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MATHEMATICAL ANALYSIS OF GOLDSTEIN'S MODEL FOR TIME-HARMONIC ACOUSTICS IN FLOWS

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Abstract. Goldstein's equations have been introduced in 1978 as an alternative model to linearized Euler equations to model acoustic waves in moving fluids. This new model is particularly attractive since it appears as a perturbation of a simple scalar model: the potential model. In this work we propose a mathematical analysis of boundary value problems associated with Goldstein's equations in the time-harmonic regime.

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1. Introduction

Aeroacoustics concerns the study of the sound propagation in presence of a fluid in flow. In this framework, we aim at determining the propagation of small perturbations of a fluid, namely the acoustic perturbations, created by a known source in an imposed flow [44]. The main motivations are in aeronautics where the noise pollution induced by aircraft engines is a major environmental issue which is addressed both by numerical simulations [28], and by experiment. Applications lie also in the car industry with the need of reducing the sound emitted by exhaust pipes [42, 43, 47], or in the domestic industry with the noise of air-conditioning devices and ventilation ducts.

The most natural model for aeroacoustics is provided by the linearized Euler equations obtained from the linearization of Euler Equations [51] around a stationary solution: $\rho_0, p_0, \boldsymbol{v}_0$, density, pressure and velocity of the so called base flow. In what follows, we shall suppose that the fluid is perfect and the flow is homentropic (constant entropy). Linearized Euler's system appears as a first order hyperbolic system with zero order governing the acoustic velocity $\boldsymbol{v}$ and the acoustic pressure p, perturbations of $\boldsymbol{v}_0$ and p_0.

Motivated by better properties regarding to their discretization by already available numerical methods, alternative models have been proposed in the literature. In the mid 1900's, Galbrun's equations have been proposed [31]: the unknown is the so-called Lagrangian displacement field $\boldsymbol{u}$ and the model looks like a vectorial convected wave equation.

In the particular case where the mean flow is potential (*i.e.* the mean velocity field $\boldsymbol{v}_0$ is the gradient of a scalar potential), one can show [12, 48], under reasonable assumptions about source terms, that the acoustic

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velocity is itself the gradient of a scalar potential φ. The model governing φ, called the potential model in what follows, consists in the scalar convected wave equation

$$\rho_0 D_t\big(c_0^{-2} D_t\varphi\big) - \operatorname{div}(\rho_0 \boldsymbol{\nabla}\varphi) = \rho_0 f,$$

where f is a source term, $D_t := \partial_t + \boldsymbol{v}_0 \cdot \boldsymbol{\nabla}$ is the convective derivative relatively to the base flow and c_0 is the sound celerity (deduced from the base flow $\rho_0, p_0, \boldsymbol{v}_0$ and the state law chosen). Because of its simplicity and its adequation to numerical approximations, this scalar model is used in many industrial applications [22, 24], for instance in the analysis of the influence of liners on the acoustic propagation [28, 30, 50].

More recently, Goldstein has proposed a new mathematical model [11, 32]. It can be seen as an extension of the potential one to the general situation where the mean flow is no longer potential. It has the advantage that the corresponding computational code can be built as a modification of existing codes for the potential case. The model couples a scalar potential φ to a vectorial unknown $\boldsymbol{\xi}$, the hydrodynamic velocity, as follow:

$$\begin{cases} \rho_0 D_t\big(c_0^{-2} D_t\varphi\big) - \operatorname{div}(\rho_0 \boldsymbol{\nabla}\varphi + \rho_0\boldsymbol{\xi}) = \rho_0 f, \\ D_t\boldsymbol{\xi} + (\boldsymbol{\xi}\cdot\boldsymbol{\nabla})\boldsymbol{v}_0 = \boldsymbol{\nabla}\varphi \times \boldsymbol{\omega}_0, \end{cases}$$

where $\boldsymbol{\omega}_0 := \boldsymbol{\nabla} \times \boldsymbol{v}_0$ is the vorticity of the base flow. As we will show in this paper, Goldstein's equations are well adapted for aeroacoustics whereas they are better known in the field of fluid dynamics and where they have been widely used to model the development of perturbations in a swirling flow [3, 20, 21, 33, 34, 37, 52].

Note that all three models can be shown to be equivalent in the case where the mean flow obeys nonlinear stationary Euler equations [51]. In particular, concerning the link between linearized Euler equations and Goldstein's equations, the equivalence is investigated in the original paper of Goldstein [32] and in the Ph.D. thesis [8]. In particular, the connection between Euler and Goldstein's unknowns reads

$$p = -\rho_0 D_t\varphi, \ \text{ and } \ \boldsymbol{v} = \boldsymbol{\nabla}\varphi + \boldsymbol{\xi}.$$

By the way, Goldstein's equations deserve to be studied independently of the fact that $\rho_0, p_0, \boldsymbol{v}_0$ satisfy stationary Euler equations. This is important for the development of numerical methods and also for true life applications for which available mean flows are not necessarily exact solutions of stationary Euler equations. The only equation that we will use in the following is the mass conservation equation $\operatorname{div}(\rho_0\boldsymbol{v}_0) = 0$.

In this article we study Goldstein's equations which did not retain much attention from mathematicians. A particularity of this work is that we are interested in the time-harmonic regime: we look for solutions that oscillate in time at a given frequency $\omega > 0$, proportionally to $e^{-i\omega t}$. Our goal is to study the well-posedness (existence and uniqueness of solutions) of the Goldstein's model in this particular case.

There are relatively few mathematical works about aeroacoustic models. In time domain, one can benefit from the well-known theory of symmetric hyperbolic systems in the sense of Friedrichs [29, 38] to prove the well-posedness of the linearized Euler equations. For instance, it has been done in the recent paper [35] where the authors also deduce the well-posedness of Galbrun's equations from the one of Euler's equations.

The analysis of the time-harmonic regime appears to be much more delicate. An important assumption, not needed for the time domain analysis, is that the mean flow is **subsonic**, *i.e.* its velocity field has an amplitude strictly less than the speed of sound. This assumption, which is presumed in all existing works, is not restrictive with respect to many applications in aeroacoustics.

For the analysis, first order Euler equations appear to be not adapted to a direct mathematical approach. As a matter of fact, the first existing results concern, to our knowledge, Galbrun's equations. More precisely the Fredholm nature of the corresponding boundary value problem has been shown in situations of increasing difficulty. In [13], the case of a 1D shear flow was considered (one benefits from the simple geometry to use explicit computations). In [14, 15], the analysis has been extended to more general 2D mean flows, first in the case of a simplified approximate model (the so called low Mach model, valid under a smallness assumption about the velocity field $\boldsymbol{v}_0$), then for the full model. In all cases, the results are obtained under some restrictions about the variations on the reference mean flow: roughly speaking $|\boldsymbol{\nabla}\boldsymbol{v}_0|$ must be small enough.

