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THERMAL FLOWS IN FRACTURED POROUS MEDIA

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Abstract. We consider the thermal flow problem occuring in a fractured porous medium. The incompressible filtration flow in the porous matrix and the viscous flow in the fractures obey the Boussinesq approximation of Darcy-Forchheimer law and respectively, the Stokes system. They are coupled by the Saffman's variant of the Beavers–Joseph condition. Existence and uniqueness properties are presented. The use of the energy norm in describing the Darcy-Forchheimer law proves to be appropriate. In the ε-periodic framework, we find the two-scale homogenized system which governs their asymptotic behaviours when $\varepsilon \to 0$ and the Forchheimer effect vanishes. The limit problem is mainly a model of two coupled thermal flows, neither of them being incompressible.

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1. Introduction

Among the issues raised by the heat and mass transfer in fractured porous media, the requirement for further construction and characterization of macroscopic models is of special interest. It is a difficult task because the two components have highly constrasting behaviours. The models of flows through fractured porous media (see [3, 4, 9, 25, 27]) are usually obtained by asymptotic methods, from the alteration of a homogeneous porous medium by a distribution of microscopic fractures/fissures. In this context, the periodic homogenization, based on the assumption of the ε-periodicity of the structure properties, is an important modelling tool for a fractured porous media process. Although it looks like an idealistic assumption, it usually authorizes a rigorous approach, yielding many of the properties which must be taken into account at macroscopic level.

Here, we consider that the heat and mass transfer takes place in a periodically structured domain consisting of two interwoven regions, separated by an interface. As the process at the microscopic scale takes place under the assumption of ε-periodicity, the study of its asymptotic behaviour (when $\varepsilon \to 0$) is amenable to the procedures of the homogenization theory. Regarding our subject, the homogenization of phenomena in fractured media could be studied in a more realistic manner only when the non-connectedness assumption of one of the components was dropped out (see [1, 23, 24]). We improve the properties of the ε-periodic biphasic structure introduced in [24], by attaching the so-called ε-domes. They are placed in the last entire ε-cells contained in the domain, near

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the boundary, and they complete the ε-periodic interface such that it can be as smooth as it is needed, all the properties of [24] remaining valid.

The first region, the only one reaching the boundary of the domain, represents a connected porous matrix, where, disregarding its pore scale, we consider the movement of an incompressible average filtration fluid governed by the Boussinesq approximation of the Darcy-Forchheimer system. The linear Darcy's law relating the flow and the pressure gradient in the porous surrounding matrix relies on the assumption of laminar flow (see [29]). Unfortunately, this assumption does not hold when high imposed pressure gradients and resistance from fracture walls lead to reduced flow rates compared to the linear Darcy relation. The standard extended model involves a Forchheimer correction term (see [10]) which introduces a non-linear coupling between pressure gradient and flow rates. This Forchheimer term was proved to be valid at higher Reynolds number by Muskat (see [21]). Exterior forces are present.

The second region, representing the fractures, which are not necessarily connected, is saturated by an incompressible viscous fluid governed by the Boussinesq approximation of the Stokes system.

These two flows are coupled on the interface by the Saffman's variant [26] of the Beavers–Joseph condition (see [5, 16]) which was confirmed by Jäger and Mikelić [15] as the limit of a homogenization process. Besides the continuity of the normal component of the velocity, it imposes the proportionality of the tangential velocity with the tangential component of the viscous stress on the fluid-side of the interface.

The tensors of thermal diffusion of the two phases are ε-periodic and not necessarily equal. At the interface, the the temperature and the heat flux are continuous. Heat sources are present in each component and a temperature distribution is imposed on the boundary of the domain.

We prove the existence and uniqueness properties of the velocity, pressure and temperature distribution, solutions of the corresponding thermal flow boundary problem. An L^∞-estimate of the temperature, uniform with respect to ε, is also presented (Thm. 3.1). The way of describing the Darcy-Forchheimer law by powers of the energy norm of the inverse permeability tensor proves to be appropriate. These results have an intrinsic interest, apart from the related homogenization result.

As the Forchheimer effect vanishes with the small period of the distribution shrinking to zero, we study the asymptotic behaviour of the flow when the Rayleigh number is of unity order, the permeability of the porous blocks of unity order and the Beavers–Joseph transfer coefficient of ε-order, balancing the measure of the interface. Our main result (Thm. 4.5) presents the two-scale system verified by the limits of the ε-solutions, the local problems and the effective coefficients of the leading homogenized system. Regarding the case of the non-vanishing Forchheimer effect, the expression of its limit seems to us untraceable by the procedures of the two-scale convergence theory.

The paper is organized as follows.

In Section 2 we present our fractured porous medium, the ε-periodic structure provided with the useful ε-domes. The direct form of the thermal flow problem is introduced.

In Section 3 we prove the existence and uniqueness properties. The weak solutions of our nonlinear problem are found by means of the Browder-Minty and Schauder fixed-point theorems. The primary estimates are also obtained.

Section 4 is devoted to the homogenization in the case when the Forchheimer effect is vanishing. We present the a priori estimates which serve as departure point for adapting the compactness results of the two-scale convergence theory (see [2, 19, 22]). Using the techniques of the two-scale convergence theory (see [2, 7, 22]), we obtain the so-called two-scale homogenized problem and the solutions of the local problems which allow us to define the effective coefficients of the homogenized system and to eliminate some of the oscillating unknowns. It is a model of two coupled thermal flows, neither of them being incompressible. This macroscopic problem takes a classic form in the case of non-oscillating permeability tensor.

At the end, in Appendix A, a usefull result of strict monotonicity is proved.

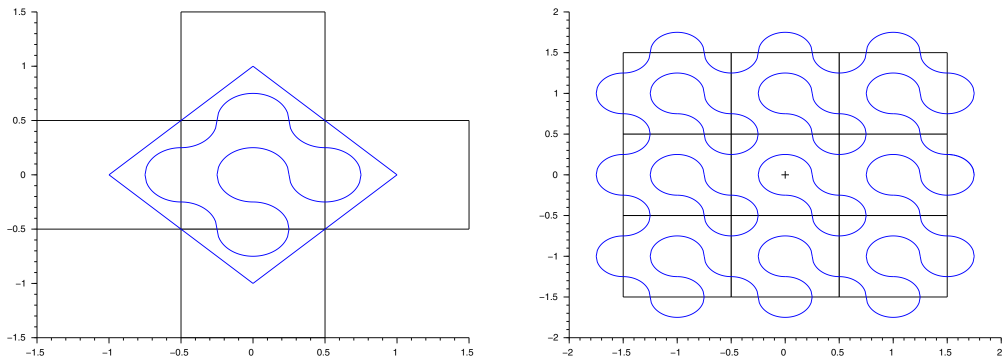

FIGURE 1. The rhombic cell including domes (*left*) and the resulting modified periodic distribution (*right*).

2. The fractured structure and the governing system

Let Ω be an open connected bounded set in $\mathbb{R}^N$, $N \in \{2, 3\}$, a manifold of class C^2 composed of a finite number of connected components, locally located on one side of the boundary $\partial\Omega$ with ν its outward normal.

We describe now the geometric structure of our fractured porous medium, similar to that introduced in [14].

Let E be the rhombic polyhedron obtained by affixing square pyramids of $1/2$ height on each face of the cube $Y =]-1/2, 1/2[^N$, that is

$$E = \operatorname{int}\left(\operatorname{Conv}\left(\overline{Y} \cup \left\{\pm\frac{1}{2}e_i,\ i = 1, 2, \ldots, N\right\}\right)\right), \tag{2.1}$$

where e_i are the unit vectors of the canonical basis in $\mathbb{R}^N$.

For $D \subset\subset E$, an open set of class C^2, and denoting $Y_f := Y \cap D$ and $\Sigma^{\pm i} = \{y \in \partial Y :\ y_i = \pm 1/2\}$, we assume that for every $i \in \{1, 2, \ldots, N\}$ it holds

$$\overline{Y}_f \cap \Sigma^{\pm i} \subset\subset \Sigma^{\pm i}. \tag{2.2}$$

We assume also that these intersections are reproduced identically on opposite faces of the cube Y, that is

$$e_i + \overline{Y}_f \cap \Sigma^{-i} = \overline{Y}_f \cap \Sigma^{+i}, \quad \forall i \in \{1, 2, \ldots, N\}. \tag{2.3}$$

For every $i \in \{1, 2, \ldots, N\}$ we define the corresponding two opposite domes of Y_f by

$$D_i^+ = \left(Y + \frac{1}{2}e_i\right) \cap D \quad \text{and} \quad D_i^- = \left(Y - \frac{1}{2}e_i\right) \cap D. \tag{2.4}$$

Denoting $Y_s := Y \setminus \overline{Y}_f$, we assume that the reunion in $\mathbb{R}^N$ of all the periodic replications of $\overline{Y}_s$ parts, denoted by $\mathbb{R}^N_s$, has a C^2 boundary; $\mathbb{R}^N_f$ is similarly defined. The characteristic functions of Y_s and Y_f are denoted by χ_s and χ_f, respectively; we also assume $m := |Y_f| \in]0, 1[$ (Fig. 1).

Without loss of the generality, we set the origin of the coordinate system in such a way that there exists $r > 0$ with the property $B(0, r) \subseteq \mathbb{R}^N_s$.

