

# $L^2$ -STABILITY OF A FINITE ELEMENT – FINITE VOLUME DISCRETIZATION OF CONVECTION-DIFFUSION-REACTION EQUATIONS WITH NONHOMOGENEOUS MIXED BOUNDARY CONDITIONS

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**Abstract.** We consider a time-dependent and a steady linear convection-diffusion-reaction equation whose coefficients are nonconstant. Boundary conditions are mixed (Dirichlet and Robin–Neumann) and nonhomogeneous. Both the unsteady and the steady problem are approximately solved by a combined finite element – finite volume method: the diffusion term is discretized by Crouzeix–Raviart piecewise linear finite elements on a triangular grid, and the convection term by upwind barycentric finite volumes. In the unsteady case, the implicit Euler method is used as time discretization. This scheme is shown to be unconditionally  $L^2$ -stable, uniformly with respect to diffusion, except if the Robin–Neumann boundary condition is inhomogeneous and the convective velocity is tangential at some points of the Robin–Neumann boundary. In that case, a negative power of the diffusion coefficient arises. As is shown by a counterexample, this exception cannot be avoided.

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## 1. INTRODUCTION

We consider the convection-diffusion-reaction equation

$$\partial_t w - \operatorname{div}_x(a \nabla_x w) + \operatorname{div}_x(w b) + m w = g \quad \text{in } \Omega \times (0, T), \quad (1.1)$$

with the boundary conditions

$$w|_{\Gamma_D} \times (0, T) = f_D, \quad -\min\{b \cdot n, 0\} w + n \cdot a \nabla_x w = f_N \quad \text{on } \Gamma_N \times (0, T), \quad (1.2)$$

and with the initial conditions

$$w(x, 0) = w^{(0)}(x) \quad \text{for } x \in \Omega, \quad (1.3)$$

where  $\Omega \subset \mathbb{R}^2$  is a connected, bounded, open polygon with Lipschitz boundary  $\partial\Omega$ . In condition (1.2), this boundary is split into a part  $\Gamma_D$  where Dirichlet boundary conditions are prescribed, and a part  $\Gamma_N$  where

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Robin–Neumann conditions are imposed. We suppose that  $\Gamma_D$  and  $\Gamma_N$  are open in  $\partial\Omega$ ,  $\partial\Omega = \overline{\Gamma_D} \cup \overline{\Gamma_N}$ ,  $\Gamma_D \cap \Gamma_N = \emptyset$ ,  $\Gamma_D \neq \emptyset$ , and  $\Gamma_D$  and  $\Gamma_N$  each consist of the union of a finite number of open segments. The letter  $n$  stands for the outward unit normal to  $\Omega$ .

We assume  $T \in (0, \infty)$ , that is, we consider (1.1)–(1.3) for a finite time interval, although all our estimates yield upper bounds independent of  $T$ . The convective velocity  $b : [0, T] \mapsto H^1(\Omega)^2 \cap C^0(\overline{\Omega})^2$  is supposed to be bounded with respect to the norm of  $H^1(\Omega)$ , with  $|b|$  as a function on  $\overline{\Omega} \times [0, T]$  being bounded as well. We further require there is  $p \in (2, \infty]$  with

$$\sup_{0 < t < T} \|\operatorname{div}_x b(t)\|_{L^p(\Omega)} < \infty. \tag{1.4}$$

Moreover, we suppose there is some real  $\underline{\beta} > 0$  and some Lipschitz continuous function ([29], Sect. 1.2.7)  $\zeta : \overline{\Omega} \times [0, T] \mapsto \mathbb{R}$  such that  $\zeta(\cdot, t) \in W_{\text{loc}}^{1,1}(\Omega)$  for  $t \in (0, T)$  and

$$\operatorname{div}_x b(x, t)/2 + m(x, t) \geq 0 \quad \text{and} \quad -b(x, t) \cdot \nabla_x \zeta(x, t) \geq \underline{\beta} \quad \text{for } x \in \overline{\Omega}, t \in [0, T]. \tag{1.5}$$

Actually the Lipschitz continuity of  $\zeta$  implies that  $\zeta(\cdot, t) \in W_{\text{loc}}^{1,1}(\Omega)$  for  $t \in [0, T]$ . But since we do not know a direct reference for this result and do not want to enter into its proof here, we introduced it as an assumption. In the case of constant  $b \neq 0$ , a suitable function  $\zeta$  is given by  $\zeta(x, t) := -b \cdot x$ .

Concerning the matrix-valued function  $a$  (diffusion coefficient), we suppose that  $a(t) : \Omega \mapsto \mathbb{R}^{2 \times 2}$  is symmetric and measurable for  $t \in [0, T]$ , and there are constants  $\nu, \overline{\nu} \in (0, \infty)$  with

$$\xi^T \cdot a(x, t) \xi \geq \nu |\xi|^2, \quad |\xi^T \cdot a(x, t) \eta| \leq \overline{\nu} |\xi| |\eta| \quad \text{for } \xi, \eta \in \mathbb{R}^2, x \in \overline{\Omega}, t \in [0, T]. \tag{1.6}$$

We further assume that  $g : [0, T] \mapsto L^2(\Omega)$  is bounded,  $w^{(0)}$  belongs to  $H^1(\Omega)$ ,  $m$  is bounded as a function from  $[0, T]$  into  $L^p(\Omega)$ , and  $f_N : [0, T] \mapsto L^2(\Gamma_N)$  is bounded, too. Concerning  $f_D$ , we require there is a function  $\tilde{f}_D \in W^{1,1}(0, T, H^{1/2}(\partial\Omega))$  with  $\tilde{f}_D|_{\Gamma_D \times (0, T)} = f_D$ . This assumption that  $f_D(t)$  may be extended to a function in  $H^{1/2}(\partial\Omega)$  for any  $t \in (0, T)$  allows us to avoid technical difficulties when we further extend  $f_D$  to a function  $f : (0, T) \mapsto H^1(\Omega)$ ; compare ([32], Thm. 1.5.2.3) or ([36], p. 84/85) in this respect.

We will also consider the steady variant of problem (1.1)–(1.3), that is,

$$-\operatorname{div}(A \nabla W) + \operatorname{div}(W B) + M W = G \quad \text{in } \Omega, \tag{1.7}$$

$$W|_{\Gamma_D} = F_D, \quad -\min\{B \cdot n, 0\} W + n \cdot A \nabla W = F_N \quad \text{on } \Gamma_N, \tag{1.8}$$

under analogous assumptions on coefficients and data, that is,  $B \in H^1(\Omega)^2 \cap C^0(\overline{\Omega})^2$ ,  $\operatorname{div} B \in L^p(\Omega)$ ,  $M \in L^p(\Omega)$ ,

$$\operatorname{div} B/2 + M \geq 0 \quad \text{and} \quad -B \cdot \nabla \phi \geq \underline{\beta} \tag{1.9}$$

for some Lipschitz continuous function  $\phi \in W_{\text{loc}}^{1,1}(\Omega)$ ,  $A : \Omega \mapsto \mathbb{R}^{2 \times 2}$  symmetric and measurable, with

$$\xi^T \cdot A(x) \xi \geq \nu |\xi|^2, \quad |\xi^T \cdot A(x) \eta| \leq \overline{\nu} |\xi| |\eta| \quad \text{for } \xi, \eta \in \mathbb{R}^2, x \in \overline{\Omega}, \tag{1.10}$$

$G \in L^2(\Omega)$ ,  $F_N \in L^2(\Gamma_N)$  and  $F_D = \tilde{F}_D|_{\Gamma_D}$  for some function  $\tilde{F}_D \in H^{1/2}(\partial\Omega)$ . Concerning the assumption  $\phi \in W_{\text{loc}}^{1,1}(\Omega)$ , an analogous remark is valid as the one with respect to  $\zeta$  in the passage following (1.5).

A remark is perhaps in order with respect to conditions (1.5)<sub>2</sub> and (1.9)<sub>2</sub>. In ([15], Sect. 5), we presented a heuristic argument indicating that assumption (1.9)<sub>2</sub> is necessary for obtaining an  $L^2$ -stability estimate independent of  $\nu$ . The condition in (1.9)<sub>2</sub> may be interpreted geometrically in the sense that the convective velocity  $B$  does not exhibit closed curves or stationary points (points  $x \in \Omega$  with  $B(x) = 0$ ). In fact, it is shown in [16] that if  $B$  is smooth and presents these geometrical properties, there exists a function  $\phi$  with (1.9)<sub>2</sub>. This result was generalized to the case  $B \in W^{1,\infty}(\Omega)^2$  in [3]. The problem of existence of a function  $\phi$  with (1.9)<sub>2</sub> is closely

related to the question as to whether the vector field  $B$  admits a potential, that is, whether there is a function  $Z$  with  $\nabla Z = B$  (see ([31], Thm. I.2.9) for example). If such a potential exists and  $|B|$  is bounded away from zero, then there is a function  $\phi$  such that (1.9)<sub>2</sub> holds. These remarks carry over to (1.5)<sub>2</sub> in an obvious way.

Both problem (1.1)–(1.3) and (1.7), (1.8) are of particular interest in the convection-dominated regime, that is, if  $\nu \ll |b|$  in the evolutionary and  $\nu \ll |B|$  in the steady case, an interest that seems to be due to the belief that the preceding problems in the convection-dominated case show some affinity (although distant) with the Navier–Stokes system in the same regime. In this spirit, numerical schemes working well for that latter system are sometimes reduced to problem (1.1)–(1.3) or (1.7), (1.8) so that they may be accessible to theoretical studies regarding stability or accuracy.

In the work at hand, we consider a discretization of (1.1)–(1.3) and (1.7), (1.8), respectively, that is motivated in this way. This scheme may be characterized by the fact that the diffusion term in (1.1) and (1.7) is approximated by piecewise linear Crouzeix–Raviart finite elements, and the convective term by an upwind finite volume method based on barycentric finite volumes on a triangular grid. Choosing an explicit time discretization, Feistauer e.a. ([18], Sect. 7, [26], Chap. 4.4) tested this FE-FV method in the case of high-speed compressible Navier–Stokes flows in complex geometries and obtained very satisfactory results.

In [15], we applied this FE-FV method to problem (1.1)–(1.3), using the implicit Euler method as time discretization. Under the assumptions  $\Gamma_D = \partial\Omega$ ,  $f_D = 0$  (homogeneous Dirichlet boundary conditions),  $m = 0$  (no reaction term) and  $b$  and  $\zeta$  independent of  $t$  (autonomous differential equation), and with (1.5)<sub>1</sub> replaced by the equation  $\operatorname{div}_x b = 0$  (solenoidal convective velocity), we showed for a shape-regular grid (minimum angle condition) that the approximate solution provided by this approach may be estimated in the  $L^\infty(L^2)$ -norm against the data, with the constant in this estimate being independent of the diffusion parameter  $\nu$ , and depending polynomially on  $\underline{\beta}^{-1}$ ,  $\|b\|_{1,2}$  and on pointwise upper bounds of  $\zeta$  and  $|\nabla\zeta|$ . An analogous result was established with respect to problem (1.7), (1.8). The results in [15] improve an earlier theory in [14], where the case of constant  $b$  was considered under a restrictive assumption ([14], (3.9)) on the grid.

In the work at hand, we extend the theory from [15] to the more general framework set up above. The elliptic operator considered in [15] consists of the Laplace operator multiplied by a constant diffusion coefficient  $\nu$ . The role of this coefficient will be played here by the ellipticity constant  $\nu$  of the matrices  $a$  and  $A$ , respectively; see (1.6) and (1.10). The condition  $\operatorname{div}_x b(\cdot, t) = 0 = m$  in [15] is replaced by the condition  $\operatorname{div}_x b/2 + m \geq 0$  (see (1.5)<sub>1</sub>; in the steady case:  $\operatorname{div} B/2 + M \geq 0$  instead of  $\operatorname{div} B=0=M$ ; see (1.9)<sub>1</sub>). But we need an additional assumption in this context: in the unsteady case, the maximal diameter of the triangles of the grid must be small with respect to the quantities  $\underline{\beta}$  and  $\sup_{0 < t < T} \|\operatorname{div}_x b(t)\|_{L^p(\Omega)}$ , and with respect to an upper bound on  $\zeta$  and the Lipschitz constant of  $\zeta$ . This situation is expressed by the requirement  $h \leq h_0$  in Theorem 2.14. A corresponding condition is imposed in the steady case.

A surprising feature appears in the context of the inhomogeneous boundary conditions in (1.2) and (1.8). If the Robin–Neumann boundary data are non-vanishing and the convective velocity does not keep a minimum angle with respect to  $\Gamma_N$ , at least in a suitable averaged sense, then a factor  $\nu^{-K}$  with  $K > 1/4$  arises in our stability estimates (2.17) (evolutionary case) and (2.20) (steady case); also see the remarks following Theorem 2.14. The appearance of this factor is not due to our method of proof. Indeed, a counterexample in Section 5 (Thm. 5.1) shows that the factor in question cannot be removed. Since it refers to the continuous problem (1.7), (1.8), this counterexample implies that stability estimates analogous to ours for any discretization of (1.7), (1.8) necessarily involve such a factor in the circumstances just described, if the constants in these estimates are independent of the mesh size. Thus the interest of our theory extends beyond FE-FV schemes for (1.7), (1.8).

Except for these special features related to inhomogeneous Robin–Neumann boundary data, the stability estimates from [15] carry over to the more general situation considered in the work at hand. More specifically, it will turn out the  $L^\infty(L^2)$ -norm and  $\nu^{1/2}$  times the  $L^2(H^1)$ -norm of the solution to the discrete evolutionary problem (2.11), (2.12) are bounded by a product involving certain norms of the data and of  $\zeta$ , the factor  $1 + \sup_{0 < t < T} \|b(t)\|_{H^1(\Omega)}$ , the quantities  $\bar{\nu}/\nu$ ,  $\bar{\nu}/\nu^{1/2}$  and  $\epsilon^{1/2}$ , with  $\epsilon$  from the condition on the minimum angle between the convective velocity and  $\Gamma_N$ , as well as a constant independent of  $\nu$ . This constant depends on the minimum angle of our grid, and polynomially on  $\underline{\beta}^{-1}$ ,  $(\operatorname{diam} \Omega)^{1/2}$  and an upper bound of  $\zeta$  and the Lipschitz

constant of  $\zeta$ . Moreover, there are some parameters entering linearly into this constant, namely the constants from certain Sobolev, interpolation and trace estimates on  $\Omega$ . A precise list of these linear dependencies is given in Section 2 (see the passage preceding (2.4)). The factors  $\bar{\nu}/\nu$  and  $\bar{\nu}/\nu^{1/2}$  do not reduce the quality of our estimates because if the elliptic operator in (1.1) reduces to  $-\nu \Delta_x$  (the benchmark case), we may take  $\bar{\nu} = \nu$ , so that  $\bar{\nu}/\nu = 1$  and  $\bar{\nu}/\nu^{1/2} = \nu^{1/2}$ , with positive powers of  $\nu$  being irrelevant in the critical case  $\nu \leq 1$ . Analogous remarks are valid in the stationary case. For a detailed statement of our results, we refer to Theorem 2.14 (evolutionary problem) and 2.15 (steady case). Our results should be expected to hold in the 3D case as well, although their proof would become more technical.

When we adapted the discretization from [15] to the inhomogeneous boundary conditions considered in the work at hand, it was not so clear how to take account of the Robin–Neumann condition in (1.2) and (1.8). We opted for a boundary term in our discrete convection operator which is easy to implement and fits in well with our stability proof. Of course, one might ask whether this term degrades accuracy. These doubts, however, are lifted by a companion paper [12], where we show that optimal error estimates hold with respect to the  $L^2(H^1)$ - and the  $L^\infty(L^2)$ -norm (in the steady case: with respect to the  $H^1$ -norm). Thus our scheme does allow to maintain accuracy. Reference [12] generalizes earlier results from [13], where the case  $b$  constant,  $m = 0$ ,  $\Gamma_D = \partial\Omega$ ,  $f_D = 0$ ,  $a = -\nu(\delta_{jk})_{1 \leq j, k \leq 2}$  is considered.

There is a vast literature dealing with stability estimates of various discretizations of (1.1)–(1.3) or (1.7), (1.8). As examples, we cite the monographs ([42], Chaps. 8 and 12, [19], Sects. 5.2.3, 5.4.4), [44], and the articles [2–4, 8–10, 18, 20–22, 24, 25, 27, 28, 30, 33, 34, 38–41, 43, 48–50]. However, the stability bounds derived in these references depend on  $T$  or exponentially on some quantity related to  $b$  or  $B$ , or the assumption  $\Gamma_D = \partial\Omega$ ,  $f_D = 0$  is essential, or condition (1.5)<sub>1</sub> or (1.9)<sub>1</sub> is replaced by the stronger assumption  $\operatorname{div}_x b/2 + m \geq \delta$  and  $\operatorname{div} B/2 + M \geq \delta$ , respectively, for some  $\delta > 0$ , or special types of grids are used, like Shishkin meshes, or the diffusion parameter  $\nu$  enters in some way into the stability bound even if no inhomogeneous Robin–Neumann boundary conditions are imposed. Here we are able to avoid all these features.

Our study was inspired by Feistauer e.a. [2, 18], who considered a scalar time-dependent nonlinear conservation law with a diffusion term. Discretizing this equation by the combined FE-FV scheme described above, with a rather general numerical flux adapted to the nonlinearity, and with a semi-implicit Euler method as time discretization, they derived  $L^2(H^1)$ - and  $L^\infty(L^2)$ -error estimates. References [5, 24, 25, 28] present results analogous to those in [2, 18], but for a combined FE-FV method involving piecewise linear conforming finite elements and dual finite volumes (triangular finite volumes in the case of [5]). Similar  $L^2(H^1)$ - and  $L^\infty(L^2)$ -error estimates as in [18] are shown in [27, 50], but with respect to various discontinuous Galerkin schemes.

The article most closely related to our work here and in [15] is reference [3], in which Ayuso and Marini study stability and accuracy of various discontinuous Galerkin approximations of (1.7), (1.8), with discrete convection terms similar to ours. These authors base their theory on condition (1.9)<sub>2</sub> – to our knowledge the only ones doing so previous to [15] in the context of convection-diffusion equations. However, their stability constant depends exponentially on  $\max \phi - \min \phi$  ([3], Lem. 4.1), and they impose additional technical conditions on  $B$  and  $M$  ([3], (H2), (H3)) we do not need. But the theory in [3] is remarkable because its assumptions are sufficient for deriving not only  $\nu$ -independent stability estimates, but also error bounds which are uniform in  $\nu$ , with the usual caveat that the bound in questions also depends on certain norms of the exact solution, and may thus depend on  $\nu$  implicitly.

