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AN *A POSTERIORI* ERROR ANALYSIS FOR DYNAMIC VISCOELASTIC PROBLEMS *

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Abstract. In this paper, a dynamic viscoelastic problem is numerically studied. The variational problem is written in terms of the velocity field and it leads to a parabolic linear variational equation. A fully discrete scheme is introduced by using the finite element method to approximate the spatial variable and an Euler scheme to discretize time derivatives. An *a priori* error estimates result is recalled, from which the linear convergence is derived under suitable regularity conditions. Then, an *a posteriori* error analysis is provided, extending some preliminary results obtained in the study of the heat equation and quasistatic viscoelastic problems. Upper and lower error bounds are obtained. Finally, some two-dimensional numerical simulations are presented to show the behavior of the error estimators.

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Introduction

In this paper, a dynamic problem involving a viscoelastic body is considered from the numerical point of view. Viscoelastic materials have been studied in the past thirty years and they are interesting because many metals or crystals can be modeled by using viscoelasticity theory. We recall, for instance, the well-known Kelvin-Voigt viscoelastic constitutive law.

Since the first results provided by [13], many works dealing with mathematical problems including viscoelastic materials have been published (see, for instance [6,11,12,14–16,23,24]) or with their numerical analysis (see, *e.g.*, [1,3,20,22,26,29]). Recently, a large number of quasistatic contact problems including a more general constitutive law have been analyzed from both points of view (see the monograph [18] and the numerous references cited therein).

In this paper, we revisite a well-known dynamic problem involving a linear viscoelastic body. An *a priori* analysis is recalled (to our knowledge, it was not published yet), by using some ideas employed in [7] for the

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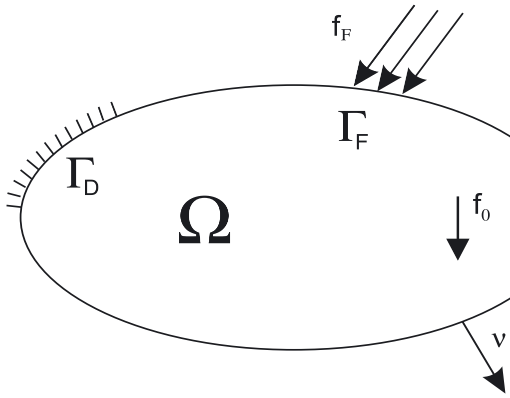

FIGURE 1. Physical setting: a viscoelastic body.

case including the contact with a deformable obstacle and the mechanical damage. However, some additional regularity conditions are required on the continuous solution. Then, an *a posteriori* error analysis is provided extending some arguments already applied in the study of the heat equation (see, *e.g.*, [25,28]), some parabolic equations [4], the Stokes equation [5] or the recently considered quasistatic case [17]. As far as we know, this is the first time when the *a posteriori* error techniques are applied to the study of dynamic problems in solid mechanics.

The paper is structured as follows. In Section 1, the mechanical model and its variational formulation are described following the notation and assumptions introduced in [8,21]. Then, a fully discrete scheme is introduced in Section 2, by using the finite element method to approximate the spatial variable and an Euler scheme to discretize the time derivatives. An *a priori* error estimates result, obtained proceeding as in the case of a contact problem with a deformable obstacle, is recalled. Then, extending some results obtained in the study of quasistatic viscoelastic problems and the heat equation, an *a posteriori* error analysis is done in Section 3, providing an upper bound for the error, Theorem 3.1, and a lower bound, Theorem 3.2. Finally, some numerical simulations, involving two-dimensional examples, are presented in Section 4.

1. Mechanical problem and its variational formulation

In this section, we present a brief description of the model (details can be found in [8,21]).

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, denote a domain occupied by a viscoelastic body with a smooth boundary $\Gamma = \partial\Omega$ decomposed into two disjoint parts Γ_D and Γ_F such that meas $(\Gamma_D) > 0$. Moreover, let $[0, T]$, $T > 0$, be the time interval of interest and denote by $\boldsymbol{\nu}$ the unit outer normal vector to Γ (see Fig. 1).

Let $\boldsymbol{x} \in \Omega$ and $t \in [0, T]$ be the spatial and time variables, respectively, and, in order to simplify the writing, we do not indicate the dependence of the functions on $\boldsymbol{x}$ and t. Moreover, a dot above a variable represents the derivative with respect to the time variable.

Let $\boldsymbol{u}$, $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u}))_{i,j=1}^d$ denote the displacement field, the stress tensor and the linearized strain tensor, respectively. We recall that

$$\varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad i, j = 1, \ldots, d.$$

The body is assumed viscoelastic and it satisfies the following constitutive law (see, for instance, [13], Chap. 3),

$$\boldsymbol{\sigma} = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}), \tag{1.1}$$

where $\mathcal{A} = (a_{ijkl})$ and $\mathcal{B} = (b_{ijkl})$ are the fourth-order viscous and elastic tensors, respectively.

We turn now to describe the boundary conditions.

On the boundary part Γ_D we assume that the body is clamped and thus the displacement field vanishes there (and so $\boldsymbol{u} = \boldsymbol{0}$ on $\Gamma_D \times (0,T)$). Moreover, we assume that a density of traction forces, denoted by $\boldsymbol{f}_F$, acts on the boundary part Γ_F; *i.e.*

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_F \quad \text{on} \quad \Gamma_F \times (0,T).$$

Denote by $\mathbb{S}^d$ the space of second order symmetric tensors on $\mathbb{R}^d$ and by "$\cdot$" and $\|\cdot\|$ the inner product and the Euclidean norms on $\mathbb{R}^d$ and $\mathbb{S}^d$.

The mechanical problem of the dynamic deformation of a viscoelastic body is then written as follows.

Problem P. Find a displacement field $\boldsymbol{u} : \Omega \times (0,T) \to \mathbb{R}^d$ and a stress field $\boldsymbol{\sigma} : \Omega \times (0,T) \to \mathbb{S}^d$ such that,

$$\boldsymbol{\sigma} = \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}) \quad \text{in} \quad \Omega \times (0,T), \tag{1.2}$$

$$\rho\ddot{\boldsymbol{u}} - \operatorname{Div}\boldsymbol{\sigma} = \boldsymbol{f}_0 \quad \text{in} \quad \Omega \times (0,T), \tag{1.3}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on} \quad \Gamma_D \times (0,T), \tag{1.4}$$

$$\boldsymbol{\sigma\nu} = \boldsymbol{f}_F \quad \text{on} \quad \Gamma_F \times (0,T), \tag{1.5}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \quad \dot{\boldsymbol{u}}(0) = \boldsymbol{v}_0 \quad \text{in} \quad \Omega. \tag{1.6}$$

Here, $\rho > 0$ is the density of the material (which is assumed constant for simplicity), $\boldsymbol{u}_0$ and $\boldsymbol{v}_0$ represent initial conditions for the displacement and velocity fields, respectively, and $\boldsymbol{f}_0$ denotes the density of body forces.

In order to obtain the variational formulation of Problem P, let us denote by $H = [L^2(\Omega)]^d$ and construct the variational spaces V and Q as follows,

$$V = \{\boldsymbol{w} \in [H^1(\Omega)]^d \,;\, \boldsymbol{w} = \boldsymbol{0} \quad \text{on} \quad \Gamma_D\},$$
$$Q = \{\boldsymbol{\tau} = (\tau_{ij})_{i,j=1}^d \in [L^2(\Omega)]^{d\times d} \;\;;\;\; \tau_{ij} = \tau_{ji}, \;\; i,j = 1,\ldots,d\}.$$

We will make the following assumptions on the problem data.

The viscosity tensor $\mathcal{A}(\boldsymbol{x}) = (a_{ijkl}(\boldsymbol{x}))_{i,j,k,l=1}^d : \boldsymbol{\tau} \in \mathbb{S}^d \to \mathcal{A}(\boldsymbol{x})(\boldsymbol{\tau}) \in \mathbb{S}^d$ satisfies:

$$\begin{aligned}
&\text{(a) } a_{ijkl} = a_{klij} = a_{jikl} \quad \text{for} \quad i,j,k,l = 1,\ldots,d.\\
&\text{(b) } a_{ijkl} \in L^\infty(\Omega) \quad \text{for} \quad i,j,k,l = 1,\ldots,d.\\
&\text{(c) There exists } m_{\mathcal{A}} > 0 \text{ such that } \mathcal{A}(\boldsymbol{x})\boldsymbol{\tau}\cdot\boldsymbol{\tau} \geq m_{\mathcal{A}}\|\boldsymbol{\tau}\|^2\\
&\qquad \forall\,\boldsymbol{\tau} \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega.
\end{aligned} \tag{1.7}$$

The elastic tensor $\mathcal{B}(\boldsymbol{x}) = (b_{ijkl}(\boldsymbol{x}))_{i,j,k,l=1}^d : \boldsymbol{\tau} \in \mathbb{S}^d \to \mathcal{B}(\boldsymbol{x})(\boldsymbol{\tau}) \in \mathbb{S}^d$ satisfies:

$$\begin{aligned}
&\text{(a) } b_{ijkl} = b_{klij} = b_{jikl} \quad \text{for} \quad i,j,k,l = 1,\ldots,d.\\
&\text{(b) } b_{ijkl} \in L^\infty(\Omega) \quad \text{for} \quad i,j,k,l = 1,\ldots,d.\\
&\text{(c) There exists } m_{\mathcal{B}} > 0 \text{ such that } \mathcal{B}(\boldsymbol{x})\boldsymbol{\tau}\cdot\boldsymbol{\tau} \geq m_{\mathcal{B}}\|\boldsymbol{\tau}\|^2\\
&\qquad \forall\,\boldsymbol{\tau} \in \mathbb{S}^d, \text{ a.e. } \boldsymbol{x} \in \Omega.
\end{aligned} \tag{1.8}$$

The following regularity is assumed on the density of volume forces and tractions:

$$\boldsymbol{f}_0 \in C([0,T];H), \quad \boldsymbol{f}_F \in C([0,T];[L^2(\Gamma_F)]^d). \tag{1.9}$$

Using Riesz' theorem, from (1.9) we can define the element $\boldsymbol{f}(t) \in V'$ given by

$$\langle \boldsymbol{f}(t), \boldsymbol{w}\rangle_{V'\times V} = \int_\Omega \boldsymbol{f}_0(t)\cdot\boldsymbol{w}\,\mathrm{d}\boldsymbol{x} + \int_{\Gamma_F} \boldsymbol{f}_F(t)\cdot\boldsymbol{w}\,\mathrm{d}\Gamma \quad \forall \boldsymbol{w} \in V,$$

and then $\boldsymbol{f} \in C([0,T];V')$. Finally, we assume that the initial displacement and velocity satisfy

$$\boldsymbol{u}_0, \boldsymbol{v}_0 \in V. \tag{1.10}$$

Plugging (1.2) into (1.3) and using the previous boundary conditions, applying a Green's formula we derive the following variational formulation of Problem P, written in terms of the velocity field $\boldsymbol{v}(t) = \dot{\boldsymbol{u}}(t)$.

Problem VP. Find a velocity field $\boldsymbol{v} : [0,T] \to V$ such that $\boldsymbol{v}(0) = \boldsymbol{v}_0$ and for a.e. $t \in (0,T)$ and for all $\boldsymbol{w} \in V$,

$$\langle \rho \dot{\boldsymbol{v}}(t), \boldsymbol{w} \rangle_{V' \times V} + (\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}(t)) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{w}))_Q = \langle \boldsymbol{f}(t), \boldsymbol{w} \rangle_{V' \times V}, \tag{1.11}$$

where the displacement field $\boldsymbol{u}(t)$ is given by

$$\boldsymbol{u}(t) = \int_0^t \boldsymbol{v}(s)\,\mathrm{d}s + \boldsymbol{u}_0. \tag{1.12}$$

Proceeding as in [21], where also the contact with a deformable obstacle, the mechanical damage and the adhesion were considered, we have the following.