For any $\varepsilon \in]0, 1[$ we denote

$$\mathbb{Z}_\varepsilon = \left\{k \in \mathbb{Z}^N :\ \varepsilon k + \varepsilon Y \subseteq \Omega\right\}, \tag{2.5}$$

$$\mathbb{I}_\varepsilon = \{k \in \mathbb{Z}_\varepsilon :\ \varepsilon k \pm \varepsilon e_i + \varepsilon Y \subseteq \Omega,\ \forall i \in \{1, 2, \ldots, N\}\}, \tag{2.6}$$

$$\Omega^{\square}_{\varepsilon f} = \cup_{k \in \mathbb{I}_\varepsilon}(\varepsilon k + \varepsilon Y_f). \tag{2.7}$$

For any $k \in \mathbb{Z}_\varepsilon \setminus \mathbb{I}_\varepsilon$, denoting by

$$\mathbb{J}^{\pm}_{\varepsilon k} = \left\{ \; i \in \{1, 2, \dots, N\}, \quad \left(\varepsilon k + \varepsilon \overline{D}^{\pm}_i\right) \cap \overline{\Omega^{\square}_{\varepsilon f}} \neq \emptyset \; \right\}, \tag{2.8}$$

we define the ε-domes which have to be attached to $\Omega^{\square}_{\varepsilon f}$ in order to regularize the interface between the free fluid saturating the fractures and the filtration fluid saturating the porous matrix, by

$$D_{\varepsilon k} = \left(\cup_{i\in\mathbb{J}^+_{\varepsilon k}} \left(\varepsilon k + \varepsilon D^+_i\right)\right) \cup \left(\cup_{i\in\mathbb{J}^-_{\varepsilon k}} \left(\varepsilon k + \varepsilon D^-_i\right)\right). \tag{2.9}$$

We consider that the free fluid takes place in

$$\Omega_{\varepsilon f} = \mathrm{int}\left(\left(\cup_{k\in\mathbb{I}_\varepsilon} \left(\varepsilon k + \varepsilon \overline{Y}_f\right)\right) \cup \left(\cup_{k\in\mathbb{Z}_\varepsilon\setminus\mathbb{I}_\varepsilon} D_{\varepsilon k}\right)\right). \tag{2.10}$$

Consequently, the porous matrix and the interface between the two components are defined by:

$$\Omega_{\varepsilon s} = \Omega \setminus \overline{\Omega}_{\varepsilon f}, \tag{2.11}$$
$$\Gamma_\varepsilon = \partial\Omega_{\varepsilon f} \cap \partial\Omega_{\varepsilon s} = \partial\Omega_{\varepsilon f}. \tag{2.12}$$

We assume that $\Omega_{\varepsilon s}$ is connected, which means that for $N = 2$ we have the classical setup with $Y_f = D \subset\subset Y$ and the ε-domes can be considered only when $N = 3$. Also, for every $\varepsilon > 0$, there exist $k_\varepsilon \in \mathbb{N}$, $k_\varepsilon \geq 1$, such that

$$\Omega_{\varepsilon f} = \cup_{k=1}^{k_\varepsilon} \Omega^k_{\varepsilon f} \tag{2.13}$$

where every $\Omega^k_{\varepsilon f}$ is a connected subdomain of $\Omega_{\varepsilon f}$ with $\mathrm{dist}(\Omega^i_{\varepsilon f}, \Omega^j_{\varepsilon f}) > 0$ if $i \neq j$. The characteristic functions of $\Omega_{\varepsilon s}$ and $\Omega_{\varepsilon f}$ are denoted by $\chi_{\varepsilon s}$ and $\chi_{\varepsilon f}$, respectively.

Denoting $\Gamma^k_\varepsilon = \partial\Omega^k_{\varepsilon f}$, it follows that

$$\Gamma_\varepsilon = \cup_{k=1}^{k_\varepsilon} \Gamma^k_\varepsilon. \tag{2.14}$$

Denoting by $\Gamma = \partial Y_f \cap \partial Y_s \subseteq \partial D$, by n the normal on ∂D (inward to D) and by n^ε the normal on Γ_ε (outward to $\Omega_{\varepsilon s}$), we have

$$n^\varepsilon(x) = n(x/\varepsilon), \text{ for any } x \in (\varepsilon k + \varepsilon\Gamma) \text{ with } k \in \mathbb{I}_\varepsilon, \tag{2.15}$$

where the Y-periodic extension of $n|_\Gamma$ is still denoted by n.

The class of the connections between $\Omega^{\square}_{\varepsilon f}$ and the corresponding ε-domes is similar to that between Y_f and its domes, that is the class of D. This is an important advantage of the structures with ε-domes: the class of Γ_ε is given by D and by the reunion of all the $\overline{Y}_s$ parts in $\mathbb{R}^N$, which can be assumed as smooth as it is needed. There is also an important feature of our periodic structure, provided with ε-domes. As the $(\varepsilon k + \varepsilon Y)$-cells containing ε-domes are of at most $(4^N - 2)$ types and the distance between Γ_ε and $\partial\Omega$ is greater than $\varepsilon/2$, they do not affect the results obtained for the classical ε-periodic structures. The present structure preserves many specific properties (see [6, 9, 13, 14, 24]).

Now we can present the thermal flow problem which corresponds to our framework. If $(u^{\varepsilon s}, p^{\varepsilon s}, \theta^{\varepsilon s})$ and $(u^{\varepsilon f}, p^{\varepsilon f}, \theta^{\varepsilon f})$ stand for the velocities, pressures and temperatures associated to the corresponding phase of our structure, then they verify the following dimensionless system:

$$\mathrm{div}\, u^{\varepsilon s} = 0 \quad \text{in} \quad \Omega_{\varepsilon s}, \quad \mathrm{div}\, u^{\varepsilon f} = 0 \quad \text{in} \quad \Omega_{\varepsilon f}, \quad u^{\varepsilon s} \cdot n^\varepsilon = u^{\varepsilon f} \cdot n^\varepsilon \quad \text{on} \quad \Gamma_\varepsilon, \tag{2.16}$$
$$\nabla p^{\varepsilon s} + (1 + d_\varepsilon |u^{\varepsilon s}|^{r-2}_{A^\varepsilon}) A^\varepsilon u^{\varepsilon s} + \alpha_\varepsilon \theta^{\varepsilon s} g = 0, \quad |u^{\varepsilon s}|_{A^\varepsilon} = (A^\varepsilon_{ij} u^{\varepsilon s}_i u^{\varepsilon s}_j)^{1/2} \quad \text{in} \quad \Omega_{\varepsilon s}, \tag{2.17}$$
$$-\,\mathrm{div}\,\Sigma^{\varepsilon i} + \alpha_\varepsilon \theta^{\varepsilon f} g_i = 0 \quad \text{in} \quad \Omega_{\varepsilon f}, \quad \forall i \in \{1, 2, \dots, N\}, \tag{2.18}$$
$$\Sigma^{\varepsilon i}_j = -p^{\varepsilon f}\delta_{ij} + e_{ij}(u^{\varepsilon f}), \quad e_{ij}(u^{\varepsilon f}) = \frac{1}{2}\left(\frac{\partial u^{\varepsilon f}_i}{\partial x_j} + \frac{\partial u^{\varepsilon f}_j}{\partial x_i}\right) \quad \text{in} \quad \Omega_{\varepsilon f}, \tag{2.19}$$

$$p^{\varepsilon s} n_i^\varepsilon + \Sigma^{\varepsilon i} n^\varepsilon + \varepsilon \beta_\varepsilon (u_i^{\varepsilon f} - (u^{\varepsilon f} \cdot n^\varepsilon) n_i^\varepsilon) = 0 \quad \text{on} \quad \Gamma_\varepsilon, \qquad \forall i \in \{1, 2, \ldots, N\}, \tag{2.20}$$

$$u^{\varepsilon f} \nabla \theta^{\varepsilon f} - \operatorname{div}(B^{\varepsilon f} \nabla \theta^{\varepsilon f}) = Q^f \quad \text{in} \quad \Omega_{\varepsilon f}, \tag{2.21}$$

$$u^{\varepsilon s} \nabla \theta^{\varepsilon s} - \operatorname{div}(B^{\varepsilon s} \nabla \theta^{\varepsilon s}) = Q^s \quad \text{in} \quad \Omega_{\varepsilon s}, \tag{2.22}$$

$$B_{ij}^{\varepsilon f} \frac{\partial \theta^{\varepsilon f}}{\partial x_j} n_i^\varepsilon = B_{ij}^{\varepsilon s} \frac{\partial \theta^{\varepsilon s}}{\partial x_j} n_i^\varepsilon, \quad \theta^{\varepsilon s} = \theta^{\varepsilon f} \quad \text{on} \quad \Gamma_\varepsilon \tag{2.23}$$

$$u^{\varepsilon s} \cdot \nu = 0 \quad \text{on} \quad \partial\Omega, \quad \nu \text{ the outward normal}, \tag{2.24}$$

$$\theta^\varepsilon = \tau \quad \text{on} \quad \partial\Omega, \tag{2.25}$$

where $\tau \in H^1(\Omega) \cap L^\infty(\Omega)$ has the property that $\exists \tau_0 > 0$ for which

$$|\tau| \leq \tau_0 \text{ on } \partial\Omega \text{ in the sense of } H^1(\Omega) \text{ (see [17])}. \tag{2.26}$$

The symmetric tensor $A^\varepsilon \in L^\infty(\Omega)^{N \times N}$, which stands for the inverse of the permeability tensor, the Beavers–Joseph coefficient $\beta_\varepsilon \in C^1(\Omega)$ and the symmetric conductivities $B^{\varepsilon f}, B^{\varepsilon s} \in L^\infty(\Omega)^{N \times N}$ are given with the property that there exist b_2 and $b_1 > 0$, $b_1 < b_2$, independent of ε, such that for any $\varepsilon > 0$ we have

$$|A^\varepsilon|_{L^\infty(\Omega)} \leq b_2, \quad |B^{\varepsilon s}|_{L^\infty(\Omega)} \leq b_2, \quad |B^{\varepsilon f}|_{L^\infty(\Omega)} \leq b_2, \qquad a.e. \text{ in } \Omega, \tag{2.27}$$

$$\beta_\varepsilon \geq b_1, \quad \left(A_{ij}^\varepsilon, B_{ij}^{\varepsilon s}, B_{ij}^{\varepsilon f} \right) \xi_i \xi_j \geq b_1 \, \xi_i \xi_i, \quad \forall \xi \in \mathbb{R}^N, \qquad a.e. \text{ in } \Omega. \tag{2.28}$$

The rest of the data are the Forchheimer coefficient $d_\varepsilon > 0$, the Rayleigh number $\alpha_\varepsilon > 0$, the exterior forces $g \in L^2(\Omega)^N$, the heat sources $Q^f, Q^s \in L^2(\Omega)$ and the Forchheimer exponent $r \in \mathbb{R}$ with the property:

$$r > 2 \quad \text{if} \quad N = 2 \quad \text{and} \quad 3 \leq r < 6 \quad \text{if} \quad N = 3. \tag{2.29}$$

3. Existence and estimates of the weak solutions

We present in this section the existence and uniqueness properties of the weak solutions of the convection problem (2.16)–(2.25), together with an L^∞-estimate of the temperature.