## 2. NOTATION. FE-FV DISCRETIZATION OF (1.1)–(1.3) AND (1.7), (1.8), RESPECTIVELY. STATEMENT OF MAIN RESULTS.

For any function  $\psi : A \mapsto \mathbb{R}$  with domain  $A \neq \emptyset$ , we put  $|\psi|_\infty := \sup\{|\psi(x)| : x \in A\}$ . Define  $\hat{K} := \{x \in (0, 1)^2 : x_1 + x_2 < 1\}$ . The Euclidean norm of  $\mathbb{R}^2$  is designated by  $|\cdot|$ . For  $A \subset \mathbb{R}^2$ , we set  $\operatorname{diam}(A) := \sup\{|x - y| : x, y \in A\}$ , and denote by  $P_k(A)$  the space of all polynomials on  $A$  of degree at most  $k$ , where  $k \in \mathbb{N}$ . If  $A$  is measurable, we write  $|A|$  for the measure of  $A$ . The usual Lebesgue space on  $A$  with exponent  $q \in [1, \infty]$  is designated by  $L^q(A)$ , and its usual norm by  $\|\cdot\|_{L^q(A)}$ . It will be convenient to use the notation  $\|\cdot\|_{L^2(A)}$  also for the  $L^2$ -norm

of (vector-valued) functions  $v = (v_1, v_2) \in L^2(A)^2$ , so  $\|v\|_{L^2(A)} := (\|v_1\|_{L^2(A)}^2 + \|v_2\|_{L^2(A)}^2)^{1/2}$  in that case. Put  $(u, v)_{L^2(A)} := \int_A u v dx$  for  $u, v \in L^2(A)$  and  $(u, v)_{L^2(A)^2} := \int_A u \cdot v dx$  for  $u, v \in L^2(A)^2$  ( $L^2$ -inner product). We define a norm  $\|\cdot\|_{L^2(S)}$  for the Lebesgue space  $L^2(S)$  on measurable subsets  $S$  of the manifold  $\partial\Omega$  by setting  $\|h\|_{L^2(S)} := \int_S h^2 do_x$  for  $h \in L^2(S)$ . This means we choose a norm which is independent of the choice of local charts. Let  $U \subset \mathbb{R}^2$  be an open set, and let  $q \in [1, \infty]$ ,  $k \in \mathbb{N}$ . Then the term  $W^{k,q}(U)$  stands for the standard Sobolev space of order  $k$  and of exponent  $q$ , and the term  $\|\cdot\|_{W^{k,q}(U)}$  for its usual norm, that is,  $\|v\|_{W^{k,q}(U)} := (\sum_{\alpha \in \mathbb{N}_0^2, \alpha_1 + \alpha_2 \leq k} \|\partial^\alpha v\|_q^q)^{1/q}$  for  $v \in W^{k,q}(U)$ . We will use the notation  $H^k(U)$  instead of  $W^{k,2}(U)$ , and  $\|\cdot\|_{H^k(U)}$  instead of  $\|\cdot\|_{W^{k,q}(U)}$ . The symbol  $W_{\text{loc}}^{1,1}(U)$  stands for the set of all functions  $v : U \mapsto \mathbb{R}$  such that  $v|_V \in W^{1,1}(V)$  for any open bounded set  $V \subset \mathbb{R}^2$  with  $\bar{V} \subset U$ . For  $s \in (0, 1)$ , the Sobolev space  $H^s(U)$  and its norm  $\|\cdot\|_{H^s(U)}$  is to be defined as in ([29], Sect. 6.8.2). For the definition of the space  $H^s(\partial\Omega)$  and its norm  $\|\cdot\|_{H^s(\partial\Omega)}$ , we refer to ([29], Sect. 6.8.6).

Let  $a, b \in \mathbb{R}$  with  $a < b$ ,  $p \in [1, \infty]$ , and  $H$  a Hilbert space. Then we will use the Lebesgue space  $L^p(a, b, H)$  and its norm  $\|\cdot\|_{L^p(a,b,H)}$ , defined as in ([52], Def. 24.4). The space  $W^{1,1}(a, b, H)$  is to consist of all functions  $v \in L^1(a, b, H)$  such that the distributional derivative  $v'$  of  $v$  ([52], Def. 25.2) is represented by a function, also denoted by  $v'$ , belonging to  $L^1(a, b, H)$ .

If  $I \subset \mathbb{R}$  and  $u : U \times I \mapsto \mathbb{R}$  is a function with suitable smoothness, the index  $x$  in the expressions  $\text{div}_x u$ ,  $\nabla_x u$  and  $\Delta_x u$  means that the differential operators in question only act on the variable  $x \in U$ . Otherwise the operators  $\text{div}$ ,  $\nabla$  and  $\Delta$  are used without index.

We recall that the bounded, open polygon  $\Omega \subset \mathbb{R}^2$  with Lipschitz boundary  $\partial\Omega$  was introduced in Section 1, as were the sets  $\Gamma_D$ ,  $\Gamma_N$ , the positive reals  $T$ ,  $p$ ,  $\underline{\beta}$ ,  $\nu$ ,  $\bar{\nu}$ , as well as the functions  $n$ ,  $a$ ,  $b$ ,  $m$ ,  $g$ ,  $f_D$ ,  $\tilde{f}_D$ ,  $f_N$ ,  $w^{(0)}$ ,  $\zeta$ ,  $A$ ,  $B$ ,  $M$ ,  $G$ ,  $F_D$ ,  $F_N$  and  $\phi$ . These sets, real numbers and functions will be kept fixed throughout.

By adding a suitable constant to  $\zeta$  and  $\phi$ , we may suppose without loss of generality that  $\zeta(x, t) \geq \text{diam } \Omega$  and  $\phi(x) \geq \text{diam } \Omega$  ( $x \in \bar{\Omega}$ ,  $t \in [0, T]$ ). For example, if  $b \neq 0$  is constant, we may choose  $\zeta(x, t) := 2 \text{diam } \Omega - |b|^{-1} b \cdot (x - x_0)$ , where  $x_0$  is some arbitrary but fixed point in  $\Omega$ . Thus, since  $\zeta$  and  $\phi$  are Lipschitz continuous, there is  $\varphi_1 > 0$  with

$$\begin{aligned} |\zeta(x, t) - \zeta(x', t')| &\leq \varphi_1 (|x - x'| + |t - t'|), \quad \text{diam } \Omega \leq \zeta(x, t) \leq \varphi_1, \\ |\phi(x) - \phi(x')| &\leq \varphi_1 |x - x'|, \quad \text{diam } \Omega \leq \phi(x) \leq \varphi_1 \quad \text{for } x, x' \in \bar{\Omega}, t, t' \in [0, T]. \end{aligned} \quad (2.1)$$

The Lipschitz continuity of  $\zeta$  further implies that  $\zeta \in C^0(\bar{\Omega} \times [0, T])$ , as well as

**Lemma 2.1.** *For  $x \in \bar{\Omega}$ , the function  $\zeta(x, \cdot)$  is differentiable at a. e.  $t \in (0, T)$ . Put  $\partial_3 \zeta(x, t) := \limsup_{h \downarrow 0} (\zeta(x, t+h) - \zeta(x, t))/h$  for  $x \in \bar{\Omega}$ ,  $t \in (0, T)$ . Then  $\partial_3 \zeta$  is bounded, in particular  $\partial_3 \zeta \in L^1(0, T, L^\infty(\Omega))$ , and  $\zeta(x, b) - \zeta(x, a) = \int_a^b \partial_3 \zeta(x, t) dt$  for  $a, b \in [0, T]$ .*

For a proof of this lemma, we refer to ([45], Problem 5.16a, pp. 104–108).

A technical problem we will have to take into account in the following is that no trace estimate of the form  $\|v|_{\partial\Omega}\|_{L^2(\Omega)} \leq C \|v\|_{H^{1/2}(\partial\Omega)}$  holds for  $v \in H^{1/2}(\Omega)$ . In view of this situation, we fix a parameter  $\delta \in (0, 1/2)$ , and we will use the ensuing trace theorem, for which we refer to [35].

**Theorem 2.2.** *There is a constant  $C_1(\delta) > 0$  such that  $\|v|_{\partial\Omega}\|_{H^\delta(\partial\Omega)} \leq C_1(\delta) \|v\|_{H^{\delta+1/2}(\Omega)}$  for  $v \in H^1(\Omega)$ .*

Moreover we will need a standard interpolation inequality for fractional order Sobolev norms. For the convenience of the reader, we state this inequality in

**Theorem 2.3.** *There is a constant  $C_2(\delta) > 0$  such that  $\|v\|_{H^{1/2+\delta}(\Omega)} \leq C_2(\delta) \|v\|_{H^1(\Omega)}^{1/2+\delta} \|v\|_{L^2(\Omega)}^{1/2-\delta}$  for  $v \in H^1(\Omega)$ .*

*Proof.* The theorem follows from ([47], inequality (1.3.3.5)) with  $A_0 = L^2(\Omega)$ ,  $A_1 = H^1(\Omega)$ ,  $\theta = 1/2 + \delta$ , in view of ([47], Rem. 4.4.1.2, 4.4.2.2, Def. 4.2.3, 4.2.1.1, Thm. 4.3.1.2, Eq. (2.4.2.16), Thm. 2.3.3 (b)).  $\square$

We will refer to the following result on Bochner integrals.

**Theorem 2.4** ([46], Lem. 3.1.1). *Let  $H$  be a Hilbert space,  $a, b \in \mathbb{R}$  with  $a < b$ , and  $v \in W^{1,1}(a, b, H)$ . Then, possibly after a modification on set with measure zero in  $[a, b]$ , the function  $v$  belongs to  $C^0([a, b], H)$ , and  $v(t) = \int_s^t v'(r) dr - v(s)$  for  $r, s \in [a, b]$ , where the preceding integral is to be understood as a Bochner integral in  $H$ .*

Due to Theorem 2.4, we may suppose without loss of generality that  $\tilde{f}_D$  belongs to the space  $C^0([0, T], H^{1/2}(\partial\Omega))$  as well as to  $W^{1,1}(0, T, H^{1/2}(\partial\Omega))$ .

Let  $E : H^{1/2}(\partial\Omega) \mapsto H^1(\Omega)$  be a linear, bounded extension operator ([29], Thm. 6.9.2). This means in particular that  $E(b)|_{\partial\Omega} = b$  for  $b \in H^{1/2}(\partial\Omega)$ . We put

$$f(t) := E(\tilde{f}_D(t)) \quad \text{for } t \in (0, T), \tag{2.2}$$

so that  $f : (0, T) \mapsto H^1(\Omega)$  and  $f(t)|_{\partial\Omega} = \tilde{f}_D(t)$  for  $t \in (0, T)$ .

**Lemma 2.5.** *The function  $f$  belongs to  $W^{1,1}(0, T, H^1(\Omega))$  and to  $C^0([0, T], H^1(\Omega))$ , and  $f'(t) = E(\tilde{f}'_D(t))$  for  $t \in (0, T)$ .*

*Proof.* Since  $\tilde{f}_D$  belongs to  $W^{1,1}(0, T, H^{1/2}(\partial\Omega))$ , and by the choice of  $E$ , we have  $f(t) \in H^1(\Omega)$  and  $E(\tilde{f}'_D(t)) \in H^1(\Omega)$  for  $t \in (0, T)$ , as well as  $\int_0^T \tilde{f}_D(t) \varphi'(t) dt = -\int_0^T \tilde{f}'_D(t) \varphi(t) dt$  for  $\varphi \in C_0^\infty((0, T))$ , with the preceding integrals being Bochner integrals in  $H^{1/2}(\partial\Omega)$ . But  $E : H^{1/2}(\partial\Omega) \mapsto H^1(\Omega)$  is linear and bounded, so a well-known result on Bochner integrals allows to deduce from the preceding equation that  $\int_0^T E(\tilde{f}'_D(t)) \varphi'(t) dt = -\int_0^T E(\tilde{f}'_D(t)) \varphi(t) dt$  for  $\varphi$  as before, where these integrals are to be understood as Bochner integrals in  $H^1(\Omega)$ .  $\square$

Let  $\sigma_0 \in (0, 1)$ , and let  $\mathfrak{T}$  be a triangulation of  $\Omega$  with the following properties. The set  $\mathfrak{T}$  consists of a finite number of open triangles  $K \subset \mathbb{R}^2$  with  $\overline{\Omega} = \cup\{\overline{K} : K \in \mathfrak{T}\}$ . If  $K_1, K_2 \in \mathfrak{T}$  with  $\overline{K_1} \cap \overline{K_2} \neq \emptyset$  and  $K_1 \neq K_2$ , the set  $\overline{K_1} \cap \overline{K_2}$  is either a common vertex or a common side of  $K_1$  and  $K_2$ . The relation

$$B_{\sigma_0 h_K}(x) \subset K \quad \text{holds for any } K \in \mathfrak{T} \text{ and some } x \in K, \tag{2.3}$$

where  $B_r(x) := \{y \in \mathbb{R}^2 : |x - y| < r\}$  for  $r > 0$ ,  $x \in \mathbb{R}^2$ , and  $h_K := \text{diam } K$  for  $K \in \mathfrak{T}$ . The parameter  $\sigma_0$  will be the only grid-related quantity entering into the constants in our estimates. In other words, we only impose a minimum angle condition on our grid (shape regularity). Set  $h := \max\{h_K : K \in \mathfrak{T}\}$ .

We write  $c$  for constants that are numerical or only depend on  $\sigma_0$ , and  $c(\gamma)$  if they are influenced by an additional parameter  $\gamma > 0$ . The symbol  $\mathfrak{C}$  stands for constants that may depend on  $\sigma_0$  and polynomially on  $\underline{\beta}^{-1}$ ,  $\varphi_1$  and  $(\text{diam } \Omega)^{1/2}$ , as well as linearly on the constant  $C_1(\delta)$  from the trace estimate in Theorem 2.2, on the constant  $C_2(\delta)$  from the interpolation inequality in Theorem 2.3, on the operator norm of the extension operator  $E$ , and on the constant from the Sobolev imbedding of  $H^1(\Omega)$  into  $L^{(1/2-1/p)^{-1}}(\Omega)$ . We recall that the parameter  $p$  was introduced in (1.4),  $\varphi_1$  in (2.1), and  $\underline{\beta}$  in (1.5) and (1.9). As a consequence of (2.3), we have

$$h_K^2 \leq c|K| \quad \text{for } K \in \mathfrak{T}. \tag{2.4}$$

Let  $\mathfrak{S}$  be the set of the sides – without their endpoints – of the triangles  $K \in \mathfrak{T}$ . Put  $J := \{1, \dots, \#\mathfrak{S}\}$ , where  $\#\mathfrak{S}$  denotes the number of elements of  $\mathfrak{S}$ . Let  $(S_i)_{i \in J}$  be an enumeration of  $\mathfrak{S}$ . We write  $Q_i$  for the midpoint of  $S_i$ , and  $l_i$  for the length of  $S_i$  ( $i \in J$ ). Put  $J^\circ := \{i \in J : Q_i \in \Omega\}$ .

Since it is Lipschitz bounded, the set  $\Omega$  is locally located on one side of  $\partial\Omega$ , so by our assumptions on  $\mathfrak{T}$ , we have  $S_i \subset \Omega$  for  $i \in J^\circ$ , and  $S_i \subset \partial\Omega$  for  $i \in J \setminus J^\circ$ . It further follows that for  $i \in J^\circ$ , there are exactly two triangles  $K \in \mathfrak{T}$ , which we denote by  $K_i^1$  and  $K_i^2$ , such that  $S_i \subset \overline{K_i^1} \cap \overline{K_i^2}$ , and for  $i \in J \setminus J^\circ$ , there is a single triangle  $K \in \mathfrak{T}$ , denoted by  $K_i$ , with  $S_i \subset \overline{K_i}$ . It will be convenient to put  $K_i^1 := K_i^2 := K_i$  for  $i \in J \setminus J^\circ$ .

We further put  $J_D := \{i \in J : Q_i \in \Gamma_D\}$ ,  $J_N := \{i \in J : Q_i \in \Gamma_N\}$ , and we require – this is another assumption on  $\mathfrak{T}$  – that either  $S_i \subset \Gamma_D$  or  $S_i \subset \Gamma_N$  for any  $i \in J \setminus J^\circ$ . Thus each boundary edge is a subset of either  $\Gamma_D$  or  $\Gamma_N$ , and  $J \setminus J^\circ = J_D \cup J_N$ .

Next we introduce a barycentric mesh  $(D_i)_{i \in J}$  on the triangular grid  $\mathfrak{T}$ . If  $i \in J^\circ$ , we join the barycenters of  $K_i^1$  and  $K_i^2$  with the endpoints of  $S_i$ . The open quadrilateral containing  $S_i$  and obtained in this way is denoted by  $D_i$ . If  $i \in J \setminus J^\circ$  (hence  $Q_i \in \partial\Omega$ ), let  $D_i$  be the open triangle whose sides are the edge  $S_i$  and the segments joining the endpoints of  $S_i$  with the barycenter of the (unique) triangle  $K \in \mathfrak{T}$  with  $S_i \subset \overline{K}$ . (This latter triangle was denoted by  $K_i$ .) If  $i, j \in J$  with  $i \neq j$  are such that the intersection  $\overline{D_i} \cap \overline{D_j}$  contains more than one point, this intersection is a common side of  $D_i$  and  $D_j$ . In this case, the sets  $D_i$  and  $D_j$  are called “adjacent”, and their common side – without its endpoints – is denoted  $\Gamma_{ij}$ . For  $i \in J$ , we set

$$s(i) := \{j \in J \setminus \{i\} : D_i \text{ and } D_j \text{ are adjacent.}\}.$$

If  $i \in J$  and  $j \in s(i)$ , let  $n_{ij}$  denote the outward unit normal to  $D_i$  on  $\Gamma_{ij}$ , so that  $n_{ij}$  points from  $D_i$  into  $D_j$ .

We find with (2.4) and the relation  $|K_i^l \cap D_i| = |K_i^l|/3$  that  $h_{K_i^l} \leq c|K_i^l|^{1/2} \leq c|K_i^l \cap D_i|^{1/2} \leq c|D_i|^{1/2}$  for  $i \in J$ ,  $l \in \{1, 2\}$ . Since  $1 - 2/p \geq 0$ , we thus see there is a constant  $c_0 > 0$ , only depending on  $\sigma_0$ , such that

$$\max\{h_{K_i^l} : l \in \{1, 2\}\} |D_i|^{1/p'} \leq c_0 |D_i| h^{1-2/p} \quad \text{for } i \in J, \tag{2.5}$$

where  $p' := (1 - 1/p)^{-1}$ , with  $p$  from (1.4). We introduce two finite element spaces by setting

$$\begin{aligned} X_h &:= \{v \in L^2(\Omega) : v|_K \in P_1(K) \text{ for } K \in \mathfrak{T}, \ v \text{ continuous at } Q_i \text{ for } i \in J\}, \\ V_h &:= \{v_h \in X_h : v_h(Q_i) = 0 \text{ for } i \in J_D\}. \end{aligned}$$

The spaces  $X_h$  and  $V_h$  are nonconforming finite element spaces of piecewise linear functions based on the Crouzeix–Raviart finite element. For  $i \in J$ , let  $\omega_i$  be the function from  $X_h$  that is uniquely determined by the requirement that  $\omega_i(Q_j) = \delta_{ij}$  for  $j \in J$ . The family  $(\omega_i)_{i \in J}$  is a basis of  $X_h$ , and

$$v_h = \sum_{i \in J} v_h(Q_i) \omega_i \quad \text{for } v_h \in X_h. \tag{2.6}$$

For  $v_h \in X_h \oplus H^1(\Omega)$ , we define the function  $\nabla_h v_h$  on  $\Omega$  by setting  $(\nabla_h v_h)|_K := \nabla(v_h|_K)$  for  $K \in \mathfrak{T}$ . A discrete Poincaré’s inequality is valid on  $V_h$ :

**Theorem 2.6** ([7], Thm. 4.1, Rem. 4.4). *For any  $v_h \in V_h$ , the relation  $\|v_h\|_{L^2(\Omega)} \leq c \|\nabla_h v_h\|_{L^2(\Omega)}$  holds.*

Similar inequalities were discussed in [17, 37, 51]. The one in [7] fits our situation best because in that latter reference, a Dirichlet boundary condition needs to hold only on part of the boundary of the domain in question. From ([2], (3.30)) and (2.6), we obtain a formula for the  $L^2$ -scalar product of  $v_h, w_h \in X_h$ , that is,

$$(v_h, w_h)_{L^2(\Omega)} = \sum_{i \in J} v_h(Q_i) w_h(Q_i) |D_i|, \quad \text{for } v_h, w_h \in X_h. \tag{2.7}$$

Next we state a Sobolev inequality on the triangles  $K \in \mathfrak{T}$ .