Theorem 1.1. *Let assumptions (1.7)–(1.10) hold. Therefore, there exists a unique solution to Problem VP. Moreover, this solution has the regularity*

$$\boldsymbol{v} \in C^1([0,T];H) \cap C([0,T];V).$$

We notice that the above regularity allows us to obtain the following relation,

$$\langle \rho \dot{\boldsymbol{v}}(t), \boldsymbol{w} \rangle_{V' \times V} = (\rho \dot{\boldsymbol{v}}(t), \boldsymbol{w})_H \quad \forall \boldsymbol{w} \in V.$$

2. Fully discrete approximations: *a priori* error estimates

In this section, we now introduce a finite element algorithm to approximate solutions to Problem VP.

The discretization of Problem VP is done as follows. First, we assume that Ω is a polyhedral domain and we consider a finite dimensional space $V^h \subset V$, approximating the variational space V, given by

$$V^h = \{\boldsymbol{w}^h \in [C(\overline{\Omega})]^d \,;\, \boldsymbol{w}^h_{|_T} \in [P_1(T)]^d \quad T \in \mathcal{T}^h, \quad \boldsymbol{w}^h = \boldsymbol{0} \text{ on } \Gamma_D\}, \tag{2.1}$$

where $P_1(T)$ represents the space of polynomials of global degree less or equal to one in T and we denote by $(\mathcal{T}^h)_{h>0}$ a regular family of triangulations of $\overline{\Omega}$ (in the sense of [9]), compatible with the decomposition of the boundary $\Gamma = \partial\Omega$ into its parts Γ_D and Γ_F; *i.e.* the finite element space V^h is composed of continuous and piecewise affine functions. Let h_T be the diameter of an element $T \in \mathcal{T}^h$ and let $h = \max_{T \in \mathcal{T}^h} h_T$ denote the spatial discretization parameter. Moreover, we assume that the discrete initial conditions, denoted by $\boldsymbol{u}_0^h$ and $\boldsymbol{v}_0^h$, are given by

$$\boldsymbol{u}_0^h = \mathcal{P}^h \boldsymbol{u}_0, \quad \boldsymbol{v}_0^h = \mathcal{P}^h \boldsymbol{v}_0, \tag{2.2}$$

where $\mathcal{P}^h$ is the $[L^2(\Omega)]^d$-projection operator on V^h.

To discretize the time derivatives, we consider a uniform partition of the time interval $[0,T]$, denoted by $0 = t_0 < t_1 < \ldots < t_N = T$, and let k be the time step size, $k = T/N$. For a continuous function $f(t)$, let $f_n = f(t_n)$ and, for a sequence $\{w_n\}_{n=0}^N$, we let $\delta w_n = (w_n - w_{n-1})/k$ be its corresponding divided differences.

Finally, in order to simplify the writing, we assume, without loss of generality, that $\rho = 1$.

Therefore, using the implicit Euler scheme, we obtain the following fully discrete approximation of Problem VP.

Problem VP^{hk}. Find a discrete velocity field $\boldsymbol{v}^{hk} = \{\boldsymbol{v}_n^{hk}\}_{n=0}^N \subset V^h$ such that $\boldsymbol{v}_0^{hk} = \boldsymbol{v}_0^h$ and for all $n = 1, \dots, N$ and $\boldsymbol{w}^h \in V^h$,

$$(\delta \boldsymbol{v}_n^{hk}, \boldsymbol{w}^h)_H + (\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk}) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_n^{hk}), \boldsymbol{\varepsilon}(\boldsymbol{w}^h))_Q = (\boldsymbol{f}_n, \boldsymbol{w}^h)_V, \tag{2.3}$$

where the discrete displacement field $\boldsymbol{u}^{hk} = \{\boldsymbol{u}_n^{hk}\}_{n=0}^N \subset V^h$ is given by

$$\boldsymbol{u}_n^{hk} = \sum_{j=1}^n k \boldsymbol{v}_j^{hk} + \boldsymbol{u}_0^h. \tag{2.4}$$

Using Lax-Milgram lemma, it is easy to obtain the following theorem which states the existence of a unique discrete solution $\boldsymbol{v}^{hk} \subset V^h$ to Problem VP^{hk}.

Theorem 2.1. *Let assumptions (1.7)–(1.10) hold. Therefore, there exists a unique solution to Problem VP^{hk}.*

Here, we use the Euler implicit method instead of the explicit one because the constitutive law is assumed linear. As it was also noticed for quasistatic problems, this scheme should be replaced by its explicit version when the constitutive functions are nonlinear, in order to avoid the use of fixed-point iterations (see [18, Chap. 9]).

Now, and in the rest of this section, we recall some *a priori* error estimates for Problem VP^{hk}. It is based on the arguments employed in [7] and we refer the reader there for details.

Proceeding like in [7,8], we have the following.

Theorem 2.2. *Let assumptions (1.7)–(1.10) hold. Let us denote by $\boldsymbol{v}$ and $\boldsymbol{v}^{hk}$ the respective solutions to Problems VP and VP^{hk}. Therefore, there exists a positive constant $c > 0$, independent of the discretization parameters h and k but depending on the continuous solution $\boldsymbol{v}$ and the problem data, such that for all $\{\boldsymbol{w}_n^h\}_{n=0}^N \subset V^h$,*

$$\begin{aligned}\max_{0\le n\le N} \left\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\right\|_H^2 &+ \sum_{j=1}^N k \left\|\boldsymbol{v}_j - \boldsymbol{v}_j^{hk}\right\|_V^2 \le c\Bigg(\max_{1\le n\le N} \left\|\boldsymbol{v}_n - \boldsymbol{w}_n^h\right\|_V^2 \\ &+ \max_{1\le n\le N} \left\|\dot{\boldsymbol{v}}_n - \delta\boldsymbol{v}_n\right\|_H^2 + \left\|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\right\|_V^2 + \left\|\boldsymbol{v}_0 - \boldsymbol{v}_0^h\right\|_H^2 \\ &+ \frac{1}{k} \sum_{n=1}^{N-1} \left\|\boldsymbol{v}_n - \boldsymbol{w}_n^h - (\boldsymbol{v}_{n+1} - \boldsymbol{w}_{n+1}^h)\right\|_H^2 \Bigg).\end{aligned} \tag{2.5}$$

We notice that the above error estimates are the basis for the analysis of the convergence rate of the algorithm. Hence, under additional regularity assumptions we obtain the linear convergence of the algorithm that we state in the following.

Corollary 2.3. *Let assumptions of Theorem 2.2 hold. Under the additional regularity conditions*

$$\boldsymbol{v} \in H^1\left(0, T; [H^1(\Omega)]^d\right) \cap C\left([0, T]; [H^2(\Omega)]^d\right) \cap H^2(0, T; H),$$

there exists a positive constant $c > 0$, independent of the discretization parameters h and k, such that

$$\max_{0\le n\le N} \left\|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\right\|_V + \max_{0\le n\le N} \left\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\right\|_H \le c(h + k). \tag{2.6}$$

The proof of the above corollary is obtained by using the well-known result on the approximation by finite elements and the projection operator $\mathcal{P}^h$ (see [9]),

$$\inf_{w_n^h\in V^h} \left\|\boldsymbol{v}_n-\boldsymbol{w}_n^h\|_V \le ch\|\boldsymbol{v}_n\right\|_{[H^2(\Omega)]^d} \le ch\|\boldsymbol{v}\|_{C([0,T];[H^2(\Omega)]^d)},$$

$$\left\|\boldsymbol{u}_0-\boldsymbol{u}_0^h\right\|_V \le ch\,\|\boldsymbol{u}_0\|_{[H^2(\Omega)]^d} \le ch\|\boldsymbol{u}\|_{C([0,T];[H^2(\Omega)]^d)},$$

$$\left\|\boldsymbol{v}_0-\boldsymbol{v}_0^h\right\|_V + \left\|\boldsymbol{v}_0-\boldsymbol{v}_0^h\right\|_H \le ch\|\boldsymbol{v}_0\|_{[H^2(\Omega)]^d} \le ch\|\boldsymbol{v}\|_{C([0,T];[H^2(\Omega)]^d)},$$

and an straightforward estimate implies that

$$\max_{1\le n\le N} \|\dot{\boldsymbol{v}}_n-\delta\boldsymbol{v}_n\|_H \le ck\|\boldsymbol{v}\|_{H^2(0,T;H)}.$$

Now, keeping in mind that

$$\left\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\right\|_V \le \left\|\int_0^{t_n}\boldsymbol{v}(s)\,\mathrm{d}s-\sum_{j=1}^n k\boldsymbol{v}_j\right\|_V + \sum_{j=1}^n k\left\|\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}\right\|_V + \left\|\boldsymbol{v}_0-\boldsymbol{v}_0^h\right\|_V,$$

we have

$$\max_{0\le n\le N} \left\|\boldsymbol{u}_n-\boldsymbol{u}_n^{hk}\right\|_V \le ck\,\|\boldsymbol{v}\|_{H^1(0,T;[H^1(\Omega)]^d)} + c\sum_{n=1}^N k\left\|\boldsymbol{v}_j-\boldsymbol{v}_j^{hk}\right\|_V + ch\|\boldsymbol{v}\|_{C([0,T];[H^2(\Omega)]^d)}.$$

Finally, we only need to apply the following estimate (see [2]),

$$\frac{1}{k}\sum_{n=1}^{N-1} \|\boldsymbol{v}_n-\boldsymbol{w}_n^h-(\boldsymbol{v}_{n+1}-\boldsymbol{w}_{n+1}^h)\|_H^2 \le ch^2\|\boldsymbol{v}\|^2_{H^1(0,T;[H^1(\Omega)]^d)}.$$

3. *A posteriori* error estimates

In this section, we will use the finite element spaces and the notations introduced in the previous section. Moreover, here we will assume that the mesh of the domain $\overline{\Omega}$ may change during the time, and so, for any $0<h<1$ and for any $n=0,1,\ldots,N$, let $\mathcal{T}^{hn}$ be a mesh of $\overline{\Omega}$ composed of closed elements T with diameter h_T less than h. We will also assume that, for each $n=1,\ldots,N$, the mesh $\mathcal{T}^{hn}$ is regular in the sense of [9] and that $\mathcal{T}^{h(n-1)}\subset\mathcal{T}^{hn}$. Thus, for any $n=1,\ldots,N$ and for any $T\in\mathcal{T}^{hn}$, let h_T (respectively ρ_T) be the diameter of the smallest (resp. largest) ball containing (resp. contained in) $(t_{n-1},t_n)\times T$. Therefore, there exists a positive constant β such that

$$\frac{h_T}{\rho_T}\le\beta \quad \forall T\in\mathcal{T}^{hn},\ n=0,1,\ldots,N.$$

In order to simplify the writing and the calculations, in this section we assume that $\boldsymbol{f}_F=\boldsymbol{0}$ and therefore $(\boldsymbol{f},\boldsymbol{w})_V=(\boldsymbol{f},\boldsymbol{w})_H$ for all $\boldsymbol{w}\in V$, where $\boldsymbol{f}=\boldsymbol{f}_0\in C([0,T];H)$. It is straightforward to extend the results presented below to more general situations.

Finally, the notation $a\lesssim b$ means that there exists a positive constant c independent of a and b (and of the time and space discretization parameters) such that $a\le c\,b$. Moreover, the notation $a\sim b$ means that $a\lesssim b$ and $b\lesssim a$ hold simultaneously.