Let us introduce the following spaces:

$$H = \left\{ v \in H(\operatorname{div}, \Omega), \quad v \in L^r(\Omega)^N, \quad v_\nu = 0 \quad \text{on} \quad \partial\Omega \right\}, \tag{3.1}$$

$$V = \{ v \in H, \quad \operatorname{div} v = 0 \quad \text{in} \quad \Omega \}, \tag{3.2}$$

$$H_\varepsilon = \left\{ v \in H, \quad v|_{\Omega_{\varepsilon f}} \in H^1(\Omega_{\varepsilon f})^N \right\}, \tag{3.3}$$

$$V_\varepsilon = \{ v \in H_\varepsilon, \quad \operatorname{div} v = 0 \quad \text{in} \quad \Omega \}, \tag{3.4}$$

$$L_0^2(\Omega) = \left\{ p \in L^2(\Omega), \quad \int_\Omega p = 0 \right\}, \tag{3.5}$$

where v_ν stands for the normal trace on $\partial\Omega$.

For any $v \in H_\varepsilon$ we denote the normal trace on Γ_ε in the $H(\operatorname{div}, \Omega)$ sense by v_{n^ε} and the trace on Γ_ε in the $H^1(\Omega_{\varepsilon f})$ sense by $\gamma_{\varepsilon f} v$. As Γ_ε is of class C^2, let us remark that

$$v_{n^\varepsilon} = (\gamma_{\varepsilon f} v) n^\varepsilon \in H^{1/2}(\Gamma_\varepsilon). \tag{3.6}$$

Introducing

$$v_{t^\varepsilon} := \gamma_{\varepsilon f} v - (v_{n^\varepsilon}) n^\varepsilon \in H^{1/2}(\Gamma_\varepsilon)^N, \tag{3.7}$$

we obviously have

$$(\gamma_{\varepsilon f} v)^2 = (v_{n^\varepsilon})^2 + (v_{t^\varepsilon})^2 \quad a.e. \text{ on} \quad \Gamma_\varepsilon. \tag{3.8}$$

We see now that H and H_ε are Banach spaces, endowed with the norms:

$$|v|_H = |v|_{L^r(\Omega)} + |\operatorname{div} v|_{L^2(\Omega)}, \tag{3.9}$$

$$|v|_{H_\varepsilon} = |v|_{L^r(\Omega_{\varepsilon s})} + |\operatorname{div} v|_{L^2(\Omega_{\varepsilon s})} + |e(v)|_{L^2(\Omega_{\varepsilon f})} + \varepsilon^{1/2}|v_{t^\varepsilon}|_{\Gamma_\varepsilon}. \tag{3.10}$$

Moreover, by rescaling some inequalities valid in Y, $E \setminus Y$ and Y_f, we obtain:

$$|v|_{L^2(\Omega_{\varepsilon f})} \le C\left(|v|_{L^2(\Omega_{\varepsilon s})} + \varepsilon|\operatorname{div} v|_{L^2(\Omega_{\varepsilon s})} + \varepsilon|e(v)|_{L^2(\Omega_{\varepsilon f})} + \varepsilon^{1/2}|v_{t^\varepsilon}|_{L^2(\Gamma_\varepsilon)}\right), \tag{3.11}$$

$$|v|_{H^1(\Omega_{\varepsilon f})} \le C|v|_{H_\varepsilon}, \quad \forall v \in H_\varepsilon, \tag{3.12}$$

where C is independent of ε.

Denoting $T^\varepsilon = \theta^\varepsilon - \tau$ in (2.16)–(2.25), we are led to the following variational problem:

To find $(u^\varepsilon, T^\varepsilon) \in V_\varepsilon \times H_0^1(\Omega)$ which verifies

$$\int_{\Omega_{\varepsilon s}} \left(1 + d_\varepsilon|u^\varepsilon|_{A^\varepsilon}^{r-2}\right) A^\varepsilon u^\varepsilon v + \int_{\Omega_{\varepsilon f}} e_{ij}(u^\varepsilon)e_{ij}(v) + \varepsilon\beta_\varepsilon \int_{\Gamma_\varepsilon} u^\varepsilon_{t_\varepsilon} v + \alpha_\varepsilon \int_\Omega (T^\varepsilon + \tau)gv = 0, \quad \forall v \in V_\varepsilon, \tag{3.13}$$

$$\int_\Omega B^\varepsilon \nabla T^\varepsilon \nabla S + \int_\Omega u^\varepsilon S \nabla T^\varepsilon = \int_\Omega QS - \int_\Omega u^\varepsilon S\nabla\tau - \int_\Omega B^\varepsilon \nabla\tau\nabla S, \qquad \forall S \in H_0^1(\Omega), \tag{3.14}$$

where we denoted

$$B^\varepsilon = \begin{cases} B^{\varepsilon s} \text{ in } \Omega_{\varepsilon s}, \\ B^{\varepsilon f} \text{ in } \Omega_{\varepsilon f} \end{cases} \quad \text{and} \quad Q = \begin{cases} Q^s \text{ in } \Omega_{\varepsilon s}, \\ Q^f \text{ in } \Omega_{\varepsilon f}. \end{cases} \tag{3.15}$$

Theorem 3.1. *There exists a solution of the problem (3.13) and (3.14). Any solution $(u^\varepsilon, T^\varepsilon)$ of (3.13) and (3.14) has the property that $T^\varepsilon \in L^\infty(\Omega)$ and that for some $c > 0$, independent of ε, we have*

$$|\nabla T^\varepsilon|_{L^2(\Omega)} + |T^\varepsilon + \tau|_{L^\infty(\Omega)} \le c, \tag{3.16}$$

$$|u^\varepsilon|_{L^2(\Omega)} + |u^\varepsilon|_{H^1(\Omega_{\varepsilon f})} + \varepsilon^{1/2}|u^\varepsilon_{t^\varepsilon}|_{L^2(\Gamma_\varepsilon)} \le c\,\alpha_\varepsilon, \tag{3.17}$$

$$|u^\varepsilon|_{L^r(\Omega_{\varepsilon s})} \le c\alpha_\varepsilon^{2/r} d_\varepsilon^{-1/r}. \tag{3.18}$$

Proof. By splitting the system according to the two distinct types of nonlinearities involved, we expect to complete the proof by the Schauder fixed-point theorem.

For $w \in V_\varepsilon$, we define $T_w \in H_0^1(\Omega)$ to be the unique solution of the problem:

$$\int_\Omega B^\varepsilon \nabla T_w \nabla S + \int_\Omega wS\nabla T_w = \int_\Omega QS - \int_\Omega wS\nabla\tau - \int_\Omega B^\varepsilon\nabla\tau\nabla S, \ \forall S \in H_0^1(\Omega). \tag{3.19}$$

First, let us examine the continuity of the convective term.

$$\left|\int_\Omega wS\nabla T_w\right| \le |w|_{L^r(\Omega)}|T_w|_{L^{2r/(r-2)}(\Omega)}|S|_{H_0^1(\Omega)}. \tag{3.20}$$

As $r \in (2, \infty)$ if $N = 2$ and $r \in [3, 6)$ if $N = 3$, then by using the corresponding Sobolev inequalities we get

$$|T_w|_{L^{2r/(r-2)}(\Omega)} \le c|T_w|_{H_0^1(\Omega)}. \tag{3.21}$$

$$|w|_{L^r(\Omega_{\varepsilon f})} \le c|w|_{H^1(\Omega_{\varepsilon f})}, \tag{3.22}$$

which obviously imply

$$\left|\int_\Omega wS\nabla T_w\right| \le c|w|_{H_\varepsilon}|T_w|_{H_0^1(\Omega)}|S|_{H_0^1(\Omega)}. \tag{3.23}$$

Thus, using again that div $w = 0$ in Ω, the existence and uniqueness results follow straightly from the Lax–Milgram Theorem.

Moreover, acting like in [8, 14], we prove that $T_w \in L^\infty(\Omega)$ and there exists $c > 0$ (independent of ε) such that

$$|\nabla T_w|_{L^2(\Omega)} + |T_w + \tau|_{L^\infty(\Omega)} \le c. \tag{3.24}$$

Setting $S = T_w$ in (3.19), we obtain

$$|\nabla T_w|_{L^2(\Omega)} \le c\left(|Q|_{L^2(\Omega)} + |\nabla\tau|_{L^2(\Omega)} + \left(|Q|_{L^2(\Omega)} + \tau_0\right)|w|_{L^2(\Omega)}\right), \tag{3.25}$$

where $c > 0$ is independent of ε.

