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SUPERCONVERGENCE BY M-DECOMPOSITIONS. PART III: CONSTRUCTION OF THREE-DIMENSIONAL FINITE ELEMENTS *

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Abstract. We apply the concept of an M-decomposition in the framework of steady-state diffusion problems to construct local spaces defining superconvergent hybridizable discontinuous Galerkin methods as well as their companion sandwiching mixed methods in $\mathbb{R}^3$ with tetrahedral, pyramidal, prismatic, and hexahedral elements.

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1. Introduction

This is the third of a series of paper in which we develop the concept of an M-decomposition as an effective tool for devising hybridizable discontinuous Galerkin (HDG) methods, and their companion sandwiching mixed methods, which superconverge on unstructured meshes of shape-regular polyhedral elements. In the first part of this series, [6], the general theory of M-decompositions was developed in the frame of steady-state diffusion problems:

$$\begin{aligned} \mathrm{c}\boldsymbol{q} + \nabla u &= 0 && \text{in } \Omega, \\ \nabla\cdot\boldsymbol{q} &= f && \text{in } \Omega, \\ u &= g && \text{on } \partial\Omega, \end{aligned}$$

where $\Omega\subset\mathbb{R}^d$ is a bounded polygonal ($d=2$) or polyhedral ($d=3$) domain, c is a uniformly bounded, uniformly positive definite symmetric matrix-valued function, $f\in L^2(\Omega)$ and $g\in H^{1/2}(\partial\Omega)$. In the second part of this series, [5], the general theory was applied to the two-dimensional case. Here we apply it to explicitly obtain new ready-for-implementation local spaces admitting M-decompositions for flat-faced pyramidal, prismatic, and hexahedral elements.

To better describe our results, let us recall the definition of the HDG (and mixed) methods under consideration; we use the notation used in Part I, [6]. The HDG methods seek an approximation to $(u,\boldsymbol{q},u|_{\mathcal{E}_h})$,

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$(u_h, \boldsymbol{q}_h, \widehat{u}_h)$, in the finite element space $W_h \times \boldsymbol{V}_h \times M_h$, of the form

$$\begin{aligned}\boldsymbol{V}_h &:= \{\boldsymbol{v} \in \boldsymbol{L}^2(\mathcal{T}_h) : \boldsymbol{v}|_K \in \boldsymbol{V}(K),\ \ K \in \mathcal{T}_h\},\\ W_h &:= \{w \in L^2(\mathcal{T}_h) : w|_K \in W(K),\ K \in \mathcal{T}_h\},\\ M_h &:= \{\mu \in L^2(\mathcal{E}_h) : \mu|_F \in M(F),\ \ F \in \mathcal{E}_h\},\end{aligned}$$

which is determined as the only solution of the following weak formulation:

$$\begin{aligned}(\mathrm{c}\,\boldsymbol{q}_h\,,\,\boldsymbol{v})_{\mathcal{T}_h} - (u_h\,,\,\nabla\cdot\boldsymbol{v})_{\mathcal{T}_h} + \langle \widehat{u}_h\,,\,\boldsymbol{v}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} &= 0,\\ -\,(\boldsymbol{q}_h\,,\,\nabla w)_{\mathcal{T}_h} \ + \langle \widehat{\boldsymbol{q}}_h\cdot\boldsymbol{n}\,,\,w\rangle_{\partial\mathcal{T}_h} &= (f\,,\,w)_{\mathcal{T}_h},\\ \langle \widehat{\boldsymbol{q}}_h\cdot\boldsymbol{n},\mu\rangle_{\partial\mathcal{T}_h\setminus\partial\Omega} &= 0,\\ \langle \widehat{u}_h,\mu\rangle_{\partial\Omega} &= \langle g,\mu\rangle_{\partial\Omega},\end{aligned}$$

for all $(w, \boldsymbol{v}, \mu) \in W_h \times \boldsymbol{V}_h \times M_h$, where

$$\widehat{\boldsymbol{q}}_h\cdot\boldsymbol{n} = \boldsymbol{q}_h\cdot\boldsymbol{n} + \alpha(u_h - \widehat{u}_h) \quad \text{on} \quad \partial\mathcal{T}_h.$$

In Part I, [6], it was shown that these HDG methods are superconvergent on unstructured meshes if, for all elements $K \in \mathcal{T}_h$, the local space $\boldsymbol{V}(K) \times W(K)$ admits an $M(\partial K)$-decomposition, where

$$M(\partial K) := \{\mu \in L^2(\partial K) :\ \mu|_F \in M(F) \text{ for all faces } F \text{ of } K\},$$

and that the resulting methods are mixed methods, that is, we can take $\alpha = 0$ since $\nabla\cdot\boldsymbol{V}(K) = W(K)$. Moreover, the construction of M-decompositions for any space $M(\partial K)$ was proven to be possible *via* solving Laplace equation with certain Neumann boundary conditions on the element K. The actual construction of read-for-implementation spaces admitting M-decompositions on polygonal meshes, *without* solving Laplace equations, was carried out in Part II, [5]. Here, we extend this effort to the three-dimensional case.

As in Part II, [5], we summarize the construction as follows. (From now on, if there is no confusion, we drop the dependence of the local spaces on the element K.) Given a space of traces M on ∂K containing the constants, we pick any *given* space $\boldsymbol{V}_g \times W_g$ satisfying the inclusion properties:

(I.1) $\gamma\boldsymbol{V}_g + \gamma W_g \subset M$,
(I.2) $\nabla W_g \times \nabla\cdot\boldsymbol{V}_g \subset \boldsymbol{V}_g \times W_g$,

where $\gamma\boldsymbol{V}_g := \{\boldsymbol{v}\cdot\boldsymbol{n}|_{\partial K} : \boldsymbol{v} \in \boldsymbol{V}_g\}$ and $\gamma W_g := \{w|_{\partial K} : w \in W_g\}$. We then construct the three spaces $\boldsymbol{V} \times W$ admitting M-decompositions described in Tables 1 and 2. The spaces in the top and bottom rows give rise to (hybridized) mix methods. The two integers in the last column of Table 2 are defined as follows:

$$\begin{aligned}I_M(\boldsymbol{V}_g \times W_g) :=\,& \dim M - \dim\{\boldsymbol{v}\cdot\boldsymbol{n}|_{\partial K} : \boldsymbol{v} \in \boldsymbol{V}_g, \nabla\cdot\boldsymbol{v} = 0\}\\ & - \dim\{w|_{\partial K} : w \in W_g, \nabla w = 0\},\\ I_S(\boldsymbol{V}_g \times W_g) :=\,& \dim W_g - \dim \nabla\cdot\boldsymbol{V}_g.\end{aligned}$$

Here, we carry out this construction for the main flat-faced polyhedral elements involved in three-dimensional meshing, namely, tetrahedra, pyramids of quadrilateral base, prisms and hexahedra of quadrilateral faces. We summarize our results in Tables 3 and 4. In Table 3, we consider tetrahedral elements, pyramidal elements with a square (or parallelogram) base, prisms with parallel faces, and cubes (or parallelepipeds). Therein, $\mathcal{Q}_k(K)$ denotes the space of tensor product polynomials of degree k defined on K, and $\mathcal{P}_{k|k}(K)$ the space of the form $\mathcal{P}_k(B)\otimes\mathcal{P}_k(e)$ whenever K is the prism $B\times e$. We denote by $\boldsymbol{\mathcal{P}}_k(K)$, respectively, $\boldsymbol{\mathcal{P}}_{k|k}(K)$ and $\boldsymbol{\mathcal{Q}}_k(K)$, the vector-valued functions whose components lie in $\mathcal{P}_k(K)$, respectively, $\mathcal{P}_{k|k}(K)$ and $\mathcal{Q}_k(K)$. In Table 4, we consider the

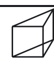
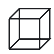
TABLE 1. Construction of spaces $\boldsymbol{V} \times W$ admitting $M(\partial K)$-decompositions; we assume that $M(\partial K)$ contains the constants. The *given space* space $\boldsymbol{V}_g \times W_g$ satisfies the inclusion properties (I). We assume the scalar space W contains the constants $\mathcal{P}_0(K)$.

$\boldsymbol{V}$	W
$\boldsymbol{V}^{\text{mix}} := \boldsymbol{V}_g \oplus \delta\boldsymbol{V}_{\text{fillM}} \oplus \delta\boldsymbol{V}_{\text{fillW}}$	$W^{\text{mix}} := W_g$
$\boldsymbol{V}^{\text{hdg}} := \boldsymbol{V}_g \oplus \delta\boldsymbol{V}_{\text{fillM}}$	$W^{\text{hdg}} := W_g$
$\boldsymbol{V}_{\text{mix}} := \boldsymbol{V}_g \oplus \delta\boldsymbol{V}_{\text{fillM}}$	$W_{\text{mix}} := \nabla \cdot \boldsymbol{V}_g$

TABLE 2. The properties of the spaces $\delta\boldsymbol{V}$.

$\delta\boldsymbol{V}$	$\nabla \cdot \delta\boldsymbol{V}$	$\gamma\delta\boldsymbol{V}$	$\dim \delta\boldsymbol{V}$	
$\delta\boldsymbol{V}_{\text{fillM}}$	$\{\mathbf{0}\}$	$\subset M, \cap\gamma\boldsymbol{V}_{g_s} = \{0\}$	$I_M(\boldsymbol{V}_g \times W_g)$	$= \dim \gamma\delta\boldsymbol{V}_{\text{fillM}}$
$\delta\boldsymbol{V}_{\text{fillW}}$	$\subset W_g, \cap\nabla \cdot \boldsymbol{V}_g = \{0\}$	$\subset M$	$I_S(\boldsymbol{V}_g \times W_g)$	$= \dim \nabla \cdot \delta\boldsymbol{V}_{\text{fillW}}$

TABLE 3. Some properties of the spaces $\boldsymbol{V} \times W$ admitting an $M(\partial K)$-decomposition when K is a *regular* polyhedral element: a tetrahedron, a pyramid with a square base, a prism with congruent, parallel faces, and a cube. If F is a rectangular face of K, then $M(F) := \mathcal{Q}_k(F)$; if not, $M(F) := \mathcal{P}_k(F)$. Here $k \geq 1$. The symbol ✓ indicates that the spaces are new.

				
$\boldsymbol{V}_g \times W_g$	$\boldsymbol{\mathcal{P}}_k \times \mathcal{P}_k$	$\boldsymbol{\mathcal{P}}_k \times \mathcal{P}_k$	$\boldsymbol{\mathcal{P}}_{k\|k} \times \mathcal{P}_{k\|k}$	$\boldsymbol{\mathcal{Q}}_k \times \mathcal{Q}_k$
$I_M(\boldsymbol{V}_g \times W_g)$	0	$\frac{1}{2}k\,(k+1)+3$	$k+2$	6
$I_S(\boldsymbol{V}_g \times W_g)$	$\frac{1}{2}(k+1)(k+2)$	$\frac{1}{2}(k+1)(k+2)$	$k+1$	1
$\boldsymbol{V}^{\text{mix}} \times W^{\text{mix}}$	$\mathbf{RT}_k$[10]	✓	✓	$\mathbf{TNT}_{[k]}$[7]
$\boldsymbol{V}^{\text{hdg}} \times W^{\text{hdg}}$	$\mathbf{HDG}_k$[8]	✓	✓	$\mathbf{HDG}^{Q}_{[k]}$[8]
$\boldsymbol{V}_{\text{mix}} \times W_{\text{mix}}$	$\mathbf{BDM}_k$[2,11]	✓	✓	✓

construction for the other elements. Therein, $\widetilde{\mathcal{P}}_\ell(K)$ denotes the homogeneous polynomials of degree ℓ in K. In these tables, we restrict ourselves to displaying the dimension of the spaces involved in the construction and in stating if they are new or already known. When the spaces we obtain are actually new but can be considered a small variation of already-known ones, we do not mark them as *new*. We remark that the three old spaces listed in Table 4 are only defined on a regular polyhedron, though.

Let us give an idea of how do we proceed to obtain our results. The construction of the space $\delta\boldsymbol{V}_{\text{fillW}}$ is fairly easy, especially when compared with that of the space $\delta\boldsymbol{V}_{\text{fillM}}$. To construct $\delta\boldsymbol{V}_{\text{fillM}}$, we proceed in two steps. In the first, since

$$\gamma\{\boldsymbol{v} \in \boldsymbol{V}_g :\ \nabla\cdot \boldsymbol{v} = 0\} \subset \{\mu \in M(\partial K) :\ \langle \mu, 1\rangle_{\partial K} = 0\},$$

we have to find a subspace $C_M \subset M(\partial K)$ such that

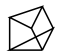
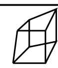
$$C_M \oplus \gamma\{\boldsymbol{v} \in \boldsymbol{V}_g :\ \nabla\cdot \boldsymbol{v} = 0\} = \{\mu \in M(\partial K) :\ \langle \mu, 1\rangle_{\partial K} = 0\}.$$

In the second step, we find a basis of C_M, $\mathcal{B}_M$, and simply set

$$\delta\boldsymbol{V}_{\text{fillM}} := \operatorname{span}\{\boldsymbol{v}_\mu :\ \mu \in \mathcal{B}_M\},$$

where the function $\boldsymbol{v}_\mu$ is a suitably chosen, divergence-free function which *lifts* the trace functions of C_M, because we have that $\boldsymbol{v}_\mu \cdot \boldsymbol{n}|_{\partial K} = \mu$, into the interior of the element K. These liftings are relatively easy to

TABLE 4. Some properties of the spaces $\boldsymbol{V} \times W$ admitting an $M(\partial K)$-decomposition when K is an irregular pyramid (of a quadrilateral base), prism or hexahedron (of quadrilateral faces). Here $M(\partial K) = \mathcal{P}_k(\partial K)$ and $\boldsymbol{V}_g \times W_g = \boldsymbol{\mathcal{P}}_k(K) \times \mathcal{P}_k(K)$. In all cases, $\boldsymbol{V}_{\text{fillW}} := \boldsymbol{x}\,\widetilde{\mathcal{P}}_k(K)$. The symbol ✓ indicates that the spaces are new.

							
		number of pairs of parallel faces		number of pairs of parallel faces			
$I_M(\boldsymbol{V}_g \times W_g)$		0	1	0	1	2	3
$k = 0$	1	1	1	2	2	2	2
$k = 1$	3	3	3	6	6	6	6
$k \geq 2$	3	3	$k+2$	9	$k+7$	$2k+5$	$3k+3$
$\boldsymbol{V}^{\text{mix}} \times W^{\text{mix}}$	✓	✓	✓	✓	✓	✓	✓
$\boldsymbol{V}^{\text{hdg}} \times W^{\text{hdg}}$	✓	✓	✓	✓	✓	✓	$\mathbf{HDG}_{[k]}$ [8]
$\boldsymbol{V}_{\text{mix}} \times W_{\text{mix}}$	✓	✓	$\mathbf{BDM}_{<k>}$ [4]	✓	✓	✓	$\mathbf{BDM}_{[k]}$ [2]

get as they have a simple, explicit formula in terms of polynomial or rational functions, or are given in terms of easily computable piecewise-polynomial functions. On the other hand, the difficulty is the characterization of the space $\gamma\{\boldsymbol{v} \in \boldsymbol{V}_g : \ \nabla\cdot \boldsymbol{v} = 0\}$. When it is not possible to get it all at once, we number the faces and use a sequential, divide-and-conquer strategy to find C_M which allows us to focus our attention on the normal traces of the space $\{\boldsymbol{v} \in \boldsymbol{V}_g : \ \nabla\cdot \boldsymbol{v} = 0\}$ on a *single* face of the polyhedron at a time. As a consequence, our construction does depend on the way we number the faces. This is a reflection of the fact that the spaces we are seeking are not uniquely defined.

Now, let us discuss why is it that we only consider *some* polyhedral elements. To illustrate the idea, we take $\boldsymbol{V}_g = \boldsymbol{\mathcal{P}}_k$. The reason is, roughly speaking, that the space

$$\{\boldsymbol{v} \in \boldsymbol{\mathcal{P}}_k : \ \boldsymbol{v} \cdot \boldsymbol{n}|_{\partial K} = 0,\ \nabla\cdot \boldsymbol{v} = 0\},$$

which we have to know well in order to carry out our construction, is very complicated for an arbitrary polyhedron, especially when k is big. Indeed, in two-space dimensions, such space, for the polygon K, can be characterized as

$$\{(\partial_y w, -\partial_x w) : \quad w \in \mathcal{P}_{k+1}(K) \text{ and } w|_{\partial K} = 0\}.$$

Thus, a basis can be readily computed since the space of H^1-bubbles $\{w \in \mathcal{P}_{k+1}(K) : \quad w|_{\partial K} = 0\}$ is very easy to get. However, in three-space dimensions, such a characterization becomes extremely involved for a polyhedron K, especially as the number of faces increases. Indeed, the space we seek is now

$$\{\nabla\times \boldsymbol{v} : \boldsymbol{v} \in \boldsymbol{\mathcal{P}}_{k+1}(K) \text{ and } \boldsymbol{n} \times (\boldsymbol{v} \times \boldsymbol{n})|_{\partial K} = 0\}\,,$$

where $\boldsymbol{n}$ is the normal to the boundary of K. The space of $H(\text{curl})$-bubbles $\{\boldsymbol{v} \in \boldsymbol{\mathcal{P}}_{k+1}(K) : \ \boldsymbol{n}\times(\boldsymbol{v}\times\boldsymbol{n})|_{\partial K} = 0\}$ is very hard to compute. In particular, as we are going to see, it depends on the number of pairs of parallel faces of the polyhedron, and even on the number of parallel edges. This is why, we restrict ourselves to the above-mentioned polyhedral elements.

Finally, to highlight the relevance of our approach, let us compare our mixed finite element spaces $\boldsymbol{V}^{\text{mix}}\times W^{\text{mix}}$ containing the give space $\boldsymbol{V}_g \times W_g := \boldsymbol{\mathcal{P}}_k \times \mathcal{P}_k$ in Table 3 on the *regular* pyramid with those of Nigam and Phillips [12,13]. The first family of high-order accurate, stable mixed finite element spaces on a *regular* pyramid was presented in [12]; they were denoted by $\mathcal{U}^{(2),k+1} \times \mathcal{U}^{(3),k+1}$. Later in [13], the same authors constructed a second family of pyramidal finite elements, $\mathcal{R}^{(2),k+1} \times \mathcal{R}^{(3),k+1}$, with significantly smaller space dimension.

The spaces were constructed by mapping certain rational functions from a reference *infinite pyramid* to the physical pyramid. They have the following inclusion property to ensure approximability:

$$\boldsymbol{\mathcal{P}}_k \times \mathcal{P}_k \subset \mathcal{R}^{(2),k+1} \times \mathcal{R}^{(3),k+1} \subset \mathcal{U}^{(2),k+1} \times \mathcal{U}^{(3),k+1}.$$

While these two pairs of spaces and ours have the same convergence rates, our spaces have significantly smaller dimension for $k \geq 1$ (the spaces are identical for $k = 0$), since

$$\begin{aligned}
\dim \mathcal{U}^{(2),k+1} - \dim \mathcal{R}^{(2),k+1} &= \frac{1}{2} k\,(k+1)(4k+7),\\
\dim \mathcal{R}^{(2),k+1} - \dim \boldsymbol{V}^{\mathrm{mix}} &= \frac{1}{2} k\,(k+1)(k+2) + k - 2,\\
\dim \mathcal{U}^{(3),k+1} - \dim \mathcal{R}^{(3),k+1} &= \frac{1}{2} k\,(k+1)(4k+5),\\
\dim \mathcal{R}^{(3),k+1} - \dim W^{\mathrm{mix}} &= \frac{1}{6} k\,(k+1)(k+2).
\end{aligned}$$

We achieve this improvement on the space dimension by directly working with the physical pyramid to augment $\boldsymbol{\mathcal{P}}_k \times \mathcal{P}_k$ with *the* minimal number of additional basis functions to ensure a space admitting an M-decomposition rather than using sophiscated mappings from an *infinite pyramid* as was done in [12, 13].

The rest of the paper is organized as follows. In Section 2, we describe and discuss our constructions of M-decompositions. In Section 3, we show how to compute the *composite liftings* used in Section 2. In Section 4, we prove the main results in Section 2. We end in Section 5 with some concluding remarks.

2. The main results

In this section, we display and discuss our main results, that is, the construction of the filling spaces $\delta \boldsymbol{V}_{\mathrm{fillM}}$ and $\delta \boldsymbol{V}_{\mathrm{fillW}}$ satisfying the properties in Table 2. We begin by introducing the notation we are going to use and then proceed according to the shape of the polyhedral element K.

2.1. Notation

We start by setting the notation we need to state our results. After describe the several elements of our polyhedra K, we introduce two objects needed for the construction of the space $\boldsymbol{V}_{\mathrm{fillM}}$: *extensions* of a trace defined in a single face of K into the whole boundary ∂K, and *liftings* of those traces into the interior of the element K.

