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OPTIMAL CONVERGENCE OF A DISCONTINUOUS-GALERKIN-BASED IMMERSED BOUNDARY METHOD *

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Abstract. We prove the optimal convergence of a discontinuous-Galerkin-based immersed boundary method introduced earlier [Lew and Buscaglia, *Int. J. Numer. Methods Eng.* **76** (2008) 427–454]. By switching to a discontinuous Galerkin discretization near the boundary, this method overcomes the suboptimal convergence rate that may arise in immersed boundary methods when strongly imposing essential boundary conditions. We consider a model Poisson's problem with homogeneous boundary conditions over two-dimensional C^2-domains. For solution in H^q for $q > 2$, we prove that the method constructed with polynomials of degree one on each element approximates the function and its gradient with optimal orders h^2 and h, respectively. When $q = 2$, we have $h^{2-\epsilon}$ and $h^{1-\epsilon}$ for any $\epsilon > 0$ instead. To this end, we construct a new interpolant that takes advantage of the discontinuities in the space, since standard interpolation estimates lead here to suboptimal approximation rates. The interpolation error estimate is based on proving an analog to Deny-Lions' lemma for discontinuous interpolants on a patch formed by the reference elements of any element and its three face-sharing neighbors. Consistency errors arising due to differences between the exact and the approximate domains are treated using Hardy's inequality together with more standard results on Sobolev functions.

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1. Introduction

Numerical methods for the approximation of solutions of partial differential equations over curved domains can be broadly classified as domain-fitting methods or immersed boundary methods. The first type of methods require the construction of a mesh over a sufficiently-accurate approximation of the exact domain, which often makes the imposition of boundary conditions simple. The second type of methods allows the boundary of the domain to cut through elements of a background mesh of a larger and simple-to-mesh domain. The task of meshing is hence simplified, at the expense of complicating the imposition of boundary conditions. This type of methods are often advantageous for problems with evolving domains, such as for shape optimization problems or for some fluid-structure interaction problems. Immersed boundary methods are also attractive because fairly

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structured meshes can be adopted. More generally, these methods provide an attractive alternative in the absence of advanced meshing tools.

Many natural strategies to impose essential boundary conditions in immersed boundary methods often result in suboptimal convergence rates, a phenomenon termed "boundary locking" (see [17] for a discussion). For example, for homogeneous boundary conditions, simply constraining all continuous P^1 functions over a mesh to be zero along an immersed boundary can only guarantee convergence as $h^{1/2}$ in H^1, where h is the mesh size. The problem can be traced back to attempting to impose too many constraints on the discretization near the boundary, leaving too few degrees of freedom to approximate the solution.

One way to circumvent this problem is by relaxing the constraint. This has led, broadly speaking, to two related classes of methods. One class of methods is represented by the fictitious domain method [6,11,12], in which a small enough space of Lagrange multipliers is designed so that optimal convergence is retained. The second class of methods appeals to some form of penalty formulation along the boundary, as in the immerse interface method [16], or to a lesser extent, Nitsche's method (see [8,20]).

A third class of methods imposes essential boundary conditions strongly by modifying the stencil near the boundary, see [17] for a discussion. In this paper we analyze the convergence of a discontinuous-Galerkin-based immersed boundary method, which belongs to this class. The method was introduced in [17] for homogeneous boundary conditions, and extended to non-homogeneous boundary conditions in [21]. It has the following features:

1. A mesh-dependent, polygonal or polyhedral approximation of the domain is constructed so that the approximate boundary intersects elements along straight segments or planar polygons only. In particular, the approximate domain is chosen as the zero sublevel set of a finite element interpolant of the signed distance function to the boundary. This approach has some advantages for three-dimensional domains.
2. A finite element space constructed with functions that can be discontinuous across element faces of elements intersected by the boundary of the approximate domain. Functions in this space are constrained to satisfy (an approximation of the) essential boundary conditions.

The introduction of the discontinuities along element faces has the effect of adding enough degrees of freedom to impose constraints on the boundary without degrading the approximation properties. Because of the discontinuities, a discontinuous Galerkin method based on the Bassi-Rebay numerical fluxes [4,18] is adopted to obtain the numerical approximation.

Perhaps less commonly found in immersed boundary methods is an approximation of the domain, which is needed here for two reasons. First, it simplifies the integration over elements cut by the boundary, and over the boundary itself. Second, the constraint imposed by the boundary conditions over the finite element space needs to be such that it leaves enough degrees of freedom in the element to approximate the solution. When the finite element space in each element is a polynomial, constraining its values on a segment (2D) or a plane (3D) leaves polynomials of the same degree to approximate the dependence of a function in the normal direction. In contrast, a similar constraint imposed on an arbitrary curve or surface generally leads to suboptimal approximation properties away from it.

We restrict the analysis in the paper to finite element spaces whose functions are polynomials of first order over quasiuniform families of meshes of triangles in two dimensions. We adopt Poisson's problem with homogeneous essential boundary conditions as a model. When the exact solution is in H^q, $q > 2$, we prove that the L^2-error in the solution and its derivatives decreases at least as h^2 and h, respectively, which are optimal convergence rates. Otherwise, for solutions that are only in H^2, the convergence rates become $h^{2-\epsilon}$ and $h^{1-\epsilon}$, respectively, for any $\epsilon > 0$. Optimal convergence rates for this method were numerically observed for elasticity problems with non-homogeneous boundary conditions in two and three dimensions [21].

The extension of these ideas to construct higher order methods for curved domains is not straightforward. Solely increasing the order of the polynomials in the finite element space is not enough. A different strategy to approximate the domain is needed in this case.

The key difference in the analysis of this method with standard analyses of DG methods is the need to construct a new interpolant that takes advantage of the discontinuities in the space. The standard continuous, piecewise linear interpolant that satisfies the homogeneous boundary conditions along the approximate boundary can only be guaranteed to approximate the exact solution at a suboptimal rate. We therefore construct a new interpolant with discontinuities in Section 4. A special and necessary feature of the interpolant is that the interpolation points in the reference element depend on the relative position of the approximate boundary with respect to the element. The interpolation error estimate is based on proving an analog to Deny-Lions' lemma for discontinuous interpolants on a patch formed by the reference elements of any element and its three face-sharing neighbors, Lemma 5.6. Crucial to the approximation result is that the resulting bound is *independent* of the choice of interpolation points. Similar considerations were needed for problems involving material interfaces, see [13].

The second crucial step in this analysis is to properly account for consistency and approximation errors arising due to the difference between the exact and approximate domains. For simplicity we considered only C^2 domains. Technical differences appear with respect to previous estimates for finite element approximations in curved domains (see, *e.g.*, [2,15,22]), since herein the exact boundary is not interpolated by the approximate one. Approximation errors are treated by defining an appropriate reference domain, while consistency errors are treated using Hardy's inequality together with more standard results on Sobolev functions. It is because of the consistency errors that we can only prove optimal convergence for exact solutions that are slightly more regular than H^2, being only almost optimal for solutions that are exclusively in H^2.

Once approximation and consistency errors are established, the convergence follows by a careful adaption of standard discontinuous Galerkin arguments. Throughout the paper we make extensive use of a number of lemmas about the geometric properties of the approximate domains, which we establish in Section A. These properties should be generally useful in problems in which domains are approximated as sublevel sets of scalar functions.

The rest of the paper is as follows. In Section 2 we introduce the continuum and discrete problems. As commonly done we consider quasiuniform families of meshes that may not fit the domain. Additionally, for the sake of simplicity, we assume that the boundary of the exact domain is at a small distance, proportional to the mesh size, of any node in the mesh family. The approximation of the domain is introduced in Section 3. We state the main results of the paper in Section 4, discuss approximation of functions in Section 5, and prove the convergence in Section 6.

2. Setting of the problem and notation

2.1. Continuum problem

Let Ω be an open, connected and bounded domain in $\mathbb{R}^2$. Assume that the boundary Γ of Ω is of class C^2 in the sense of [14] (see Sect. 3). Under these assumptions the length $|\Gamma|$ is bounded. In what follows $|\cdot|$ indicates the Euclidean norm in $\mathbb{R}^2$, $|\cdot|_{m,\Omega}$ indicates the $W^{m,2}(\Omega)$-seminorm, $\|\cdot\|_{m,\Omega}$ the $W^{m,2}(\Omega)$-norm, and $\|\cdot\|_{m,p,\Omega}$ the $W^{m,p}(\Omega)$-norm.

As a model problem for our analysis we consider a simple elliptic PDE, *i.e.*

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{in } \Gamma. \end{cases} \tag{2.1}$$

The above problem, despite its simplicity, is perfectly suitable for our purposes, since it clearly presents all the main features of the method. The (weak) variational formulation the problem reads: find $u \in H^1_0(\Omega)$ such that

$$a(u,v) = F(v) \qquad \text{for every } v \in H^1_0(\Omega), \tag{2.2}$$

where

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x, \qquad F(v) = \int_\Omega f v \, \mathrm{d}x \quad \text{for } f \in L^2(\Omega).$$

Existence and uniqueness of the solution are trivial by Riesz Representation Theorem or by Lax-Milgram Lemma. Moreover, since $f \in L^2(\Omega)$ and Ω is at least of class C^2 we have that $u \in H^2(\Omega) \cap H^1_0(\Omega)$ and that

$$\|u\|_{2,\Omega} \leq C\|f\|_{0,\Omega} \tag{2.3}$$

for some constant $C > 0$ independent of f. In particular, if $f \in L^p(\Omega)$ for $p > 2$ then $u \in W^{2,p}(\Omega)$.

To conclude this section we recall a standard extension result for Sobolev function (*e.g.* [1]) of frequent use throughout the paper. Given $u \in H^2(\Omega)$ there exists an extension to $H^2(\mathbb{R}^2)$, denoted with $\tilde{u}$, in such a way that

$$\|\tilde{u}\|_{2,\mathbb{R}^2} \leq C\|u\|_{2,\Omega}, \tag{2.4}$$

for some constant $C > 0$ independent of u.

2.2. Discrete problem

Let $\{\mathcal{T}_h\}_h$ be a family of triangulations in the plane. We assume that each element E of $\mathcal{T}_h$ is open and that

$$\sup_{E\in\mathcal{T}_h} \operatorname{diam}(E) \leq h. \tag{2.5}$$

In particular, the length of each edge e of E is less than or equal to h.

Furthermore, we assume that $\{\mathcal{T}_h\}_h$ is quasiuniform, *i.e.* that there exists $\rho > 0$ (independent of h) such that

$$\rho h < \inf_{E\in\mathcal{T}_h} \operatorname{diam}(B_E), \tag{2.6}$$

where B_E denote the circle inscribed in the triangle E.

The next step is the definition of the discrete domain Ω_h for each h. This is actually rather delicate and is developed in full detail in the next section. For the moment, consider that Ω_h is an open polygonal set, such that the intersection of $\Gamma_h = \partial\Omega_h$ with each element is either empty or a straight segment. Moreover let $\mathcal{E}_h$ denote the set of all edges e of the triangles $E \in \mathcal{T}_h$ and let

$$\Gamma_h^i = \bigcup_{e\in\mathcal{E}_h} e \cap \Omega_h \tag{2.7}$$

denote the union of the "internal" edges.

Let V_h^c be the finite element space of continuous piecewise linear functions on $\mathcal{T}_h$. Let V_h be the finite element space made of functions whose restriction to each element E of $\mathcal{T}_h$ is affine. Note that $V_h^c \subsetneq V_h$ and that the inclusion is strict since the functions of V_h are not necessarily continuous across element boundaries. Finally, define $V_h^0 = \{v_h \in V_h \colon v_h|_{\Gamma_h} = 0\}$.

