ESAIM: M2AN 49 (2015) 1549–1550
DOI: 10.1051/m2an/2015081

ESAIM: Mathematical Modelling and Numerical Analysis
www.esaim-m2an.org

PREFACE

Bertrand Maury and Filippo Santambrogio

The history of Optimal Transportation is not a long, quiet river. Its birth is commonly dated back to the end of the 18th century, with the problem set by G. Monge in his *Mémoire sur la théorie des déblais et des remblais*, where the motivation for the problem was the optimal way of moving sand piles for building or military applications. For many modern mathematicians, Monge is the father of OT theory, but his work did not answer to all the questions that we would consider as important issues nowadays, and in particular Monge did not even wonder about a rigorous proof of the existence of an optimizer.

A long period of sleep followed, until Kantorovich, who, facing optimization problems related to railway supply chain during World War II, proposed an alternative, relaxed formulation of the transport problem set by Monge. In particular, his work allowed to give both existence and duality results, and paved the way to a modern mathematical treatise of the problem.

Initially motivated by those two engineering issues, the problem turned out to be incredibly rich from the mathematical point of view, thereby triggering a huge amount of theoretical questions, some of which are still open. Its connections with many other branches of pure mathematics, and in particular the geometry of manifolds and metric spaces are a very important feature of the theory. In the invited preface by Cédric Villani, the reader will find the point of view on the various topics that are developed in this volume of a mathematician who has been highly involved in the research on these applications to "pure" mathematics.

However, in parallel to a huge activity around existence, regularity of transport maps, geometric properties of the Wasserstein space, *etc.*.. many connections with real world application have surfaced in the last decades. Those links have taken different forms, that we aim at highlighting in the present volume.

Firstly, OT has been identified as a natural framework to various existing models (in particular in physics or life sciences), allowing to establish new mathematical properties, or to suggest numerical strategies.

In particular, OT allows to study several types of evolution PDEs describing the motion of a density of particles, as it is the case for many (typically parabolic) equations which can be interpreted as gradient flows of suitable energy functionals with respect to optimal transport distances. In [1] the authors study the parabolic-parabolic Keller Segel model in chemotaxis *via* these methods, while in [3] a higher-order optimal transport problem involving gradients is proposed in connection with combinatorics, and a gradient-flow equation is derived from it.

Among other models from physics which fit in an OT framework, a very important role is played by the multi-electron problems of the Density Functional Theory, which can be interpreted as a multi-marginal Kantorovich problem. In [6] the reader will find new duality results on this model, and in [12] a general introduction to multi-marginal problems and their applications, which include also many models from economics.

Secondly, OT is now gaining more and more attention as a tool to tackle "real life" problems. Some of them involve transport networks with possible traffic concentration effects, as we can see in the congested transport model of [4], where the concentration of mass on a same path is discouraged, and its "branched" counterpart, where the economy of scale, on the contrary, favors it ([13]: note that recent applications of branched transport to medical problems are briefly introduced in this paper, mainly devoted to a general survey of this theory). We

Article published by EDP Sciences

can't help observing the strong "Monge" flavor of [7], where optimal pit digging is studied in terms of a convex problem involving optimal transport tools.

In the framework of real life applications, image processing deserves a separate mention. In image processing OT is often used as a tool to compare images (*via* Wasserstein distances), to process them (when modifying colors, for instance), or to interpolate between them (*via* different sorts of geodesics) and the pertinence of these choices can be evaluated from the practical results. This is different from the case of applications to PDEs, where OT is a tool to prove existence or uniqueness results, or to provide interpretations of the corresponding evolutions, and also from the case of other real-life applications where OT and its variants are mainly a modeling tool. The reader will find many different applications to different issues of interest for the image community in [2, 5, 8, 10–12].

Finally, this volume also contains many contributions on numerical methods to compute optimal transport maps, plans and costs.

The actual computation of transport costs together with transportation maps and plans still remains a challenge in many situations, but the situation is rapidly evolving. Some years ago, only the fluid mechanics approach by Benamou and Brenier and few discrete combinatorial algorithms were available, but often too costly for real data. Today, semi-discrete methods based on Voronoi cells and efficient PDE approaches based on suitable discretization of the Monge Ampère equation have appeared, and many other strategies have been considerably improved. Think that one of the most spectacular applications of optimal transport, the reconstruction of the early universe[1], required days of computational time for a set of 10 000 target points, while the methods presented, for instance, in [9] allow to deal with data sets 100 times larger in some minutes. Different methodologies are addressed in this volume, including non-smooth methods [5, 9], and methods based on, or extending, the Benamou–Brenier approach [2, 8, 10, 11].

Orsay, October 2015
Bertrand Maury
Filippo Santambrogio

References

[1] A. Blanchet, José Antonio Carrillo, David Kinderlehrer, Michal Kowalczyk, Philippe Laurençot and Stefano Lisini, A hybrid variational principle for the Keller–Segel system in $\mathbb{R}^2$. [Link]

[2] Afaf Bouharguane, Angelo Iollo and Lisl Weynans, Numerical solution of the Monge-Kantorovich problem by density lift-up continuation. [Link]

[3] Yann Brenier, Connections between optimal transport, combinatorial optimization and hydrodynamics. [Link]

[4] Giuseppe Buttazzo, Guillaume Carlier and Serena Guarino Lo Bianco, Optimal regions for congested transport. [Link]

[5] Guillaume Carlier, Adam Oberman and Edouard Oudet, Numerical methods for matching for teams and Wasserstein barycenters. [Link]

[6] Luigi De Pascale, Optimal transport with Coulomb cost. Approximation and duality. [Link]

[7] Ivar Ekeland and Maurice Queyranne, Optimal pits and optimal transportation. [Link]

[8] Romain Hug, Emmanuel Maitre and Nicolas Papadakis, Multi-physics optimal transportation and image interpolation. [Link]

[9] Bruno Lévy, A numerical algorithm for L_2 semi-discrete optimal transport in 3D. [Link]

[10] Damiano Lombardi and Emmanuel Maitre, Eulerian models and algorithms for unbalanced optimal transport. [Link]

[11] Jan Maas, Martin Rumpf, Carola Schönlieb and Stefan Simon, A generalized model for optimal transport of images including dissipation and density modulation. [Link]

[12] Brendan Pass, Multi-marginal optimal transport: theory and applications. [Link]

[13] Qinglan Xia, Motivations, ideas and applications of ramified optimal transportation. [Link]

[1] Only occurrence of OT in a journal such as Nature, see Frisch *et al.* A reconstruction of the initial conditions of the Universe by optimal mass transportation. *Nature* **417** (2002) 260–262.