**Lemma 2.7.** *Let  $r \in (1, \infty)$ . Then the inequality*

$$\|v\|_{L^r(K)} \leq c(r) (h_K^{2/r-1} \|v\|_{L^2(K)} + h_K^{2/r} \|\nabla v\|_{L^2(K)})$$

*holds for  $v \in H^1(K)$  and  $K \in \mathfrak{T}$ .*

*Proof.* Let  $K \in \mathfrak{T}$ , and let  $c_1^{(1)}, c^{(2)}, c^{(3)} \in \mathbb{R}^2$  be the vertex points of  $K$ . Put  $C := (c^{(1)} - c^{(3)}, c^{(2)} - c^{(3)}) \in \mathbb{R}^{2 \times 2}$ , where  $c^{(1)} - c^{(3)}$  and  $c^{(2)} - c^{(3)}$  are to be considered as columns. Put  $T(x) := C \cdot x + c^{(3)}$  for  $x \in \mathbb{R}^2$ . Then  $\int_K v \, dx = 2|K| \int_{\hat{K}} v(T(x)) \, dx$  for  $v \in L^1(K)$ , with the reference triangle  $\hat{K}$  introduced at the beginning of this section. This equation, the Sobolev inequality  $\|\hat{v}\|_{L^r(\hat{K})} \leq c(r) \|\hat{v}\|_{H^1(\hat{K})}$  for  $\hat{v} \in H^1(\hat{K})$ , and inequality (2.4) imply Lemma 2.7.  $\square$

We further need a trace inequality on triangles, and an inverse inequality.

**Lemma 2.8.** *The estimate  $\|v|_{\partial K}\|_{L^2(\partial K)} \leq c(h_K^{-1/2} \|v\|_{L^2(K)} + h_K^{1/2} \|\nabla v\|_{L^2(K)})$  holds for  $K \in \mathfrak{T}$ ,  $v \in H^1(K)$ .*

*Proof.* Well known: by using a scaling argument as in the proof of Lemma 2.7, we may reduce Lemma 2.8 to a trace estimate on the reference triangle  $\hat{K}$ ; see ([12], proof of Lem. 2.1) for details.  $\square$

**Lemma 2.9** ([6], Lem. 4.5.3). *We have  $h_K \|\nabla v\|_{L^2(K)} \leq c \|v\|_{L^2(K)}$  for  $v \in P_2(K)$  and  $K \in \mathfrak{T}$ .*

Put  $W_h := \{v \in C^0(\bar{\Omega}) : v|_K \in P_2(K) \text{ for } K \in \mathfrak{T}\}$  ( $P_2$  Lagrangian finite element space). Note that  $W_h \subset H^1(\Omega)$ . Functions from  $X_h$  may be approximated by functions from  $W_h$  in the following way.

**Lemma 2.10.** *There is a linear operator  $\mathfrak{E}_h : X_h \mapsto W_h$  such that for  $v_h \in X_h$ ,*

$$\begin{aligned} \sum_{K \in \mathfrak{T}} h_K^{-1} \|(\mathfrak{E}_h(v_h) - v_h)|_K\|_{L^2(K)}^2 &\leq c \sum_{K \in \mathfrak{T}} h_K \|\nabla(v_h|_K)\|_{L^2(K)}^2, \\ \|\mathfrak{E}_h(v_h)\|_{H^1(\Omega)} &\leq c(\|v_h\|_{L^2(\Omega)} + \|\nabla_h v_h\|_{L^2(\Omega)}), \quad \|\mathfrak{E}_h(v_h)\|_{L^2(\Omega)} \leq c \|v_h\|_{L^2(\Omega)}. \end{aligned}$$

*Proof.* The second and third estimate in Lemma 2.10 hold according to ([7], Cor. 3.3). As for the first, we observe that the maximal number of triangles sharing a common vertex is bounded by some number  $n \in \mathbb{N}$  only depending on the constant  $c_0$  in (2.3), or in other words, on the minimum angle of the triangles  $K \in \mathfrak{T}$ . Also by (2.3), if  $K, L \in \mathfrak{T}$  share a common vertex or midpoint, then  $h_K \leq ch_L$ ; compare ([7], (3.8)). These observations and the estimate in ([7], (3.7)) imply the first inequality in Lemma 2.10.  $\square$

Let  $I_h : H^1(\Omega) \mapsto X_h$  be the interpolation operator introduced in ([23], Eq. (8.9.79)), that is,  $I_h(v) := \sum_{i \in J} l_i^{-1} \int_{S_i} v(x) \, do_x \omega_i$  for  $v \in H^1(\Omega)$ . Note that a function  $v \in H^1(\Omega)$  admits a trace on  $S_i$  for any  $i \in J$ , so the operator  $I_h$  is well defined. For  $\psi \in L^1(\Gamma_D)$ , we put  $I_{h,D}(\psi) := \sum_{i \in J_D} l_i^{-1} \int_{S_i} \psi(x) \, do_x \omega_i$ . This operator is well defined because  $S_i \subset \Gamma_D$  for  $i \in J_D$ . Note that since  $\omega_j(Q_i) = 0$  for  $i, j \in J$  with  $i \neq j$ , we have

$$I_h(v)(Q_i) = I_{h,D}(v|_{\Gamma_D})(Q_i) \quad \text{for } i \in J_D, v \in H^1(\Omega). \tag{2.8}$$

We will need the following interpolation property of  $I_h$ .

**Theorem 2.11.** *For  $v \in H^1(\Omega)$ ,  $K \in \mathfrak{T}$ , we have*

$$h_K^{-1} \|v - I_h(v)|_K\|_{L^2(K)} + \|\nabla(v - I_h(v)|_K)\|_{L^2(K)} \leq c \|\nabla v|_K\|_{L^2(K)},$$

*in particular  $\|I_h(v)\|_{L^2(\Omega)} + \|\nabla_h I_h(v)\|_{L^2(\Omega)} \leq \mathfrak{C} \|v\|_{H^1(\Omega)}$ .*

*Proof.* See ([23], Lem. 8.9.81) and its proof. Alternatively, this theorem may be deduced from ([11], Thm. 3.1.4); see ([12], proof of Thm. 2.2).  $\square$

**Corollary 2.12.** *Let  $r \in (1, \infty)$ ,  $v \in H^1(\Omega)$ . Then the inequality*

$$\|I_h(v) - v\|_{L^r(\Omega)} \leq c(r) (1 + \text{diam } \Omega)^2 \|\nabla v\|_{L^2(\Omega)}$$

*is valid.*



*Proof.* By Lemma 2.7, Theorems 2.11 and (2.4), we get

$$\begin{aligned} \|I_h(v) - v\|_{L^r(\Omega)} &= \left( \sum_{K \in \mathfrak{T}} \|I_h(v) - v|_K\|_{L^r(K)}^r \right)^{1/r} \\ &\leq c(r) \left[ \sum_{K \in \mathfrak{T}} \left( h_K^{2/r-1} \|I_h(v) - v|_K\|_{L^2(K)} + h_K^{2/r} \|\nabla(I_h(v) - v|_K)\|_{L^2(K)} \right)^r \right]^{1/r} \\ &\leq c(r) \left( \sum_{K \in \mathfrak{T}} (h_K^{2/r} \|\nabla v|_K\|_{L^2(K)})^r \right)^{1/r} \leq c(r) \|\nabla v\|_{L^2(\Omega)} \left( \sum_{K \in \mathfrak{T}} h_K^2 \right)^{1/r} \\ &\leq c(r) \|\nabla v\|_{L^2(\Omega)} \left( \sum_{K \in \mathfrak{T}} |K| \right)^{1/r} = c(r) \|\nabla v\|_{L^2(\Omega)} |\Omega|^{1/r} \leq c(r) (1 + \text{diam } \Omega)^2 \|\nabla v\|_{L^2(\Omega)}. \end{aligned}$$

□

It will be useful to introduce another interpolation operator besides  $I_h$ . In fact, for  $v \in L^2(\Omega)$  with  $v|_K \in C^0(K)$  for  $K \in \mathfrak{T}$ ,  $v$  continuous at  $Q_i$  for  $i \in J$ , we set  $\varrho_h(v) := \sum_{i \in J} v(Q_i) \omega_i$ . Abbreviate

$$\vartheta_{ij}^+(t) := \int_{\Gamma_{ij}} \max\{b(x, t) \cdot n_{ij}, 0\} \, do_x, \quad \bar{m}_i(t) := \int_{D_i} m(x, t) \, dx \tag{2.9}$$

for  $i \in J, j \in s(i), t \in (0, T)$ . We define a discrete convection term  $d_h$  by setting

$$\begin{aligned} d_h(t, v_h, w_h) &:= \sum_{i \in J} w_h(Q_i) \left( \sum_{j \in s(i)} (\vartheta_{ij}^+(t) v_h(Q_i) - \vartheta_{ji}^+(t) v_h(Q_j)) + \bar{m}_i(t) v_h(Q_i) \right) \\ &+ \sum_{i \in J_N} v_h(Q_i) w_h(Q_i) \int_{S_i} \max\{b(x, t) \cdot n(x), 0\} \, do_x \quad \text{for } v_h, w_h \in X_h, t \in (0, T). \end{aligned} \tag{2.10}$$

This definition means that our numerical flux is based on an upwind finite volume method on the barycentric grid  $(D_i)_{i \in J}$ .

In order discretize the time variable, we fix  $Z \in \mathbb{N}$  and choose  $t_1, \dots, t_Z \in (0, T)$  with  $t_1 < \dots < t_Z$ . Put  $t_0 := 0, t_{Z+1} := T, \tau_k := t_k - t_{k-1}$  for  $1 \leq k \leq Z + 1$ .

Now we are in a position to introduce the finite element – finite volume discretization of problem (1.1)–(1.3) we want to study in the work at hand. This problem consists in finding  $w_h^{(1)}, \dots, w_h^{(Z+1)} \in X_h$  with

$$\begin{aligned} \tau_{k+1}^{-1} (w_h^{(k+1)} - w_h^{(k)}, v_h)_{L^2(\Omega)} + \left( a(t_{k+1}) \nabla_h w_h^{(k+1)}, \nabla_h v_h \right)_{L^2(\Omega)} + d_h(t_{k+1}, w_h^{(k+1)}, v_h) \\ = (g(t_{k+1}), v_h)_{L^2(\Omega)} + \sum_{i \in J_N} v_h(Q_i) \int_{S_i} f_N(x, t_{k+1}) \, do_x \end{aligned} \tag{2.11}$$

for  $v_h \in V_h, k \in \{0, \dots, Z\}, w_h^{(0)} := I_h(w^{(0)})$ ,

$$w_h^{(k+1)}(Q_i) = I_{h,D}(f_D(t_{k+1}))(Q_i) \quad \text{for } i \in J_D, 0 \leq k \leq Z. \tag{2.12}$$

This scheme is implicit because both the diffusion and the convection term are discretized on the same time level. In (2.11), (2.12), our discrete problem is stated in a way which is suited for implementation. However,

for our theoretical studies, a variant of (2.11), (2.12) with homogeneous Dirichlet boundary conditions on  $\Gamma_D$  is more appropriate. To this end, we put

$$f_h(t) := I_h(f(t)) \quad \text{for } t \in (0, T), \tag{2.13}$$

with  $f$  from (2.2).

**Lemma 2.13.** *The relations  $f_h \in W^{1,1}(0, T, X_h)$  and  $f_h \in C^0([0, T], X_h)$  hold, where  $X_h$  is considered as a space equipped with the  $L^2$ -norm. We further have  $f_h'(t) = I_h(f'(t))$  for  $t \in (0, T)$ .*

*Proof.* By the second inequality in Theorem 2.11, the operator  $I_h : H^1(\Omega) \mapsto X_h$  is bounded. Thus Lemma 2.5 follows by a reasoning as in the proof of Lemma 2.5.  $\square$

The variant of (2.11), (2.12) we have in mind may now be stated as follows: Find  $u_h^{(1)}, \dots, u_h^{(Z+1)} \in V_h$  such that

$$\begin{aligned} & \tau_{k+1}^{-1} (u_h^{(k+1)} - u_h^{(k)}, v_h)_{L^2(\Omega)} + (a(t_{k+1}) \nabla_h u_h^{(k+1)}, \nabla_h v_h)_{L^2(\Omega)} + d_h(t_{k+1}, u_h^{(k+1)}, v_h) \\ & = (g(t_{k+1}), v_h)_{L^2(\Omega)} + \sum_{i \in J_N} v_h(Q_i) \int_{S_i} f_N(x, t_{k+1}) \, do_x - (a(t_{k+1}) \nabla_h f_h(t_{k+1}), \nabla_h v_h)_{L^2(\Omega)} \\ & \quad - d_h(t_{k+1}, f_h(t_{k+1}), v_h) - \tau_{k+1}^{-1} (f_h(t_{k+1}) - f_h(t_k), v_h)_{L^2(\Omega)}, \end{aligned} \tag{2.14}$$

with  $v_h \in V_h$ ,  $0 \leq k \leq Z$ ,  $u_h^{(0)} := I_h(w^{(0)} - f(0))$ . The relation  $u_h^{(1)}, \dots, u_h^{(Z+1)} \in V_h$  means that the functions  $u_h^{(1)}, \dots, u_h^{(Z+1)}$  satisfy homogeneous Dirichlet boundary conditions on  $\Gamma_D$ . Due to (2.8), problem (2.11), (2.12) on the one hand and (2.14) on the other are equivalent: if  $w_h^{(1)}, \dots, w_h^{(Z+1)} \in X_h$  and  $u_h^{(1)}, \dots, u_h^{(Z+1)} \in V_h$  with  $w_h^{(i)} = u_h^{(i)} + f_h(t_i)$  for  $1 \leq i \leq Z + 1$ , then the family  $(w_h^{(1)}, \dots, w_h^{(Z+1)})$  solves (2.11), (2.12) if and only if  $(u_h^{(1)}, \dots, u_h^{(Z+1)})$  is a solution to (2.14). We state our stability estimate for our finite element – finite volume discretization of (1.1)–(1.3) in terms of solutions to (2.11), (2.12). To this end, we introduce the abbreviations

$$\begin{aligned} \mathfrak{A}(v_h^{(1)}, \dots, v_h^{(Z+1)}) & := \left( \sum_{l=1}^{Z+1} \tau_l \|v_h^{(l)}\|_{L^2(\Omega)}^2 \right)^{1/2} + \max_{1 \leq l \leq Z+1} \|v_h^{(l)}\|_{L^2(\Omega)} \\ & \quad + \nu^{1/2} \left( \sum_{l=1}^{Z+1} \tau_l \|\nabla_h v_h^{(l)}\|_{L^2(\Omega)}^2 \right)^{1/2} + \left( \sum_{l=1}^{Z+1} \tau_l \sum_{i \in J_N} v_h^{(l)}(Q_i)^2 \int_{S_i} |b(x, t) \cdot n(x)| \, do_x \right)^{1/2} \end{aligned} \tag{2.15}$$

for  $v_h^{(1)}, \dots, v_h^{(Z+1)} \in X_h$ ,

$$h_0 := \min \left\{ \left( \frac{\beta}{\left[ 4 c_0 \varphi_1 \left( 1 + \sup_{0 < t < T} \|\operatorname{div}_x b(t)\|_{L^p(\Omega)} \right) \right]} \right)^{p/(p-2)}, (\operatorname{diam} \Omega) / \varphi_1 \right\}, \tag{2.16}$$

with  $c_0$  from (2.5),  $\beta$  from (1.5),  $\varphi_1$  from (2.1) and  $p$  from (1.4). Then our theory with respect to discrete solutions to (1.1)–(1.3) reads as follows.

**Theorem 2.14.** *Problem (2.11), (2.12) admits a unique solution  $w_h^{(1)}, \dots, w_h^{(Z+1)} \in X_h$ .*

Let  $\epsilon \in (0, \infty)$  and put  $J_N^{(\epsilon)} := \{i \in J_N : l_i^{-1} \int_{S_i} |b(x, t) \cdot n(x)| \, do_x \geq \epsilon \text{ for any } t \in [0, T]\}$ . Then, if  $h \leq h_0$ ,

$$\begin{aligned} \mathfrak{A}(w_h^{(1)}, \dots, w_h^{(Z+1)}) &\leq \mathfrak{C} \left( 1 + \bar{\nu}/\nu + \|\partial_3 \zeta\|_{L^1(0, T, L^\infty(\Omega))} \right) \left( 1 + \max_{1 \leq l \leq Z+1} \|b(t_l)\|_{H^1(\Omega)} \right) \\ &\left[ \left( \sum_{l=1}^{Z+1} \tau_l \|g(t_l)\|_{L^2(\Omega)}^2 \right)^{1/2} + \|w^{(0)}\|_{H^1(\Omega)} + \max_{0 \leq l \leq Z+1} \|\tilde{f}_D(t_l)\|_{H^{1/2}(\partial\Omega)} \right. \\ &+ \|\tilde{f}'_D\|_{L^1(0, T, H^{1/2}(\partial\Omega))} + \mathfrak{B}(\nu, \bar{\nu}, b, m) \left( \sum_{l=1}^{Z+1} \tau_l \|\tilde{f}_D(t_l)\|_{H^{1/2}(\partial\Omega)}^2 \right)^{1/2} \\ &\left. + \epsilon^{-1/2} \left( \sum_{l=1}^{Z+1} \tau_l \sum_{i \in J_N^{(\epsilon)}} \|f_N(t_l)|_{S_i}\|_{L^2(S_i)}^2 \right)^{1/2} + \nu^{-1/4-\delta/2} \left( \sum_{l=1}^{Z+1} \tau_l \sum_{i \in J_N \setminus J_N^{(\epsilon)}} \|f_N(t_l)|_{S_i}\|_{L^2(S_i)}^2 \right)^{1/2} \right], \end{aligned} \tag{2.17}$$

with  $\delta$  from Theorem 2.2,  $\nu, \bar{\nu}$  from (1.6),  $\zeta$  from (1.5),  $\partial_3 \zeta$  from Lemma 2.1, and with the term  $\mathfrak{B}(\nu, \bar{\nu}, b, m)$  defined by

$$\mathfrak{B}(\nu, \bar{\nu}, b, m) := \max_{1 \leq l \leq Z+1} (1 + |b(t_l)|_\infty + \|\operatorname{div}_x b(t_l) + m(t_l)\|_{L^p(\Omega)}) + \bar{\nu}/\nu^{1/2},$$

where  $p$  was introduced in (1.4).

The constant  $\mathfrak{C}$  in (2.17) is to be understood as a generic constant of the type introduced further above, depending on parameters as described there.