When considering spaces of functions that may be discontinuous across element boundaries, it is convenient to choose (arbitrarily) a unit normal n for each edge in $\mathcal{E}_h$, and then to introduce the jump operator on V_h given by

$$[\![v_h]\!] = v_h^- - v_h^+, \tag{2.8}$$

and the average operator over V_h^2, given by

$$\{\gamma_h\} = \frac{1}{2}\left(\gamma_h^+ + \gamma_h^-\right). \tag{2.9}$$

Here $v_h^\pm$ and $\gamma_h^\pm$ denote the traces on both sides of each edge, labeled in such a way that $v_h^\pm = \lim_{\lambda\to 0^+} v_h(x \pm n\lambda)$ and similarly for $\gamma_h^\pm$. Next, we introduce the DG-derivative as the linear operator $D_{\mathrm{DG}} : V_h \to V_h^2$ such that for all $E \in \mathcal{T}_h$ we have

$$D_{\mathrm{DG}} u_h = \nabla u_h + R([\![u_h]\!]) \qquad \text{in } E. \tag{2.10}$$

Here the lifting operator $R: L^2(\Gamma_h^i) \to V_h^2$ is defined by

$$\int_{\Omega_h} R(v_h) \cdot \gamma_h \, \mathrm{d}x = -\int_{\Gamma_h^i} v_h \{\gamma_h\} \cdot n \, \mathrm{d}s \qquad \text{for all } \gamma_h \in V_h^2. \tag{2.11}$$

The discrete approximation consists in finding $u_h \in V_h^0$ such that

$$a_h(u_h, v_h) = F_h(v_h) \tag{2.12}$$

for all $v_h \in V_h^0$. Here

$$a_h(u_h, v_h) = \sum_{E \in \mathcal{T}_h} \int_{E \cap \Omega_h} D_{\mathrm{DG}} u_h \cdot D_{\mathrm{DG}} v_h \, \mathrm{d}x + \beta \int_{\Omega_h} R([\![u_h]\!]) \cdot R([\![v_h]\!]) \, \mathrm{d}x \tag{2.13}$$

where $\beta > 0$ is a stabilization parameter, and

$$F_h(v_h) = \int_{\Omega_h} \tilde{f} \, v_h \, \mathrm{d}x, \tag{2.14}$$

where $\tilde{f}$ is any extension of f to $L^2(\mathbb{R}^2)$ such that

$$\|\tilde{f}\|_{0,\mathbb{R}^2} \leq C \|f\|_{0,\Omega} \tag{2.15}$$

for a constant $C > 0$ independent of f.

Finally, we remark that the functions in V_h^0 are defined in the whole $\mathbb{R}^2$, therefore (strictly speaking) the solution of problem (2.12) is not unique. However, its restriction to Ω_h will be.

3. Approximation of the domain

We specify now the definition of the approximate domain. We do not adopt the standard interpolation of Γ based on its intersections with element edges. This approach works well in two dimensions, but is not trivially extended to three-dimensional immersed domains over tetrahedral meshes. Instead, we analyze the alternative approach specified below based on level sets, which has proved to be very easy to use in both two- and three-dimensional problems (see [21]).

Following Definition 1.2 in [14], an open domain Ω has a C^2-regular boundary if there exists $\phi \in C^2(\mathbb{R}^2, \mathbb{R})$ such that $\Omega = \{x\colon \phi(x) < 0\}$, and such that $|\nabla \phi| \geq 1$ on $\partial\Omega$. Theorem 1.3 therein states that Ω has a C^2-regular boundary if and only if there exist constants $M, r > 0$ such that given any open ball $B \subset \mathbb{R}^2$ of radius r, after a proper translation and rotation to new coordinates $x = (y_1, y_2)$, we have that

$$\begin{aligned} B \cap \Omega &= \big\{x \in \mathbb{R}^2 \colon y_2 > \Phi(y_1)\big\} \cap B \\ B \cap \partial\Omega &= \big\{x \in \mathbb{R}^2 \colon y_2 = \Phi(y_1)\big\} \cap B, \end{aligned} \tag{3.1}$$

for some $\Phi \in C^2(\mathbb{R}, \mathbb{R})$ such that $\|\Phi\|_{W^{2,\infty}(\mathbb{R})} \leq M$.

Let $d\colon \mathbb{R}^2 \to \mathbb{R}$ be the signed distance function to $\partial\Omega$, with $d < 0$ in Ω. It follows from Theorem 1.5 in [14] that there exists $r > 0$ such that the distance function d is C^2 in $U_r(\partial\Omega) = \{x \in \mathbb{R}^2 \colon |d(x)| < r\}$, and $|\nabla d| = 1$ therein.

In order to obtain optimal order of convergence we need to assume that for some $0 < \eta < \rho$, independent of h, we have

$$|d(x_a)| \geq \eta h, \tag{3.2}$$

where x_a is any node in $\mathcal{T}_h$. This condition prevents the approximate boundary from cutting slices with arbitrarily small aspect ratios from an element. In practice it can generally be enforced by simply moving the nodes slightly away from the exact boundary.

In the most general case we could assume that

$$\text{either } d(x_a) = 0 \text{ or } |d(x_a)| \geq \eta h,$$

for every node x_a in $\mathcal{T}_h$. In this case the exact boundary can cross through nodes in the mesh. However, for sake of clarity and simplicity we shall assume that condition (3.2) is satisfied though all the results should hold for the most general case as well.

Let d_h be the nodal interpolant of d in V_h^c. Then, the approximate domain Ω_h is defined in a simple way as

$$\Omega_h = \{x \in \mathbb{R}^2 \colon d_h(x) < 0\}. \tag{3.3}$$

Notice that Ω_h is open since d_h is continuous. Let $\Gamma_h = \partial\Omega_h$ (note that under assumption (3.2), $\Gamma_h = \{x \in \mathbb{R}^2 \colon d_h(x) = 0\}$).

Finally, we denote the symmetric difference between Ω and Ω_h with

$$\Omega \triangle \Omega_h = (\Omega_h \setminus \Omega) \cup (\Omega \setminus \Omega_h). \tag{3.4}$$

In terms of distance functions, we have the following characterization

$$\Omega \triangle \Omega_h = \{x \in \mathbb{R}^2 : d_h(x) d(x) < 0\}.$$

4. Main results

We now state the main results of the paper, and introduce a few necessary definitions.

The first interesting result is the construction of an approximation operator that takes advantage of the discontinuities in V_h to optimally approximate functions that are equal to zero on Γ_h. As mentioned earlier, such operator cannot generally be constructed with functions in V_h^c that are equal to zero on Γ_h.

We define the interpolation operator by first selecting interpolation points in the discrete domain Ω_h. Remember that in general Ω_h is the union of triangles and quadrilaterals and that the finite element space V_h^0 is made of piecewise affine functions vanishing on Γ_h. Additionally, if $E \cap \Gamma_h \neq \emptyset$ then one and only one face of $E \cap \Omega_h$ lies on the approximate boundary Γ_h.

Definition 4.1 (approximation operator)**.** Let $E \in \mathcal{T}_h$ be such that $E \cap \Omega_h \neq \emptyset$. We define the interpolation points for $E \cap \Omega_h$ as (*cf.* Fig. 1):

(1) the three vertices if $E \cap \Omega_h$ is a triangle;
(2) the two vertices on Γ_h and the middle point of the opposite edge if $E \cap \Omega_h$ is a quadrilateral.

Then, let $\Pi_h^0 : H^2(\Omega) \cap H_0^1(\Omega) \to V_h^0$ be the interpolation operator such that $\Pi_h^0 v|_E$ is affine and $\Pi_h^0 v(x) = v(x)$ if the interpolation point $x \notin \Gamma_h$.

Notice that in this definition it is implicitly stated that $\Pi_h^0 v(x) = 0$ if $x \in \Gamma_h$, since $\Pi_h^0 v \in V_h^0$, and that if and element E is such that $E \cap \Gamma_h \neq \emptyset$ then $\Pi_h^0 v$ has just one degree of freedom on E.

In these non-conforming spaces we will adopt the so-called broken norm $\||\cdot\||_\Omega$, defined as

$$\||v\||_\Omega^2 = \sum_{E \in \mathcal{T}_h} \|\nabla v\|_{0,E\cap\Omega}^2 + \frac{1}{h} \sum_{e \in \mathcal{E}_h} \| [\![v]\!] \|_{0,e\cap\Omega}^2 \tag{4.1}$$

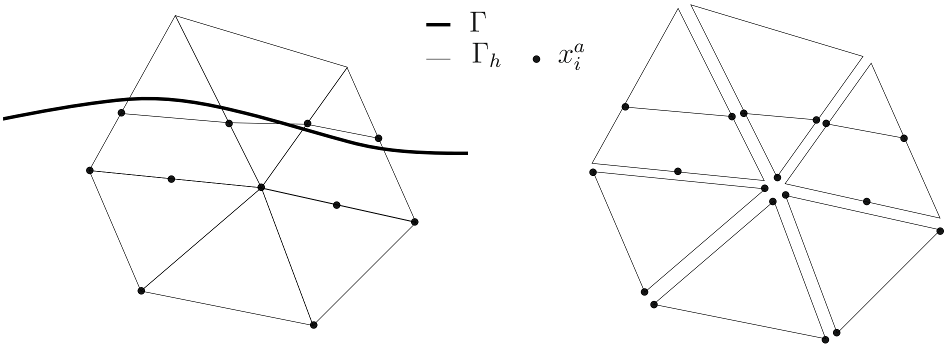

FIGURE 1. Location of the interpolation points for the operator Π_h^0, marked with black circles, in Ω_h (left) and element by element (right).

and similarly for any other domain in $\mathbb{R}^2$; in particular, to simplify the notation we will use $\|v_h\|_h$ for $\|v_h\|_{\Omega_h}$. Given $\mathcal{T}_h$, let us denote by $\tilde{\Omega}_h$ the "smallest union of elements that contains Ω", defined as the interior of

$$\bigcup_{\substack{E\in\mathcal{T}_h\\ E\cap\Omega\neq\emptyset}} \overline{E}. \tag{4.2}$$

Accordingly, let $\tilde{\Gamma}_h^i$ be the union of the edges of triangles $E \in \mathcal{T}_h$ such that $E\cap\Omega \neq \emptyset$.

The key property of this interpolation operator is contained in the following theorem.

Theorem 4.2. *Let $\Pi_h^0\colon H^2(\Omega)\cap H_0^1(\Omega) \to V_h^0$ be the interpolation operator of Definition 4.1. For h sufficiently small, the following interpolation estimate holds:*

$$\|\Pi_h^0 u - \tilde{u}\|_h \leq \|\Pi_h^0 u - \tilde{u}\|_{\tilde{\Omega}_h} \leq Ch\|u\|_{2,\Omega}, \tag{4.3}$$

for some $C > 0$ independent of u, $\tilde{u}$ and h.

The second key result of the paper is the optimal convergence of the method in the broken norm, as stated next. To this end, recall that u_h is well defined in $\mathbb{R}^2$ and thus in $\Omega \setminus \Omega_h$.

Theorem 4.3. *Let u be the solution of (2.1) and u_h the solution of (2.12). Then, for h small enough*

$$\|u - u_h\|_\Omega \leq Ch\|u\|_{2,\Omega}, \tag{4.4}$$

and

$$\|\tilde{u} - u_h\|_h \leq Ch\|u\|_{2,\Omega}, \tag{4.5}$$

for a constant $C > 0$ independent of h and u.

Finally, the last important result is the almost optimal convergence of the approximation in $L^2(\Omega)$.

Theorem 4.4. *Let u be the solution of (2.1) and u_h the solution of (2.12). For every $\epsilon > 0$ there exists a constant $C_\epsilon > 0$ independent of h and u such that*

$$\|u - u_h\|_{0,\Omega} \leq C_\epsilon h^{2-\epsilon}\|u\|_{2,\Omega}, \tag{4.6}$$

for h small enough. Additionally, if $u \in W^{2,p}(\Omega)$ for $p > 2$, then there exists a constant $C_0 > 0$ independent of h and u such that

$$\|u - u_h\|_{0,\Omega} \leq C_0 h^2\|u\|_{2,p,\Omega}, \tag{4.7}$$

for h small enough.

We would like to remark that the optimal h^2-convergence can be achieved not only when u is more regular than $W^{2,2}$, but also when the computational domain Ω_h approximates better the physical domain Ω, *e.g.* when $|\Omega_h \triangle \Omega|$ is of order h^q for $q > 2$ (for further details see the proof of Lem. 6.7). However, with our definition of Ω_h we can only guarantee $q = 2$, see Corollary A.3.

5. Approximation of functions

This section is concerned with proving the approximation result in Theorem 4.2.