The analysis simplifies drastically when one considers the potential model which can be studied with the same tools and method as the classical Helmholtz equation although the medium is anisotropic and non homogenous. In [17] (see also [24]), the well-posedness of this model is shown under the only assumption that the mean flow is subsonic, using Fredholm's alternative.

Very recently, in [36], a work relatively close to ours, the authors study the time-harmonic damped Galbrun's equations in the context of helioseismology. In particular, their model contains absorption terms that we shall do not consider in this paper. This allows to consider more general flows, without using the Ω-filling assumption, see the Definition 3.15. The method of analysis is based on some original *ad hoc* Helmholtz decomposition of vector fields. The results are obtained under the assumption that the absorption is large enough but do not require any other assumption on the mean flow, apart its subsonic nature.

In this paper, we do not consider any intrinsic absorption. It complicates the analysis in particular through the time-harmonic vectorial transport equation satisfied by the unknown $\boldsymbol{\xi}$. The consequence is that a new restrictive assumption must be done on the reference flow: the Ω-filling condition. In this work, we prove the existence and uniqueness of solutions to Goldstein's equations for subsonic Ω-filling flows satisfying an additional condition similar to the one in [15] but more explicit (in particular easy to check) and this condition can be moreover interpreted as a low vorticity condition. From the methodological point of view, our method can be seen more as a modification of the analysis made in [17] for the potential model (this is another advantage of Goldstein's model) and uses in an essential manner our previous work on the time-harmonic transport equation [9] where the Ω-filling condition, already introduced in [4] for the scalar stationary transport equation, plays a fundamental role.

The outline of this paper is as follows. In Section 2, we present the problem under consideration, beginning with the assumptions on the mean flow (Sect. 2.1), then presenting the governing equations (Sect. 2.2) and finally the boundary conditions (Sect. 2.3). The full problem is presented in a mathematically oriented manner in Section 2.4. The main section of the paper is Section 3. Our main results are the object of Section 3.1 in which we present and discuss the important notion of admissible flows. In Section 3.2, we explain the difficulties of the problem and present the approach we have chosen. In Section 3.3, we give a recap of the analysis of the potential model. Section 3.4 is devoted to the proof of our main theorem, based on analytic Fredholm theory. Finally, in Section 3.5, we explain in which sense our admissibility condition for the mean flow can be interpreted as a low vorticity condition. The paper is completed by three appendices devoted to a justification of the boundary conditions chosen in Section 2.3 (Appendix A), to the proof of a technical lemma related to Section 3.5 (Appendix B), and to a discussion about a possible alternative approach to the well-posedness analysis (Appendix C).

2. Equations of the problem

2.1. Geometry and mean flow

We consider a mean flow occupying $\mathbb{R}^d \backslash \mathcal{O}_R$, $d = 2$ or 3, where the set $\mathcal{O}_R$ represents a rigid body inside which acoustic waves will not penetrate. The stationary mean flow is characterized by its pressure p_0 and velocity vector field $\boldsymbol{v}_0$, all measurable function of the space variable $\boldsymbol{x}$. The constitutive law for a barotropic fluid, namely $p = F(\rho)$, where $F : \mathbb{R} \mapsto \mathbb{R}$ is a smooth non decreasing function, then determines the mean flow density $\rho_0 = F^{-1}(p_0)$ as well as its speed of sound $c_0 > 0$ *via* $c_0^2 := F'(\rho_0)$ [51]. The quantities $(\rho_0, p_0, \boldsymbol{v}_0)$ are supposed to satisfy stationary Euler equations [51] (see however Rem. 2.1) and in particular the mass conservation:

$$\operatorname{div}(\rho_0 \boldsymbol{v}_0) = 0, \tag{2.1}$$

which is the only equation on the mean flow that we shall use explicitly in this paper.

Remark 2.1. Real life computations in aeroacoustics are often done with idealized mean flows that are not necessary exact solutions of stationary Euler equations but result of various approximations (constant density for instance) that mays due to physical simplifications or due to approximate numerical calculations. For this paper, the mass conservation condition (2.1) is the only equation that we shall use explicitly.

For the application to acoustics, we assume that the velocity field is smooth enough, $\boldsymbol{v}_0 \in \mathrm{C}^1(\mathbb{R}^d \setminus O_R; \mathbb{R}^d)$ and that, as $\mathcal{O}_R$ is rigid, the flow is sliding along $\partial\mathcal{O}_R$, that is to say, $\boldsymbol{n}(\boldsymbol{x})$ denoting the unit normal on $\partial\mathcal{O}_R$,

$$\forall\, \boldsymbol{x} \in \partial\mathcal{O}_R, \quad \boldsymbol{v}_0(\boldsymbol{x}) \cdot \boldsymbol{n}(\boldsymbol{x}) = 0.$$

We also assume that the density ρ_0 and the velocity c_0 are bounded from below and above by two strictly positive constants:

$$\forall\, \boldsymbol{x} \in \mathbb{R}^d \backslash \mathcal{O}_R, \quad 0 < \rho_- \leq \rho_0(\boldsymbol{x}) \leq \rho_+, \quad 0 < c_- \leq c_0(\boldsymbol{x}) \leq c_+. \tag{2.2}$$

We are interested to study the propagation of acoustic waves in a connected and bounded domain $\Omega \subset \mathbb{R}^d \setminus \mathcal{O}_R$ whose boundary $\partial\Omega$ is split into two parts:

$$\partial\Omega = \Gamma \cup \Gamma_R,$$

where $\Gamma_R := \partial\Omega \cap \partial\mathcal{O}_R$ is the rigid part and Γ is the outer boundary. It will also be useful, for formulating boundary conditions, to separate Γ into three parts:

$$\Gamma = \Gamma_+ \cup \Gamma_- \cup \Gamma_0,$$

where, by definition

$$\begin{aligned} \Gamma_+ &:= \big\{\boldsymbol{x} \in \Gamma,\ \boldsymbol{n}(\boldsymbol{x}) \cdot \boldsymbol{v}_0(\boldsymbol{x}) > 0\big\} \text{ is the outflow boundary,} \\ \Gamma_- &:= \big\{\boldsymbol{x} \in \Gamma,\ \boldsymbol{n}(\boldsymbol{x}) \cdot \boldsymbol{v}_0(\boldsymbol{x}) < 0\big\} \text{ is the inflow boundary,} \\ \Gamma_0 &:= \big\{\boldsymbol{x} \in \Gamma,\ \boldsymbol{n}(\boldsymbol{x}) \cdot \boldsymbol{v}_0(\boldsymbol{x}) = 0\big\} \text{ is the sliding boundary.} \end{aligned} \tag{2.3}$$

For the mathematical analysis, we shall assume that the inflow and outflow boundaries are well separated, namely (such a condition appears, for instance, in most mathematical works about the stationary transport equation)

$$d(\Gamma_-, \Gamma_+) > 0. \tag{2.4}$$

Remark 2.2. As seen in (2.3), the boundaries are defined by the flow so that they should be denoted $\Gamma_\pm(\boldsymbol{v}_0)$ and $\Gamma_0(\boldsymbol{v}_0)$. We did not do so for avoiding heavy notation.