Let us define the continuous and piecewise linear approximation in time given by

$$\boldsymbol{v}^{h\tau}(\boldsymbol{x},t)=\frac{t-t_{n-1}}{k}\boldsymbol{v}_n^{hk}(\boldsymbol{x})+\frac{t_n-t}{k}\boldsymbol{v}_{n-1}^{hk}(\boldsymbol{x})\quad t_{n-1}<t\leq t_n,\quad \boldsymbol{x}\in\overline{\Omega},$$

and an approximation of the displacement field as follows,

$$\boldsymbol{u}^{h\tau}(t)=\int_0^t \boldsymbol{v}^{h\tau}(s)\,\mathrm{d}s+\boldsymbol{u}_0^h.$$

According to [28], let us define the residual $R(\boldsymbol{v}^{h\tau})\in L^2(0,T;V')$ as follows,

$$\langle R(\boldsymbol{v}^{h\tau}),\boldsymbol{w}\rangle_{V'\times V}=(\boldsymbol{f},\boldsymbol{w})_H-(\dot{\boldsymbol{v}}^{h\tau},\boldsymbol{w})_H-(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}^{h\tau})+\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}^{h\tau}),\boldsymbol{\varepsilon}(\boldsymbol{w}))_Q$$

for all $\boldsymbol{w}\in V$ and $t\in[0,T]$, and decompose it into the temporal residual $R_\tau(\boldsymbol{v}^{h\tau})\in L^2(0,T;V')$ given by

$$\langle R_\tau(\boldsymbol{v}^{h\tau}),\boldsymbol{w}\rangle_{V'\times V}=(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk}-\boldsymbol{v}^{h\tau})+\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_n^{hk}-\boldsymbol{u}^{h\tau}),\boldsymbol{\varepsilon}(\boldsymbol{w}))_Q \tag{3.1}$$

on $(t_{n-1},t_n]$ for all $\boldsymbol{w}\in V$, and into the spatial residual $R_h(\boldsymbol{v}^{h\tau})\in L^2(0,T;V')$ defined as

$$\langle R_h(\boldsymbol{v}^{h\tau}),\boldsymbol{w}\rangle_{V'\times V}=(\boldsymbol{f}_{h\tau},\boldsymbol{w})_H-(\dot{\boldsymbol{v}}^{h\tau},\boldsymbol{w})_H-(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk})+\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_n^{hk}),\boldsymbol{\varepsilon}(\boldsymbol{w}))_Q$$

on $(t_{n-1},t_n]$ for all $\boldsymbol{w}\in V$, where we used the notation $\boldsymbol{f}_{h\tau}$ for the function which is piecewise constant on the time intervals and which, on each interval $(t_{n-1},t_n]$, is equal to the L^2-projection of $\boldsymbol{f}_n$ onto the finite element space V^h.

Obviously, it is easy to check that $R(\boldsymbol{v}^{h\tau})=\boldsymbol{f}-\boldsymbol{f}_{h\tau}+R_\tau(\boldsymbol{v}^{h\tau})+R_h(\boldsymbol{v}^{h\tau})$.

First, let us estimate the spatial residual. From its definition, it follows that

$$\langle R_h(\boldsymbol{v}^{h\tau}),\boldsymbol{w}^h\rangle_{V'\times V}=0\quad\forall \boldsymbol{w}^h\in V^h.$$

Hence, for each $\boldsymbol{w}\in V$, let us define by $\boldsymbol{w}^h=\Pi_C^h\boldsymbol{w}$, where Π_C^h is the Clément's interpolant on the triangulation $\mathcal{T}^{hn}$ (see [10]). We recall that this operator satisfies:

$$\|\boldsymbol{w}-\Pi_C^h\boldsymbol{w}\|_{[L^2(T)]^d}\leq ch_T\|\boldsymbol{w}\|_{[H^1(\Delta T)]^d}, \tag{3.2}$$

$$\|\boldsymbol{w}-\Pi_C^h\boldsymbol{w}\|_{[L^2(E)]^d}\leq ch_E^{1/2}\|\boldsymbol{w}\|_{[H^1(\Delta T)]^d}, \tag{3.3}$$

where c is a positive constant which depends on the given constant β, ΔT denotes the set of elements having a common vertex, edge or face with T, E represents an edge (if $d=2$) or a face (if $d=3$) of T and h_E denotes the size of the edge or face E.

Integrating in Ω and using Green's formula, we find that

$$\begin{aligned}\langle R_h\left(\boldsymbol{v}^{h\tau}\right),\boldsymbol{w}\rangle_{V'\times V}=&\sum_{T\in\mathcal{T}^{hn}}\left(\int_T\left(-\dot{\boldsymbol{v}}^{h\tau}+\mathrm{Div}\left(\mathcal{A}\boldsymbol{\varepsilon}\left(\boldsymbol{v}_n^{hk}\right)+\mathcal{B}\boldsymbol{\varepsilon}\left(\boldsymbol{u}_n^{hk}\right)\right)\right)\cdot\boldsymbol{w}\,\mathrm{d}\boldsymbol{x}\right.\\&\left.+\int_T\boldsymbol{f}_{h\tau}\cdot\boldsymbol{w}\,\mathrm{d}\boldsymbol{x}-\sum_{E\in\mathcal{E}_T^{hn}}\int_E\left[\left(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk})+\mathcal{B}\boldsymbol{\varepsilon}\left(\boldsymbol{u}_n^{hk}\right)\right)\boldsymbol{\nu}_E\right]\cdot\boldsymbol{w}\,\mathrm{d}\boldsymbol{x}\right),\end{aligned}$$

where $\mathcal{E}_T^{hn}$ is the set of interior edges or faces of the element T, and $[\boldsymbol{\tau\nu}]$ denotes the jump of $\boldsymbol{\tau\nu}$ across the edge or face E.

Therefore, using properties (3.2) and (3.3) for operator Π_C^h it follows that

$$
\begin{aligned}
&\langle R_h(\boldsymbol{v}^{h\tau}),\boldsymbol{w}\rangle_{V'\times V} = \langle R_h(\boldsymbol{v}^{h\tau}),\boldsymbol{w}-\Pi_C^h\boldsymbol{w}\rangle_{V'\times V}\\
&\lesssim \sum_{T\in\mathcal{T}^{hn}}\Bigg(h_T\left\|\boldsymbol{f}_{h\tau}-\dot{\boldsymbol{v}}^{h\tau}+\mathrm{Div}\Big(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk})+\mathcal{B}\boldsymbol{\varepsilon}\left(\boldsymbol{u}_n^{hk}\right)\Big)\right\|_{[L^2(T)]^d}\|\boldsymbol{w}\|_{[H^1(\Delta T)]^d}\\
&+\sum_{E\in\mathcal{E}_T^{hn}} h_E^{1/2}\left\|\Big[\Big(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk})+\mathcal{B}\boldsymbol{\varepsilon}\left(\boldsymbol{u}_n^{hk}\right)\Big)\boldsymbol{\nu}_E\Big]\right\|_{[L^2(E)]^d}\|\boldsymbol{w}\|_{[H^1(\Delta T)]^d}\Bigg)\\
&\lesssim \left(\sum_{T\in\mathcal{T}^{hn}} h_T^2\left\|\boldsymbol{f}_{h\tau}-\dot{\boldsymbol{v}}^{h\tau}+\mathrm{Div}\Big(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk})+\mathcal{B}\boldsymbol{\varepsilon}\left(\boldsymbol{u}_n^{hk}\right)\Big)\right\|^2_{[L^2(T)]^d}\right)^{1/2}\times\left(\sum_{T\in\mathcal{T}^{hn}}\|\boldsymbol{w}\|^2_{[H^1(\Delta T)]^d}\right)^{1/2}\\
&+\left(\sum_{E\in\mathcal{E}^{hn}} h_E\left\|\Big[\left(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk})+\mathcal{B}\boldsymbol{\varepsilon}\left(\boldsymbol{u}_n^{hk}\right)\right)\boldsymbol{\nu}_E\Big]\right\|^2_{[L^2(E)]^d}\right)^{1/2}\times\left(\sum_{T\in\mathcal{T}^{hn}}\|\boldsymbol{w}\|^2_{[H^1(\Delta T)]^d}\right)^{1/2},
\end{aligned}
$$

where $\mathcal{E}^{hn}$ denotes the set of interior edges or faces that do not belong to Γ_D.

Since $\left(\sum_{T\in\mathcal{T}^{hn}}\|\boldsymbol{w}\|^2_{[H^1(\Delta T)]^d}\right)^{1/2}\lesssim\|\boldsymbol{w}\|_V$ and the element $\boldsymbol{w}$ was chosen arbitrarily, keeping in mind that $\mathrm{Div}(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk})+\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_n^{hk}))=\boldsymbol{0}$ we then conclude that, for any $t\in(t_{n-1},t_n]$,

$$
\begin{aligned}
\|R_h(\boldsymbol{v}^{h\tau})\|_{V'} &\lesssim \left(\sum_{T\in\mathcal{T}^{hn}} h_T^2\|\boldsymbol{f}_{h\tau}-\dot{\boldsymbol{v}}^{h\tau}\|^2_{[L^2(T)]^d}\right)^{1/2}+\left(\sum_{E\in\mathcal{E}^{hn}} h_E\left\|\Big[\Big(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk})+\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_n^{hk})\Big)\boldsymbol{\nu}_E\Big]\right\|^2_{[L^2(E)]^d}\right)^{1/2}\\
&\lesssim\left\{\sum_{T\in\mathcal{T}^{hn}}\left(h_T\|\boldsymbol{f}_{h\tau}-\dot{\boldsymbol{v}}^{h\tau}\|_{[L^2(T)]^d}+\sum_{E\in\mathcal{E}_T^{int}} h_E^{1/2}\left\|\Big[\Big(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk})+\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_n^{hk})\Big)\boldsymbol{\nu}_E\Big]\right\|_{[L^2(E)]^d}\right)^2\right\}^{1/2}\\
&=\eta_1^{hn},
\end{aligned}
$$

where $\mathcal{E}_T^{int}$ denotes the set of interior edges or faces of element T. As a consequence, we deduce that

$$
\begin{aligned}
\left\|R_h(\boldsymbol{v}^{h\tau})\right\|_{L^2(0,T;V')} &\lesssim\left(\sum_{n=1}^N k\left(\eta_1^{hn}\right)^2\right)^{1/2}\\
&=\Bigg\{\sum_{n=1}^N\sum_{T\in\mathcal{T}^{hn}} k\Bigg(h_T\|\boldsymbol{f}_{h\tau}-\dot{\boldsymbol{v}}^{h\tau}\|_{[L^2(T)]^d}\\
&\quad+\sum_{E\in\mathcal{E}_T^{int}} h_E^{1/2}\left\|\left[\left(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk})+\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_n^{hk})\right)\boldsymbol{\nu}_E\right]\right\|_{[L^2(E)]^d}\Bigg)^2\Bigg\}^{1/2}\\
&=\eta_1^h. \qquad (3.4)
\end{aligned}
$$

Let us bound now the time residual. From (3.1) we immediately have

$$\left\|R_\tau(\boldsymbol{v}^{h\tau})\right\|_{V'} \lesssim v\left\|\boldsymbol{v}_n^{hk}-\boldsymbol{v}^{h\tau}\right\|_V+\left\|\boldsymbol{u}_n^{hk}-\boldsymbol{u}^{h\tau}\right\|_V \quad \text{on} \quad (t_{n-1},t_n].$$