5.1. Equivalence of norms

In the sequel we will often need the equivalence of norms in different domains. Let us start with the L^2-norm.

Lemma 5.1. *There exists $C > 0$, independent of h, such that*

$$\|v_h\|_{0,\Omega_h} \leq \|v_h\|_{0,\tilde{\Omega}_h} \leq C\|v_h\|_{0,\Omega_h} \tag{5.1}$$

$$\|\gamma_h\|_{0,\Gamma_h^i} \leq \|\gamma_h\|_{0,\tilde{\Gamma}_h^i} \leq C\|\gamma_h\|_{0,\Gamma_h^i} \tag{5.2}$$

for all $v_h \in V_h$ and $\gamma_h \in V_h^2$.

Proof. Since $\Omega_h \subseteq \tilde{\Omega}_h$ the first inequality in (5.1) is trivial. Next, for each element $E \in \mathcal{T}_h$ let $\Psi_E \colon \widehat{E} \to E$ be an affine map, where $\widehat{E}$ is the reference triangle with vertices $(0,0)$, $(0,1)$ and $(1,0)$ in a set of Cartesian coordinates. For any $v_h \in P_1(E)$, we have that $\hat{v}_h = v_h \circ \Psi_E$ is a linear polynomial over $\widehat{E}$. Without loss of generality, and because of (3.2), we can then assume that $\Psi_E(\hat{x}) \in E \cap \Omega_h$ for any $\hat{x}$ in the triangle $\widehat{E}_\eta$ defined by the vertices $(0,0)$, $(0,\eta)$ and $(\eta,0)$. It then holds that there exists a constant C such that

$$\|\hat{v}_h\|_{0,\widehat{E}} \leq C\|\hat{v}_h\|_{0,\widehat{E}_\eta}$$

for all $\hat{v}_h \in P_1(\widehat{E})$. It follows that

$$\|v_h\|_{0,E} = (2|E|)^{1/2}\|\hat{v}_h\|_{0,\widehat{E}} \leq C(2|E|)^{1/2}\|\hat{v}_h\|_{0,\widehat{E}_\eta} \leq C\|v_h\|_{E\cap\Omega_h}$$

for all $v_h \in P_1(E)$, from where (5.1) is obtained by adding over all of the elements in the mesh.

The proof of (5.2) is constructed with the same arguments. □

Arguing as in Lemma 5.1 we easily obtain that:

Corollary 5.2. *There exists $C > 0$, independent of h, such that*

$$|\!|\!| v_h |\!|\!|_h \leq |\!|\!| v_h |\!|\!|_{\tilde{\Omega}_h} \leq C |\!|\!| v_h |\!|\!|_h \tag{5.3}$$

for all $v_h \in V_h$.

5.2. An auxiliary interpolation operator

Strictly speaking the points $x \in \Gamma_h$ in Definition 4.1 are not interpolation points, because it may happen that $\Pi_h^0 v(x) = 0 \neq \tilde{v}(x)$. Precisely because $x \in \Gamma_h$ are not effective interpolation points, for the proof of (4.3) it is convenient to introduce an auxiliary interpolation operator $\acute{\Pi}_h^0$, defined below. This auxiliary interpolation operator provides a simple way to estimate the error introduced by the approximate boundary conditions satisfied by $\Pi_h^0 v$ on Γ.

To this end, we first introduce additional interpolation points $\acute{x}^a$. For each triangle E denote by x^a (for $a = 1,2,3$) the three interpolation points of Definition 4.1, with $x^a \in \Gamma_h$ for $a = 1,2$, and by y^a (for $a = 1,2,3$)

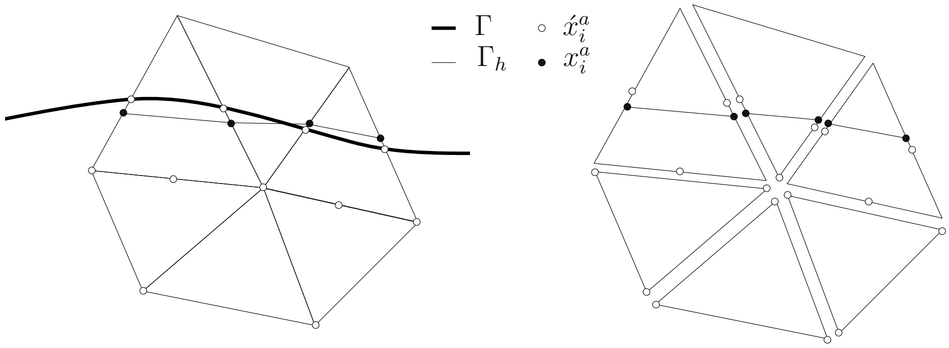

FIGURE 2. Location of the interpolation points for the operator $\acute{\Pi}_h^0$, marked with hollow circles, in Ω_h (left) and element by element (right). In each element intersected by the boundary, black circles mark the location of the two interpolation points for Π_h^0 that do not coincide with those of $\acute{\Pi}_h^0$.

the three vertices. For E such that $E \cap \Omega \neq \emptyset$ and h small enough we define $\acute{x}^a$ as follows. If $E \cap \Gamma = \emptyset$ we set $\acute{x}^a$ (for $a = 1, 2, 3$) to be the vertices of E, and hence $\acute{x}^a = x^a$. If $E \cap \Gamma \neq \emptyset$ then, by Lemma A.2, Γ intersects ∂E exactly in two points which belong to different edges. We denote these points by $\acute{x}^a$ (for $a = 1, 2$), so that x^a and $\acute{x}^a$ belong to the same edge, and set $\acute{x}^3 = x^3$ (see Fig. 2).

Note that the positions of the points x^a and $\acute{x}^a$ depend on Γ_h and Γ, respectively, *i.e.* on the intersection between E and Ω_h and between E and Ω. In particular, if $x^a \in \Gamma_h$ is on the edge with vertices y^{a_1} and y^{a_2} then by (3.2) and Lemma A.2 for h sufficiently small we have $d(x^a, y^{a_i}) \geq \eta' h$ for $i = 1, 2$ and $0 < \eta' < \eta$. Hence $x^a = \lambda^1 y^{a_1} + \lambda^2 y^{a_2}$ where $\lambda^i \in [\eta', 1 - \eta']$ (clearly $\eta' < 1/2$). Similarly, by (3.2) we have that $d(\acute{x}^a, y^{a_i}) \geq \eta h$, for $i = 1, 2$. Finally, from Lemma A.2 we know that $d(x^a, \acute{x}^a) < Ch^2$.

Next, we define $\acute{\Pi}_h^0$. Let $\acute{\Gamma}_h$ be the polygonal line obtained by interpolation of the points $\acute{x}^a$ for $a = 1, 2$. In analogy with V_h^0 let $\acute{V}_h^0 = \{v_h \in V_h \colon v_h|_{\acute{\Gamma}_h} = 0\}$. Finally, let the auxiliary interpolation operator $\acute{\Pi}_h^0 : H^2(\Omega) \cap H_0^1(\Omega) \to \acute{V}_h^0$ be such that $\acute{\Pi}_h^0 v|_E$ is affine and $\acute{\Pi}_h^0 v(\acute{x}) = v(\acute{x})$ in every interpolation point $\acute{x}$ defined above, and every element E that has a nonempty intersection with Ω. As with the points $\acute{x}^a$, the auxiliary interpolation is only well defined for h small enough.

In view of the proof of Theorem 4.2, we conclude this part with a geometric result useful to estimate the difference between $\Pi_h^0 u$ and $\acute{\Pi}_h^0 u$. Let T be the affine map such that $Tx^a = \acute{x}^a$ for $a = 1, 2, 3$. (Clearly, T is the identity if $E \cap \Gamma = \emptyset$.) Let $A : \mathbb{R}^2 \to \mathbb{R}$ be the affine function such that $\acute{\Pi}_h^0 u|_E = A|_E$. Then $\Pi_h^0 u|_E = A \circ T|_E$. Note that in general $\Pi_h^0 u \neq \acute{\Pi}_h^0 u \circ T$ since when $E \cap \Omega_h$ is a quadrilateral it may happen that $T(E \cap \Omega_h) \not\subset E$. Let us prove that the map T is close to the identity.

Lemma 5.3. *Let T be the map defined above. Then there exist $C > 0$, independent of h and of the points x^a and $\acute{x}^a$, such that*

$$|I - T| \leq Ch, \tag{5.4}$$

where I denotes the identity.

Proof. Note that for each element with $E \cap \Omega \neq \emptyset$ the interpolation points $\acute{x}^3$ and x^3 always coincide. Consider a system of coordinates centered in $\acute{x}^3 = x^3$ so that the map T is linear. Let M be a linear map from E onto a reference triangle $\hat{E}$ (obtained after a suitable translation) and consider the map $\widehat{T} = MTM^{-1}$. By a standard property of quasi-uniform triangulations $|M| \leq Ch^{-1}$ and $|M^{-1}| \leq Ch$, where C is independent of h. Therefore

$$|I - T| = |M^{-1}(I - \widehat{T})M| \leq C|I - \widehat{T}|.$$

Since $d(\acute{x}^a, x^a) \leq Ch^2$, we have $d(M\acute{x}^a, Mx^a) \leq C'h$ in the reference element $\hat{E}$, for some $C' > 0$ independent of h, x^a and $\acute{x}^a$. Then, due to (3.2) it is not hard to see that $|I - \widehat{T}| \leq C''h$, for $C'' > 0$ independent of h, x^a and $\acute{x}^a$. □

5.3. **A Deny-Lions lemma on patches of elements**

Our proof of (4.3) consists in an original generalization to DG-FEM of the classical Deny-Lions lemma (see, *e.g.*, [9]). For simplicity, we restrict the lemma to elemental spaces made of polynomials of degree one. The main technical difference, due to the presence of jump terms in the triple norm, lies in the use of patches of neighboring elements rather than a single element.

Definition 5.4 (element patch)**.** Let E be an element in $\mathcal{T}_h$, E_1, E_2, E_3 be its three neighbors and denote $E_0 = E$. The element patch of E is

$$\mathcal{P}_E = \bigcup_{i=0}^{3} E_i. \tag{5.5}$$

As usual the estimate is proved first in a reference patch and then by change of variable in the real patch.

Definition 5.5 (reference patch of an element)**.** Given a patch $\mathcal{P}_E$, let $M : \overline{\mathcal{P}_E} \to \mathbb{R}^2$ be a continuous and one-to-one map such that, for $i = 0, \ldots, 3$, the restriction $M|_{E_i}$ is affine and $M(E_i)$ is equilateral with unit side. Then, the reference patch of $\mathcal{P}_E$ is $\widehat{\mathcal{P}}_E = M(\mathcal{P}_E)$, and we let $\widehat{E}_i = M(E_i)$.

It is possible to select a single reference patch for all the elements in $\{\mathcal{T}_h\}_h$. For this reason, we shall henceforth drop the subindex E from $\widehat{\mathcal{P}}$. Notice that since the elements are open, the patch $\mathcal{P}_E$ is not connected; however, its closure $\overline{\mathcal{P}_E}$ is.

For each E_i ($i = 0, \ldots, 4$) let $\acute{x}_i^a$ (for $a = 1, 2, 3$) be the three interpolation points in E_i defined in Section 5.2. We recall once more that the position of the points $\acute{x}_i^a$ in the patch depends on Γ. Denote by y_i^a the vertices of the triangle E_i. Let $\hat{x}_i^a = M(\acute{x}_i^a)$ and $\hat{y}_i^a = M(y_i^a)$ be the corresponding points in the reference patch.

Next, we prove our Deny-Lions estimate in the reference patch. For convenience, denote by $\hat{e}_i$ (for $i = 1, 2, 3$) the edges $\partial\widehat{E}_i \cap \partial\widehat{E}_0$.