To illustrate our purpose, let us consider two examples of “real life” applications.

Application 1. Propagation of acoustic waves in a deformed duct.

Denoting $\boldsymbol{x} = (\boldsymbol{x}_T, x_d)$, with $\boldsymbol{x}_T \in \mathbb{R}^{d-1}$ the transverse variable and $x_d \in \mathbb{R}$ the longitudinal one, we consider that the fluid domain $\mathbb{R}^d \setminus \mathcal{O}_R$ is an infinite “deformed cylinder”, *i.e.* a infinite connected domain that is transversally bounded

$$\exists\, R_0 > 0 \quad \text{s.t.} \quad \mathbb{R}^d \setminus \mathcal{O}_R \subset \big\{\boldsymbol{x} \in \mathbb{R}^d \;/\; |\boldsymbol{x}_T| < R_0\big\},$$

and, outside a bounded set, is perfectly cylindrical, namely $S_\pm$ denoting two bounded domains of $\mathbb{R}^{d-1}$

$$\exists\, L > 0 \quad \text{s.t} \quad \mathbb{R}^d \setminus \mathcal{O}_R \cap \{\pm\, x_d > L\} = (S_\pm \times \mathbb{R}) \cap \{\pm\, x_d > L\}.$$

We also assume that the flow is homogeneous outside a bounded domain: there exists positive constants $\rho_\infty^\pm, c_\infty^\pm$ and $v_\infty^\pm$ such that, $\boldsymbol{e}_d$ being the unit vector in the direction x_d,

$$\pm\, x_d > L \quad \Longrightarrow \quad \rho_0(\boldsymbol{x}) = \rho_\infty^\pm, \quad c_0(\boldsymbol{x}) = c_\infty^\pm \quad \text{and} \quad \boldsymbol{v}_0(\boldsymbol{x}) = v_\infty^\pm\, \boldsymbol{e}_d.$$

In this case, the domain of interest for the propagation of acoustic waves will be typically

$$\Omega := \big\{\boldsymbol{x} \in \mathbb{R}^d \setminus \mathcal{O}_R \;/\; |x_d| < L\big\},$$

whose outer boundary Γ is such that $\Gamma_- = S_- \times \{-L\}$ and $\Gamma_+ = S_+ \times \{L\}$.

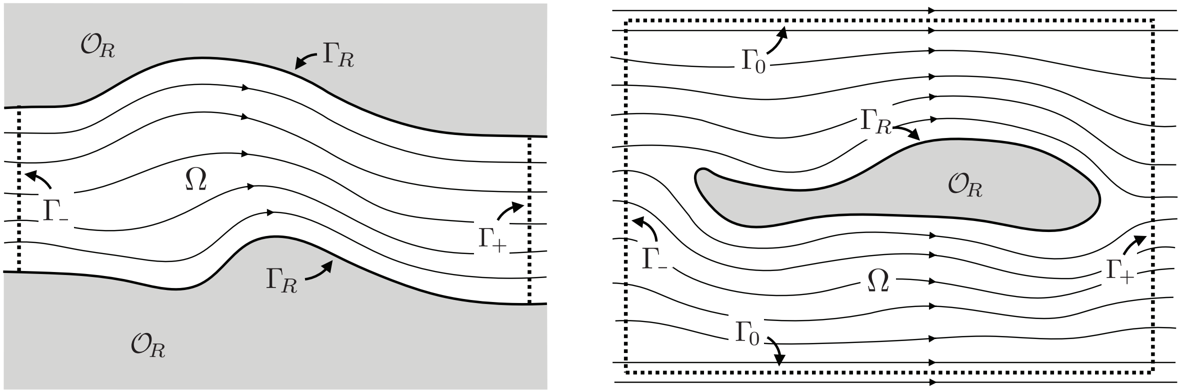

FIGURE 1. Typical mean flows. Application 1 (*left*). Application 2 (*right*).

Application 2. A model problem in aeronautics ($d = 3$).

This concerns a more industrial application about modeling the noise produced by airplanes during their flight. In cruise regime, the airplane moves at a constant speed $v_\infty \, \boldsymbol{e}_d$ and, for the modeling, the idea is to stand in the attached moving frame. In this way, everything happens as if the plane, which will be typically the rigid body $\mathcal{O}_R$, created a mean flow whose velocity would be constant "at infinity", which means sufficiently far, equal to $v_\infty \, \boldsymbol{e}_d$. This flow is typically obtained from a CFD calculation solving stationary Euler (or Navier–Stokes) equations. More precisely, one generally assumes that there exists a (sufficiently large) parallelepipedic box B (which will contain the computational domain), outside which the mean flow is supposed to be homogeneous, namely

$$\forall \, \boldsymbol{x} \in \mathbb{R}^d \backslash \Omega, \quad \rho_0(\boldsymbol{x}) = \rho_\infty, \quad c_0(\boldsymbol{x}) = c_\infty \quad \text{and} \quad \boldsymbol{v}_0(\boldsymbol{x}) = v_\infty \, \boldsymbol{e}_d.$$

In that case, the computational domain is $\Omega = B \setminus \mathcal{O}_R$ and the outer boundary Γ is ∂B. The boundary Γ_0 is the union of the four faces of B that are parallel to $\boldsymbol{e}_d$, the inflow boundary Γ_- is the face of B that has $\boldsymbol{e}_d$ as incoming normal vector and the outflow boundary Γ_+ is the face of B that has $\boldsymbol{e}_d$ as outgoing normal vector. Both applications are represented on Figure 1.