Geometry

We begin by displaying in Figure 1 the notation we are going to follow to describe the faces of each of the four polyhedral elements we are going to consider. The faces are taken to be flat and so the points $\boldsymbol{x}$ on the face F_i lie on the hyperplane $\lambda_i(\boldsymbol{x}) = 0$. The outward unit normal at the face F_i is denoted by $\boldsymbol{n}_i$ and is parallel to $\nabla \lambda_i$. When $\kappa_{ijk} := \boldsymbol{n}_i \times \boldsymbol{n}_j \cdot \boldsymbol{n}_k$ is not equal to zero, we know that there is a unique point $\boldsymbol{v}_{ijk}$ defined by $\lambda_\ell(\boldsymbol{v}_{ijk}) = 0$, $\ell \in \{i, j, k\}$. In general, $\boldsymbol{v}_{ijk}$ lies outside the element K, but when it does not, it is nothing but the vertex shared by the faces F_i, F_j and F_k. We indicate that the face F_i is, respectively, is not, parallel to the face F_j by writing, $F_i \parallel F_j$, repectively, $F_i \nparallel F_j$.

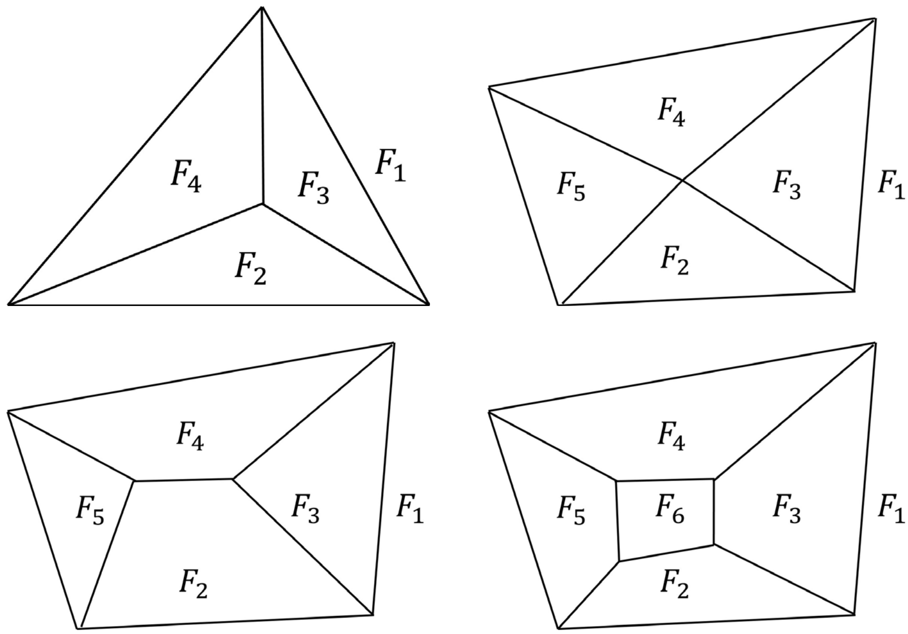


FIGURE 1. The faces of a tetrahedron (*top left*), pyramid (*top right*), prism (*bottom left*), hexahedron (*bottom right*). The face F_1 is the face at the bottom of the polyhedral.

Extensions of traces

To be able to obtain the space of traces C_M described in the Introduction, we need to define two extensions to ∂K of functions defined on a single face F_i of K. The extensions are the following:

$$\eta_i(\zeta) = \begin{cases} \zeta & \text{on } F_i, \\ -\frac{\int_{F_i}\zeta}{\int_{F_{i+1}} 1} & \text{on } F_{i+1}, \\ 0 & \text{on } \partial K \setminus (F_i \cup F_{i+1}), \end{cases} \tag{2.1a}$$

$$\eta_i(\zeta_1, \zeta_2) = \begin{cases} \zeta_1 - \zeta_2 \frac{\int_{F_i}\zeta_1}{\int_{F_i}\zeta_2} & \text{on } F_i. \\ 0 & \text{on } \partial K \setminus F_i. \end{cases} \tag{2.1b}$$

Notice for any $\zeta \in \mathcal{P}_k(F_i)$, then $\mu := \eta_i(\zeta) \in \mathcal{P}_k(\partial K)$ and $\int_{\partial K} \mu \, \mathrm{d}s = 0$. Similarly properties hold for any $\zeta_i \in \mathcal{P}_k(F_i), i = 1, 2$ and $\mu := \eta_i(\zeta_1, \zeta_2)$.

Liftings of traces

Finally, we introduce some special functions needed to describe our spaces $\boldsymbol{V}_{\mathrm{fillM}}$. They are devised to *lift* the space of traces C_M into the interior of K. For some elements, those liftings can be polynomial functions but, just as in the two-dimensional case [5], it turns out that it is not possible to carry out the construction of the spaces under consideration by using only polynomials for more general elements K, as we see in the next result.

Theorem 2.1. *Let K be a polyhedron which is not a tetrahedron, a prism with parallel congruent triangular bases, or a parallelepiped. If $\boldsymbol{V} \times W$ admits an M-decomposition with $M := \mathcal{P}_k(\partial K)$ for some $k \geq 0$ and $\mathcal{P}_0 \subset W$, then $\boldsymbol{V}$ must include some non-polynomial elements.*

Because of this result, whose proof is detailed in Section 4, we have to rely on functions we construct as follows. Given a flat-faced polyhedron K, let us triangulate it by using the non-overlapping tetrahedra $\{T_i\}_{i=1}^{nt}$.

For any given $k \geq 0$, we set the space

$$\boldsymbol{\mathcal{V}}^{cmp}_{k,nt}(K) := \big\{\boldsymbol{v} \in H_0(\mathrm{div}; K) : \ \boldsymbol{v}|_{T_i} \in \boldsymbol{\mathcal{P}}_k(T_i)\ 1 \leq i \leq nt\big\}, \tag{2.2}$$

where $H_0(\mathrm{div}; K) := \{\boldsymbol{v} \in H(\mathrm{div}; K) : \ \nabla\cdot \boldsymbol{v} = 0\}$. Since the functions in this space are piecewise polynomials, we call them *composite* functions, hence the superscript *cmp*. As we see in the next result, whose proof is provided in Section 3, we can use these functions as liftings of (normal) traces on ∂K.

Theorem 2.2. *Let K be a flat-faced polyhedron. Then, for any $\mu \in \mathcal{P}_k(\partial K)$ with $\int_{\partial K} \mu\, ds = 0$, there exists a function $\boldsymbol{v}_\mu$ in the space $\boldsymbol{\mathcal{V}}^{cmp}_{k,nt}(K)$ such that*

$$\boldsymbol{v}_\mu \cdot \boldsymbol{n}|_{\partial K} = \mu.$$

It is easy to see that the tetrahedral triangulation of the polyhedral element K is not unique. This non-uniqueness leads to the non-uniqueness of the composite lifting function $\boldsymbol{v}_\mu$. However, as will be clear in the proof of Theorem 2.2 in Section 3, the lifting function $\boldsymbol{v}_\mu$, which serves as the building block of our explicit construction of M-decompositions, can be uniquely computed for a given tetrahedral triangulation. Moreover, a good criterion to single out a tetrahedral triangulation in practice is to require the number of tetrahedra nt to be minimal. This leads to $nt = 2$ for pyramids, $nt = 3$ for prisms, and $nt = 5$ for hexahedra.

We are now ready to begin the presentation of our results.

2.2. Tetrahedra

We begin by considering the simplest polyhedron. The following result is well-known in the literature.

Theorem 2.3. *Let K be a tetrahedron. Then, for $M := \mathcal{P}_k(\partial K)$ and $\boldsymbol{V}_g \times W_g := \boldsymbol{\mathcal{P}}_k(K) \times \mathcal{P}_k(K)$, we have that*

$$I_M(\boldsymbol{V}_g \times W_g) = 0 \quad \text{and} \quad I_S(\boldsymbol{V}_g \times W_g) = (k+1)(k+2)/2.$$

Moreover, the spaces $\delta\boldsymbol{V}_{\mathrm{fillM}} := \{\boldsymbol{0}\}$ and $\delta\boldsymbol{V}_{\mathrm{fillW}} := \boldsymbol{x}\,\widetilde{\mathcal{P}}_k$ satisfy the properties in Table 2.

The space $\boldsymbol{V}^{\mathrm{mix}} \times W^{\mathrm{mix}}$ is nothing but the Raviart–Thomas space $\mathbf{RT}_k$ of index k, [10], the space $\boldsymbol{V}^{\mathrm{hdg}} \times W^{\mathrm{hdg}}$ is the space $\mathbf{HDG}_k$, [8], and the space $\boldsymbol{V}_{\mathrm{mix}} \times W_{\mathrm{mix}}$ is nothing but the space $\mathbf{BDM}_k$ of index k, [2,11].

2.3. Pyramids

Here, we consider two cases for which we provide entirely new spaces. In the first, we assume that the base of the pyramid, F_1, is a parallelogram so that the space of traces there can be taken to be $\mathcal{Q}_k(F_1)$. We then consider pyramids for which F_1 is a general quadrilateral and take the space of traces there to be $\mathcal{P}_k(F_1)$.

2.3.1. Pyramids with a parallelogram as base

For simplicity, we consider K being a unit pyramid, that is, the pyramid whose base (in the xy-plane) is the unit square $(0,1)^2$, and whose faces are unit triangles on the xz- and yz-planes. We set

$$\lambda_1 = z, \lambda_2 = x, \lambda_3 = y, \lambda_4 = 1 - x - z, \lambda_5 = 1 - y - z.$$

Our construction is contained in the following result.

Theorem 2.4. *Let K be the unit pyramid. Then, for*

$$M(\partial K) := \{\mu \in L^2(\partial K) : \ \mu|_{F_1} \in \mathcal{Q}_k(F_1),\ \mu|_{F_i} \in \mathcal{P}_k(F_i) \ \text{ for } 2 \leq i \leq 5\},$$

and $\boldsymbol{V}_g \times W_g = \boldsymbol{\mathcal{P}}_k(K) \times \mathcal{P}_k(K)$ with $k \geq 1$, we have that

$$I_M(\boldsymbol{V}_g \times W_g) = k(k+1)/2 + 3 \quad \text{and} \quad I_S(\boldsymbol{V}_g \times W_g) = \ (k+1)(k+2)/2.$$

Moreover, the spaces

$$\begin{aligned}
\delta \boldsymbol{V}_{\mathrm{fillM}} :=& \nabla\times \operatorname{span}\{\lambda_2^{\alpha}\lambda_3^{\beta+1}\lambda_4\lambda_5 \nabla \lambda_2 :\ 0\le \alpha,\beta\le k-1, \alpha+\beta\ge k-1\}\\
&\oplus \nabla\times \operatorname{span}\{\xi\lambda_3^{k}\nabla\lambda_1, \xi\lambda_2^{k-1}\lambda_4\nabla\lambda_1, \xi\lambda_1\nabla\lambda_4\},\\
\delta \boldsymbol{V}_{\mathrm{fillW}} :=& \boldsymbol{x}\,\widetilde{\mathcal{P}}_k(K),
\end{aligned}$$

satisfy the properties in Table 2. Here $\xi\in H^1(K)$ *can be either the piecewise-linear function defined by* $\xi(\boldsymbol{v}_{145})=1$, $\xi|_{T_1}=0$, $\xi|_{T_2}\in\mathcal{P}_1(T_2)$, *or the rational function* $\xi = x\,y/(1-z)$. *Here* $K=\{T_i\}_{i=1}^2$, *with* T_1 *being the tetrahedron with vertices* $\boldsymbol{v}_{123},\boldsymbol{v}_{125},\boldsymbol{v}_{134}$ *and* $\boldsymbol{v}_{234}$ *and* T_2 *being the tetrahedron with vertices* $\boldsymbol{v}_{125},\boldsymbol{v}_{134},\boldsymbol{v}_{145}$ *and* $\boldsymbol{v}_{234}$.

Note that, for any $k\ge 1$, most of the functions in the resulting vector space $\boldsymbol{V}=\boldsymbol{V}_g\oplus\delta\boldsymbol{V}_{\mathrm{fillM}}$ are polynomials except *three* rational or composite functions. This has to be contrasted with the spaces obtained in [12,13] for which almost all the basis functions are rational functions. Moreover, as discussed in the Introduction, they have significantly bigger spaces, for the same accuracy, than ours.

2.3.2. Pyramids with a quadrilateral base

Theorem 2.5. *Let* K *be a quadrilateral-based pyramid. Then, for* $M:=\mathcal{P}_k(\partial K)$ *and* $\boldsymbol{V}_g\times W_g := \boldsymbol{\mathcal{P}}_k(K)\times\mathcal{P}_k(K)$, *we have that*

$$I_M(\boldsymbol{V}_g\times W_g) = 1+2\min\{k,1\} \ \ \textit{and}\ \ I_S(\boldsymbol{V}_g\times W_g) = (k+1)(k+2)/2.$$

Moreover, the spaces

$$\begin{aligned}
\delta \boldsymbol{V}_{\mathrm{fillM}} :=& \begin{cases} \nabla\times\operatorname{span}\{\xi\nabla\lambda_1\} & \textit{if } k=0,\\ \nabla\times\operatorname{span}\{\xi\lambda_3^k\nabla\lambda_1, \xi\lambda_2^{k-1}\lambda_4\nabla\lambda_1, \xi\lambda_1\nabla\lambda_4\} & \textit{if } k\ge 1,\end{cases}\\
\delta \boldsymbol{V}_{\mathrm{fillW}} :=& \ \boldsymbol{x}\,\widetilde{\mathcal{P}}_k(K),
\end{aligned}$$

satisfy the properties in Table 2. Here $\xi\in H^1(K)$ *can be either the piecewise-linear function defined by* $\xi(\boldsymbol{v}_{145})=1$, $\xi|_{T_1}=0$, $\xi|_{T_2}\in\mathcal{P}_1(T_2)$, *or the rational function* $\xi=\lambda_2\lambda_3/\lambda_0$, *where* $\lambda_0\in\widetilde{\mathcal{P}}_1(\lambda_2,\lambda_4)\cap\widetilde{\mathcal{P}}_1(\lambda_3,\lambda_5)$.

Again, we notice that most of the basis functions in the resulting vector space $\boldsymbol{V}=\boldsymbol{V}_g\oplus\delta\boldsymbol{V}_{\mathrm{fillM}}$ are polynomials except for *one* rational or composite function for $k=0$ and *three* rational or composite functions for $k\ge 1$. Note also that, since piecewise polynomial functions are amenable to simpler numerical integration, perhaps the choice of ξ as a piecewise-linear function could be more advantageous.

2.4. Prisms

For prisms, we have three different cases according whether the bases F_3 and F_5 are congruent and parallel, non-congruent and parallel, and not parallel. In the case of congruent and parallel faces, the faces F_1, F_2 and F_4 being parallelograms, we consider the case in which the space of traces is $\mathcal{Q}_k(F_j)$ or $\mathcal{P}_k(F_j)$ for $j=1,2,4$.

2.4.1. Prisms with congruent, parallel faces

For simplicity, we take K to be the unit prism, that is, the prism whose basis (in the planes $z=0$ and $z=1$) are unit triangles, and whose faces on the xz- and yz-planes are unit squares. We set

$$\lambda_1 = x, \lambda_2 = y, \lambda_3 = z, \lambda_4 = 1-x-y, \lambda_5 = 1-z.$$

Our first result is for the case in which the space of traces on the rectangular faces is a tensor product, that is, the case in which $M(F_1)=\mathcal{Q}_k(F_1)$.

Theorem 2.6. *Let K be the unit prism. Then, for*

$$M(\partial K) := \{\mu \in L^2(\partial K) : \ \mu|_{F_i} \in \mathcal{Q}_k(F_1), \ \ \textit{for } i = 1,2,4, \ \mu|_{F_i} \in \mathcal{P}_k(F_i) \ \ \textit{for } i = 3,5\},$$

and $\boldsymbol{V}_g \times W_g = \boldsymbol{\mathcal{P}}_{k|k}(K) \times \mathcal{P}_{k|k}(K)$ with $k \geq 1$, we have that

$$I_M(\boldsymbol{V}_g \times W_g) = \ k+2 \ \ \textit{and} \ \ I_S(\boldsymbol{V}_g \times W_g) = \ k+1.$$

Moreover, the spaces

$$\begin{aligned}\delta \boldsymbol{V}_{\text{fillM}} &:= \nabla\times \ \text{span}\{z^{k+1}(x \nabla y - y \nabla x), \ y^k z\lambda_4 \nabla x, \ xz\lambda_4 \widetilde{\mathcal{P}}_{k-1}(x,z) \nabla y\},\\ \delta \boldsymbol{V}_{\text{fillW}} &:= \text{span}\{z^{k+1} \widetilde{\mathcal{P}}_k(x,y) \nabla z\},\end{aligned}$$

satisfy the properties in Table 2.

With this result, it is also easy to check that the following more *symmetric* choice of the filling space

$$\delta \boldsymbol{V}_{\text{fillM}} := \nabla\times \ \text{span}\{z^{k+1}(x \nabla y - y \nabla x), z\widetilde{\mathcal{P}}_k(x,y)(x \nabla y - y \nabla x)\}$$

ensures an M-decomposition.

There are two other family of spaces (defining mixed methods) on the unit prism admitting M-decompositions for the trace space M given in Theorem 2.6 available in the literature. The first was introduced in [4] for $k \geq 1$

$$\boldsymbol{V}^1 \times W^1 := \begin{pmatrix} \mathcal{P}_{k|k} \\ \mathcal{P}_{k|k} \\ \mathcal{P}_{k-1|k+1} \end{pmatrix} \times \mathcal{P}_{k-1|k},$$

and the second, which is a **RT**-like variation of the first, was recently presented in [9] for $k \geq 0$,

$$\boldsymbol{V}^2 \times W^2 := \begin{pmatrix} \begin{matrix}\mathcal{P}_{k|k} \\ \mathcal{P}_{k|k}\end{matrix} & \oplus & \begin{pmatrix} x \\ y \end{pmatrix} \widetilde{\mathcal{P}}_k(x,y) \otimes \mathcal{P}_k(z) \\ \mathcal{P}_{k|k} & \oplus & z \ \mathcal{P}_k(x,y) \otimes \widetilde{\mathcal{P}}_k(z) \end{pmatrix} \times \mathcal{P}_{k|k}.$$

Since $\boldsymbol{\mathcal{P}}_k \times \mathcal{P}_k \not\subset \boldsymbol{V}^1 \times W^1$, the approximation properties of $\boldsymbol{V}^1 \times W^1$ is expected to be worse than that for the resulting spaces in Theorem 2.6 and that of $\boldsymbol{V}^2 \times W^2$. On the other hand, the dimension of $\boldsymbol{V}^2$ is bigger than that of our space $\boldsymbol{V}^{\text{mix}}$ by $\frac{3}{2}(k^2+k)-1$ for $k \geq 1$. These two spaces are exactly the same when $k=0$, as we point out right after stating the next theorem.

Our second result on the unit prism is for the case in which $M(\partial K) := \mathcal{P}_k(\partial K)$.

Theorem 2.7. *Let K be the unit prism. Then, for $M(\partial K) = \mathcal{P}_k(\partial K)$ and $\boldsymbol{V}_g \times W_g = \boldsymbol{\mathcal{P}}_k(K) \times \mathcal{P}_k(K)$, we have that*

$$I_M(\boldsymbol{V}_g \times W_g) = \ k+1+\min\{k,1\} \ \ \textit{and} \ \ I_S(\boldsymbol{V}_g \times W_g) = \ (k+1)(k+2)/2.$$

Moreover, the spaces

$$\begin{aligned}\delta \boldsymbol{V}_{\text{fillM}} &:= \begin{cases} \nabla\times \ \text{span}\{zx \nabla y - zy \nabla x\} & \textit{if } k=0,\\ \nabla\times \ \text{span}\{z^{k+1}(x \nabla y - y \nabla x), y^k z\lambda_4 \nabla x, \\ \qquad\qquad xz\lambda_4 \widetilde{\mathcal{P}}_{k-1}(x,y) \nabla y\} & \textit{if } k \geq 1,\end{cases}\\ \delta \boldsymbol{V}_{\text{fillW}} &:= \text{span}\{z \widetilde{\mathcal{P}}_k \nabla z\},\end{aligned}$$

satisfy the properties in Table 2.