Now, keeping in mind that

$$\left\|\boldsymbol{u}_n^{hk}-\boldsymbol{u}^{h\tau}\right\|_{L^2(0,T;V)} \lesssim \max_{1\le n\le N}\max_{t_{n-1}<t\le t_n} v\left\|\boldsymbol{u}_n^{hk}-\boldsymbol{u}^{h\tau}(t)\right\|_V$$

and (see [19]),

$$\begin{aligned}\left\{\sum_{n=1}^N\int_{t_{n-1}}^{t_n}\left\|\boldsymbol{v}_n^{hk}-\boldsymbol{v}^{h\tau}\right\|_V^2\mathrm{d}t\right\}^{1/2} &= \left\{\sum_{n=1}^N\int_{t_{n-1}}^{t_n}\left(\frac{t_n-t}{k}\right)^2\left\|\boldsymbol{v}_n^{hk}-\boldsymbol{v}_{n-1}^{hk}\right\|_V^2\mathrm{d}t\right\}^{1/2}\\ &= \left\{\sum_{n=1}^N\frac{k}{3}\left\|\boldsymbol{v}_n^{hk}-\boldsymbol{v}_{n-1}^{hk}\right\|_V^2\right\}^{1/2}\\ &= \left\{\sum_{n=1}^N\sum_{T\in\mathcal{T}^{hn}}\frac{k}{3}\left\|\boldsymbol{v}_n^{hk}-\boldsymbol{v}_{n-1}^{hk}\right\|_{[H^1(T)]^d}^2\right\}^{1/2}\\ &= \left(\sum_{n=1}^N k\left(\eta_2^{hn}\right)^2\right)^{1/2}\\ &= \eta_2^h,\end{aligned} \tag{3.5}$$

where $\eta_2^{hn}=\dfrac{1}{\sqrt{3}}\left\|\boldsymbol{v}_n^{hk}-\boldsymbol{v}_{n-1}^{hk}\right\|_V$, we find that

$$\left\|R_\tau(\boldsymbol{v}^{h\tau})\right\|_{L^2(0,T;V')} \lesssim \left(\sum_{n=1}^N k(\eta_2^{hn})^2\right)^{1/2}+\max_{1\le n\le N}\max_{t_{n-1}<t\le t_n}\eta_3(t). \tag{3.6}$$

Here, we denoted by

$$\eta_3(t)=\|\boldsymbol{u}_n^{hk}-\boldsymbol{u}^{h\tau}(t)\|_V. \tag{3.7}$$

Now, combining (3.4) and (3.6) we obtain the following estimate for the residual:

$$\|R\left(\boldsymbol{v}^{h\tau}\right)\|_{L^2(0,T;V')} \lesssim \|\boldsymbol{f}-\boldsymbol{f}_{h\tau}\|_{L^2(0,T;V')}+\eta_1^h+\eta_2^h+\max_{1\le n\le N}\max_{t_{n-1}<t\le t_n}\eta_3(t).$$

Finally, let us prove a relation between the residual $R(\boldsymbol{v}^{h\tau})$ and the error $\boldsymbol{v}-\boldsymbol{v}^{h\tau}$. From the definition of the residual, it follows that

$$\left(\dot{\boldsymbol{v}}-\dot{\boldsymbol{v}}^{h\tau},\boldsymbol{w}\right)_H+\left(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}-\boldsymbol{v}^{h\tau})+\mathcal{B}\boldsymbol{\varepsilon}\left(\boldsymbol{u}-\boldsymbol{u}^{h\tau}\right),\boldsymbol{\varepsilon}(\boldsymbol{w})\right)_Q=\left\langle R\left(\boldsymbol{v}^{h\tau}\right),\boldsymbol{w}\right\rangle_{V'\times V} \tag{3.8}$$

for all $\boldsymbol{w}\in V$ and $t\in(t_{n-1},t_n]$, $n=1,\dots,N$.

If we take $\boldsymbol{w}=\boldsymbol{v}-\boldsymbol{v}^{h\tau}$ in the previous variational equation and we employ assumptions (1.7)-(1.8), by using the ellipticity of $\mathcal{A}$ and Young's inequality, we immediately get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|\boldsymbol{v}-\boldsymbol{v}^{h\tau}\right\|_H^2+\left\|\boldsymbol{v}-\boldsymbol{v}^{h\tau}\right\|_V^2 \lesssim \left\|R(\boldsymbol{v}^{h\tau})\right\|_{V'}^2+\left\|\boldsymbol{u}-\boldsymbol{u}^{h\tau}\right\|_V^2.$$

Integrating in time between 0 and t the last expression, we find that

$$\|(\boldsymbol{v}-\boldsymbol{v}^{h\tau})(t)\|_H^2+\|\boldsymbol{v}-\boldsymbol{v}^{h\tau}\|_{L^2(0,t;V)}^2\lesssim\|R(\boldsymbol{v}^{h\tau})\|_{L^2(0,t;V')}^2+\|\boldsymbol{v}_0-\boldsymbol{v}_0^h\|_H^2+\|\boldsymbol{u}-\boldsymbol{u}^{h\tau}\|_{L^2(0,t;V)}^2,$$

and therefore,

$$\begin{aligned}\left\|(\boldsymbol{v}-\boldsymbol{v}^{h\tau})(t)\right\|_H^2+\left\|\boldsymbol{v}-\boldsymbol{v}^{h\tau}\right\|_{L^2(0,t;V)}^2\lesssim\ &\left\|R(\boldsymbol{v}^{h\tau})\right\|_{L^2(0,t;V')}^2+\left\|\boldsymbol{v}_0-\boldsymbol{v}_0^h\right\|_H^2\\&+\left\|\int_0^s\boldsymbol{v}(r)-\boldsymbol{v}^{h\tau}(r)\,\mathrm{d}r\right\|_{L^2(0,t;V)}^2+\left\|\boldsymbol{u}_0-\boldsymbol{u}_0^h\right\|_V^2.\end{aligned}$$

Finally, from the properties of the $[L^2(\Omega)]^d$-projection operator, we have

$$\|\boldsymbol{f}-\boldsymbol{f}_{h\tau}\|_{V'}\le h\|\boldsymbol{f}-\boldsymbol{f}_{h\tau}\|_H.$$

Summarizing the previous results and using classical Gronwall's lemma, it leads to the following theorem which provides an upper bound for the error.

Theorem 3.1. *Let the assumptions of Theorem 1.1 hold. Denote by $\boldsymbol{v}$ and $\boldsymbol{v}^{h\tau}$ the solution to Problem VP and the continuous piecewise linear approximation of the solution to Problem VP^{hk}, respectively. If we denote by $\eta=\sqrt{(\eta_1^h)^2+(\eta_2^h)^2}$, then we have*

$$\begin{aligned}\left\|\boldsymbol{v}-\boldsymbol{v}^{h\tau}\right\|_{C([0,T];H)}+\left\|\boldsymbol{v}-\boldsymbol{v}^{h\tau}\right\|_{L^2(0,T;V)}\lesssim\ &\eta+h\left\|\boldsymbol{f}-\boldsymbol{f}_{h\tau}\right\|_{L^2(0,T;H)}\\&+\left\|\boldsymbol{u}_0-\boldsymbol{u}_0^h\right\|_V+\left\|\boldsymbol{v}_0-\boldsymbol{v}_0^h\right\|_H+\max_{1\le n\le N}\ \max_{t\in(t_{n-1},t_n]}\eta_3(t),\end{aligned}\tag{3.9}$$

where the error estimators η_1^h, η_2^h and η_3 were defined in (3.4), (3.5) and (3.7), respectively.

Next, in the following theorem we prove a lower bound for these error estimators.

Theorem 3.2. *Let the assumptions of Theorem 3.1 hold. For all elements $T\in\mathcal{T}^{hn}$, the following local lower error bounds are obtained for $n=1,\ldots,N$:*

$$\begin{aligned}\eta_{1T}^{hn}&\lesssim\left\|\boldsymbol{v}(t)-\boldsymbol{v}^{h\tau}(t)\right\|_{[H^1(\Delta T)]^d}+\left\|\boldsymbol{u}(t)-\boldsymbol{u}_n^{hk}\right\|_{[H^1(\Delta T)]^d}\\&\quad+h_T\left\|\boldsymbol{f}(t)-\boldsymbol{f}_{h\tau}(t)\right\|_{[L^2(\Delta T)]^d}+h_T\left\|\dot{\boldsymbol{v}}^{h\tau}-\dot{\boldsymbol{v}}\right\|_{[L^2(\Delta T)]^d},\\ \eta_{2T}^{hn}&\lesssim\left\|\boldsymbol{v}(t)-\boldsymbol{v}_n^{hk}\right\|_{[H^1(T)]^d}+\left\|\boldsymbol{v}(t)-\boldsymbol{v}_{n-1}^{hk}\right\|_{[H^1(T)]^d},\\ \eta_{3T}(t)&\le\left\|\boldsymbol{u}(t)-\boldsymbol{u}_n^{hk}\right\|_{[H^1(T)]^d}+\left\|\boldsymbol{u}(t)-\boldsymbol{u}^{h\tau}(t)\right\|_{[H^1(T)]^d},\end{aligned}$$

where η_{1T}^{hn}, η_{2T}^{hn} and $\eta_{3T}(t)$ are the local errors in space given by

$$\begin{aligned}\eta_{1T}^{hn}&=h_T\left\|\boldsymbol{f}_{h\tau}-\dot{\boldsymbol{v}}^{h\tau}\right\|_{[L^2(T)]^d}+\sum_{E\in\mathcal{E}_T^{hn}}h_E^{1/2}\left\|\left[\left(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}_n^{hk})+\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}_n^{hk})\right)\boldsymbol{\nu}_E\right]\right\|_{[L^2(E)]^d},\\ \eta_{2T}^{hn}&=\frac{1}{\sqrt{3}}\left\|\boldsymbol{v}_n^{hk}-\boldsymbol{v}_{n-1}^{hk}\right\|_{[H^1(T)]^d},\\ \eta_{3T}(t)&=\left\|\boldsymbol{u}^{h\tau}(t)-\boldsymbol{u}_n^{hk}\right\|_{[H^1(T)]^d},\end{aligned}$$

and $\mathcal{E}_T^{hn}$ represents the set of interior edges or faces of T which do not belong to Γ_D.

