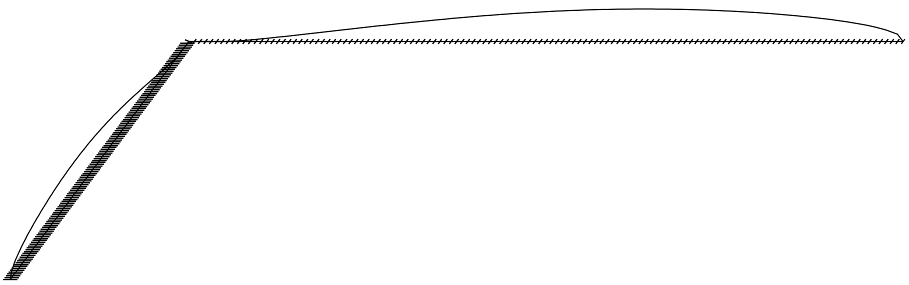

FIGURE 6. Gap variable λ (Ex. 7.2).

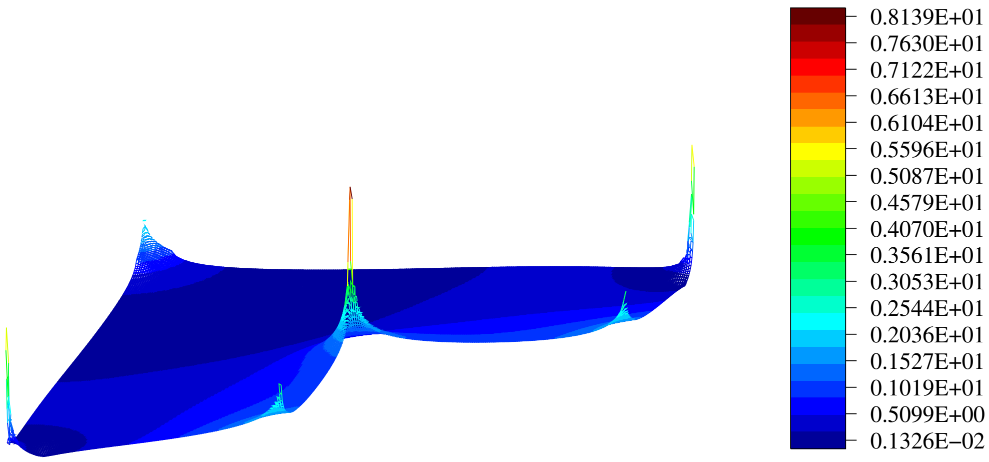

FIGURE 7. Stress q (Ex. 7.2). (Figure in color available online at www.esaim-m2an.org.)

that is $\langle 1, t_0 \rangle = 0$, whence the corresponding uniqueness requirement (1.6) is fulfilled. The extrapolated value for $\tilde{\Psi}$ is $\tilde{\Psi}(\mathbf{q}_{ex}) = -0.187364$.

Furthermore, Table 2 gives the experimental convergence rates $\alpha \sim -0.4$ which corresponds to $h^{0.8}$ due to $N \sim h^{-2}$. Again the number of Uzawa iterations is bounded. Figure 6 shows that the gap variable λ vanishes near the corner $(-\frac{1}{4}, -\frac{1}{4})$, *i.e.* in this example there is unilateral contact. The approximated displacements and stresses are shown in Figures 5 and 7, respectively.

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