2.2. Time-harmonic Goldstein's equations

The goal of the modeling of acoustics in a stationary mean flow (characterized by $(\boldsymbol{v}_0, p_0)$) is to compute the perturbations $(\boldsymbol{v}(\boldsymbol{x}, t), p(\boldsymbol{x}, t))$ induced by a small perturbative source term. If this source term varies in time proportionally to $e^{-i\omega t}$, for a given frequency $\omega > 0$ and if we look for the first order term of the perturbation (with respect to the small amplitude of the source term), it is natural to look for acoustic perturbations of the form $\boldsymbol{v}(\boldsymbol{x}) \, e^{-i\omega t}$ for the velocity and $p(\boldsymbol{x}) \, e^{-i\omega t}$ for the pressure where $(\boldsymbol{v}, p)$ are complex valued functions. The unknowns $(\boldsymbol{v}, p)$ are naturally governed by time-harmonic linearized Euler equations [51]. However, the Goldstein's model is better adapted for taking into account the fact that the nature of the acoustic perturbations depends on the characteristics of the flow and in particular on its vorticity $\boldsymbol{\omega}_0 := \boldsymbol{\nabla} \times \boldsymbol{v}_0$ (see also Rem. 2.3).

When the flow is potential ($\boldsymbol{\omega}_0 = \boldsymbol{0}$) and homentropic (constant entropy), $\boldsymbol{v}$ is found to be potential, *i.e.* $\boldsymbol{v} = \boldsymbol{\nabla} \varphi$, where the velocity potential φ satisfies the convected Helmholtz equation [12,48], in which the acoustic source is represented by the right hand side f:

$$D_\omega \big(c_0^{-2} D_\omega \varphi \big) - \rho_0^{-1} \operatorname{div} (\rho_0 \boldsymbol{\nabla} \varphi) = f. \tag{2.5}$$

In this equation, $D_\omega := -i\omega + \boldsymbol{v}_0 \cdot \boldsymbol{\nabla}$, is the harmonic convective derivative. Equation (2.5) is the form of the convected wave equation that is the most commonly used in the literature [12, 17]. However, exploiting the mass conservation equation (2.1), this equation can be rewritten in divergence form, more suitable for the mathematical and numerical analysis. This exploits the fact that, for any scalar function ψ,

$$\rho_0\, D_\omega \psi = -i\omega\, \rho_0\, \psi + \rho_0\, \boldsymbol{v}_0 \cdot \boldsymbol{\nabla}\psi = -i\omega\, \rho\, \psi + \operatorname{div}(\rho_0\, \boldsymbol{v}_0\, \psi).$$

Applying the above with $\psi = c_0^{-2} D_\omega \phi$, (2.5) (multiplied by ρ_0) can be rewritten as

$$-\operatorname{div}\left(\rho_0\big(\boldsymbol{\nabla}\varphi - c_0^{-2} D_\omega \varphi\, \boldsymbol{v}_0\big)\right) - i\omega\, \rho_0\, c_0^{-2} D_\omega \varphi = \rho_0\, f. \tag{2.6}$$

For a general flow of vorticity $\boldsymbol{\omega}_0 \neq \boldsymbol{0}$, the acoustic perturbations are also found vortical. That is why, in addition to the potential φ, one has to introduce a new (vector valued) unknown: the hydrodynamic vector field $\boldsymbol{\xi}$. These unknowns are found to satisfy the Goldstein equations [11,32], that we write below in divergence form, in conformity with (2.6):

$$\begin{cases} -\operatorname{div}\left(\rho_0\big(\boldsymbol{\nabla}\varphi + \boldsymbol{\xi} - c_0^{-2} D_\omega \varphi\, \boldsymbol{v}_0\big)\right) - i\omega\, \rho_0\, c_0^{-2} D_\omega \varphi = \rho_0\, f, & \text{(i)} \\ D_\omega \boldsymbol{\xi} + (\boldsymbol{\xi} \cdot \boldsymbol{\nabla})\boldsymbol{v}_0 = \boldsymbol{\nabla}\varphi \times \boldsymbol{\omega}_0, & \text{(ii)} \end{cases} \tag{2.7}$$

and are linked to the Euler's unknowns, velocity $\boldsymbol{v}$ and pressure p, by

$$\begin{cases} \boldsymbol{v} = \boldsymbol{\nabla}\varphi + \boldsymbol{\xi}, & \text{(i)} \\ p = -\rho_0\, D_\omega \varphi. & \text{(ii)} \end{cases} \tag{2.8}$$

Let us interpret each equation. First (2.7)(i) means that, given $\boldsymbol{\xi}$, φ is a solution of the convected Helmholtz equation, with source term $f + \rho_0^{-1} \operatorname{div}(\rho_0 \boldsymbol{\xi})$. Next (2.7)(ii) means that, given φ, $\boldsymbol{\xi}$ is solution of a time-harmonic transport equation, namely of the form

$$\boldsymbol{\mathcal{T}}_0(\omega)\, \boldsymbol{\xi} = \boldsymbol{g}, \quad \text{with} \quad \boldsymbol{\mathcal{T}}_0(\omega)\, \boldsymbol{\xi} := D_\omega\, \boldsymbol{\xi} + (\boldsymbol{\xi} \cdot \boldsymbol{\nabla})\boldsymbol{v}_0, \tag{2.9}$$

by definition the time-harmonic transport operator, with source term $\boldsymbol{g} = \boldsymbol{\nabla}\varphi \times \boldsymbol{\omega}_0$. Of course, (2.7) needs to be completed with boundary conditions (object of the next section).

Remark 2.3. Another drawback of time-harmonic linearized Euler equations is that, contrary to what happens in time domain, there are not well-established numerical methods for their resolution. Although no well-established numerical methods to solve the Goldstein's equations exists neither, one can rely on the much more numerically studied equations which constitute the Goldstein's coupling: the convected Helmholtz equation and a time-harmonic transport equation. See [17] for the first one and [16,27] for the second one. As a consequence, one can build a numerical method for Goldstein's equations: this is what has been done for instance in [8].

2.3. Boundary conditions for the perturbations

As we work with a system of two equations (2.7), we need two different boundary conditions. The first one, called the acoustic condition, will be seen as a boundary condition for the potential φ and naturally attached to (2.7)(i) while the second one, called the hydrodynamic condition, will be seen as a boundary condition for the hydrodynamic velocity $\boldsymbol{\xi}$ and naturally attached to (2.7)(ii). However, in fact, as it is the case of the two equations in (2.7), these conditions (the acoustic condition to be more precise) couple the two unknowns.

2.3.1. Acoustic condition

This condition writes differently depending on the part of the boundary, Γ_R or Γ, one is looking at.