It is interesting to see that, for $k \geq 1$, we have exactly the same space $\delta \boldsymbol{V}_{\text{fillM}}$ for the two given spaces in Theorem 2.6 and 2.7. Again, we can change $\delta \boldsymbol{V}_{\text{fillM}}$ to be the following more *symmetric* one,

$$\delta \boldsymbol{V}_{\text{fillM}} := \nabla\times \;\text{span}\{z^{k+1}(x \nabla y - y \nabla x), z\widetilde{\mathcal{P}}_k(x,y)(x\nabla y - y \nabla x)\}.$$

We can also change this space to the following

$$\delta \boldsymbol{V}_{\text{fillM}} = \;\nabla\times \;\text{span}\{z^{k+1}x\nabla y - z^{k+1} y \nabla x,\; y^{k+1} z \nabla x, x^2 z\widetilde{\mathcal{P}}_{k-1}(x,y)\nabla y\}$$

for $k \geq 1$ so that the resulting space for the lower mixed method is exactly the same as the original prismatic **BDDF** elements introduced in [4] (which is $\mathbf{BDM}_{<k>}$ in our notation; see [8]).

Finally, we remark that after a simple calculation, we get

$$\delta \boldsymbol{V}_{\text{fillM}} \oplus \delta \boldsymbol{V}_{\text{fillW}} = \text{span}\,\{x\nabla x + y\nabla y, z\nabla z\}\;\text{ for } k = 0.$$

So the spaces for the upper mixed method for $k = 0$ is exactly the same as $\boldsymbol{V}^2 \times W^2$ presented in the previous subsection which is originally from [9].

2.4.2. Prisms with non-congruent, parallel bases

Now, let us consider a prismatic element K with non-congruent, parallel bases F_3 and F_5.

Theorem 2.8. *Let K be a prism with its face F_3 parallel to its face F_5. Then, for $M := \mathcal{P}_k(\partial K)$ and $\boldsymbol{V}_g \times W_g := \boldsymbol{\mathcal{P}}_k(K) \times \mathcal{P}_k(K)$ with $k \geq 0$, we have that*

$$I_M(\boldsymbol{V}_g \times W_g) = \; k+1+\min\{k,1\}, \quad \textit{and} \quad I_S(\boldsymbol{V}_g \times W_g) = \; (k+1)(k+2)/2,$$

and the spaces

$$\begin{aligned}
\delta \boldsymbol{V}_{\text{fillM}} &:= \begin{cases} \text{span}\{\boldsymbol{v}_{\mu_4^0}\} & \textit{if } k = 0,\\ \text{span}\{\boldsymbol{v}_{\mu_4^k}\} \oplus \nabla\times\text{span}\{\lambda_2^k\lambda_3\lambda_4\nabla\lambda_1, \lambda_1\lambda_3\lambda_4\widetilde{\mathcal{P}}_{k-1}(\lambda_1,\lambda_2)\nabla\lambda_2\} & \textit{if } k \geq 1.\end{cases}\\
\delta \boldsymbol{V}_{\text{fillW}} &:= \boldsymbol{x}\,\widetilde{\mathcal{P}}_k(K),
\end{aligned}$$

satisfy the properties in Table 2, where $\mu_4^k := \eta_4(\lambda_3^k)$ is the extension defined in (2.1a), and $\boldsymbol{v}_{\mu_4^k}$ is the lifting defined in Theorem 2.2.

Note that this result is very similar to that in Theorem 2.7. There, the function $\boldsymbol{v}_{\mu_4^k}$ is replaced by a divergence-free polynomial defined on the unit prism K; such polynomial does not exist when the triangular bases of K are not congruent to each other.

2.4.3. Prisms with non-parallel bases

Finally, keeping the notation of the previous subsection, we consider a prism with non-parallel bases.

Theorem 2.9. *Le K be a prism with its face F_3 is not parallel to its face F_5, and $\kappa_{235} \neq 0$. Then, for $M := \mathcal{P}_k(\partial K)$ and $\boldsymbol{V}_g \times W_g := \boldsymbol{\mathcal{P}}_k(K) \times \mathcal{P}_k(K)$ with $k \geq 0$, we have that*

$$I_M(\boldsymbol{V}_g \times W_g) = \; 1 + 2\,\min\{k,1\} \;\; \textit{and} \;\; I_S(\boldsymbol{V}_g \times W_g) = \; (k+1)(k+2)/2..$$

Moreover, the spaces

$$\begin{aligned}
\delta \boldsymbol{V}_{\text{fillM}} &:= \begin{cases} \text{span}\{\boldsymbol{v}_{\mu_4^0}\} & \textit{if } k = 0,\\ \text{span}\{\boldsymbol{v}_{\mu_4^k}\} \oplus \nabla\times\text{span}\{\lambda_2^k\xi\nabla\lambda_1, \lambda_1\lambda_2^{k-1}\xi\nabla\lambda_2\} & \textit{if } k \geq 1.\end{cases}\\
\delta \boldsymbol{V}_{\text{fillW}} &:= \boldsymbol{x}\,\widetilde{\mathcal{P}}_k(K),
\end{aligned}$$

satisfy the properties in Table 2, where $\mu_4^k := \eta_4(\lambda_3^k)$ *is the extension in (2.1a),* $\boldsymbol{v}_{\mu_4^k}$ *is the lifting defined in Theorem 2.2, and* $\xi \in H^1(K)$ *is the piecewise-linear function defined by* $\xi(\boldsymbol{v}_{125}) = 1$, $\xi|_{T_1 \cup T_2} = 0$, $\xi|_{T_3} \in \mathcal{P}_1(T_3)$. *Here* $K := \{T_i\}_{i=1}^3$, *with* T_1 *being the tetrahedron with vertices* $\boldsymbol{v}_{123}, \boldsymbol{v}_{134}, \boldsymbol{v}_{145}$ *and* $\boldsymbol{v}_{234}$, T_2 *being the tetrahedron with vertices* $\boldsymbol{v}_{123}, \boldsymbol{v}_{145}, \boldsymbol{v}_{234}$ *and* $\boldsymbol{v}_{245}$, *and* T_3 *being the tetrahedron with vertices* $\boldsymbol{v}_{123}, \boldsymbol{v}_{125}, \boldsymbol{v}_{145}$ *and* $\boldsymbol{v}_{245}$.

Note that if $\kappa_{235} = 0$, we must have $\kappa_{135} \neq 0$ since $F_3 \not\parallel F_5$. In this case, we switch faces F_1 and F_2 so that for the new face ordering, we do have $\kappa_{235} \neq 0$. Note also that for $k = 0$ and $k = 1$, we can again choose the above filling space $\delta \boldsymbol{V}_{\text{fillW}}$ for the previous considered prisms with parallel faces since for $k \leq 1$ the M-index and the space of traces C_M are the same in all three cases. So, for $k \leq 1$, the filling spaces can be made to be independent of the geometry of the prism.

2.5. Hexahedra

For hexahedral elements (with quadrilateral faces), the influence of the geometric shape of the element induces many cases. We have four cases, according to whether the hexahedron has 3, 2,1 or 0 pairs of parallel faces. We take the space of traces to be $\mathcal{P}_k(\partial K)$, except in the case in which the hexahedron is a parallelepiped, case in which we also consider the choice $\mathcal{Q}_k(\partial K)$. It is interesting to note that, for the case of 1 pair of parallel faces, we must distinguish between the cases in which the parallel faces are parallelograms or not. Moreover, when the parallel faces are parallelograms, different spaces are obtained according to whether the hexagon was obtained by cutting a pyramid or not. Finally, when we have 1 or 0 pairs of parallel faces, we also obtain different spaces according to whether the normals of three faces lie on a single plane or not.

2.5.1. Hexahedra with three pairs of parallel faces

For simplicity, we take K to be the unit cube. We start with the case $M(\partial K) := \mathcal{Q}_k(\partial K)$. As we see next, we find spaces closely related to the $\mathbf{HDG}^{Q}_{[k]}$ and $\mathbf{TNT}_{[k]}$ spaces obtained in [7,8].

Theorem 2.10. *Let* K *be the unit cube. Then, for* $M := \mathcal{Q}_k(\partial K)$ *and* $\boldsymbol{V}_g \times W_g := \boldsymbol{\mathcal{Q}}_k(K) \times \mathcal{Q}_k(K)$ *with* $k \geq 1$, *we have that*

$$I_M(\boldsymbol{V}_g \times W_g) = 6 \quad \textit{and} \quad I_S(\boldsymbol{V}_g \times W_g) = 1.$$

Moreover, the spaces

$$\begin{aligned}
\delta \boldsymbol{V}_{\text{fillM}} &:= \nabla \times \ \text{span} \left\{ \begin{array}{c} x^k y z^{k+1} \nabla x, \ x^{k+1} z \nabla y, \ x^{k+1} y^k z \nabla y, \\ (1-x)x(1-z)z^k \nabla y, \ (1-x)x(1-y)y^k \nabla z, \\ (1-x)x(1-y)y^k z^k \nabla z \end{array} \right\}, \\
\delta \boldsymbol{V}_{\text{fillW}} &:= \text{span}\{x^{k+1} y^k z^k \nabla x\},
\end{aligned}$$

satisfy the properties in Table 2.

Here let us compare the spaces $\boldsymbol{V}^{\text{mix}} \times W^{\text{mix}}$ defining the upper mixed method comparing to the ones obtained in [7,8]. The discussion below amplifies the non-uniqueness of the filling spaces for obtaining M-decompositions.

Our space $\boldsymbol{V}^{\text{mix}} \times W^{\text{mix}}$ defining the upper mixed method can be easily recasted into the following form by applying the curl operator on $\delta \boldsymbol{V}_{\text{fillM}}$. We get

$$\boldsymbol{V}^{\text{mix}} \times W^{\text{mix}} := \boldsymbol{\mathcal{Q}}_k \oplus \text{span} \left\{ \begin{array}{c} x^{k+1} \nabla x, \ x^{k+1} y^k \nabla x, \ z^{k+1} x^k \nabla z, \\ (1-2x) y^{k+1} \nabla y, \ (1-2x) z^{k+1} \nabla z, \\ (1-2x) y^{k+1} z^k \nabla y \\ x^{k+1} y^k z^k \nabla x \end{array} \right\} \times \mathcal{Q}_k.$$

In [7], two family of spaces containing $\boldsymbol{\mathcal{Q}}_k \times \mathcal{Q}_k$ and admitting M-decompositions with $M(\partial K) := \mathcal{Q}_k(\partial K)$ were introduced. One defines the HDG method $\mathbf{HDG}^{Q}_{[k]}$ while the other defines the mixed method $\mathbf{TNT}_{[k]}$ (TNT is the acronym for the TiNiest spaces containing Tensor product spaces). Their dimensions are the same as those of the spaces $\boldsymbol{V}^{\mathrm{hdg}} \times W^{\mathrm{hdg}}$ and $\boldsymbol{V}^{\mathrm{mix}} \times W^{\mathrm{mix}}$, respectively, resulting from Theorem 2.10. The space defining the $\mathbf{TNT}_{[k]}$ mixed method is,

$$\boldsymbol{V}^1 \times W^1 := \boldsymbol{\mathcal{Q}}_k \oplus \operatorname{span}\left\{\begin{array}{c} x^{k+1}\nabla x, y^{k+1}\nabla y, z^{k+1}\nabla z, \\ x^{k+1}y^k\nabla x, y^{k+1}z^k\nabla y, z^{k+1}z^k\nabla z, \\ x^{k+1}y^kz^k\nabla x \end{array}\right\} \times \mathcal{Q}_k.$$

Later in [7], another space

$$\boldsymbol{V}^2 \times W^2 := \boldsymbol{\mathcal{Q}}_k \oplus \operatorname{span}\left\{\begin{array}{c} x^{k+1}\nabla x, y^{k+1}\nabla y, z^{k+1}\nabla z, \\ y^kz^k(y\nabla y + z\nabla z), \\ z^kx^k(z\nabla z + x\nabla x), \\ x^ky^k(x\nabla x + y\nabla y), \\ x^ky^kz^k(x\nabla x + y\nabla y + z\nabla z) \end{array}\right\} \times \mathcal{Q}_k$$

was used to define the $\mathbf{TNT}_{[k]}$ mixed method. It is clear that the three spaces are very close to each other, but are not exactly the same. We note that the additional space for $\boldsymbol{V}^{\mathrm{mix}}$ depends on a particular order of the coordinates (x, y, z). It is less *symmetric* than the defnition of $\boldsymbol{V}^1$ which is invariant under the permutation $(x, y, z) \to (y, z, x)$ and $(x, y, z) \to (z, x, y)$. In turn, the definition of $\boldsymbol{V}^1$ is less *symmetric* than that of $\boldsymbol{V}^2$ which is invariant under any permutation of the coordinates (x, y, z). The reason for the loss of symmetry in the definition of the space $\boldsymbol{V}^{\mathrm{mix}}$ is due to our face-by-face construction; see Section 4 below. On the other hand, we can take the filling spaces as follows

$$\begin{aligned} \delta\boldsymbol{V}_{\mathrm{fillM}} &:= \nabla\times\ \operatorname{span}\left\{\begin{array}{c} x(y^{k+1}\nabla z - z^{k+1}\nabla y),\ y(z^{k+1}\nabla x - x^{k+1}\nabla z), \\ z(x^{k+1}\nabla y - y^{k+1}\nabla x),\ xy^kz^k(y\nabla z - z\nabla y), \\ yz^kx^k(z\nabla x - x\nabla z),\ zx^ky^k(x\nabla y - y\nabla x) \end{array}\right\} \\ \delta\boldsymbol{V}_{\mathrm{fillM}} &:= \operatorname{span}\{x^ky^kz^k(x\nabla x + y\nabla y + z\nabla z)\} \end{aligned}$$

so that the resulting spaces for the upper mixed method is identical to the (most symmetric) $\mathbf{TNT}_{[k]}$ spaces in [7].

Also, note that the space for the $\mathbf{RT}_{[k]}$ method in [10], namely,

$$\boldsymbol{V} \times W := \begin{pmatrix} \mathcal{P}_{k+1,k,k} \\ \mathcal{P}_{k,k+1,k} \\ \mathcal{P}_{k,k,k+1} \end{pmatrix} \times \mathcal{Q}_k,$$

admits an M-decomposition for $M := \mathcal{Q}_k(\partial K)$. However, the dimension of the vector-valued functions is bigger than our space $\boldsymbol{V}^{\mathrm{mix}}$ by $3k^2 + 6k - 4$ for $k \geq 1$.

Now, we consider the case $M(\partial K) := \mathcal{P}_k(\partial K)$. For this case, we obtain spaces strongly related to the spaces of the $\mathbf{BDM}_{[k]}$ (or $\mathbf{BDDF}_{[k]}$) method in [2], see also [1], and the spaces $\mathbf{HDG}_{[k]}$ obtained in [8]. We also compare our spaces to those of the $\mathbf{BDFM}_{[k]}$ method in [3].

Theorem 2.11. *Let K be the unit cube. Then, for $M := \mathcal{P}_k(\partial K)$ and $\boldsymbol{V}_g \times W_g := \boldsymbol{\mathcal{P}}_k(K) \times \mathcal{P}_k(K)$, we have that*

$$I_M(\boldsymbol{V}_g \times W_g) = \ 3k + 2 + \min\{k, 1\} \ \ \textit{and}\ \ I_S(\boldsymbol{V}_g \times W_g) = \ (k+1)(k+2)/2.$$

Moreover, the spaces

$$\delta \boldsymbol{V}_{\text{fillM}} := \begin{cases} \nabla\times\ \operatorname{span}\Big\{ yz\,\nabla\, x,\ xz\,\nabla\, y \Big\} & \textit{if } k=0,\\ \nabla\times\ \operatorname{span}\Big\{ yz\widetilde{\mathcal{P}}_k(x,z)\,\nabla\, x,\ xz\widetilde{\mathcal{P}}_k(x,y)\,\nabla\, y, & \textit{if } k\geq 1 \\ \qquad\qquad xy(1-y)\widetilde{\mathcal{P}}_{k-1}(y,z)\,\nabla\, z,\ xz^k(1-z)\,\nabla\, y \Big\}, & \end{cases}$$
$$\delta \boldsymbol{V}_{\text{fillW}} := \operatorname{span}\{x\,\widetilde{\mathcal{P}}_k\,\nabla\, x\}$$

satisfy the properties in Table 2.

Note that $\mathbf{RT}_{[0]}$ space in [10] is exactly the same as the space $\boldsymbol{V}^{\text{mix}}$ for $k=0$, since $\mathcal{P}_0 = \mathcal{Q}_0$ and

$$\operatorname{span}\big\{\nabla\times\big(yz\,\nabla\, x\big),\ \nabla\times\big(xz\,\nabla\, y\big),\ x\,\nabla\, x\big\} = \operatorname{span}\{x\,\nabla\, x,\ y\,\nabla\, y,\ z\,\nabla\, z\}\,.$$

Note also that the filling space $\delta\boldsymbol{V}_{\text{fillM}}$ can be changed in two ways to render its definition more *symmetric*. The first was presented in [8] and the second in [1]:

$$\delta\boldsymbol{V}^{[8]}_{\text{fillM}} = \nabla\times\ \operatorname{span}\left\{\begin{array}{c} xy\widetilde{\mathcal{P}}_k(y,z)\,\nabla\, z,\\ yz\widetilde{\mathcal{P}}_k(z,x)\,\nabla\, x,\\ zx\widetilde{\mathcal{P}}_k(x,y)\,\nabla\, y \end{array}\right\},$$
$$\delta\boldsymbol{V}^{[1]}_{\text{fillM}} = \nabla\times\ \operatorname{span}\left\{\begin{array}{c} x\widetilde{\mathcal{P}}_k(y,z)(y\,\nabla\, z - z\,\nabla\, y),\\ y\widetilde{\mathcal{P}}_k(z,x)(z\,\nabla\, x - x\,\nabla\, z),\\ z\widetilde{\mathcal{P}}_k(x,y)(x\,\nabla\, y - y\,\nabla\, x) \end{array}\right\}.$$

We can also modify the other filling space to be $\delta\boldsymbol{V}_{\text{fillW}} := \operatorname{span}\{\boldsymbol{x}\,\widetilde{\mathcal{P}}_k\}$. Again, these modifications do not change the dimension of spaces, but the resulting vector spaces are slightly different than those in Theorem 2.11.

Also, note that the space for the $\mathbf{BDFM}_{[k]}$ method in [3], namely,

$$\boldsymbol{V}\times W := \begin{pmatrix} \mathcal{P}_{k+1}\backslash\widetilde{\mathcal{P}}_{k+1}(y,z)\\ \mathcal{P}_{k+1}\backslash\widetilde{\mathcal{P}}_{k+1}(x,z)\\ \mathcal{P}_{k+1}\backslash\widetilde{\mathcal{P}}_{k+1}(x,y)\end{pmatrix}\times\mathcal{P}_k,$$

admits an M-decomposition for $M := \mathcal{P}_k(\partial K)$, and that the dimension of the space of vector-valued functions is bigger than that of $\boldsymbol{V}^{\text{mix}}$ by k^2-1.

2.5.2. *Hexahedra with two pairs of parallel faces*

In our next result, we assume that the hexahedron has two pairs of parallel faces. All the spaces obtained here are new.

Theorem 2.12. *Let K be a hexahedron for which $F_2 \nparallel F_4$, $F_3 \parallel F_5$, $F_1 \parallel F_6$, and $\kappa_{234}\neq 0$. Then, for $M := \mathcal{P}_k(\partial K)$ and $\boldsymbol{V}_g\times W_g := \boldsymbol{\mathcal{P}}_k(K)\times\mathcal{P}_k(K)$ with $k\geq 0$, we have that*

$$I_M(\boldsymbol{V}_g\times W_g) = 2k+5-\delta_{k,1}-3\,\delta_{k,0},$$
$$I_S(\boldsymbol{V}_g\times W_g) = (k+1)(k+2)/2.$$

Here $\delta_{i,j}$ is the Kronecker delta. Moreover, the spaces

$$\delta \boldsymbol{V}_{\text{fillM}} := \begin{cases} \operatorname{span}\{\boldsymbol{v}_{\mu^0_{4,1}}, \boldsymbol{v}_{\mu^0_{5,1}}\} & \textit{if } k=0, \\ \operatorname{span}\{\boldsymbol{v}_{\mu^1_{4,1}}, \boldsymbol{v}_{\mu^0_{5,1}}, \boldsymbol{v}_{\mu^1_{6,1}}\} & \\ \quad \oplus \nabla\times \operatorname{span}\{\lambda_2\lambda_3\lambda_4 \nabla \lambda_1,\ \lambda_1\lambda_3\lambda_4 \nabla \lambda_2, \lambda_1\lambda_2\lambda_4 \nabla \lambda_3\} & \textit{if } k=1, \\ \operatorname{span}\{\boldsymbol{v}_{\mu^k_{4,1}}, \boldsymbol{v}_{\mu^{k-1}_{5,1}}, \boldsymbol{v}_{\mu^{k-1}_{6,1}}\} & \\ \quad \oplus \nabla\times \operatorname{span}\{\lambda_2^k\lambda_3\lambda_4 \nabla \lambda_1, \lambda_1\lambda_3^{k-1}\lambda_4\lambda_5 \nabla \lambda_2, & \\ \qquad\qquad \lambda_1\lambda_3\lambda_4\widetilde{\mathcal{P}}_{k-1}(\lambda_1,\lambda_2) \nabla \lambda_2, & \\ \qquad\qquad \lambda_1\lambda_2\lambda_4\widetilde{\mathcal{P}}_{k-1}(\lambda_2,\lambda_3) \nabla \lambda_3\} & \textit{if } k\geq 2. \end{cases}$$

$$\delta \boldsymbol{V}_{\text{fillW}} := \boldsymbol{x}\, \widetilde{\mathcal{P}}_k(K),$$

satisfy the properties in Table 2, where

$$\mu^m_{4,1} := \eta_4(\lambda^m_3),\ \mu^m_{5,1} := \eta_5(\lambda^m_2) \textit{ and } \mu^m_{6,1} := \eta_6(\lambda^m_3, \lambda^{m-1}_3)$$

are the eextensions defined by (2.1a) and (2.1b), and $\boldsymbol{v}_\mu$ is the lifting defined in Theorem 2.2.