Lemma 5.6. *Let $\widehat{\mathcal{P}}$ be a reference patch. There exists a constant $C > 0$ such that for all $w \in H^2(\widehat{\mathcal{P}})$ and all choices of the interpolation points $\hat{x}_i^a$ we have*

$$\sum_{i=0}^{3} \|w\|_{2,\widehat{E}_i} + \sum_{i=1}^{3} \| [\![w]\!] \|_{0,\hat{e}_i} \leq C \left[\sum_{i=0}^{3} |w|_{2,\widehat{E}_i} + \sum_{i=0}^{3} \sum_{a=1}^{3} |w(\hat{x}_i^a)| \right]. \tag{5.6}$$

Proof. We proceed by contradiction. We assume that there exists a sequence $w_k \in H^2(\widehat{\mathcal{P}})$ and a sequence of points $\hat{x}_{i,k}^a$ such that for all k,

$$\sum_{i=0}^{3} \|w_k\|_{2,\widehat{E}_i} + \sum_{i=1}^{3} \| [\![w_k]\!] \|_{0,\hat{e}_i} = 1 \tag{5.7}$$

$$\sum_{i=0}^{3} |w_k|_{2,\widehat{E}_i} + \sum_{i=0}^{3} \sum_{a=1}^{3} |w_k(\hat{x}_{i,k}^a)| \to 0. \tag{5.8}$$

Hence, from (5.7) w_k is bounded in $H^2(\widehat{\mathcal{P}})$, and up to a subsequence, w_k converges strongly in $H^1(\widehat{\mathcal{P}})$. By (5.8) we also have that $|w_m - w_n|_{2,\widehat{E}_i} \leq |w_m|_{2,\widehat{E}_i} + |w_n|_{2,\widehat{E}_i} \to 0$ as $m, n \to \infty$. Hence w_k is a Cauchy sequence in $H^2(\widehat{\mathcal{P}})$ and it converges strongly to $w \in H^2(\widehat{\mathcal{P}})$. Consequently by (5.8) we have $|w|_{2,\widehat{E}_i} = 0$, hence $w|_{\widehat{E}_i}$ is affine for each i. Moreover, by Sobolev inclusions w_k converges uniformly to w. Upon extracting a subsequence we can also assume that the points $\hat{x}_{i,k}^a$ converge to $\hat{x}_i^a$ for each index. Hence $w_k(\hat{x}_{i,k}^a) \to w(\hat{x}_i^a) = 0$ by (5.8), from where it follows that $w = 0$ in $\widehat{\mathcal{P}}$.

Next, by (5.7) we have that

$$\sum_{i=1}^{3} \| [\![w_k]\!] \|_{0,\hat{e}_i} \to 1. \tag{5.9}$$

However, by the continuity of the trace operator in $H^1(\widehat{\mathcal{P}})$

$$\sum_{i=1}^{3} \| [\![w_k]\!] \|_{0,\hat{e}_i} \leq C \sum_{i=0}^{3} \|w_k\|_{1,\widehat{E}_i} \to 0,$$

which contradicts (5.9). □

We remark that Lemma 5.6 generalizes Deny-Lions lemma to bound the jumps of w across elements in the patch. With minor changes further generalization are possible to a more general class of patches, *e.g.* domains with a finite number of connected components.

We next provide the Deny-Lions estimate in the patch $\mathcal{P}_E$.

Lemma 5.7. *Let $\mathcal{P}_E$ be the patch of the element E. There exists a constant $C > 0$ independent of h such that for h small enough and for all $w \in H^2(\mathcal{P}_E)$ the following estimate holds*

$$\sum_{i=0}^{3} \left[|w|_{1,E_i} + h^{-1}\|w\|_{0,E_i} \right] + h^{-1/2} \sum_{i=1}^{3} \| [\![w]\!] \|_{0,e_i} \leq C \left[h \sum_{i=0}^{3} |w|_{2,E_i} + \sum_{i=0}^{3}\sum_{a=1}^{3} |w(\acute{x}_i^a)| \right]. \tag{5.10}$$

Proof. Let M be the map of Definition 5.5. For any function $w \in H^2(\mathcal{P})$, let $\hat{w} = w \circ M^{-1}$. Then, *e.g.* by Theorem 3.1.2 in [7], there exists $C > 0$ independent of h and w such that

$$|\hat{w}|_{2,\widehat{E}_i} \leq Ch|w|_{2,E_i}, \qquad \|w\|_{0,E_i} \leq Ch|\hat{w}|_{0,\widehat{E}_i}, \qquad |w|_{1,E_i} \leq C|\hat{w}|_{1,\widehat{E}_i}, \tag{5.11}$$

from where we have that

$$|w|_{1,E_i} + h^{-1}\|w\|_{0,E_i} \leq C\|\hat{w}\|_{1,\widehat{E}_i} \leq C\|\hat{w}\|_{2,\widehat{E}_i}. \tag{5.12}$$

The last bound needed is the one for the discontinuity terms. Using again Theorem 3.1.2 in [7] we have

$$\| [\![w]\!] \|_{0,e_i} \leq h^{1/2} \| [\![\hat{w}]\!] \|_{0,\hat{e}_i}$$

and hence

$$h^{-1/2} \| [\![w]\!] \|_{0,e_i} \leq \| [\![\hat{w}]\!] \|_{0,\hat{e}_i}. \tag{5.13}$$

Using (5.12), (5.13), Lemma 5.6 and (5.11) we get

$$\begin{aligned}\sum_{i=0}^{3} \left[|w|_{1,E_i} + h^{-1}\|w\|_{0,E_i} \right] + h^{-1/2} \sum_{i=1}^{3} \| [\![w]\!] \|_{0,e_i} &\leq C \left[\sum_{i=0}^{3} \|\hat{w}\|_{2,\widehat{E}_i} + \sum_{i=1}^{3} \| [\![\hat{w}]\!] \|_{0,\hat{e}_i} \right] \\ &\leq C \left[\sum_{i=0}^{3} |\hat{w}|_{2,\widehat{E}_i} + \sum_{i=0}^{3}\sum_{a=1}^{3} |\hat{w}(\hat{x}_i^a)| \right] \\ &\leq C \left[h \sum_{i=0}^{3} |w|_{2,E_i} + \sum_{i=0}^{3}\sum_{a=1}^{3} |w(\acute{x}_i^a)| \right]\end{aligned}$$

since $\hat{w}(\hat{x}_i^a) = w(M^{-1}\hat{x}_i^a) = w(\acute{x}_i^a)$. Here $C > 0$ is independent of h and w and may change from line to line in the equations above. □

5.4. **Proof of the approximation error estimate**

This section contains the proof of Theorem 4.2. Using Lemma 5.7 with $w = \tilde{u} - \acute{\Pi}_h^0 u$ we deduce that for every element patch

$$|\tilde{u} - \acute{\Pi}_h^0 u|_{1,E_0} + h^{-1/2} \sum_{i=1}^{3} \| \left[\!\left[\tilde{u} - \acute{\Pi}_h^0 u \right]\!\right] \|_{0,e_i} \le Ch \sum_{i=0}^{3} |\tilde{u}|_{2,E_i},$$

since $\acute{\Pi}_h^0 u$ is affine and $\tilde{u}(\acute{x}_i^a) = \acute{\Pi}_h^0 u(\acute{x}_i^a)$. Therefore, taking the sum over all the elements with $E \cap \Omega \neq \emptyset$ we get

$$\|\tilde{u} - \acute{\Pi}_h^0 u\|_{\tilde{\Omega}_h} \le Ch|u|_{2,\Omega}. \tag{5.14}$$

It remains to show that

$$\|\acute{\Pi}_h^0 u - \Pi_h^0 u\|_{\tilde{\Omega}_h} \le Ch\|u\|_{2,\Omega}. \tag{5.15}$$

As in Section 5.2, let us write $\acute{\Pi}_h^0 u|_E = A|_E$ and $\Pi_h^0 u|_E = A \circ T|_E$ where A is affine and T linear (in a suitable system of coordinates). Then $\nabla \Pi_h^0 u = \nabla \acute{\Pi}_h^0 u \; T$ where, by abuse of notation, T denotes both the matrix and the map. Hence

$$\int_E |\nabla \Pi_h^0 u - \nabla \acute{\Pi}_h^0 u|^2 \, \mathrm{d}x \quad \le \quad |I - T|^2 \int_E |\nabla \acute{\Pi}_h^0 u|^2 \, \mathrm{d}x \le Ch^2 |\acute{\Pi}_h^0 u|_{1,E}^2. \tag{5.16}$$

Let us consider the norm of the jump. Let e be an edge of E. Then,

$$\left| \left[\!\left[\Pi_h^0 u - \acute{\Pi}_h^0 u \right]\!\right] \right|^2 \le 2 \left| [\Pi_h^0 u - \acute{\Pi}_h^0 u]^+ \right|^2 + 2 \left| [\Pi_h^0 u - \acute{\Pi}_h^0 u]^- \right|^2.$$

Now we will estimate only the term with the left traces which depends only the values on the actual element E; the other term will be considered taking the sum over all the elements. Then, by the representation $\acute{\Pi}_h^0 u|_E = A|_E$ and $\Pi_h^0 u|_E = A \circ T|_E$ it follows that

$$\acute{\Pi}_h^0 u|_E = \nabla(\acute{\Pi}_h^0 u|_E) \cdot x + c, \qquad \Pi_h^0 u|_E = \nabla(\acute{\Pi}_h^0 u|_E) T \cdot x + c.$$

Then, using Lemma 5.3 we get

$$\begin{aligned} \int_e |\Pi_h^0 u - \acute{\Pi}_h^0 u|^2 \, \mathrm{d}s \quad &\le \quad |I - T|^2 \int_e |\nabla \acute{\Pi}_h^0 u|^2 |x|^2 \, \mathrm{d}s \\ &\le \quad Ch^4 \int_e |\nabla \acute{\Pi}_h^0 u|^2 \, \mathrm{d}s. \end{aligned}$$

Considering that $\nabla \acute{\Pi}_h^0 u$ is constant in E and that $|E| \ge Ch^2$, the previous inequality becomes

$$\int_e |\Pi_h^0 u - \acute{\Pi}_h^0 u|^2 \, \mathrm{d}s \quad \le \quad Ch^3 \int_E |\nabla \acute{\Pi}_h^0 u|^2 \, \mathrm{d}x \le Ch^3 |\acute{\Pi}_h^0 u|_{1,E}^2. \tag{5.17}$$

Finally, taking the sum over all the elements $E \in \mathcal{T}_h(\Omega)$ by (5.16) and (5.17) we get

$$\begin{aligned} \|\Pi_h^0 u - \acute{\Pi}_h^0 u\|_{\tilde{\Omega}_h}^2 \quad &\le \quad C \left[\sum_{E \in \mathcal{T}_h} |\Pi_h^0 u - \acute{\Pi}_h^0 u|_{1,E\cap\Omega}^2 + h^{-1/2} \sum_{e \in \mathcal{E}_h} \left\| \left[\!\left[\Pi_h^0 u - \acute{\Pi}_h^0 u \right]\!\right] \right\|_{0,e\cap\Omega}^2 \right] \\ &\le \quad Ch^2 |\acute{\Pi}_h^0 u|_{1,\tilde{\Omega}_h \setminus \tilde{\Gamma}_h^i}^2. \end{aligned}$$

Then, by (5.14)

$$|\acute{\Pi}_h^0 u|_{1,\tilde{\Omega}_h \setminus \tilde{\Gamma}_h^i} - |u|_{1,\tilde{\Omega}_h} \le |\acute{\Pi}_h^0 u - u|_{1,\tilde{\Omega}_h \setminus \tilde{\Gamma}_h^i} \le Ch|u|_{2,\tilde{\Omega}_h}.$$

Hence, for h sufficiently small, $|\acute{\Pi}_h^0 u|_{1,\tilde{\Omega}_h\setminus\tilde{\Gamma}_h^i} \leq 2\|u\|_{2,\tilde{\Omega}_h} \leq C\|u\|_{2,\Omega}$ and thus

$$|\!|\!|\Pi_h^0 u - \acute{\Pi}_h^0 u|\!|\!|^2_{\tilde{\Omega}_h} \leq Ch^2\|u\|^2_{2,\Omega}.$$

6. Convergence

We can now prove the convergence of the method, Theorems 4.3 and 4.4. The essential step here is the proof of the asymptotic consistency of the method, Lemma 6.7, which embodies the errors due to the differences between domains.

6.1. Properties of the lifting operator

Lemma 6.1. *There exist constants $C_1, C_2 > 0$ independent of h such that*

$$C_1\|R([\![v_h]\!])\|_{0,\Omega_h} \leq h^{-1/2}\|\,[\![v_h]\!]\,\|_{0,\Gamma_h^i} \leq C_2\|R([\![v_h]\!])\|_{0,\Omega_h} \tag{6.1}$$

for all $v_h \in V_h$.