Now we introduce $F(w) \in V_\varepsilon$ as the unique solution of the problem:

$$\int_{\Omega_{\varepsilon s}} \left(1 + d_\varepsilon |F(w)|_{A^\varepsilon}^{r-2}\right) A^\varepsilon F(w) v + \int_{\Omega_{\varepsilon f}} e_{ij}(F(w)) e_{ij}(v) + \varepsilon\beta_\varepsilon \int_{\Gamma_\varepsilon} F(w)_{t_\varepsilon} v = -\alpha_\varepsilon \int_\Omega (T_w + \tau) g v, \quad \forall v \in V_\varepsilon. \tag{3.26}$$

The existence and the uniqueness can be proved by the Browder-Minty Theorem (see [30]) applied to the strictly monotone map (see Cor. A.3 in the Appendices) $G_\varepsilon : V_\varepsilon \to V_\varepsilon'$ defined by

$$\langle G_\varepsilon u, v\rangle_{V_\varepsilon, V_\varepsilon'} = \int_{\Omega_{\varepsilon s}} \left(1 + d_\varepsilon |u|_{A^\varepsilon}^{r-2}\right) A^\varepsilon u v + \int_{\Omega_{\varepsilon f}} e_{ij}(u) e_{ij}(v) + \varepsilon\beta_\varepsilon \int_{\Gamma_\varepsilon} u_{t_\varepsilon} v, \tag{3.27}$$

which is also bounded and hemicontinuous. As $r \ge 2$ and as for any $u \in H_\varepsilon$ we have

$$\langle G_\varepsilon u, u\rangle_{V_\varepsilon, V_\varepsilon'} \ge c_\varepsilon \left(|u|_{L^r(\Omega_{\varepsilon s})}^r + |e(u)|_{L^2(\Omega_{\varepsilon f})}^2 + |u_{t^\varepsilon}|_{L^2(\Gamma_\varepsilon)}^2\right), \tag{3.28}$$

for some $c_\varepsilon > 0$ independent of u, the coercivity of G_ε follows.

Next, we estimate the range of $F(w)$ with respect to $w \in V_\varepsilon$. Setting $v = F(w)$ in (3.26) and calling (3.16) we get for some $c > 0$ independent of ε

$$d_\varepsilon |F(w)|_{L^r(\Omega_{\varepsilon s})}^r + |F(w)|_{L^2(\Omega_{\varepsilon s})}^2 + |e(F(w))|_{L^2(\Omega_{\varepsilon f})}^2 + \varepsilon |F(w)_{t_\varepsilon}|_{L^2(\Gamma_\varepsilon)}^2 \le c\alpha_\varepsilon \left(\tau_0 + |Q|_{L^2(\Omega)}\right) |F(w)|_{L^2(\Omega)}. \tag{3.29}$$

Using (3.11) we finally obtain:

$$|F(w)|_{L^2(\Omega_{\varepsilon s})} + |e(F(w))|_{L^2(\Omega_{\varepsilon f})} + \varepsilon^{1/2} |F(w)_{t_\varepsilon}|_{L^2(\Gamma_\varepsilon)} \le c\alpha_\varepsilon \left(\tau_0 + |Q|_{L^2(\Omega)}\right), \tag{3.30}$$

$$|F(w)|_{L^r(\Omega_{\varepsilon s})} \le c\alpha_\varepsilon^{2/r} \left(\tau_0 + |Q|_{L^2(\Omega)}\right)^{2/r} d_\varepsilon^{-1/r}, \tag{3.31}$$

that is, there exists $c_F > 0$ independent of ε such that

$$|F(w)|_{H_\varepsilon} \le c_F \left(\alpha_\varepsilon + \alpha_\varepsilon^{2/r} d_\varepsilon^{-1/r}\right). \tag{3.32}$$

Thus we have defined a mapping $w \in M_\varepsilon \mapsto F(w) \in M_\varepsilon$, where

$$M_\varepsilon = \left\{ v \in V_\varepsilon, \quad |v|_{H_\varepsilon} \le c_F \left(\alpha_\varepsilon + \alpha_\varepsilon^{2/r} d_\varepsilon^{-1/r}\right)\right\}. \tag{3.33}$$

We check now that F is compact. Let $(w_k)_{k\in\mathbb{N}}$ be bounded in V_ε; then, using (3.25) and (3.11), we see that $(\nabla T_{w_k})_{k\in\mathbb{N}}$ is bounded in $L^2(\Omega)$. As $H_0^1(\Omega)$ is compactly included in $L^r(\Omega)$, we find that there exists a subsequence $(T_{w_{k'}})_{k'\in\mathbb{N}}$ which is a Cauchy sequence in $L^r(\Omega)$. Using the strict monotony of G_ε, it follows from (3.26) that $(F(w_{k'}))_{k'\in\mathbb{N}}$ is a Cauchy sequence in V_ε.

We see that the Schauder fixed-point theorem can be applied. Thus we obtain an element $u \in V_\varepsilon$ such that $u = F(u)$ and obviously $(u, T_u) \in V_\varepsilon \times H_0^1(\Omega)$ is a solution of the problem (3.13) and (3.14).

The rest of the proof is straightforward. □

Remark 3.2. Problem (3.13) and (3.14) has a unique solution only if we assume the Rayleigh number $\alpha_\varepsilon > 0$ to be small enough.

We proceed by recovering the pressure which was hidden by the (3.13) and (3.14) formulation. Let us introduce the spaces

$$\mathcal{V}(\Omega_{\varepsilon h}) = \left\{ v \in \mathcal{D}(\Omega_{\varepsilon h})^N, \ \operatorname{div} v = 0 \text{ in } \Omega_{\varepsilon h} \right\}, \quad h = s \text{ or } f, \tag{3.34}$$

$$L_\varepsilon = \left\{ q \in L^2_0(\Omega), \ q|_{\Omega_{\varepsilon s}} \in W^{1,r'}(\Omega_{\varepsilon s}) \right\}, \quad \frac{1}{r'} + \frac{1}{r} = 1. \tag{3.35}$$

Remark 3.3. $W^{1,r'}(\Omega_{\varepsilon s}) \subset\subset L^2(\Omega_{\varepsilon s})$.

Theorem 3.4. *Let $(u^\varepsilon, T^\varepsilon) \in V_\varepsilon \times H^1_0(\Omega))$ be a solution of (3.13) and (3.14). Then there exists $p^\varepsilon \in L_\varepsilon$ such that*

$$\int_{\Omega_{\varepsilon s}} \left(1 + d_\varepsilon |u^\varepsilon|^{r-2}_{A^\varepsilon}\right) A^\varepsilon u^\varepsilon v + \int_{\Omega_{\varepsilon f}} e_{ij}(u^\varepsilon) e_{ij}(v) + \varepsilon\beta_\varepsilon \int_{\Gamma_\varepsilon} u^\varepsilon_{t_\varepsilon} v + \alpha_\varepsilon \int_\Omega (T^\varepsilon + \tau) g v = \int_\Omega p^\varepsilon \operatorname{div} v, \quad \forall v \in H_\varepsilon. \tag{3.36}$$

Moreover, there exists $c > 0$ independent of ε such that

$$|p^\varepsilon|_{L^2(\Omega)} \le c \left(d_\varepsilon^{1/r} \alpha_\varepsilon^{2/r'} + \alpha_\varepsilon + \alpha_\varepsilon \beta_\varepsilon \right) \quad \text{and} \quad |\nabla p^\varepsilon|_{L^{r'}(\Omega_{\varepsilon s})} \le c \left(\alpha_\varepsilon + \alpha_\varepsilon^{2/r'} d_\varepsilon^{1/r'} \right). \tag{3.37}$$

Proof. For some $w \in \mathcal{V}(\Omega_{\varepsilon s})$, we set in (3.13)

$$v = \begin{cases} 0 & \text{in } \Omega_{\varepsilon f} \\ w & \text{in } \Omega_{\varepsilon s} \end{cases}. \tag{3.38}$$

Applying the corresponding version of the De Rham theorem we find that $\exists p^{\varepsilon s} \in W^{1,r'}(\Omega_{\varepsilon s})$, unique up to an additive constant, such that

$$-\nabla p^{\varepsilon s} = \left(1 + d_\varepsilon |u^{\varepsilon s}|^{r-2}_{A^\varepsilon}\right) A^\varepsilon u^{\varepsilon s} + \alpha_\varepsilon (T^\varepsilon + \tau) g \quad \text{in} \quad L^{r'}(\Omega_{\varepsilon s}). \tag{3.39}$$

The corresponding Green formula follows:

$$\int_{\Omega_{\varepsilon s}} \left(1 + d_\varepsilon |u^\varepsilon|^{r-2}_{A^\varepsilon}\right) A^\varepsilon u^\varepsilon v + \alpha_\varepsilon \int_{\Omega_{\varepsilon s}} (T^\varepsilon + \tau) g v = \int_{\Omega_{\varepsilon s}} p^{\varepsilon s} \operatorname{div} v + \int_{\Gamma_\varepsilon} p^{\varepsilon s} v_{n^\varepsilon}, \quad \forall v \in H_\varepsilon. \tag{3.40}$$

Next, let $w \in \mathcal{V}(\Omega_{\varepsilon f})$ and set in (3.13)

$$v = \begin{cases} 0 & \text{in } \Omega_{\varepsilon s} \\ w & \text{in } \Omega_{\varepsilon f} \end{cases}. \tag{3.41}$$

Using again De Rham theorem, we find that $\exists p^{\varepsilon f} \in L^2(\Omega_{\varepsilon f})$, unique up to additive constants corresponding to each connected component of $\Omega_{\varepsilon f}$, and such that

$$-\frac{\partial p^{\varepsilon f}}{\partial x_i} = \alpha_\varepsilon \left(T^\varepsilon + \tau\right) g_i - \frac{\partial e_{ij}\left(u^\varepsilon\right)}{\partial x_j} \quad \text{in} \quad H^{-1}\left(\Omega_{\varepsilon f}\right). \tag{3.42}$$

Defining $\Sigma^{\varepsilon i} \in L^2(\Omega_{\varepsilon f})^N$ by

$$\Sigma^{\varepsilon i}_j = -p^{\varepsilon f} \delta_{ij} + e_{ij}\left(u^\varepsilon\right) \tag{3.43}$$

we see that $\operatorname{div}\left(\Sigma^{\varepsilon i}\right) = \alpha_\varepsilon \left(T^\varepsilon + \tau\right) g_i \in L^2(\Omega_{\varepsilon f})$ and the Green formula follows:

$$\int_{\Omega_{\varepsilon f}} e_{ij}(u^\varepsilon) e_{ij}(v) + \alpha_\varepsilon \int_{\Omega_{\varepsilon f}} \left(T^\varepsilon + \tau\right) g v = \int_{\Omega_{\varepsilon f}} p^{\varepsilon f} \operatorname{div} v + \langle \Sigma^{\varepsilon i}_{n^\varepsilon}, v_i \rangle_{H^{-1/2}, H^{1/2}(\Gamma_\varepsilon)}, \quad \forall v \in H_\varepsilon. \tag{3.44}$$

From (3.40) and (3.44) we deduce that

$$\langle \Sigma^{\varepsilon i}_{n^\varepsilon}, v_i \rangle_{H^{-1/2},H^{1/2}(\Gamma_\varepsilon)} + \int_{\Gamma_\varepsilon} p^{\varepsilon s} v_{n^\varepsilon} + \varepsilon\beta_\varepsilon \int_{\Gamma_\varepsilon} u^\varepsilon_{t^\varepsilon} v = 0, \quad \forall v \in V_\varepsilon. \tag{3.45}$$

We shall prove now that for a certain choice of the free constants, (3.45) holds for any $v \in H_\varepsilon$.