If we denote by η^n *the error estimator at time step* n*:*

$$\eta^n = k^{1/2} \left(\left(\eta_1^{hn}\right)^2 + (\eta_2^{hn})^2 \right)^{1/2},$$

then

$$\begin{aligned}\eta^n \lesssim\ & \left\|\boldsymbol{u} - \boldsymbol{u}^{h\tau}\right\|_{L^2(t_{n-1},t_n;V)} + \left\|\boldsymbol{v} - \boldsymbol{v}^{h\tau}\right\|_{L^2(t_{n-1},t_n;V)} + \left\|\boldsymbol{u} - \boldsymbol{u}_n^{hk}\right\|_{L^2(t_{n-1},t_n;V)} \\ & + h \left\|\dot{\boldsymbol{v}} - \dot{\boldsymbol{v}}^{h\tau}\right\|_{L^2(t_{n-1},t_n;H)} + h \left\|\boldsymbol{f} - \boldsymbol{f}_{h\tau}\right\|_{L^2(t_{n-1},t_n;H)}, \\ \eta_3(t) \leq\ & \left\|\boldsymbol{u}(t) - \boldsymbol{u}_n^{hk}\right\|_V + \left\|\boldsymbol{u}(t) - \boldsymbol{u}^{h\tau}(t)\right\|_V.\end{aligned}$$

Obviously, it follows that

$$\eta = \left(\sum_{n=1}^{N} (\eta^n)^2 \right)^{1/2}.$$

Proof. From the definition of the local error estimators η_{2T}^{hn} and $\eta_{3T}(t)$ we easily find that

$$\begin{aligned}\eta_{2T}^{hn} &\lesssim \left\|\boldsymbol{v}(t) - \boldsymbol{v}_n^{hk}\right\|_{[H^1(T)]^d} + \left\|\boldsymbol{v}(t) - \boldsymbol{v}_{n-1}^{hk}\right\|_{[H^1(T)]^d}, \\ \eta_{3T}(t) &\leq \left\|\boldsymbol{u}(t) - \boldsymbol{u}_n^{hk}\right\|_{[H^1(T)]^d} + \left\|\boldsymbol{u}(t) - \boldsymbol{u}^{h\tau}(t)\right\|_{[H^1(T)]^d},\end{aligned}$$

and therefore

$$\begin{aligned}\eta_2^{hn} &\lesssim \left\|\boldsymbol{v}(t) - \boldsymbol{v}_n^{hk}\right\|_V + \left\|\boldsymbol{v}(t) - \boldsymbol{v}_{n-1}^{hk}\right\|_V, \\ \eta_3(t) &\leq \left\|\boldsymbol{u}(t) - \boldsymbol{u}_n^{hk}\right\|_V + \left\|\boldsymbol{u}(t) - \boldsymbol{u}^{h\tau}(t)\right\|_V.\end{aligned}$$

From equation (3.8) we deduce, for any $t \in [0,T]$,

$$\left\|R(\boldsymbol{v}^{h\tau})\right\|_{V'} \lesssim \left\|\boldsymbol{u} - \boldsymbol{u}^{h\tau}\right\|_V + \left\|\boldsymbol{v} - \boldsymbol{v}^{h\tau}\right\|_V + \left\|\dot{\boldsymbol{v}} - \dot{\boldsymbol{v}}^{h\tau}\right\|_H,$$

and therefore,

$$\left\|R(\boldsymbol{v}^{h\tau})\right\|_{L^2(t_1,t_2;V')} \lesssim \left\|\boldsymbol{u} - \boldsymbol{u}^{h\tau}\right\|_{L^2(t_1,t_2;V)} + \left\|\boldsymbol{v} - \boldsymbol{v}^{h\tau}\right\|_{L^2(t_1,t_2;V)} + \left\|\dot{\boldsymbol{v}} - \dot{\boldsymbol{v}}^{h\tau}\right\|_{L^2(t_1,t_2;H)},$$

for any t_1, t_2 in $[0,T]$. Next we bound η^n. We begin with the second term given by $k^{1/2}\|\boldsymbol{v}_n^{hk} - \boldsymbol{v}_{n-1}^{hk}\|_V$. We have, for any $t \in [t_{n-1}, t_n]$,

$$\begin{aligned}\left(\frac{t_n - t}{k}\right)^2 \left\|\boldsymbol{v}_n^{hk} - \boldsymbol{v}_{n-1}^{hk}\right\|_V^2 &= \left\|\boldsymbol{v}_n^{hk} - \boldsymbol{v}^{h\tau}\right\|_V^2 \\ &\lesssim \left(\mathcal{A}\boldsymbol{\varepsilon}\left(\boldsymbol{v}_n^{hk} - \boldsymbol{v}^{h\tau}\right), \boldsymbol{\varepsilon}\left(\boldsymbol{v}_n^{hk} - \boldsymbol{v}^{h\tau}\right)\right)_Q \\ &= \langle R_\tau\left(\boldsymbol{v}^{h\tau}\right), \boldsymbol{v}_n^{hk} - \boldsymbol{v}^{h\tau}\rangle_{V'\times V} - \left(\mathcal{B}\boldsymbol{\varepsilon}\left(\boldsymbol{u}_n^{hk} - \boldsymbol{u}^{h\tau}\right), \boldsymbol{\varepsilon}\left(\boldsymbol{v}_n^{hk} - \boldsymbol{v}^{h\tau}\right)\right)_Q \\ &= \langle R\left(\boldsymbol{v}^{h\tau}\right), \boldsymbol{v}_n^{hk} - \boldsymbol{v}^{h\tau}\rangle_{V'\times V} - \langle R_h\left(\boldsymbol{v}^{h\tau}\right), \boldsymbol{v}_n^{hk} - \boldsymbol{v}^{h\tau}\rangle_{V'\times V} \\ &\quad - \left(\mathcal{B}\boldsymbol{\varepsilon}\left(\boldsymbol{u}_n^{hk} - \boldsymbol{u}^{h\tau}\right), \boldsymbol{\varepsilon}\left(\boldsymbol{v}_n^{hk} - \boldsymbol{v}^{h\tau}\right)\right)_Q - \left(\boldsymbol{f} - \boldsymbol{f}_{h\tau}, \boldsymbol{v}_n^{hk} - \boldsymbol{v}^{h\tau}\right)_{V'\times V}.\end{aligned}$$

Using Cauchy-Schwarz inequality and integrating the last expression from t_{n-1} to t_n we get

$$\frac{k}{3}\left\|\boldsymbol{v}_n^{hk}-\boldsymbol{v}_{n-1}^{hk}\right\|_V^2 \lesssim \Big(\left\|R(\boldsymbol{v}^{h\tau})\right\|_{L^2(t_{n-1},t_n;V')}+\left\|R_h(\boldsymbol{v}^{h\tau})\right\|_{L^2(t_{n-1},t_n;V')} + \|\boldsymbol{u}_n^{hk}-\boldsymbol{u}^{h\tau}\|_{L^2(t_{n-1},t_n;V)}+\|\boldsymbol{f}-\boldsymbol{f}_{h\tau}\|_{L^2(t_{n-1},t_n;V')}\Big)\left\|\boldsymbol{v}_n^{hk}-\boldsymbol{v}^{h\tau}\right\|_{L^2(t_{n-1},t_n;V)}.$$

Keeping in mind that

$$\begin{aligned}\left\|\boldsymbol{v}_n^{hk}-\boldsymbol{v}^{h\tau}\right\|_{L^2(t_{n-1},t_n;V)} &= \left(\int_{t_{n-1}}^{t_n}\left\|\boldsymbol{v}_n^{hk}-\boldsymbol{v}^{h\tau}\right\|_V^2\right)^{1/2}\\ &= \left(\int_{t_{n-1}}^{t_n}\left(\frac{t_n-t}{k}\right)^2\left\|\boldsymbol{v}_n^{hk}-\boldsymbol{v}_{n-1}^{hk}\right\|_V^2\right)^{1/2}\\ &= \left(\frac{k}{3}\right)^{1/2}\left\|\boldsymbol{v}_n^{hk}-\boldsymbol{v}_{n-1}^{hk}\right\|_V,\end{aligned}$$

it follows that

$$\begin{aligned}\left(\frac{k}{3}\right)^{1/2}\left\|\boldsymbol{v}_n^{hk}-\boldsymbol{v}_{n-1}^{hk}\right\|_V \lesssim\; & \left\|R\left(\boldsymbol{v}^{h\tau}\right)\right\|_{L^2(t_{n-1},t_n;V')}+\left\|R_h\left(\boldsymbol{v}^{h\tau}\right)\right\|_{L^2(t_{n-1},t_n;V')}\\ & +\|\boldsymbol{f}-\boldsymbol{f}_{h\tau}\|_{L^2(t_{n-1},t_n;V')}+\|\boldsymbol{u}_n^{hk}-\boldsymbol{u}^{h\tau}\|_{L^2(t_{n-1},t_n;V)}\\ \lesssim\; & \|\boldsymbol{u}-\boldsymbol{u}^{h\tau}\|_{L^2(t_{n-1},t_n;V)}+\|\boldsymbol{v}-\boldsymbol{v}^{h\tau}\|_{L^2(t_{n-1},t_n;V)}+\|\dot{\boldsymbol{v}}-\dot{\boldsymbol{v}}^{h\tau}\|_{L^2(t_{n-1},t_n;H)}\\ & +\left\|R_h\left(\boldsymbol{v}^{h\tau}\right)\right\|_{L^2(t_{n-1},t_n;V')}+\|\boldsymbol{f}-\boldsymbol{f}_{h\tau}\|_{L^2(t_{n-1},t_n;V')}+\|\boldsymbol{u}-\boldsymbol{u}_n^{hk}\|_{L^2(t_{n-1},t_n;V)}\\ \lesssim\; & \|\boldsymbol{u}-\boldsymbol{u}^{h\tau}\|_{L^2(t_{n-1},t_n;V)}+\|\boldsymbol{v}-\boldsymbol{v}^{h\tau}\|_{L^2(t_{n-1},t_n;V)}+\|\dot{\boldsymbol{v}}-\dot{\boldsymbol{v}}^{h\tau}\|_{L^2(t_{n-1},t_n;H)}\\ & +k^{1/2}\eta_1^{hn}+\|\boldsymbol{f}-\boldsymbol{f}_{h\tau}\|_{L^2(t_{n-1},t_n;V')}+\|\boldsymbol{u}-\boldsymbol{u}_n^{hk}\|_{L^2(t_{n-1},t_n;V)}.\end{aligned}$$

Again, from the properties of the $[L^2(\Omega)]^d$-projection operator, we have

$$\|\boldsymbol{f}-\boldsymbol{f}_{h\tau}\|_{V'}\le h\,\|\boldsymbol{f}-\boldsymbol{f}_{h\tau}\|_H\,.$$

Thus, it only remains to bound $k^{1/2}\eta_1^{hn}$. Recalling that

$$\eta_1^{hn}=\left(\sum_{T\in\mathcal{T}^{hn}}\left(h_T\|\boldsymbol{f}_{h\tau}-\dot{\boldsymbol{v}}^{h\tau}\|_{[L^2(T)]^d} + \sum_{E\in\mathcal{E}_T^{int}}h_E^{1/2}\left\|\left[(\mathcal{A}\boldsymbol{\varepsilon}\left(\boldsymbol{v}_n^{hk}\right)+\mathcal{B}\boldsymbol{\varepsilon}\Big(\boldsymbol{u}_n^{hk}\Big))\boldsymbol{\nu}_E\right]\right\|_{[L^2(E)]^d}\right)^2\right)^{1/2}$$

this is done in the following, when the estimate of the estimator η_{1T}^{hn} is obtained. Let w_T be the bubble function associated with the element T (for instance, in the two-dimensional setting, we have $w_T=\lambda_{a1}\lambda_{a2}\lambda_{a3}$, where λ_{ai}, $i=1,2,3$ denote the barycentric coordinates and a_1, a_2 and a_3 are the three nodes of the element T). We notice that $w_T\in H_0^1(T)$. Let us define $\boldsymbol{w}_T\in[H_0^1(T)]^d$ which is constructed as $w_i=w_T$ for $i=1,\ldots,d$.