Proof. The proof follows closely that in [5], the key difference being that integrals are performed over intersections of elements with the approximate domain.

We first prove the left inequality in (6.1). We will need the fact that there exists a constant $C > 0$ independent of h such that

$$\|v_h\|_{0,e} \leq Ch^{-1/2}\|v_h\|_{0,E} \tag{6.2}$$

for any $v_h \in V_h$, see *e.g.* Lemma 3.2 in [18]. Next,

$$\begin{aligned}
\|R([\![v_h]\!])\|^2_{0,\Omega_h} &= -\int_{\Gamma_h^i} [\![v_h]\!]\,\{R([\![v_h]\!])\}\cdot n\,\mathrm{d}s && \text{by (2.11)}\\
&\leq \|\,[\![v_h]\!]\,\|_{0,\Gamma_h^i}\|\{R([\![v_h]\!])\}\|_{0,\Gamma_h^i} && \text{by Cauchy-Schwartz}\\
&\leq \|\,[\![v_h]\!]\,\|_{0,\Gamma_h^i}\|\{R([\![v_h]\!])\}\|_{0,\tilde{\Gamma}_h^i} && \text{by Lemma 5.1}\\
&\leq Ch^{-1/2}\|\,[\![v_h]\!]\,\|_{0,\Gamma_h^i}\|R([\![v_h]\!])\|_{0,\tilde{\Omega}_h} && \text{by (6.2)}\\
&\leq Ch^{-1/2}\|\,[\![v_h]\!]\,\|_{0,\Gamma_h^i}\|R([\![v_h]\!])\|_{0,\Omega_h} && \text{by Lemma 5.1.}
\end{aligned}$$

For the second inequality we will need the space of BDM elements of order 1 over $\mathcal{T}_h$ [3]. These are vector fields that are affine over each element and whose normal components are continuous across element boundaries. For this element it holds that

$$\|w_h\|_{0,E} \leq Ch^{1/2}\|w_h\cdot n\|_{0,e} \tag{6.3}$$

for every $w_h \in BDM_1(\mathcal{T}_h)$, where $C > 0$ is a constant independent of h. In particular, let $w_h \in BDM_1(\mathcal{T}_h)$ be defined by

$$w_h\cdot n = [\![v_h]\!] \quad \text{in } \tilde{\Gamma}_h^i. \tag{6.4}$$

It then follows from (6.3) and Lemma 5.1 that

$$\|w_h\|_{0,\Omega_h} \leq Ch^{1/2}\|w_h\cdot n\|_{0,\tilde{\Gamma}_h^i}. \tag{6.5}$$

Then, since $w_h \in V_h^2$, we have

$$\begin{aligned}
\| [\![v_h]\!] \|^2_{0,\Gamma_h^i} &= \int_{\Gamma_h^i} [\![v_h]\!]\ w_h \cdot n \,\mathrm{d}s = -\int_{\Omega_h} R([\![v_h]\!]) \cdot w_h \,\mathrm{d}x \\
&\le \|R([\![v_h]\!])\|_{0,\Omega_h} \|w_h\|_{0,\Omega_h} \\
&\le Ch^{1/2} \|R([\![v_h]\!])\|_{0,\Omega_h} \|w_h \cdot n\|_{0,\tilde{\Gamma}_h^i} && \text{by (6.5)} \\
&\le Ch^{1/2} \|R([\![v_h]\!])\|_{0,\Omega_h} \| [\![v_h]\!] \|_{0,\tilde{\Gamma}_h^i} && \text{by (6.4)} \\
&\le Ch^{1/2} \|R([\![v_h]\!])\|_{0,\Omega_h} \| [\![v_h]\!] \|_{0,\Gamma_h^i} && \text{by Lemma 5.1,}
\end{aligned}$$

which concludes the proof. □

6.2. Properties of the bilinear form

Lemma 6.2 (continuity and coercivity)**.** *For any $\beta > 0$, there exists $m, M > 0$ independent of h such that*

$$|a_h(u_h, v_h)| \le M |\!|\!| u_h |\!|\!|_h |\!|\!| v_h |\!|\!|_h, \tag{6.6}$$

$$m |\!|\!| u_h |\!|\!|_h^2 \le a_h(u_h, u_h) \tag{6.7}$$

for any $u_h, v_h \in V_h$.

Proof. To prove continuity we note that because of Lemma 6.1 there exists $C > 0$ independent of h such that

$$\|R([\![u_h]\!])\|_{0,\Omega_h} \le Ch^{-1/2} \| [\![u_h]\!] \|_{0,\Gamma_h^i} \le C |\!|\!| u_h |\!|\!|_h$$

and

$$\begin{aligned}
\|D_{\mathrm{DG}} u_h\|^2_{0,\Omega_h} &= \|\nabla u_h + R([\![u_h]\!])\|^2_{0,\Omega_h \setminus \Gamma_h^i} \le 2\|\nabla u_h\|^2_{0,\Omega_h \setminus \Gamma_h^i} + 2\|R([\![u_h]\!])\|^2_{0,\Omega_h} \\
&\le 2\|\nabla u_h\|^2_{0,\Omega_h \setminus \Gamma_h^i} + Ch^{-1} \| [\![u_h]\!] \|^2_{0,\Gamma_h^i} \le C |\!|\!| u_h |\!|\!|_h^2
\end{aligned}$$

for any $u_h \in V_h$. Consequently,

$$\begin{aligned}
|a_h(u_h, v_h)| &= \left| \int_{\Omega_h} D_{\mathrm{DG}} u_h \cdot D_{\mathrm{DG}} v_h \,\mathrm{d}x + \beta \int_{\Omega_h} R([\![u_h]\!]) \cdot R([\![v_h]\!]) \,\mathrm{d}x \right| \\
&\le \|D_{\mathrm{DG}} u_h\|_{0,\Omega_h} \|D_{\mathrm{DG}} v_h\|_{0,\Omega_h} + \beta \|R([\![u_h]\!])\|_{0,\Omega_h} \|R([\![u_h]\!])\|_{0,\Omega_h} \\
&\le M |\!|\!| u_h |\!|\!|_h |\!|\!| v_h |\!|\!|_h.
\end{aligned}$$

To see the coercivity, we shall take advantage of Young's inequality, *i.e.*, $|2ab| \le \mu a^2 + b^2/\mu$, for any $\mu > 0$ and $a, b \in \mathbb{R}$. We then have

$$\begin{aligned}
\|D_{\mathrm{DG}} u_h\|^2_{0,\Omega_h} &= \|\nabla u_h\|^2_{0,\Omega_h \setminus \Gamma_h^i} + \|R([\![u_h]\!])\|^2_{0,\Omega_h} + 2\int_{\Omega_h \setminus \Gamma_h^i} \nabla u_h \cdot R([\![u_h]\!]) \,\mathrm{d}x \\
&\ge (1-\mu)\|\nabla u_h\|^2_{0,\Omega_h \setminus \Gamma_h^i} + (1 - 1/\mu) \|R([\![u_h]\!])\|^2_{0,\Omega_h}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
a_h(u_h, u_h) &= \|D_{\mathrm{DG}} u_h\|^2_{0,\Omega_h} + \beta \|R([\![u_h]\!])\|^2_{0,\Omega_h} \\
&\ge (1-\mu)\|\nabla u_h\|^2_{0,\Omega_h \setminus \Gamma_h^i} + (\beta + 1 - 1/\mu) \|R([\![u_h]\!])\|^2_{0,\Omega_h} \\
&\ge \min(1-\mu, \beta + 1 - 1/\mu) |\!|\!| u_h |\!|\!|_h^2.
\end{aligned}$$

Given $\beta > 0$, it is enough to set $\mu \in (1/(\beta+1), 1)$ for the bilinear form to be coercive. □

A similar result holds true in $V_h + H^1(\Omega_h)$.

Corollary 6.3. *For any $\beta > 0$, there exists $m, M > 0$ independent of h such that*

$$|a_h(u,v)| \le M \|u\|_h \|v\|_h, \tag{6.8}$$

$$m\|u\|_h^2 \le a_h(u,v) \tag{6.9}$$

for any $u, v \in V_h + H^1(\Omega_h)$.

To prove (6.8) and (6.9) it is sufficient to follow step by step the previous proof, remembering that if $u = u_h + w$, with $u_h \in V_h$ and $w \in H^1$, then

$$[\![u]\!] = [\![u_h]\!], \quad R([\![u]\!]) = R([\![u_h]\!]), \quad D_{DG}u = \nabla u + R([\![u_h]\!]) = D_{DG}u_h + \nabla w.$$

6.3. Consistency

Consistency errors in the formulation fall into two types: errors due to the difference between Ω and Ω_h, and errors because the bilinear form a_h is asymptotically consistent even when the two domains are equal.

We will need the following trace inequality for a quasi-uniform family of triangulations: there exists a constant $C > 0$ independent of h such that

$$\|v\|_{0,e}^2 \le C\left(h^{-1}\|v\|_{0,E}^2 + h|v|_{1,E}^2\right) \tag{6.10}$$

for any $v \in H^1(E)$, for any $E \in \mathcal{T}_h$ and $e \in \partial E$. It then follows that

$$\|\{v\}\|_{0,\Gamma_h^i}^2 \le Ch^{-1}\left(\|v\|_{0,\tilde{\Omega}_h}^2 + h^2|v|_{1,\tilde{\Omega}_h\setminus\tilde{\Gamma}_h^i}^2\right) \tag{6.11}$$

for all $v \in H^1(\tilde{\Omega}_h) + V_h$.

Lemma 6.4 (L^2-estimate on the domain differences)**.** *The exists $C > 0$ independent of h such that*

$$\begin{aligned} \|v_h\|_{0,\Omega_h\setminus\Omega} &\le Ch^2\left(\sum_{E\in\mathcal{T}_h}|v_h|_{1,E\cap(\Omega\setminus\Omega_h)}^2\right)^{1/2} \\ \|v_h\|_{0,\Omega\setminus\Omega_h} &\le Ch^2\left(\sum_{E\in\mathcal{T}_h}|v_h|_{1,E\cap(\Omega_h\setminus\Omega)}^2\right)^{1/2} \end{aligned} \tag{6.12}$$

for any $v_h \in V_h^0$ and for h small enough.

Proof. The lemma relies heavily on the facts that the gradient of v_h is constant in each element E and that $v_h = 0$ on Γ_h. Then, in every element E such that $E \cap (\Omega \triangle \Omega_h) \ne \emptyset$ we have

$$\sup_{E\cap(\Omega\triangle\Omega_h)} |v_h| \le Ch^2\left|\nabla v_{h|E}\cdot n\right| \le Ch^2\left|\nabla v_{h|E}\right|$$

where n is the unit normal to Γ_h, constant in E, and we have used the bound on the distance of points in $E \cap (\Omega\triangle\Omega_h)$ to Γ_h in Lemma A.2. Here the constant $C > 0$ is independent of h and the element $E \in \mathcal{T}_h$, for h small enough.

Next,

$$\begin{aligned}\|v_h\|^2_{0,\Omega_h\setminus\Omega} &\leq \sum_{E\in\mathcal{T}_h} \sup_{E\cap(\Omega_h\setminus\Omega)} |v_h|^2 |E\cap(\Omega_h\setminus\Omega)| \\ &\leq \sum_{E\in\mathcal{T}_h} Ch^4 |\nabla v_{h|E}|^2 |E\cap(\Omega_h\setminus\Omega)| \\ &\leq \sum_{E\in\mathcal{T}_h} Ch^4 |v_h|^2_{1,E\cap(\Omega_h\setminus\Omega)},\end{aligned}$$

which gives the first inequality in (6.12). The second inequality follows in a similar way. □

For sake of clarity we state also the following lemma; its proof is a direct consequence of Hardy's inequality in $H^1_0(\Omega)$ (see, *e.g.* [19]) and of Corollary A.3.

