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A NON-LOCAL MACROSCOPIC MODEL FOR TRAFFIC FLOW

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Abstract. In this work, we propose a non-local Hamilton–Jacobi model for traffic flow and we prove the existence and uniqueness of the solution of this model. This model is justified as the limit of a rescaled microscopic model. We also propose a numerical scheme and we prove an estimate error between the continuous solution of this problem and the numerical one. Finally, we provide some numerical illustrations.

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1. Introduction

Traffic flow modelling is an important challenge and has known an important development in the last decades. The goal of this paper is to propose a new non-local macroscopic model for traffic flow. Macroscopic models consider quantities that describe the collective behaviour of traffic flow. At the macroscopic scale, the most popular model is the LWR model (see [28,31]). This model, expressed in the Eulerian coordinates, describes the dynamics of the density of vehicles. Since these pioneering works, a lot of models have been proposed and we refer to [17] for an overview of these models. More recently other approaches have been proposed. First, using the lagrangian coordinates, the LWR model can be reformulated to describe the dynamics of the spacing (see [27]). Moreover, all these models could be reformulated using a Hamilton–Jacobi equation. Indeed, the link between conservation laws and Hamilton–Jacobi equations has been known to mathematicians for decades [21,25,30], but was brought up to the attention of the traffic flow theory community just recently by [10,11] (see also [24]). At the Hamilton–Jacobi level, the link between eulerian and lagrangian coordinates is given in [20] (see also [26]).

More recently, non-local LWR models have been proposed in order to take into account the action of drivers to the surrounding density of other vehicles, see [3]. We also refer to [4–8,12,18].

In the present paper, we propose a new non-local macroscopic model. This model is expressed in the lagrangian coordinates at the Hamilton–Jacobi level. It then describes the dynamics of the position of the vehicles. In our model, drivers will adapt their velocity to the downstream traffic, assigning greater importance to close vehicles.

Our model is obtained by rescaling a microscopic model, which describes the dynamics of each vehicle individually. We recall in particular that the main advantage of microscopic models is that they are easily justifiable but, with these models, it is difficult to model the traffic at the scale of a road or a city since the number of vehicles becomes too large.

Keywords and phrases. Traffic flow, macroscopic models, non-local model, homogenization, viscosity solutions, Hamilton–Jacobi equations.

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1.1. Description of the model and assumptions

The non-local macroscopic model we propose is the following:

$$\begin{cases} u_t = V\left(\dfrac{1}{\int_0^{+\infty} g(z)\mathrm{d}z}\displaystyle\int_0^{\infty}\left(\frac{u(t,x+z)-u(t,x)}{z}\right)g(z)\mathrm{d}z\right) & \text{on }]0,T[\times\mathbb{R}; \\ u(0,x)=u_0(x) & \text{on } \mathbb{R}. \end{cases} \tag{1.1}$$

In this model, expressed in the Lagrangian coordinates, u represents the position, at time t of the car x. The velocity of the car is given by u_t while the spacing is given by u_x. Moreover, using the link between Lagrangian and Eulerian coordinates, the density of cars is given by $1/u_x$. In this model, the function g, called the weight, is decreasing and takes into account the fact that drivers will adapt their velocity to the downstream traffic, assigning a greater importance to close vehicles. The non-local term $\dfrac{u(t,x+z)-u(t,x)}{z}$ represents the average spacing between vehicles x and $x+z$. Finally, the function V is called an optimal velocity function and is non-negative, non-decreasing and bounded. Precise assumptions on g and V are given below:

(**H1**) $V:\mathbb{R}\to[0,+\infty[$ is Lipschitz continuous and non-negative.
(**H2**) V is non-decreasing on $\mathbb{R}$.
(**H3**) There exists $h_0\in(0,+\infty)$ such that for all $h\le h_0$, $V(h)=0$.
(**H4**) There exists $h_{\max}\in(h_0,+\infty)$ such that for all $h\ge h_{\max}$, $V(h)=V(h_{\max})=V_{\max}$.
(**H5**) $g:[0;+\infty[\to[0;+\infty[$ is $L^1(]0,+\infty[)$.
(**H6**) There exists $\delta>0$ such that for all $x\in[0,1]$ we have that

$$g(x)\ge\delta.$$

(**H7**) The function $z\mapsto z\cdot g(z)$ is $L^1(]0;+\infty[)$.

We denote by (**H**) the set of assumptions (**H1**)–(**H7**).

1.2. Main results

The first main result of the paper is an existence and uniqueness result for the macroscopic non local model (1.1).

Theorem 1.1 (Existence and uniqueness for the macroscopic model)**.** *Assume* (**H**) *and let* u_0 *be a Lipschitz continuous and non-decreasing function. Then Problem* (1.1) *admits a unique solution* u *which is Lipschitz continuous and non-decreasing.*

The second main result is a justification of the macroscopic model. In fact, we will show that our macroscopic model can been obtained by rescaling a microscopic model. More precisely, we consider the following microscopic model:

$$\dot{U}_i(t)=V\left(\frac{1}{\sum_{k=1}^{\infty} g^{\varepsilon}(k)}\sum_{j=1}^{\infty} g^{\varepsilon}(j)\frac{U_{i+j}(t)-U_i(t)}{j}\right) \tag{1.2}$$

where $g^{\varepsilon}(j)=g(\varepsilon j)$. The rescaling in the function g takes into account that the weight of the closer cars is more and more important as ε goes to zero. In this model, the term $\dfrac{U_{i+j}(t)-U_i(t)}{j}$ represents the average distance of two successive vehicles comprised between vehicles i and $i+j$. Then the velocity of the vehicle i depends on the weighted average of these average distances.

Concerning the initial condition, let us assume that, at initial time, vehicles satisfy

$$U_i(0)=\varepsilon^{-1}u_0(i\varepsilon)$$

for some $\varepsilon > 0$ and where u_0 is a Lipschitz continuous function (we denote by L_0 its Lipschitz constant). From a traffic point of view, we also assume that u_0 is non-decreasing.

Then, if we define $\overline{u}^\varepsilon$ by

$$\overline{u}^\varepsilon(t,x) := \varepsilon U_{\lfloor \frac{x}{\varepsilon} \rfloor}\left(\frac{t}{\varepsilon}\right), \tag{1.3}$$

we have the following result.

Theorem 1.2 (Micro-macro limit)**.** *Under assumption* (**H**)*, if moreover u_0 is Lipschitz continuous, then the function $\overline{u}^\varepsilon$ defined in* (1.3) *converges to the unique solution u of* (1.1)*.*

In fact, the idea of the proof of this theorem is inspired by [15, 16] and consists in injecting the microscopic models into a partial differential equation. More precisely, we have that the function $\overline{u}^\varepsilon$ is solution of the following problem

$$\overline{u}^\varepsilon_t = V\left(\frac{1}{\sum_{k=1}^\infty g^\varepsilon(k)}\sum_{j=1}^\infty g^\varepsilon(j)\frac{\overline{u}^\varepsilon(t,x+\varepsilon j)-\overline{u}^\varepsilon(t,x)}{\varepsilon j}\right) \quad \text{on }]0,T[\times\mathbb{R};$$

the result is then obtained by homogenization (*i.e.* a passage as $\varepsilon \to 0$) of this non-local PDE. We refer to [13, 29] for homogenization results in the local case and to [14–16, 22] for the non-local case.

In the last Section of the paper, we also consider the numerical analysis of the macroscopic model (1.1). We propose a finite difference numerical scheme and we prove an error estimate between the solution of the macroscopic model (1.1) and its numerical approximation. We also provide some numerical simulations.

Organization of the paper. This article is organized as follow. In Section 2, we present the microscopic problem. The existence and uniqueness of the solution of this problem are proved. Section 3 is devoted to the study of the macroscopic problem where a comparison principle is proved. The proof of the homogenization result is given in Section 4. Finally in Section 5, we present a numerical scheme with an error estimate result. We also present some numerical simulations.

2. Well-posedness of the microscopic problem

The goal of this section is to give some preliminary results for the following problem

$$\begin{cases} u^\varepsilon_t = V\left(\frac{1}{\sum_{k=1}^\infty g^\varepsilon(k)}\sum_{j=1}^\infty g^\varepsilon(j)\frac{u^\varepsilon(t,x+\varepsilon j)-u^\varepsilon(t,x)}{\varepsilon j}\right) & \text{on }]0,T[\times\mathbb{R}; \\ u^\varepsilon(0,x) = u_0(x) & \text{on } \mathbb{R}. \end{cases} \tag{2.1}$$

We will show in particular a comparison principle as well as an existence result. We shall recall first the definition of a viscosity solution.

Definition 2.1. Let $u : [0,T[\times\mathbb{R} \to \mathbb{R}$ be an upper semi-continuous function and v be a lower semi-continuous function. We assume that there exists a constant C_0 such that

$$|u(t,x) - u_0(x)| \le C_0 t \quad \text{and} \quad |v(t,x) - v_0(x)| \le C_0 t.$$

We say that u is a viscosity sub-solution of (2.1) if $u(0,\cdot) \le u_0$ and if, for all $\phi \in C^1(]0,T[\times\mathbb{R})$ such that $u-\phi$ attends a maximum point in $(\bar{t},\bar{x}) \in]0,T[\times\mathbb{R}$ we have that

$$\phi_t(\bar{t},\bar{x}) \le V\left(\frac{1}{\sum_{k=1}^\infty g^\varepsilon(k)}\sum_{j=1}^\infty g^\varepsilon(j)\frac{u(\bar{t},\bar{x}+\varepsilon j)-u(\bar{t},\bar{x})}{\varepsilon j}\right).$$

We say that v is a viscosity super-solution of equation (2.1) if $v(0,\cdot) \geq u_0$ and if, for all $\phi \in C^1(]0,T[\times\mathbb{R})$ such that $u-\phi$ attends a minimum point in $(\bar{t},\bar{x}) \in]0,T[\times\mathbb{R}$ we have that

$$\phi_t(\bar{t},\bar{x}) \geq V\left(\frac{1}{\sum_{k=1}^{\infty} g^{\varepsilon}(k)} \sum_{j=1}^{\infty} g^{\varepsilon}(j) \frac{u(\bar{t},\bar{x}+\varepsilon j) - u(\bar{t},\bar{x})}{\varepsilon j}\right).$$

Finally, we say that u is a solution of (2.1) if u^* is a sub-solution and u_* is a super-solution of (2.1).

Remark 2.2. We recall that u^* and u_* are respectively the upper and lower semi-continuous envelope of u defined by

$$u^*(t,x) = \limsup_{(s,y)\to(t,x)} u(s,y) \quad \text{and} \quad u_*(t,x) = \liminf_{(s,y)\to(t,x)} u(s,y).$$

We now give the comparison principle for (2.1).