Let us show that it is possible to carry out a reordering of the faces in such a way that we get that $F_2 \nparallel F_4$, $F_3 \parallel F_5$, $F_1 \parallel F_6$, and $\kappa_{234} \neq 0$. We proceed as follows. First, we first reorder the faces in such a way that $F_2 \nparallel F_4$, $F_3 \parallel F_5$, $F_1 \parallel F_6$. Now, if $\kappa_{234} = 0$ we must have that $\kappa_{214} \neq 0$ since $F_2 \nparallel F_4$. Then, we switch the faces F_1 and F_3, and the faces F_5 and F_6 to get the desired ordering.

2.5.3. Hexahedra with one pair of parallel faces

In this case, we have to treat differently the case in which the parallel faces are paralleograms or not. All the spaces are new.

The parallel basis are parallelograms

Here, we consider an hexahedron K for which $F_2 \nparallel F_4$, $F_3 \nparallel F_5$, $F_1 \parallel F_6$, where F_1 and F_6 are parallelograms. It is interesting to see that, according to whether $\lambda_5(\boldsymbol{v}_{234})$ is equal to zero or not, we get spaces that differ by one basis function. The condition $\lambda_5(\boldsymbol{v}_{234}) = 0$ means that the hexahedron in question is obtained buy cutting a pyramid whose vertex is $\boldsymbol{v}_{234}$.

Theorem 2.13. *Let K be a hexahedron such that $F_2 \nparallel F_4$, $F_3 \nparallel F_5$, $F_1 \parallel F_6$ where F_1 and F_6 are parallelograms. Then, for $M := \mathcal{P}_k(\partial K)$ and $\boldsymbol{V}_g \times W_g := \boldsymbol{\mathcal{P}}_k(K) \times \mathcal{P}_k(K)$ with $k \geq 0$, we have that*

$$\begin{aligned} I_M(\boldsymbol{V}_g \times W_g) &= \ k + 7 - 2\,\delta_{k,1} - 5\,\delta_{k,0}, \\ I_S(\boldsymbol{V}_g \times W_g) &= \ (k+1)(k+2)/2. \end{aligned}$$

Moreover, the space

$$\delta \boldsymbol{V}_{\text{fillM}} := \begin{cases} \operatorname{span}\{\boldsymbol{v}_{\mu^0_{4,1}}, \boldsymbol{v}_{\mu^0_{5,1}}\} & \textit{if } k=0, \\ \operatorname{span}\{\boldsymbol{v}_{\mu^1_{4,1}}, \boldsymbol{v}_{\mu^1_{5,1}}, \boldsymbol{v}_{\mu^1_{5,2}}, \boldsymbol{v}_{\mu^0_{5,1}}, \boldsymbol{v}_{\mu^1_{6,1}}, \boldsymbol{v}_{\mu^1_{6,2}}\} & \textit{if } k=1, \\ \operatorname{span}\{\boldsymbol{v}_{\mu^k_{4,1}}, \boldsymbol{v}_{\mu^k_{5,1}}, \boldsymbol{v}_{\mu^k_{5,2}}, \boldsymbol{v}_{\mu^{k-1}_{6,1}}, \boldsymbol{v}_{\mu^{k-1}_{6,2}}\} & \\ \quad \oplus \nabla\times \operatorname{span}\{\lambda_1\lambda_2^{k-1}\lambda_4\lambda_5 \nabla \lambda_3, \lambda_1\lambda_2^{k-1}\lambda_3\lambda_5 \nabla \lambda_4, & \\ \qquad\qquad \lambda_1\lambda_3\lambda_4\lambda_5\widetilde{\mathcal{P}}_{k-2}(\lambda_2,\lambda_3) \nabla \lambda_2\} & \\ \quad \oplus \begin{cases} \operatorname{span}\{\boldsymbol{v}_{\mu^{k-1}_{5,1}}\} & \textit{if } k\geq 2 \textit{ and } \lambda_5(\boldsymbol{v}_{234}) \neq 0, \\ \operatorname{span}\{\boldsymbol{v}_{\mu^0_{5,1}}\} & \textit{if } k\geq 2 \textit{ and } \lambda_5(\boldsymbol{v}_{234}) = 0, \end{cases} & \end{cases}$$

$$\delta \boldsymbol{V}_{\text{fillW}} := \boldsymbol{x}\, \widetilde{\mathcal{P}}_k(K),$$

satisfy the properties in Table 2, where

$$\begin{aligned}&\mu_{4,1}^m := \eta_4(\lambda_3^m),\\&\mu_{5,1}^m := \eta_5(\lambda_2^m),\ \mu_{5,2}^m := \eta_5(\lambda_2^{m-1}\lambda_1),\\&\mu_{6,1}^m := \eta_6(\lambda_3^m, \lambda_3^{m-1}),\ \mu_{6,2}^m := \eta_6(\lambda_2^m, \lambda_2^{m-1}).\end{aligned}$$

are the eextensions defined by (2.1a) and (2.1b), and $\boldsymbol{v}_\mu$ is the lifting defined in Theorem 2.2.

The parallel faces are not parallelograms

Next, we consider the hexahedron with one pair of parallel faces, and the parallel faces are not parallelograms.

Theorem 2.14. *Let K be a hexahedron such that $F_2 \nparallel F_4$, $F_3 \nparallel F_5$, $F_1 \parallel F_6$, where F_1 and F_6 are not parallelograms, and $\kappa_{135} \neq 0$. Then, for $M := \mathcal{P}_k(\partial K)$ and $\boldsymbol{V}_g \times W_g := \boldsymbol{\mathcal{P}}_k(K) \times \mathcal{P}_k(K)$ with $k \geq 0$, we have that*

$$\begin{aligned}I_M(\boldsymbol{V}_g \times W_g) &= \ k + 7 - 2\,\delta_{k,1} - 5\,\delta_{k,0},\\ I_S(\boldsymbol{V}_g \times W_g) &= \ (k+1)(k+2)/2.\end{aligned}$$

Moreover, the spaces

$$\begin{aligned}\delta\boldsymbol{V}_{\rm fillM} &:= \begin{cases}\operatorname{span}\{\boldsymbol{v}_{\mu_{4,1}^0}, \boldsymbol{v}_{\mu_{5,1}^0}\} & \text{if } k=0,\\ \operatorname{span}\{\boldsymbol{v}_{\mu_{5,3}^1}, \boldsymbol{v}_{\mu_{5,4}^1}, \boldsymbol{v}_{\mu_{5,3}^0}, \boldsymbol{v}_{\mu_{6,1}^1}, \boldsymbol{v}_{\mu_{6,2}^1}\} \\ \quad \oplus \begin{cases}\operatorname{span}\{\boldsymbol{v}_{\mu_{4,1}^1}\} & \text{if } k=1 \text{ and } \kappa_{234} \neq 0,\\ \operatorname{span}\{\boldsymbol{v}_{\mu_{4,2}^1}\} & \text{if } k=1 \text{ and } \kappa_{234} = 0,\end{cases}\\ \operatorname{span}\{\boldsymbol{v}_{\mu_{5,3}^k}, \boldsymbol{v}_{\mu_{5,4}^k}, \boldsymbol{v}_{\mu_{5,3}^{k-1}}, \boldsymbol{v}_{\mu_{6,1}^{k-1}}, \boldsymbol{v}_{\mu_{6,3}^{k-1}}\} \\ \quad \oplus \nabla\times \operatorname{span}\{\lambda_1\lambda_2^{k-1}\lambda_4\lambda_5\,\nabla\lambda_3, \lambda_1\lambda_2^{k-1}\lambda_3\lambda_5\,\nabla\lambda_4,\\ \qquad\qquad \lambda_1\lambda_3\lambda_4\lambda_5\widetilde{\mathcal{P}}_{k-2}(\lambda_2,\lambda_3)\,\nabla\lambda_2\}\\ \quad \oplus \begin{cases}\operatorname{span}\{\boldsymbol{v}_{\mu_{4,1}^k}\} & \text{if } k\geq 2 \quad \text{and } \kappa_{234} \neq 0,\\ \operatorname{span}\{\boldsymbol{v}_{\mu_{4,2}^k}\} & \text{if } k \geq 2 quad \text{ and } \kappa_{234} = 0,\end{cases}\end{cases}\\ \delta\boldsymbol{V}_{\rm fillW} &:= \boldsymbol{x}\,\widetilde{\mathcal{P}}_k(K),\end{aligned}$$

satisfy the properties in Table 2, where,

$$\begin{aligned}&\mu_{4,1}^m := \eta_4(\lambda_3^m),\ \mu_{4,2}^m := \eta_4(\lambda_1^m),\\&\mu_{5,1}^m := \eta_5(\lambda_2^m),\ \mu_{5,3}^m := \eta_5(\lambda_1^m),\ \mu_{5,4}^m := \eta_5(\lambda_1^{m-1}\lambda_2),\\&\mu_{6,1}^m := \eta_6(\lambda_3^m, \lambda_3^{m-1}),\ \mu_{6,2}^m := \eta_6(\lambda_2^m, \lambda_2^{m-1}),\ \mu_{6,3}^m := \eta_6(\lambda_3^{m-1}\lambda_2, \lambda_3^{m-1})\end{aligned}$$

are the eextensions defined by (2.1a) and (2.1b), and $\boldsymbol{v}_\mu$ is the lifting defined in Theorem 2.2.

Note that we can indeed reorder the faces in such a way that $F_2 \nparallel F_4$, $F_3 \nparallel F_5$, $F_1 \parallel F_6$, and $\kappa_{135} \neq 0$. To see this, we proceed was follows. We first reorder the faces such that $F_2 \nparallel F_4$, $F_3 \nparallel F_5$, $F_1 \parallel F_6$. Now, if $\kappa_{135} = 0$, we must have that $\kappa_{124} \neq 0$ since F_1 is not a parallelogram. Then, we switch the faces F_2 and F_3, and the faces F_4 and F_5 to get the desired ordering.

2.5.4. Hexahedra with no parallel faces

Our last case is when the hexahedra has no pair of parallel faces. Let us note that the spaces $\delta\boldsymbol{V}_{\rm fillM}$ differ by a single basis function provided κ_{234} is or is not equal to zero, that is, according to whether the edges $e_{23} := F_2 \cap F_3$ and $e_{34} := F_3 \cap F_4$ are parallel to each other or not. All the spaces are new.

Theorem 2.15. *Let K be a hexahedron such that $F_2 \not\parallel F_4$, $F_3 \not\parallel F_5$, $F_1 \not\parallel F_6$, and $\kappa_{135} \neq 0, \kappa_{136} \neq 0, \kappa_{356} \neq 0$. Then, for $M := \mathcal{P}_k(\partial K)$ and $\boldsymbol{V}_g \times W_g := \boldsymbol{\mathcal{P}}_k(K) \times \mathcal{P}_k(K)$ with $k \geq 0$, we have that*

$$\begin{aligned} I_M(\boldsymbol{V}_g \times W_g) = &\ 9 - 3\,\delta_{k,1} - 7\,\delta_{k,0}, \\ I_S(\boldsymbol{V}_g \times W_g) = &\ (k+1)(k+2)/2. \end{aligned}$$

Moreover, the space

$$\delta \boldsymbol{V}_{\text{fillM}} := \begin{cases} \operatorname{span}\{\boldsymbol{v}_{\mu^0_{4,1}}, \boldsymbol{v}_{\mu^0_{5,1}}\} & \text{if } k = 0, \\ \operatorname{span}\{\boldsymbol{v}_{\mu^1_{5,3}}, \boldsymbol{v}_{\mu^1_{5,4}}, \boldsymbol{v}_{\mu^0_{5,3}}, \boldsymbol{v}_{\mu^1_{6,1}}, \boldsymbol{v}_{\mu^1_{6,2}}\} & \\ \quad \oplus \begin{cases} \operatorname{span}\{\boldsymbol{v}_{\mu^1_{4,1}}\} & \text{if } k = 1 \text{ and } \kappa_{234} \neq 0, \\ \operatorname{span}\{\boldsymbol{v}_{\mu^1_{4,2}}\} & \text{if } k = 1 \text{ and } \kappa_{234} = 0, \end{cases} & \\ \operatorname{span}\{\boldsymbol{v}_{\mu^k_{5,3}}, \boldsymbol{v}_{\mu^k_{5,4}}, \boldsymbol{v}_{\mu^{k-1}_{5,3}}, \boldsymbol{v}_{\mu^{k-1}_{6,1}}, \boldsymbol{v}_{\mu^{k-1}_{6,3}}, \boldsymbol{v}_{\mu^k_{6,1}}, \boldsymbol{v}_{\mu^k_{6,3}}, \boldsymbol{v}_{\mu^k_{6,4}}\} & \\ \quad \oplus \begin{cases} \operatorname{span}\{\boldsymbol{v}_{\mu^k_{4,1}}\} & \text{if } k \geq 2 \text{ and } \kappa_{234} \neq 0, \\ \operatorname{span}\{\boldsymbol{v}_{\mu^k_{4,2}}\} & \text{if } k \geq 2 \text{ and } \kappa_{234} = 0, \end{cases} & \end{cases}$$

$$\delta \boldsymbol{V}_{\text{fillW}} := \boldsymbol{x}\, \widetilde{\mathcal{P}}_k(K),$$

satisfy the properties in Table 2, where

$$\begin{aligned} &\mu^m_{4,1} := \eta_4(\lambda_3^m),\ \mu^m_{4,2} := \eta_4(\lambda_1^m), \\ &\mu^m_{5,1} := \eta_5(\lambda_2^m),\ \mu^m_{5,2} := \eta_5(\lambda_2^{m-1}\lambda_1),\ \mu^m_{5,3} := \eta_5(\lambda_1^m),\ \mu^m_{5,4} := \eta_5(\lambda_1^{m-1}\lambda_2), \\ &\mu^m_{6,1} := \eta_6(\lambda_3^m, \lambda_3^{m-1}),\ \mu^m_{6,2} := \eta_6(\lambda_2^m, \lambda_2^{m-1}), \\ &\mu^m_{6,3} := \eta_6(\lambda_3^{m-1}\lambda_2, \lambda_3^{m-1}) \text{ and } \mu^m_{6,4} := \eta_6(\lambda_3^m, \lambda_3^{m-2}) \end{aligned}$$

are the eextensions defined by (2.1a) and (2.1b), and $\boldsymbol{v}_\mu$ is the lifting defined in Theorem 2.2.

Our result is for a reordering of the faces in such a way that $\kappa_{135} \neq 0$, $\kappa_{136} \neq 0$, $\kappa_{356} \neq 0$. Let us argue that we can always obtain such a reordering. If we have $\kappa_{135} = 0$ for a given face ordering, then $\kappa_{235} \neq 0$ and $\kappa_{345} \neq 0$ because $F_3 \not\parallel F_5$. Now, since $F_2 \not\parallel F_4$, the constants κ_{234} and κ_{245} can not be both zero. Without loss of generality, we assume $\kappa_{234} \neq 0$. Now, we switch the faces F_1 and F_2, and the faces F_6 and F_4 to get the desired ordering.

As for the prisms, for $k = 0$ and $k = 1$, we can choose the above filling space $\delta \boldsymbol{V}_{\text{fillW}}$ for the previously considered hexahedra with any number of parallel faces. This is true because, for $k \leq 1$ the M-index and the space of traces C_M are the same in all cases. So, for $k \leq 1$, the filling spaces can be made to be independent of the geometry of the hexahedron.

This concludes the presentation of our spaces.

3. Properties of the composite liftings

3.1. Proof of the existence of composite liftings, Theorem 2.2

Let us first prove Theorem 2.2 in the case in which K is a tetrahedron. By Theorem 2.3, we have that $\boldsymbol{\mathcal{P}}_k \times \mathcal{P}_k$ admits an $\mathcal{P}_k(\partial K)$-decomposition for a tetrahedral element K. Then, by the kernels' trace decomposition in ([6], Thm. 2.4), we get

$$\{\boldsymbol{v} \cdot \boldsymbol{n}|_{\partial K} :\ \boldsymbol{v} \in \boldsymbol{\mathcal{P}}_k,\ \nabla \cdot \boldsymbol{v} = 0\} = \left\{\mu \in \mathcal{P}_k(\partial K) :\ \int_{\partial K} \mu \, \mathrm{d}s = 0\right\},$$

and the result follows.

Let us now prove the general case. Suppose that the polyhedron K is triangulated by the tetrahedra $\{T_i\}_{i=1}^{nt}$ where we assume that we have numbered the tetrahedra in such a way that T_ℓ has three of its faces on the surface of the polyhedron $\cup_{i=\ell}^{nt} T_i$, for $\ell = 1, \ldots, nt-1$. Then, for $j = 1, \ldots, nt-1$, let F_j be the only face of T_j interior to $K_j = \cup_{i=j}^{nt} T_i$, set the value of therein to be $\mu_j := -\int_{\partial T_j \setminus \mathsf{F}_j} \mu \,\mathrm{d}s / \int_{\mathsf{F}_j} 1 \,\mathrm{d}s$, and set the value of $\mu \in \mathcal{P}_k(K_{j+1})$ on the face F_j equal to $-\mu_j$. By construction, and from the fact that $\int_{\partial K} \mu \,\mathrm{d}s = 0$, we have that $\int_{\partial K_j} \mu \, ds = 0$, for $j = 1, \ldots, nt$. We can now apply the above result to each tetrahedron T_j, $j = 1, \ldots, nt$. This completes the proof of Theorem 2.2.

3.2. Computing the lifting for a tetrahedron

Although Theorem 2.2 ensures the existence of a composite lifting for zero-average piecewise polynomial trace functions, a computable formulation of those lifting functions is highly desirable. It is now clear from the previous proof that in order to compute a composite lifting on a general polyhedron, we only need to do so for a tetrahedron. In this subsection, we present one way to do such computation.

To this end, we assume K is a tetrahedron, $\{\lambda_i\}_{i=1}^4$ its barycentric coordinates, F_i its face lying on $\lambda_i = 0$, for $1 \le i \le 4$. To simplify the notation, given a finite element space $S(D)$ in a domain D, we set $\overset{\circ}{S}(D) := \{\phi \in S(D) : \int_D \phi = 0\}$ to be its *mean zero* subspace.

Our goal is to lift any trace function in $\overset{\circ}{\mathcal{P}}_k(\partial K) = \{\mu \in \mathcal{P}_k(\partial K) : \int_{\partial K} \mu = 0\}$ into the element K by a divergence-free function in $\boldsymbol{\mathcal{P}}_k$. We achieve this goal by finding a set of basis for $\overset{\circ}{\mathcal{P}}_k(\partial K)$ consisting of the normal trace of divergence-free functions in $\boldsymbol{\mathcal{P}}_k$. To do so, we use the following lemma. Its proof is straightforward and hence omitted.

Lemma 3.1. *Let K be a tetrahedron with $\{\lambda_i\}_{i=1}^4$ its barycentric coordinates. Then, we have*

$$\begin{aligned}
\gamma(\nabla \lambda_3 \times \nabla \lambda_4) &= c_1\, \eta_1(1),\\
\gamma(\nabla \lambda_4 \times \nabla \lambda_1) &= c_2\, \eta_2(1),\\
\gamma(\nabla \lambda_1 \times \nabla \lambda_2) &= c_3\, \eta_3(1),
\end{aligned}$$

where

$$c_1 = |\nabla \lambda_3|\,|\nabla \lambda_4|\kappa_{123}, \quad c_2 = -|\nabla \lambda_4|\,|\nabla \lambda_1|\kappa_{123}, \quad c_3 = |\nabla \lambda_1|\,|\nabla \lambda_2|\kappa_{123}.$$

Moreover,

$$\left\{\gamma\left(\nabla\times\left(\lambda_{i+1}^{\alpha+1}\lambda_{i+2}^{\beta+1}\nabla\lambda_{i+3}\right)\right)\right\}_{\alpha,\beta\ge 0}^{\alpha+\beta\le k-1}, \ \textit{and}\ \left\{\gamma\left(\nabla\times\left(\lambda_{i+1}^{\alpha+1}\lambda_{i+3}\nabla\lambda_{i+2}\right)\right)\right\}_{\alpha=0}^{k-1}$$

form a basis for $\operatorname{span}\left\{\eta_i(\zeta_i) :\ \zeta_i \in \overset{\circ}{\mathcal{P}}_k(F_i)\right\}$.