– On the rigid boundary Γ_R, the boundary condition to be chosen is clear: as for the mean flow, the velocity of the acoustic disturbances is tangential. According to (2.8)(i), this condition, namely $\boldsymbol{v} \cdot \boldsymbol{n} = 0$, simply reads:

$$(\boldsymbol{\nabla}\varphi + \boldsymbol{\xi}) \cdot \boldsymbol{n} = 0 \quad \text{on} \quad \Gamma_R. \tag{2.10}$$

– On the artificial boundary Γ, the choice of good boundary conditions, which are supposed to represent the fact that acoustic waves want to leave the domain Ω through Γ, is a delicate issue: this is where approximate modeling enters into account. In the context of this paper, we will content ourselves with a first order absorbing boundary condition (or first order radiation condition in the spirit of [25], see also Remark 2.4 for more accurate and sophisticated alternatives). This condition writes

$$(\nabla\varphi + \boldsymbol{\xi}) \cdot \boldsymbol{n} - (\boldsymbol{v}_0 \cdot \boldsymbol{n})\, c_0^{-1} \left(\boldsymbol{\xi} \cdot \boldsymbol{n} + c_0^{-1} D_\omega \varphi\right) = i\, \frac{\omega}{c_0}\, \varphi \quad \text{on} \quad \Gamma, \tag{2.11}$$

and its derivation is explained in Appendix A: as for the standard Helmholtz equation, it is designed in order to perfectly absorb the waves that strike the artificial boundary Γ with normal incidence. By the way, in the absence of flow (*i.e.* $\boldsymbol{v}_0$ and $\boldsymbol{\xi}$ vanishing everywhere), one recovers the well known first order (or Sommerfeld) absorbing condition $\nabla\varphi \cdot \boldsymbol{n} = i\, \frac{\omega}{c_0}\, \varphi$.

The reader will note that, as $\boldsymbol{v}_0 \cdot \boldsymbol{n} = 0$ on Γ_R, introducing the function $\lambda : \partial\Omega \to \mathbb{R}^+$ such that $\lambda = 0$ on Γ_R and $\lambda = 1$ on Γ the conditions (2.10) and (2.11) can be gathered in the following unified form,

$$\rho_0(\nabla\varphi + \boldsymbol{\xi} - c_0^{-2} D_\omega\varphi\, \boldsymbol{v}_0) \cdot \boldsymbol{n} - \rho_0\, c_0^{-1}(\boldsymbol{v}_0 \cdot \boldsymbol{n})(\boldsymbol{\xi} \cdot \boldsymbol{n}) = i\, \lambda\, \frac{\omega}{c_0}\, \rho_0\, \varphi \quad \text{on } \partial\Omega, \tag{2.12}$$

which is, thanks to the multiplication by ρ_0, compatible with the divergence form of the equation (2.7)(i). In the rest of the paper, we will consider the boundary condition (2.12) in the more general case where λ is an impedance function along $\partial\Omega$ satisfying

$$\lambda : \partial\Omega \to \mathbb{R}^+ \in \mathrm{L}^\infty(\Gamma) \quad / \quad \exists\, \gamma \subset \partial\Omega,\, \boldsymbol{b} \in \partial\Omega,\, \varrho > 0, \text{ s.t. } \partial\Omega \cap B(\boldsymbol{b}, \varrho) \subset \gamma \text{ and } \lambda > 0 \text{ on } \gamma, \tag{2.13}$$

where $B(\boldsymbol{b}, \varrho)$ is the ball of center $\boldsymbol{b}$ and radius ϱ. In particular, this includes the case of a boundary partially made of an absorbing wall with impedance λ.

Remark 2.4. There are several approaches to define alternative radiation conditions which are more efficient than the first order that we used. Let us quote two of them.

– In the situation described in Application 2, Section 2.1, where the mean flow is supposed to be homogeneous at the outer boundary, one can construct transparent boundary conditions *via* an integral representation formula [17]. Such a condition is thus non local. The corresponding boundary value problem has been analyzed in [17] in the case where the flow is potential everywhere (so that φ is the only unknown) and at rest outside Ω. Note that for Application 1, the boundary condition is much less precise due to multiple reflections on the walls of the waveguide.
– An alternative to radiation conditions is provided by Perfectly Matched Layers (PMLs): the absorption of waves is realized inside an absorbing layer instead across the absorbing boundary (see [10,40] for introductory papers). There is not much analysis of PMLs for acoustics in flow. It is nevertheless worthwhile mentioning the works [6] for the convected Helmholtz equation or [7] for Galbrun's equations, in the case of a waveguide (a situation that enters the framework of Application 1, Sect. 2.1).

2.3.2. Hydrodynamic condition

Because of the nature of the equation (2.14)(ii) satisfied by $\boldsymbol{\xi}$, transport type equation, we need a boundary condition only on the inflow boundary Γ_-. We suppose that this boundary is located in such a way that there does not exist any acoustic source upstream and that the vorticity of the mean flow vanishes in the upstream area. As a consequence, we prescribe that $\boldsymbol{\xi}$ vanishes on Γ_- (this can be interpreted as a causal boundary condition)

$$\boldsymbol{\xi} = \mathbf{0} \quad \text{on} \quad \Gamma_-.$$

2.4. Mathematical formulation of the boundary value problem

We now describe the mathematical problem to be solved, making precise in which functional spaces the unknowns are searched and the data are taken:

Given $f \in \mathrm{L}^2(\Omega)$, find $(\varphi, \boldsymbol{\xi}) \in \mathrm{H}^1(\Omega) \times \mathrm{L}^2(\Omega)^d$, such that

$$\begin{cases} -\operatorname{div}\big(\rho_0\big(\boldsymbol{\nabla}\varphi + \boldsymbol{\xi} - c_0^{-2} D_\omega \varphi\, \boldsymbol{v}_0\big)\big) - i\omega\, \rho_0\, c_0^{-2} D_\omega \varphi = \rho_0 f, & \text{in } \Omega, \quad \text{(i)} \\ D_\omega \boldsymbol{\xi} + (\boldsymbol{\xi}\cdot\boldsymbol{\nabla})\boldsymbol{v}_0 = \boldsymbol{\nabla}\varphi \times \boldsymbol{\omega}_0, & \text{in } \Omega, \quad \text{(ii)} \end{cases} \tag{2.14}$$

with the following acoustic and hydrodynamic boundary conditions

$$\begin{cases} \rho_0(\boldsymbol{\nabla}\varphi + \boldsymbol{\xi} - c_0^{-2} D_\omega \varphi\, \boldsymbol{v}_0)\cdot \boldsymbol{n} - \rho_0\, c_0^{-1}(\boldsymbol{v}_0\cdot\boldsymbol{n})(\boldsymbol{\xi}\cdot\boldsymbol{n}) = i\,\lambda\, c_0^{-1}\,\rho_0\,\varphi, \quad \text{on } \partial\Omega, & \text{(i)} \\ \boldsymbol{\xi} = \boldsymbol{0}, \quad \text{on } \Gamma_-. & \text{(ii)} \end{cases} \tag{2.15}$$

Let us comment the physical pertinence of the functional framework and the sense to give to the boundary conditions (2.15), which is related to the existence of appropriate traces. First note that looking for $\varphi \in \mathrm{H}^1(\Omega)$ and $\boldsymbol{\xi} \in \mathrm{L}^2(\Omega)^d$ implies, *via* $\boldsymbol{v} = \boldsymbol{\nabla}\varphi + \boldsymbol{\xi}$, that

$$\int_\Omega \rho_0\, |\boldsymbol{v}|^2 < +\infty,$$

namely that the solution has a finite kinetic energy.