Theorem 2.3 (Comparison principle for (2.1))**.** *Let u and v be respectively a sub- and a super-solution of* (2.1)*. We assume that there exists a constant C_0 such that*

$$|u(t,x) - u_0(x)| \leq C_0 t \quad \text{and} \quad |v(t,x) - u_0(x)| \leq C_0 t.$$

Then

$$u \leq v \quad \text{on } [0,T]\times\mathbb{R}.$$

Proof. For $\eta > 0$, let

$$M = \sup_{(t,x)} \left\{ u(t,x) - v(t,x) - \frac{\eta}{T-t} \right\}.$$

We assume, by contradiction, that $M > 0$. For $\alpha, \theta, \delta > 0$, we duplicate the variable by considering

$$M_{\theta,\delta} = \sup_{(t,s,x,y)} \left\{ u(t,x) - v(s,y) - \alpha x^2 - \frac{\eta}{T-t} - \frac{(t-s)^2}{2\delta} - \frac{(x-y)^2}{2\theta} \right\}.$$

Let $(\tilde{t},\tilde{s},\tilde{x},\tilde{y})$ be a point of maximum of $M_{\theta,\delta}$. By assumption, we have, for α small enough, that $M_{\theta,\delta} > 0$. This implies in particular that

$$\begin{aligned}
\alpha\widetilde{x}^2 + \frac{(\widetilde{t}-\widetilde{s})^2}{2\delta} + \frac{(\widetilde{x}-\widetilde{y})^2}{4\theta} &\leq u(\widetilde{t},\widetilde{x}) - v(\widetilde{s},\widetilde{y}) - \frac{(\widetilde{x}-\widetilde{y})^2}{4\theta} \\
&\leq u_0(\widetilde{x}) - u_0(\widetilde{y}) + 2C_0 T - \frac{(\widetilde{x}-\widetilde{y})^2}{4\theta} \\
&\leq C,
\end{aligned}$$

where C is a positive constant. We then deduce that

$$\alpha\widetilde{x} \to 0, \quad |\widetilde{t}-\widetilde{s}| \to 0, \quad |\widetilde{x}-\widetilde{y}| \to 0$$

respectively as α, δ and θ go to zero. We now claim that $\widetilde{t},\widetilde{s} > 0$ for δ and θ small enough. Indeed, by contradiction, assume that $\widetilde{t} = 0$ (the proof for $\widetilde{s}$ is similar). We then have

$$\frac{\eta}{T} \leq u(0,\widetilde{x}) - v(\widetilde{s},\widetilde{y}) \leq u_0(\widetilde{x}) - u_0(\widetilde{y}) + C_0\widetilde{s} \leq L_0|\widetilde{x}-\widetilde{y}| + C_0\widetilde{s}.$$

Taking δ and θ small enough, we get a contradiction.

We are now able to use the equation satisfied by u and v. We consider

$$\Phi_1(t,x) = v(\tilde{s},\tilde{y}) + \alpha x^2 + \frac{\eta}{T-t} + \frac{(t-\tilde{s})^2}{2\delta} + \frac{(x-\tilde{y})^2}{2\theta}$$

and

$$\Phi_2(s,y) = u\left(\tilde{t},\tilde{x}\right) - \alpha\tilde{x}^2 - \frac{\eta}{T-\tilde{t}} - \frac{\left(\tilde{t}-s\right)^2}{2\delta} - \frac{(\tilde{x}-y)^2}{2\theta}.$$

We can easily see that Φ_1 and Φ_2 belong to $C^1(]0,T[\times\mathbb{R})$ and that $u-\Phi_1$ has a maximum point in $\left(\tilde{t},\tilde{x}\right)\in]0,T[\times\mathbb{R}$ and $v-\Phi_2$ has a minimum point in $(\tilde{s},\tilde{y})\in]0,T[\times\mathbb{R}$.

Hence, by definition of viscosity solutions, we get that

$$\frac{\eta}{\left(T-\tilde{t}\right)^2} + \frac{\left(\tilde{t}-\tilde{s}\right)}{\delta} \le V\left(\frac{1}{\sum_{k=1}^{\infty} g^{\varepsilon}(k)} \sum_{j=1}^{\infty} g^{\varepsilon}(j) \frac{u\left(\tilde{t},\tilde{x}+\varepsilon j\right) - u\left(\tilde{t},\tilde{x}\right)}{\varepsilon j}\right)$$

and

$$\frac{\left(\tilde{t}-\tilde{s}\right)}{\delta} \ge V\left(\frac{1}{\sum_{k=1}^{\infty} g^{\varepsilon}(k)} \sum_{j=1}^{\infty} g^{\varepsilon}(j) \frac{v(\tilde{s},\tilde{y}+\varepsilon j) - v(\tilde{s},\tilde{y})}{\varepsilon j}\right).$$

We subtract the two previous inequalities and we get that

$$\begin{aligned}\frac{\eta}{T^2} \le \frac{\eta}{\left(T-\tilde{t}\right)^2} \le{}& V\left(\frac{1}{\sum_{k=1}^{\infty} g^{\varepsilon}(k)} \sum_{j=1}^{\infty} g^{\varepsilon}(j) \frac{u\left(\tilde{t},\tilde{x}+\varepsilon j\right) - u\left(\tilde{t},\tilde{x}\right)}{\varepsilon j}\right)\\ &- V\left(\frac{1}{\sum_{k=1}^{\infty} g^{\varepsilon}(k)} \sum_{j=1}^{\infty} g^{\varepsilon}(j) \frac{v(\tilde{s},\tilde{y}+\varepsilon j) - v(\tilde{s},\tilde{y})}{\varepsilon j}\right). \end{aligned} \tag{2.2}$$

Using that $\left(\tilde{t},\tilde{s},\tilde{x},\tilde{y}\right)$ is a point of maximum of $M_{\theta,\delta}$, we have, for all (t,s,x,y) that

$$\begin{aligned} & u(t,x) - v(s,y) - \alpha x^2 - \frac{\eta}{T-t} - \frac{(t-s)^2}{2\delta} - \frac{(x-y)^2}{2\theta} \\ &\quad \le u\left(\tilde{t},\tilde{x}\right) - v(\tilde{s},\tilde{y}) - \alpha\tilde{x}^2 - \frac{\eta}{T-\tilde{t}} - \frac{\left(\tilde{t}-\tilde{s}\right)^2}{2\delta} - \frac{(\tilde{x}-\tilde{y})^2}{2\theta}. \end{aligned}$$

Choosing $x=\tilde{x}+\varepsilon j$, $y=\tilde{y}+\varepsilon j$, $t=\tilde{t}$ and $s=\tilde{s}$, we get that

$$u\left(\tilde{t},\tilde{x}+\varepsilon j\right) - u\left(\tilde{t},\tilde{x}\right) - \left(v(\tilde{s},\tilde{y}+\varepsilon j) - v(\tilde{s},\tilde{y})\right) \le -\alpha\tilde{x}^2 + \alpha(\tilde{x}+\varepsilon j)^2 = \alpha\left(2\tilde{x}\varepsilon j + \varepsilon^2 j^2\right)$$

which means that

$$\frac{u\left(\tilde{t},\tilde{x}+\varepsilon j\right) - u\left(\tilde{t},\tilde{x}\right)}{\varepsilon j} \le \frac{\left(v(\tilde{s},\tilde{y}+\varepsilon j) - v(\tilde{s},\tilde{y})\right)}{\varepsilon j} + \alpha(2\tilde{x}+\varepsilon j).$$

Injecting this in (2.2) and using the monotonicity of V, we get

$$\begin{aligned}\frac{\eta}{T^2} \le{}& V\left(\frac{1}{\sum_{k=1}^{\infty} g^{\varepsilon}(k)} \sum_{j=1}^{\infty} g^{\varepsilon}(j)\left(\frac{v(\tilde{s},\tilde{y}+j\varepsilon) - v\left(\tilde{s},\tilde{t}\right)}{\varepsilon j} + \alpha(2\widetilde{x}+\varepsilon j)\right)\right)\\ &- V\left(\frac{1}{\sum_{k=1}^{\infty} g^{\varepsilon}(k)} \sum_{j=1}^{\infty} g^{\varepsilon}(j) \frac{v(\tilde{s},\tilde{y}+j\varepsilon) - v(\tilde{s},\tilde{y})}{\varepsilon j}\right)\\ \le{}& C\alpha \sum_{j=1}^{\infty} g^{\varepsilon}(j)(2\widetilde{x}+\varepsilon j).\end{aligned}$$

Using that g and $z \mapsto zg(z)$ are in L^1, we get a contradiction for α small enough. □

We also give a comparison principle in a bounded set that will be useful later. Since the proof is similar to the previous one, we skip it.

Theorem 2.4 (Comparison in a bounded set)**.** *Let Ω be a subset of $[0,T]\times\mathbb{R}$ and u and v be respectively a sub-solution and a super-solution of (2.1) in Ω. We assume that there exists a constant C_0 such that*

$$|u\,(t,x)-u_0\,(x)|\le C_0t \quad \text{and} \quad |v\,(t,x)-u_0\,(x)|\le C_0t.$$

We also assume that $u\le v$ outside Ω. Then

$$u\le v \quad \text{on } [0,T]\times\mathbb{R}.$$

We can now give the existence and uniqueness result for problem (2.1)

Theorem 2.5. *Let $u_0\in Lip\,(\mathbb{R})$. For all $\varepsilon>0$, there exists a unique solution u_ε of (2.1) such that there is a positive constant C_0 (depending only on the Lipschitz constant of u_0) such that*

$$|u_\varepsilon\,(t,x)-u_0\,(t,x)|\le C_0t,\quad \forall t,x\in[0,T[\times\mathbb{R}. \tag{2.3}$$

Proof. We use Perron's method (see [23]) to prove the existence of u_ε, the uniqueness being a direct consequence of Theorem 2.3. It is sufficient to show that $u_0\pm C_0t$ are respectively super- and sub-solution of (2.1) for a suitable choice of C_0.

We start by verifying that u_0+C_0t is a super-solution. Since u_0 is assumed to be Lipschitz continuous with Lipschitz constant L_0, we have that

$$u_0\,(x+\varepsilon j)-u_0\,(x)\le L_0\varepsilon j.$$

Using that $\overline{g}_\varepsilon\,(j)=\frac{g^\varepsilon(j)}{\sum_{k=1}^\infty g^\varepsilon(k)}$ is positive, we obtain that

$$\sum_{j=1}^{\infty}\overline{g}^\varepsilon\,(j)\,\frac{u_0\,(t,x+j\varepsilon)-u_0\,(t,x)}{\varepsilon j}\le\sum_{j=1}^{\infty}\overline{g}^\varepsilon\,(j)\,L_0=L_0.$$

Since V is non-decreasing we get

$$\begin{aligned}V&\left(\frac{1}{\sum_{k=1}^\infty g^\varepsilon\,(k)}\sum_{j=1}^{\infty}g^\varepsilon\,(j)\,\frac{u_0\,(t,x+j\varepsilon)+C_0t-u_0\,(t,x)-C_0t}{\varepsilon j}\right)\\&=V\left(\sum_{j=1}^{\infty}\overline{g}^\varepsilon\,(j)\,\frac{u_0\,(t,x+j\varepsilon)-u_0\,(t,x)}{\varepsilon j}\right)\le V\,(L_0)\,.\end{aligned}$$

Choosing $C_0=V\,(L_0)$, we get that u_0+C_0t is a super-solution of (2.1).