It follows that the function $\boldsymbol{\psi}_T = \boldsymbol{w}_T \cdot (\boldsymbol{f}_{h\tau} - \dot{\boldsymbol{v}}^{h\tau})$ verifies (see [27], Chap. 3),

$$\left\|\boldsymbol{f}_{h\tau} - \dot{\boldsymbol{v}}^{h\tau}\right\|^2_{[L^2(T)]^d} \lesssim \int_T (\boldsymbol{f}_{h\tau} - \boldsymbol{f}) \cdot \boldsymbol{\psi}_T \,\mathrm{d}\boldsymbol{x} - \int_T \left(\dot{\boldsymbol{v}}^{h\tau} - \dot{\boldsymbol{v}}\right) \cdot \boldsymbol{\psi}_T \,\mathrm{d}\boldsymbol{x} + \int_T \left(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v} - \boldsymbol{v}^{h\tau}) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u} - \boldsymbol{u}_n^{hk})\right) \cdot \boldsymbol{\varepsilon}(\boldsymbol{\psi}_T) \,\mathrm{d}\boldsymbol{x}.$$

Using an inverse inequality, it follows that

$$\|\boldsymbol{\varepsilon}(\boldsymbol{\psi}_T)\|_{[L^2(T)]^{d\times d}} \lesssim h_T^{-1} \|\boldsymbol{\psi}_T\|_{[L^2(T)]^d},$$

and therefore,

$$h_T\left\|\boldsymbol{f}_{h\tau} - \dot{\boldsymbol{v}}^{h\tau}\right\|_{[L^2(T)]^d} \lesssim \left\|\boldsymbol{v}(t) - \boldsymbol{v}^{h\tau}(t)\right\|_{[H^1(T)]^d} + h_T\left\|\boldsymbol{f}(t) - \boldsymbol{f}_{h\tau}(t)\right\|_{[L^2(T)]^d} + h_T\left\|\dot{\boldsymbol{v}}^{h\tau} - \dot{\boldsymbol{v}}\right\|_{[L^2(T)]^d} + \left\|\boldsymbol{u}(t) - \boldsymbol{u}_n^{hk}\right\|_{[H^1(T)]^d}. \tag{3.10}$$

We turn now to estimate the second term of error estimator η_{1T}^{hn}. Proceeding in a similar way that in the previous estimate, let us consider the bubble function w_E associated with the edge or face E. Hence, taking now $\boldsymbol{w}_E = [w_E]^d$ we deduce that (see again [27], Chap. 3),

$$\left\|\left[\left(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}^{h\tau}) + \mathcal{B}\boldsymbol{\varepsilon}\left(\boldsymbol{u}_n^{hk}\right)\right)\boldsymbol{\nu}_E\right]\right\|^2_{[L^2(E)]^d} \lesssim \Big(\|\boldsymbol{f}(t) - \boldsymbol{f}_{h\tau}(t)\|_{[L^2(\Delta T)]^d} + h_E^{-1}\left(\left\|\boldsymbol{v}(t) - \boldsymbol{v}^{h\tau}(t)\right\|_{[H^1(\Delta T)]^d} + \left\|\boldsymbol{u}_n^{hk}(t) - \boldsymbol{u}(t)\right\|_{[H^1(\Delta T)]^d}\right) + \left\|\dot{\boldsymbol{v}}^{h\tau} - \dot{\boldsymbol{v}}\right\|_{[L^2(\Delta T)]^d} + \left\|\boldsymbol{f}_{h\tau} - \dot{\boldsymbol{v}}^{h\tau}\right\|_{[L^2(\Delta T)]^d} \Big) \|\boldsymbol{\psi}_E\|_{[L^2(\Delta T)]^d},$$

where ΔT stands for the set of elements of $\mathcal{T}^{hn}$ sharing the common edge or face E. From the definition of $\boldsymbol{w}_E$ we conclude that

$$\begin{aligned} h_E^{1/2}\left\|\left[(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}^{h\tau}) + \mathcal{B}\boldsymbol{\varepsilon}\left(\boldsymbol{u}_n^{hk}\right))\boldsymbol{\nu}_E\right]\right\|_{[L^2(E)]^d} &\lesssim h_E\left\|\boldsymbol{f}(t) - \boldsymbol{f}_{h\tau}(t)\right\|_{[L^2(\Delta T)]^d} + h_E\left\|\dot{\boldsymbol{v}}^{h\tau} - \dot{\boldsymbol{v}}\right\|_{[L^2(\Delta T)]^d} \\ &\quad + \left\|\boldsymbol{v}(t) - \boldsymbol{v}^{h\tau}(t)\right\|_{[H^1(\Delta T)]^d} + \left\|\boldsymbol{u}_n^{hk}(t) - \boldsymbol{u}(t)\right\|_{[H^1(\Delta T)]^d} \\ &\quad + h_E\left\|\boldsymbol{f}_{h\tau} - \dot{\boldsymbol{v}}^{h\tau}\right\|_{[L^2(\Delta T)]^d} \\ &\lesssim h_E\left\|\boldsymbol{f}(t) - \boldsymbol{f}_{h\tau}(t)\right\|_{[L^2(\Delta T)]^d} + h_E\left\|\dot{\boldsymbol{v}}^{h\tau} - \dot{\boldsymbol{v}}\right\|_{[L^2(\Delta T)]^d} \\ &\quad + \left\|\boldsymbol{v}(t) - \boldsymbol{v}^{h\tau}(t)\right\|_{[H^1(\Delta T)]^d} + \left\|\boldsymbol{u}_n^{hk}(t) - \boldsymbol{u}(t)\right\|_{[H^1(\Delta T)]^d}. \end{aligned}$$

Keeping in mind (3.10) and the previous estimate, we obtain, for all $T \in \mathcal{T}^{hn}$,

$$\begin{aligned} \eta_{1T}^{hn} &= h_T\left\|\boldsymbol{f}_{h\tau} - \dot{\boldsymbol{v}}^{h\tau}\right\|_{[L^2(T)]^d} + \sum_{E\in\mathcal{E}_T^{hn}} h_E^{1/2}\left\|\left[\left(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}^{h\tau}) + \mathcal{B}\boldsymbol{\varepsilon}\left(\boldsymbol{u}_n^{hk}\right)\right)\boldsymbol{\nu}_E\right]\right\|_{[L^2(E)]^d} \\ &\lesssim \left\|\boldsymbol{v}(t) - \boldsymbol{v}^{h\tau}(t)\right\|_{[H^1(\Delta T)]^d} + \left\|\boldsymbol{u}(t) - \boldsymbol{u}_n^{hk}\right\|_{[H^1(\Delta T)]^d} \\ &\quad + h_T\left\|\boldsymbol{f}(t) - \boldsymbol{f}_{h\tau}(t)\right\|_{[L^2(\Delta T)]^d} + h_T\left\|\dot{\boldsymbol{v}}^{h\tau} - \dot{\boldsymbol{v}}\right\|_{[L^2(\Delta T)]^d}, \end{aligned}$$

and therefore,

$$\eta_1^{hn} \lesssim \left\|\boldsymbol{v}(t)-\boldsymbol{v}^{h\tau}(t)\right\|_V + \left\|\boldsymbol{u}(t)-\boldsymbol{u}_n^{hk}\right\|_V + h_T\left\|\boldsymbol{f}(t)-\boldsymbol{f}_{h\tau}(t)\right\|_H + h_T\left\|\dot{\boldsymbol{v}}^{h\tau}-\dot{\boldsymbol{v}}\right\|_H.$$

Thus, we find that

$$k^{1/2}\eta_1^{hn} \lesssim \left\|\boldsymbol{v}-\boldsymbol{v}^{h\tau}\right\|_{L^2(t_{n-1},t_n;V)} + \left\|\boldsymbol{u}-\boldsymbol{u}_n^{hk}\right\|_{L^2(t_{n-1},t_n;V)} + h\left\|\boldsymbol{f}-\boldsymbol{f}_{h\tau}\right\|_{L^2(t_{n-1},t_n;H)} + h\left\|\dot{\boldsymbol{v}}^{h\tau}-\dot{\boldsymbol{v}}\right\|_{L^2(t_{n-1},t_n;H)},$$

and, combining all these results and taking into account the definitions (3.4) and (3.5), it leads to the desired lower error bounds of η^n. □

We observe that, from Theorem 3.2, under some additional regularity conditions, we can prove a similar convergence order as provided in the *a priori* error analysis (see Cor. 2.3) which we state in the following.

Corollary 3.3. *Let the assumptions of Theorem 3.2 hold. If the continuous solution has the regularity $\boldsymbol{v} \in H^1(0,T;[H^2(\Omega)]^d)$ and we assume that the density of volume forces satisfies $\boldsymbol{f}_0 \in C([0,T];[H^1(\Omega)]^d)$, we have*

$$\eta + \max_{1\le n\le N}\max_{t_{n-1}<t\le t_n}\eta_3(t) \le c(h+k),$$

for a positive constant c which depends on the given data and the continuous solution $\boldsymbol{v}$.

Proof. The proof of this corollary is obtained taking into account the following straightforward estimate

$$\left\|\boldsymbol{f}-\boldsymbol{f}_{h\tau}\right\|_{L^2(0,T;H)} \le ch\left\|\boldsymbol{f}\right\|_{C([0,T];[H^1(\Omega)]^d)}.$$

Using estimates (2.6), under the required regularity we conclude that

$$\left\|\boldsymbol{v}-\boldsymbol{v}^{h\tau}\right\|_{C([0,T];H)} + \left\|\boldsymbol{u}-\boldsymbol{u}^{h\tau}\right\|_{C([0,T];V)} \le c(h+k),$$

and we easily obtain

$$\max_{1\le n\le N}\max_{t_{n-1}<t\le t_n}\eta_3(t) \le c(h+k).$$

Using again (3.8) we find that, for $n=1,\ldots,N$,

$$\left(\dot{\boldsymbol{v}}(t)-\dot{\boldsymbol{v}}^{h\tau}(t),\boldsymbol{w}^h\right)_H + \left(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}(t)-\boldsymbol{v}^{h\tau}(t)) + \mathcal{B}\boldsymbol{\varepsilon}\left(\boldsymbol{u}(t)-\boldsymbol{u}^{h\tau}\right),\boldsymbol{\varepsilon}\left(\boldsymbol{w}^h\right)\right)_Q = 0$$

for all $\boldsymbol{w}^h \in V^h$ and $t_{n-1} < t \le t_n$. Thus, since $\dot{\boldsymbol{v}}^{h\tau}(t) \in V^h$, we have

$$\begin{aligned}&\left(\dot{\boldsymbol{v}}(t)-\dot{\boldsymbol{v}}^{h\tau}(t),\dot{\boldsymbol{v}}(t)-\dot{\boldsymbol{v}}^{h\tau}(t)\right)_H + \left(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}(t)-\boldsymbol{v}^{h\tau}(t)) + \mathcal{B}\boldsymbol{\varepsilon}\left(\boldsymbol{u}(t)-\boldsymbol{u}^{h\tau}\right),\boldsymbol{\varepsilon}\left(\dot{\boldsymbol{v}}(t)-\dot{\boldsymbol{v}}^{h\tau}(t)\right)\right)_Q\\ &= \left(\dot{\boldsymbol{v}}(t)-\dot{\boldsymbol{v}}^{h\tau}(t),\dot{\boldsymbol{v}}(t)-\boldsymbol{w}^h\right)_H + \left(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}(t)-\boldsymbol{v}^{h\tau}(t)) + \mathcal{B}\boldsymbol{\varepsilon}\left(\boldsymbol{u}(t)-\boldsymbol{u}^{h\tau}\right),\boldsymbol{\varepsilon}\left(\dot{\boldsymbol{v}}(t)-\boldsymbol{w}^h\right)\right)_Q \quad \forall \boldsymbol{w}^h \in V^h,\end{aligned}$$

for $t_{n-1} < t \le t_n$. Using properties (1.7) and (1.8), applying several times inequality

$$ab \le \epsilon a^2 + \frac{1}{4\epsilon}b^2, \quad a,b,\epsilon \in \mathbb{R},\ \epsilon > 0, \tag{3.11}$$

keeping in mind that

$$\begin{aligned}\left(\mathcal{B}\boldsymbol{\varepsilon}\left(\boldsymbol{u}(t)-\boldsymbol{u}^{h\tau}\right),\boldsymbol{\varepsilon}\left(\dot{\boldsymbol{v}}(t)-\dot{\boldsymbol{v}}^{h\tau}(t)\right)\right)_Q &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)-\boldsymbol{u}^{h\tau}),\boldsymbol{\varepsilon}\left(\boldsymbol{v}(t)-\boldsymbol{v}^{h\tau}(t)\right)\right)_Q \\ &\quad -\left(\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{v}(t)-\boldsymbol{v}^{h\tau}),\boldsymbol{\varepsilon}\left(\boldsymbol{v}(t)-\boldsymbol{v}^{h\tau}(t)\right)\right)_Q, \\ \left(\mathcal{A}\boldsymbol{\varepsilon}(\boldsymbol{v}(t)-\boldsymbol{v}^{h\tau}(t)),\boldsymbol{\varepsilon}\left(\dot{\boldsymbol{v}}(t)-\dot{\boldsymbol{v}}^{h\tau}(t)\right)\right)_Q &\sim \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{v}(t)-\boldsymbol{v}^{h\tau}(t)\|_V^2,\end{aligned}$$

and integrating in time between 0 and t, it follows that

$$\begin{aligned}\left\|\dot{\boldsymbol{v}}-\dot{\boldsymbol{v}}^{h\tau}\right\|^2_{L^2(0,t;H)}+\left\|\boldsymbol{v}(t)-\boldsymbol{v}^{h\tau}(t)\right\|_V^2 &\le c\Bigg(\left\|\boldsymbol{u}_0-\boldsymbol{u}_0^h\right\|_V^2+\left\|\boldsymbol{v}_0-\boldsymbol{v}_0^h\right\|_V^2 \\ &\quad +\left\|\dot{\boldsymbol{v}}-\boldsymbol{w}^h\right\|^2_{L^2(0,t;V)}+\int_0^t\left\|\boldsymbol{v}(s)-\boldsymbol{v}^{h\tau}(s)\right\|_V^2\,\mathrm{d}s\Bigg)\quad\forall\boldsymbol{w}^h\in V^h.\end{aligned}$$