As Ω^ε_f is of class C^2, we can introduce $\Sigma^\varepsilon_{n^\varepsilon n^\varepsilon} \in H^{-1/2}(\Gamma_\varepsilon)$ by

$$\langle \Sigma^\varepsilon_{n^\varepsilon n^\varepsilon}, u \rangle_{H^{-1/2},H^{1/2}(\Gamma_\varepsilon)} = \langle \Sigma^{\varepsilon i}_{n^\varepsilon}, u\, n^\varepsilon_i \rangle_{H^{-1/2},H^{1/2}(\Gamma_\varepsilon)}, \quad \forall u \in H^{1/2}(\Gamma_\varepsilon). \tag{3.46}$$

Also, for $k \in \{1, 2, \dots, k_\varepsilon\}$, we define $Q^k_\varepsilon :\ H^{1/2}(\Gamma^k_\varepsilon) \to H^{1/2}(\Gamma_\varepsilon)$ as the natural extension with zero:

$$Q^k_\varepsilon w(x) = \begin{cases} w(x), & x \in \Gamma^k_\varepsilon \\ 0, & x \in \Gamma^i_\varepsilon, \quad i \neq k. \end{cases} \tag{3.47}$$

First, let $w \in H^{1/2}(\Gamma_\varepsilon)^N$; we set in (3.45) $v \in V_\varepsilon$ with the properties

$$v = 0 \quad \text{in} \quad \Omega_{\varepsilon s} \quad \text{and} \quad v = w - w_{n^\varepsilon} n^\varepsilon \quad \text{on} \quad \Gamma_\varepsilon. \tag{3.48}$$

Thus we obtain

$$\langle \Sigma^{\varepsilon i}_{n^\varepsilon}, w_i \rangle_{H^{-1/2},H^{1/2}(\Gamma_\varepsilon)} - \langle \Sigma^\varepsilon_{n^\varepsilon n^\varepsilon}, w_{n^\varepsilon} \rangle_{H^{-1/2},H^{1/2}(\Gamma_\varepsilon)} + \varepsilon\beta_\varepsilon \int_{\Gamma_\varepsilon} u^\varepsilon_{t^\varepsilon} w = 0, \quad \forall w \in V_\varepsilon. \tag{3.49}$$

Next, let $w \in H^{1/2}(\Gamma^k_\varepsilon)$ with $\int_{\Gamma^k_\varepsilon} w = 0$; obviously, there exists $v \in V_\varepsilon$ such that $v = wn^\varepsilon$ on Γ^k_ε and $v = 0$ in $\Omega \setminus \Omega^k_\varepsilon$. By setting such a v in (3.45) we get

$$\langle \Sigma^{\varepsilon k}_{n^\varepsilon n^\varepsilon}, w \rangle_{H^{-1/2},H^{1/2}(\Gamma^k_\varepsilon)} + \int_{\Gamma^k_\varepsilon} p^{\varepsilon s} w = 0, \tag{3.50}$$

where $\Sigma^{\varepsilon k}_{n^\varepsilon n^\varepsilon} \in H^{-1/2}(\Gamma^k_\varepsilon)$ is defined by

$$\langle \Sigma^{\varepsilon k}_{n^\varepsilon n^\varepsilon}, v \rangle_{H^{-1/2},H^{1/2}(\Gamma^k_\varepsilon)} = \langle \Sigma^\varepsilon_{n^\varepsilon n^\varepsilon}, Q^k_\varepsilon v \rangle_{H^{-1/2},H^{1/2}(\Gamma_\varepsilon)}, \quad \forall v \in H^{1/2}(\Gamma^k_\varepsilon). \tag{3.51}$$

Classic manipulations of (3.50) yield

$$\Sigma^{\varepsilon k}_{n^\varepsilon n^\varepsilon} + p^{\varepsilon s} = \frac{1}{|\Gamma^k_\varepsilon|} \left(\langle \Sigma^\varepsilon_{n^\varepsilon n^\varepsilon}, Q^k_\varepsilon 1 \rangle_{H^{-1/2},H^{1/2}(\Gamma_\varepsilon)} + \int_{\Gamma^k_\varepsilon} p^{\varepsilon s} \right) \quad \text{in} \quad H^{-1/2}(\Gamma^k_\varepsilon). \tag{3.52}$$

Choosing the free constants of $p^{\varepsilon f}$ and $p^{\varepsilon s}$ such that

$$\langle \Sigma^\varepsilon_{n^\varepsilon n^\varepsilon}, Q^k_\varepsilon 1 \rangle_{H^{-1/2},H^{1/2}(\Gamma_\varepsilon)} + \int_{\Gamma^k_\varepsilon} p^{\varepsilon s} = 0, \quad \forall k \in \{1, 2, \dots, k_\varepsilon\}, \tag{3.53}$$

$$\int_{\Omega_{\varepsilon f}} p^{\varepsilon f} + \int_{\Omega_{\varepsilon s}} p^{\varepsilon s} = 0 \tag{3.54}$$

we find that

$$\begin{aligned} \langle \Sigma^\varepsilon_{n^\varepsilon n^\varepsilon}, w_{n^\varepsilon} \rangle_{H^{-1/2},H^{1/2}(\Gamma_\varepsilon)} &= \sum_{k=1}^{k_\varepsilon} \left\langle \Sigma^\varepsilon_{n^\varepsilon n^\varepsilon}, Q^k_\varepsilon \left(w_{n^\varepsilon}|_{\Gamma^k_\varepsilon} \right) \right\rangle_{H^{-1/2},H^{1/2}(\Gamma_\varepsilon)} \\ &= \sum_{k=1}^{k_\varepsilon} \left\langle \Sigma^{\varepsilon k}_{n^\varepsilon n^\varepsilon}, w_{n^\varepsilon}|_{\Gamma^k_\varepsilon} \right\rangle_{H^{-1/2},H^{1/2}(\Gamma^k_\varepsilon)} = -\sum_{k=1}^{k_\varepsilon} \left\langle p^{\varepsilon s}, w_{n^\varepsilon}|_{\Gamma^k_\varepsilon} \right\rangle_{H^{-1/2},H^{1/2}(\Gamma^k_\varepsilon)} \quad (3.55) \\ &= -\sum_{k=1}^{k_\varepsilon} \int_{\Gamma^k_\varepsilon} p^{\varepsilon s} \left(w_{n^\varepsilon}|_{\Gamma^k_\varepsilon} \right) = -\int_{\Gamma_\varepsilon} p^{\varepsilon s} w_{n^\varepsilon}, \quad \forall w \in H_\varepsilon \quad (3.56) \end{aligned}$$

and hence (3.45) holds for any $v \in H_\varepsilon$.

Also, by adding (3.40) and (3.44), it follows that $p^\varepsilon \in L^\varepsilon$, defined by

$$p^\varepsilon = \begin{cases} p^{\varepsilon f} \text{ in } \Omega_{\varepsilon f}, \\ p^{\varepsilon s} \text{ in } \Omega_{\varepsilon s}, \end{cases} \tag{3.57}$$

satisfies (3.36).

As Ω is of class C^2 and $p^\varepsilon \in L_0(\Omega)$, the unique solution of the following Laplace equation with Neumann boundary condition which has zero mean value belongs to $H^2(\Omega)$; we denote it by v^ε:

$$\Delta v^\varepsilon = p^\varepsilon \qquad \text{in} \quad \Omega, \tag{3.58}$$

$$\frac{\partial v^\varepsilon}{\partial n^\varepsilon} = 0 \qquad \text{on} \quad \partial\Omega. \tag{3.59}$$

Moreover (see [20] Chap. 4, Sect. 2.3), that there exists $C > 0$, independent of p^ε, such that

$$|v^\varepsilon|_{H^2(\Omega} \le C|p^\varepsilon|_{L^2(\Omega)}. \tag{3.60}$$

Setting $v = \nabla v^\varepsilon \in H_\varepsilon$ in (3.36), we prove the estimate (3.37) by using (3.16)–(3.18) and (3.60) in a straightforward manner. □