Arguing in a similar way, we get that u_0-C_0t is a sub-solution and the proof is complete. □

3. Well-posedness of the macroscopic model

This section is devoted to useful results concerning problem (1.1). We begin by the definition of the viscosity solution. This definition and the main properties of the solution are inspired from [2].

In order to have the non-local term well defined and as well as for equation (2.1), we will assume that all the sub- and super-solutions (and hence solutions) u satify the following property:

$$\exists C_0>0 \text{ such that } |u\,(t,x)-u_0\,(x)|\le C_0t. \tag{3.1}$$

Definition 3.1 (Definition of $\overline{\alpha}$-viscosity solution for (1.1))**.** Let $\overline{\alpha}$ be a positive constant. We say that an upper semi-continuous function $u : [0,T]\times\mathbb{R}\to\mathbb{R}$ satisfying (3.1) is a $\overline{\alpha}$-sub-solution of (1.1) if $u(0,\cdot)\le u_0$ and if, for all $\phi\in C^1\left(]0,T[\times\mathbb{R}\right)$ such that $u-\phi$ has a maximum point in $\left(\overline{t},\overline{x}\right)\in]0,T[\times\mathbb{R}$, we have that

$$\phi_t(\left(\overline{t},\overline{x}\right)\le V\left(\frac{1}{\int_{\mathbb{R}_+} g\left(z\right)\mathrm{d}z}\left\{\int_0^{\overline{\alpha}}\frac{\phi\left(\overline{t},\overline{x}+z\right)-\phi\left(\overline{t},\overline{x}\right)}{z}g\left(z\right)\mathrm{d}z\right.\right.$$
$$\left.\left.+\int_{\overline{\alpha}}^{+\infty}\frac{u\left(\overline{t},\overline{x}+z\right)-u\left(\overline{t},\overline{x}\right)}{z}g\left(z\right)\mathrm{d}z\right\}\right).$$

We say that a lower semi-continuous function $v : [0,T[\times\mathbb{R}\to\mathbb{R}$ satisfying (3.1) is a $\overline{\alpha}$-super-solution of (1.1) if $v(0,\cdot)\ge u_0$ and if, for all $\phi\in C^1\left(]0,T[\times\mathbb{R}\right)$ such that $v-\phi$ has a minimum point in $\left(\overline{t},\overline{x}\right)\in]0,T[\times\mathbb{R}$, we have that

$$\phi_t\left(\overline{t},\overline{x}\right)\ge V\left(\frac{1}{\int_{\mathbb{R}_+} g\left(z\right)\mathrm{d}z}\left\{\int_0^{\overline{\alpha}}\frac{\phi\left(\overline{t},\overline{x}+z\right)-\phi\left(\overline{t},\overline{x}\right)}{z}g\left(z\right)\mathrm{d}z\right.\right.$$
$$\left.\left.+\int_{\overline{\alpha}}^{+\infty}\frac{v\left(\overline{t},\overline{x}+z\right)-v\left(\overline{t},\overline{x}\right)}{z}g\left(z\right)\mathrm{d}z\right\}\right).$$

We say that u is a $\overline{\alpha}$-solution of (1.1) if it is a $\overline{\alpha}$-sub- and super-solution.

Remark 3.2. For simplicity of presentation, given $\Phi:\mathbb{R}\to\mathbb{R}$, we define

$$I^{1,\alpha}[\Phi](x)=\frac{1}{\int_0^{+\infty}g(z)\mathrm{d}z}\int_0^{\alpha}\left(\Phi(x+z)-\Phi(x)\right)\frac{g(z)}{z}\mathrm{d}z$$

and

$$I^{2,\alpha}[\Phi(x)]=\frac{1}{\int_0^{+\infty}g(z)\mathrm{d}z}\int_{\alpha}^{+\infty}\left(\Phi(x+z)-\Phi(x)\right)\frac{g(z)}{z}\mathrm{d}z.$$

Theorem 3.3 (Equivalence of α-solutions)**.** *Let $\alpha_1,\alpha_2>0$. We have that u is a α_1-sub-solution (resp. a α_1-super-solution) if and only if it is a α_2-sub-solution (resp. a α_2-super-solution).*

Proof. We only prove the result for sub-solution, the super-solution case being similar. We assume that u is a α_1-sub-solution and we want to show that it is a α_2-sub-solution. Let $\psi\in C^1(]0,T[\times\mathbb{R})$ be such that $u-\psi$ reaches a global maximum in (t_0,x_0). We suppose that $u(t_0,x_0)=\psi(t_0,x_0)$. For $\varepsilon_1>0$, there exists $\phi_1\in C^1(]0,T[\times\mathbb{R})$ such that $\phi_1\ge u$ and $||u-\phi_1||_{L^\infty}\le\varepsilon_1$. We then define a test function $\phi\in C^1(]0,T[\times\mathbb{R})$, which will be used for the α_1-sub-solution, by

$$\phi(t,x)=\begin{cases}\phi_1(t,x) & \text{if } x\ge x_0+\alpha_2;\\ \psi(t,x) & \text{if } x\le x_0+\dfrac{\alpha_2}{2}.\end{cases}$$

Since u is α_1-viscosity sub-solution of (1.1) and (t_0,x_0) is maximum point of $u-\phi$, we have

$$\phi_t(t_0,x_0)\le V\left(I^{1,\alpha_1}[\phi(t_0,\cdot)](x_0)+I^{2,\alpha_1}[u(t_0,\cdot)](x_0)\right).$$

Using that $\phi_t(t_0,x_0)=\psi_t(t_0,x_0)$, we then deduce that

$$\begin{aligned}\psi_t(t_0,x_0) &\le V\left(I^{1,\alpha_1}[\phi(t_0,\cdot)](x_0)+I^{2,\alpha_1}[u(t_0,\cdot)](x_0)\right)\\ &\le V\Bigg(I^{1,\alpha_2}[\phi(t_0,\cdot)](x_0)+I^{2,\alpha_2}[u(t_0,\cdot)](x_0)\\ &\quad+\frac{1}{\int_0^{+\infty}g(z)\mathrm{d}z}\int_{\alpha_2}^{\alpha_1}\left(\phi(t_0,x_0+z)-\phi(t_0,x_0)\right)\frac{g(z)}{z}\mathrm{d}z\\ &\quad-\frac{1}{\int_0^{+\infty}g(z)\mathrm{d}z}\int_{\alpha_2}^{\alpha_1}\left(u(t_0,x_0+z)-u(t_0,x_0)\right)\frac{g(z)}{z}\mathrm{d}z\Bigg)\\ &\le V\Bigg(I^{1,\alpha_2}[\phi(t_0,\cdot)](x_0)+I^{2,\alpha_2}[u(t_0,\cdot)](x_0)\\ &\quad+\frac{1}{\int_0^{+\infty}g(z)\mathrm{d}z}\int_{\alpha_2}^{\alpha_1}\left(\phi(t_0,x_0+z)-u(t_0,x_0+z)\right)\frac{g(z)}{z}\mathrm{d}z\Bigg)\\ &\le V\left(I^{1,\alpha_2}[\phi(t_0,\cdot)](x_0)+I^{2,\alpha_2}[u(t_0,\cdot)](x_0)+\frac{\varepsilon_1}{\int_0^{+\infty}g(z)\mathrm{d}z}\int_{\alpha_2}^{\alpha_1}\frac{g(z)}{z}\mathrm{d}z\right)\end{aligned}$$

where for the last inequality, we have used the fact that V is non-decreasing. Sending $\varepsilon_1\to 0$ and using that g and $z\mapsto zg(z)$ are in L^1, we get the result. □

Theorem 3.4 (Comparison principle for (1.1))**.** *Let u be a sub-solution and v be a super-solution of* (1.1)*. We assume that u and v satisfy* (3.1)*. Then*

$$u\le v.$$

Proof. We assume by contradiction that

$$M=\sup_{(t,x)\in(0,T)\times\mathbb{R}}\{u(t,x)-v(t,x)\}>0.$$

We then duplicate the variable, by considering, for $\eta,\alpha,\varepsilon,\delta>0$,

$$M_{\varepsilon,\delta}=\sup_{(t,s,x,y)}\left\{u(t,x)-v(s,y)-\frac{(t-s)^2}{2\delta}-\frac{(x-y)^2}{2\varepsilon}-\frac{\eta}{T-t}-\alpha x^2\right\}.$$

We denote by $(\bar t,\bar s,\bar x,\bar y)$ a point of maximum of $M_{\varepsilon,\delta}$. For η and α small enough, we have $M_{\varepsilon,\delta}>0$ and we deduce as in the proof of Theorem 2.3 that

$$\alpha\bar x\to 0,\quad |\bar t-\bar s|\to 0,\quad |\bar x-\bar y|\to 0,$$

respectively as α, δ and ε go to zero, and that $\bar t,\bar s>0$ for δ and ε small enough. We can then use the equations satisfied by u and v. We have that $u-\phi_1$ reaches a maximum in $(\bar t,\bar x)$, with ϕ_1 given by

$$\phi_1=v(\bar s,\bar y)+\frac{(t-\bar s)^2}{2\delta}+\frac{(x-\bar y)^2}{2\varepsilon}+\frac{\eta}{T-t}+\alpha x^2.$$

So by definition of viscosity solution we obtain, for every $\bar\alpha$, that

$$\frac{(\bar t-\bar s)}{\delta}+\frac{\eta}{(T-\bar t)^2}\le V\left(I^{1,\bar\alpha}[\phi_1(\bar t,\cdot)](\bar x)+I^{2,\bar\alpha}[u(\bar t,\cdot)](\bar x)\right).$$

Since $v-\phi_2$ reaches a minimum point in $(\bar s,\bar y)$, with ϕ_2 given by

$$\phi_2=u(\bar t,\bar x)-\frac{(s-\bar t)^2}{2\delta}-\frac{(y-\bar x)^2}{2\varepsilon}-\frac{\eta}{T-\bar t}-\alpha\bar x^2,$$

we also have

$$\frac{(\bar{t}-\bar{s})}{\delta} \geq V\left(I^{1,\overline{\alpha}}[\phi_2(\bar{t},\cdot)](\overline{x}) + I^{2,\overline{\alpha}}[v(\bar{t},\cdot)](\overline{x})\right).$$

We subtract the two viscosity inequalities, and we obtain

$$\frac{\eta}{T^2} \leq V\left(I^{1,\overline{\alpha}}\left[\phi_1(\bar{t},\cdot)\right](\overline{x}) + I^{2,\overline{\alpha}}\left[u(\bar{t},\cdot)\right](\overline{x})\right) - V\left(I^{1,\overline{\alpha}}\left[\phi_2(\bar{s},\cdot)\right](\overline{y}) + I^{2,\overline{\alpha}}\left[v(\bar{s},\cdot)\right](\overline{y})\right). \tag{3.2}$$