We draw the attention of the reader to the factor  $\nu^{-1/4-\delta/2}$ , which is a surprising feature of (2.17). This is the factor  $\nu^{-K}$  with  $K > 1/4$  mentioned in Section 1. Note that it comes to bear only if  $f_n \neq 0$  (non-vanishing Robin–Neumann boundary data) and if the index set  $J_N \setminus J_N^{(\epsilon)}$  is not empty. But an index  $i \in J$  belongs to  $J_N^{(\epsilon)}$  if, for example,  $|b(x, t) \cdot n(x)| \geq \epsilon$  for any  $x \in S_i$  and  $t \in [0, T]$ . Then  $\max_{x \in S_i} |b(x, t)| > 0$ ,  $0 < \epsilon / \max_{x \in S_i} |b(x, t)| \leq 1$  for  $t \in [0, T]$ , and the angle between  $b(y, t)$  and  $S_i$  is bounded from below by  $\pi/2 - \arccos(\epsilon / \max_{x \in S_i} |b(x, t)|)$ , for any  $y \in S_i$  and  $t \in [0, T]$ . Therefore the case  $J_N \setminus J_N^{(\epsilon)} \neq \emptyset$  for any  $\epsilon > 0$  may be interpreted in the sense that the convective velocity does not keep a minimum angle with  $\Gamma_N$ . On the other hand, if such an angle exists, it enters into the stability bound *via* the parameter  $\epsilon$ .

As concerns the steady problem (1.7), (1.8), we define a discrete convection term  $D_h(v_h, w_h)$  in an analogous way as in the evolutionary case. Then we consider an approximate solution  $W_h \in X_h$  of (1.7), (1.8) satisfying the equations

$$(A \nabla_h W_h, \nabla_h v_h)_{L^2(\Omega)} + D_h(W_h, v_h) = (G, v_h)_{L^2(\Omega)} + \sum_{i \in J_N} v_h(Q_i) \int_{S_i} F_N(x) \, do_x, \quad \text{for } v_h \in V_h, \tag{2.18}$$

$$W_h(Q_i) = I_{h,D}(F_D)(Q_i) \quad \text{for } i \in J_D. \tag{2.19}$$

Then the ensuing stability result for  $W_h$  holds.

**Theorem 2.15.** *There is a unique solution  $W_h \in X_h$  to (2.18), (2.19).*

Put  $h_1 := \min\left\{\left(\frac{\underline{\beta}}{2c_0\varphi_1}\left(1 + \|\operatorname{div} B\|_{L^p(\Omega)}\right)\right)^{p/(p-2)}, (\operatorname{diam} \Omega)/\varphi_1\right\}$ , with  $c_0, \underline{\beta}, \varphi_1$ , and  $p$  as in (2.16). Suppose that  $h \leq h_1$ . Let  $\epsilon \in (0, \infty)$ . Then

$$\begin{aligned} & \|W_h\|_{L^2(\Omega)} + \nu^{1/2} \|\nabla_h W_h\|_{L^2(\Omega)} + \left(\sum_{i \in J_N} W_h(Q_i)^2 \int_{S_i} |B(x) \cdot n(x)| \, d\omega_x\right)^{1/2} \\ & \leq \mathfrak{C} (1 + \bar{\nu}/\nu) (1 + \|B\|_{H^1(\Omega)}) \\ & \quad \left[ \|G\|_{L^2(\Omega)} + \left(1 + |B|_\infty + \|\operatorname{div} B + M\|_{L^p(\Omega)} + \bar{\nu}/\nu^{1/2}\right) \|\tilde{F}_D\|_{H^{1/2}(\partial\Omega)} \right. \\ & \quad \left. + \epsilon^{-1/2} \left(\sum_{i \in \tilde{J}_N^{(\epsilon)}} \|F_N|_{S_i}\|_{L^2(S_i)}^2\right)^{1/2} + \nu^{-1/4-\delta/2} \left(\sum_{i \in J_N \setminus \tilde{J}_N^{(\epsilon)}} \|F_N|_{S_i}\|_{L^2(S_i)}^2\right)^{1/2} \right], \end{aligned} \tag{2.20}$$

with  $\tilde{J}_N^{(\epsilon)} := \{i \in J_N : l_i^{-1} \int_{S_i} |B(x) \cdot n(x)| \, d\omega_x \geq \epsilon\}$ .

We will give a proof only of Theorem 2.14. The same argument, but with considerable simplifications, may be used to establish Theorem 2.15.

### 3. COERCIVITY OF THE DISCRETE CONVECTION TERM

In this section, we derive a coercivity relation for  $d_h(t, \cdot, \cdot)$ . We begin by fixing  $t \in (0, T)$  and abbreviating  $\beta := b(\cdot, t)$ ,  $b_h := d_h(t, \cdot, \cdot)$ ,  $\theta_{ij}^+ := \vartheta_{ij}^+(t) = \int_{\Gamma_{ij}} \max\{\beta(x) \cdot n(x), 0\} \, d\omega_x$  for  $i \in J, j \in s(i)$ ; see (2.10), (2.9) for the definition of  $d_h$  and  $\vartheta_{ij}^+(t)$ , respectively. Moreover, we set  $\mu := m(\cdot, t)$ ,  $\varphi := \zeta(\cdot, t)$ , with  $\zeta$  from (1.5),  $\bar{\mu}_i := \int_{D_i} \mu(x) \, dx = \bar{m}_i(t)$  for  $i \in J$ , with  $\bar{m}_i(t)$  defined in (2.9). Note that by (2.10),

$$\begin{aligned} b_h(v_h, w_h) &= \sum_{i \in J} w_h(Q_i) \left( \sum_{j \in s(i)} (\theta_{ij}^+ v_h(Q_i) - \theta_{ji}^+ v_h(Q_j)) + \bar{\mu}_i v_h(Q_i) \right) \\ & \quad + \sum_{i \in J_N} v_h(Q_i) w_h(Q_i) \int_{S_i} \max\{(\beta \cdot n)(x), 0\} \, d\omega_x \quad \text{for } v_h, w_h \in X_h. \end{aligned}$$

We further observe that by our assumptions in Section 1, we have  $\beta \in H^1(\Omega)^2$ ,  $\operatorname{div} \beta \in L^p(\Omega)$ ,  $\mu \in L^p(\Omega)$  with  $p$  from (1.4),

$$\operatorname{div} \beta(x)/2 + \mu(x) \geq 0, \quad -\nabla \varphi(x) \cdot \beta(x) \geq \underline{\beta} \quad \text{for } x \in \bar{\Omega}. \tag{3.1}$$

Since  $n_{ij} = -n_{ji}$  for  $i \in J, j \in s(i)$ , we get  $\theta_{ij}^+ - \theta_{ji}^+ = \int_{\Gamma_{ij}} (\beta \cdot n)(x) \, d\omega_x$  for  $i \in J, j \in s(i)$ , so

$$\sum_{j \in s(i)} (\theta_{ij}^+ - \theta_{ji}^+) = \int_{D_i} \operatorname{div} \beta(x) \, dx - \delta_{J \setminus J^o}(i) \int_{S_i} (\beta \cdot n)(x) \, d\omega_x \quad (i \in J), \tag{3.2}$$

with  $\delta_{J \setminus J^o}(i) = 1$  if  $i \in J \setminus J^o$ , and  $\delta_{J \setminus J^o}(i) = 0$  else. Moreover, by (2.1),

$$|\varphi(x) - \varphi(x')| \leq \varphi_1 |x - x'|, \quad \operatorname{diam} \Omega \leq \varphi(x) \leq \varphi_1 \quad \text{for } x, x' \in \bar{\Omega}. \tag{3.3}$$

**Lemma 3.1.** *Let  $v_h, w_h \in X_h$ . Then, with  $\varphi_1$  from (3.3),*

$$\begin{aligned} (v_h, \varrho_h(w_h \varphi)) &\leq (v_h, \varrho_h(v_h \varphi))^{1/2} (w_h, \varrho_h(w_h \varphi))^{1/2}, \\ (v_h, \varrho_h(v_h \varphi)) &\leq \varphi_1 \|v_h\|_2^2, \quad \text{and if } v_h \in V_h : \|\nabla_h \varrho_h(v_h \varphi)\|_{L^2(\Omega)} \leq c \varphi_1 \|\nabla_h v_h\|_{L^2(\Omega)}. \end{aligned}$$

*Proof.* The first two estimates in the lemma are an immediate consequence of (2.7), (3.3)<sub>2</sub> and the Cauchy–Schwarz inequality for sums. As for the third, it follows by modifying the proof of ([15], Lem. 3.2). In fact, let  $K \in \mathfrak{T}$ , and let  $i_1, i_2, i_3 \in J$  be such that the points  $Q_{i_1}, Q_{i_2}, Q_{i_3}$  are the midpoints of the edges of  $K$ . Then by (2.6) and the definition of  $\varrho_h(v_h \varphi)$ ,

$$\|\nabla(\varrho_h(v_h \varphi)|K)\|_{L^2(K)} = \int_K \left| \sum_{r=1}^3 (v_h \varphi)(Q_{i_r}) \nabla(w_{i_r}|K) \right|^2 dx \leq \mathfrak{A} + 2 \varphi(Q_{i_1})^2 \|\nabla(v_h|K)\|_{L^2(K)}^2, \quad (3.4)$$

with  $\mathfrak{A} := 2 \int_K |\sum_{r=1}^3 (\varphi(Q_{i_r}) - \varphi(Q_{i_1})) v_h(Q_{i_r}) \nabla(w_{i_r}|K)|^2 dx$ . But  $|\varphi(Q_{i_r}) - \varphi(Q_{i_1})| \leq \varphi_1 h_K$  by (3.3), and  $\|\nabla(w_{i_r}|K)\|_{L^2(K)} \leq c h_K^{-1} \|w_{i_r}|K\|_{L^2(K)}$  by Lemma 2.9. Moreover the inequality  $\|w_{i_r}|K\|_{L^2(K)} \leq |K| = 3 |D_{i_r} \cap K|$  holds for  $1 \leq r \leq 3$ . These relations put together yield that  $\mathfrak{A} \leq c \varphi_1 \sum_{i \in J, Q_i \in \overline{K}} v_h(Q_i)^2 |K \cap D_i|$ . Thus, taking the sum with respect to  $K \in \mathfrak{T}$  in (3.4), we obtain that  $\|\nabla_h \varrho_h(v_h \varphi)\|_{L^2(\Omega)}$  is bounded by the term  $c \varphi_1 \sum_{i \in J} v_h(Q_i)^2 |D_i| + 2 \varphi_1 \|\nabla_h v_h\|_{L^2(\Omega)}^2$ . The third inequality in Lemma 3.1 now follows with (2.7).  $\square$

The next lemma shows in particular that  $b_h(v_h, v_h)$  is non-negative.

**Lemma 3.2.** *Let  $v_h \in V_h$ . Then*

$$b_h(v_h, v_h) \geq \mathfrak{K}_h/2 + \sum_{i \in J_N} v_h(Q_i)^2 \int_{S_i} |\beta \cdot n|(x) \, do_x / 2,$$

where  $\mathfrak{K}_h := \mathfrak{K}_h(v_h) := \sum_{i \in J} \sum_{j \in s(i)} \theta_{ji}^+ (v_h(Q_i) - v_h(Q_j))^2$ . In particular  $b_h(v_h, v_h) \geq \mathfrak{K}_h/2 \geq 0$ .

*Proof.* As in ([12], proof of Lem. 3.1), we obtain

$$b_h(v_h, v_h) = \mathfrak{K}_h/2 + \sum_{i \in J} v_h(Q_i)^2 \left( \sum_{j \in s(i)} (\theta_{ij}^+ - \theta_{ji}^+)/2 + \bar{\mu}_i \right) + \sum_{i \in J_N} v_h(Q_i)^2 \int_{S_i} \max\{(\beta \cdot n)(x), 0\} \, do_x. \quad (3.5)$$

Recalling (3.2), (3.1)<sub>1</sub> and the relations  $v_h(Q_i) = 0$  for  $i \in J_D$ ,  $J \setminus J^o = J_N \cup J_D$ , we may deduce from (3.5) that

$$\begin{aligned} b_h(v_h, v_h) &\geq \mathfrak{K}_h/2 - \sum_{i \in J_N} v_h(Q_i)^2 \int_{S_i} (\beta \cdot n)(x) \, do_x / 2 + \sum_{i \in J_N} v_h(Q_i)^2 \int_{S_i} \max\{(\beta \cdot n)(x), 0\} \, do_x \\ &= \mathfrak{K}_h/2 + \sum_{i \in J_N} v_h(Q_i)^2 \int_{S_i} |(\beta \cdot n)(x)| \, do_x / 2. \end{aligned} \quad \square$$

**Lemma 3.3.** *Let  $v_h \in V_h$ . Then  $b_h(v_h, \varrho_h(v_h \varphi)) \geq \mathfrak{A}_h/2 + \mathfrak{R}_h$ , where*

$$\mathfrak{A}_h := \mathfrak{A}_h(v_h) := \sum_{j \in J} v_h(Q_i)^2 \sum_{j \in s(i)} (\theta_{ji}^+ \varphi(Q_i) - \theta_{ij}^+ \varphi(Q_j)),$$

and  $\mathfrak{R}_h := \mathfrak{R}_h(v_h)$  is an abbreviation for the expression

$$\sum_{i \in J} v_h(Q_i)^2 \varphi(Q_i) \int_{D_i} \operatorname{div} \beta(x) \, dx / 2 - \sum_{i \in J_N} v_h(Q_i)^2 \varphi(Q_i) \int_{S_i} \min\{(\beta \cdot n)(x), 0\} \, do_x.$$

*Proof.* We modify the proof of ([15], Lem. 3.4), starting with the equation  $b_h(v_h, \varrho_h(v_h \varphi)) = A_1 + A_2 + A_3$ , where

$$\begin{aligned} A_1 &:= \sum_{i \in J} v_h(Q_i) \varphi(Q_i) \sum_{j \in s(i)} (v_h(Q_i) - v_h(Q_j)) \theta_{ji}^+, \\ A_2 &:= \sum_{i \in J} v_h(Q_i)^2 \varphi(Q_i) \left( \sum_{j \in s(i)} (\theta_{ij}^+ - \theta_{ji}^+) + \bar{\mu}_i \right), \\ A_3 &:= \sum_{i \in J_N} v_h(Q_i)^2 \varphi(Q_i) \int_{S_i} \max\{(\beta \cdot n)(x), 0\} \, do_x. \end{aligned}$$

Proceeding as in the proof of ([15], Lem. 3.4), we find that  $A_1 \geq \mathfrak{A}_h$ . On the other hand, by (3.2), (3.1)<sub>1</sub> and the equations  $v_h(Q_i) = 0$  for  $i \in J_D$ ,  $J \setminus J^o = J_N \cup J_D$ , we obtain

$$A_2 \geq \sum_{i \in J} v_h(Q_i)^2 \varphi(Q_i) \int_{D_i} \operatorname{div} \beta(x) \, dx / 2 - \sum_{i \in J_N} v_h(Q_i)^2 \varphi(Q_i) \int_{S_i} (\beta \cdot n)(x) \, do_x.$$

Now the lemma follows from the relation  $b_h(v_h, \varrho_h(v_h \varphi)) = A_1 + A_2 + A_3$ . □

**Lemma 3.4.** *Let  $v_h \in V_h$ , and put*

$$\mathfrak{B}_h := \mathfrak{B}_h(v_h) := - \sum_{i \in J} v_h(Q_i)^2 \sum_{j \in s(i)} \int_{\Gamma_{ij}} \varphi(x) \beta(x) \cdot n_{ij} \, do_x.$$

*Then*

$$\mathfrak{B}_h \geq \underline{\beta} \|v_h\|_{L^2(\Omega)}^2 - \sum_{i \in J} v_h(Q_i)^2 \int_{D_i} \operatorname{div} \beta(x) \varphi(x) \, dx + \sum_{i \in J_N} v_h(Q_i)^2 \int_{S_i} (\beta \cdot n)(x) \varphi(x) \, do_x.$$

*Proof.* For  $i \in J$ , we find

$$- \sum_{j \in s(i)} \int_{\Gamma_{ij}} \varphi \beta \cdot n_{ij} \, do_x = - \int_{D_i} \beta \cdot \nabla \varphi \, dx - \int_{D_i} \operatorname{div} \beta \varphi \, dx + \delta_{J \setminus J^o}(i) \int_{S_i} (\beta \cdot n) \varphi \, do_x,$$

where  $\delta_{J \setminus J^o}(i)$  is defined as in (3.2). But  $-\int_{D_i} \beta \cdot \nabla \varphi \, dx \geq \underline{\beta} |D_i|$  for  $i \in J$  by (3.1)<sub>2</sub>, so Lemma 3.4 follows with (2.7) and the equations  $v_h(Q_i) = 0$  for  $i \in J_D$ ,  $J \setminus J^o = \bar{J}_D \cup J_N$ . □

**Lemma 3.5.** *The estimate  $|\mathfrak{A}_h - \mathfrak{B}_h| \leq c \varphi_1 \|\beta\|_{H^1(\Omega)}^{1/2} \|v_h\|_{L^2(\Omega)} \mathfrak{K}_h^{1/2}$  holds for  $v_h \in V_h$ , with  $\mathfrak{B}_h$  from Lemma 3.4,  $\mathfrak{A}_h$  from Lemma 3.3, and  $\mathfrak{K}_h$  from Lemma 3.2.*

*Proof.* We refer to the proof of ([15], Lem. 3.6). □

The term  $\sum_{i \in J_N} v_h(Q_i) w_h(Q_i) \int_{S_i} \max\{b(x, t) \cdot n(x), 0\} \, do_x$  in the definition of  $d_h(t, v_h, w_h)$ , a term which is linked to the Robin–Neumann boundary condition (1.2)<sub>2</sub>, causes most of the difficulties in the proof of the next theorem. Actually this term was inserted into the definition of the term  $d_h(t, v_h, w_h)$  because otherwise we could not see how to carry through this proof. Since this term was not present in [15], we elaborate this proof in more detail.