Now, for $1 \le i \le 4$, if we define the space

$$\begin{aligned}
\mathcal{V}_k^i := {} & \nabla\times\left(\lambda_{i+1}\lambda_{i+2}\mathcal{P}_{k-1}(\lambda_{i+1},\lambda_{i+2})\nabla\lambda_{i+3}\right)\\
& \oplus \nabla\times\left(\lambda_{i+1}\lambda_{i+3}\widetilde{\mathcal{P}}_{k-1}(\lambda_{i+1})\nabla\lambda_{i+2}\right),
\end{aligned}$$

we have that the trace operator $\begin{array}{rcl} \gamma:\ \mathcal{V}_k^i & \longrightarrow & \operatorname{span}\{\eta_i(\zeta_i) :\ \zeta_i \in \overset{\circ}{\mathcal{P}}_k(F_i)\} \\ \boldsymbol{v} & \longmapsto & \boldsymbol{v}\cdot\boldsymbol{n}|_{\partial K} \end{array}$ is an isomorphism by Lemma 3.1. Here the subindexes are integers modulo 4.

Then, we define

$$\begin{aligned}\mathcal{V}_k :=&\ \oplus_{i=1}^4 \mathcal{V}_k^i \oplus_{i=1}^3 \nabla\times(\lambda_{i+2}\nabla\lambda_{i+3})\\ =&\ \oplus_{i=1}^4 \nabla\times(\lambda_{i+1}\lambda_{i+2}\mathcal{P}_{k-1}(\lambda_{i+1},\lambda_{i+2})\nabla\lambda_{i+3})\\ &\ \oplus_{i=1}^4 \nabla\times\left(\lambda_{i+1}\lambda_{i+3}\widetilde{\mathcal{P}}_{k-1}(\lambda_{i+1})\nabla\lambda_{i+2}\right)\\ &\ \oplus_{i=1}^3 \nabla\times(\lambda_{i+2}\nabla\lambda_{i+3}).\end{aligned}$$

Using the fact that

$$\overset{\circ}{\mathcal{P}}_k(\partial K) = \operatorname{span}\{\eta_1(1),\eta_2(1),\eta_3(1)\} \oplus_{i=1}^4 \operatorname{span}\left\{\eta_i(\zeta_i) : \forall\zeta_i \in \overset{\circ}{\mathcal{P}}_k(F_i)\right\},$$

we have that any trace function in $\overset{\circ}{\mathcal{P}}_k(\partial K)$ can be uniquely expressed as the normal trace of a function in $\mathcal{V}_k$, whose actual computation requires solving *four* linear system of dimension $\dim \overset{\circ}{\mathcal{P}}_k(F) = (k^2+3k)/2$. When coding, such computation can be done on a reference tetrahedron first and then Poila-transformed back to the physical tetrahedron.

Last but not least, let us point out that the actual computation of each *composite lifting* on the prism and hexahedron in Theorems 2.8 and 2.9, and Theorems 2.12 and 2.15, respectively, only requires solving *one* linear system of dimension $(k^2+3k)/2$ as its trace is expected to be non-constant on at most one face of each tetrahedron resulting from the tetrahedral subdivision of the prism or hexahedron under consideration. On the other hand, the *composite liftings* for a pyramid in Theorems 2.5 and 2.4 have closed-form representations and no linear system needs to be solved.

4. Proof of main results in Section 2

In this section, we first prove the negative result of Theorem 2.1, then prove the main results on the justification of the construction of the spaces $\delta\boldsymbol{V}_{\text{fillM}}$ in Theorem 2.3 to Theorem 2.15. The corresponding justification of the construction of the space $\delta\boldsymbol{V}_{\text{fillW}}$ is quite easy, hence omitted.

4.1. Proof of Theorem 2.1

Here we prove the negative result in Theorem 2.1.

The proof is similar to that for the two-dimensional case in [5]. If an element K is not a tetrahedron, a prism with parallel congruent triangular bases, or a parallelepiped, we can reorder the faces such that the first four faces extends to form a tetrahedron, denoted as T. We proceed by contradiction. If there is a polynomial space $\boldsymbol{V}\times W$ admits a $\mathcal{P}_k(\partial K)$-decomposition, we can find a divergence-free polynomial function $\boldsymbol{v}_\mu \in \boldsymbol{V}$ such that $\boldsymbol{v}\cdot\boldsymbol{n}|_{\partial K} = \mu$ for a trace function $\mu \in \mathcal{P}_k(\partial K)$ with average *zero* on ∂K, $\mu|_{F_i} = 0$ for $1 \le i \le 3$ and $\mu|_{F_4} = 1$. Now, since $\boldsymbol{v}_\mu$ is a divergence-free polynomial function, it is defined on the whole space $\mathbb{R}^3$. By restricting the function $\boldsymbol{v}_\mu$ on the tetrahedron T, we have $\int_{\partial T} \boldsymbol{v}_\mu\cdot\boldsymbol{n}\,\mathrm{d}s \neq 0 = \int_T \nabla\cdot\boldsymbol{v}_\mu\,\mathrm{d}x$, which is a contradiction. This completes the proof of Theorem 2.1. □

4.2. An algorithm to construct the space $\delta\boldsymbol{V}_{\text{fillM}}$

Next, we justify the construction of the spaces $\delta\boldsymbol{V}_{\text{fillM}}$ in Section 2. We first recall the algorithm introduced in Part II, [5]. We then apply the algorithm to treat of the general case in which $M(\partial K) := \mathcal{P}_k(\partial K)$ and $\boldsymbol{V}_g \times W_g := \boldsymbol{\mathcal{P}}_k(K)\times\mathcal{P}_k(K)$, and then sketch the proof for the other (similar and simpler) special cases.

As pointed out in the Introduction, we proceed as follows. For a given polyhedral element K, a space of traces M and the given space $\boldsymbol{V}_g\times W_g$, we begin by finding a space of traces $C_M \subset M$ such that

$$C_M \oplus \{\boldsymbol{v}\cdot\boldsymbol{n}|_{\partial K} :\ \boldsymbol{v}\in\boldsymbol{V}_g, \nabla\cdot\boldsymbol{v} = 0\} = \overset{\circ}{M}(\partial K), \tag{4.1}$$

where $\overset{\circ}{M}(\partial K) := \{\mu \in M(\partial K) : \int_{\partial K} \mu \, \mathrm{d}s = 0\}$. Then, if $\mathcal{B}$ is a basis of C_M, the space $\delta \boldsymbol{V}_{\mathrm{fillM}}(K) := \mathrm{span}\{\boldsymbol{v}_\mu : \mu \in \mathcal{B}\}$, immediately satisfies the properties in Table 2. To find C_M, we apply the same algorithm introduced in Part II, [5]; it allows us to restrict our attention to a single face at a time. To introduce it, we need some notation.

4.2.1. Notation

For $i = 1, \dots, n_f + 1$, we define $\boldsymbol{V}_{g_{s,i}}$ to be the divergence-free subspace of $\boldsymbol{V}_g$ with vanishing normal traces on the first $i-1$ faces, that is,

$$\boldsymbol{V}_{g_{s,i}} := \{\boldsymbol{v} \in \boldsymbol{V}_g : \nabla \cdot \boldsymbol{v} = 0, \ \boldsymbol{v} \cdot \boldsymbol{n}|_{F_j} = 0, \ 1 \le j \le i-1\}.$$

We also use the gradient-free subspace of W_g, $W_{g_{cst}} := \{w \in W_g : \nabla w = \boldsymbol{0}\}$. By the inclusion property $\mathcal{P}_0(K) \subset W_g$, we have that $W_{g_{cst}} = \mathcal{P}_0(K)$ is just the space of constants on K.

For $i = 1, \dots, n_f$, we define $\gamma_i(\boldsymbol{V}) := \{\boldsymbol{v} \cdot \boldsymbol{n}|_{F_i} : \boldsymbol{v} \in \boldsymbol{V}\}$ to be the normal trace of $\boldsymbol{V}$ on F_i, and $\gamma_i(W) := \{w|_{F_i} : w \in W\}$ to be the trace of W on F_i. Note that $\gamma_i(W_{g_{cst}}) = \mathcal{P}_0(F_i)$ is just the space of constants on F_i.

We define the M-index for each face as follows.

Definition 4.1 (The M-index for each face)**.** The M-index of the space $\boldsymbol{V}_g \times W_g$ for the ith face F_i is the number

$$I_{M,i}(\boldsymbol{V}_g \times W_g) := \dim M(F_i) - \dim \gamma_i(\boldsymbol{V}_{g_{s,i}}) - \delta_{i,n_f} \dim \gamma_{n_f}(W_{g_{cst}}),$$

where δ_{i,n_f} is the Kronecker delta.

4.2.2. A possible construction of $\delta \boldsymbol{V}_{\mathrm{fillM}}$

Now, we have the following result on a construction of $\delta \boldsymbol{V}_{\mathrm{fillM}}$.

Theorem 4.2. *Set $\delta \boldsymbol{V}_{\mathrm{fillM}} := \oplus_{i=1}^{n_f} \delta \boldsymbol{V}^i_{\mathrm{fillM}}$ where*

(α) $\gamma(\delta \boldsymbol{V}^i_{\mathrm{fillM}}) \subset M$,
(β) $\nabla \cdot \delta \boldsymbol{V}^i_{\mathrm{fillM}} = \{0\}$,
(γ.1) $\gamma_j(\delta \boldsymbol{V}^i_{\mathrm{fillM}}) = \{0\}$, *for* $1 \le j \le i-1$,
(γ.2) $\gamma_i(\boldsymbol{V}_{g_{s,i}}) \cap \gamma_i(\delta \boldsymbol{V}^i_{\mathrm{fillM}}) = \{0\}$,
(δ) $\dim \delta \boldsymbol{V}^i_{\mathrm{fillM}} = \dim \gamma_i(\delta \boldsymbol{V}^i_{\mathrm{fillM}}) = I_{M,i}(\boldsymbol{V}_g \times W_g)$.

Then $\delta \boldsymbol{V}_{\mathrm{fillM}}$ satisfies the properties in Table 2, that is,

(a) $\gamma \delta \boldsymbol{V}_{\mathrm{fillM}} \subset M$,
(b) $\nabla \cdot \delta \boldsymbol{V}_{\mathrm{fillM}} = \{0\}$,
(c) $\gamma \boldsymbol{V}_{g_{s,1}} \cap \gamma \delta \boldsymbol{V}_{\mathrm{fillM}} = \{0\}$,
(d) $\dim \delta \boldsymbol{V}_{\mathrm{fillM}} = \dim \gamma \delta \boldsymbol{V}_{\mathrm{fillM}} = I_M(\boldsymbol{V}_g \times W_g)$.

This result implies that $(\boldsymbol{V}_g \oplus \delta \boldsymbol{V}_{\mathrm{fillM}}) \times W_g$ admits an M-decompositon (see [6], Prop. 5.1).

4.2.3. The divide-and-conquer algorithm

Based on this result, we have that the following algorithm provides a practical construction of the filling space $\delta \boldsymbol{V}_{\mathrm{fillM}}$.

A practical construction of $\delta \boldsymbol{V}_{\text{fillM}}$

Input: An ordering of the n_f faces of the polyhedron K, $\{F_i\}_{i=1}^{n_f}$.
Input: The space of traces M.
Input: A space $\boldsymbol{V}_g \times W_g$ satisfying the inclusion properties (I).
Output: The space $\delta \boldsymbol{V}_{\text{fillM}}$.

For each $i = 1, \ldots, n_f$,
(1) Find the auxiliary spaces $\boldsymbol{V}_{g_{s,i}}$.
(2) Find an $I_{M,i}(\boldsymbol{V}_g \times W_g)$-dimensional complement space $C_{M,i}$ on face F_i:

$$\gamma_i(\boldsymbol{V}_{g_{s,i}}) \oplus C_{M,i} = \widetilde{M}(F_i),$$

where $\widetilde{M}(F_i) := M(F_i)$ if $i < n_f$, and $\widetilde{M}(F_{n_f}) := \gamma_{n_f}(W_{g_{cst}})$ is the subspace of $M(F_{n_f})$ that is $L^2(F_{n_f})$-orthogonal to $\gamma_{n_f}(W_{g_{cst}}) = \mathcal{P}_0(F_{n_f})$.
(3) Find an $I_{M,i}(\boldsymbol{V}_g \times W_g)$-dimensional, divergence-free filling space $\delta \boldsymbol{V}^i_{\text{fillM}}$ on K:

$$\begin{aligned} &(3.1)\ \ \gamma_j(\delta \boldsymbol{V}^i_{\text{fillM}}) = \{0\}, && \text{for } 1 \le j \le i-1,\\ &(3.2)\ \ \gamma_i(\delta \boldsymbol{V}^i_{\text{fillM}}) = C_{M,i}, &&\\ &(3.3)\ \ \gamma_j(\delta \boldsymbol{V}^i_{\text{fillM}}) \subset M(F_j), && \text{for } i+1 \le j \le ne. \end{aligned}$$

(The space $\delta \boldsymbol{V}^i_{\text{fillM}}$ satisfies properties (α)–(δ) of Theorem 4.2.)
return $\delta \boldsymbol{V}_{\text{fillM}} := \oplus_{i=1}^{n_f} \delta \boldsymbol{V}^i_{\text{fillM}}$.

4.3. The general case $\boldsymbol{V_g} \times \boldsymbol{W_g} \times \boldsymbol{M} := \boldsymbol{\mathcal{P}_k(K)} \times \mathcal{P}_{\boldsymbol{k}}(\boldsymbol{K}) \times \mathcal{P}_{\boldsymbol{k}}(\boldsymbol{\partial K})$

Now we apply algorithm 4.2.3, and proceed in the following three steps.

(1). Finding the spaces $\boldsymbol{V}_{g_{s,i}}$.
We begin with a characterization of the auxiliary spaces $\boldsymbol{V}_{g_{s,i}}$ which is valid for a general flat-faced polyhedron K. It is stated in terms of bubble functions associated to the faces of the polyhedron. We define them as follows. For any given numbering of the faces of a polyhedron K, we set, for $i = 1, \ldots, n_f$,

$$b_{i-1} := \prod_{k=1}^{i-1} \lambda_k \quad \text{ and } \quad b_{i-1,j} := \prod_{\substack{k=1\\ \boldsymbol{n}_k \times \boldsymbol{n}_j \neq \boldsymbol{0}}}^{i-1} \lambda_k,$$

where $b_0 = 1$.
The proof of this result is quite technical and so is provided in the Appendix.

Proposition 4.3. *Let K be a polyhedron of n_f faces with no pair of faces lying on the same hyperplane. We number its faces in such a way that $\boldsymbol{n}_1 \cdot (\boldsymbol{n}_2 \times \boldsymbol{n}_3) \neq 0$. Then we have that, for $\boldsymbol{V}_g := \boldsymbol{\mathcal{P}}_k(K)$,*

$$\boldsymbol{V}_{g_{s,i}} = \nabla\times\ \boldsymbol{\Phi}_i,$$

where

$$\boldsymbol{\Phi}_i := \{b_{i-1}\boldsymbol{\xi} + \sum_{j \in Z_{i-1}} b_{i-1,j}\phi_j \boldsymbol{n}_j : b_{i-1}\boldsymbol{\xi} \in \boldsymbol{\mathcal{P}}_{k+1},\ b_{i-1,j}\phi_j \in \mathcal{P}_{k+1}\ \forall j \in Z_{i-1}\},$$

where $Z_{i-1} := \{1 \le j \le i-1 :\ \boldsymbol{n}_j \times \boldsymbol{n}_k \neq \boldsymbol{0}$ for $k = 1, \ldots, j-1\}$. Moreover, we have

$$\dim \boldsymbol{V}_{g_{s,i}} = \dim \boldsymbol{\Phi}_i - \dim \mathcal{P}_{k+3-i}(\mathbb{R}^3) + \delta_{1,i}.$$

Here we denote $\mathcal{P}_m(\mathbb{R}^d)$ to be the polynomial of total degree no greater than m in $\mathbb{R}^d$, with the convention that $\mathcal{P}_m(\mathbb{R}^d) = \{0\}$ for a negative integer m, and $\mathcal{P}_m(\mathbb{R}^0) = \text{span}\{1\}$ for $m \geq 0$. We have, for $m \geq 0$,

$$\dim \mathcal{P}_m(\mathrm{R}^d) = \begin{pmatrix} m+d \\ m \end{pmatrix} = \frac{(m+d)!}{m!\, d!}\cdot$$

With this result, we are now ready to characterize the spaces $\gamma_i(\boldsymbol{V}_{g_{s,i}})$ and the complement spaces $C_{M,i}$ for the four polyhedral elements considered in Section 2.

(2). Finding the complement spaces $\boldsymbol{C_{M,i}}$.

We know that the space $C_{M,i}$ is any subspace of $\widetilde{M}(F_i)$ such that $\gamma_i(\boldsymbol{V}_{g_{s,i}}) \oplus C_{M,i} = \widetilde{M}(F_i)$; see the definition of $\widetilde{M}(F_i)$ in algorithm 4.2.3. Since $M(F_i) = \mathcal{P}_k(F_i)$, we need first to characterize $\gamma_i(\boldsymbol{V}_{s,i})$ then to find a choice of $C_{M,i}$, which is not necessarily unique. Currently, we do not have a systematic characterization of the bases of $\gamma_i(\boldsymbol{V}_{g_{s,i}})$ for a general polyhedron due to the complexity of the space $\boldsymbol{V}_{g_{s,i}}$. Let us turn to the simpler (and very important) cases of polyhedral elements considered in Section 2, namely when K is a tetrahedron (4 vertices, 4 faces), a quadrilateral-based pyramid (5 vertices, 5 faces), a triangle-based prism (6 vertices, 5 faces), and a quadrilateral-based hexahedron (8 vertices, 6 faces). We further assume each quadrilateral face is convex and does not degenerate into a triangle.

It turns out that the results for a tetrahedral, pyramidal, or prismatic element would follow immediately from that for a hexahedral element. For this reason, let us first find the spaces $C_{M,i}$ for a hexahedron.

K is a hexahedron. Let the skeleton of the hexahedron with its faces numbered be given in Figure 1.

Our first result, a direct consequence of Proposition 4.3, gives the dimension of the spaces $\boldsymbol{V}_{g_{s,i}}$ for different types of hexahedra.

Lemma 4.4. *Le K be a hexahedron with its faces ordered as in Figure 1. Then, we have*

$$\begin{aligned}
&\dim \boldsymbol{V}_{g_{s,1}} = 2\,\dim \mathcal{P}_k(\mathbb{R}^3) + \dim \mathcal{P}_k(\mathbb{R}^2),\\
&\dim \boldsymbol{V}_{g_{s,2}} = 2\,\dim \mathcal{P}_k(\mathbb{R}^3),\\
&\dim \boldsymbol{V}_{g_{s,3}} = 2\,\dim \mathcal{P}_k(\mathbb{R}^3) - \dim \mathcal{P}_k(\mathbb{R}^2),\\
&\dim \boldsymbol{V}_{g_{s,4}} = 2\,\dim \mathcal{P}_{k-1}(\mathbb{R}^3),\\
&\dim \boldsymbol{V}_{g_{s,5}} = \begin{cases} 2\,\dim \mathcal{P}_{k-2}(\mathbb{R}^3) + \dim \mathcal{P}_{k-2}(\mathbb{R}^2) & \text{if } F_4 \nparallel F_2,\\ \dim \mathcal{P}_{k-1}(\mathbb{R}^3) + \dim \mathcal{P}_{k-2}(\mathbb{R}^3) & \text{if } F_4 \parallel F_2,\end{cases}
\end{aligned}$$

$$\dim \boldsymbol{V}_{g_{s,6}} = \begin{cases} 2\,\dim \mathcal{P}_{k-3}(\mathbb{R}^3) + 2\,\dim \mathcal{P}_{k-3}(\mathbb{R}^2) & \text{if } F_4 \nparallel F_2,\ F_5 \nparallel F_3,\\ \dim \mathcal{P}_{k-2}(\mathbb{R}^3) + \dim \mathcal{P}_{k-3}(\mathbb{R}^3) & \\ \qquad + \dim \mathcal{P}_{k-3}(\mathbb{R}^2) & \text{if } F_4 \nparallel F_2,\ F_5 \parallel F_3,\\ 2\,\dim \mathcal{P}_{k-2}(\mathbb{R}^3) & \text{if } F_4 \parallel F_2,\ F_5 \parallel F_3,\end{cases}$$

$$\dim \boldsymbol{V}_{g_{s,7}} = \begin{cases} 2\,\dim \mathcal{P}_{k-4}(\mathbb{R}^3) + 3\,\dim \mathcal{P}_{k-4}(\mathbb{R}^2) & \text{if } F_4 \nparallel F_2,\ F_5 \nparallel F_3,\ F_6 \nparallel F_1,\\ \dim \mathcal{P}_{k-3}(\mathbb{R}^3) + \dim \mathcal{P}_{k-4}(\mathbb{R}^3) & \\ \qquad +2\,\dim \mathcal{P}_{k-4}(\mathbb{R}^2) & \text{if } F_4 \nparallel F_2,\ F_5 \nparallel F_3,\ F_6 \parallel F_1,\\ 2\,\dim \mathcal{P}_{k-3}(\mathbb{R}^3) + \dim \mathcal{P}_{k-4}(\mathbb{R}^2) & \text{if } F_4 \nparallel F_2,\ F_5 \parallel F_3,\ F_6 \parallel F_1,\\ 2\,\dim \mathcal{P}_{k-3}(\mathbb{R}^3) + \dim \mathcal{P}_{k-3}(\mathbb{R}^2) & \text{if } F_4 \parallel F_2,\ F_5 \parallel F_3,\ F_6 \parallel F_1.\end{cases}$$

Proof. The results for $i \leq 3$ is easy to verify. Let us prove the results for $i \geq 4$. From Proposition 4.3, it is clear that we only need to count the dimension of $\boldsymbol{\Phi}_i$ to get the dimension of $\boldsymbol{V}_{g_{s,i}}$.