Using that $(\bar{t},\bar{s},\overline{x},\overline{y})$ is a maximum point we obtain

$$u(\bar{t},\overline{x}+z) - v(\bar{s},\overline{y}+z) \leq u(\bar{t},\overline{x}) - v(\bar{s},\overline{y}) + \alpha(\overline{x}+z)^2 - \alpha\overline{x}^2$$

and so

$$\frac{u(\bar{t},\overline{x}+z)-u(\bar{t},\overline{x})}{z} \leq \frac{v(\bar{s},\overline{y}+z)-v(\bar{s},\overline{y})}{z} + \alpha(2\overline{x}+z).$$

We then deduce that

$$I^{2,\overline{\alpha}}[u(\bar{t},\cdot)](\overline{x}) \leq I^{2,\overline{\alpha}}[v(\bar{s},\cdot)](\overline{y}) + 2\alpha\overline{x}||g||_{L^1} + \alpha||zg||_{L^1} \leq I^{2,\overline{\alpha}}[v(\bar{s},\cdot)](\overline{y}) + C\alpha(\overline{x}+1). \tag{3.3}$$

We now compare $I^{1,\overline{\alpha}}[\phi_1(\bar{t},\cdot)](\overline{x})$ and $I^{1,\overline{\alpha}}[\phi_2(\bar{s},\cdot)](\overline{y})$. We have

$$\begin{aligned}
I^{1,\overline{\alpha}}[\phi_1(\bar{t},\cdot)](\overline{x}) - I^{1,\overline{\alpha}}[\phi_2(\bar{s},\cdot)](\overline{y}) &= \frac{1}{\int_0^{+\infty} g(z)\mathrm{d}z}\left(\frac{1}{2\varepsilon}\int_0^{\overline{\alpha}} \frac{(\overline{x}-\overline{y}+z)^2-(\overline{x}-\overline{y})^2}{z} g(z)\mathrm{d}z\right.\\
&\quad - \frac{1}{2\varepsilon}\int_0^{\overline{\alpha}} \frac{(\overline{y}-\overline{x})^2-(\overline{y}-\overline{x}+z)^2}{z} g(z)\mathrm{d}z\\
&\quad \left. + \alpha\int_0^{\overline{\alpha}} \frac{(\overline{x}+z)^2-\overline{x}^2}{z} g(z)\mathrm{d}z\right)\\
&= \frac{1}{\int_0^{+\infty} g(z)\mathrm{d}z}\left(\frac{1}{\varepsilon}\int_0^{\overline{\alpha}} zg(z)\mathrm{d}z + \alpha\int_0^{\overline{\alpha}}(2\overline{x}+z)g(z)\mathrm{d}z\right)\\
&\leq C\left(\alpha(\overline{x}+1) + \frac{1}{\varepsilon}\int_0^{\overline{\alpha}} zg(z)\mathrm{d}z\right).
\end{aligned}$$

Injecting the previous estimate and (3.3) in (3.2) and using the fact that V is non-decreasing, we get

$$\begin{aligned}
\frac{\eta}{T^2} &\leq V\left(I^{1,\overline{\alpha}}[\phi_2(\bar{s},\cdot)](\overline{y}) + I^{2,\overline{\alpha}}[v(\bar{s},\cdot)](\overline{y}) + C\left(\alpha(\overline{x}+1) + \frac{1}{\varepsilon}\int_0^{\overline{\alpha}} zg(z)\mathrm{d}z\right)\right)\\
&\quad - V\left(I^{1,\overline{\alpha}}[\phi_2(\bar{s},\cdot)](\overline{y}) + I^{2,\overline{\alpha}}[v(\bar{s},\cdot)](\overline{y})\right)\\
&\leq C\left(\alpha(\overline{x}+1) + \frac{1}{\varepsilon}\int_0^{\overline{\alpha}} zg(z)\mathrm{d}z\right).
\end{aligned}$$

Sending $\alpha,\overline{\alpha}\to 0$, we get that

$$\frac{\eta}{T^2} \leq 0$$

and we obtain a contradiction since $\eta > 0$. This ends the proof. □

We are now able to give the proof of Theorem 1.1

Proof of Theorem 1.1. As in the proof of Theorem 2.5, we have that $u_0 \pm C_0 t$ are respectively sub- and super-solution of (1.1), so the existence follows by Perron's method. The uniqueness is a direct consequence of Theorem 3.4. It just remains to show that u is Lipschitz continuous.

To do that, let us define u^h by $u^h(t,x) = u(t,x+h) - L_0 h$. In particular, we have

$$u^h(0,x) = u_0(x+h) - L_0 h \leq u_0(x).$$

Moreover, since equation (1.1) is invariant by translation in space and by addition of constant, we get that u^h is a sub-solution of (1.1). Hence, the comparison principle yields that

$$u(t,x+h) - L_0 h = u^h(t,x) \leq u(t,x),$$

i.e.

$$\frac{u(t,x+h) - u(t,x)}{h} \leq L_0.$$

To show that u is non-decreasing, we argue in a similar way by using the fact that $u(t,x+h)$ (recall that u_0 is non decreasing) is a super-solution of (1.1). This shows that u is Lipschitz continuous in space. The fact that u is also Lipschitz continuous in times is a direct consequence of the fact that V is bounded.

□

4. Homogenization result

This section is devoted to the proof of the following convergence result. We will see that Theorem 1.2 is a corollary of this result.

Theorem 4.1 (Homogenization result)**.** *Assume* (**H**) *and let* u_0 *be a Lipschitz continuous function. Then, the viscosity solution* u^ε *of* (2.1)*, given by Theorem 2.5, converges as* $\varepsilon \to 0$*, locally uniformly in* (t,x)*, to the unique solution* u *of* (1.1).

Proof. We define $\overline{u}$ and $\underline{u}$ by

$$\overline{u} = \limsup_{\varepsilon\to 0,(y,s)\to(t,x)} u^\varepsilon(y,s) \text{ and } \underline{u} = \liminf_{\varepsilon\to 0,(y,s)\to(t,x)} u^\varepsilon(y,s).$$

We are going to prove that $\overline{u}$ is a sub-solution of (1.1) on $[0,T[\times\mathbb{R}$. Similarly, we can prove that $\underline{u}$ is a super-solution of the same equation. Then, using the comparison principle Theorem 3.4, we will get $\overline{u} \leq \underline{u}$ and so $\overline{u} = \underline{u} = u$ which implies the convergence of u^ε to u.

First, by (2.3), we have that $\overline{u}(0,\cdot) = u_0$. We argue by contradiction by assuming that $\overline{u}$ is not a sub-solution on $]0,T[\times\mathbb{R}$. Then there exists $(t_0,x_0) \in]0,T[\times\mathbb{R}$ and a test function $\Phi \in C^1$ such that $\overline{u} - \Phi$ reaches a strict maximum in (t_0,x_0) and such that

$$\Phi_t(t_0,x_0) - V\left(I^{1,+\infty}[\Phi(t_0,\cdot)](x_0)\right) = \theta > 0. \tag{4.1}$$

We also assume that $\overline{u}(t_0,x_0) = \Phi(t_0,x_0)$. We now apply the perturbed test function method introduced by Evans [13] in terms here of hull functions. We refer to [14,16] for the use of hull functions as correctors. We recall that in these papers, the perturbed test function is essentially defined by

$$\Phi^\varepsilon(t,x) = \varepsilon h\left(\frac{\Phi(t,x)}{\varepsilon}\right),$$

where h is the hull function. In fact in our case, the hull function is simpler (since our Hamilton–Jacobi equation is independent of (t,x)) and is given by $h = Id$. So the perturbed test function reduces to

$$\Phi^\varepsilon(t,x) = \begin{cases} \Phi(t,x) - \eta_r & \text{if } (t,x) \in Q_{1,1}(t_0,x_0) \\ u^\varepsilon(t,x) & \text{if not} \end{cases}$$

where η_r is chosen later and $Q_{r,R}(t_0,x_0) = (t_0 - r, t_0 + r) \times (x_0 - R, x_0 + R)$. We want to prove that Φ^ε is solution in $Q_{r,r}(t_0,x_0)$, of

$$\Phi^\varepsilon_t - V\left(\frac{1}{\sum_{k\geq 1} g^\varepsilon(k)} \sum_{j\geq 1} g^\varepsilon(j) \frac{\Phi^\varepsilon(t,x+\varepsilon j) - \Phi^\varepsilon(t,x)}{\varepsilon j}\right) \geq 0$$

and that $\Phi^\varepsilon \geq u^\varepsilon$ outside $Q_{r,r}(t_0,x_0)$. In particular r is chosen smaller that 1 so that $Q_{r,r}(t_0,x_0) \subset Q_{1,1}(t_0,x_0)$.

Let us first focus on the "boundary conditions". Since $\overline{u} - \Phi$ reaches a strict maximum at (t_0,x_0), we can ensure that

$$u^\varepsilon(t,x) \leq \Phi(t,x) - \eta_r \quad \text{for} \quad (t,x) \in Q_{2,2}(t_0,x_0)\backslash Q_{r,r}(t_0,x_0)$$

for $\eta_r = o_r(1) > 0$. Hence we conclude that $\Phi^\varepsilon \geq u^\varepsilon$ outside $Q_{r,r}(t_0,x_0)$.

We now turn to the equation. Since Φ^ε is smooth, we can check this property pointwise. Let $(\overline{t},\overline{x}) \in Q_{r,r}(t_0,x_0)$. We have

$$\begin{aligned}
&\Phi^\varepsilon_t(\overline{t},\overline{x}) - V\left(\frac{1}{\sum_{k\geq 1} g^\varepsilon(k)} \sum_{j\geq 1} g^\varepsilon(j) \frac{\Phi^\varepsilon(\overline{t},\overline{x}+\varepsilon j) - \Phi^\varepsilon(\overline{t},\overline{x})}{\varepsilon j}\right)\\
&\quad = \theta + \Phi_t(\overline{t},\overline{x}) - \Phi_t(t_0,x_0) - V\left(\frac{1}{\sum_{k\geq 1} g^\varepsilon(k)} \sum_{j\geq 1} g^\varepsilon(j) \frac{\Phi(\overline{t},\overline{x}+\varepsilon j) - \Phi(\overline{t},\overline{x})}{\varepsilon j}\right)\\
&\qquad + V\left(I^{1,+\infty}[\Phi(t_0,\cdot)](x_0)\right)\\
&\quad \geq \frac{3}{4}\theta - C\left|\frac{1}{\sum_{k\geq 1} g^\varepsilon(k)} \sum_{j\geq 1} g^\varepsilon(j) \frac{\Phi(\overline{t},\overline{x}+\varepsilon j) - \Phi(\overline{t},\overline{x})}{\varepsilon j} - I^{1,+\infty}[\Phi(t_0,\cdot)](x_0)\right|
\end{aligned}$$

for r small enough. Using that, by convergence of Riemann sum,

$$\frac{1}{\sum_{k\geq 1} g^\varepsilon(k)} \sum_{j\geq 1} g^\varepsilon(j) \frac{\Phi(t,x+\varepsilon j) - \Phi(t,x)}{\varepsilon j} \to I^{1,+\infty}[\Phi(\overline{t},\cdot)](\overline{x})$$

we deduce, for r small enough, that

$$\Phi^\varepsilon_t(\overline{t},\overline{x}) - V\left(\frac{1}{\sum_{k\geq 1} g^\varepsilon(k)} \sum_{j\geq 1} g^\varepsilon(j) \frac{\Phi^\varepsilon(\overline{t},\overline{x}+\varepsilon j) - \Phi^\varepsilon(\overline{t},\overline{x})}{\varepsilon j}\right) \geq \frac{\theta}{2}$$

and show that Φ^ε is a super-solution in $Q_{r,r}(t_0,x_0)$. Recalling that $u^\varepsilon \leq \Phi^\varepsilon$ outside $Q_{r,r}(t_0,x_0)$ and using the comparison principle on bounded set Theorem 2.4, we get that

$$u^\varepsilon(x,t) \leq \Phi^\varepsilon(x,t).$$

Passing to the limit as ε goes to zero, and as $(t,x) \to (t_0,x_0)$ we get that

$$\Phi(t_0,x_0) = \overline{u}(t_0,x_0) \leq \Phi(t_0,x_0) - \eta,$$

which gives a contradiction with $\eta > 0$. Therefore $\overline{u}$ is a sub-solution of (1.1) on $[0,T[\times\mathbb{R}$ and this ends the proof of the theorem.