4. Homogenizing the case of negligeable Forchheimer effect

In this section we shall study the asymptotic behaviour (when $\varepsilon \to 0$) of $(u^\varepsilon, p^\varepsilon, T^\varepsilon) \in V_\varepsilon \times L_\varepsilon \times H_0^1(\Omega)$ verifying (3.14) and (3.36), as the Forchheimer effect is vanishing, that is,

$$d_\varepsilon \to 0. \tag{4.1}$$

In the framework of the homogenization procedure, we assume that there exist $A \in L^\infty(\Omega, L^\infty_{\text{per}}(Y))^{N\times N}$, $\beta \in C^1_{\text{per}}(Y)$, B^f and $B^s \in L^\infty_{\text{per}}(Y)^{N\times N}$ such that

$$\left(\beta^\varepsilon, B^{\varepsilon s}, B^{\varepsilon f}\right)(x) = \left(\beta, B^s, B^f\right)\left(\frac{x}{\varepsilon}\right), \; A^\varepsilon(x) = A\left(x, \frac{x}{\varepsilon}\right), \text{ for } a.a.\ x \in \Omega, \tag{4.2}$$

$$\beta \ge b_1, \quad \left(A, B^s, B^f\right)\xi_i\xi_j \ge b_1\,\xi_i\xi_i, \quad \forall \xi \in \mathbb{R}^N,\ a.e. \text{ in } \Omega \times Y. \tag{4.3}$$

Also, there exists $\alpha > 0$ such that

$$\alpha_\varepsilon \to \alpha \quad \text{when} \quad \varepsilon \to 0. \tag{4.4}$$

Under these conditions, the estimates (3.16), (3.18), (3.37) and the relation (3.39) yield

$$|u^\varepsilon|_{L^r(\Omega_{\varepsilon s})} \le C d_\varepsilon^{-1/r}, \tag{4.5}$$

$$|u^\varepsilon|_{L^2(\Omega)} + |\nabla u^\varepsilon|_{L^2(\Omega_{\varepsilon f})} + \varepsilon |u^\varepsilon_{t^\varepsilon}|_{L^2(\Gamma_\varepsilon)} \le C, \tag{4.6}$$

$$|\nabla T^\varepsilon|_{L^2(\Omega)} + |T^\varepsilon|_{L^\infty(\Omega)} \le C, \tag{4.7}$$

$$|p^\varepsilon|_{L^2(\Omega)} + |\nabla p^\varepsilon|_{L^{r'}(\Omega_{\varepsilon s})} \le C, \tag{4.8}$$

for some $C > 0$ independent of ε.

From (4.5) we obtain immediately

$$\int_{\Omega_{\varepsilon s}} d_\varepsilon |u^\varepsilon|^{r-2} A^\varepsilon u^\varepsilon v \to 0, \quad \forall v \in H_\varepsilon, \tag{4.9}$$

that is, the Forchheimer term has no macroscopic influence in this case.

For any $h \in \{s, f\}$ and for any function φ defined on $\Omega \times Y$, let us introduce the following notations.

$$H_{\text{per}}(\text{div}, Y) = \left\{\varphi \in H_{\text{loc}}\left(\text{div}, \mathbb{R}^N\right),\ \varphi \text{ is } Y\text{-periodic}\right\}, \tag{4.10}$$

$$V_{\text{per}}\,(\text{div}, Y) = \{\varphi \in H_{\text{per}}(\text{div}, Y),\ \text{div}_y\varphi = 0 \text{ in } Y\ \}, \tag{4.11}$$

$$\varphi^h = \varphi|_{\Omega\times Y_h}, \quad \tilde{\varphi}^h = \frac{1}{|Y_h|}\int_{Y_h}\varphi(\cdot, y)\mathrm{d}y, \quad h \in \{s, f\}, \tag{4.12}$$

$$\tilde{\varphi} = \int_Y \varphi(\cdot, y)\mathrm{d}y, \quad \text{that is} \quad \tilde{\varphi} = (1-m)\tilde{\varphi}^s + m\tilde{\varphi}^f. \tag{4.13}$$

$$H^1_{\text{per}}\,(Y_h) = \left\{\varphi \in H^1_{\text{loc}}\left(\mathbb{R}^N_h\right), \varphi \text{ is } Y\text{-periodic}\right\}, \tag{4.14}$$

$$\tilde{H}^1_{\text{per}}(Y_h) = \left\{\varphi \in H^1_{\text{per}}(Y_h), \quad \tilde{\varphi} = 0\right\}. \tag{4.15}$$

Also, for any sequence $(\varphi^\varepsilon)_\varepsilon$, bounded in $L^p(\Omega \times Y)$, $1 < p < \infty$, we denote

$$\varphi^\varepsilon \overset{p}{\rightharpoonup} \varphi$$

when φ^ε is two-scale convergent to $\varphi \in L^p(\Omega \times Y)$ in the sense of [19] and as usual

$$H_0(\text{div}, \Omega) = \{v \in H(\text{div}, \Omega), \quad v_\nu = 0 \quad \text{on} \quad \partial\Omega\}, \tag{4.16}$$

$$V_0(\text{div}, \Omega) = \{v \in H_0(\text{div}, \Omega), \quad \text{div}\, v = 0 \quad \text{in} \quad \Omega\}. \tag{4.17}$$

From (4.6), it follows that $\exists u \in L^2(\Omega \times Y)^N$ such that, on some subsequence

$$u^\varepsilon \overset{2}{\rightharpoonup} u, \tag{4.18}$$

$$u^\varepsilon \rightharpoonup \int_Y u(\cdot, y)\mathrm{d}y \in V_0(\text{div}, \Omega) \quad \text{weakly in} \quad L^2(\Omega)^N. \tag{4.19}$$

Also, we see that $(\chi_{\varepsilon s}u^\varepsilon)_\varepsilon$, $(\chi_{\varepsilon f}u^\varepsilon)_\varepsilon$ and $\left(\chi_{\varepsilon f}\dfrac{\partial u^\varepsilon}{\partial x_i}\right)_\varepsilon$ are bounded in $(L^2(\Omega))^N$, $\forall i \in \{1, 2, \ldots, N\}$. This situation was already studied in [12] and we recall the results proved there.

Theorem 4.1. *There exist* $u \in L^2\left(\Omega, V_{\text{per}}(\text{div}, Y)\right)$, $w \in L^2\left(\Omega, (H^1_{\text{per}}(Y_f)/\mathbb{R})^N\right)$ *such that the following convergences hold on some subsequence:*

$$u^\varepsilon \overset{2}{\rightharpoonup} u, \tag{4.20}$$

$$\chi_{\varepsilon f}\nabla u^\varepsilon_i \overset{2}{\rightharpoonup} \chi_f\left(\nabla u^f_i + \nabla_y w_i\right), \quad \forall i \in \{1, 2, \ldots, N\}. \tag{4.21}$$

Moreover, u^f *is independent of the microscopic variable, namely:*

$$u^f = \tilde{u}^f \in H^1_0(\Omega), \tag{4.22}$$

$$\tilde{u} \in V_0(\text{div}, \Omega), \tag{4.23}$$

$$\text{div}_y w + \text{div}\, u^f = 0 \quad \text{in} \quad \Omega \times Y_f. \tag{4.24}$$

Concerning the temperature behaviour, from (4.7), and using the compactness result of [2], we get

Theorem 4.2. *There exist* $T \in H^1_0(\Omega)$ *and* $R \in L^2\left(\Omega, H^1_{\text{per}}(Y)/\mathbb{R}\right)$ *such that*

$$T^\varepsilon \overset{2}{\rightharpoonup} T, \tag{4.25}$$

$$\frac{\partial T^\varepsilon}{\partial x_i} \overset{2}{\rightharpoonup} \left(\frac{\partial T}{\partial x_i} + \frac{\partial R}{\partial y_i}\right), \quad \forall i \in \{1, 2, \ldots, N\}. \tag{4.26}$$

Moreover, $T \in L^\infty(\Omega)$ *and we have*

$$T^\varepsilon \rightharpoonup T \quad \text{weakly in} \quad H^1_0(\Omega) \quad \text{and weakly star in} \quad L^\infty(\Omega). \tag{4.27}$$

Theorem 4.3. *There exists* $p \in L_0^2(\Omega \times Y)$ *with* $p^s = \tilde{p}^s \in W^{1,r'}(\Omega)$*, such that on some subsequence we have*

$$p^\varepsilon \overset{2}{\rightharpoonup} p. \tag{4.28}$$

Proof. Calling (4.8), the compactness result of [2] implies the existence of some $p \in L_0^2(\Omega \times Y)$ such that (4.28) holds on some subsequence.

By rescaling the corresponding Rellich-Kondrachov inequality in Y_s, we have

$$|q|_{L^r(\Omega_{\varepsilon s})} \leq C\,\varepsilon\,|q|_{W_0^{1,r}(\Omega_{\varepsilon s})}, \quad \forall q \in W_0^{1,r}(\Omega_{\varepsilon s}). \tag{4.29}$$

Thus, taking (3.39) into account, we obtain

$$\left|\langle \nabla p^{\varepsilon s}, q\rangle_{W^{-1,r'},W_0^{1,r}(\Omega_{\varepsilon s})}\right| = \left|\int_{\Omega_{\varepsilon s}} q \nabla p^{\varepsilon s}\right| \leq |q|_{L^r(\Omega_{\varepsilon s})} |\nabla p^{\varepsilon s}|_{L^{r'}(\Omega_{\varepsilon s})} \leq C\varepsilon |q|_{W_0^{1,r}(\Omega_{\varepsilon s})}, \tag{4.30}$$

that is,

$$|\nabla p^{\varepsilon s}|_{W^{-1,r'}(\Omega_{\varepsilon s})} \leq C\varepsilon. \tag{4.31}$$

Then, using the extension operator $Q_{\varepsilon s} \in \mathcal{L}\left(L^2(\Omega_{\varepsilon s}), L^2(\Omega)\right)$ of Lipton–Avellaneda (see [18]), defined by

$$Q_{\varepsilon s}\pi = \begin{cases} \pi(x) & \text{in } \Omega_{\varepsilon s}, \\ \dfrac{1}{|\varepsilon Y_s|} \displaystyle\int_{\varepsilon k + \varepsilon Y_s} \pi(y)\,\mathrm{d}y & \text{in } \Omega_{\varepsilon f}, \end{cases} \tag{4.32}$$

Theorem 3.2 of [24] implies that there exists $q^s \in L^2(\Omega)$ such that

$$Q_{\varepsilon s} p^{\varepsilon s} \to q^s \qquad \text{in} \quad L^2(\Omega), \tag{4.33}$$

$$\chi_{\varepsilon s} p^{\varepsilon s} \overset{2}{\rightharpoonup} \chi_s(y) q^s(x) \qquad \text{in} \quad L^2(\Omega \times Y). \tag{4.34}$$

Passing the equality

$$\chi_{\varepsilon s} Q_{\varepsilon s} p^{\varepsilon s} = \chi_{\varepsilon s} p^\varepsilon \quad \text{in} \quad L^2(\Omega), \tag{4.35}$$

at the two-scale limit, we obtain

$$\chi_s(y) q^s(x) = \chi_s(y) p(x,y) \quad \text{for } a.a.\ (x,y) \in \Omega \times Y, \tag{4.36}$$

that is, $\tilde{p}^s = p^s \in L^2(\Omega)$.