We denote $\mathcal{P}_k(\mathbb{R}^3)/\{\lambda_j\}$ be the polynomial space of degree k that is independent of λ_j. Then, we have the following characterization of $\boldsymbol{\Phi}_i$ on the hexahedron,

$$\boldsymbol{\Phi}_i = \sum_{j=1}^{3} \boldsymbol{n}_j b_{i-1,j} \mathcal{P}_{k+1-\#_j^{i-1}}(\mathbb{R}^3) + \sum_{\substack{j\in Z_{i-1} \\ j\geq 4}} \boldsymbol{n}_j b_{i-1,j} \mathcal{P}_{k+1-\#_j^{i-1}}(\mathbb{R}^3)/\{\lambda_j\},$$

where $\#_j^{i-1}$ is the polynomial degree of $b_{i-1,j}$. After verifying that the above function are linearly independence, we get

$$\begin{aligned}
\dim \boldsymbol{\Phi}_i &= \sum_{j=1}^{3} \dim \mathcal{P}_{k+1-\#_j^{i-1}}(\mathbb{R}^3) + \sum_{\substack{j\in Z_{i-1} \\ j\geq 4}} \dim \mathcal{P}_{k+1-\#_j^{i-1}}(\mathbb{R}^3)/\{\lambda_j\} \\
&= \sum_{j=1}^{3} \dim \mathcal{P}_{k+1-\#_j^{i-1}}(\mathbb{R}^3) + \sum_{\substack{j\in Z_{i-1} \\ j\geq 4}} \dim \mathcal{P}_{k+1-\#_j^{i-1}}(\mathbb{R}^2).
\end{aligned}$$

The dimension of $\boldsymbol{V}_{g_{s,i}}$ for $i \geq 4$ immediately follows from this result. □

This result indicates that, the dimension $\gamma_i(\boldsymbol{V}_{g_{s,i}})$ for an hexahedron, and hence that of the space $C_{M,i}$, depends on how many parallel faces does the hexahedron have.

As an immediate consequence, we can compute the M-indexes for each face using the fact that $\dim \gamma_i \boldsymbol{V}_{g_{s,i}} = \dim \boldsymbol{V}_{g_{s,i}} - \dim \boldsymbol{V}_{g_{s,i+1}}$. The results are collected in the following, whose proof is straightforward and omitted.

Corollary 4.5. *Le K be a hexahedron with its faces ordered as in Figure 1. Then, we have*

$$\begin{aligned}
\dim \gamma_i(\boldsymbol{V}_{g_{s,i}}) &= \dim \mathcal{P}_k(\mathbb{R}^2) && \textit{for } i \in \{1,2,3\}, \\
\dim \gamma_4(\boldsymbol{V}_{g_{s,4}}) &= \begin{cases} \dim \mathcal{P}_{k-1}(\mathbb{R}^2) + \dim \mathcal{P}_{k-1}(\mathbb{R}) & \textit{if } F_4 \nparallel F_2, \\ \dim \mathcal{P}_{k-1}(\mathbb{R}^2) & \textit{if } F_4 \parallel F_2, \end{cases} \\
\dim \gamma_5(\boldsymbol{V}_{g_{s,5}}) &= \begin{cases} \dim \mathcal{P}_{k-2}(\mathbb{R}^2) + 2\dim \mathcal{P}_{k-2}(\mathbb{R}) & \textit{if } F_4 \nparallel F_2,\ F_5 \nparallel F_3, \\ \dim \mathcal{P}_{k-2}(\mathbb{R}^2) + \dim \mathcal{P}_{k-2}(\mathbb{R}) & \textit{if } F_4 \nparallel F_2,\ F_5 \parallel F_3, \\ \dim \mathcal{P}_{k-1}(\mathbb{R}^2) & \textit{if } F_4 \parallel F_2,\ F_5 \parallel F_3, \end{cases}
\end{aligned}$$

$$\dim \gamma_6(\boldsymbol{V}_{g_{s,6}}) = \begin{cases} \dim \mathcal{P}_{k-3}(\mathbb{R}^2) + 3\dim \mathcal{P}_{k-3}(\mathbb{R}) & \textit{if } F_4 \nparallel F_2,\ F_5 \nparallel F_3,\ F_6 \nparallel F_1, \\ \dim \mathcal{P}_{k-3}(\mathbb{R}^2) + 2\dim \mathcal{P}_{k-3}(\mathbb{R}) & \textit{if } F_4 \nparallel F_2,\ F_5 \nparallel F_3,\ F_6 \parallel F_1, \\ \dim \mathcal{P}_{k-2}(\mathbb{R}^2) + \dim \mathcal{P}_{k-3}(\mathbb{R}) & \textit{if } F_4 \nparallel F_2,\ F_5 \parallel F_3,\ F_6 \parallel F_1, \\ \dim \mathcal{P}_{k-2}(\mathbb{R}^2) + \dim \mathcal{P}_{k-2}(\mathbb{R}) & \textit{if } F_4 \parallel F_2,\ F_5 \parallel F_3,\ F_6 \parallel F_1. \end{cases}$$

Moreover, we have

$$
\begin{aligned}
&I_{M,i} = \dim C_{M,i} = 0, \quad \text{if } 1 \le i \le 3,\\
&I_{M,4} = \dim C_{M,4} = \begin{cases} \dim \mathcal{P}_k(\mathbb{R}^0) & \text{if } F_4 \not\parallel F_2,\\ \dim \mathcal{P}_k(\mathbb{R}) & \text{if } F_4 \parallel F_2, \end{cases}\\
&I_{M,5} = \dim C_{M,5} = \begin{cases} \dim \mathcal{P}_k(\mathbb{R}^0) + 2\dim \mathcal{P}_{k-1}(\mathbb{R}^0) & \text{if } F_4 \not\parallel F_2,\ F_5 \not\parallel F_3,\\ \dim \mathcal{P}_k(\mathbb{R}) + \dim \mathcal{P}_{k-1}(\mathbb{R}^0) & \text{if } F_4 \not\parallel F_2,\ F_5 \parallel F_3,\\ \dim \mathcal{P}_k(\mathbb{R}) & \text{if } F_4 \parallel F_2,\ F_5 \parallel F_3, \end{cases}\\
&I_{M,6} = \dim C_{M,6} = \begin{cases} 2\dim \mathcal{P}_{k-1}(\mathbb{R}^0) + 3\dim \mathcal{P}_{k-2}(\mathbb{R}^0) & \text{if } F_4 \not\parallel F_2, F_5 \not\parallel F_3, F_6 \not\parallel F_1,\\ \dim \mathcal{P}_{k-1}(\mathbb{R}) + \dim \mathcal{P}_{k-1}(\mathbb{R}^0) & \\ \quad +2\dim \mathcal{P}_{k-2}(\mathbb{R}^0) & \text{if } F_4 \not\parallel F_2, F_5 \not\parallel F_3, F_6 \parallel F_1,\\ \dim \mathcal{P}_{k-1}(\mathbb{R}) + \dim \mathcal{P}_{k-1}(\mathbb{R}^0) & \\ \quad + \dim \mathcal{P}_{k-2}(\mathbb{R}^0) & \text{if } F_4 \not\parallel F_2, F_5 \parallel F_3, F_6 \parallel F_1,\\ \dim \mathcal{P}_{k-1}(\mathbb{R}) + \dim \mathcal{P}_{k-1}(\mathbb{R}^0) & \text{if } F_4 \parallel F_2, F_5 \parallel F_3, F_6 \parallel F_1. \end{cases}
\end{aligned}
$$

This result shows that $C_{M,i} = \{0\}$ for $i = 1, 2, 3$. The corresponding spaces for the fourth, fifth, and sixth faces are given next.

Lemma 4.6. *Let K be a hexahedron with its faces ordered as in Figure 1. Then, for the fourth face, we can take*

$$
C_{M,4} = \begin{cases} \operatorname{span}\{\lambda_1^k\} & \text{if } \quad F_4 \not\parallel F_2 \text{ and } \kappa_{124} \neq 0,\\ \operatorname{span}\{\lambda_3^k\} & \text{if } \quad F_4 \not\parallel F_2 \text{ and } \kappa_{234} \neq 0,\\ \operatorname{span}\{\lambda_1^\alpha \lambda_3^{k-\alpha}\}_{0\le\alpha\le k} & \text{if } \quad F_4 \parallel F_2. \end{cases}
$$

For the fifth face, we can take

$$
C_{M,5} = \begin{cases} \operatorname{span}\{\lambda_1^k, \lambda_1^{k-1}\lambda_2, \lambda_1^{k-1}\} & \text{if } \quad F_4 \not\parallel F_2, F_5 \not\parallel F_3 \text{ and } \quad \kappa_{135} \neq 0,\\ \operatorname{span}\{\lambda_2^k, \lambda_2^{k-1}\lambda_1, \lambda_2^{k-1}\} & \text{if } \quad F_4 \not\parallel F_2, F_5 \not\parallel F_3,\ \kappa_{235} \neq 0,\ \kappa_{245} \neq 0\\ & \text{and } \quad F_2 \cap F_3 \cap F_5 \notin F_4\\ \operatorname{span}\{\lambda_2^k, \lambda_2^{k-1}\lambda_1, 1\} & \text{if } \quad F_4 \not\parallel F_2, F_5 \not\parallel F_3,\ \kappa_{235} \neq 0,\ \kappa_{245} \neq 0\\ & \text{and } \quad F_2 \cap F_3 \cap F_5 \in F_4\\ \operatorname{span}\{\lambda_2^{k-1}\} \oplus \operatorname{span}\{\lambda_1^\alpha \lambda_2^{k-\alpha} + p_\alpha\}_{0\le\alpha\le k} & \text{if } \quad F_4 \not\parallel F_2, F_5 \parallel F_3\\ & \text{and } \quad \kappa_{245} \neq 0,\\ \operatorname{span}\{\lambda_1^\alpha \lambda_2^{k-\alpha}\}_{0\le\alpha\le k} & \text{if } \quad F_4 \parallel F_2, F_5 \parallel F_3. \end{cases}
$$

Here $p_\alpha \in \mathcal{P}_{k-1}(F_5)$ is any polynomial on F_5 of degree less than k.

And for the sixth face, we can take

$$
C_{M,6} = \begin{cases} \operatorname{span}\{\lambda_3^{k-2}(\lambda_2 - c_1), \lambda_3^{k-2}(\lambda_3 - c_2), & \text{if } \quad F_4 \not\parallel F_2, F_5 \not\parallel F_3, F_6 \not\parallel F_1,\\ \quad \lambda_3^{k-2}(\lambda_3^2 - c_3), \lambda_3^{k-1}(\lambda_2 - c_4), \lambda_3^{k-1}(\lambda_3 - c_5)\} & \text{and } \quad \kappa_{136} \neq 0, \kappa_{356} \neq 0,\\ \operatorname{span}\{\lambda_3^{k-2}(\lambda_2 - c_1), \lambda_3^{k-2}(\lambda_3 - c_2)\} & \text{if } \quad F_4 \not\parallel F_2,\ F_5 \not\parallel F_3,\ F_6 \parallel F_1,\\ \quad \oplus \operatorname{span}\{\lambda_3^\alpha \lambda_2^{k-\alpha} + p_\alpha\}_{0\le\alpha\le k} & \text{and } \quad \kappa_{356} \neq 0.\\ \operatorname{span}\{\lambda_2^{k-2}(\lambda_2 - c_6), \lambda_3^{k-2}(\lambda_3 - c_2)\} & \text{if } \quad F_4 \not\parallel F_2,\ F_5 \not\parallel F_3,\ F_6 \parallel F_1,\\ \quad \oplus \operatorname{span}\{\lambda_3^\alpha \lambda_2^{k-\alpha} + p_\alpha\}_{0\le\alpha\le k} & \text{and } \quad \kappa_{356} = \kappa_{246} = 0.\\ \operatorname{span}\{\lambda_3^{k-2}(\lambda_3 - c_2)\} & \\ \quad \oplus \operatorname{span}\{\lambda_3^\alpha \lambda_2^{k-\alpha} + p_\alpha\}_{0\le\alpha\le k} & \text{if } \quad F_4 \not\parallel F_2,\ F_5 \parallel F_3,\ F_6 \parallel F_1.\\ \operatorname{span}\{\lambda_3^\alpha \lambda_2^{k-\alpha} + p_\alpha\}_{0\le\alpha\le k} & \text{if } \quad F_4 \parallel F_2,\ F_5 \parallel F_3,\ F_6 \parallel F_1. \end{cases}
$$

Here $p_\alpha \in \mathcal{P}_{k-1}(F_6)$, and the constants $\{c_i\}_{i=1}^6$ make sure that the related functions have average zero on F_6.

Here we use the convention that we only take the above function when its power index is not negative. For example, if $k=0$, $C_{M,4} = \operatorname{span}\{1\}$, $C_{M,5} = \operatorname{span}\{1\}$, and $C_{M,6} = \{0\}$.

Proof. In this proof, we set $\mathrm{k}_{ijk} = \nabla\lambda_i \cdot (\nabla\lambda_j \times \nabla\lambda_k)$, this constant simply differ with the constant k_{ijk} defined in Section 2 by a scaling factor.

Let us first prove the result for the fourth face. We give a detailed proof for the case in which $F_2 \nparallel F_4$ and $\mathrm{k}_{124} \neq 0$, and sketch those of the other two cases.

If $F_2 \nparallel F_4$ and $\mathrm{k}_{124} \neq 0$, we claim that

$$\begin{aligned}\gamma_4\left(\nabla\times\left(\lambda_1\lambda_2\widetilde{\mathcal{P}}_{k-1}(\lambda_1,\lambda_3)\,\nabla\lambda_3\right)\right)&\\ \oplus\,\gamma_4\left(\nabla\times\left(\lambda_1\lambda_3\widetilde{\mathcal{P}}_{k-1}(\lambda_1)\,\nabla\lambda_2\right)\right)&\oplus\operatorname{span}\{\lambda_1^k\} = \mathcal{P}_k(F_4).\end{aligned}$$

This implies that we can take $C_{M,4} = \operatorname{span}\{\lambda_1^k\}$, since

$$\nabla\times\left(\lambda_1\lambda_2\widetilde{\mathcal{P}}_{k-1}(\lambda_1,\lambda_3)\,\nabla\lambda_3\right)\oplus\nabla\times\left(\lambda_1\lambda_3\widetilde{\mathcal{P}}_{k-1}(\lambda_1)\,\nabla\lambda_2\right)\subset \boldsymbol{V}_{s,4}.$$

To prove the claim, we show that the following functions form a set of basis for $\mathcal{P}_k(F_4)$,

$$\left\{\gamma_4\left(\nabla\times\left(\lambda_1^{\alpha+1}\lambda_2\lambda_3^\beta\,\nabla\lambda_3\right)\right)\right\}_{\alpha\geq0,\beta\geq0}^{\alpha+\beta\leq k-1},\ \left\{\gamma_4\left(\nabla\times\left(\lambda_1^{\alpha+1}\lambda_3\,\nabla\lambda_2\right)\right)\right\}_{\alpha=0}^{k-1},\ \{\lambda_1^k\}.$$

We have

$$\gamma_4\left(\nabla\times\left(\lambda_1^{\alpha+1}\lambda_2\lambda_3^\beta\,\nabla\lambda_3\right)\right) = \lambda_1^\alpha\lambda_3^\beta(\mathrm{k}_{234}\,\lambda_1 + (\alpha+1)\mathrm{k}_{134}\,\lambda_2),$$

and

$$\gamma_4\left(\nabla\times\left(\lambda_1^{\alpha+1}\lambda_3\,\nabla\lambda_2\right)\right) = \lambda_1^\alpha(\mathrm{k}_{324}\,\lambda_1 + (\alpha+1)\mathrm{k}_{124}\,\lambda_3).$$

Now, assume that there exist constants $\{C_{\alpha\beta}\}_{\alpha\geq0,\beta\geq0}^{\alpha+\beta\leq k-1}$, $\{D_\alpha\}_{\alpha\geq0}^{\alpha\leq k-1}$, and E such that

$$\begin{aligned}&\sum_{\alpha\geq0,\beta\geq0}^{\alpha+\beta\leq k-1} C_{\alpha\beta}\,\lambda_1^\alpha\lambda_3^\beta(\mathrm{k}_{234}\,\lambda_1+(\alpha+1)\mathrm{k}_{134}\,\lambda_2)\\ &\qquad+\sum_{\alpha\geq0}^{\alpha\leq k-1} D_\alpha\,\lambda_1^\alpha(\mathrm{k}_{324}\,\lambda_1+(\alpha+1)\mathrm{k}_{124}\,\lambda_3)+E\,\lambda_1^k = 0 \quad \text{on} \quad F_4.\end{aligned} \tag{4.2}$$

We are going to prove that all the constants are zero. Evaluating the expression on the edge $e_{14} := F_1\cap F_4$, we get

$$\sum_{\beta\geq0}^{\beta\leq k-1} C_{0\beta}\,\lambda_3^\beta\mathrm{k}_{134}\,\lambda_2 + D_0\,\mathrm{k}_{124}\,\lambda_3 = 0 \quad \text{on } e_{14},$$

since $\lambda_1|_{e_{14}} = 0$. Now, evaluating the above expression on the node $\boldsymbol{v}_{124} := F_1\cap F_2\cap F_4$, we get $D_0 = 0$, since $\lambda_2|_{\boldsymbol{v}_{124}} = 0$, $\lambda_3|_{\boldsymbol{v}_{124}} \neq 0$, and $\mathrm{k}_{124}\neq 0$. This implies

$$\sum_{\beta\geq0}^{\beta\leq k-1} C_{0\beta}\,\lambda_3^\beta\mathrm{k}_{134}\,\lambda_2 = 0 \quad \text{on } e_{14}.$$

Hence $C_{0\beta} = 0$ for $0 \le \beta \le k-1$ because λ_2 and λ_3 are linear functions on e_{14} and $\mathrm{k}_{134} \neq 0$. Then, consequently dividing equation (4.2) by λ_1 and using the same argument, we obtain $D_\alpha = 0$ and $C_{\alpha\beta} = 0$ for $0 \le \beta \le k-1-\alpha$ for all $1 \le \alpha \le k-1$, and finally, $E = 0$. This completes the proof of the claim.

Now, if $F_2 \nparallel F_4$ and $\mathrm{k}_{234} \neq 0$, we can use similar argument to prove

$$\begin{aligned}&\gamma_4 \left(\nabla\times \left(\lambda_2\lambda_3\widetilde{\mathcal{P}}_{k-1}(\lambda_1,\lambda_3)\,\nabla\,\lambda_1\right)\right)\\ &\quad\oplus \gamma_4 \left(\nabla\times \left(\lambda_1\lambda_3\widetilde{\mathcal{P}}_{k-1}(\lambda_3)\,\nabla\,\lambda_2\right)\right) \oplus \mathrm{span}\{\lambda_3^k\} = \mathcal{P}_k(F_4),\end{aligned}$$

where we need to evaluate the expression on the edge $e_{34} := F_3 \cap F_4$, and consequently dividing the expression by λ_3. And if $F_2 \parallel F_4$, we can easily get

$$\gamma_4 \left(\nabla\times \left(\lambda_1\lambda_2\widetilde{\mathcal{P}}_{k-1}(\lambda_1,\lambda_3)\,\nabla\,\lambda_3\right)\right) \oplus \widetilde{\mathcal{P}}_k(\lambda_1,\lambda_3) = \mathcal{P}_k(F_4),$$

because λ_2 is a constant on F_4 and $\gamma_4 \left(\nabla\times \left(\lambda_1\lambda_2\widetilde{\mathcal{P}}_{k-1}(\lambda_1,\lambda_3)\,\nabla\,\lambda_3\right)\right) \subset \mathcal{P}_{k-1}(F_4)$. This completes the proof for the fourth face.