□

We are now able to give the proof of Theorem 1.2.

Proof of Theorem 1.2. We recall that the initial condition of $\overline{u}^\varepsilon$ is given by

$$\overline{u}^\varepsilon(0,x) = \varepsilon U_{\lfloor \frac{x}{\varepsilon} \rfloor}(0) = u_0\left(\left\lfloor \frac{x}{\varepsilon}\right\rfloor\right).$$

Hence

$$u_0(x) - L_0\varepsilon \le u_0(x-\varepsilon) \le \overline{u}^\varepsilon(0,x) \le u_0(x).$$

Using the comparison principle Theorem 3.4, we then get

$$u^\varepsilon(t,x) - L_0\varepsilon \le \overline{u}^\varepsilon(t,x) \le u^\varepsilon(t,x) \quad \text{on } [0,T]\times\mathbb{R}.$$

By convergence of u^ε to u given in Theorem 4.1, we deduce the convergence of $\overline{u}^\varepsilon$ to u. This ends the proof of the theorem. □

5. Numerical tests

In this section, we present a numerical scheme in order to compute the solution of the macroscopic model. In a first subsection, we present the scheme. Then, we give some properties of the solution of the scheme and we prove an error estimate between the numerical solution and the solution of the macroscopic model. We end the paper with some numerical results in Section 5.3.

5.1. Discretization aspects and numerical scheme

Since problem (1.1) is defined on the unbounded line $\mathbb{R}$, we need to consider a bounded computational domain $[a,b]$ with $a<b$ two real parameters. The treatment of the boundary condition on the right side, on $x=b$, will be detailed below. To implement this problem, we consider the spatial discretization $x_i = i\Delta x + a$, $i \in 0,\cdots,N$ and $\Delta x = \frac{(b-a)}{N}$ with $N \in \mathbb{N}^*$ a parameter, so that $x_0 = a$ and $x_N = b$. For the time variable, we set $t_n = n\Delta t$ with $\Delta t = T/N_t$ and $N_t \in \mathbb{N}^*$ a second parameter. We will denote by u_i^n the approximation of $u(t^n, x_i)$.

Then, the main point is to handle the non-local integral term. To do so, the idea is to decompose the integral as follows:

$$\begin{aligned}\int_0^{+\infty}\left(\frac{u(t^n,x_i+z)-u(t^n,x_i)}{z}\right)g(z)\mathrm{d}z &= \underbrace{\int_0^{A}\left(\frac{u(t^n,x_i+z)-u(t^n,x_i)}{z}\right)g(z)\mathrm{d}z}_{=I_A(u(t^n,\cdot))(x_i)} \\ &\quad + \underbrace{\int_A^{B}\left(\frac{u(t^n,x_i+z)-u(t^n,x_i)}{z}\right)g(z)\mathrm{d}z}_{=I(u(t^n,\cdot))(x_i)} \\ &\quad + \underbrace{\int_B^{+\infty}\left(\frac{u(t^n,x_i+z)-u(t^n,x_i)}{z}\right)g(z)\mathrm{d}z}_{=I_\infty(u(t^n,\cdot))(x_i)}\end{aligned}$$

where $A \ge \Delta x$ is a small parameter (of order $\sqrt{\Delta x}$) and B is a big one. In the numerical implementation, we simply neglect the terms $I_A(u(t^n,\cdot))(x_i)$ to avoid the treatment of the division by 0, which, as we will see in the analysis of the scheme, induces an error of order A, and the last term $I_\infty(u(t^n,\cdot))(x_i)$ taking $B>b$ sufficiently large, which introduces a (small) consistency error. In particular, if the weight function g is compactly supported, this term is exactly 0 by choosing properly B.

For the remaining integral $I(u(t^n,\cdot))(x_i)$, we use a simple trapezoidal quadrature rule based on the discretization points x_i, see Figure 1, in order to include directly the quantity u_i^n. Yet, as we can see, x_i+z can be greater

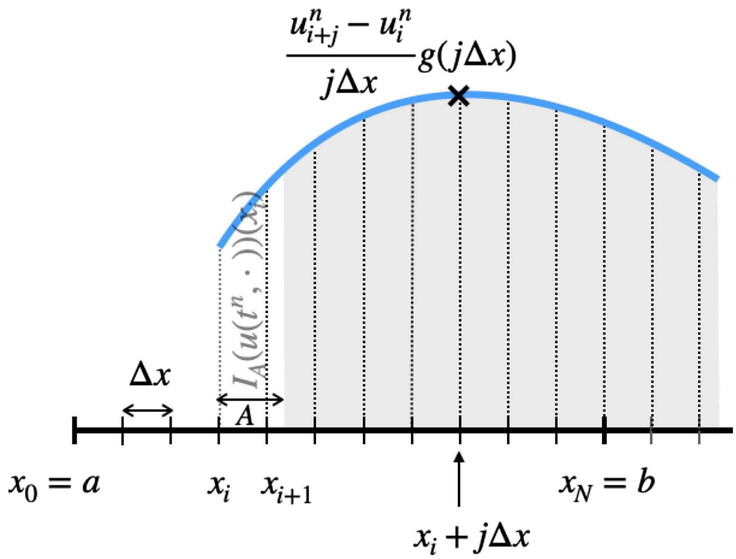

FIGURE 1. Illustration of the integral approximation based on the spatial discretization x_i.

than b, so it involves also values of u outside the computational domain $[a, b]$. To get these values, we will assume that we have a constant "outgoing" flux so that

$$u(t, b + \tilde{z}) = u(t, b) + \frac{\tilde{z}}{\rho_\infty} \quad \forall \tilde{z} \geq 0 \tag{5.1}$$

where $\rho_\infty = \partial_x u(t, b) > 0$ represents the constant density in the outgoing direction $x > b$. Thus, we obtain the following approximation:

$$I(u(t^n,\cdot))(x_i) \simeq \underbrace{\frac{1}{2}\left(\frac{\tilde{u}_{i+N_A}^n - \tilde{u}_i^n}{N_A} g(N_A\Delta x) + \frac{\tilde{u}_{i+N_B}^n - \tilde{u}_i^n}{N_B} g(N_B\Delta x)\right) + \sum_{j=N_A+1}^{N_B-1} \frac{\tilde{u}_{i+j}^n - \tilde{u}_i^n}{j} g(j\Delta x)}_{:=\mathbb{I}(\tilde{u}^n)(x_i)} \tag{5.2}$$

where $N_A = \lfloor \frac{A}{\Delta x} \rfloor$, $N_B = \lfloor \frac{B}{\Delta x} \rfloor$ and

$$\tilde{u}_k^n = \begin{cases} u_k^n & \text{if} \quad k \leq N, \\ u_N^n + (k - N)\Delta x \frac{u_N^n - u_{N-1}^n}{\Delta x} & \text{if} \quad k > N. \end{cases}$$

Note that we approximate $\partial_x u(t, b) = \frac{u_N^n - u_{N-1}^n}{\Delta x}$. To sum up, the numerical scheme is then given by:

$$u_i^{n+1} = u_i^n + \Delta t V\left(\frac{\mathbb{I}(\tilde{u}^n)(x_i)}{I_g}\right) \quad \text{where} \quad I_g = \int_0^{+\infty} g(z)\mathrm{d}z, \tag{5.3}$$

and $\mathbb{I}(\tilde{u}^n)(x_i)$ is given by (5.2).

5.2. Analysis of the scheme

This approximation of the integral term is justified by the following lemma

Lemma 5.1. *There exists a constant $K \geq 0$ such that*

$$|I(u(t,\cdot))(x) - \mathbb{I}(u(t,\cdot)(x)| \leq \frac{KB}{A}\Delta x. \tag{5.4}$$

Proof. Using that the trapezoidal quadrature error for a Lipschitz function f is of order $LB\Delta x$, where L is the Lipschitz constant of f, it suffices to show that the function

$$z \mapsto \frac{u(t,x+z)-u(t,x)}{z} g(z)$$

is Lipschitz continuous in $[A,B]$, with a Lipschitz constant given by $\frac{K}{A}$. Since $z \mapsto \frac{u(t,x+z)-u(t,x)}{z}$ and g are bounded and g is Lipschitz continuous, it suffices to show that $z \mapsto \frac{u(t,x+z)-u(t,x)}{z}$ is Lipschitz continuous. Let $z_1, z_2 \in [A,B]$. We have

$$\left|\frac{u(t,x+z_1)-u(t,x)}{z_1} - \frac{u(t,x+z_2)-u(t,x)}{z_2}\right| \leq \left|\frac{1}{z_1}(u(t,x+z_1)-u(t,x+z_2))\right| + \left|\frac{u(t,x+z_2)-u(t,x)}{z_2}\cdot\frac{z_2-z_1}{z_1}\right| \leq \frac{2L_0}{A}|z_1-z_2|.$$

□

We now prove, under the following CFL condition

$$\Delta t \leq \frac{I_g}{L\mathcal{I}} \quad \text{where} \quad \mathcal{I} = \frac{1}{2N_A} g(N_A\Delta x) + \left[\sum_{j=N_A+1}^{N_B-1} \frac{g(j\Delta x)}{j}\right] + \frac{1}{2N_B} g(N_B\Delta x) \tag{5.5}$$

and L is the Lipschitz constant associated to V, that the scheme is monotone.

Remark 5.2. We can notice that the CFL condition does not explicitly depend on the space step Δx. Indeed, we can remark that the term $\mathcal{I}$ corresponds to the trapezoidal quadrature rule applied to the function $\frac{g(z)}{z}$ on the interval (A,B). Also, it is worth recalling that for the "local" equation $u_t = V(u_x)$ discretized with the simple scheme

$$u_i^{n+1} = u_i^n + \Delta t V\left(\frac{u_{i+1}^n - u_i^n}{\Delta x}\right)$$

the CLF condition is $\Delta t \leq \frac{\Delta x}{L}$. This condition is usually worse than the one obtained in the non local case if $g(N_A\Delta x)$ is small.