The question of traces is a little bit more delicate. First, we remark that (2.14)(i) implies

$$\rho_0\big(\boldsymbol{\nabla}\varphi + \boldsymbol{\xi} - c_0^{-2} D_\omega\varphi\, \boldsymbol{v}_0\big) \in \mathrm{H}(\operatorname{div};\Omega),$$

so that the usual trace theorem in $\mathrm{H}(\operatorname{div};\Omega)$ ensures that

$$\rho_0\big(\boldsymbol{\nabla}\varphi + \boldsymbol{\xi} - c_0^{-2} D_\omega\varphi\, \boldsymbol{v}_0\big)\cdot\boldsymbol{n} \in \mathrm{H}^{-\frac{1}{2}}(\partial\Omega). \tag{2.16}$$

Note also that (2.14)(ii) implies that $\boldsymbol{\xi}$ belongs to an anisotropic Sobolev space, namely

$$\boldsymbol{\xi} \in \mathrm{H}(\Omega, \boldsymbol{v}_0) := \big\{\boldsymbol{\zeta} \in \mathrm{L}^2(\Omega)^d \;/\; (\boldsymbol{v}_0\cdot\boldsymbol{\nabla})\boldsymbol{\zeta} \in \mathrm{L}^2(\Omega)^d\big\},$$

where $\mathrm{H}(\Omega, \boldsymbol{v}_0)$ is a Hilbert space for the natural graph norm

$$\|\boldsymbol{\zeta}\|^2_{\mathrm{H}(\Omega,\boldsymbol{v}_0)} := \int_\Omega \Big(|\boldsymbol{\zeta}|^2 + |(\boldsymbol{v}_0\cdot\boldsymbol{\nabla})\boldsymbol{\zeta}|^2\Big),$$

see [26] for the proof (with weaker assumption on $\boldsymbol{v}_0$ regularity). Next, it is well known [23,39] that, under the separation condition (2.4), the trace $\boldsymbol{\zeta}|_{\Gamma_\pm}$ on $\Gamma_\pm$ of any $\boldsymbol{\zeta} \in \mathrm{H}(\Omega, \boldsymbol{v}_0)$ is well defined and that

$$\boldsymbol{\zeta}|_{\Gamma_\pm} \in \mathrm{L}^2(\Gamma_\pm, |\boldsymbol{v}_0\cdot\boldsymbol{n}|)^d := \left\{\boldsymbol{\zeta} : \Gamma_\pm \to \mathbb{C}^d \;/\; \int_{\Gamma_\pm} |\boldsymbol{v}_0\cdot\boldsymbol{n}|\, |\boldsymbol{\zeta}|^2 < +\infty\right\}. \tag{2.17}$$

Moreover, the trace map $\boldsymbol{\zeta} \mapsto \boldsymbol{\zeta}|_{\Gamma_\pm}$ is continuous from $\mathrm{H}(\Omega, \boldsymbol{v}_0)$ in $\mathrm{L}^2(\Gamma_\pm, |\boldsymbol{v}_0\cdot\boldsymbol{n}|)^d$. This gives a sense to (2.15)(ii) which can also be rewritten as

$$\boldsymbol{\xi} \in \mathrm{H}^-(\Omega, \boldsymbol{v}_0) := \big\{\boldsymbol{\zeta} \in \mathrm{H}(\Omega, \boldsymbol{v}_0) \;/\; \boldsymbol{\zeta} = \boldsymbol{0} \text{ on } \Gamma_-\big\} \quad (\text{closed in } \mathrm{H}(\Omega, \boldsymbol{v}_0)).$$

This also gives a sense to the trace $(\boldsymbol{\zeta}\cdot\boldsymbol{n})(\boldsymbol{v}_0\cdot\boldsymbol{n})$, when $\boldsymbol{\zeta} \in \mathrm{H}^-(\Omega, \boldsymbol{v}_0)$, as an element of $\mathrm{L}^2(\partial\Omega)$, that vanishes along $\Gamma_0 \cup \Gamma_R$ (because $\boldsymbol{v}_0\cdot\boldsymbol{n} = 0$) and Γ_- (because $\boldsymbol{\xi} = 0$). Indeed, it suffices to check that $(\boldsymbol{\zeta}\cdot\boldsymbol{n})(\boldsymbol{v}_0\cdot\boldsymbol{n}) \in \mathrm{L}^2(\Gamma_+)$ which follows from the following estimate

$$\int_{\Gamma_+} |(\boldsymbol{\zeta}\cdot\boldsymbol{n})(\boldsymbol{v}_0\cdot\boldsymbol{n})|^2 \le \int_{\Gamma_+} |\boldsymbol{\zeta}\cdot\boldsymbol{n}|^2\, |\boldsymbol{v}_0\cdot\boldsymbol{n}|^2 \le \|\boldsymbol{v}_0\|_{\mathrm{L}^\infty} \int_{\Gamma_+} |\boldsymbol{\zeta}\cdot\boldsymbol{n}|^2\, |\boldsymbol{v}_0\cdot\boldsymbol{n}|$$

which is finite thanks to (2.17). Then, by continuity of the trace operator $\boldsymbol{\zeta} \mapsto \boldsymbol{\zeta}|_{\Gamma_+}$, we know that there exists a constant $C_+ > 0$, that depends only on Ω, Γ_+ and $\boldsymbol{v}_0$ such that

$$\forall\, \boldsymbol{\zeta} \in \mathrm{H}^-(\Omega, \boldsymbol{v}_0), \quad \|(\boldsymbol{\zeta} \cdot \boldsymbol{n})(\boldsymbol{v}_0 \cdot \boldsymbol{n})\|_{\mathrm{L}^2(\Gamma_+)} \le C_+ \, \|\boldsymbol{\zeta}\|_{\mathrm{H}(\Omega, \boldsymbol{v}_0)}. \tag{2.18}$$

Finally (2.16) and (2.18) give a sense to the boundary condition (2.15)(i) in $\mathrm{H}^{-\frac{1}{2}}(\partial\Omega)$ *a priori*. However, since $\varphi|_{\partial\Omega} \in \mathrm{H}^{\frac{1}{2}}(\partial\Omega) \subset \mathrm{L}^2(\partial\Omega)$ and $\lambda \in \mathrm{L}^\infty(\partial\Omega)$, $\lambda\, \rho_0\, {c_0}^{-1}\, \varphi \in \mathrm{L}^2(\partial\Omega)$ so that (2.15)(i) implies that the trace (2.16) belongs to $\mathrm{L}^2(\partial\Omega)$ and (2.15)(i) holds in $\mathrm{L}^2(\partial\Omega)$.