Moreover, (3.20)–(3.21) of [24] reads:

$$Q_{\varepsilon s} p^{\varepsilon s} \to p^s \qquad \text{in} \quad L^r(\Omega)/\mathbb{R}, \tag{4.37}$$

$$\nabla(Q_{\varepsilon s} p^{\varepsilon s}) \to \nabla p^s \qquad \text{in} \quad W^{-1,r'}(\Omega). \tag{4.38}$$

Noticing that

$$|\nabla(Q_{\varepsilon s} p^{\varepsilon s})|_{L^{r'}(\Omega)} + |\nabla p^{\varepsilon s}|_{L^{r'}(\Omega)} \leq C, \tag{4.39}$$

we infer that (4.38) implies

$$\nabla(Q_{\varepsilon s} p^{\varepsilon s}) \to \nabla p^s \quad \text{in} \quad L^{r'}(\Omega), \tag{4.40}$$

that is, $\tilde{p}^s = p^s \in W^{1,r'}(\Omega)$. □

Now, we can present the so-called two-scale homogenized problem, verified by the limits given by Theorems 4.1–4.3. We find this problem to be well-posed at least for α sufficiently small. Hence, the asymptotic behaviour of u^ε, T^ε and p^ε is completely described by the solutions of this problem, via (4.20)–(4.21), (4.25)–(4.27) and (4.28).

Denoting

$$H(\Omega\times Y)=\left\{u\in L^2(\Omega\times Y),\ \mathrm{div}_y u=0 \text{ in } \Omega\times Y,\ u^f=\tilde u^f\in H_0^1(\Omega)^N,\ \tilde u\in H_0(\mathrm{div},\Omega)\right\}, \tag{4.41}$$

$$V(\Omega\times Y)=\{u\in H(\Omega\times Y),\ \ \mathrm{div}\,\tilde u=0 \text{ in } \Omega\}, \tag{4.42}$$

we see that

$$X=H(\Omega\times Y)\times L^2\left(\Omega,\tilde H^1_{\mathrm{per}}(Y_f)^N\right) \tag{4.43}$$

is a Hilbert space endowed with the scalar product

$$((u,w),(\varphi,\psi))_X=\int_{\Omega\times Y_s}u\cdot\varphi+\int_\Omega \mathrm{div}\,\tilde u\,\mathrm{div}\,\tilde\varphi+\int_{\Omega\times Y_f}(e(u)+e_y(w))\ (e(\varphi)+e_y(\psi)). \tag{4.44}$$

We also have to introduce the following spaces

$$M=\left\{q\in L_0^2(\Omega\times Y),\quad q^s=\tilde q^s\in H^1(\Omega)\right\},$$
$$X_0=\left\{(u,w)\in X,\ \ \mathrm{div}\,\tilde u=0\ \text{ in }\ \Omega,\ \ \mathrm{div}_y w+\mathrm{div}\,u^f=0\ \text{ in }\ \Omega\times Y_f\right\}.$$

Theorem 4.4. *The limits of the convergences (4.20)–(4.21), (4.25)–(4.27) and (4.28), that is* $(u,w)\in X_0$, $(T,R)\in H_0^1(\Omega)\times H^1_{\mathrm{per}}(Y)/\mathbb{R}$ *and* $p\in M$, *verify the following system:*

$$\int_{\Omega\times Y}B\left(\nabla(T+\tau)+\nabla_y R\right)\left(\nabla\Phi+\nabla_y\Psi\right)+\int_\Omega\tilde u\,\Phi\nabla(T+\tau)$$
$$=\int_\Omega\tilde Q\Phi,\ \forall\,(\Phi,\Psi)\in H_0^1(\Omega)\times H^1_{\mathrm{per}}(Y)/\mathbb{R}. \tag{4.45}$$

$$\int_{\Omega\times Y_s}Au\varphi+\int_{\Omega\times Y_f}(e(u)+e_y(w))\,(e(\varphi)+e_y(\psi))+\int_{\Omega\times\Gamma}\beta(u^f-u^f_n n)\varphi+\alpha\int_\Omega(T+\tau)g\tilde\varphi$$
$$=\int_\Omega p^s\mathrm{div}\,\tilde\varphi+\int_{\Omega\times Y_f}(p^f-p^s)\left(\mathrm{div}\,\varphi+\mathrm{div}_y\psi\right),\quad\forall(\varphi,\psi)\in X. \tag{4.46}$$

Proof. First, for some $\Phi\in\mathcal{D}(\Omega)$ and $\Psi\in\mathcal{D}(\Omega,C^\infty_{\mathrm{per}}(Y))$, we set $S=\Phi+\varepsilon\Psi^\varepsilon$ in (3.14), where $\Psi^\varepsilon(x)=\Psi(x,x/\varepsilon)$ for *a.a.* $x\in\Omega$. Using (4.20), (4.21) and (4.25)–(4.27) we easily obtain (4.45), even the convergence of the convective term, as

$$\int_\Omega u^\varepsilon\Phi\nabla T^\varepsilon=-\int_\Omega T^\varepsilon u^\varepsilon\nabla\Phi\quad\text{and}\quad u^\varepsilon\rightharpoonup\tilde u\quad\text{weakly in}\quad L^2(\Omega). \tag{4.47}$$

Next, let $\varphi\in\mathcal{D}(\Omega,C^\infty_{\mathrm{per}}(Y))^N$ and $\psi\in\mathcal{D}(\Omega,C^\infty_{\mathrm{per}}(Y_f))^N$ such that $(\varphi,\psi)\in X$. Let $\hat\psi$ a prolongation of ψ to $\mathcal{D}(\Omega,\tilde H_{\mathrm{per}}(\mathrm{div},Y))$, which can be done, for instance, by considering a certain Neumann problem in Y_s. Denoting, as usual, $\varphi^\varepsilon(x)=\varphi\left(x,\dfrac{x}{\varepsilon}\right)$ and $\psi^\varepsilon(x)=\hat\psi\left(x,\dfrac{x}{\varepsilon}\right)$, we can set $v(x)=\varphi^\varepsilon(x)+\varepsilon\psi^\varepsilon(x)$ in (3.36). Passing to the limit with $\varepsilon\to0$ and using the two-scale convergences of Theorems 4.1–4.3, we obtain:

$$\int_\Omega p^\varepsilon\mathrm{div}(\varphi^\varepsilon+\varepsilon\psi^\varepsilon)\to\int_{\Omega\times Y_f}p^f\left(\mathrm{div}_x\varphi+\mathrm{div}_y\psi\right)+\int_{\Omega\times Y_s}p^s\left(\mathrm{div}_x\varphi+\mathrm{div}_y\hat\psi\right)$$

As $p\in M$, we have also

$$\int_{\Omega\times Y_s}p^s\mathrm{div}_y\hat\psi=-\int_{\Omega\times\Gamma}p^s\psi_n=-\int_{\Omega\times Y_f}p^s\mathrm{div}_y\psi$$

and the convergence of the right-hand side term of (3.36) is proved. All the other convergences are straightforward, except that on $\Omega \times \Gamma_\varepsilon$, which is similar to that in [14]. □

The system (4.45) and (4.46) will provide all the local solutions of our problem, allowing us to successively eliminate some of the rapidly oscillating unknowns from the governing system.

First, denoting

$$V_f = \left\{ \varphi \in \left(H^1_{\mathrm{per}}(Y_f)/\mathbb{R}\right)^N,\ \mathrm{div}_y\varphi = 0 \right\}, \tag{4.48}$$

$$K_f = \left\{ \varphi \in \left(H^1_{\mathrm{per}}(Y_f)/\mathbb{R}\right)^N,\ \mathrm{div}_y\varphi = -1 \right\}, \tag{4.49}$$

for any $k, h \in \{1, 2, \ldots, N\}$ we define $R^k \in H^1_{\mathrm{per}}(Y)/\mathbb{R}$, $(W^{kh}, q^{kh}) \in V_f \times L^2_0(Y_f)$ and $W \in K_f$ as the unique solutions of the following three problems:

$$\int_Y B\nabla(y_k + R^k)\nabla\psi = 0, \quad \forall\psi \in H^1_{\mathrm{per}}(Y)/\mathbb{R}, \tag{4.50}$$

where $B = \begin{cases} B^s & \text{in } Y_s, \\ B^f & \text{in } Y_f, \end{cases}$

$$\begin{cases} \int_{Y_f} \left(\delta_{ik}\delta_{jh} + e_{y,ij}(W^{kh})\right) e_{y,ij}(\psi) = \int_{Y_f} q^{kh}\mathrm{div}_y\psi,\ \forall\psi \in (H^1_{\mathrm{per}}(Y_f)/\mathbb{R})^N, \\ \int_{Y_f} q\,\mathrm{div}_y(W^{hk}) = 0, \quad \forall q \in L^2_0(Y_f), \end{cases} \tag{4.51}$$

$$\int_{Y_f} e_y(W)\, e_y(\psi) = 0, \quad \forall\psi \in V_f. \tag{4.52}$$

The existence and uniqueness results for (4.50) and (4.51) are obtained by the Lax–Milgram Theorem. Regarding (4.52), we notice that W is the projection of 0 on the closed convex $K_f \neq \emptyset$ in $(H^1_{\mathrm{per}}(Y_f)/\mathbb{R})^N$.