Theorem 5.3 (Monotonicity of the scheme)**.** *Assume that* (5.5) *holds and let u and v be respectively a sub and a super-solution of the scheme, i.e. such that*

$$u_i^{n+1} \leq u_i^n + \Delta t V\left(\frac{\mathbb{I}(\tilde{u}^n)(x_i)}{I_g}\right) \quad \text{and} \quad v_i^{n+1} \geq v_i^n + \Delta t V\left(\frac{\mathbb{I}(\tilde{v}^n)(x_i)}{I_g}\right).$$

Assume also that $u_i^0 \leq v_i^0$ for all $i \in \{0,\dots,N\}$. Then

$$u_i^n \leq v_i^n \quad \forall i \in \{0,\dots,N\}, n \in \{0,\dots,N_T\}.$$

Proof. The proof is made by induction. The initialization is true by hypothesis. Assume that for a certain n, we have

$$u_i^n \leq v_i^n \quad \forall i \in \{0,\dots,N\}.$$

We have

$$u_i^{n+1} - v_i^{n+1} \le u_i^n - v_i^n + \Delta t \left(V\left(\frac{\mathbb{I}(\tilde{u}^n)(x_i)}{I_g}\right) - V\left(\frac{\mathbb{I}(\tilde{v}^n)(x_i)}{I_g}\right)\right).$$

Using that

$$\mathbb{I}(\tilde{u}^n)(x_i) \le \mathbb{I}(\tilde{v}^n)(x_i) - (u_i^n - v_i^n)\mathcal{I}$$

and since V is monotone, we deduce that

$$\begin{aligned} u_i^{n+1} - v_i^{n+1} &\le u_i^n - v_i^n + \Delta t \left(V\left(\frac{\mathbb{I}(\tilde{v}^n)(x_i) - (u_i^n - v_i^n)\mathcal{I}}{I_g}\right) - V\left(\frac{\mathbb{I}(\tilde{v}^n)(x_i)}{I_g}\right)\right) \\ &\le (u_i^n - v_i^n)\left(1 - \frac{\Delta t}{I_g} L\mathcal{I}\right) \\ &\le 0. \end{aligned}$$

This ends the proof. □

We now show that $(u_i^n)_i$ is non decreasing and "Lipschitz continuous".

Theorem 5.4 (Monotonicity and Lipschitz bounds on the numerical solution)**.** *For all* $i \in \{0, \dots, N\}$ *and* $n \in \{0, \dots, N_T - 1\}$*, we have*

$$0 \le \frac{u_i^{n+1} - u_i^n}{\Delta t} \le V_{\max}. \tag{5.6}$$

Moreover, if we assume that (5.5) *holds and that*

$$0 \le \frac{u_{i+1}^0 - u_i^0}{\Delta x} \le L_0$$

for all $i \in \{0, \dots, N-1\}$*, then, for all* $i \in \{0, \dots, N-1\}$ *and* $n \in \{0, \dots, N_T\}$*, we have*

$$0 \le \frac{u_{i+1}^n - u_i^n}{\Delta x} \le L_0.$$

Proof. The estimate (5.6) is a direct consequence of the definition of the scheme and the fact that $0 \le V \le V_{\max}$. Let us show the second result by induction. The initialization is true by hypothesis. Assume that, for a given n, we have

$$0 \le \frac{u_{i+1}^n - u_i^n}{\Delta x} \le L_0 \quad \forall i \in \{0, \cdots, N-1\}.$$

We set $\overline{u}_i^n = u_{i+1}^n$ and

$$\overline{u}_i^{n+1} = \overline{u}_i^n + \Delta t V\left(\frac{\mathbb{I}(\tilde{\overline{u}}^n)(x_i)}{I_g}\right).$$

In particular, we have $\overline{u}_i^{n+1} = u_{i+1}^{n+1}$ for all $i = 0, \dots, N-1$. By monotonicity of the scheme, since $\overline{u}_i^n \ge u_i^n$, we deduce that

$$u_{i+1}^{n+1} = \overline{u}_i^{n+1} \ge u_i^{n+1} \quad \forall i = 0, \dots, N-1.$$

In the same way, if we define $\underline{u}_i^n = u_{i+1}^n - L_0 \Delta x$ and

$$\underline{u}_i^{n+1} = \underline{u}_i^n + \Delta t V\left(\frac{\mathbb{I}(\tilde{\underline{u}}^n)(x_i)}{I_g}\right),$$

then we have $\underline{u}_i^{n+1} = u_{i+1}^{n+1} - L_0 \Delta x$ for all $i = 0, \dots, N-1$ and $\underline{u}_i^n \le u_i^n$. By monotonicity of the scheme, we then get

$$u_{i+1}^{n+1} - L_0 \Delta x = \underline{u}_i^{n+1} \le u_i^{n+1}.$$

This ends the proof of the theorem. □

We now give the convergence result, with the estimate of convergence. We first define

$$Q_T^\Delta = \{(t_n, x_i), t_n = n\Delta t, x_i = a + i\Delta x, n \in \{0, \dots, N_T\}, i \in \{0, \dots, N\}\}$$

and

$$Q_0^\Delta = \{x_i = a + i\Delta x, i \in \{0, \dots, N\}\}\,.$$

Theorem 5.5 (Discrete-continuous error estimate for (1.1))**.** *Assume that $T \le 1$ and $\Delta x + \Delta t \le 1$. Assume also that the CFL condition (5.5) is satisfied. Then, there exists a constant $K > 0$, depending only on the Lipschitz constant of u_0 and on g, such that the error estimate between the continuous solution u of (1.1) and the discrete solution v of (5.3) (with initial condition given by v_0) is given by*

$$\sup_{Q_T^\Delta} |u - v| \le K\left(\sqrt{\Delta t} + \frac{B}{A}\Delta x + R_B + \left(A + \frac{A^2}{\sqrt{\Delta t}}\right)\right) + \sup_{Q_0^\Delta} |u_0 - v_0|, \tag{5.7}$$

if $K\left(\sqrt{\Delta t} + \frac{B}{A}\Delta x + R_B + \left(A + \frac{A^2}{\sqrt{\Delta t}}\right)\right) \le 1$ and $\sup_{Q_0^\Delta} |u_0 - v_0| \le 1$ and where

$$R_B = \int_B^{+\infty} g(z)\mathrm{d}z.$$

Remark 5.6. For example, if g has compact support, B can be chosen large enough so that $R_B = 0$ and we can take A of the order of $\sqrt{\Delta x}$. In that case, we recover a estimate of order $\sqrt{\Delta x + \Delta t}$. If, we take $g(z) = e^{-z}$, then we can take $B = -\ln(\Delta x)$ and $A = \sqrt{\Delta x}$ and we get an estimate of order $-\ln(\Delta x)\sqrt{\Delta x + \Delta t}$.

Proof. The proof is inspired by the one of Crandall Lions [9], revisited by Alvarez *et al.* [1] and is an adaptation of the comparison principle.

We define μ by

$$\mu = \sup_{Q_T^\Delta} |u(t_n, x_i) - v_i^n|.$$

We want to prove that μ is bounded by $f(\Delta x, \Delta t)$, f being defined later. We first assume that $u_0(x_i) \ge v_i^0$ and we set

$$\mu_0 = \sup_{Q_0^\Delta} |u_0(x_i) - v_i^0| \ge 0.$$

We define for $0 < \alpha, \varepsilon \le 1$ and $\sigma > 0$, the function

$$\begin{array}{rcl} \psi : [0,T] \times \mathbb{R} \times Q_T^\Delta & \to & \mathbb{R} \\ (t, x, t_n, x_i) & \mapsto & \psi(t, x, t_n, x_i) \end{array},$$

by

$$\psi(t, x, t_n, x_i) = u(t, x) - v(t_n, x_i) - \frac{|t - t_n|^2}{2\varepsilon} - \frac{|x - x_i|^2}{2\varepsilon} - \sigma t - \alpha|x|^2 - \alpha|x_i|^2.$$

This function has a maximum denoted by $M_{\varepsilon,\alpha}^\sigma$. Indeed, u is Lipschitz continuous then

$$|u(t, x) - u(t, 0)| \le L_0|x|,$$

and since, $T \le 1$,

$$|u(t, x)| \le L_0|x| + Ct + |u_0(0)| \le K(1 + |x|). \tag{5.8}$$

In the same way, using Theorem 5.4, we also have that

$$|v(t_n, x_i)| \le K(1 + |x_i|). \tag{5.9}$$

Let us underline that the constant K is not necessarily the same on each estimation.

This implies that the maximum $M^{\sigma}_{\varepsilon,\alpha}$ is reached in a point denoted by $(t^\star, x^\star, t_n^\star, x_i^\star)$. We now give some estimates on the maximum point. We have

$$\alpha|x^\star| + \alpha|x_i^\star| \le K \tag{5.10}$$

and

$$|x^\star - x_i^\star| \le K\varepsilon \text{ and } |t^\star - t_n^\star| \le (K + 2\sigma)\varepsilon. \tag{5.11}$$

Indeed, since $(t^\star, x^\star, t_n^\star, x_i^\star)$ is a maximum point of ψ, we have

$$\psi(t^\star, x^\star, t_n^\star, x_i^\star) \ge \psi(0,0,0,0) \ge 0.$$

Then we obtain, by (5.8) and (5.9), and using Young's inequality that

$$\begin{aligned}\alpha|x^\star|^2 + \alpha|x_i^\star|^2 &\le u(t^\star, x^\star) - u(0,0) - v(t_n^\star, x_i^\star) + v(0,0)\\ &\le K(1 + |x^\star| + |x_i^\star|) \le K + \frac{K^2}{\alpha} + \frac{\alpha}{2}|x^\star|^2 + \frac{\alpha}{2}|x_i^\star|^2.\end{aligned}$$

This implies (5.10) since $\alpha \le 1$.