**Theorem 3.6.** *Suppose that  $h \leq h_0$ , with  $h_0$  introduced in (2.16). Let  $v_h \in V_h$ . Then*

$$\|v_h\|_{L^2(\Omega)}^2 \leq \mathfrak{C} (1 + \|\beta\|_{H^1(\Omega)}) b_h(v_h, v_h) + (4/\underline{\beta}) b_h(v_h, \varrho_h(v_h \varphi)).$$

*Proof.* Lemma 3.4 yields  $\underline{\beta} \|v_h\|_{L^2(\Omega)}^2 \leq \mathfrak{B}_h + A$ , with

$$A := \sum_{i \in J} v_h(Q_i)^2 \int_{D_i} \operatorname{div} \beta(x) \varphi(x) \, dx - \sum_{i \in J_N} v_h(Q_i)^2 \int_{S_i} (\beta \cdot n)(x) \varphi(x) \, do_x.$$

Therefore

$$\underline{\beta} \|v_h\|_{L^2(\Omega)}^2 \leq |\mathfrak{A}_h - \mathfrak{B}_h| + \mathfrak{A}_h + A \leq |\mathfrak{A}_h - \mathfrak{B}_h| + 2b_h(v_h, \varrho_h(v_h \varphi)) - 2\mathfrak{R}_h + A,$$

where we used Lemma 3.3 in the last inequality. We may conclude with Lemmas 3.5 and 3.2 that

$$\underline{\beta} \|v_h\|_{L^2(\Omega)}^2 \leq \mathfrak{C} \|\beta\|_{H^1(\Omega)}^{1/2} \|v_h\|_{L^2(\Omega)} b_h(v_h, v_h)^{1/2} + 2b_h(v_h, \varrho_h(v_h \varphi)) - 2\mathfrak{R}_h + A. \tag{3.6}$$

But

$$\begin{aligned} -2\mathfrak{R}_h + A &= \sum_{i \in J} v_h(Q_i)^2 \int_{D_i} \operatorname{div} \beta(x) (\varphi(x) - \varphi(Q_i)) \, dx \\ &+ \sum_{i \in J_N} v_h(Q_i)^2 \int_{S_i} (2\varphi(Q_i) \min\{(\beta \cdot n)(x), 0\} - \varphi(x) (\beta \cdot n)(x)) \, do_x. \end{aligned} \tag{3.7}$$

For  $i \in J_N$ ,  $x \in S_i$ , we find

$$\begin{aligned} 2\varphi(Q_i) \min\{(\beta \cdot n)(x), 0\} - \varphi(x) (\beta \cdot n)(x) &= (2\varphi(Q_i) - \varphi(x)) \min\{(\beta \cdot n)(x), 0\} - \varphi(x) \max\{(\beta \cdot n)(x), 0\} \\ &\leq (2\varphi(Q_i) - \varphi(x)) \min\{(\beta \cdot n)(x), 0\}, \end{aligned}$$

where the preceding inequality holds because  $-\varphi(x) \max\{(\beta \cdot n)(x), 0\} \leq 0$ . Taking account of (3.3) and the relation  $\min\{(\beta \cdot n)(x), 0\} \leq 0$  for any  $x \in \partial\Omega$ , we may thus conclude that

$$\begin{aligned} 2\varphi(Q_i) \min\{(\beta \cdot n)(x), 0\} - \varphi(x) (\beta \cdot n)(x) &\leq (\operatorname{diam} \Omega - |\varphi(Q_i) - \varphi(x)|) \min\{(\beta \cdot n)(x), 0\} \\ &\leq (\operatorname{diam} \Omega - \varphi_1 |Q_i - x|) \min\{(\beta \cdot n)(x), 0\} \leq (\operatorname{diam} \Omega - \varphi_1 h) \min\{(\beta \cdot n)(x), 0\} \leq 0 \end{aligned}$$

for  $i \in J_N$ ,  $x \in S_i$ , where we used the relation  $h \leq h_0 \leq (\operatorname{diam} \Omega)/\varphi_1$  (see (2.16)) in the last inequality. Therefore from (3.7),

$$-2\mathfrak{R}_h + A \leq \sum_{i \in J} v_h(Q_i)^2 \int_{D_i} \operatorname{div} \beta(x) (\varphi(x) - \varphi(Q_i)) \, dx. \tag{3.8}$$

Using (3.3), we further observe that  $|\varphi(x) - \varphi(Q_i)| \leq \max\{h_{K_i^l} : l \in \{1, 2\}\}$  for  $i \in J$ ,  $x \in S_i$ . Therefore with (2.5),

$$\begin{aligned} \left| \int_{D_i} \operatorname{div} \beta(x) (\varphi(x) - \varphi(Q_i)) \, dx \right| &\leq \varphi_1 \max\{h_{K_i^l} : l \in \{1, 2\}\} \int_{D_i} |\operatorname{div} \beta(x)| \, dx \\ &\leq \varphi_1 \max\{h_{K_i^l} : l \in \{1, 2\}\} |D_i|^{1/p'} \|\operatorname{div} \beta\|_{L^p(\Omega)} \leq c_0 \varphi_1 |D_i| h^{1-2/p} \|\operatorname{div} \beta\|_{L^p(\Omega)}, \end{aligned}$$

where  $c_0$  was introduced in (2.5) and  $p$  in (1.4). It follows with (3.8) and (2.7) that

$$-2\mathfrak{R}_h + A \leq c_0 \varphi_1 h^{1-2/p} \|\operatorname{div} \beta\|_{L^p(\Omega)} \sum_{i \in J} v_h(Q_i)^2 |D_i| \leq c_0 \varphi_1 h^{1-2/p} \|\operatorname{div} \beta\|_{L^p(\Omega)} \|v_h\|_{L^2(\Omega)}^2.$$

On the other hand, due to the assumptions  $h \leq h_0$  and  $p > 2$ , and by the choice of  $h_0$  in (2.16), we find  $c_0 \varphi_1 h^{1-2/p} \|\operatorname{div} \beta\|_{L^p(\Omega)} \leq \underline{\beta}/4$ , so  $-2\mathfrak{R}_h + A \leq \underline{\beta} \|v_h\|_{L^2(\Omega)}^2/4$ . Hence by (3.6),

$$\underline{\beta} \|v_h\|_{L^2(\Omega)}^2 \leq \mathfrak{C} \|\beta\|_{H^1(\Omega)}^{1/2} \|v_h\|_{L^2(\Omega)} b_h(v_h, v_h)^{1/2} + 2b_h(v_h, \varrho_h(v_h \varphi)) + \underline{\beta} \|v_h\|_{L^2(\Omega)}^2/4.$$

As a consequence,

$$\underline{\beta} \|v_h\|_{L^2(\Omega)}^2 \leq \mathfrak{C} \|\beta\|_{H^1(\Omega)} b_h(v_h, v_h) + 2 b_h(v_h, \varrho_h(v_h \varphi)) + \underline{\beta} \|v_h\|_{L^2(\Omega)}^2/2.$$

Theorem 3.6 follows from this inequality. □

**Corollary 3.7.** *Suppose that  $h \leq h_0$ , and let  $v_h \in V_h$ . Then*

$$\|v_h\|_{L^2(\Omega)}^2 + \sum_{i \in J_N} v_h(Q_i)^2 \int_{S_i} |(\beta \cdot n)(x)| \, do_x \leq \mathfrak{C} (1 + \|\beta\|_{H^1(\Omega)}) b_h(v_h, v_h) + (4/\underline{\beta}) b_h(v_h, \varrho_h(v_h \varphi)).$$

*Proof.* Lemma 3.2 yields that  $\sum_{i \in J_N} v_h(Q_i)^2 \int_{S_i} |(\beta \cdot n)(x)| \, do_x \leq 2 b_h(v_h, v_h)$ . Thus Corollary 3.7 follows with Theorem 3.6. □

#### 4. PROOF OF THEOREM 2.14.

The proof of the next theorem is based on the argument in ([15], p. 521–522). However, since the function  $\zeta$  (see (1.5)<sub>2</sub>) is time-dependent, the reasoning becomes more complicated, so we present it some detail.

**Theorem 4.1.** *Let  $(\alpha_i^{(l)})_{i \in J} \in \mathbb{R}^J$  for  $l \in \{1, \dots, Z + 1\}$ . Suppose that  $h \leq h_0$ , with  $h_0$  defined in (2.16), and let  $u_h^{(0)}, \dots, u_h^{(Z+1)} \in V_h$  satisfy*

$$\tau_{k+1}^{-1} (u_h^{(k+1)} - u_h^{(k)}, v_h)_{L^2(\Omega)} + (a(t_{k+1}) \nabla_h u_h^{(k+1)}, \nabla_h v_h)_{L^2(\Omega)} + d_h(t_{k+1}, u_h^{(k+1)}, v_h) = \sum_{i \in J} \alpha_i^{(k+1)} v_h(Q_i) \quad (4.1)$$

for  $0 \leq k \leq Z$ ,  $v_h \in V_h$ . Then, with  $\mathfrak{A}(u^{(1)}, \dots, u_h^{(Z+1)})$  introduced in (2.15),  $\zeta$  in (1.5) and  $\partial_3 \zeta$  in Lemma 2.1,

$$\begin{aligned} \mathfrak{A}(u_h^{(1)}, \dots, u_h^{(Z+1)})^2 &\leq \mathfrak{C} (1 + \bar{\nu}/\nu + \|\partial_3 \zeta\|_{L^1(0,T, L^\infty(\Omega))}) \left( 1 + \max_{1 \leq l \leq Z+1} \|b(t_l)\|_{H^1(\Omega)} \right) \\ &\cdot \left( \left| \sum_{l=1}^{Z+1} \tau_l \sum_{i \in J} \alpha_i^{(l)} u_h^{(l)}(Q_i) \right| + \left| \sum_{l=1}^{Z+1} \tau_l \sum_{i \in J} \alpha_i^{(l)} \varrho_h(u_h^{(l)} \zeta(t_l))(Q_i) \right| + \|u_h^{(0)}\|_{L^2(\Omega)}^2 \right). \end{aligned} \quad (4.2)$$

*Proof.* Choosing  $v_h = u_h^{(k+1)}$  in (4.1), we get with (1.6) and (4.1) that

$$\begin{aligned} &\nu \|\nabla_h u_h^{(k+1)}\|_{L^2(\Omega)}^2 + \tau_{k+1}^{-1} \|u_h^{(k+1)}\|_{L^2(\Omega)}^2 + d_h(t_{k+1}, u_h^{(k+1)}, u_h^{(k+1)}) \\ &\leq (a(t_{k+1}) \nabla_h u_h^{(k+1)}, \nabla_h u_h^{(k+1)})_{L^2(\Omega)} + \tau_{k+1}^{-1} \|u_h^{(k+1)}\|_{L^2(\Omega)}^2 + d_h(t_{k+1}, u_h^{(k+1)}, u_h^{(k+1)}) \\ &\leq \sum_{i \in J} \alpha_i^{(k+1)} u_h^{(k+1)}(Q_i) + (2\tau_{k+1})^{-1} \|u_h^{(k+1)}\|_{L^2(\Omega)}^2 + (2\tau_{k+1})^{-1} \|u_h^{(k)}\|_{L^2(\Omega)}^2 \quad \text{for } 0 \leq k \leq Z. \end{aligned}$$

From this we may conclude as in the proof of ([15], (4.1), (4.2)) that

$$\begin{aligned} &\sum_{l=1}^{Z+1} \tau_l d_h(t_l, u_h^{(l)}, u_h^{(l)}) + \nu \sum_{l=1}^{Z+1} \tau_l \|\nabla_h u_h^{(l)}\|_{L^2(\Omega)}^2 + \max_{1 \leq l \leq Z+1} \|u_h^{(l)}\|_{L^2(\Omega)}^2 \\ &\leq 4 \left( \left| \sum_{l=1}^{Z+1} \tau_l \sum_{i \in J} \alpha_i^{(l)} u_h^{(l)}(Q_i) \right| + \|u_h^{(0)}\|_{L^2(\Omega)}^2/2 \right). \end{aligned} \quad (4.3)$$



In the following, we use the abbreviation  $\tilde{u}_h^{(k)} := \varrho_h(u_h^{(k)} \zeta(t_k))$  for  $0 \leq k \leq Z + 1$ , with  $\zeta$  from (1.5)<sub>2</sub>. Referring to (4.1) again, this time with  $v_h = \tilde{u}_h^{(k+1)}$ , we find that

$$\begin{aligned} & \tau_{k+1} \left( a(t_{k+1}) \nabla_h u_h^{(k+1)}, \nabla_h \tilde{u}_h^{(k+1)} \right)_{L^2(\Omega)} \\ & + \tau_{k+1} d_h \left( t_{k+1}, u_h^{(k+1)}, \tilde{u}_h^{(k+1)} \right) + (u_h^{(k+1)}, \tilde{u}_h^{(k+1)})_{L^2(\Omega)} \\ & \leq \tau_{k+1} \sum_{i \in J} \alpha_i^{(k+1)} \tilde{u}_h^{(k+1)}(Q_i) + \left( u_h^{(k)}, \varrho_h \left( u_h^{(k)} \zeta(t_{k+1}) \right) \right)_{L^2(\Omega)}^{1/2} (u_h^{(k+1)}, \tilde{u}_h^{(k+1)})_{L^2(\Omega)}^{1/2} \\ & \leq \tau_{k+1} \sum_{i \in J} \alpha_i^{(k+1)} \tilde{u}_h^{(k+1)}(Q_i) + (u_h^{(k+1)}, \tilde{u}_h^{(k+1)})_{L^2(\Omega)}/2 + \left( u_h^{(k)}, \varrho_h \left( u_h^{(k)} \zeta(t_{k+1}) \right) \right)_{L^2(\Omega)}/2 \end{aligned} \tag{4.4}$$

for  $0 \leq k \leq Z$ , where we used Lemma 3.1 in the first inequality. But by (2.7) and Lemma 2.1,

$$\begin{aligned} & \left| \left( u_h^{(k)}, \varrho_h \left( u_h^{(k)} [\zeta(t_{k+1}) - \zeta(t_k)] \right) \right)_{L^2(\Omega)} \right| \leq \sum_{i \in J} u_h^{(k)}(Q_i)^2 |\zeta(Q_i, t_{k+1}) - \zeta(Q_i, t_k)| |D_i| \\ & \leq \sum_{i \in J} u_h^{(k)}(Q_i)^2 \int_{t_k}^{t_{k+1}} |\partial_3 \zeta(Q_i, t)| dt |D_i| \leq \int_{t_k}^{t_{k+1}} \|\partial_3 \zeta(t)\|_{L^\infty(\Omega)} dt \sum_{i \in J} u_h^{(k)}(Q_i)^2 |D_i| \\ & = \int_{t_k}^{t_{k+1}} \|\partial_3 \zeta(t)\|_{L^\infty(\Omega)} dt \|u_h^{(k)}\|_{L^2(\Omega)}^2 \quad \text{for } 1 \leq k \leq Z + 1. \end{aligned}$$

As a consequence

$$\begin{aligned} & \left( u_h^{(k)}, \varrho_h \left( u_h^{(k)} \zeta(t_{k+1}) \right) \right)_{L^2(\Omega)} \leq \left| \left( u_h^{(k)}, \varrho_h \left( u_h^{(k)} [\zeta(t_{k+1}) - \zeta(t_k)] \right) \right)_{L^2(\Omega)} \right| \\ & + \left( u_h^{(k)}, \tilde{u}_h^{(k)} \right)_{L^2(\Omega)} \leq \int_{t_k}^{t_{k+1}} \|\partial_3 \zeta(t)\|_{L^\infty(\Omega)} dt \max_{0 \leq l \leq Z} \|u_h^{(l)}\|_{L^2(\Omega)}^2 + \left( u_h^{(k)}, \tilde{u}_h^{(k)} \right)_{L^2(\Omega)}. \end{aligned} \tag{4.5}$$

Thus, by applying (4.5) on the right-hand side of (4.4), taking the sum with respect to  $k \in \{0, \dots, Z\}$  on both sides of the resulting inequality, then subtracting the term

$$\sum_{k=0}^Z \left( u_h^{(k+1)}, \tilde{u}_h^{(k+1)} \right)_{L^2(\Omega)} / 2 + \sum_{k=1}^Z \left( u_h^{(k)}, \tilde{u}_h^{(k)} \right)_{L^2(\Omega)} / 2,$$

and taking account of the fact that  $(u_h^{(Z+1)}, \tilde{u}_h^{(Z+1)})_{L^2(\Omega)} \geq 0$  (Lem. 3.1), we arrive at the estimate

$$\begin{aligned} & \sum_{k=0}^Z \tau_{k+1} \left( a(t_{k+1}) \nabla_h u_h^{(k+1)}, \nabla_h \tilde{u}_h^{(k+1)} \right)_{L^2(\Omega)} + \sum_{k=0}^Z \tau_{k+1} d_h \left( t_{k+1}, u_h^{(k+1)}, \tilde{u}_h^{(k+1)} \right) \\ & \leq \sum_{k=0}^Z \tau_{k+1} \sum_{i \in J} \alpha_i^{(k+1)} \tilde{u}_h^{(k+1)}(Q_i) + (u_h^{(0)}, \tilde{u}_h^{(0)})_{L^2(\Omega)} / 2 \\ & + \int_0^T \|\partial_3 \zeta(t)\|_{L^\infty(\Omega)} dt \max_{0 \leq l \leq Z} \|u_h^{(l)}\|_{L^2(\Omega)}^2. \end{aligned}$$

Note that  $(u_h^{(0)}, \tilde{u}_h^{(0)})_{L^2(\Omega)} \leq \varphi_1 \|u^{(0)}\|_{L^2(\Omega)}^2$  by Lemma 3.1. Thus we may conclude that

$$\begin{aligned} \sum_{l=1}^{Z+1} \tau_l d_h(t_l, u_h^{(l)}, \tilde{u}_h^{(l)}) &\leq \sum_{l=1}^{Z+1} \tau_l \left| (a(t_l) \nabla_h u_h^{(l)}, \nabla_h \tilde{u}_h^{(l)})_{L^2(\Omega)} \right| + \left| \sum_{l=1}^{Z+1} \tau_l \sum_{i \in J} \alpha_i^{(l)} \tilde{u}_h^{(l)}(Q_i) \right| \\ &+ \mathfrak{C} \left( 1 + \|\partial_3 \zeta\|_{L^1(0, T, L^\infty(\Omega))} \right) \max_{0 \leq l \leq Z} \|u_h^{(l)}\|_{L^2(\Omega)}^2. \end{aligned} \tag{4.6}$$

On the other hand, Corollary 3.7 with  $v_h = u_h^{(l)}$  yields

$$\begin{aligned} \|u_h^{(l)}\|_{L^2(\Omega)}^2 + \sum_{i \in J_N} u_h^{(l)}(Q_i)^2 \int_{S_i} |b(x, t_l) \cdot n(x)| \, do_x \\ \leq \mathfrak{C} (1 + \|b(t_l)\|_{H^1(\Omega)}) d_h(t_l, u_h^{(l)}, u_h^{(l)}) + (4/\underline{\beta}) d_h(t_l, u_h^{(l)}, \tilde{u}_h^{(l)}) \quad \text{for } 1 \leq l \leq Z + 1. \end{aligned}$$

We may deduce from (4.3), (4.6) and the preceding inequality that

$$\sum_{l=1}^{Z+1} \tau_l \|u_h^{(l)}\|_{L^2(\Omega)}^2 + \sum_{l=1}^{Z+1} \tau_l \sum_{i \in J_N} u_h^{(l)}(Q_i)^2 \int_{S_i} |b(x, t_l) \cdot n(x)| \, do_x \leq \mathfrak{C} \left( 1 + \max_{1 \leq l \leq Z+1} \|b(t_l)\|_{H^1(\Omega)} \right) \mathfrak{R}, \tag{4.7}$$

with

$$\begin{aligned} \mathfrak{R} := &\left| \sum_{l=1}^{Z+1} \tau_l \sum_{i \in J} \alpha_i^{(l)} \tilde{u}_h^{(l)}(Q_i) \right| + \sum_{l=1}^{Z+1} \tau_l \left| (a(t_l) \nabla_h u_h^{(l)}, \nabla_h \tilde{u}_h^{(l)}) \right| \\ &+ (1 + \|\partial_3 \zeta\|_{L^1(0, T, L^\infty(\Omega))}) \left( \|u_h^{(0)}\|_{L^2(\Omega)}^2 + \left| \sum_{l=1}^{Z+1} \tau_l \sum_{i \in J} \alpha_i^{(l)} u_h^{(l)}(Q_i) \right| \right). \end{aligned}$$

Proceeding as in the proof of ([15], (4.5)), we add the left- and right-hand side of (4.3) to respectively the left- and right-hand side of (4.7). On taking account of the fact that  $d_h(t, v_h, v_h) \geq 0$  for  $t \in (0, T)$ ,  $v_h \in V_h$  (Lem. 3.2), we get

$$\mathfrak{A}(u_h^{(1)}, \dots, u_h^{(Z+1)})^2 \leq \mathfrak{C} \left( 1 + \max_{1 \leq l \leq Z+1} \|b(t_l)\|_{H^1(\Omega)} \right) \mathfrak{R}. \tag{4.8}$$

Moreover, as a consequence of (1.6) and Lemma 3.1,

$$\begin{aligned} \sum_{l=1}^{Z+1} \tau_l \left| (a(t_l) \nabla_h u_h^{(l)}, \nabla_h \tilde{u}_h^{(l)}) \right| &\leq \bar{\nu} \sum_{l=1}^{Z+1} \tau_l \|\nabla_h u_h^{(l)}\|_{L^2(\Omega)} \|\nabla_h \tilde{u}_h^{(l)}\|_{L^2(\Omega)} \\ &\leq \mathfrak{C} \bar{\nu} \sum_{l=1}^{Z+1} \tau_l \|\nabla_h u_h^{(l)}\|_{L^2(\Omega)}^2 \leq \mathfrak{C} (\bar{\nu}/\nu) \left( \left| \sum_{l=1}^{Z+1} \tau_l \sum_{i \in J} \alpha_i^{(l)} u_h^{(l)}(Q_i) \right| + \|u_h^{(0)}\|_2^2 \right), \end{aligned} \tag{4.9}$$

where the last inequality follows from (4.3). Theorem 4.1 follows from (4.8) and (4.9). □

Due to (2.7) and (2.6), problem (4.1) corresponds to (2.14) if we put

$$\begin{aligned} \alpha_i^{(l)} := &(g(t_l), \omega_i)_{L^2(\Omega)} - (a(t_l) \nabla_h f_h(t_l), \nabla_h \omega_i)_{L^2(\Omega)} - d_h(t_l, f_h(t_l), \omega_i) \\ &+ \int_{S_i} f_N(x, t_l) \, do_x \delta_{J_N}(i) - \tau_l^{-1} (f_h(t_l) - f_h(t_{l-1}))(Q_i) |D_i| \end{aligned} \tag{4.10}$$

for  $i \in J$ ,  $1 \leq l \leq Z + 1$ , where  $\delta_{J_N}(i) := 0$  for  $i \in J \setminus J_N$ ,  $\delta_{J_N}(i) := 1$  for  $i \in J_N$ . The function  $f$  was introduced in (2.2), and  $f_h$  in (2.13). In the ensuing lemmas, we will estimate terms which may be written in the form  $\left| \sum_{l=1}^{Z+1} \tau_l \sum_{i \in J} \beta_i^{(l)} v_h^{(l)}(Q_i) \right|$ , with certain coefficients  $\beta_i^{(l)}$  and given functions  $v_h^{(l)} \in V_h$ , and which are relevant if the coefficients  $\alpha_i^{(l)}$  are chosen as in (4.10). At the end of this section (Proof of Thm. 2.14), we will then show how Theorem 2.14 may be deduced from (4.10), Theorem 4.1 and the ensuing lemmas. We begin by an obvious result.