Finally, applying again Gronwall's lemma we find that

$$\left\|\dot{\boldsymbol{v}}-\dot{\boldsymbol{v}}^{h\tau}\right\|^2_{L^2(0,T;H)}+\left\|\boldsymbol{v}-\boldsymbol{v}^{h\tau}\right\|^2_{C([0,T];V)}\le c\left(\left\|\boldsymbol{u}_0-\boldsymbol{u}_0^h\right\|_V^2+\left\|\boldsymbol{v}_0-\boldsymbol{v}_0^h\right\|_V^2+\left\|\dot{\boldsymbol{v}}-\boldsymbol{w}^h\right\|^2_{L^2(0,T;V)}\right)\quad\forall\boldsymbol{w}^h\in V^h.$$

Now, using the regularity condition $\boldsymbol{v}\in H^1(0,T;[H^2(\Omega)]^d)$, we conclude that (see [9]),

$$\inf_{w^h\in V^h}\left\|\dot{\boldsymbol{v}}-\boldsymbol{w}^h\right\|_{L^2(0,T;V)}\le ch\left\|\boldsymbol{v}\right\|_{H^1(0,T;[H^2(\Omega)]^d)}.$$

It implies the linear convergence. □

4. Numerical results

4.1. Numerical scheme

First, we recall that the variational space V is approximated by using the finite element space V^h defined by (2.1).

Let $\boldsymbol{u}_{n-1}^{hk}\in V^h$ and $\boldsymbol{v}_{n-1}^{hk}\in V^h$ be known. For $n=1,\dots,N$, the fully discrete problem VP^{hk} can be written in the following form,

$$\begin{aligned}\left(\boldsymbol{v}_n^{hk},\boldsymbol{w}^h\right)_H+k\left(\mathcal{A}\boldsymbol{\varepsilon}\left(\boldsymbol{v}_n^{hk}\right)+k\mathcal{B}\boldsymbol{\varepsilon}\left(\boldsymbol{v}_n^{hk}\right),\boldsymbol{\varepsilon}(\boldsymbol{w}^h)\right)_Q &= k\left(\boldsymbol{f}_n,\boldsymbol{w}^h\right)_V \\ &\quad +\left(\boldsymbol{v}_{n-1}^{hk},\boldsymbol{w}^h\right)_H-k\left(\mathcal{B}\boldsymbol{\varepsilon}\left(\boldsymbol{u}_{n-1}^{hk}\right),\boldsymbol{\varepsilon}(\boldsymbol{w}^h)\right)_Q\quad\forall\boldsymbol{w}^h\in V^h,\end{aligned}$$

where we recall that the discrete displacement field $\boldsymbol{u}^{hk}=\{\boldsymbol{u}_n^{hk}\}_{n=0}^N\subset V^h$ is updated from the equation

$$\boldsymbol{u}_n^{hk}=\sum_{j=1}^n k\boldsymbol{v}_j^{hk}+\boldsymbol{u}_0^h=\boldsymbol{u}_{n-1}^{hk}+k\boldsymbol{v}_n^{hk}.$$

This leads to a linear variational equation which is solved by using the classical Cholesky method.

Remark 4.1. By standard calculations and induction it can be found that estimator $\eta_3(t)$ can be bounded by

$$\eta_3(t) \leq \max_{t_{n-1}<t\leq t_n} \left\| \mathbf{a}t^2 + \mathbf{b}t + \mathbf{c} \right\|_V,$$

where

$$\mathbf{a} = \frac{\boldsymbol{v}_{n-1}^{hk} - \boldsymbol{v}_n^{hk}}{2k}, \quad \mathbf{b} = \frac{t_{n-1}\boldsymbol{v}_n^{hk} - t_n\boldsymbol{v}_{n-1}^{hk}}{k},$$
$$\mathbf{c} = \frac{(t_n^2 + k^2)}{2k}\boldsymbol{v}_{n-1}^{hk} - \frac{(t_{n-1}^2 - 2k^2)}{2k}\boldsymbol{v}_n^{hk} - \frac{k}{2}\boldsymbol{v}_0^{hk}.$$

That is, it can be bounded by a second order polynomial on t for each spatial coordinate.

In order to compute the quantity $\max_{t_{n-1}<t\leq t_n} \left\| \mathbf{a}t^2 + \mathbf{b}t + \mathbf{c} \right\|_V$, we evaluate the function $a_i t^2 + b_i t + c_i$ at t_{n-1}, t_n and $-\frac{b_i}{2a_i}$ if $-\frac{b_i}{2a_i} \in [t_{n-1}, t_n]$, to take the maximum in each coordinate, and we then calculate its V-norm.

The numerical scheme was implemented on a Intel(R) Core(TM)2 Quad CPU Q6600 @ 2.40GHz PC using MATLAB, and a typical 2D run ($h = k = 0.05$), including the calculation of the three error estimators, took about 27 seconds of CPU time.

4.2. **A first 2D-example: error estimators with respect to the exact error**

As a first two-dimensional example, the following problem is considered.

Problem T2D. Find a displacement field $\boldsymbol{u} : [0,1] \times [0,1] \times [0,1] \to \mathbb{R}^2$ and a stress tensor $\boldsymbol{\sigma} : [0,1] \times [0,1] \times [0,1] \to \mathbb{S}^2$ such that,

$$\begin{aligned}
\boldsymbol{\sigma} &= \mathcal{A}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}) \quad \text{in} \quad [0,1] \times [0,1] \times (0,1),\\
\rho\ddot{\boldsymbol{u}} - \operatorname{Div}\boldsymbol{\sigma} &= \boldsymbol{f}_0 \quad \text{in} \quad [0,1] \times [0,1] \times (0,1),\\
\boldsymbol{u} &= \boldsymbol{0} \quad \text{on} \quad \{0\} \times [0,1] \times (0,1),\\
\boldsymbol{\sigma\nu} &= \boldsymbol{f}_F \quad \text{on} \quad ([0,1] \times \{1\} \cup [0,1] \times \{0\} \cup \{1\} \times [0,1]) \times (0,1),\\
\boldsymbol{u}(0) &= \boldsymbol{v}(0) = (x^2, xy) \quad \text{in} \quad [0,1] \times [0,1],
\end{aligned}$$

where we have chosen the following data:

- traction forces $\boldsymbol{f}_F$ are given by

$$\boldsymbol{f}_F(x,y,t) = \begin{cases} \left(-\frac{1}{2}e^t y, -4e^t x\right) \text{ if } x \in [0,1],\ y = 0,\\ \left(\frac{1}{2}e^t y, 4e^t x\right) \text{ if } x \in [0,1],\ y = 1,\\ \left(5e^t x, \frac{1}{2}e^t y\right) \text{ if } x = 1,\ y \in [0,1], \end{cases}$$

- volume force $\boldsymbol{f}_0$ is taken as $\boldsymbol{f}_0(x,y,y) = (e^t(x^2 - \frac{11}{2}), e^t xy)$,
- the elastic tensor $\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u})$ satisfies the classical Hooke's law and it has the following form,

$$\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}) = 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}) + \lambda \mathrm{trace}(\boldsymbol{\varepsilon}(\boldsymbol{u}))I,$$

  where Lame's coefficients $\lambda = \frac{2}{3}$ and $\mu = \frac{1}{3}$ are used,
- the viscoelastic tensor $\mathcal{A}$ is given by $\mathcal{A} = \mathcal{B}/2$ and the material density is assumed $\rho = 1$.

TABLE 1. Example T2D: numerical errors for some h and k.

h	k	η_1	η_2	η_3	η_{Tot}	e	$e.i.$
0.250000	0.10	1.26856e+0	1.52379E-1	6.96377E-1	1.45513e+0	3.06251e-2	47.51
0.125000	0.10	6.46085e-1	1.52361e-1	6.96886e-1	9.62440e-1	2.85772e-2	33.68
0.062500	0.10	3.25794e-1	1.52387e-1	6.97049e-1	7.84372e-1	2.84570e-2	27.56
0.031250	0.10	1.63545e-1	1.52397e-1	6.97095e-1	7.32061e-1	2.84610e-2	25.72
0.015625	0.10	8.19286e-2	1.52401e-1	6.97108e-1	7.18261e-1	2.84668e-2	25.23
0.250000	0.05	1.23308e+0	7.73608e-2	3.61124e-1	1.28719e+0	1.85108e-2	69.54
0.125000	0.05	6.28065e-1	7.72995e-2	3.61372e-1	7.28718e-1	1.47082e-2	49.55
0.062500	0.05	3.16727e-1	7.73040e-2	3.61450e-1	4.86763e-1	1.44006e-2	33.80
0.031250	0.05	1.59000e-1	7.73078e-2	3.61472e-1	4.02392e-1	1.43794e-2	27.98
0.015625	0.05	7.96530e-2	7.73092e-2	3.61478e-1	3.78137e-1	1.43790e-2	26.30
0.250000	0.025	1.21556e+0	3.89967e-2	1.83950e-1	1.23001e+0	1.36924e-2	89.83
0.125000	0.025	6.19170e-1	3.89359e-2	1.84073e-1	6.47124e-1	7.87045e-3	82.22
0.062500	0.025	3.12251e-1	3.89335e-2	1.84111e-1	3.64573e-1	7.27980e-3	50.08
0.031250	0.025	1.56756e-1	3.89348e-2	1.84122e-1	2.44927e-1	7.23210e-3	33.87
0.015625	0.025	7.85296e-2	3.89353e-2	1.84125e-1	2.03923e-1	7.22770e-3	28.21
0.250000	0.0125	1.20686e+0	1.95886e-2	9.28418e-2	1.21058e+0	1.21160e-2	99.92
0.125000	0.0125	6.14751e-1	1.95420e-2	9.29030e-2	6.22038e-1	4.74492e-3	131.10
0.062500	0.0125	3.10028e-1	1.95379e-2	9.29220e-2	3.24243e-1	3.72001e-3	87.16
0.031250	0.0125	1.55641e-1	1.95381e-2	9.29273e-2	1.82322e-1	3.63256e-3	50.19
0.015625	0.0125	7.79716e-2	1.95383e-2	9.29287e-2	1.22870e-1	3.62440e-3	33.90
0.250000	0.00625	1.20252e+0	9.82179e-3	4.66402e-2	1.20346e+0	1.16554e-2	103.25
0.125000	0.00625	6.12549e-1	9.79079e-3	4.66707e-2	6.14403e-1	3.52314e-3	174.39
0.062500	0.00625	3.08920e-1	9.78695e-3	4.66801e-2	3.12581e-1	1.98549e-3	157.43
0.031250	0.00625	1.55086e-1	9.78680e-3	4.66828e-2	1.62255e-1	1.82937e-3	88.69
0.015625	0.00625	7.76935e-2	9.78687e-3	4.66835e-2	9.11669e-2	1.81590e-3	50.20