Lemma 6.5 (Hardy's inequality on the domain differences)**.** *There exists $C>0$ independent of h such that*

$$\|\tilde{v}\|_{0,\Omega_h\setminus\Omega} + \|v\|_{0,\Omega\setminus\Omega_h} \leq Ch^2|v|_{1,\Omega}, \tag{6.13}$$

for every $v\in H^1_0(\Omega)$.

Another useful estimate about traces is the following. For its proof see *e.g.* Sections 4.3 and 5.3 in [10].

Lemma 6.6. *For $r>0$ let $B_r(\Gamma)=\{x\in\mathbb{R}^2 : d(x,\Gamma)<r\}$ be the neighborhood of Γ with radius r. For r sufficiently small there exists $C>0$ such that for every $v\in H^2(\Omega)$ the following inequality holds true*

$$\left|(2r)^{-1}\int_{B_r(\Gamma)} |\nabla\tilde{v}|\,\mathrm{d}x - \int_\Gamma |\nabla\tilde{v}|\,\mathrm{d}s\right| \leq C\int_{B_r(\Gamma)} |D^2\tilde{v}(x)|\,\mathrm{d}x.$$

Lemma 6.7 (asymptotic consistency)**.** *Given $f\in L^2$ let $\tilde{u}$ denote an extension to $H^2(\mathbb{R})$ of u, the exact solution of* (2.1).

There exists a constant $C>0$ (independent of f and u) such that for h small enough and for every $v_h\in V_h^0$ the following estimate holds true:

$$|a_h(\tilde{u},v_h)-F_h(v_h)| \leq Ch\left(\|u\|_{2,\Omega}+\|\tilde{f}\|_{0,\Omega_h\setminus\Omega}\right)\left(h\Big(\sum_{E\in\mathcal{T}_h}|v_h|^2_{1,E\cap(\Omega_h\setminus\Omega)}\Big)^{1/2} + h^{-1/2}\|\,[\![v_h]\!]\,\|_{0,\Gamma_h^i}\right). \tag{6.14}$$

Moreover for any $\epsilon>0$ there exists $C_\epsilon>0$ (independent of f and u) such that for h small enough and every $v\in H^1_0(\Omega)\cap H^2(\Omega)$ the following estimate holds true:

$$|a_h(\tilde{u},\tilde{v})-F_h(\tilde{v})| \leq C_\epsilon h\left(\|u\|_{2,\Omega}+\|\tilde{f}\|_{0,\Omega_h\setminus\Omega}\right)\left(h\|v\|_{1,\Omega}+h^{1-\epsilon}\|v\|_{2,\Omega}\right), \tag{6.15}$$

where $\tilde{v}$ is an extension of v to $H^2(\mathbb{R}^2)$. Additionally, for every $q>2$ there exists $C_q>0$ (independent of f and u) such that for h small enough and for every $v\in H^1_0(\Omega)\cap W^{2,q}(\Omega)$ the following estimate holds true:

$$|a_h(\tilde{u},\tilde{v})-F_h(\tilde{v})| \leq Ch^2\left(\|u\|_{2,\Omega}+\|\tilde{f}\|_{0,\Omega_h\setminus\Omega}\right)\|v\|_{2,q,\Omega}. \tag{6.16}$$

Proof. We begin by manipulating the following expression:

$$\begin{aligned}\sum_{E\in\mathcal{T}_h}\int_{E\cap\Omega_h}\nabla\tilde{u}\cdot R([\![v_h]\!])\,\mathrm{d}x &= \sum_{E\in\mathcal{T}_h}\int_{E\cap\Omega_h}(\nabla\tilde{u}-\nabla\Pi^0_h u)\cdot R([\![v_h]\!])\,\mathrm{d}x + \sum_{E\in\mathcal{T}_h}\int_{E\cap\Omega_h}\nabla\Pi^0_h u\cdot R([\![v_h]\!])\,\mathrm{d}x \\ &= \sum_{E\in\mathcal{T}_h}\int_{E\cap\Omega_h}(\nabla\tilde{u}-\nabla\Pi^0_h u)\cdot R([\![v_h]\!])\,\mathrm{d}x - \int_{\Gamma_h^i}[\![v_h]\!]\,\{\nabla\Pi^0_h u\}\cdot n\,\mathrm{d}s.\end{aligned}$$

Since $[\![u]\!] = 0$, and thus $R([\![u]\!]) = 0$, we have that

$$a_h(\tilde{u}, v_h) = \sum_{E\in\mathcal{T}_h} \int_{E\cap\Omega_h} (\nabla\tilde{u}\cdot\nabla v_h + \nabla\tilde{u}\cdot R([\![v_h]\!]))\ \mathrm{d}x$$
$$= -\sum_{E\in\mathcal{T}_h} \int_{E\cap\Omega_h} \Delta\tilde{u}\ v_h\ \mathrm{d}x + \int_{\Gamma_h^i} [\![v_h]\!] \left(\nabla\tilde{u} - \{\nabla\Pi_h^0 u\}\right)\cdot n\ \mathrm{d}s + \sum_{E\in\mathcal{T}_h} \int_{E\cap\Omega_h} \left(\nabla\tilde{u} - \nabla\Pi_h^0 u\right)\cdot R([\![v_h]\!])\ \mathrm{d}x.$$

Since u is the exact solution, from Lemma 6.4 we have that

$$\left|\sum_{E\in\mathcal{T}_h} \int_{E\cap\Omega_h} (\Delta\tilde{u} + \tilde{f})\ v_h\ \mathrm{d}x\right| = \left|\sum_{E\in\mathcal{T}_h} \int_{E\cap(\Omega_h\setminus\Omega)} (\Delta\tilde{u} + \tilde{f})\ v_h\ \mathrm{d}x\right|$$
$$\le C(\|u\|_{2,\Omega} + \|\tilde{f}\|_{0,\Omega_h\setminus\Omega})\|v_h\|_{0,\Omega_h\setminus\Omega}$$
$$\le Ch^2(\|u\|_{2,\Omega} + \|\tilde{f}\|_{0,\Omega_h\setminus\Omega}) \left(\sum_{E\in\mathcal{T}_h} |v_h|^2_{1,E\cap(\Omega\setminus\Omega_h)}\right)^{1/2}.$$

From the trace inequality (6.11) and the interpolation estimate in Theorem 4.2 it follows that

$$\left|\int_{\Gamma_h^i} [\![v_h]\!] \left(\nabla\tilde{u} - \{\nabla\Pi_h^0 u\}\right)\cdot n\ \mathrm{d}s\right| \le \|\,[\![v_h]\!]\,\|_{0,\Gamma_h^i} \|\nabla\tilde{u} - \{\nabla\Pi_h^0 u\}\|_{0,\Gamma_h^i}$$
$$\le Ch^{-1/2}\|\,[\![v_h]\!]\,\|_{0,\Gamma_h^i} \left(\|\nabla\tilde{u} - \nabla\Pi_h^0 u\|_{0,\tilde{\Omega}_h\setminus\Gamma_h^i} + h|\nabla\tilde{u} - \nabla\Pi_h^0 u|_{1,\tilde{\Omega}_h\setminus\Gamma_h^i}\right)$$
$$\le Ch^{1/2}\|\,[\![v_h]\!]\,\|_{0,\Gamma_h^i}\|u\|_{2,\Omega}.$$

Finally, from (6.1) and Theorem 4.2 we have

$$\left|\sum_{E\in\mathcal{T}_h} \int_{E\cap\Omega_h} \left(\nabla\tilde{u} - \nabla\Pi_h^0 u\right)\cdot R([\![v_h]\!])\ \mathrm{d}x\right| \le \sum_{E\in\mathcal{T}_h} \|\nabla\tilde{u} - \nabla\Pi_h^0 u\|_{0,E\cap\Omega_h}\|R([\![v_h]\!])\|_{0,E\cap\Omega_h}$$
$$\le C|\!|\!|\tilde{u} - \Pi_h^0 u|\!|\!|_h h^{-1/2}\|\,[\![v_h]\!]\,\|_{0,\Gamma_h^i}$$
$$\le Ch^{1/2}\|u\|_{2,\Omega}\|\,[\![v_h]\!]\,\|_{0,\Gamma_h^i}.$$

Consequently, (6.14) follows.

Let us prove (6.15). Since u is the exact solution, integration by parts gives

$$a_h(\tilde{u}, \tilde{v}) - F_h(\tilde{v}) = \int_{\Omega_h} \nabla\tilde{u}\cdot\nabla\tilde{v}\ - \tilde{f}\tilde{v}\ \mathrm{d}x$$
$$= \int_{\Gamma_h} \tilde{v}\nabla\tilde{u}\cdot n\ \mathrm{d}s - \int_{\Omega_h} (\Delta\tilde{u} + \tilde{f})\tilde{v}\ \mathrm{d}x$$
$$= \int_{\Gamma_h} \tilde{v}\nabla\tilde{u}\cdot n\ \mathrm{d}s - \int_{\Omega_h\setminus\Omega} (\Delta\tilde{u} + \tilde{f})\tilde{v}\ \mathrm{d}x.$$

Then

$$\left|\int_{\Omega_h\setminus\Omega} (\Delta\tilde{u} + \tilde{f})\tilde{v}\ \mathrm{d}x\right| \le C(\|u\|_{2,\Omega} + \|\tilde{f}\|_{0,\Omega_h\setminus\Omega})\|\tilde{v}\|_{0,\Omega_h\setminus\Omega}$$
$$\le Ch^2(\|u\|_{2,\Omega} + \|\tilde{f}\|_{0,\Omega_h\setminus\Omega})|v|_{1,\Omega}$$

where the bound on $\|\tilde{v}\|_{0,\Omega_h\setminus\Omega}$ follows from Hardy's inequality (6.13).

It remains to consider the boundary integral. Remember that $v = 0$ on Γ, so that

$$\left|\int_{\Gamma_h} \tilde{v}\nabla\tilde{u}\cdot n \,\mathrm{d}s\right| = \left|\int_{\Gamma_h} \tilde{v}\nabla\tilde{u}\cdot n \,\mathrm{d}s + \int_{\Gamma} \tilde{v}\nabla\tilde{u}\cdot n \,\mathrm{d}s\right| \leq \int_{\Omega_h\triangle\Omega} |\nabla\tilde{u}\cdot\nabla\tilde{v}| + |\Delta\tilde{u}\tilde{v}| \,\mathrm{d}x,$$

where we have used the divergence theorem in each connected component of $\Omega\triangle\Omega_h$.

Using Hardy's inequality (6.13) the second term is bounded by

$$\int_{\Omega_h\triangle\Omega} |\Delta\tilde{u}\tilde{v}| \,\mathrm{d}x \leq C\|u\|_{2,\Omega}\|v\|_{0,\Omega_h\triangle\Omega} \leq Ch^2\|u\|_{2,\Omega}|v|_{1,\Omega}.$$

In general, by Sobolev extension and embedding we have $\|\nabla\tilde{v}\|_{q,\mathbb{R}^2} \leq C\|v\|_{2,\Omega}$ for C depending on Ω and $q \in [1,+\infty)$. Let $\chi_{\Omega_h\triangle\Omega}$ be the characteristic function of $\Omega_h\triangle\Omega$. Let us choose p, q, r such that $p^{-1}+q^{-1}+s^{-1} = 1$, then, by Hölder's inequality

$$\begin{aligned}\int_{\Omega_h\triangle\Omega} |\nabla\tilde{u}\cdot\nabla\tilde{v}| \,\mathrm{d}x = \int_{\mathbb{R}^2} \chi_{\Omega_h\triangle\Omega}|\nabla\tilde{u}\cdot\nabla\tilde{v}| \,\mathrm{d}x &\leq \|\nabla\tilde{u}\|_{p,\mathbb{R}^2}\|\nabla\tilde{v}\|_{q,\mathbb{R}^2}\|\chi\|_{s,\mathbb{R}^2} \\ &\leq C_{p,q}\|v\|_{2,\Omega}\|u\|_{2,\Omega}|\Omega_h\setminus\Omega|^{1/s} \\ &\leq C_{p,q}\|v\|_{2,\Omega}\|u\|_{2,\Omega}h^{2/s},\end{aligned} \tag{6.17}$$

where $C_{p,q}$ depends on Ω, p and q. As the above inequality holds true for every $p, q \in [1,+\infty)$ we can choose s arbitrarily close to 1, from which our assertion follows.