**Lemma 4.2.** *For functions  $v_h^{(1)}, \dots, v_h^{(Z+1)} \in V_h$ , the following inequality holds:*

$$\left| \sum_{l=1}^{Z+1} \tau_l \left( g(t_l), v_h^{(l)} \right)_{L^2(\Omega)} \right| \leq \left( \sum_{l=1}^{Z+1} \tau_l \|g(t_l)\|_{L^2(\Omega)}^2 \right)^{1/2} \left( \sum_{l=1}^{Z+1} \tau_l \|v_h^{(l)}\|_{L^2(\Omega)}^2 \right)^{1/2}.$$

In Lemmas 4.3, 4.5 and Theorem 4.4, we deal with the terms generated by the nonhomogeneous Dirichlet boundary conditions (1.2)<sub>1</sub>. We recall that  $f$  was fixed in (2.2) and  $f_h$  in (2.13).

**Lemma 4.3.** *Let  $v_h^{(1)}, \dots, v_h^{(Z+1)} \in V_h$ . Then*

$$\left| \sum_{l=1}^{Z+1} \left( f_h(t_l) - f_h(t_{l-1}), v_h^{(l)} \right)_{L^2(\Omega)} \right| \leq \mathfrak{C} \cdot \|\tilde{f}'_D\|_{L^1(0,T,H^{1/2}(\partial\Omega))} \max_{1 \leq l \leq Z+1} \|v_h^{(l)}\|_{L^2(\Omega)}.$$

*Proof.* Obviously

$$\left| \sum_{l=1}^{Z+1} \left( f_h(t_l) - f_h(t_{l-1}), v_h^{(l)} \right)_{L^2(\Omega)} \right| \leq \sum_{l=1}^{Z+1} \|f_h(t_l) - f_h(t_{l-1})\|_{L^2(\Omega)} \max_{1 \leq k \leq Z+1} \|v_h^{(k)}\|_{L^2(\Omega)}.$$

But by Theorem 2.4 and Lemma 2.13,

$$\sum_{l=1}^{Z+1} \|f_h(t_l) - f_h(t_{l-1})\|_{L^2(\Omega)} = \sum_{l=1}^{Z+1} \left\| \int_{t_{l-1}}^{t_l} f'_h(s) \, ds \right\|_{L^2(\Omega)} \leq \int_0^T \|f'_h(s)\|_{L^2(\Omega)} \, ds.$$

On the other hand,  $f'_h(s) = I_h(f'(s)) = I_h(E[\tilde{f}'_D(s)])$  for  $s \in (0, T)$  according to Lemmas 2.13 and 2.5, with the extension operator  $E$  and the interpolation operator  $I_h$  introduced in Section 2. Thus we get  $\|f'_h(s)\|_{L^2(\Omega)} \leq \mathfrak{C} \|E[\tilde{f}'_D(s)]\|_{H^1(\Omega)} \leq \mathfrak{C} \|\tilde{f}'_D(s)\|_{H^{1/2}(\partial\Omega)}$  by the second inequality in Theorem 2.11 and the boundedness of  $E : H^{1/2}(\partial\Omega) \mapsto H^1(\Omega)$ . Lemma 4.3 follows from the preceding estimates.  $\square$

In the ensuing theorem, we estimate the term  $d_h(t_l, f_h(t_l), v_h)$ , with  $d_h$  defined in (2.10).

**Theorem 4.4.** *For  $v_h^{(1)}, \dots, v_h^{(Z+1)} \in V_h$ , with  $f_h$  from (2.13) and  $p$  from (1.4), we have*

$$\begin{aligned} & \left| \sum_{l=1}^{Z+1} \tau_l d_h(t_l, f_h(t_l), v_h^{(l)}) \right| \\ & \leq \mathfrak{C} \max_{1 \leq l \leq Z+1} \left( |b(t_l)|_\infty + |b(t_l)|_\infty^{1/2} + \|\operatorname{div}_x b(t_l) + m(t_l)\|_{L^p(\Omega)} \right) \left( \sum_{l=1}^{Z+1} \tau_l \|\tilde{f}_D(t_l)\|_{H^{1/2}(\partial\Omega)}^2 \right)^{1/2} \\ & \quad \cdot \left[ \left( \sum_{l=1}^{Z+1} \tau_l \|v_h^{(l)}\|_{L^2(\Omega)}^2 \right)^{1/2} + \left( \sum_{l=1}^{Z+1} \tau_l \sum_{i \in J_N} v_h^{(l)}(Q_i)^2 \int_{S_i} |\min\{b(x, t_l) \cdot n(x), 0\}| \, do_x \right)^{1/2} \right]. \end{aligned}$$

*Proof.* Let  $t \in (0, T)$ ,  $v_h \in V_h$ . Then  $d_h(t, f_h(t), v_h) = A_1 + A_2$ , with

$$\begin{aligned} A_1 &:= \sum_{i \in J} v_h(Q_i) \sum_{j \in s(i)} (f_h(t)(Q_i) - f_h(t)(Q_j)) \vartheta_{ji}^+(t), \\ A_2 &:= \sum_{i \in J} v_h(Q_i) f_h(t)(Q_i) \left( \sum_{j \in s(i)} (\vartheta_{ij}^+(t) - \vartheta_{ji}^+(t)) + \overline{m}_i(t) \right) \\ &\quad + \sum_{i \in J_N} v_h(Q_i) f_h(t)(Q_i) \int_{S_i} \max\{b(x, t) \cdot n(x), 0\} \, do_x. \end{aligned}$$

We find

$$\begin{aligned} |A_1| &\leq \sum_{i \in J} |v_h(Q_i)| \sum_{k \in \{1, 2\}} |\nabla(f_h(t)|K_i^k)| \sum_{j \in s(i), Q_j \in \overline{K_i^k}} |Q_j - Q_i| |\vartheta_{ji}^+(t)| \\ &\leq \sum_{i \in J} |v_h(Q_i)| \sum_{k \in \{1, 2\}} |\nabla(f_h(t)|K_i^k)| h_{K_i^k}^2 |b(t)|_\infty, \end{aligned}$$

where the last inequality follows from the definition of  $\vartheta_{ji}^+(t)$  (see (2.9)) and the fact that the length of  $\Gamma_{ij}$  is bounded by  $\text{diam } K_i^k$ , for  $i \in J$ ,  $j \in s(i)$ ,  $k \in \{1, 2\}$  with  $Q_j \in \overline{K_i^k}$ . Note that for each index  $j \in s(i)$ , we have either  $Q_j \in \overline{K_i^1}$  or  $Q_j \in \overline{K_i^2}$ . For  $i \in J$ ,  $k \in \{1, 2\}$ , we use (2.4) and the relation  $|K_i^k|/3 = |K_i^k \cap D_i|$  to obtain  $h_{K_i^k}^2 \leq c|K_i^k| \leq c|K_i^k \cap D_i| \leq c|D_i|$ . Thus we get

$$\begin{aligned} |A_1| &\leq c|b(t)|_\infty \sum_{i \in J} |v_h(Q_i)| |D_i|^{1/2} \sum_{k \in \{1, 2\}} |K_i^k|^{1/2} |\nabla(f_h(t)|K_i^k)| \\ &\leq c|b(t)|_\infty \left( \sum_{i \in J} v_h(Q_i)^2 |D_i| \right)^{1/2} \left( \sum_{i \in J} \sum_{k \in \{1, 2\}} |K_i^k| |\nabla(f_h(t)|K_i^k)|^2 \right)^{1/2}, \end{aligned}$$

hence with (2.7), the second estimate in Theorem 2.11 and the choice of  $f_h$  and  $f$  (see (2.13) and (2.2), respectively),

$$|A_1| \leq \mathfrak{C} |b(t)|_\infty \|v_h\|_{L^2(\Omega)} \|\nabla_h f_h(t)\|_{L^2(\Omega)} \leq \mathfrak{C} |b(t)|_\infty \|v_h\|_{L^2(\Omega)} \|\widetilde{f}_D(t)\|_{H^{1/2}(\partial\Omega)}. \tag{4.11}$$

Moreover, by (3.2) and the equation  $v_h(Q_i) = 0$  for  $i \in J_D$ , we have  $A_2 = A_{2,1} + A_{2,2}$ , where

$$\begin{aligned} A_{2,1} &:= \sum_{i \in J} v_h(Q_i) f_h(t)(Q_i) \int_{D_i} (\text{div}_x b(x, t) + m(x, t)) \, dx, \\ A_{2,2} &:= - \sum_{i \in J_N} v_h(Q_i) f_h(t)(Q_i) \int_{S_i} \min\{b(x, t) \cdot n(x), 0\} \, do_x. \end{aligned}$$

But  $|A_{2,1}| \leq \sum_{i \in J} |v_h(Q_i) f_h(t)(Q_i)| |D_i|^{1/p'}$   $\|\text{div}_x b(t) + m(t)\|_{L^p(\Omega)}$ , with  $p$  from (1.4) and  $p' := (1 - 1/p)^{-1}$ . Since  $p > 2$ , hence  $1 > 1/2 - 1/p > 0$ , we may conclude that

$$|A_{2,1}| \leq \left( \sum_{i \in J} v_h(Q_i)^2 |D_i| \right)^{1/2} \left( \sum_{i \in J} |f_h(t)(Q_i)|^{(1/2-1/p)^{-1}} |D_i| \right)^{1/2-1/p} \cdot \left( \sum_{i \in J} \|\text{div}_x b(t) + m(t)\|_{L^p(\Omega)}^p |D_i| \right)^{1/p}. \tag{4.12}$$

Referring to (2.7), we observe that  $(\sum_{i \in J} v_h(Q_i)^2 |D_i|)^{1/2} = \|v_h\|_{L^2(\Omega)}$  and

$$\left( \sum_{i \in J} |f_h(t)(Q_i)|^{(1/2-1/p)^{-1}} |D_i| \right)^{1/2-1/p} = \| |f_h(t)|^{(1/2-1/p)^{-1}/2} \|_{L^2(\Omega)}^{1-2/p} = \|f_h(t)\|_{L^r(\Omega)},$$

with  $r := (1/2 - 1/p)^{-1}$ . The preceding equation, Corollary 2.12, (2.13) and the Sobolev imbedding of  $H^1(\Omega)$  into  $L^r(\Omega)$  imply

$$\left( \sum_{i \in J} |f_h(t)(Q_i)|^{(1/2-1/p)^{-1}} |D_i| \right)^{1/2-1/p} \leq \|f_h(t) - f(t)\|_{L^r(\Omega)} + \|f(t)\|_{L^r(\Omega)} \leq \mathfrak{C} \|f(t)\|_{H^1(\Omega)} \leq \mathfrak{C} \|\tilde{f}_D(t)\|_{H^{1/2}(\partial\Omega)},$$

where the last inequality follows from (2.2) and the boundedness of the extension operator  $E$ . Since  $\overline{D_i} \cap \overline{D_j}$  is a set of measure zero for  $i, j \in J$  with  $i \neq j$ , and because of the relation  $\overline{\Omega} = \cup\{\overline{D_i} : i \in J\}$ , we obtain the equation  $\sum_{i \in J} \|\operatorname{div}_x b(t) + m(t)|D_i\|_{L^p(\Omega)}^p = \|\operatorname{div}_x b(t) + m(t)\|_{L^p(\Omega)}^p$ . Now we may conclude from (4.12) that

$$|A_{2,1}| \leq \mathfrak{C} \|v_h\|_{L^2(\Omega)} \|\tilde{f}_D(t)\|_{H^{1/2}(\partial\Omega)} \|\operatorname{div}_x b(t) + m(t)\|_{L^p(\Omega)}. \tag{4.13}$$

In addition we have  $|A_{2,2}| \leq B_1 B_2$ , with  $B_k := \left( \sum_{i \in J_N} \gamma_{k,i} \int_{S_i} |\min\{b(x, t) \cdot n(x), 0\}| \, \mathrm{d}o_x \right)^{1/2}$  for  $k \in \{1, 2\}$ , where  $\gamma_{1,i} := v_h(Q_i)^2$  and  $\gamma_{2,i} := f_h(t)(Q_i)^2$  for  $i \in J_N$ . By (2.13), the definition of the operator  $I_h$  and the choice of the functions  $\omega_i$ , we get  $f_h(t)(Q_i) = l_i^{-1} \int_{S_i} f(x, t) \, \mathrm{d}o_x$  for  $i \in J$ . But  $f(t)|S_i = \tilde{f}_D(t)|S_i$  for  $i \in J \setminus J^\circ$  by (2.2), so

$$\begin{aligned} B_2 &\leq |b(t)|_\infty^{1/2} \left( \sum_{i \in J_N} l_i^{-1} \left( \int_{S_i} \tilde{f}_D(x, t) \, \mathrm{d}o_x \right)^2 \right)^{1/2} \leq |b(t)|_\infty^{1/2} \left( \sum_{i \in J_N} \int_{S_i} \tilde{f}_D(x, t)^2 \, \mathrm{d}o_x \right)^{1/2} \\ &\leq |b(t)|_\infty^{1/2} \|\tilde{f}_D(t)\|_{L^2(\partial\Omega)}. \end{aligned}$$

Thus we have shown that

$$|A_{2,2}| \leq B_1 |b(t)|_\infty^{1/2} \|\tilde{f}_D(t)\|_{L^2(\partial\Omega)}. \tag{4.14}$$

Recalling that  $d_h(t, f_h(t), v_h) = A_1 + A_2$ ,  $A_2 = A_{2,1} + A_{2,2}$  and  $|A_{2,2}| \leq B_1 B_2$ , and referring to (4.11), (4.13) and (4.14), we obtain the estimate stated in Theorem 4.4.  $\square$

**Lemma 4.5.** For  $v_h^{(1)}, \dots, v_h^{(Z+1)} \in V_h$ , the term  $\left| \sum_{l=1}^{Z+1} \tau_l (a(t_l) \nabla_h f_h(t_l), \nabla_h v_h^{(l)})_{L^2(\Omega)} \right|$  admits the upper bound

$$\mathfrak{C} (\overline{\nu}/\nu^{1/2}) \left( \sum_{l=1}^{Z+1} \tau_l \|\tilde{f}_D(t_l)\|_{H^{1/2}(\partial\Omega)}^2 \right)^{1/2} \left( \nu \sum_{l=1}^{Z+1} \tau_l \|\nabla_h v_h^{(l)}\|_{L^2(\Omega)}^2 \right)^{1/2}.$$

*Proof.* Let  $t \in (0, T)$ ,  $v_h \in V_h$ . Then with (1.6), (2.13) and Theorem 2.11,

$$\begin{aligned} |(a(t_l) \nabla_h f_h(t_l), \nabla_h v_h)_{L^2(\Omega)}| &= \left| \sum_{K \in \mathfrak{T}} \int_K \nabla (f_h(t)|K) \cdot a(x, t) \nabla (v_h|K) \, \mathrm{d}x \right| \\ &\leq \sum_{K \in \mathfrak{T}} \overline{\nu} \int_K |\nabla (f_h(t)|K)| |\nabla (v_h|K)| \, \mathrm{d}x = \overline{\nu} \int_\Omega |\nabla_h f_h(t)| |\nabla_h v_h| \, \mathrm{d}x \\ &\leq \overline{\nu} \|\nabla_h f_h(t)\|_{L^2(\Omega)} \|\nabla_h v_h\|_{L^2(\Omega)} \leq \mathfrak{C} \overline{\nu} \|f(t)\|_{H^1(\Omega)} \|\nabla_h v_h\|_{L^2(\Omega)}. \end{aligned}$$

Since  $\|f(t)\|_{H^1(\Omega)} \leq \mathfrak{C} \|\tilde{f}_D(t)\|_{H^{1/2}(\partial\Omega)}$  by (2.2) and the boundedness of the extension operator  $E$ , Lemma 4.5 follows.  $\square$

The term  $\int_{S_i} f_N(x, t_i) \, do_x \delta_{J_N}(i)$  in (4.10) is induced by the nonhomogeneous Robin–Neumann boundary conditions (1.2)<sub>2</sub>, and gives rise to a sum  $|\sum_{l=1}^{Z+1} \tau_l \sum_{i \in J_N} u_h^{(l)}(Q_i) \int_{S_i} f_N(x, t_l) \, do_x|$  on the right-hand side of (4.2) if the coefficients  $\alpha_i^{(l)}$  are chosen as in (4.10). The aim of the two ensuing lemmas consists in estimating this sum against the data times the expression  $\mathfrak{A}(u_h^{(1)}, \dots, u_h^{(Z+1)})$  (see (2.15)), which coincides with the left-hand side of our stability result (2.17). Such an estimate inevitably gives rise to a factor  $\nu^{-K}$  for some  $K \geq 1/4$ , as is shown by a counterexample in Section 5. The challenge then is to arrive at an exponent  $K > 1/4$  as close to  $1/4$  as possible. We will obtain  $K = 1/4 + \delta/2$ , for some  $\delta > 0$  which may be chosen as small as we wish, but must be kept fixed. A key role in our proof is played by the trace estimate in Theorem 2.2, which we use instead of the standard estimate of the  $H^{1/2}$ -norm on  $\partial\Omega$  against the  $H^1$ -norm on  $\Omega$ . On the basis of that latter estimate, we would end up with  $K = 1/2$  instead of  $K = 1/4 + \delta/2$ . We further have to deal with the difficulty that we cannot see why for  $v_h \in X_h$ , the relation  $v_h|_{\partial\Omega} \in H^{1/2+\delta/2}(\partial\Omega)$  should hold. For this reason, we approximate functions from  $X_h$  by functions from  $W_h$ , using Lemma 2.10 for that purpose. The space  $W_h$  is a subspace of  $H^1(\Omega)$  so that the trace estimate from Theorem 2.2 may be applied to functions from  $W_h$ .