3. Analysis of Goldstein's equations

3.1. The main result

Our main result will rely on a particular assumption of the flow $\boldsymbol{v}_0$ with respect to the domain Ω. The first important condition is related to the following definition, that we choose first to express in "physical" terms (see Def. 3.15 for the proper mathematical definition).

Definition 3.1 (Ω-filling flow and lifetime – *informal*)**.** The vector field $\boldsymbol{v}_0$ is said Ω-filling if there exists a time upper bound $t^* > 0$ such that any point inside Ω is reached before t^* by a particle that is transported by the flow from a point on the inflow boundary Γ_- at time $t = 0$. In the latter, we call the (global) lifetime of $\boldsymbol{v}_0$ in Ω, denoted $t^*(\boldsymbol{v}_0, \Omega)$, the smallest of such upper bounds t^*.

Remark 3.2. The Ω-filling condition excludes in particular two situations:

- the existence of closed streamlines (also called recirculations or periodic orbits) for the flow $\boldsymbol{v}_0$ inside Ω. These would correspond to the existence of particles that are transported by the flow and stay indefinitely inside Ω.
- the existence of stopping points, *i.e.* points $\boldsymbol{x}_s \in \overline{\Omega}$ where $\boldsymbol{v}_0(\boldsymbol{x}_s) = \boldsymbol{0}$: this corresponds to the existence of particles that take an arbitrarily large time to reach a point arbitrarily closed to $\boldsymbol{x}_s$.

In dimension 2, $d = 2$, there exists a particularly simple characterization of Ω-filling flows provided that Ω is simply connected. The result, proven in [9], is linked to Brouwer and Poincaré–Bendixson theorems [19], exploits the fact that, roughly speaking, the existence of periodic orbits implies the existence of a stopping point. The precise statement is the following

$$\text{If } \Omega \subset \mathbb{R}^2 \text{ is simply connected}, \quad \boldsymbol{v}_0 \text{ is } \Omega\text{-filling} \quad \Longleftrightarrow \quad \inf_{\boldsymbol{x}\in\Omega} |\boldsymbol{v}_0(\boldsymbol{x})| > 0. \tag{3.1}$$

Remark 3.3. Obviously, if the statement of Definition 3.1 is satisfied for some $t^* > 0$, it remains true for any larger time. In the sequel, we shall denote $t^*(\boldsymbol{v}_0, \Omega)$ the infimum of such times and we shall call it the lifetime of the flow in Ω.

The Ω-filling condition in Definition 3.1 will be stated in a more mathematical form when we shall use it, in Section 3.4.2. We shall need a more elaborate condition that relies on the introduction of the following functions

$$\Psi(s) := \sqrt{\frac{e^s - 1}{s}}, \quad \Phi(s) := \frac{\sqrt{e^s - 1 - s}}{|s|}, \quad s \in \mathbb{R}^* := \mathbb{R}\backslash\{0\}, \quad \Psi(0) = 1, \quad \Phi(0) = 1/\sqrt{2}, \tag{3.2}$$

whose useful properties are summarized in the following lemma:

Lemma 3.4. *The functions Ψ and Φ are continuous and strictly increasing from $0\,(s \to -\infty)$ to $+\infty\,(s \to +\infty)$.*

Proof. The proof is easy but given for completeness as the result is important for the rest of the paper.

The function $f(s) := (e^s - 1)/s$ is analytic on $\mathbb{R}$. Given $s \in \mathbb{R}^*$, $f'(s) = h(s)/s^2$ where $h(s) = e^s(s-1)+1$. Providing $h'(s) = se^s$, h is decreasing on $\mathbb{R}_-$ and increasing on $\mathbb{R}_+$, and thanks to $h(0) = 0$, one deduces that $f'(s) = h(s)/s^2$ is positive on $\mathbb{R}^*$ and one computes $f'(0) = 1/2$. Thus, f is strictly increasing, nonnegative as $f(s) \to 0$ when $s \to -\infty$, so is $\Psi = \sqrt{f}$.

In the same fashion, for Φ, the function $g(s) := (e^s - 1 - s)/s^2$ is analytic on $\mathbb{R}$. Given $s \in \mathbb{R}^*$, $g'(s) = H(s)/s^3$ where $H(s) = e^s(s-2)+s+2$ of derivative $H'(s) = h(s)$. We have just seen that $h(s) > 0$ on $\mathbb{R}^*$ and $h(0) = 0$, then, H is increasing on $\mathbb{R}$ and from $H(0) = 0$, one deduces that $g'(s) = H(s)/s^3$ is positive on $\mathbb{R}^*$ and thus g is strictly increasing, nonnegative as $g(s) \to 0$ when $s \to -\infty$, and so is $\Phi = \sqrt{g}$. □

Definition 3.5 (Admissible flow)**.** An flow $\boldsymbol{v}_0$ is admissible if and only if it is Ω-filling, with lifetime $t^*(\boldsymbol{v}_0, \Omega)$, and denoting $\boldsymbol{\omega}_0 := \boldsymbol{\nabla} \times \boldsymbol{v}_0$ (its vorticity) and $\boldsymbol{M}_0 := \boldsymbol{v}_0/c_0$ (its Mach number), satisfies the following inequality

$$\|\boldsymbol{\omega}_0\|_{\mathrm{L}^\infty}\ t^*(\boldsymbol{v}_0,\Omega)\,\Phi(2t^*(\boldsymbol{v}_0,\Omega)\|\boldsymbol{\nabla}\boldsymbol{v}_0\|_{\mathrm{L}^\infty}) < 1 - \|\boldsymbol{M}_0\|^2_{\mathrm{L}^\infty}. \tag{3.3}$$

At first glance, equation (3.3) appears as an upper bound for the L^∞ norm of the vorticity $\boldsymbol{\omega}_0$. This is at second glance not so obvious since the lifetime $t^*(\boldsymbol{v}_0, \Omega)$ depends on $\boldsymbol{v}_0$ in a complicated implicit manner while $\boldsymbol{\omega}_0$ is also related to $\boldsymbol{v}_0$. To emphasize that (3.3) does indeed correspond to a smallness of the vorticity, we refer the reader to Section 3.5.