Next, we prove the result for the fifth face. The key part of the proof has already been shown for the fourth face, so we just sketch the main idea for the first case, namely $F_2 \nparallel F_4$, $F_3 \nparallel F_5$, and $\mathrm{k}_{135} \neq 0$. In this case, we claim that, for $k \ge 1$,

$$\begin{aligned}&\gamma_5 \left(\nabla\times \left(\lambda_1\lambda_3\lambda_4\widetilde{\mathcal{P}}_{k-2}(\lambda_1,\lambda_2)\,\nabla\,\lambda_2\right)\right)\\ &\quad\oplus \gamma_5 \left(\nabla\times \left(\lambda_1\lambda_2\lambda_4\widetilde{\mathcal{P}}_{k-2}(\lambda_1)\,\nabla\,\lambda_3\right)\right)\\ &\qquad\oplus \gamma_5 \left(\nabla\times \left(\lambda_1\lambda_2\lambda_3\widetilde{\mathcal{P}}_{k-2}(\lambda_1)\,\nabla\,\lambda_4\right)\right) \oplus \mathrm{span}\{\lambda_1^k, \lambda_1^{k-1}\lambda_2, \lambda_1^{k-1}\} = \mathcal{P}_k(F_5).\end{aligned}$$

To prove the claim, we show that the following functions form a set of basis for $\mathcal{P}_k(F_5)$,

$$\begin{aligned}&\left\{\gamma_5 \left(\nabla\times \left(\lambda_1^{\alpha+1}\lambda_2^{\beta}\lambda_3\lambda_4\,\nabla\,\lambda_2\right)\right)\right\}_{\alpha\ge 0,\beta\ge 0}^{\alpha+\beta\le k-2},\ \left\{\gamma_5 \left(\nabla\times \left(\lambda_1^{\alpha+1}\lambda_2\lambda_4\,\nabla\,\lambda_3\right)\right)\right\}_{\alpha\ge 0}^{\alpha\le k-2},\\ &\quad\left\{\gamma_5 \left(\nabla\times \left(\lambda_1^{\alpha+1}\lambda_2\lambda_3\,\nabla\,\lambda_4\right)\right)\right\}_{\alpha\ge 0}^{\alpha\le k-2},\ \{\lambda_1^k, \lambda_1^{k-1}\lambda_2, \lambda_1^{k-1}\}.\end{aligned}$$

Then, to show that, we perform a linear independence check of these functions by first evaluating an expression similar to (4.2) at certain edges and nodes, and then do a mathematical induction on the index α.

Finally, we sketch the proof for the last face for the first case when $F_2 \nparallel F_4$, $F_3 \nparallel F_5$, $F_1 \nparallel F_6$, and $\mathrm{k}_{136} \neq 0$, $\mathrm{k}_{356} \neq 0$. In this case, we claim that, for $k \ge 2$,

$$\begin{aligned}&\gamma_6 \left(\nabla\times \left(\lambda_2\lambda_3\lambda_4\lambda_5\widetilde{\mathcal{P}}_{k-3}(\lambda_2,\lambda_3)\,\nabla\,\lambda_1\right)\right)\\ &\quad\oplus \gamma_6 \left(\nabla\times \left(\lambda_1\lambda_3\lambda_4\lambda_5\widetilde{\mathcal{P}}_{k-3}(\lambda_3)\,\nabla\,\lambda_2\right)\right)\\ &\qquad\oplus \gamma_6 \left(\nabla\times \left(\lambda_1\lambda_2\lambda_3\lambda_5\widetilde{\mathcal{P}}_{k-3}(\lambda_3)\,\nabla\,\lambda_4\right)\right)\\ &\qquad\quad\oplus \gamma_6 \left(\nabla\times \left(\lambda_1\lambda_2\lambda_3\lambda_4\widetilde{\mathcal{P}}_{k-3}(\lambda_3)\,\nabla\,\lambda_5\right)\right) \oplus \lambda_3^{k-2}\mathcal{P}_2(\lambda_2,\lambda_3) = \mathcal{P}_k(F_6),\end{aligned}$$

whose proof is similar to that for the fourth face. Using this claim, we easily get

$$\gamma_6(\boldsymbol{V}_{s,6}) \oplus C_{M,6} = \{\mu \in \mathcal{P}_k(F_6) : \int_{F_6} \mu \,\mathrm{d}s = 0\}.$$

This completes the proof $\square$

Now, the choices of $C_{M,i}$ for a tetrahedron, a pyramid, or a prism follow immediately from Lemma 4.6. We collect the results in the following without giving the proof.

K is a tetrahedron, a pyramid, or a prism.

Corollary 4.7. *Le K be a tetrahedron. Then, we have $C_{M,i} = \{0\}$ for $1 \le i \le 4$.*

Corollary 4.8. *Le K be a pyramid with its faces ordered as in Figure 1. Then, we have $C_{M,i} = \{0\}$ for $1 \le i \le 3$, and we can take*

$$C_{M,4} = \operatorname{span}\{\lambda_3^k\},$$

and

$$C_{M,5} = \operatorname{span}\{\lambda_2^{k-1} p_1, p_2\}.$$

Here $p_1, p_2 \in \mathcal{P}_1(F_5)$ satisfy

$$\int_{F_5} \lambda_2^{k-1} p_1 = 0 \quad \text{and} \quad p_1(\boldsymbol{v}_{234}) = 0,$$
$$\int_{F_5} p_2 = 0 \quad \text{and} \quad p_2(\boldsymbol{v}_{234}) \neq 0,$$

where we recall $\boldsymbol{v}_{234} := F_2 \cap F_3 \cap F_4$ is the vertex of the pyramid opposite to the quadrilateral base.

Corollary 4.9. *Le K be a prism with its faces ordered as in Figure 1. Then, we have $C_{M,i} = \{0\}$ for $1 \le i \le 3$, and we can take*

$$C_{M,4} = \operatorname{span}\{\lambda_3^k\},$$

and

$$C_{M,5} = \begin{cases} \operatorname{span}\{\lambda_2^{k-1} p_1, \lambda_2^{k-1} p_2\}, & \text{if } F_5 \nparallel F_3 \text{ and } \kappa_{235} \neq 0, \\ \operatorname{span}\{\lambda_1^{\alpha} \lambda_2^{k-\alpha} + p_\alpha\}_{0 \le \alpha \le k}, & \text{if } F_5 \parallel F_3. \end{cases}$$

Here $p_1, p_2 \in \mathcal{P}_1(F_5)$ are linear independent satisfying

$$\int_{F_5} \lambda_2^{k-1} p_1 = 0, \quad \int_{F_5} \lambda_2^{k-1} p_2 = 0,$$

and $p_\alpha \in \mathcal{P}_{k-1}(F_5)$ satisfies

$$\int_{F_5} \lambda_1^{\alpha} \lambda_2^{k-\alpha} + p_\alpha = 0.$$

(3). Validation of the spaces $\boldsymbol{\delta V}_{\text{fillM}}$.

Now, let us prove the choice of $\delta \boldsymbol{V}_{\text{fillM}}$ in Theorem 2.3 for a tetrahedron, Theorem 2.5 for a pyramid, Theorem 2.7 for a regular prism and Theorems 2.8 and 2.9 for different types of prisms, and Theorem 2.11 for a cube and Theorems 2.12 and 2.15 for different types of hexahedra satisfy the conditions in Table 2. For simplicity, we say the choice of $\delta \boldsymbol{V}_{\text{fillM}}$ is vaild if it satisfy the conditions in Table 2.

First, if K is a tetrahedron, we have $C_{M,i} = \{0\}$ for all the faces from Corollary 4.7. So, $\delta \boldsymbol{V}_{\text{fillM}} = \{\boldsymbol{0}\}$ in Theorem 2.3 is valid.

Next, if K is a pyramid and $k \geq 1$, it is easy to show that the spaces

$$\begin{aligned}
&\delta \boldsymbol{V}^i_{\text{fillM}} = \{\boldsymbol{0}\} \qquad \text{for } i \in \{1,2,3\},\\
&\delta \boldsymbol{V}^4_{\text{fillM}} = \nabla\times \operatorname{span}\{\xi\lambda_3^k \nabla \lambda_1\},\\
&\delta \boldsymbol{V}^5_{\text{fillM}} = \nabla\times \operatorname{span}\{\xi\lambda_2^{k-1}\lambda_4 \nabla \lambda_1, \xi\lambda_1 \nabla \lambda_4\}
\end{aligned}$$

where ξ is defined in Theorem 2.5, satisfy the conditions in algorithm 4.2.3 with $C_{M,i}$ given in Corollary 4.8. Indeed, by the definition of ξ, we have $\xi|_{F_2\cup F_3} = 0$, $\xi|_{F_4}$ is proportional to $\lambda_3|_{F_4}$, and $\xi|_{F_5}$ is proportional to $\lambda_2|_{F_5}$. This implies $\gamma_4(\nabla\times(\xi\lambda_3^k \nabla \lambda_1))$ is proportional to $\lambda_3^k|_{F_4}$, and $\gamma_5(\nabla\times(\xi\lambda_2^{k-1}\lambda_4 \nabla \lambda_1))$ is proportional to $\lambda_2^{k-1}(k\lambda_4\kappa_{215} + \lambda_2\kappa_{415})|_{F_5}$ and $\gamma_5(\nabla\times(\xi\lambda_1 \nabla \lambda_4))$ is proportional to $(\lambda_1\kappa_{245} + \lambda_2\kappa_{145})|_{F_5}$. So $\delta\boldsymbol{V}_{\text{fillM}}$ is valid in Theorem 2.8 for a pyramid for $k \geq 1$. The proof for the case $k = 0$ is simpler and omitted.

Then, if K is a prism, we prove that $\delta\boldsymbol{V}_{\text{fillM}}$ is valid in Theorem 2.9 for a prism without parallel faces when $k \geq 1$. The proof for the case with parallel faces in Theorems 2.7 and 2.8, and the case when $k = 0$ are simpler and omitted. We show that the spaces

$$\begin{aligned}
&\delta \boldsymbol{V}^i_{\text{fillM}} = \{\boldsymbol{0}\} \qquad \text{for } i \in \{1,2,3\},\\
&\delta \boldsymbol{V}^4_{\text{fillM}} = \operatorname{span}\{\boldsymbol{v}_{\mu_4^k}\},\\
&\delta \boldsymbol{V}^5_{\text{fillM}} = \nabla\times \operatorname{span}\{\xi\lambda_2^k \nabla \lambda_1, \xi\lambda_1\lambda_2^{k-1} \nabla \lambda_2\}
\end{aligned}$$

where ξ is defined in Theorem 2.9, satisfy the conditions in algorithm 4.2.3 with $C_{M,i}$ given in Corollary 4.9. Indeed, by the definition of ξ, we have $\xi|_{F_3\cup F_4} = 0$, $\xi|_{F_5}$ is proportional to $\lambda_4|_{F_5}$. This implies $\gamma_5(\nabla\times(\xi\lambda_2^k \nabla \lambda_1))$ is proportional to $\lambda_2^{k-1}(k\lambda_4\kappa_{215} + \lambda_2\kappa_{415})|_{F_5}$ and $\gamma_5(\nabla\times(\xi\lambda_1\lambda_2^{k-1} \nabla \lambda_2))$ is proportional to $\lambda_2^{k-1}(\lambda_1\kappa_{425}+\lambda_4\kappa_{125})|_{F_5}$. Also, by definition, we have $\gamma_4(\boldsymbol{v}_{\mu_4^k}) = \mu_4^k|_{F_4} = \lambda_3^k|_{F_4}$. So $\delta\boldsymbol{V}_{\text{fillM}}$ is valid in Theorem 2.8 for a pyramid for $k \geq 1$.

Finally, if K is a hexahedron, we show that $\delta\boldsymbol{V}_{\text{fillM}}$ is valid in Theorem 2.14 for a hexahedron with one pair of parallel, non-parallelepipedal faces when $k \geq 2$. The other cases, namely, three pair of parallel faces in Theorem 2.11, two pair of parallel faces in Theorem 2.12, one pair of parallel, parallelepipedal faces in Theorem 2.13, no pair of parallel faces in Theorem 2.15, and the cases when $k = 0$ or $k = 1$ are similar and omitted. To even simplify presentation, we assume that $\kappa_{234} \neq 0$. Then, it is easy to check that the spaces

$$\begin{aligned}
&\delta \boldsymbol{V}^i_{\text{fillM}} = \{\boldsymbol{0}\} \qquad \text{for } i \in \{1,2,3\},\\
&\delta \boldsymbol{V}^4_{\text{fillM}} = \operatorname{span}\{\boldsymbol{v}_{\mu_{4,1}^k}\},\\
&\delta \boldsymbol{V}^5_{\text{fillM}} = \operatorname{span}\{\boldsymbol{v}_{\mu_{5,3}^k}, \boldsymbol{v}_{\mu_{5,1}^k}, \boldsymbol{v}_{\mu_{5,3}^{k-1}}\},\\
&\delta \boldsymbol{V}^6_{\text{fillM}} = \operatorname{span}\{\boldsymbol{v}_{\mu_{6,1}^{k-1}}, \boldsymbol{v}_{\mu_{6,3}^{k-1}}\}\\
&\qquad\qquad \oplus \nabla\times \operatorname{span}\{\lambda_1\lambda_2^{k-1}\lambda_4\lambda_5 \nabla \lambda_3,\\
&\qquad\qquad \lambda_1\lambda_2^{k-1}\lambda_3\lambda_5 \nabla \lambda_4, \lambda_1\lambda_3\lambda_4\lambda_5 \widetilde{\mathcal{P}}_{k-2}(\lambda_2,\lambda_3) \nabla \lambda_2\}
\end{aligned}$$

satisfy the conditions in algorithm 4.2.3 with $C_{M,i}$ given in Lemma 4.6, and the hexahedron has faces ordered such that $F_2 \nparallel F_4$, $F_3 \nparallel F_5$, $F_1 \parallel F_6$, and $\mu_{135} \neq 0$.

This completes the validation of $\delta\boldsymbol{V}_{\text{fillM}}$ on the *four* polyhedron considered in Section 2 for the given space $\boldsymbol{V}_g \times W_g := \boldsymbol{\mathcal{P}}_k(K) \times \mathcal{P}_k(K)$ and trace space $M := \mathcal{P}_k(\partial K)$.

4.4. Three special cases

Now, we consider other choices of the given spaces for some special elements.

The first special case is a square based pyramid with trace space

$$M := \{\mu \in L^2(\partial K): \ \ \mu|_{F_1} \in \mathcal{Q}_k(F_1),\ \mu|_{F_i} \in \mathcal{P}_k(F_i) \text{ for } 2 \le i \le 5\},$$

and given space $\boldsymbol{V}_g \times W_g := \boldsymbol{\mathcal{P}}_k(K) \times \mathcal{P}_k(K)$. The result is shown in Theorem 2.4. The second special case is a regular prism with given spaces $\boldsymbol{V}_g \times W_g \times M := \boldsymbol{\mathcal{P}}_{k|k}(K) \times \mathcal{P}_{k|k}(K) \times M_k^{pq}(\partial K)$. The result is shown in Theorem 2.6. And the last special case is a cube with given spaces $\boldsymbol{V}_g \times W_g \times M := \boldsymbol{\mathcal{Q}}_k(K) \times \mathcal{Q}_k(K) \times \mathcal{Q}(\partial K)$. The result is shown in Theorem 2.10.

Let us first prove Theorem 2.4 for a square based pyramid.

Proof. The result follows directly from Theorem 2.5, and the fact that

$$\begin{aligned} \mathcal{P}_k(F_1) \oplus \gamma_1(\delta \boldsymbol{V}^1_{\text{fillM}}) &= \mathcal{Q}_k(F_1),\\ \gamma_i(\delta \boldsymbol{V}^1_{\text{fillM}}) &= \{0\} \quad \text{ for } 2 \le i \le 5, \end{aligned}$$

where

$$\delta \boldsymbol{V}^1_{\text{fillM}} := \text{span}\{\lambda_2^{\alpha}\lambda_3^{\beta+1}\lambda_4\lambda_5 \nabla \lambda_2 : \ k-1-\beta \le \alpha \le k-1, 0 \le \beta \le k-1\}. \qquad \square$$

Let us now prove Theorem 2.6 for a regular prism.

Proof. We start with proving that the following polynomial sequence is exact on $\mathbb{R}^3$ for $k \ge 1$.

$$\mathbb{R} \xrightarrow{i} H \xrightarrow{\nabla} \boldsymbol{E} \xrightarrow{\nabla\times} \boldsymbol{V} \xrightarrow{\nabla\cdot} W \xrightarrow{o} 0,$$

where

$$\begin{aligned} H &:= \mathcal{P}_{k+1|k} \oplus \widetilde{\mathcal{P}}_{k+1}(z), & \boldsymbol{E} &:= \begin{pmatrix} \mathcal{P}_{k|k} \oplus \widetilde{\mathcal{P}}_k(x,y)y \\ \mathcal{P}_{k|k} \\ \mathcal{P}_{k+1|k} \end{pmatrix},\\ \boldsymbol{V} &:= \boldsymbol{\mathcal{P}}_{k|k}, & W &:= \nabla\cdot \boldsymbol{\mathcal{P}}_{k|k} = \mathcal{P}_{k-1|k} + \mathcal{P}_{k,k-1}. \end{aligned}$$

We need to show that

$$\text{Ker}_{\nabla} H = \mathbb{R}, \quad \text{Ker}_{\nabla\times} \boldsymbol{E} = \nabla H, \quad \text{Ker}_{\nabla\cdot} \boldsymbol{V} = \nabla\times \boldsymbol{E}.$$

The first and second equalities are straightforward. And for the third one, it is easy to show that $\nabla\times \boldsymbol{E} \subset \text{Ker}_{\nabla\cdot} \boldsymbol{V}$ and that

$$\begin{aligned} \dim \nabla\cdot \boldsymbol{V} + \dim \nabla\times \boldsymbol{E} &= \dim W + \dim \boldsymbol{E} - \dim \nabla H\\ &= \dim \boldsymbol{V}. \end{aligned}$$

This result implies that any divergence-free function in $\boldsymbol{V} = \boldsymbol{\mathcal{P}}_{k|k}$ is a curl of a function in $\boldsymbol{E}$, and we will use this fact in what follows.

First, we have

$$
\begin{aligned}
\boldsymbol{V}_{s,1} &= \nabla\times \begin{pmatrix} \mathcal{P}_{k|k} \oplus y\,\widetilde{\mathcal{P}}_k(x,y) \\ \mathcal{P}_{k|k} \\ \mathcal{P}_{k+1|k} \end{pmatrix}, \\
\boldsymbol{V}_{s,2} &= \nabla\times \begin{pmatrix} \mathcal{P}_{k|k} \oplus y\,\widetilde{\mathcal{P}}_k(x,y) \\ x\mathcal{P}_{k-1|k} \\ x\mathcal{P}_{k|k} \end{pmatrix}, \\
\boldsymbol{V}_{s,3} &= \nabla\times \begin{pmatrix} y\,\mathcal{P}_{k-1|k} \oplus y\,\widetilde{\mathcal{P}}_k(x,y) \\ x\,\mathcal{P}_{k-1|k} \\ xy\,\mathcal{P}_{k-1|k} \end{pmatrix}, \\
\boldsymbol{V}_{s,4} &= \nabla\times \begin{pmatrix} yz\,\mathcal{P}_{k-1|k-1} \\ xz\,\mathcal{P}_{k-1|k-1} \\ xy\,\mathcal{P}_{k-1|k} \end{pmatrix}, \\
\boldsymbol{V}_{s,5} &= \nabla\times \begin{pmatrix} yz\lambda_4\,\mathcal{P}_{k-2|k-1} \\ xz\lambda_4\,\mathcal{P}_{k-2|k-1} \\ xy\lambda_4\,\mathcal{P}_{k-2|k} \end{pmatrix} \oplus \nabla\times(xyz\,\mathcal{P}_{k-2,k-1}\,\nabla\,\lambda_4), \\
\boldsymbol{V}_{s,6} &= \nabla\times \begin{pmatrix} yz\lambda_4\lambda_5\,\mathcal{P}_{k-2|k-2} \\ xz\lambda_4\lambda_5\,\mathcal{P}_{k-2|k-2} \\ xy\lambda_4\,\mathcal{P}_{k-2|k} \end{pmatrix} \oplus \nabla\times(xyz\lambda_5\,\mathcal{P}_{k-2,k-2}\,\nabla\,\lambda_4).
\end{aligned}
$$

Then, we apply algorithm 4.2.3. It is now routine to check that $\dim\gamma_1(\boldsymbol{V}_{s,1}) = \dim \mathcal{Q}_k(F_1)$, $\dim\gamma_2(\boldsymbol{V}_{s,2}) = \dim \mathcal{Q}_k(F_2)$, $\dim\gamma_3(\boldsymbol{V}_{s,3}) = \dim \mathcal{P}_k(F_3)$, $\dim\gamma_4(\boldsymbol{V}_{s,4}) = \dim \mathcal{Q}_k(F_4) - 1$, $\dim\gamma_5(\boldsymbol{V}_{s,5}) = \dim \mathcal{P}_{k-1}(F_5) - 1$, and the trace spaces $C_{M,i}$ defined in Corollary 4.9 satisfy the properties in algorithm 4.2.3. So, we need to find one function related to the fourth face, and $k+1$ functions related to the fifth face. And the space $\delta\boldsymbol{V}_{\text{fillM}}$ in Theorem 2.6 is valid. □

Finally, let us sketch the proof of Theorem 2.10 for a cube.