In the case $\tilde{v} \in W^{2,q}(\Omega)$, $q > 2$, by Sobolev embedding we have $\|\nabla\tilde{v}\|_{\infty,\mathbb{R}^2} \leq C\|v\|_{2,q,\Omega}$. Therefore

$$\int_{\Omega_h\triangle\Omega} |\nabla\tilde{u}\cdot\nabla\tilde{v}| \,\mathrm{d}x \leq C\|v\|_{2,q,\Omega}\int_{\Omega_h\triangle\Omega} |\nabla\tilde{u}| \,\mathrm{d}x.$$

By Corollary A.3 we know that for h sufficiently small $(\Omega_h\triangle\Omega) \subset B_{Ch^2}(\Gamma)$. Then, invoking Lemma 6.6 for $r = Ch^2$ we obtain

$$\int_{\Omega_h\triangle\Omega} |\nabla\tilde{u}| \,\mathrm{d}x \leq \int_{B_{Ch^2}(\Gamma)} |\nabla\tilde{u}| \,\mathrm{d}x \leq Ch^2\int_{\Gamma} |\nabla u| \,\mathrm{d}s + Ch^2\int_{B_{Ch^2}(\Gamma)} |D^2\tilde{u}| \,\mathrm{d}x.$$

The continuity of the trace operator and Hölder's inequality yield

$$\int_{\Omega_h\triangle\Omega} |\nabla\tilde{u}| \,\mathrm{d}x \leq Ch^2\|u\|_{2,\Omega},$$

which concludes the proof. □

Note that the sub-optimal order of convergence in (6.15) is only due to the use of Sobolev embedding in (6.17). Instead of considering higher regularity, as it is for (6.16), it would be possible to recover the optimal order also with better approximation properties for the computational domain Ω_h, for instance when $|\Omega_h\triangle\Omega|$ is of order h^q for $q > 2$.

6.4. Convergence in the broken norm

We can now prove Theorem 4.3.

Lemma 6.8. *Let u be the solution of (2.1) and u_h the solution of (2.12). Then, for h small enough*

$$\|u_h - \Pi_h^0 u\|_h \leq Ch\|u\|_{2,\Omega}, \tag{6.18}$$

for a constant $C > 0$ independent of h and u.

Proof. From the coercivity and continuity of the bilinear form, the asymptotic consistency (6.14) and the approximation error in Theorem 4.2, we have that

$$\begin{aligned}
m\|\Pi_h^0 u - u_h\|_h^2 &\leq a_h(\Pi_h^0 u - u_h, \Pi_h^0 u - u_h) && \text{by (6.7)}\\
&= a_h(\Pi_h^0 u - \tilde{u}, \Pi_h^0 u - u_h) + a_h(\tilde{u} - u_h, \Pi_h^0 u - u_h)\\
&\leq M\|\Pi_h^0 u - \tilde{u}\|_h \|\Pi_h^0 u - u_h\|_h && \text{by (6.8)}\\
&\quad + |a_h(\tilde{u}, \Pi_h^0 u - u_h) - F_h(\Pi_h^0 u - u_h)| && \text{by (2.12)}\\
&\leq Ch\|u\|_{2,\Omega}\|\Pi_h^0 u - u_h\|_h && \text{by (4.3), (6.14)}
\end{aligned}$$

and the proof is concluded. □

Proof of Theorem 4.3. From Theorem 4.2 we have that

$$\|\tilde{u} - \Pi_h^0 u\|_{\tilde{\Omega}_h} \leq Ch\|u\|_{2,\Omega}.$$

Also, from Corollary 5.2 and Lemma 6.8 it follows that

$$\|u_h - \Pi_h^0 u\|_{\tilde{\Omega}_h} \leq C\|u_h - \Pi_h^0 u\|_h \leq C' h\|u\|_{2,\Omega}.$$

We then conclude that

$$\|\tilde{u} - u_h\|_{\tilde{\Omega}_h} \leq \|\Pi_h^0 u - u_h\|_{\tilde{\Omega}_h} + \|\Pi_h^0 u - \tilde{u}\|_{\tilde{\Omega}_h} \leq Ch\|u\|_{2,\Omega},$$

from where (4.4) and (4.5) follow. □

6.5. Convergence in the L^2-norm

We prove convergence in the L^2 norm, Theorem 4.4, using a classical duality argument. To this end, let us introduce the auxiliary elliptic problem

$$\begin{cases} -\triangle w = u - u_h & \text{in } \Omega\\ w = 0 & \text{on } \Gamma. \end{cases}$$

Note that u_h is well defined in Ω and that $u - u_h$ belongs to $L^2(\Omega)$, therefore the above problem is of the same type as (2.1), for $f = u - u_h$. The variational formulation is the following: find $w \in H_0^1(\Omega)$ such that

$$a(w, v) = \int_\Omega (u - u_h)v \,\mathrm{d}x \qquad \text{for every } v \in H_0^1(\Omega). \tag{6.19}$$

It is well known that the solution of the above problem is unique, it belongs to $H^2(\Omega)$ and $\|w\|_{2,\Omega} \leq C\|u - u_h\|_{0,\Omega}$, being Ω of class C^2. The discrete problem consists in finding $w_h \in V_h^0$ such that

$$a_h(w_h, v_h) = \int_{\Omega_h} (\tilde{u} - u_h) v_h \, \mathrm{d}x \qquad \text{for every } v_h \in V_h^0, \tag{6.20}$$

obtained by setting $\tilde{f} = \tilde{u} - u_h$.

Proof of Theorem 4.4. In the following we denote with C and C_ϵ positive constants independent of h and u that may change from line to line.

The approximation estimate of Theorem 4.3 leads to

$$\begin{aligned}\left(\sum_{E \in \mathcal{T}_h} |u_h|^2_{1, E \cap (\Omega_h \setminus \Omega)} \right)^{1/2} &\leq \| \tilde{u} - u_h \|_h + \| \tilde{u} \|_h \\ &\leq C' h \|u\|_{2,\Omega} + C' \|u\|_{2,\Omega} \leq C \|u\|_{2,\Omega},\end{aligned} \tag{6.21}$$

and similarly

$$\left(\sum_{E \in \mathcal{T}_h} |u_h|^2_{1, E \cap (\Omega \setminus \Omega_h)} \right)^{1/2} \leq \| \tilde{u} - u_h \|_\Omega + \| \tilde{u} \|_\Omega \leq C \|u\|_{2,\Omega}, \tag{6.22}$$

where the two inequalities hold true for h sufficiently small. Using again Theorem 4.3 we estimate the jump term with

$$h^{-1/2} \| [\![u_h]\!] \|_{0, \Gamma_h^i} \leq \| \tilde{u} - u_h \|_h \leq C h \|u\|_{2,\Omega}. \tag{6.23}$$

From (6.21), (6.22), Hardy's inequality (6.13) and Lemma 6.4 we can write

$$\begin{aligned}\|\tilde{u} - u_h\|_{0, \Omega \triangle \Omega_h} &\leq \|\tilde{u}\|_{0, \Omega \triangle \Omega_h} + \|u_h\|_{0, \Omega \triangle \Omega_h} \\ &\leq C h^2 \left(\|u\|_{1,\Omega} + \left(\sum_{E \in \mathcal{T}_h} |u_h|^2_{1, E \cap (\Omega \triangle \Omega_h)} \right)^{1/2} \right) \leq C h^2 \|u\|_{2,\Omega}.\end{aligned} \tag{6.24}$$

Using the estimates (6.14) and (6.15) of the consistency error we have

$$\begin{aligned}\left| a_h(\tilde{w}, \tilde{u} - u_h) - \int_{\Omega_h} (\tilde{u} - u_h)^2 \, \mathrm{d}x \right| &\leq C_\epsilon h \left(\|w\|_{H^2(\Omega)} + \|\tilde{u} - u_h\|_{0, \Omega_h \setminus \Omega} \right) \\ &\times \left(h \left(\sum_{E \in \mathcal{T}_h} |u_h|^2_{1, E \cap (\Omega_h \setminus \Omega)} \right)^{1/2} + h^{-1/2} \| [\![u_h]\!] \|_{0, \Gamma_h^i} + h \|u\|_{1,\Omega} + h^{1-\epsilon} \|u\|_{2,\Omega} \right).\end{aligned}$$

Now, the elliptic estimate for w, (6.21), (6.23) and (6.24) give that for h sufficiently small

$$\begin{aligned}\left| a_h(\tilde{w}, \tilde{u} - u_h) - \int_{\Omega_h} (\tilde{u} - u_h)^2 \, \mathrm{d}x \right| &\leq C_\epsilon h \left(\|u - u_h\|_{0,\Omega} + h^2 \|u\|_{2,\Omega} \right) \left(h \|u\|_{2,\Omega} + h^{1-\epsilon} \|u\|_{2,\Omega} \right) \\ &\leq C_\epsilon h^{2-\epsilon} (\|u - u_h\|_{0,\Omega} + h^2 \|u\|_{2,\Omega}) \|u\|_{2,\Omega}.\end{aligned} \tag{6.25}$$

Let w_h be the solution of (6.20), then by (6.14), Theorem 4.3 and the elliptic estimate for w we obtain

$$\begin{aligned} |a_h(w_h, \tilde{u} - u_h)| &= |a_h(\tilde{u}, w_h) - F_h(w_h)| \\ &\le Ch\left(\|u\|_{2,\Omega} + \|\tilde{f}\|_{0,\Omega_h\setminus\Omega}\right)\left(h\left(\sum_{E\in\mathcal{T}_h} |w_h|^2_{1,E\cap(\Omega_h\setminus\Omega)}\right)^{1/2} + h^{-1/2}\| [\![w_h]\!] \|_{0,\Gamma_h^i}\right). \end{aligned}$$

Arguing as above and using the elliptic estimate for w yields

$$h\left(\sum_{E\in\mathcal{T}_h} |w_h|^2_{1,E\cap(\Omega_h\setminus\Omega)}\right)^{1/2} + h^{-1/2}\| [\![w_h]\!] \|_{0,\Gamma_h^i} \le Ch\|w\|_{2,\Omega} \le Ch\|u - u_h\|_{0,\Omega},$$

which together with (2.15) gives

$$|a_h(w_h, \tilde{u} - u_h)| \le Ch^2\|u\|_{2,\Omega}\|u - u_h\|_{0,\Omega}.$$

By continuity, the elliptic estimate for w, and by the error estimates in the triple norm we also have

$$\begin{aligned} |a_h(\tilde{w} - w_h, \tilde{u} - v_h)| &\le C|\!|\!|\tilde{w} - w_h|\!|\!|_h |\!|\!|\tilde{u} - u_h|\!|\!|_h \\ &\le Ch^2\|w\|_{2,\Omega}\|u\|_{2,\Omega} \le Ch^2\|u - u_h\|_{0,\Omega}\|u\|_{2,\Omega}. \end{aligned}$$

Therefore, from (6.25) and the two previous estimates we get for h sufficiently small

$$\begin{aligned} \|\tilde{u} - u_h\|^2_{0,\Omega_h} &\le |a_h(\tilde{w}, \tilde{u} - u_h)| + C_\epsilon h^{2-\epsilon}(\|u - u_h\|_{0,\Omega} + h^2\|u\|_{2,\Omega})\|u\|_{2,\Omega} \\ &\le |a_h(\tilde{w} - w_h, \tilde{u} - u_h)| + |a_h(w_h, \tilde{u} - u_h)| + C_\epsilon h^{2-\epsilon}\|u\|_{2,\Omega}\left(\|u - u_h\|_{0,\Omega} + h^2\|u\|_{2,\Omega}\right) \\ &\le C_\epsilon h^{2-\epsilon}\|u\|_{2,\Omega}(\|u - u_h\|_{0,\Omega} + h^2\|u\|_{2,\Omega}). \end{aligned}$$

Hence

$$\|u - u_h\|^2_{0,\Omega} - \|u - u_h\|^2_{0,\Omega\setminus\Omega_h} \le \|\tilde{u} - u_h\|^2_{0,\Omega_h} \le C_\epsilon h^{2-\epsilon}\|u\|_{2,\Omega}\|u - u_h\|_{0,\Omega}.$$

Using (6.24) we can bound $\|u - u_h\|_{0,\Omega\setminus\Omega_h}$ from below as follows

$$\|u - u_h\|^2_{0,\Omega} - C\|u\|^2_{2,\Omega}h^4 \le C_\epsilon h^{2-\epsilon}\|u\|_{2,\Omega}\|u - u_h\|_{0,\Omega}, \tag{6.26}$$

or equivalently,

$$\|u - u_h\|^2_{0,\Omega} \le C_\epsilon h^{2-\epsilon}\|u\|_{2,\Omega}\|u - u_h\|_{0,\Omega} + C\|u\|^2_{2,\Omega}h^4.$$

Then, for h sufficiently small a simple algebraic computation leads to

$$\|u - u_h\|_{0,\Omega} \le C_\epsilon h^{2-\epsilon}\|u\|_{2,\Omega},$$

where C_ϵ is the constant in (6.26), and hence independent of u and h. A similar calculation when $u \in W^{2,p}(\Omega)$, $p > 2$, leads to the slightly better estimate (4.7). □

A. Appendix: Lemmas about the approximate domain

Let E such that $E \cap \Gamma_h \neq \emptyset$. Denote with v_1, v_2, v_3 its vertices. Because of (3.2), the signs of $d_h(v_i) = d(v_i)$, $i = 1, 2, 3$, cannot all be equal.