The exact solution to Problem $T2D$ can be obtained after some easy algebra and it has the following form,

$$\boldsymbol{u}(x,y,t) = \left(e^t x^2, e^t xy\right).$$

If we denote by

$$\eta_1 = \left(\sum_{n=1}^N k(\eta_1^n)^2\right)^{\frac{1}{2}}, \qquad \eta_2 = \left(\sum_{n=1}^N k(\eta_2^n)^2\right)^{\frac{1}{2}},$$

$$\eta_3 = \max_{1\leq n\leq N} \max_{t_{n-1}<t\leq t_n} \eta_3(t), \qquad \eta_{Tot} = \sqrt{\eta_1^2+\eta_2^2+\eta_3^2},$$

$$e = \max_{1\leq n\leq N} \left\|\boldsymbol{v}_n - \boldsymbol{v}_n^{hk}\right\|_H + \max_{1\leq n\leq N} \|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V,$$

and *e.i.* the so-called effectivity index (which equals to η_{Tot}/e), in Table 1 the results obtained for several discretization parameters h and k are shown.

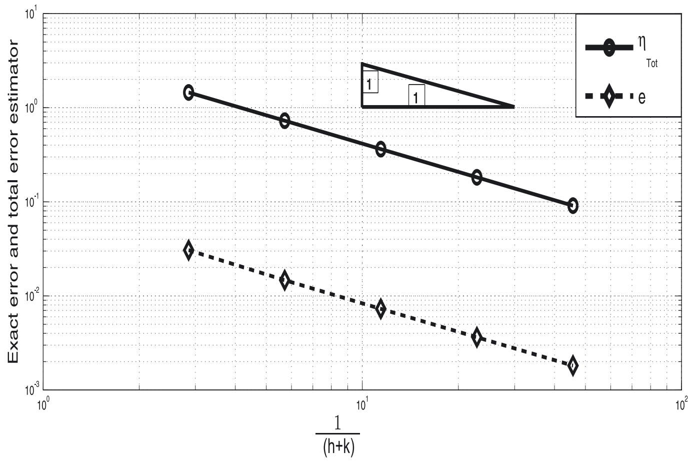

FIGURE 2. $\frac{1}{h+k}$ *vs.* e and η (log-log scale). Linear order of convergence.

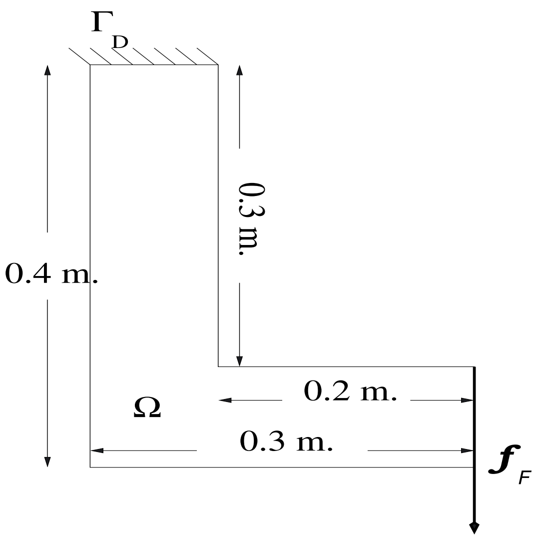

FIGURE 3. Example 2D-2: physical setting.

As it can be seen in Figure 2, the linear convergence of the discrete solution is clearly observed when the discretization parameters h and k tend to zero (the well-known log-log scale is employed). Effectivity index is good because, for instance, when both discretization parameters are reduced simultaneously, its value stays between 47 and 51.

4.3. A second 2D-example: a viscoelastic L-shaped body

As a second two-dimensional example, we consider a viscoelastic body, made of steel, with the geometry shown in Figure 3.

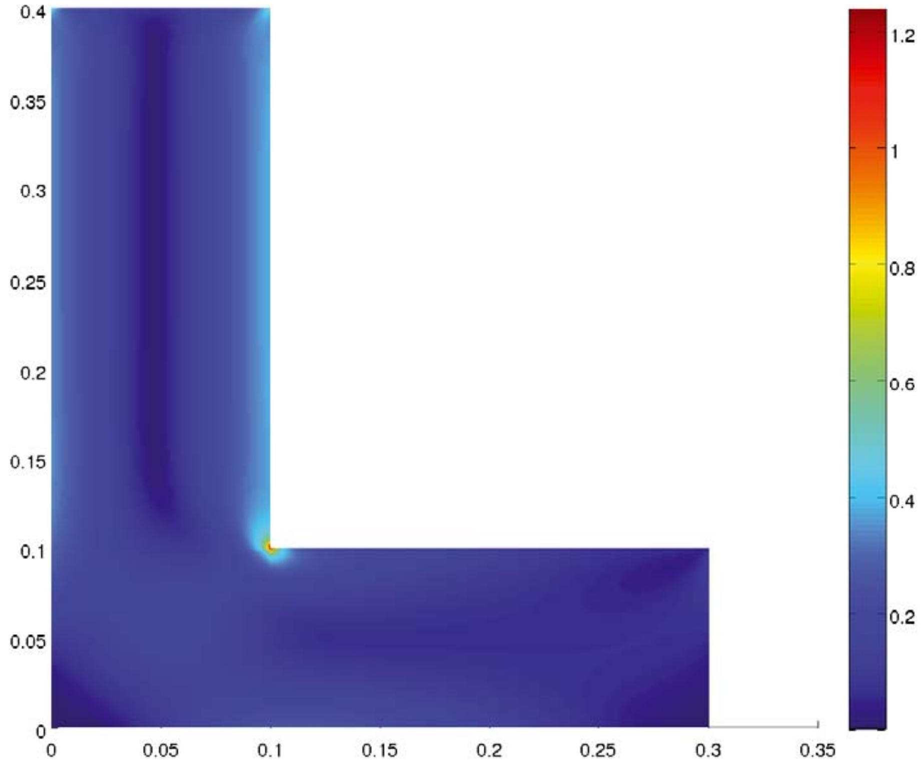

FIGURE 4. Example 2D-2: von Mises stress norm at final time. (Figure in color available online at www.esaim-m2an.org.)

No volume forces are supposed to act in the body and vertical constant tractions are applied on the boundary part $\{0.3\} \times [0, 0.1]$. Finally, the body is supposed to be clamped on the top $[0, 0.1] \times \{0.4\}$.

The following data have been employed in these simulations:

- traction forces $\boldsymbol{f}_F$ are given by

$$\boldsymbol{f}_F(x, y, t) = \begin{cases} (0, -e^t xy) & \text{on } \{0.3\} \times [0, 0.1], \\ \mathbf{0} & \text{otherwise,} \end{cases}$$

- the elastic tensor $\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u})$ satisfies the classical Hooke's law and it has the following form,

$$\mathcal{B}\boldsymbol{\varepsilon}(\boldsymbol{u}) = 2\mu\boldsymbol{\varepsilon}(\boldsymbol{u}) + \lambda \text{trace}(\boldsymbol{\varepsilon}(\boldsymbol{u}))I,$$

  where Lame's coefficients $\lambda = 1.2444 \times 10^{11}$ and $\mu = 5.3333 \times 10^{10}$ are now used,
- the viscoelastic tensor $\mathcal{A}$ is given by $\mathcal{A} = \mathcal{B}/2$ and the material density is assumed $\rho = 7700$,
- the initial conditions are taken as $\boldsymbol{v}_0 = \boldsymbol{u}_0 = \mathbf{0}$ and the final time is $T = 1$.

Using discretization parameters $h = 0.00166$ and $k = 0.05$, in Figure 4 the von Mises stress norm is plotted at final time. As expected, the highest stressed area is located near the inner corner of the L-domain. Finally, the error estimators η_1, η_2 and η_3 have the following values:

$$\eta_1 = 2.95934 \times 10^{-3}, \qquad \eta_2 = 7.25289 \times 10^{-13}, \qquad \eta_3 = 3.17194 \times 10^{-13}.$$

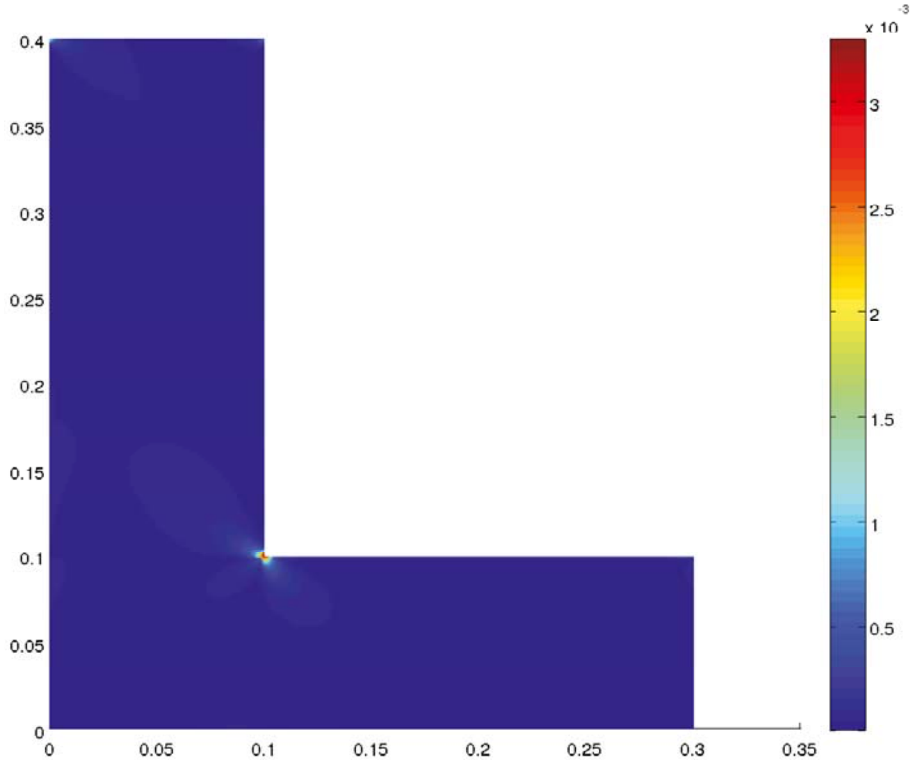

FIGURE 5. Example 2D-2: estimator η_1^{hn} (element-wise contributions) at final time. (Figure in color available online at www.esaim-m2an.org.)

We notice that, even if the exact solution is unknown, these estimates give us an idea of the error approximation and this constitutes no doubt one of the main aspects of this *a posteriori* error analysis. Finally, in Figure 5 estimator η_1^{hn} (element-wise contributions) is plotted at final time. The remaining estimators are small and they can be neglected. As can be seen, we obtain a similar behavior than for the von Mises stress norm and so, its highest values are located near the reentrant corner.

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