Proof. as in the proof of Theorem 2.6, we start with an exact sequence.

$$\mathbb{R} \xrightarrow{i} H \xrightarrow{\nabla} \boldsymbol{E} \xrightarrow{\nabla\times} \boldsymbol{V} \xrightarrow{\nabla\cdot} W \xrightarrow{o} 0,$$

where

$$
\begin{aligned}
H &:= \mathcal{P}_{k,k,k} \oplus \operatorname{span}\{x^{k+1}, y^{k+1}, z^{k+1}\}, & \boldsymbol{E} &:= \begin{pmatrix} \mathcal{P}_{k,k,k} \oplus \widetilde{\mathcal{P}}_{k+1}(z) \\ \mathcal{P}_{k,k,k} \oplus \widetilde{\mathcal{P}}_{k+1}(x) \\ \mathcal{P}_{k,k,k} \oplus \widetilde{\mathcal{P}}_{k+1}(y) \end{pmatrix}, \\
\boldsymbol{V} &:= \boldsymbol{\mathcal{P}}_{k,k,k}, & W &:= \mathcal{P}_{k,k,k} \backslash \{x^k y^k z^k\}.
\end{aligned}
$$

We then use it to explicitly construct the spaces $\boldsymbol{V}_{s,i}$ for $1 \le i \le 7$, and compute the trace space $C_{M,i}$. In particular, we have $C_{M,i} = \{0\}$ for $1 \le i \le 3$, $C_{M,4} = \operatorname{span}\{x^k z^k\}$, $C_{M,5} = \operatorname{span}\{x^k, x^k y^k\}$, and $C_{M,6} = \operatorname{span}\{\partial_y(y^k(1-y)), \partial_z(z^k(1-z)), \partial_y(y^k(1-y))z^k\}$. To end, we lift the trace spaces into the element using polynomial functions thanks to the face parallelism.

The details of the proof are left out. □

This completes the proofs of the main results on the justification of the construction of the spaces $\delta\boldsymbol{V}_{\text{fillM}}$ in Theorem 2.3 to Theorem 2.15.

5. Concluding remarks

We have applied the theory of M-decomposition to systematically construct HDG and their sandwiching mixed methods on four basic polyhedral elements, namely, tetrahedra, pyramids, prisms, and hexahedra. For other flat-faced polyhedra, we want to let the computer to automatically find the spaces. How to do so consists the subject of ongoing research.

Appendix A. Proof of Proposition 4.3

In this appendix, we prove the characterization of divergence-free polynomial functions in Proposition 4.3. We use the following lemma to prove this result.

Lemma A.1. *Let $\boldsymbol{\phi}$ be a vector field in $\boldsymbol{\mathcal{P}}_k(\mathbb{R}^3)$, and let F be a hyperplane in $\mathbb{R}^3$ whose normal we denoted by $\boldsymbol{n}$. Suppose that the tangential trace of ϕ on the face F, $\gamma_t\,\boldsymbol{\phi}|_F := (\boldsymbol{\phi} - (\boldsymbol{\phi}\cdot\boldsymbol{n})\boldsymbol{n})|_F$, lies in $\boldsymbol{\mathcal{P}}_r(F)$. Then the following two statements are equivalent.*

(a) $(\nabla\times\boldsymbol{\phi})\cdot\boldsymbol{n}|_F = 0$.
(b) *There exist functions $\psi \in \mathcal{P}_{r+1}(\mathbb{R}^3)$ and $\eta \in \mathcal{P}_r(\mathbb{R}^3)$ such that*

$$\boldsymbol{\phi}|_F = (\nabla\,\psi + \eta\,\boldsymbol{n})|_F\,.$$

Proof. Let us just prove (a) $\Longrightarrow$ (b); the reverse implication is easy to see. First, we have

$$0 = (\nabla\times\boldsymbol{\phi})\cdot\boldsymbol{n}|_F = (\boldsymbol{n}\times\nabla)\cdot\boldsymbol{\phi}|_F = (\boldsymbol{n}\times\nabla)\cdot\big(\boldsymbol{\phi} - (\boldsymbol{\phi}\cdot\boldsymbol{n})\boldsymbol{n}\big)\big|_F\,,$$

where $(\boldsymbol{n}\times\nabla)\cdot$ is the *surface divergence* operator on F. This implies the existence of a surface polynomial function $\psi_F \in \mathcal{P}_{r+1}(F)$ such that

$$\big(\boldsymbol{\phi} - (\boldsymbol{\phi}\cdot\boldsymbol{n})\boldsymbol{n}\big)\big|_F = (\boldsymbol{n}\times\nabla\times\boldsymbol{n})\psi_F,$$

where $\boldsymbol{n}\times\nabla\times\boldsymbol{n}$ is the *surface curl* operator on F. Now, taking $\psi \in \mathcal{P}_{r+1}(\mathbb{R}^3)$ such that $\psi|_F = \psi_F$, we get

$$\big(\boldsymbol{\phi} - (\boldsymbol{\phi}\cdot\boldsymbol{n})\boldsymbol{n}\big)\big|_F = \big(\nabla\,\psi - (\nabla\,\psi\cdot\boldsymbol{n})\boldsymbol{n}\big)\big|_F$$

because $\nabla\,\psi = (\boldsymbol{n}\times\nabla\times\boldsymbol{n})\psi + (\nabla\,\psi\cdot\boldsymbol{n})\boldsymbol{n}$. Hence,

$$\boldsymbol{\phi}|_F = \big(\nabla\,\psi - (\nabla\,\psi\cdot\boldsymbol{n} - \boldsymbol{\phi}\cdot\boldsymbol{n})\boldsymbol{n}\big)\big|_F\,.$$

Since $\eta := \nabla\,\psi\cdot\boldsymbol{n} - \boldsymbol{\phi}\cdot\boldsymbol{n} \in \mathcal{P}_k(\mathbb{R}^3)$, the result follows. This completes the proof. $\square$

Now, let us prove Proposition 4.3.

Proof. We proceed by induction. If $i = 1$, we have $\boldsymbol{\Phi}_1 = \boldsymbol{\mathcal{P}}_{k+1}$, and the result follows from the exactness of the following polynomial sequence

$$\mathbb{R} \xrightarrow{i} \mathcal{P}_{k+2}(\mathbb{R}^3) \xrightarrow{\nabla} \boldsymbol{\mathcal{P}}_{k+1}(\mathbb{R}^3) \xrightarrow{\nabla\times} \boldsymbol{\mathcal{P}}_k(\mathbb{R}^3) \xrightarrow{\nabla\cdot} \mathcal{P}_{k-1}(\mathbb{R}^3) \xrightarrow{o} 0.$$

Now, assume that the result is true for $i = m$, for some integer $1 \le m \le n_f - 1$, we need to show that it also holds for $i = m+1$. Since it is trivial to check that $\nabla\times\boldsymbol{\Phi}_{m+1} \subset \boldsymbol{V}_{g_s,m+1}$, let us just prove the reverse inclusion. We proceed by the following four steps.

Step 1: *Notation*

Let us introduce the new bubble function $b_m^{\|} := \prod_{k\in S_m^{\|}} \lambda_k$, where

$$S_m^{\|} := \{1 \le k \le m-1 : \quad \boldsymbol{n}_k \times \boldsymbol{n}_m = \boldsymbol{0}\},$$
$$S_m := \{1 \le k \le m-1 : \quad \boldsymbol{n}_k \times \boldsymbol{n}_m \neq \boldsymbol{0}\}.$$

The relevance of this function lies in the following simple identities:

$$b_{i-1} = \prod_{k=1}^{i-1} \lambda_k \quad = \prod_{k\in S_i} \lambda_k \cdot \prod_{k\in S_i^{\|}} \lambda_k \quad = b_{i-1,i}\, b_i^{\|}, \tag{A.1a}$$

$$b_{i-1,j} = \prod_{\substack{k=1\\ \boldsymbol{n}_k\times\boldsymbol{n}_j\neq\boldsymbol{0}}}^{i-1} \lambda_k = \prod_{\substack{\boldsymbol{n}_k\times\boldsymbol{n}_j\neq\boldsymbol{0}\\ k\in S_i}} \lambda_k \cdot \prod_{\substack{\boldsymbol{n}_k\times\boldsymbol{n}_j\neq\boldsymbol{0}\\ k\in S_i^{\|}}} \lambda_k, = \prod_{\substack{\boldsymbol{n}_k\times\boldsymbol{n}_j\neq\boldsymbol{0}\\ k\in S_i}} \lambda_k \cdot b_i^{\|}, \quad \text{for } j \in S_i. \tag{A.1b}$$

Step 2: *Using the inductive hypothesis*

Let $\boldsymbol{v}$ be an element in $\boldsymbol{V}_{g_{s,m+1}}$. Since $\boldsymbol{v} \in \boldsymbol{V}_{g_{s,m}}$, by induction, there exists a function $\boldsymbol{\phi} \in \boldsymbol{\Phi}_m$ such that $\boldsymbol{v} = \nabla\times \boldsymbol{\phi}$, and so, we can write that

$$\boldsymbol{\phi} = b_{m-1}\boldsymbol{\xi} + \sum_{j\in Z_{m-1}} b_{m-1,j}\phi_j \boldsymbol{n}_j. \tag{A.2}$$

This implies that, on the face F_m, we have

$$\gamma_t\, \boldsymbol{\phi} = \boldsymbol{n}_m \times \left(b_{m-1}\boldsymbol{\xi} + \sum_{j\in Z_{m-1}\cap S_m} b_{m-1,j}\phi_j \boldsymbol{n}_j \right) \times \boldsymbol{n}_m$$

since $\boldsymbol{n}_j$ is parallel to $\boldsymbol{n}_m$ for $j \in S_m^{\|}$. Moreover, by the identities (A.1),

$$\gamma_t\, \boldsymbol{\phi} = \boldsymbol{n}_m \times \left(b_{m-1,m}\boldsymbol{\xi} + \sum_{j\in Z_{m-1}\cap S_m} \prod_{\substack{\boldsymbol{n}_k\times\boldsymbol{n}_j\neq\boldsymbol{0}\\ k\in S_m}} \lambda_k \cdot \phi_j \boldsymbol{n}_j \right) \times \boldsymbol{n}_m\, b_m^{\|},$$

and, since $b_m^{\|}$ is a constant over the face F_m, we conclude that

$$\gamma_t\, \boldsymbol{\phi}|_{F_m} \in \boldsymbol{\mathcal{P}}_{k+1-s}(F_m),$$

where s is the polynomial degree of $b_m^{\|}$. Finally, since $\nabla \times \boldsymbol{\phi} \cdot \boldsymbol{n}|_{F_m} = 0$, by Lemma A.1, we get that

$$\boldsymbol{\phi}|_{F_m} = (\nabla\, \psi + \eta\, \boldsymbol{n}_m)|_{F_m} \tag{A.3}$$

for some polynomials $\psi \in \mathcal{P}_{k+2-s}(K)$ and $\eta \in \mathcal{P}_{k+1}(K)$.

Step 3: *Transforming $\gamma_t\boldsymbol{\phi}|_{F_m}$ into zero*

Next, we show that we can modify $\boldsymbol{\phi} \in \boldsymbol{\Phi}_m$ so that we can set $\gamma_t\boldsymbol{\phi}|_{F_m} = 0$. We achieve this goal by exploring some properties of the function ψ introduced in (A.3). Note that, for $j \in S_m$, the hyperplane containing the face F_j, still denoted as F_j for simplicity, is not parallel to that containing F_m. Then the intersection of these two hyperplanes, $e_{j,m} := F_j \cap F_m$, exists. Dotting equation (A.3) with $\boldsymbol{n}_j \times \boldsymbol{n}_m$, and noting that, by the form of ϕ in (A.2), $\phi|_{e_{j,m}} = 0$, we get

$$0 = \nabla\psi \cdot (\boldsymbol{n}_j \times \boldsymbol{n}_m)|_{e_{j,m}}.$$

This implies that ψ is a constant on the line $e_{j,m}$ since the left hand side of the expression is nothing but the tangential derivative of ψ along line $e_{j,m}$. By fixing the value of ψ on one line to be zero, we would then get ψ is zero on all the lines $e_{j,m}$ where $j \in S_m$. For this to happen, we need at least two of those lines to intersect each other on the hyperplane F_j. This is guaranteed to be true because we are assuming that $\boldsymbol{n}_1 \cdot \boldsymbol{n}_2 \times \boldsymbol{n}_3 \neq 0$.

Now, since when a polynomial vanishes on a line, it should vanish on a face contains that line, there exists constants α_j and β_j, for $j \in S_m$, such that $\alpha_j\lambda_j + \beta_j\lambda_m$ divides ψ, where $\alpha_j\lambda_j + \beta_j\lambda_m = 0$ is the general formula of a face containing the line $e_{j,m}$. This implies the existence of constants $\{\alpha_j, \beta_j\}_{j\in S_m}$ such that

$$\psi = \xi \prod_{j\in S_m} (\alpha_j\lambda_j + \beta_j\lambda_m) \in \mathcal{P}_{k+2-s}$$

with some polynomial function ξ. Hence, we have

$$\psi = \xi_1\lambda_m + \xi_2 \prod_{j\in S_m} \lambda_j$$

for some polynomial functions ξ_1 and ξ_2 such that $\xi_1\lambda_m \in \mathcal{P}_{k+2-s}$ and $\xi_2 \prod_{j\in S_m} \lambda_j \in \mathcal{P}_{k+2-s}$. Now, if we set

$$\widetilde{\psi} := \xi_2 b_{m-1} = \xi_2 \prod_{j\in S_m} \lambda_j \cdot b_m^{\|},$$

we have $\widetilde{\psi} \in \mathcal{P}_{k+2}$ and, since $\lambda_m = 0$ on F_m and $b_m^{\|}$ is a constant therein, there exists a constant α such that

$$(\psi - \alpha\,\widetilde{\psi})\Big|_{F_m} = 0.$$

Hence

$$\gamma_t\boldsymbol{\phi}|_{F_m} = \gamma_t(\nabla\psi)|_{F_m} = \gamma_t(\alpha\nabla\widetilde{\psi})\Big|_{F_m}.$$

Now, set $\widetilde{\boldsymbol{\phi}} := \boldsymbol{\phi} - \alpha\nabla\widetilde{\psi}$, we have $\widetilde{\boldsymbol{\phi}} \in \boldsymbol{\Phi}_m$, and $\gamma_t\widetilde{\boldsymbol{\phi}}\Big|_{F_m} = 0$.

Without loss of generality, we assume that the given choice $\boldsymbol{\phi} \in \boldsymbol{\Phi}_m$ satisfies $\gamma_t\boldsymbol{\phi}|_{F_m} = 0$.

Step 4: *Conclusion*

We have $\gamma_t\boldsymbol{\phi}|_{F_j} = 0$ for $1 \le j \le m$ by the previous assumption on $\boldsymbol{\phi}$ and its formula in (A.2). This implies that for any $j \in S_m$, we have $\boldsymbol{\phi}|_{e_{j,m}} = 0$ where $e_{j,m} = F_j \cap F_m$.

From now on, we assume that $S_m^{\|} = \emptyset$, that is, $\boldsymbol{n}_j \times \boldsymbol{n}_m \neq 0$ for all $1 \le j \le m-1$. The proof for the case $S_m^{\|} \neq \emptyset$ is similar and omitted.

Fixing $j \in Z_{m-1} = Z_{m-1} \cap S_m$, we have that, for any index k in the set

$$\{k \in S_m : \; n_k \times n_j = 0\},$$

$\boldsymbol{\phi}|_{e_{k,m}} = b_{m-1,j}\phi_j\boldsymbol{n}_j|_{e_{k,m}} = 0$. This implies that $\phi_j = 0$ on $e_{k,m}$ for all k in $\{k \in S_m : \; n_k \times n_j = 0\}$. Hence,

$$\phi_j = \widetilde{\phi}_j\lambda_m + \xi_j \prod_{\substack{\boldsymbol{n}_k\times\boldsymbol{n}_j=0\\ k\in S_m}} \lambda_k$$

for some polynomial functions $\widetilde{\phi}_j$ and ξ_j. Hence,

$$\boldsymbol{\phi} = b_{m-1}\left(\boldsymbol{\xi} + \sum_{j\in Z_{m-1}} \xi_j \boldsymbol{n}_j\right) + \sum_{j\in Z_{m-1}} b_{m,j}\widetilde{\phi}_j \boldsymbol{n}_j$$

by the fact that, for all $j \in S_m$,

$$\begin{aligned} b_{m,j} &= b_{m-1,j}\lambda_m, \\ b_{m-1} &= b_{m-1,j} \prod_{\substack{\boldsymbol{n}_k\times\boldsymbol{n}_j=0 \\ k\in S_m}} \lambda_k. \end{aligned}$$

To simplify notation, we denote $\widetilde{\boldsymbol{\xi}} := \boldsymbol{\xi} + \sum_{j\in Z_{m-1}} \xi_j \boldsymbol{n}_j$. Then,

$$\begin{aligned} \boldsymbol{\phi} &= b_{m-1}\widetilde{\boldsymbol{\xi}} + \sum_{j\in Z_{m-1}} b_{m,j}\widetilde{\phi}_j \boldsymbol{n}_j \\ &= b_{m-1}\boldsymbol{n}_m \times \widetilde{\boldsymbol{\xi}} \times \boldsymbol{n}_m + b_{m-1}\left(\widetilde{\boldsymbol{\xi}}\cdot \boldsymbol{n}_m\right)\boldsymbol{n}_m + \sum_{j\in Z_{m-1}} b_{m,j}\widetilde{\phi}_j \boldsymbol{n}_j \end{aligned}$$

where the polynomial functions $\widetilde{\boldsymbol{\xi}}$ and ϕ_j guarantee that each term in the above right hand side belongs to $\mathcal{P}_{k+1}$. By the assumption that $S_m^{\|} = \emptyset$, we get $b_{m,m} = b_{m-1}$, hence

$$b_{m-1}\left(\widetilde{\boldsymbol{\xi}}\cdot \boldsymbol{n}_m\right)\boldsymbol{n}_m + \sum_{j\in Z_{m-1}} b_{m,j}\widetilde{\phi}_j \boldsymbol{n}_j = \sum_{j\in Z_m} b_{m,j}\widetilde{\phi}_j \boldsymbol{n}_j$$

with $\widetilde{\phi}_m := \widetilde{\boldsymbol{\xi}}\cdot\boldsymbol{n}_m$. Finally, since $\gamma_t\boldsymbol{\phi}|_{F_m} = 0$, we have λ_m divides $\boldsymbol{n}_m \times \widetilde{\boldsymbol{\xi}} \times \boldsymbol{n}_m$, which implies $\boldsymbol{\phi} \in \boldsymbol{\Phi}_{m+1}$. This completes the proof of $\boldsymbol{V}_{g_{s,m+1}} \subset \nabla\times \boldsymbol{\Phi}_{m+1}$.

It remains to compute the dimension of $\boldsymbol{V}_{g_{s,i}}$. We have

$$\dim \boldsymbol{V}_{g_{s,i}} = \dim \nabla\times\boldsymbol{\Phi}_i = \dim \boldsymbol{\Phi}_i - \dim \mathrm{Ker}_{\nabla\times}\boldsymbol{\Phi}_i.$$

We also have

$$\mathrm{Ker}_{\nabla\times}\boldsymbol{\Phi}_i = \nabla\left(b_{i-1}\mathcal{P}_{k+3-i}\right).$$

So,

$$\dim \boldsymbol{V}_{g_{s,i}} = \dim \boldsymbol{\Phi}_i - \dim \mathcal{P}_{k+3-i}(\mathbb{R}^3) + \delta_{1,i}.$$

This completes the proof. □

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