**Lemma 4.6.** *Recall the parameter  $\delta$  introduced in Theorem 2.2. Let  $\psi \in L^2(\Gamma_N)$ ,  $v_h \in V_h$  and  $\tilde{J} \subset J_N$ . Then*

$$\left| \sum_{i \in \tilde{J}} \int_{S_i} \psi v_h \, do_x \right| \leq \mathfrak{C} \nu^{-1/4-\delta/2} \left( \sum_{i \in \tilde{J}} \|\psi|_{S_i}\|_{L^2(S_i)}^2 \right)^{1/2} (\|v_h\|_{L^2(\Omega)} + \nu^{1/2} \|\nabla_h v_h\|_{L^2(\Omega)}).$$

*Proof.* We use the operator  $\mathfrak{E}_h$  introduced in Lemma 2.10. Applying Lemma 2.8, we get

$$\begin{aligned} & \left( \sum_{i \in \tilde{J}} \|v_h|_{S_i}\|_{L^2(S_i)}^2 \right)^{1/2} \leq \left( \sum_{i \in J_N} \|v_h - \mathfrak{E}_h(v_h)|_{S_i}\|_{L^2(S_i)}^2 \right)^{1/2} + \left( \sum_{i \in J_N} \|\mathfrak{E}_h(v_h)|_{S_i}\|_{L^2(S_i)}^2 \right)^{1/2} \\ & \leq c \left( \sum_{i \in J_N} [h_{K_i}^{-1} \|v_h - \mathfrak{E}_h(v_h)|_{K_i}\|_{L^2(K_i)}^2 + h_{K_i} \|\nabla(v_h - \mathfrak{E}_h(v_h)|_{K_i})\|_{L^2(K_i)}^2] \right)^{1/2} \\ & \quad + \|\mathfrak{E}_h(v_h)|_{\partial\Omega}\|_{L^2(\partial\Omega)} \\ & \leq c \left( \sum_{K \in \mathfrak{I}} [h_K^{-1} \|v_h - \mathfrak{E}_h(v_h)|_K\|_{L^2(K)}^2 + h_K \|\nabla(v_h|_K)\|_{L^2(K)}^2 + h_K \|\nabla \mathfrak{E}_h(v_h)|_K\|_{L^2(K)}^2] \right)^{1/2} \\ & \quad + \|\mathfrak{E}_h(v_h)|_{\partial\Omega}\|_{H^\delta(\partial\Omega)}. \end{aligned} \tag{4.15}$$

It follows with the first inequality in Lemma 2.10 that

$$\left( \sum_{i \in \tilde{J}} \|v_h|_{S_i}\|_{L^2(S_i)}^2 \right)^{1/2} \leq c \left( \sum_{K \in \mathfrak{I}} (h_K \|\nabla(v_h|_K)\|_{L^2(K)}^2 + h_K \|\nabla \mathfrak{E}_h(v_h)|_K\|_{L^2(K)}^2) \right)^{1/2} + \|\mathfrak{E}_h(v_h)|_{\partial\Omega}\|_{H^\delta(\partial\Omega)}.$$

Due to the inverse inequality from Lemma 2.9, we may now conclude that  $(\sum_{i \in \tilde{J}} \|v_h|_{S_i}\|_{L^2(S_i)}^2)^{1/2}$  is bounded by

$$c \left( \sum_{K \in \mathfrak{I}} (\|v_h|_K\|_{L^2(K)} \|\nabla(v_h|_K)\|_{L^2(K)} + \|\mathfrak{E}_h(v_h)|_K\|_{L^2(K)} \|\nabla \mathfrak{E}_h(v_h)|_K\|_{L^2(K)}) \right)^{1/2} + \|\mathfrak{E}_h(v_h)|_{\partial\Omega}\|_{H^\delta(\partial\Omega)},$$

and so by

$$c (\|v_h\|_{L^2(\Omega)} \|\nabla_h v_h\|_{L^2(\Omega)} + \|\mathfrak{E}_h(v_h)\|_{L^2(\Omega)} \|\nabla_h \mathfrak{E}_h(v_h)\|_{L^2(\Omega)})^{1/2} + \|\mathfrak{E}_h(v_h)|_{\partial\Omega}\|_{H^\delta(\partial\Omega)}.$$

Hence by the second and third inequality in Lemma 2.10, and by Theorems 2.2, 2.3 and 2.6,

$$\begin{aligned}
& \left( \sum_{i \in \tilde{\mathcal{J}}} \|v_h|_{S_i}\|_{L^2(\Omega)}^2 \right)^{1/2} \leq \mathfrak{C} (\|v_h\|_{L^2(\Omega)}^{1/2} \|\nabla_h v_h\|_{L^2(\Omega)}^{1/2} + \|v_h\|_{L^2(\Omega)} + \|\mathfrak{E}_h(v_h)\|_{H^{\delta+1/2}(\Omega)}) \\
& \leq \mathfrak{C} (\|v_h\|_{L^2(\Omega)}^{1/2} \|\nabla_h v_h\|_{L^2(\Omega)}^{1/2} + \|\mathfrak{E}_h(v_h)\|_{L^2(\Omega)}^{1/2-\delta} \|\mathfrak{E}_h(v_h)\|_{H^1(\Omega)}^{\delta+1/2}) \\
& \leq \mathfrak{C} (\|v_h\|_{L^2(\Omega)}^{1/2} \|\nabla_h v_h\|_{L^2(\Omega)}^{1/2} + \|v_h\|_{L^2(\Omega)}^{1/2-\delta} (\|v_h\|_{L^2(\Omega)} + \|\nabla_h v_h\|_{L^2(\Omega)})^{1/2+\delta}) \\
& \leq \mathfrak{C} \|v_h\|_{L^2(\Omega)}^{1/2-\delta} \|\nabla_h v_h\|_{L^2(\Omega)}^{1/2+\delta}.
\end{aligned}$$

By Young's inequality for real numbers as in ([1], p. 23, (4)), we may conclude for  $\kappa > 0$  that

$$\nu^\kappa \left( \sum_{i \in \tilde{\mathcal{J}}} \|v_h|_{S_i}\|_{L^2(S_i)}^2 \right)^{1/2} \leq \mathfrak{C} (\|v_h\|_{L^2(\Omega)} + \nu^{\kappa(1/2+\delta)-1} \|\nabla_h v_h\|_{L^2(\Omega)}).$$

Choosing  $\kappa = 1/4 + \delta/2$ , we now obtain

$$\begin{aligned}
& \left| \sum_{i \in \tilde{\mathcal{J}}} \int_{S_i} \psi v_h \, dx \right| \leq \mathfrak{C} \nu^{-1/4-\delta/2} \left( \sum_{i \in \tilde{\mathcal{J}}} \|\psi|_{S_i}\|_{L^2(S_i)}^2 \right)^{1/2} \nu^{1/4+\delta/2} \left( \sum_{i \in \tilde{\mathcal{J}}} \|v_h|_{S_i}\|_{L^2(S_i)}^2 \right)^{1/2} \\
& \leq \mathfrak{C} \nu^{-1/4-\delta/2} \left( \sum_{i \in \tilde{\mathcal{J}}} \|\psi|_{S_i}\|_{L^2(S_i)}^2 \right)^{1/2} (\|v_h\|_{L^2(\Omega)} + \nu^{1/2} \|\nabla_h v_h\|_{L^2(\Omega)}). \quad \square
\end{aligned}$$

**Lemma 4.7.** *Let  $\psi \in L^2(\Gamma_N)$ ,  $v_h \in V_h$  and  $\tilde{\mathcal{J}} \subset J_N$ . Then*

$$\left| \sum_{i \in \tilde{\mathcal{J}}} v_h(Q_i) \int_{S_i} \psi \, dx \right| \leq \mathfrak{C} \nu^{-1/4-\delta/2} \left( \sum_{i \in \tilde{\mathcal{J}}} \|\psi|_{S_i}\|_{L^2(S_i)}^2 \right)^{1/2} (\|v_h\|_{L^2(\Omega)} + \nu^{1/2} \|\nabla_h v_h\|_{L^2(\Omega)}).$$

*Proof.* We find with Lemma 4.6 that

$$\begin{aligned}
& \left| \sum_{i \in \tilde{\mathcal{J}}} v_h(Q_i) \int_{S_i} \psi \, dx \right| \leq \left| \sum_{i \in \tilde{\mathcal{J}}} \int_{S_i} \psi v_h \, dx \right| + \left| \sum_{i \in \tilde{\mathcal{J}}} \int_{S_i} \psi(x) (v_h(Q_i) - v_h(x)) \, dx \right| \\
& \leq \mathfrak{C} \nu^{-1/4-\delta/2} \left( \sum_{i \in \tilde{\mathcal{J}}} \|\psi|_{S_i}\|_{L^2(S_i)}^2 \right)^{1/2} (\|v_h\|_{L^2(\Omega)} + \nu^{1/2} \|\nabla_h v_h\|_{L^2(\Omega)}) + \mathfrak{A}, \quad (4.16)
\end{aligned}$$

where  $\mathfrak{A} := \sum_{i \in \tilde{\mathcal{J}}} \int_{S_i} |\psi(x)| |\nabla(v_h|_{K_i})| |Q_i - x| \, dx$ . But with (2.4) and since  $l_i \leq h_{K_i}$ ,

$$\begin{aligned}
\mathfrak{A} & \leq \sum_{i \in \tilde{\mathcal{J}}} \int_{S_i} |\psi(x)| \, dx |\nabla(v_h|_{K_i})| h_{K_i} \leq \sum_{i \in \tilde{\mathcal{J}}} \|\psi|_{S_i}\|_{L^2(S_i)} |\nabla(v_h|_{K_i})| h_{K_i}^{3/2} \\
& \leq c \sum_{i \in \tilde{\mathcal{J}}} \|\psi|_{S_i}\|_{L^2(S_i)} (|K_i| |\nabla(v_h|_{K_i})|^2)^{1/2} h_{K_i}^{1/2} \leq c \sum_{i \in \tilde{\mathcal{J}}} \|\psi|_{S_i}\|_{L^2(S_i)} \|\nabla(v_h|_{K_i})\|_{L^2(K_i)} h_{K_i}^{1/2} \\
& \leq c \left( \sum_{i \in \tilde{\mathcal{J}}} \|\psi|_{S_i}\|_{L^2(S_i)}^2 \right)^{1/2} \left( \sum_{K \in \mathfrak{T}} \|\nabla(v_h|_K)\|_{L^2(K)}^2 h_K \right)^{1/2}.
\end{aligned}$$

Using Lemma 2.9 (inverse inequality), we thus see that

$$\begin{aligned} \mathfrak{A} &\leq c \left( \sum_{i \in \tilde{\mathcal{J}}} \|\psi|_{S_i}\|_{L^2(S_i)}^2 \right)^{1/2} \left( \sum_{K \in \tilde{\mathcal{T}}} \|v_h|_K\|_{L^2(K)} \|\nabla(v_h|_K)\|_{L^2(K)} \right)^{1/2} \\ &\leq c \left( \sum_{i \in \tilde{\mathcal{J}}} \|\psi|_{S_i}\|_{L^2(S_i)}^2 \right)^{1/2} \|v_h\|_{L^2(\Omega)}^{1/2} \|\nabla_h v_h\|_{L^2(\Omega)}^{1/2}. \end{aligned}$$

On the other hand, as in the proof of Lemma 4.6, we observe that by Poincaré’s inequality (Thm. 2.6) and Young’s inequality for real numbers,

$$\nu^{1/4+\delta/2} \|v_h\|_{L^2(\Omega)}^{1/2} \|\nabla_h v_h\|_{L^2(\Omega)}^{1/2} \leq c \nu^{1/4+\delta/2} \|v_h\|_{L^2(\Omega)}^{1/2-\delta} \|\nabla_h v_h\|_{L^2(\Omega)}^{1/2+\delta} \leq c (\|v_h\|_{L^2(\Omega)} + \nu^{1/2} \|\nabla_h v_h\|_{L^2(\Omega)}).$$

This estimate and the preceding one yield that

$$\mathfrak{A} \leq c \nu^{-1/4-\delta/2} \left( \sum_{i \in \tilde{\mathcal{J}}} \|\psi|_{S_i}\|_{L^2(S_i)}^2 \right)^{1/2} \left( \|v_h\|_{L^2(\Omega)} + \nu^{1/2} \|\nabla_h v_h\|_{L^2(\Omega)} \right),$$

so Lemma 4.7 follows from (4.16). □

**Corollary 4.8.** *Let  $v_h^{(1)}, \dots, v_h^{(Z+1)} \in V_h$  and  $\tilde{\mathcal{J}} \subset J_N$ . Then*

$$\begin{aligned} \left| \sum_{l=1}^{Z+1} \tau_l \sum_{i \in \tilde{\mathcal{J}}} v_h^{(l)}(Q_i) \int_{S_i} f_N(x, t_l) \, do_x \right| &\leq c \nu^{-1/4-\delta/2} \left( \sum_{l=1}^{Z+1} \tau_l \sum_{i \in \tilde{\mathcal{J}}} \|f_N(t_l)|_{S_i}\|_{L^2(S_i)}^2 \right)^{1/2} \\ &\cdot \left[ \left( \sum_{l=1}^{Z+1} \tau_l \|v_h^{(l)}\|_{L^2(\Omega)}^2 \right)^{1/2} + \nu^{1/2} \left( \sum_{l=1}^{Z+1} \tau_l \|\nabla_h v_h\|_{L^2(\Omega)}^2 \right)^{1/2} \right]. \end{aligned}$$

*Proof.* Lemma 4.7. □

**Lemma 4.9.** *Let  $\epsilon > 0$  and let  $v_h^{(1)}, \dots, v_h^{(Z+1)} \in V_h$ . Then, with  $J_N^{(\epsilon)}$  defined in Theorem 2.14,*

$$\begin{aligned} &\left| \sum_{l=1}^{Z+1} \tau_l \sum_{i \in J_N^{(\epsilon)}} v_h^{(l)}(Q_i) \int_{S_i} f_N(x, t_l) \, do_x \right| \\ &\leq \epsilon^{-1/2} \left( \sum_{l=1}^{Z+1} \tau_l \sum_{i \in J_N^{(\epsilon)}} \|f_N(t_l)|_{S_i}\|_{L^2(S_i)}^2 \right)^{1/2} \left( \sum_{l=1}^{Z+1} \tau_l \sum_{i \in J_N} v_h^{(l)}(Q_i)^2 \int_{S_i} |b(x, t_l) \cdot n(x)| \, do_x \right)^{1/2}. \end{aligned}$$

*Proof.* Let  $v_h \in V_h$ ,  $t \in (0, T)$ . Then by the definition of  $J_N^{(\epsilon)}$ ,

$$\begin{aligned} &\left| \sum_{i \in J_N^{(\epsilon)}} v_h(Q_i) \int_{S_i} f_N(x, t) \, do_x \right| \leq \sum_{i \in J_N^{(\epsilon)}} \left( \int_{S_i} |f_N(x, t)|^2 \, do_x \right)^{1/2} l_i^{1/2} |v_h(Q_i)| \\ &\leq \epsilon^{-1/2} \sum_{i \in J_N^{(\epsilon)}} \left( \int_{S_i} |f_N(x, t)|^2 \, do_x \right)^{1/2} \left( \int_{S_i} |b(x, t) \cdot n(x)| \, do_x \right)^{1/2} |v_h(Q_i)| \end{aligned}$$



$$\leq \epsilon^{-1/2} \left( \sum_{i \in J_N^{(\epsilon)}} \|f_N(t)|_{S_i}\|_{L^2(S_i)}^2 \right)^{1/2} \left( \sum_{i \in J_N} v_h(Q_i)^2 \int_{S_i} |b(x, t) \cdot n(x)| \, d\sigma_x \right)^{1/2}.$$

This proves the lemma. □

Now we are in a position to carry out the

*Proof of Theorem 2.14.* Due to Lemma 3.2 and (1.6), it follows as in the proof of ([15], (4.1), (4.2)) that problem (2.14), and thus also problem (2.11), (2.12), admits a unique solution.

Suppose that  $h \leq h_0$ , with  $h_0$  from (2.16). Let  $u_h^{(1)}, \dots, u_h^{(Z+1)} \in V_h$  be a solution of (2.14), and let  $\epsilon > 0$ . Recall that  $J_N^{(\epsilon)}$  was defined in Theorem 2.14. Choose the coefficients  $\alpha_i^{(l)}$ , for  $i \in J$ ,  $1 \leq l \leq Z + 1$ , as in (4.10). Then by (2.6) and (2.7), for  $v_h \in V_h$ ,  $0 \leq k \leq Z$ , the sum  $\sum_{i \in J} \alpha_i^{(k+1)} v_h(Q_i)$  coincides with the right-hand side of (2.14). Thus the family  $(u_h^{(1)}, \dots, u_h^{(Z+1)})$  solves (4.1). Recall the definition of  $\mathfrak{A}(v_h^{(1)}, \dots, v_h^{(Z+1)})$  in (2.15), for  $v_h^{(1)}, \dots, v_h^{(Z+1)} \in V_h$ . Let  $\mathfrak{R}$  denote the expression appearing inside brackets on the right-hand side of (2.17), but with the term  $\|w^{(0)}\|_{H^1(\Omega)} + \max_{0 \leq l \leq Z+1} \|\tilde{f}_D(t_l)\|_{H^{1/2}(\partial\Omega)}$  omitted. Then equation (4.1), Theorem 4.1 and 4.4, Lemma 4.2, 4.3 and 4.5, Corollary 4.8 with  $\tilde{J} = J_N \setminus J_N^{(\epsilon)}$ , and Lemma 4.9 yield

$$\mathfrak{A}(u_h^{(1)}, \dots, u_h^{(Z+1)})^2 \leq \mathfrak{C} \gamma \left[ \left( \mathfrak{A}(u_h^{(1)}, \dots, u_h^{(Z+1)}) + \mathfrak{A}(\tilde{u}_h^{(1)}, \dots, \tilde{u}_h^{(Z+1)}) \right) \mathfrak{R} + \|w^{(0)} - f(0)\|_{H^1(\Omega)}^2 \right], \tag{4.17}$$

where  $\tilde{u}_h^{(l)} := \varrho(u_h^{(l)} \zeta(t_l))$  for  $1 \leq l \leq Z + 1$ , and where we used the abbreviation

$$\gamma := (1 + \bar{\nu}/\nu + \|\partial_3 \zeta\|_{L^1(0, T, L^\infty(\Omega))}) \left( 1 + \max_{1 \leq l \leq Z+1} \|b(t_l)\|_{H^1(\Omega)} \right).$$

Note that  $u_h^{(0)} = I_h(w^{(0)} - f(0))$  by (2.14), and  $\|I_h(w^{(0)} - f(0))\|_{L^2(\Omega)} \leq \mathfrak{C}\|w^{(0)} - f(0)\|_{H^1(\Omega)}$  by Theorem 2.11.