Setting $\Phi = 0$ in (4.45) and $\varphi = 0$ in (4.46), we find that R, w and p^f have closed expressions with respect to u^f, T and p^s:

$$R(x,y) = R^i(y)\frac{\partial T}{\partial x_i}(x), \tag{4.53}$$

$$w(x,y) = W^{ij}(y)e_{ij}(u^f)(x) + W(y)\mathrm{div}(u^f)(x), \tag{4.54}$$

$$p^f(x,y) = p^s(x) + q^{ij}(y)e_{ij}(u^f)(x), \quad \text{for } a.a.\ (x,y) \in \Omega \times Y. \tag{4.55}$$

Using (4.53)–(4.55), we elimitate R, w and p^f by an appropriate choice of test functions, respectively

$$\Psi = R^i\frac{\partial\Phi}{\partial x_i} \ \text{ in (4.45)} \quad \text{and} \quad \psi = W^{ij}e_{ij}(\varphi) \ \text{ in (4.46)}.$$

Thus we find the system which determines the leading limits of our homogenisation process.

Theorem 4.5. *If $u \in V(\Omega \times Y)$, $T \in H^1_0(\Omega)$ and $p \in M$ are the limits given by Theorems 4.1–4.3, then they verify the following system:*

$$\int_{\Omega\times Y} B^H\nabla(T+\tau)\nabla\Phi + \int_\Omega \tilde{u}\Phi\nabla(T+\tau) = \int_\Omega \tilde{Q}\Phi, \quad \forall\Phi \in H^1_0(\Omega), \tag{4.56}$$

$$\int_{\Omega\times Y_s} Au^s\varphi^s + m\mu^H_{ijkh}\int_\Omega e_{ij}(u^f)e_{kh}(\varphi^f) + m\beta^H_{ij}\int_\Omega u^f_i\varphi^f_j + \alpha\int_\Omega (T+\tau)g\tilde{\varphi}$$

$$= \int_\Omega p^s \mathrm{div}\, \tilde{\varphi}, \quad \forall \varphi \in H(\Omega \times Y), \tag{4.57}$$

The so-called effective coefficients which appear in (4.56) and (4.57) are the following:

$$B^H_{ij} = \int_Y B_{kh} \left(\delta_{ik} + \frac{\partial R^k}{\partial y_i}\right)\left(\delta_{jh} + \frac{\partial R^h}{\partial y_j}\right), \tag{4.58}$$

$$\mu^H_{ijkh} = \lambda \delta_{ik}\delta_{jh} + \frac{1}{|Y_f|}\int_{Y_f} \left(\delta_{\ell k}\delta_{mh} + e_{y,\ell m}(W^{kh})\right)\left(\delta_{\ell i}\delta_{mj} + e_{y,\ell m}(W^{ij})\right), \tag{4.59}$$

$$\beta^H_{ij} = \frac{1}{|Y_f|}\int_\Gamma \beta(y)(\delta_{ij} - \nu_i(y)\nu_j(y))d\sigma_y, \tag{4.60}$$

where $\lambda > 0$ is given by

$$\lambda = \frac{1}{|Y_f|}\int_{Y_f} e_y(W)\, e_y(W). \tag{4.61}$$

Remark 4.6. The tensors B^H and μ^H are positive-definite and have the usual symmetry properties $B^H_{ij} = B^H_{ji}$ and $\mu^H_{ijkh} = \mu^H_{khij} = \mu^H_{jikh}$; β^H is also symmetric and has the property:

$$\beta^H_{ij}\int_\Omega \varphi_i\varphi_j = \int_{\Omega\times\Gamma} \beta(\gamma\varphi - (\gamma_\nu\varphi)\nu)^2 \geq 0, \quad \forall\varphi \in H^1_0(\Omega)^N. \tag{4.62}$$

Remark 4.7. In the case when A is independent of y, that is $A \in L^\infty(\Omega)^{N\times N}$, we can go further. The system (4.56)–(4.57) yields:

$$u^s_i = u^f_i - \left(\frac{1}{|Y_s|}\int_{Y_s} U^k_i(y)\right)\left(A_{kj}u^f_j + \alpha(T+\tau)g_i + \frac{\partial p^s}{\partial x_k}\right) \quad \text{in} \quad L^2(\Omega\times Y_s), \tag{4.63}$$

where $U^k \in H_0(\mathrm{div}, Y_s)$ is the unique solution of

$$\int_{Y_s} AU^k\Theta = \int_{Y_s}\Theta_k, \quad \forall\Theta \in H_0(\mathrm{div}, Y_s). \tag{4.64}$$

Noticing that $\left(\frac{1}{|Y_s|}\int_{Y_s} U^k_i(y)\right)$ are the elements of a symmetric and positive-definite matrix, we define its inverse by A^H. Thus, redenoting $\theta = T + \tau$, we find that
$(\tilde{u}^s, u^f, p^s, \theta) \in H_0(\mathrm{div},\Omega) \times H^1_0(\Omega) \times W^{1,r'}(\Omega)/\mathbb{R} \times H^1(\Omega)$ is weak solution of the system

$$(1-m)\mathrm{div}\,\tilde{u}^s + m\mathrm{div}\, u^f = 0 \quad \text{in} \quad \Omega, \tag{4.65}$$

$$\nabla p^s + A^H\tilde{u}^s + \alpha\theta g = (A^H - A)u^f \quad \text{in} \quad \Omega, \tag{4.66}$$

$$\nabla p^s - \mathrm{div}(\mu^H e(u^f)) + \alpha\theta g = -\beta^H u^f \quad \text{in} \quad \Omega, \tag{4.67}$$

$$-\mathrm{div}\left(B^H\nabla\theta\right) + \tilde{u}\nabla\theta = (1-m)\,Q^s + m\,Q^f \quad \text{in} \quad \Omega. \tag{4.68}$$

$$\theta = \tau \quad \text{on} \quad \partial\Omega. \tag{4.69}$$

This is a model of two coupled thermal flows, neither of them being incompressible. The terms of the right-hand sides of (4.66) and (4.67) come from the Beavers–Joseph and the incompressible transfer conditions on the vanished interface.

Appendix A. A result of strict monotonicity

We present here the inequality claimed in the proof of Theorem 3.1.

Theorem A.1. *Let $(\cdot,\cdot)_V$ be an inner product on a vector space V over $\mathbb{R}$ and let $|\cdot|_V$ be the associated norm. Then, for every $p \geq 2$, it holds:*

$$|u+v|_V^p \leq \left(|2u|_V^{p-2}u + |2v|_V^{p-2}v, u+v\right)_V, \quad \forall u,v \in V. \tag{A.1}$$

Proof. The cases when $p=2$ or $u=0$ or $v=0$ are obvious. Then, let $p>2$, $u \neq 0$, $v \neq 0$; denoting $|u|_V = a > 0$, $|v|_V = b > 0$ and $|u+v|_V = t$ and defining $f : [0,+\infty) \to \mathbb{R}$ by:

$$f(t) = t^p - 2^{p-3}\left(a^{p-2}+b^{p-2}\right)t^2 - 2^{p-3}\left(a^{p-2}-b^{p-2}\right)\left(a^2-b^2\right) \tag{A.2}$$

we see that (A.1) is equivalent to:

$$f(t) \leq 0, \quad \forall t \in [\,|a-b|, a+b\,]. \tag{A.3}$$

As f is decreasing on $[0,t_0]$ and increasing on $[t_0,+\infty[$ where

$$t_0 = 2\left(\frac{a^{p-2}+b^{p-2}}{p}\right)^{\frac{1}{p-2}}, \tag{A.4}$$

the proof is completed by the following two inequalities:

$$f(0) = -2^{p-3}(a^{p-2}-b^{p-2})(a^2-b^2) \leq 0, \tag{A.5}$$

$$f(a+b) = 2^p\left(\frac{a+b}{2}\right)\left(\left(\frac{(a+b)}{2}\right)^{p-1} - \frac{a^{p-1}+b^{p-1}}{2}\right) \leq 0 \tag{A.6}$$

which hold for any $a,b>0$ as $p>2$. □

When $A = I$, the following results have been already proved in $\mathbb{R}^2$(see [11]) and in $\mathbb{R}^N$(see [28]).

Corollary A.2. *Let A be a positive-definite matrix on $\mathbb{R}^N$, $N \geq 1$, and $p \geq 2$. Denoting by $(x,y)_A = (y^T Ax)^{1/2}$, $\forall x,y \in \mathbb{R}^N$, and by $|\cdot|_A$ the associated norm, we have*

$$|x-y|_A^p \leq 2^{p-2}(x-y)^T\left(|x|_A^{p-2}Ax - |y|_A^{p-2}Ay\right), \quad \forall x,y \in \mathbb{R}^N. \tag{A.7}$$

Corollary A.3. *Let $A \in L^\infty(\Omega)$ be symmetric with the property that $\exists \alpha > 0$ such that*

$$A_{ij}(x)\xi_i\xi_j \geq \alpha|\xi|^2, \quad \forall \xi \in \mathbb{R}^N, \quad \text{for } a.e. \quad x \in \Omega, \tag{A.8}$$

with Ω a bounded domain in $\mathbb{R}^N$, $N \geq 1$. Then, there exists $m > 0$ such that for any $p \geq 2$ it holds:

$$\int_\Omega \left(|u|_A^{p-2}Au - |v|_A^{p-2}Av, u-v\right)\mathrm{d}x \geq m\int_\Omega |u-v|^p \mathrm{d}x \tag{A.9}$$

where $(\cdot,\cdot)$ denotes the Euclidean inner product on $\mathbb{R}^N$.

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