Lemma A.1 (intersection between elements and boundary)**.** *There exists $h_0 > 0$ such that, whenever $h < h_0$, we have*

$$E \cap \Gamma \neq \emptyset \iff E \cap \Gamma_h \neq \emptyset \tag{A.1}$$

for any $E \in \mathcal{T}_h$. Under these conditions, let v_2 and v_3 be two vertices of the triangle E with $d(v_2)d(v_3) > 0$ and let e_{23} be the edge with end points v_2 and v_3. Then $d(x)d(v_2) > 0$ for all $x \in e_{23}$.

Proof. Clearly, the implication $E \cap \Gamma_h \neq \emptyset \Longrightarrow E \cap \Gamma \neq \emptyset$ is trivial, since $E \cap \Gamma_h \neq \emptyset$ implies that the values of the signed distance function d at the three nodes of E cannot all have the same sign. Then, the continuity of d ensures that $d(x) = 0$ for some point $x \in E$.

To prove the converse, let $r > 0$ be a constant for which the representation in (3.1) is valid for any open ball B of radius r, and let $h \in (0, r)$. Consider $E \in \mathcal{T}_h$ such that $E \cap \Gamma \neq \emptyset$, and let $x_m \in E \cap \Gamma$. In the ball $B_r(x_m) = \{x \in \mathbb{R}^2 \colon |x - x_m| < r\}$ we can write $x_m = (y_1^m, \Phi(y_1^m))$. Clearly $E \subset B_h(x_m) \subset B_r(x_m)$, and it follows from (3.1) that

$$(E \cap \Gamma) \subset (B_h(x_m) \cap \Gamma) = B_h(x_m) \cap \left\{y \in \mathbb{R}^2 \colon y_2 = \Phi(y_1)\right\}.$$

Let

$$y_1^0 = \inf\{y_1 \in \mathbb{R} \colon (y_1, \Phi(y_1)) \in B_h(x_m) \cap \Gamma\}$$
$$y_1^1 = \sup\{y_1 \in \mathbb{R} \colon (y_1, \Phi(y_1)) \in B_h(x_m) \cap \Gamma\}.$$

Clearly

$$|y_1^i - y_1^m| \leq h \qquad \text{for } i = 0, 1. \tag{A.2}$$

For

$$\bar{\Phi}' = \frac{\Phi(y_1^1) - \Phi(y_1^0)}{y_1^1 - y_1^0}$$

let

$$\bar{\Phi}(y_1) = \Phi(y_1^0) + \bar{\Phi}'(y_1 - y_1^0)$$

be the affine interpolant in y_1^i for $i = 0, 1$. By a standard inequality for linear interpolation in $W^{2,\infty}$ we get

$$\left|\Phi(y_1) - \bar{\Phi}(y_1)\right| \leq CMh^2 \tag{A.3}$$

for any $y_1 \in (y_1^0, y_1^1)$, where C does not depend on h and M. Now we write (A.3) in a more convenient way as

$$\bar{\Phi}(y_1) - CMh^2 \leq \Phi(y_1) \leq \bar{\Phi}(y_1) + CMh^2$$

for all $y_1 \in (y_1^0, y_1^1)$. Geometrically, $B_r(x_m) \cap \Gamma$ is contained between two straight lines, represented by $\bar{\Phi} \pm CMh^2$.

The nodes of the triangle E can only be located at a distance larger than or equal to ηh from Γ. This implies that for h small enough the nodes of the triangles cannot lie in the region delimited by these two straight lines. More precisely, if $h < h_0 = \eta/2CM$ then $0 < CMh^2 < \eta h - CMh^2$, hence

$$\bar{\Phi}(y_1) - \eta h + CMh^2 \leq \Phi(y_1) \leq \bar{\Phi}(y_1) + \eta h - CMh^2$$

for all $y_1 \in (y_1^0, y_1^1)$. Moreover, the nodes of E need to lie in $A^+ \cup A^-$, where

$$A^{\pm} = B_h(x_m) \cap \left\{x = (y_1, y_2) \in \mathbb{R}^2 \colon \pm y_2 \geq \pm \bar{\Phi}(y_1) + (\eta h - CMh^2)\right\}.$$

Because of (3.1), $A^+ \subset \Omega$ and $A^- \subset (\mathbb{R}^2 \setminus \overline{\Omega})$ and clearly $A^{\pm} \cap \Gamma = \emptyset$. Consequently, the signed distance function d is negative in A^+ and positive in A^-. Since $A^{\pm}$ is a convex region, it follows that $E \in A^{\pm}$ whenever its three vertices are in $A^{\pm}$. But $E \cap \Gamma \neq \emptyset$ and so there is at least one vertex of E on each region. It then follows that the signed distance function cannot have the same sign at all three vertices, and hence that $E \cap \Gamma_h \neq \emptyset$. Additionally, since two of the vertices lie in the same convex region $A^{\pm}$, the segment that joins them belongs there as well. Consequently all points in the segment have the same sign of the distance function as the end points. This concludes the proof. □

Under the assumptions and notation of Lemma A.1, given $x \in \overline{E} \setminus \{v_1\}$, let L_x be the straight line that joins v_1 with x. We denote by $y_x^h = \overline{E} \cap \Gamma_h \cap L_x$, which clearly exists and is unique, and by y_x one of the points in the set $\overline{E} \cap \Gamma \cap L_x$. This set is clearly non-empty, since the sign of $d(v_1)$ is different than the sign of d on the edge e_{23} with end points v_2 and v_3.

Lemma A.2 (polar parameterization for $\Omega \triangle \Omega_h$)**.** *There exists $h_0 > 0$ and $C > 0$ such that if $h < h_0$ then for any $E \in \mathcal{T}_h$ such that $E \cap \Gamma \neq \emptyset$:*

(1) *the set $\overline{E} \cap \Gamma \cap L_x$ has only one element, and hence y_x is well defined for any $x \in \overline{E} \setminus \{v_1\}$;*
(2) *the set $L_x \cap \Omega \triangle \Omega_h \cap \overline{E}$ is a line segment with end points y_x and y_x^h and $|y_x - y_x^h| \leq Ch^2$ for any $x \in \overline{E} \setminus \{v_1\}$.*

Proof. Since Ω is assumed to be C^2-regular, the signed distance function d is C^2 in $U_r = \{x \in \mathbb{R}^2 \colon |d(x)| \leq r\}$ for some $r > 0$, and we denote $M = \sup_{U_r} |D^2 d|$. Let $0 < h_0^* < r$ be such that Lemma A.1 holds. Consider $E \in \mathcal{T}_h$ for $h < \min(h_0^*, \eta/M)$ such that $E \cap \Gamma \neq \emptyset$ and hence $E \cap \Gamma_h \neq \emptyset$.

Because $h < r$, we have that $E \subset U_r$, and hence that d is C^2 in $\overline{E}$. In particular, by Taylor expansion and Lipschitz continuity

$$|d(x_2) - d(x_1) - \nabla d(x_1) \cdot (x_2 - x_1)| \leq Mh^2 \tag{A.4}$$

$$|\nabla d(x_2) - \nabla d(x_1)| \leq M|x_2 - x_1| \leq Mh \tag{A.5}$$

for any $x_1, x_2 \in \overline{E}$.

Let $\lambda_i \in [0, 1]$, $i = 1, 2, 3$, $\lambda_1 < 1$, be the baricentric coordinates of $x \in \overline{E} \setminus \{v_1\}$, which satisfy $\lambda_1 + \lambda_2 + \lambda_3 = 1$ and $x = \sum_{i=1}^{3} \lambda_i v_i$. We can write $x - v_1 = \lambda_2(v_2 - v_1) + \lambda_3(v_3 - v_1)$, and hence the directional derivative of d at x along L_x is

$$\nabla d(x) \cdot (x - v_1) = \lambda_2 \nabla d(x) \cdot (v_2 - v_1) + \lambda_3 \nabla d(x) \cdot (v_3 - v_1). \tag{A.6}$$

Since $d(v_2)$ and $d(v_3)$ have the same sign and opposite to that of $d(v_1)$, from (3.2) and (A.4) we have that for $i = 2, 3$

$$0 < 2\eta h - Mh^2 < |d(v_i) - d(v_1)| - Mh^2 \leq |\nabla d(v_1) \cdot (v_i - v_1)|, \tag{A.7}$$

and it is simple to see from (A.4) that $\nabla d(v_1) \cdot (v_i - v_1)$ has the same sign for $i = 2, 3$. Combining (A.6) with (A.7) and (A.5) and using the fact that $(1 - \lambda_1)h \geq |x - v_1|$ we get

$$\begin{aligned}
|\nabla d(x) \cdot (x - v_1)| &\geq |\lambda_2 \nabla d(v_1) \cdot (v_2 - v_1) + \lambda_3 \nabla d(v_1) \cdot (v_3 - v_1)| - |\nabla d(x) - \nabla d(v_1)||x - v_1| \\
&\geq (1 - \lambda_1)(2\eta h - Mh^2) - M|x - v_1|^2 \\
&\geq 2|x - v_1| \, (\eta - Mh) \\
&> 0
\end{aligned}$$

so the distance function is monotone along each line joining vertex v_1 with a point in E. In particular, this shows that $d(x) = 0$ has a unique solution therein. Consequently, the set $\overline{E} \cap \Gamma \cap L_x$ has only one element and y_x is well defined for any $x \in E$. The monotonicity of d_h and d on L_x implies that $\{x \in L_x : d_h(x)d(x) < 0\}$ is the line segment with end points y_x and y_x^h. Remember that $\|d - d_h\|_{\infty,E} \leq Ch^2$. Hence $d_h(y_x^h) = 0$ and $|d_h(y_x)| \leq Ch^2$. Let $\hat{e} = (x - v_1)/|x - v_1|$, and set $z = e_{23} \cap L_x$. Since $|d_h(z) - d_h(v_1)| \geq \eta h$ and since $|z - v_1| \leq h$ we get $|\nabla d_h \cdot \hat{e}| \geq \eta$. By linearity,

$$|d_h(y)| = |\nabla d_h \cdot \hat{e} \, (y - y_x^h)| \geq \eta |y - y_x^h|$$

for $y \in L_x$. Then,

$$Ch^2 \geq |d_h(y_x)| \geq \eta |y_x - y_x^h|$$

which gives the last assertion. □

Corollary A.3. *There exists $C > 0$ such that for h sufficiently small we have*

$$d_H(\Gamma, \Gamma_h) \leq Ch^2,$$

where d_H denote the Hausdorff distance.

Proof. From Lemma A.2 there exists $C > 0$ independent of $h < h_0$ such that

$$|d(y_h, \Gamma)| = \inf_{z \in \Gamma} |y_h - z| \leq Ch^2 \qquad \text{and} \qquad |d(y, \Gamma_h)| = \inf_{z_h \in \Gamma_h} |z_h - y| \leq Ch^2,$$

for any $y \in \Gamma$ and $y_h \in \Gamma_h$. □

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