Using that $\psi(t^\star, x^\star, t_n^\star, x_i^\star) \ge \psi(t^\star, x_i^\star, t_n^\star, x_i^\star)$, (5.10) and the fact that u is Lipschitz continuous, we get

$$\begin{aligned}\frac{|x^\star - x_i^\star|^2}{2\varepsilon} &\le u(t^\star, x^\star) - u(t^\star, x_i^\star) - \alpha|x^\star|^2 + \alpha|x_i^\star|^2\\ &\le K|x^\star - x_i^\star| + \alpha\underbrace{(|x_i^\star| - |x^\star|)}_{\le |x^\star - x_i^\star|}(|x^\star| + |x_i^\star|) \le K|x^\star - x_i^\star|.\end{aligned}$$

This implies the first inequality of (5.11). We obtain the second one in the same way, using that $\psi(t^\star, x^\star, t_n^\star, x_i^\star) \ge \psi(t_n^\star, x^\star, t_n^\star, x_i^\star)$ and the fact that u is Lipschitz continuous with respect to t. Inequality (5.10) can be strengthened to

$$\alpha|x^\star|^2 + \alpha|x_i^\star|^2 \le K. \tag{5.12}$$

Indeed, using (5.6) and (5.11), the facts that $\psi(t^\star, x^\star, t_n^\star, x_i^\star) \ge \psi(0,0,0,0)$, $u_0(x_i) \ge v_i^0$, and that u is Lipschitz continuous with respect to x and t, we obtain

$$\alpha|x^\star|^2 + \alpha|x_i^\star|^2 \le u(t^\star, x^\star) - u(0, x_i^\star) + u(0, x_i^\star) - v(t_n^\star, x_i^\star) \le K(|x^\star - x_i^\star| + t^\star) + Kt_n^\star \le K.$$

We assume now that for σ large enough, we have either $t^\star = 0$ or $t_n^\star = 0$. We argue by contradiction and we suppose that $t^\star$ and $t_n^\star$ are positive. Since u is a sub-solution of (1.1), and $u - \phi_1$ reaches a maximum point in $(t^\star, x^\star)$, with ϕ_1 given by

$$\phi_1(t,x) = v(t_n^\star, x_i^\star) + \frac{|t - t_n^\star|^2}{2\varepsilon} + \frac{|x - x_i^\star|^2}{2\varepsilon} + \sigma t + \alpha|x|^2 + \alpha|x_i^\star|^2.$$

Then by definition of A-viscosity solution of (1.1), we have that,

$$\begin{aligned}\sigma + \frac{t^\star - t_n^\star}{\varepsilon} &\le V\left(\frac{1}{I_g}\left(\int_0^A \frac{\phi_1(t^\star, x^\star + z) - \phi_1(t^\star, x^\star)}{z} g(z)\mathrm{d}z + \int_A^{+\infty} \frac{u(t^\star, x^\star + z) - u(t^\star, x^\star)}{z} g(z)\mathrm{d}z\right)\right)\\ &\le V\Bigg(\frac{1}{I_g}\Bigg(\int_0^A \left(\frac{x^\star - x_i^\star}{\varepsilon} + \frac{z}{2\varepsilon} + \alpha(2x^\star + z)\right) g(z)\mathrm{d}z\\ &\quad + \int_A^{+\infty} \frac{u(t^\star, x^\star + z) - u(t^\star, x^\star)}{z} g(z)\mathrm{d}z\Bigg)\Bigg).\end{aligned} \tag{5.13}$$

On the other hand, since $t_n^\star > 0$, and $(t^\star, x^\star, t_n^\star, x_i^\star)$ is a maximum point of ψ, we have that for all $x_i \in Q^\Delta$,

$$\psi(t^\star, x^\star, t_n^\star, x_i^\star) \geq \psi(t^\star, x^\star, t_n^\star - \Delta t, x_i).$$

Then, for all $x_i \in Q^\Delta$, we obtain that,

$$v(t_n^\star - \Delta t, x_i) - v(t_n^\star, x_i^\star) \geq \phi_2(t_n^\star - \Delta t, x_i) - \phi_2(t_n^\star, x_i^\star).$$

where $\phi_2(t,x) = -\left(\frac{|t^\star - t|^2}{2\varepsilon} + \frac{|x^\star - x|^2}{2\varepsilon} + \alpha|x^\star|^2\right)$. Since v_i^n is a solution of the discrete problem (5.3), then we obtain, for $x_i = x_i^\star$

$$\begin{aligned}\frac{\phi_2(t_n^\star, x_i^\star) - \phi_2(t_n^\star - \Delta t, x_i^\star)}{\Delta t} &\geq \frac{v(t_n^\star, x_i^\star) - v(t_n^\star - \Delta t, x_i^\star)}{\Delta t}\\ &\geq V\left(\frac{1}{I_g}\mathbb{I}\left(v^{n-1}\right)(x_i^\star)\right).\end{aligned}$$

That is

$$\frac{t^\star - t_n^\star}{\varepsilon} + \frac{\Delta t}{2\varepsilon} \geq V\left(\frac{1}{I_g}\mathbb{I}\left(v^{n-1}\right)(x_i^\star)\right).$$

Moreover, by (5.6), we have

$$v(t_n^* - \Delta t, x_i) - v(t_n^* - \Delta t, x_i^*) \geq -V_{\max}\Delta t + v(t_n^*, x_i) - v(t_n^*, x_i^*).$$

This implies that (with $\mathcal{I}$ defined in (5.5))

$$\mathbb{I}(v^{n-1})(x_i^\star) \geq \mathbb{I}(v^n)(x_i^\star) - \mathcal{I}V_{\max}\Delta t$$

and so

$$\frac{t^\star - t_n^\star}{\varepsilon} + \frac{\Delta t}{2\varepsilon} \geq V\left(\frac{1}{I_g}\mathbb{I}(v^n)(x_i^\star)\right) - K\Delta t. \tag{5.14}$$

Subtracting inequality (5.14) to (5.13) and using the fact that V is a Lipschitz continuous, we get

$$\begin{aligned}\sigma \leq K &\left(\int_0^A \left|\frac{x^\star - x_i^\star}{\varepsilon} + \frac{z}{2\varepsilon} + \alpha(2x^\star + z)\right| g(z)\mathrm{d}z + \int_B^{+\infty} \left|\frac{u(t^\star, x^\star + z) - u(t^\star, x^\star)}{z} g(z)\right| \mathrm{d}z\right)\\ &+ V\left(\frac{1}{I_g}\int_A^B \frac{u(t^\star, x^\star + z) - u(t^\star, x^\star)}{z} g(z)\mathrm{d}z\right) - V\left(\frac{\mathbb{I}(v^n)(x_i^\star)}{I_g}\right) + \frac{\Delta t}{2\varepsilon} + K\Delta t.\end{aligned} \tag{5.15}$$

Note that, by (5.10) and (5.11), we have

$$\int_0^A \left|\frac{x^\star - x_i^\star}{\varepsilon} + \frac{z}{2\varepsilon} + \alpha(2x^\star + z)\right| g(z)\mathrm{d}z \leq K\left(A + \frac{A^2}{\varepsilon}\right)$$

and

$$\int_B^{+\infty} \left|\frac{u(t^\star, x^\star + z) - u(t^\star, x^\star)}{z} g(z)\right| \mathrm{d}z \leq L_0 R_B.$$

Hence inequality (5.15) becomes

$$\sigma \leq V\left(\frac{1}{I_g}\int_A^B \frac{u(t^\star, x^\star + z) - u(t^\star, x^\star)}{z} g(z)\mathrm{d}z\right) - V\left(\frac{\mathbb{I}(v^n)(x_i^\star)}{I_g}\right) + K\frac{\Delta t}{\varepsilon} + KR_B + K\left(A + \frac{A^2}{\varepsilon}\right)\cdot \tag{5.16}$$

Moreover, by (5.4), we have

$$\left|\int_A^B \frac{u(t^\star,x^\star+z)-u(t^\star,x^\star)}{z}g(z)\mathrm{d}z - \mathbb{I}(u(t^\star,\cdot))(x^\star)\right| \leq \frac{KB}{A}\Delta x,$$

hence

$$V\left(\frac{1}{I_g}\int_A^B \frac{u(t^\star,x^\star+z)-u(t^\star,x^\star)}{z}g(z)\mathrm{d}z\right) \leq V\left(\frac{\mathbb{I}(u(t^\star,\cdot))(x^\star)}{I_g}\right) + \frac{KB}{A}\Delta x. \tag{5.17}$$

We now want to bound

$$V\left(\frac{\mathbb{I}(u(t^\star,\cdot))(x^\star)}{I_g}\right) - V\left(\frac{\mathbb{I}(v^n)(x_i^\star)}{I_g}\right).$$

Since $\psi(t^\star,x^\star,t_n^\star,x_i^\star) \geq \psi(t^\star,x^\star+z,t_n^\star,x_i^\star+z)$, we get that

$$\frac{u(t^\star,x^\star+z)-u(t^\star,x^\star)}{z} \leq \frac{v(t_n^\star,x_i^\star+z)-v(t_n^\star,x_i^\star)}{z} + 2\alpha(x^\star+x_i^\star+z).$$

This implies that

$$\begin{aligned}
\mathbb{I}(u(t^\star,\cdot))(x^\star) &\leq \mathbb{I}(v^n)(x_i^\star) + 2\alpha(x^\star+x_i^\star)\Delta x\left(\frac{1}{2}g(N_A\Delta x)+\frac{1}{2}g(N_B\Delta x)+\sum_{j=N_A+1}^{N_B-1} g(j\Delta x)\right)\\
&\quad + 2\alpha\Delta x\left(\frac{1}{2}N_A\Delta x.g(N_A\Delta x)+\frac{1}{2}N_B\Delta x.g(N_B\Delta x)+\sum_{j=N_A+1}^{N_B-1} j\Delta x.g(j\Delta x)\right)\\
&\leq \mathbb{I}(v^n)(x_i^\star) + K\sqrt{\alpha}\int_A^B g(z)\mathrm{d}z + K\alpha\int_A^B zg(z)\mathrm{d}z + B(\sqrt{\alpha}+\alpha)\Delta x\\
&\leq \mathbb{I}(v^n)(x_i^\star) + K(\sqrt{\alpha}+\alpha)(1+B\Delta x).
\end{aligned}$$

We then get

$$\begin{aligned}
V\left(\frac{\mathbb{I}(u(t^\star,\cdot))(x^\star)}{I_g}\right) - V\left(\frac{\mathbb{I}(v^n)(x_i^\star)}{I_g}\right) &\leq V\left(\frac{\mathbb{I}(v^n)(x_i^\star)}{I_g} + K(\sqrt{\alpha}+\alpha)(1+B\Delta x)\right) - V\left(\frac{\mathbb{I}(v^n)(x_i^\star)}{I_g}\right)\\
&\leq K(\sqrt{\alpha}+\alpha)(1+B\Delta x)\\
&\leq K\sqrt{\alpha}(1+B\Delta x).
\end{aligned}$$

Injecting this in (5.16) and using (5.17), we get

$$\sigma \leq K\sqrt{\alpha}(1+B\Delta x) + \frac{KB}{A}\Delta x + K\frac{\Delta t}{\varepsilon} + KR_B + K\left(A+\frac{A^2}{\varepsilon}\right).$$

Let $\sigma^\star = K\sqrt{\alpha}(1+B\Delta x) + \frac{KB}{A}\Delta x + K\frac{\Delta t}{\varepsilon} + KR_B + K\left(A+\frac{A^2}{\varepsilon}\right)$, then we have a contradiction for all $\sigma \geq \sigma^\star$, and then $t^\star = 0$ or $t_n^\star = 0$ for all σ such that $\sigma \geq \sigma^\star$.