By (2.1) and (2.7) we have  $\|\tilde{u}_h^{(l)}\|_{L^2(\Omega)} \leq \varphi_1 \|u_h^{(l)}\|_{L^2(\Omega)}$  for  $1 \leq l \leq Z + 1$ , with  $\varphi_1$  from (2.1). Obviously  $\tilde{u}_h^{(l)}(Q_i)^2 \leq \varphi_1^2 u_h^{(l)}(Q_i)^2$  ( $i \in J$ ,  $1 \leq l \leq Z + 1$ ). These relations and Lemma 3.1 yield  $\mathfrak{A}(\tilde{u}_h^{(1)}, \dots, \tilde{u}_h^{(Z+1)}) \leq \mathfrak{C} \mathfrak{A}(u_h^{(1)}, \dots, u_h^{(Z+1)})$ , so we may deduce from (4.17)

$$\mathfrak{A}(u_h^{(1)}, \dots, u_h^{(Z+1)})^2 \leq \mathfrak{C} \gamma \left( \mathfrak{A}(u_h^{(1)}, \dots, u_h^{(Z+1)}) \mathfrak{R} + \|w^{(0)} - f(0)\|_{H^1(\Omega)}^2 \right).$$

Hence  $\mathfrak{A}(u_h^{(1)}, \dots, u_h^{(Z+1)})^2 \leq \mathfrak{C} \gamma^2 \mathfrak{R}^2 + \mathfrak{C} \gamma \|w^{(0)} - f(0)\|_{H^1(\Omega)}^2$ . As a consequence

$$\mathfrak{A} \left( u_h^{(1)} + f_h(t_1), \dots, u_h^{(Z+1)} + f_h(t_{Z+1}) \right) \leq \mathfrak{C} \gamma \left( \mathfrak{R} + \|w^{(0)} - f(0)\|_{H^1(\Omega)} \right) + \mathfrak{A}(f_h(t_1), \dots, f_h(t_{Z+1})). \tag{4.18}$$

In order to estimate the term  $\mathfrak{A}(f_h(t_1), \dots, f_h(t_{Z+1}))$ , we first observe that by (2.13), (2.2), Theorem 2.11 and the continuity of the extension operator  $E$ ,

$$\|\nabla_h f_h(t)\|_{L^2(\Omega)} + \|f_h(t)\|_{L^2(\Omega)} \leq \mathfrak{C} \|\tilde{f}_D(t)\|_{H^{1/2}(\partial\Omega)} \quad \text{for } t \in (0, T). \tag{4.19}$$

Moreover, for  $t \in (0, T)$ ,

$$\begin{aligned} \mathfrak{A} &:= \left( \sum_{i \in J_N} f_h(t)(Q_i)^2 \int_{S_i} |b(x, t) \cdot n(x)| \, do_x \right)^{1/2} \leq |b(t)|_\infty^{1/2} \left( \sum_{i \in J_N} f_h(t)(Q_i)^2 l_i \right)^{1/2} \\ &\leq |b(t)|_\infty^{1/2} \left[ \left( \sum_{i \in J_N} \int_{S_i} (f_h(t)(Q_i) - f_h(t)(x))^2 \, do_x \right)^{1/2} \right. \\ &\quad \left. + \left( \sum_{i \in J_N} \int_{S_i} (f_h(t)(x) - f(x, t))^2 \, do_x \right)^{1/2} + \left( \sum_{i \in J_N} \|f(t)|_{S_i}\|_{L^2(S_i)}^2 \right)^{1/2} \right]. \end{aligned} \tag{4.20}$$

But for  $t \in (0, T)$ ,  $i \in J_N$ , by (2.4),

$$\begin{aligned} \int_{S_i} (f_h(t)(Q_i) - f_h(t)(x))^2 \, do_x &\leq |\nabla(f_h(t)|_{K_i})|^2 \int_{S_i} |Q_i - x|^2 \, do_x \leq |\nabla(f_h(t)|_{K_i})|^2 h_{K_i}^3 \\ &\leq \mathfrak{C} \int_{K_i} |\nabla(f_h(t)|_{K_i})|^2 \, do_x \leq \mathfrak{C} \|\nabla f(t)|_{K_i}\|_{L^2(K_i)}^2, \end{aligned}$$

where the last inequality follows from (2.13) and Theorem 2.11, and where we estimated a factor  $h_{K_i}$  by  $\text{diam } \Omega$ . Moreover, for  $i, t$  as before, we find with (2.13) and Lemma 2.8 that the integral  $\int_{S_i} (f_h(t)(x) - f(x, t))^2 \, do_x$  is bounded by

$$c \left[ h_{K_i}^{-1} \|f_h(t) - f(t)|_{K_i}\|_{L^2(K_i)}^2 + h_{K_i} \|\nabla(f_h(t) - f(t)|_{K_i})\|_{L^2(K_i)}^2 \right]$$

and hence by  $c h_{K_i} \|\nabla f(t)|_{K_i}\|_{L^2(K_i)}^2$  according to Theorem 2.11. Again estimating a factor  $h_{K_i}$  by  $\text{diam } \Omega$ , we may thus conclude from (4.20) that

$$\begin{aligned} \mathfrak{A} &\leq \mathfrak{C} |b(t)|_\infty^{1/2} \left( \sum_{i \in J_N} \|\nabla f(t)|_{K_i}\|_{L^2(K_i)}^2 + \|f(t)|_{\Gamma_N}\|_{L^2(\Gamma_N)}^2 \right)^{1/2} \\ &\leq \mathfrak{C} |b(t)|_\infty^{1/2} (\|\nabla f(t)\|_{L^2(\Omega)} + \|f(t)|_{\partial\Omega}\|_{L^2(\partial\Omega)}) \leq \mathfrak{C} |b(t)|_\infty^{1/2} \|\tilde{f}_D(t)\|_{H^{1/2}(\partial\Omega)}, \end{aligned}$$

where the last inequality follows from (2.2) and the continuity of the extension operator  $E$ . From (4.19) and the preceding estimate we may deduce that  $\mathfrak{A}(f_h(t_1), \dots, f_h(t_{Z+1}))$  is bounded by

$$\mathfrak{C} \left( 1 + \max_{1 \leq l \leq Z+1} |b(t_l)|_\infty + \nu^{1/2} \right) \left( \sum_{l=1}^{Z+1} \tau_l \|\tilde{f}_D(t_l)\|_{H^{1/2}(\partial\Omega)}^2 \right)^{1/2} + \mathfrak{C} \max_{1 \leq l \leq Z+1} \|\tilde{f}_D(t_l)\|_{H^{1/2}(\partial\Omega)}.$$

Inequality (4.18) now implies inequality (2.17). □

### 5. A COUNTEREXAMPLE

We want to show that the factor  $\nu^{-1/4-\delta/2}$  in Theorem 2.15 cannot be replaced by  $\nu^{-K}$  with some  $K < 1/4$ . Of course, this means such a modification should not be possible in Theorem 2.14 either. We consider the case that  $\Omega$  is a square, with two sides corresponding to  $\Gamma_N$ , and the other two to  $\Gamma_D$ . We choose  $G = 0$ ,  $F_D = 0$ ,  $A = \nu(\delta_{jk})_{1 \leq j, k \leq 2}$ ,  $B = (0, -1)$ . This convective velocity  $B$  is orthogonal to one edge – denoted by  $\Gamma_N^{(1)}$  – constituting  $\Gamma_N$ , and parallel to the other one – denoted by  $\Gamma_N^{(2)}$ . In this situation, for any  $\epsilon \in (0, 1)$ , the index set  $\tilde{J}_N^{(\epsilon)}$  defined in Theorem 2.15 is such that  $\cup_{i \in \tilde{J}_N^{(\epsilon)}} S_i = \Gamma_N^{(1)}$  and  $\cup_{i \in J_N \setminus \tilde{J}_N^{(\epsilon)}} S_i = \Gamma_N^{(2)}$ .

(Recall that  $S_i \subset \Gamma_N$  for  $i \in J_N$ ,  $S_i \subset \Gamma_D$  for  $i \in J_D$ , and  $J \setminus J^o = J_N \cup J_D$ .) Thus the term  $\|F_N|_{\Gamma_N^{(1)}}\|_{L^2(\Gamma_N^{(1)})}$  in Theorem 5.1 coincides with the term  $(\sum_{i \in \tilde{J}_N^{(e)}} \|F_N|_{S_i}\|_{L^2(S_i)}^2)^{1/2}$  in Theorem 2.15, and  $\|F_N|_{\Gamma_N^{(2)}}\|_{L^2(\Gamma_N^{(2)})}$  equals  $(\sum_{i \in J_N \setminus \tilde{J}_N^{(e)}} \|F_N|_{S_i}\|_{L^2(S_i)}^2)^{1/2}$ . It is true that Theorem 5.1 relates to the continuous problem (1.7), (1.8) and not to the discrete one (2.18), (2.19). But since the right-hand side of (2.20) is uniform in  $h \in (0, h_0]$ , and in view of the preceding remarks as to which terms in Theorem 5.1 and 2.15 coincide, the negative result in Theorem 5.1 should carry over to (2.20) in the sense specified above, that is, the factor  $\nu^{-1/4-\delta/2}$  cannot be replaced by  $\nu^{-K}$  with some  $K < 1/4$ .

**Theorem 5.1.** *Suppose that  $\Omega = (0, 1)^2$ ,  $\Gamma_D = ((0, 1) \times \{0\}) \cup (\{0\} \times (0, 1))$ ,  $\Gamma_N = \Gamma_N^{(1)} \cup \Gamma_N^{(2)}$  with  $\Gamma_N^{(1)} := (0, 1) \times \{1\}$ ,  $\Gamma_N^{(2)} := \{1\} \times (0, 1)$ . Let  $\kappa \in (0, 1)$ . Then there is no constant  $C_0 > 0$  such that*

$$\|W\|_{L^2(\Omega)} \leq C_0 (\|F_N|_{\Gamma_N^{(1)}}\|_{L^2(\Gamma_N^{(1)})} + \nu^{-1/4+\kappa} \|F_N|_{\Gamma_N^{(2)}}\|_{L^2(\Gamma_N^{(2)})}) \quad (5.1)$$

for each  $\nu \in (0, \infty)$ ,  $W \in C^\infty(\overline{\Omega})$  satisfying (1.7) in the case  $B = (0, -1)$ ,  $M = 0$ ,  $A = \nu(\delta_{jk})_{1 \leq j, k \leq 2}$ ,  $G = 0$ , and (1.8) with  $F_D = 0$ ,  $B = (0, -1)$ ,

$$F_N(x) = -\min\{B(x) \cdot n(x), 0\} W(x) + \nu \partial W(x) / \partial n(x) \quad \text{for } x \in \Gamma_N.$$

*Proof.* Let  $\nu_1 \in (0, 1]$  be so small that  $4\nu^{1-\kappa} \leq 1$  for  $\nu \in (0, \nu_1]$ . Let  $\nu \in (0, \nu_1]$  and put

$$\begin{aligned} v_\nu(t) &:= e^{t\nu^{-(1+\kappa)/2}} - e^{-t\nu^{-(1+\kappa)/2}} \quad (t \in \mathbb{R}), \quad z_1^{(\nu)} := (2\nu)^{-1} \left(-1 + \sqrt{1 - 4\nu^{1-\kappa}}\right), \\ z_2^{(\nu)} &:= (2\nu)^{-1} \left(-1 - \sqrt{1 - 4\nu^{1-\kappa}}\right), \quad w_\nu(t) := e^{z_1^{(\nu)} t} - e^{z_2^{(\nu)} t} \quad (t \in \mathbb{R}), \end{aligned}$$

$W_\nu(x) := v_\nu(x_1) w_\nu(x_2)$  for  $x \in \overline{\Omega}$ . Then  $W_\nu \in C^\infty(\overline{\Omega})$ ,  $-\nu \Delta W_\nu - \partial_2 W_\nu = 0$  and  $W_\nu|_{\Gamma_D} = 0$ . Let us determine a lower bound of  $\|W_\nu\|_{L^2(\Omega)}$  in terms of  $\|v_\nu\|_{L^2(0,1)}$ . To this end we observe that  $z_1^{(\nu)} + z_2^{(\nu)} = -\nu^{-1}$ ,  $-\nu^{-1} \leq z_2^{(\nu)} \leq -\nu^{-1}$  and  $z_1^{(\nu)} = -2\nu^{-\kappa}/(1 + \sqrt{1 - 4\nu^{1-\kappa}})$ , so  $-2\nu^{-\kappa} \leq z_1^{(\nu)} \leq -\nu^{-\kappa}$ . Thus

$$\begin{aligned} \|w_\nu\|_{L^2(0,1)}^2 &= \left(e^{2z_1^{(\nu)}} - 1\right) / \left(2z_1^{(\nu)}\right) + \left(e^{2z_2^{(\nu)}} - 1\right) / \left(2z_2^{(\nu)}\right) + 2 \left(1 - e^{z_1^{(\nu)} + z_2^{(\nu)}}\right) / \left(z_1^{(\nu)} + z_2^{(\nu)}\right) \\ &\geq -1 / \left(2z_1^{(\nu)}\right) - e^{2z_1^{(\nu)}} / \left(2|z_1^{(\nu)}|\right) - 3\nu \geq \nu^\kappa / 4 - \nu^\kappa e^{-2\nu^{-\kappa}} / 2 - 3\nu. \end{aligned}$$

We may choose  $\nu_2 \in (0, \nu_1]$  so small that  $\nu^\kappa / 4 - \nu^\kappa e^{-2\nu^{-\kappa}} / 2 - 3\nu \geq \nu^\kappa / 8$  for  $\nu \in (0, \nu_2]$ . Thus, for such  $\nu$ , we have  $\|w_\nu\|_{L^2(0,1)} \geq \nu^{\kappa/2} 8^{-1/2}$ , hence  $\|W_\nu\|_{L^2(\Omega)} \geq \nu^{\kappa/2} \|v_\nu\|_{L^2(0,1)} 8^{-1/2}$ .

Put  $B(x) := (0, -1)$  for  $x \in \overline{\Omega}$ ,  $F_N^{(\nu)}(x) := \nu \partial W_\nu(x) / \partial n(x) - \min\{B(x) \cdot n(x), 0\} W_\nu(x)$  for  $x \in \Gamma_N$ . Let  $\nu_3 \in (0, \nu_2]$  with  $2\nu \leq \nu^\kappa$  for  $\nu \in (0, \nu_3]$ . Then, for  $x \in \Gamma_N^{(1)}$ ,  $\nu \in (0, \nu_3]$ , we find

$$|F_N^{(\nu)}(x)| \leq |v_\nu(x_1)| (\nu |w_\nu'(1)| + |w_\nu(1)|), \quad |w_\nu(1)| \leq e^{z_1^{(\nu)}} + e^{z_2^{(\nu)}} \leq 2e^{-\nu^{-\kappa}},$$

and  $\nu |w_\nu'(1)| \leq \nu (|z_1^{(\nu)}| e^{z_1^{(\nu)}} + |z_2^{(\nu)}| e^{z_2^{(\nu)}}) \leq 3e^{-\nu^{-\kappa}}$ , so that  $|F_N^{(\nu)}(x)| \leq 5e^{-\nu^{-\kappa}} |v_\nu(x_1)|$ . Therefore

$$\|F_N^{(\nu)}|_{\Gamma_N^{(1)}}\|_{L^2(\Gamma_N^{(1)})} \leq 5e^{-\nu^{-\kappa}} \|v_\nu\|_{L^2(0,1)} \leq 58^{1/2} e^{-\nu^{-\kappa}} \nu^{-\kappa/2} \|W_\nu\|_{L^2(\Omega)} \quad \text{for } \nu \in (0, \nu_3], \quad (5.2)$$

where we used the lower bound of  $\|W_\nu\|_{L^2(\Omega)}$  determined above. For  $x \in \Gamma_N^{(2)}$ , we have  $B(x) \cdot n(x) = 0$ , so  $F_N^{(\nu)}(x) = \nu v_\nu'(1) w_\nu(x_2)$ . But  $|\nu v_\nu'(1)| \leq 2\nu^{(1-\kappa)/2} e^{\nu^{-(1+\kappa)/2}}$ , hence

$$\nu^{-1/4+\kappa} \|F_N^{(\nu)}|_{\Gamma_N^{(2)}}\|_{L^2(\Gamma_N^{(2)})} \leq 2\nu^{1/4+\kappa/2} e^{\nu^{-(1+\kappa)/2}} \|w_\nu\|_{L^2(0,1)} \quad (\nu \in (0, \nu_1]). \quad (5.3)$$

On the other hand, for  $\nu \in (0, \nu_1]$ , we get  $\|v_\nu\|_{L^2(0,1)}^2 = \nu^{(1+\kappa)/2} (e^{2\nu^{-(1+\kappa)/2}} - e^{-2\nu^{-(1+\kappa)/2}}) / 2 - 2$ . We choose  $\nu_4 \in (0, \nu_3]$  so small that  $(1/2)e^{2\nu^{-(1+\kappa)/2}} \geq e^{-2\nu^{-(1+\kappa)/2}}$  and  $\nu^{(1+\kappa)/2} e^{2\nu^{-(1+\kappa)/2}} / 8 \geq 2$  for  $\nu \in (0, \nu_4]$ . For such  $\nu$ , we may conclude that  $\|v_\nu\|_{L^2(0,1)}^2 \geq \nu^{(1+\kappa)/2} e^{2\nu^{-(1+\kappa)/2}} / 8$ , hence  $\|W_\nu\|_{L^2(\Omega)} \geq \nu^{(1+\kappa)/4} e^{\nu^{-(1+\kappa)/2}} \|w_\nu\|_{L^2(0,1)} 8^{-1/2}$ . With this estimate we return to (5.3), to obtain

$$\nu^{-1/4+\kappa} \|F_N^{(\nu)} | \Gamma_N^{(2)}\|_{L^2(\Gamma_N^{(2)})} \leq 2\sqrt{8}\nu^{\kappa/4} \|W_\nu\|_{L^2(\Omega)} \quad \text{for } \nu \in (0, \nu_4].$$

This estimate and (5.2) imply

$$\|F_N^{(\nu)} | \Gamma_N^{(1)}\|_{L^2(\Gamma_N^{(1)})} + \nu^{-1/4+\kappa} \|F_N^{(\nu)} | \Gamma_N^{(2)}\|_{L^2(\Gamma_N^{(2)})} \leq \tilde{C} \left( e^{-\nu^{-\kappa}} \nu^{-\kappa/2} + \nu^{\kappa/4} \right) \|W_\nu\|_{L^2(\Omega)}$$

for  $\nu \in (0, \nu_4]$ . Here and in the following, we write  $\tilde{C}$  for numerical constants. There is  $\nu_5 \in (0, \nu_4]$  such that  $\nu^{-\kappa/2} e^{-\nu^{-\kappa}} \leq \nu^{\kappa/4}$  for  $\nu \in (0, \nu_5]$ . Therefore

$$\|F_N^{(\nu)} | \Gamma_N^{(1)}\|_{L^2(\Gamma_N^{(1)})} + \nu^{-1/4+\kappa} \|F_N^{(\nu)} | \Gamma_N^{(2)}\|_{L^2(\Gamma_N^{(2)})} \leq \tilde{C} \nu^{\kappa/4} \|W_\nu\|_{L^2(\Omega)} \quad \text{for } \nu \in (0, \nu_5]. \quad (5.4)$$

Now suppose there is  $C_0 > 0$  such that inequality (5.1) holds. It follows with (5.4) that  $\|W_\nu\|_{L^2(\Omega)} \leq C_0 \tilde{C} \nu^{\kappa/4} \|W_\nu\|_{L^2(\Omega)}$  for  $\nu \in (0, \nu_5]$ , hence  $1 \leq C_0 \tilde{C} \nu^{\kappa/4}$  for such  $\nu$ , which is impossible.  $\square$

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