If $t^\star = 0$, using (5.6), (5.11) and the fact that u_0 is Lipschitz continuous, we obtain that

$$M_{\varepsilon,\alpha}^\sigma \leq u(0,x^\star) - v(t_n^\star,x_i^\star) \leq u_0(x^\star) - u_0(x_i^\star) + Kt_n^\star + \mu_0 \leq K(|x^\star - x_i^\star| + t_n^\star) + \mu_0 \leq K(1+\sigma)\varepsilon + \mu_0.$$

In the same way, if $t_n^\star = 0$, we get

$$M_{\varepsilon,\alpha}^\sigma \leq u(t^\star,x^\star) - v(0,x_i^\star) \leq u(t^\star,x^\star) - u_0(x_i^\star) + \mu_0 \leq K(|x^\star - x_i^\star| + t^\star) + \mu_0 \leq K(1+\sigma)\varepsilon + \mu_0.$$

We conclude that for all $\sigma^\star \leq \sigma$, we have

$$M^{\sigma}_{\varepsilon,\alpha} \leq K(1+\sigma)\varepsilon + \mu_0.$$

We then deduce that, for every $(t_n, x_i) \in Q_T^{\Delta}$, and for $\sigma = \sigma^\star \leq 1$, we have that

$$u(t_n,x_i) - v(t_n,x_i) - \left(K\sqrt{\alpha}(1+B\Delta x) + \frac{KB}{A}\Delta x + K\frac{\Delta t}{\varepsilon} + KR_B + K\left(A + \frac{A^2}{\varepsilon}\right)\right)T \leq K\varepsilon + \mu_0.$$

Sending $\alpha, \eta \to 0$, and choosing $\varepsilon = \sqrt{\Delta t}$, we obtain that

$$\mu \leq K\left(\sqrt{\Delta t} + \frac{B}{A}\Delta x + R_B + \left(A + \frac{A^2}{\sqrt{\Delta t}}\right)\right) + \mu_0,$$

provided that $\mu_0 \leq 1$ and Δx, Δt small enough (so that $\sigma^* \leq 1$).

Using the same arguments of Alvarez *et al.* ([1], Thm. 2), we easily deduce the result in the general case ($u_0(x_i) \not\geq v_i^0$).This ends the proof. □

Remark 5.7. If $T \geq 1$, then since we have the estimation in each time interval of length 1, we then obtain that

$$\mu \leq K.T\left(\sqrt{\Delta t} + \frac{B}{A}\Delta x + R_B + \left(A + \frac{A^2}{\sqrt{\Delta t}}\right)\right) + \mu_0.$$

5.3. Some numerical illustrations

In the numerical experiments below, we take the following discretization parameters $\Delta t = 0.005$, $T = 0.5$, $\Delta x = 0.05$, $b = -a = 3$ and $B = 10$. For the velocity V, inspired by [19] we consider the two following cases:

$$\text{(Greenshield): } V(x) = \begin{cases} 0 & \text{if } x \leq x_0 \\ V_{\max}\left(1 - \left(\frac{x_0}{x}\right)^p\right) & \text{if } x \in (x_0, x_{\max}) \\ V_{\max}\left(1 - \left(\frac{x_0}{x_{\max}}\right)^p\right) & \text{if } x \geq x_{\max} \end{cases}$$

$$\text{(Underwood): } V(x) = \begin{cases} 0 & \text{if } x \leq x_0 \\ V_{\max}\left(1 - e^{-(x-x_0)^p}\right) & \text{if } x \in (x_0, x_{\max}) \\ V_{\max}\left(1 - e^{-(x_{\max}-x_0)^p}\right) & \text{if } x \geq x_{\max} \end{cases}$$

where $x_0 > 0$, $x_{\max} > x_0$ and $V_{\max}$ are real parameters and $p \in \mathbb{N}^*$. For the interpretation, we recall that x_0 corresponds to the minimal distance between two successive cars and $x_{\max}$ the distance up to which a driver will not increase his speed. One can easily check that these two examples satisfy assumptions **(H)**. In the sequel, we take $x_0 = 0.2$, $x_{\max} = 10$, $p = 1$ and $V_{\max} = 90$.

For the weighting function g, we take $g(z) = \frac{1}{\eta}e^{-\frac{z}{\eta}}$ with $\eta > 0$ so that $I_g = 1$, see equation (5.3). It can be also verified that it satisfies assumptions **(H)**.

Riemann initial data. The initial data is given by

$$u_0(x) = \begin{cases} \frac{x}{\rho_L} & \text{if } x < 0 \\ \frac{x}{\rho_R} & \text{if } x \geq 0 \end{cases} \quad \Leftrightarrow \quad \rho_0(x) = \begin{cases} \rho_L & \text{if } x < 0 \\ \rho_R & \text{if } x \geq 0 \end{cases} \tag{5.18}$$

where $\rho_L > 0$ and $\rho_R > 0$ are the initial density on left and on the right respectively. On Figure 2, we represent the density $\rho = \frac{1}{u_x}$ at time $t = 0.2$ for different values of $\eta \in \{0.5, 1, 5\}$ taking $\rho_L = 0.2$ and $\rho_R = 0.8$. We also represent in dotted line the "local" case corresponding to the situation where we solve $u_t = V(u_x)$ (meaning that the driver at position x take only into account the car directly in front of him to adapt his speed). As we

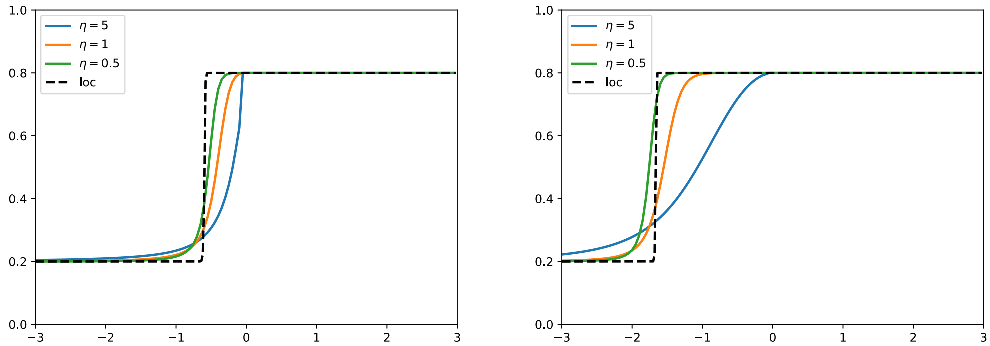

FIGURE 2. Density ρ at time $t = 0.2$ for different value of η using Greenshield velocity (on the *left*) and Underwood velocity (on the *right*) using (5.18) initial density with $\rho_L = 0.2$ and $\rho_R = 0.8$.

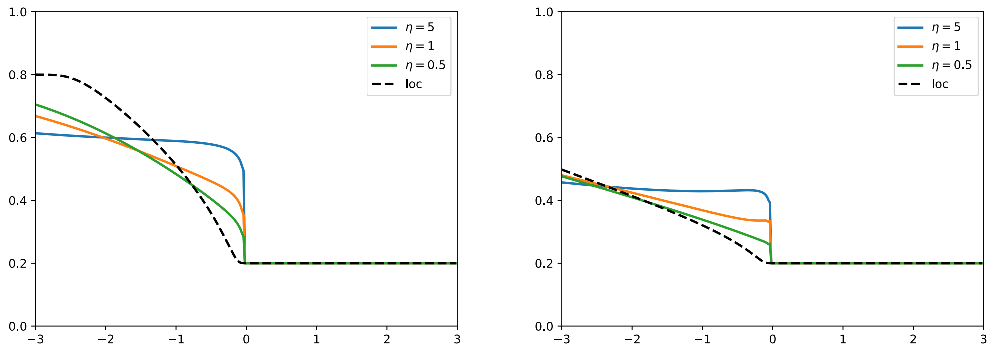

FIGURE 3. Density ρ at time $t = 0.2$ for different value of η using Greenshield velocity (on the *left*) and Underwood velocity (on the *right*) using (5.18) initial density with $\rho_L = 0.8$ and $\rho_R = 0.2$.

can see, and as we can expect, taking into account downstream traffic on a wider distance (meaning big η) leads to a smoother repartition of the density and somehow delays the progression of the front.

Similarly, we represent on Figure 3 the comparison for $\eta \in \{0.5, 1, 5\}$ of the density $\frac{1}{u_x}$ at time $t = 0.2$ taking this time $\rho_L = 0.8$ and $\rho_R = 0.2$. We can see that, even thought the downstream density is lower, taking into account downstream traffic on large distance has a strong impact on the result. In particular for large η, the "decompression" is more uniform in the sense that the density on the left is almost flat.

Oscillating initial data. Let us consider a last situation with an oscillating initial density:

$$\rho_0(x) = \begin{cases} \rho_L = 0.5 & \text{if } x < -2 \\ 0.4\sin((x+2)\pi) + 0.5 & \text{if } x \in (-2, 2) \\ \rho_R = 0.5 & \text{if } x \geq 2. \end{cases} \tag{5.19}$$

The associated initial state u_0 is recovered by simple integration. On Figure 4, we represent the density profile also at time $t = 0.2$ for $\eta \in \{0.5, 1, 5\}$ and in dotted line the "local" case. Once again, we can remark that the bigger η is, the "smoother" the density is.

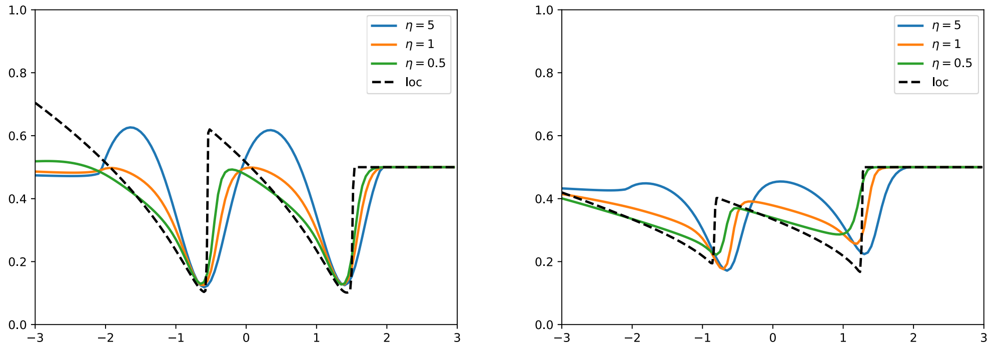

FIGURE 4. Density ρ at time $t = 0.2$ for different value of η using Greenshield velocity (on the *left*) and Underwood velocity (on the *right*) using (5.19) initial density.

6. Concluding remarks

In this paper, we proposed and study a non-local traffic flow model starting from the microscopic model to build the homogenized macroscopic model. Using Hamilton–Jacobi framework, we are able to prove the convergence of the homogenization process, and existence and uniqueness of the solution for the macroscopic problem. We believe that this approach is relevant in the context of traffic flow modeling since it enables to consider more general assumptions (such as the regularity of V) compared to what we can find for models based on conservation laws for which the analysis is more complicated and particular. Finally, we proposed and analyzed a numerical scheme to solve the proposed non local macro-model, and illustrated with some results the impact of the non-local aspect of the model. In future works, we would like to extend this model to traffic flow on network.

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