Theorem 3.6. *Assume that $\boldsymbol{v}_0$ is an admissible flow in the sense of Definition 3.5 and that the impedance function λ satisfies* (2.13) *and that the inflow and outflow boundaries are well separated, i.e.* (2.4)*. Then, for any real frequency, the problem is of Fredholm type and there exists a subset of $\mathcal{R}_{ex} \subset \mathbb{R}$, with no limit point in $\mathbb{R}$, of exceptional frequencies such that, for any $\omega \notin \mathcal{R}_{ex}$, the boundary value problem* ((2.14), (2.15)) *is well-posed. More precisely, for such a frequency and for any $f \in \mathrm{L}^2(\Omega)$, equations* ((2.14), (2.15)) *has a unique solution $(\varphi, \boldsymbol{\xi}) \in \mathrm{H}^1(\Omega) \times \mathrm{L}^2(\Omega)^d$ that depends continuously on f: for some constant $C(\boldsymbol{v}_0, \Omega)$*

$$\|\varphi\|_{\mathrm{H}^1(\Omega)} + \|\boldsymbol{\xi}\|_{\mathrm{L}^2(\Omega)^d} \leq C(\boldsymbol{v}_0,\Omega)\ \|f\|_{\mathrm{L}^2(\Omega)}.$$

Any frequency in the set $\mathcal{R}_{ex}$ would correspond to a resonance, a frequency for which there exists a non zero finite energy solution to the homogeneous problem. For a time source with this frequency, the solution of the evolution problem would blow up when $t \to +\infty$ instead of "converging" to a time-harmonic solution (limiting amplitude principle). The existence of such frequencies – which cannot be excluded *a priori* because of our method of proof – remains an open question (see also the point (d) in the discussion below).

It is interesting to question the importance/relevance of the assumptions of Theorem 3.6:

(a) The Ω-filling condition, which is implicitly included in the admissibility condition, will be used in our analysis to eliminate the unknown $\boldsymbol{\xi}$ (*cf.* Sect. 3.4.2). Violating this condition disqualifies our approach but is not *a priori* an obstacle to the well-posedness of ((2.14), (2.15)). However, it has been shown in the Ph.D thesis [8] that, in some specific situations, the existence of closed orbits (closed streamlines) was, for a wide range of frequencies, an obstacle to the well-posedness of ((2.14), (2.15)), at least in the framework adopted in Section 2.4. For such frequencies, it would be necessary to adopt a new notion of weak solution that would require to accept more singular solutions than $(\varphi, \boldsymbol{\xi}) \in \mathrm{H}^1(\Omega) \times \mathrm{L}^2(\Omega)$. Similar phenomena appear for the propagation of electromagnetic waves in magnetized plasmas [46].

(b) The separation condition (2.4) appears regularly in the theory of transport equations (see for instance [23]). Clearly for us, it will appear again as a technical condition for eliminating $\boldsymbol{\xi}$. However, contrary to the Ω-filling condition, our feeling is that this condition is not really essential and could be removed.

(c) The admissibility condition of Definition 3.5 is stronger than simply saying that the mean flow is strictly subsonic, $\|\boldsymbol{M}_0\|_{\mathrm{L}^\infty} < 1$, a condition that is made in most mathematical studies in aeroacoustics. If the flow became supersonic in parts of Ω, the nature of the equation (2.14) governing φ would change since the ellipticity of the principal part of the differential operator would be lost. Then, the mathematical analysis would fall under the application of completely different techniques, far beyond the domain of competence of the authors.

(d) The admissibility condition will be used in our approach to show that the reduced problem falls under Fredholm's alternative. In this paper, this condition will clearly appear as a technical condition not connected to physics. However, it is not so surprising to see appear a condition of this nature, which corresponds to imposing an upper bound on the velocity field $\boldsymbol{v}_0$ space variations, and more precisely an upper bound on the vorticity $\boldsymbol{\omega}_0$. Indeed, it is well-known, in particular in the case of a laminar flow inside a waveguide, that a too strong vorticity is the cause of the development of hydrodynamic instabilities, in particular instabilities of Kelvin–Helmholtz type [45] which questions the soundness of the time-harmonic model [2]. This is clearly very close to the question of the existence of resonances as evoked previously.

3.2. Orientation and difficulties

The method we shall follow is to consider the coupled problem ((2.14), (2.15)) as a perturbation of the problem obtained by taking $\boldsymbol{\omega}_0 = \mathbf{0}$ in ((2.14), (2.15)), which is the problem to be solved in the case of a potential flow, *i.e.* when $\nabla \times \boldsymbol{v}_0 = \mathbf{0}$. Assume for a while that, given φ, the transport equation (2.14)(ii) in $\boldsymbol{\xi}$, completed by the boundary condition (2.15)(ii), is well-posed. In such a case, one easily infers that, since the source term $\nabla\varphi \times \boldsymbol{\omega}_0$ vanishes, $\boldsymbol{\xi} = \mathbf{0}$. As a consequence, the only unknown is the potential φ which solves the convected Helmholtz problem

$$\begin{cases} -\operatorname{div}\big(\rho_0\big(\nabla\varphi - c_0^{-2} D_\omega\varphi\, \boldsymbol{v}_0\big)\big) - i\omega\, \rho_0\, c_0^{-2} D_\omega\varphi = \rho_0 f, & \text{in } \Omega, \quad \text{(i)} \\ \rho_0\big(\nabla\varphi - c_0^{-2} D_\omega\varphi\, \boldsymbol{v}_0\big)\cdot \boldsymbol{n} = i\,\lambda\, c_0^{-1}\,\rho_0\,\varphi & \text{on } \partial\Omega. \quad \text{(ii)} \end{cases} \tag{3.4}$$

In the context of subsonic flows to which we restrict ourselves in this paper, the analysis on the above problem is well known (see for instance [17]) but for the sake of completeness and pedagogy, we shall recall the main ideas and results in Section 3.3.

To solve the Goldstein's problem ((2.14), (2.15)), we shall use a perturbative analysis. The idea is to eliminate $\boldsymbol{\xi}$ *via* the solution of the transport problem ((2.14)(ii), (2.15)(ii)). This, at least formally for the moment (this will be made precise and rigorous in Sect. 3.4.2, see (3.34) and (3.35)), allows us to express $\boldsymbol{\xi}$ as a function of φ

$$\boldsymbol{\xi} = \boldsymbol{\mathcal{S}}_0(\omega; \nabla\varphi). \tag{3.5}$$

As already seen in Section 3.1, a sufficient condition (and probably necessary) for the solvability of (2.14)(ii) is that the flow is Ω-filling (Def. 3.1). This was demonstrated in [9] and will be recalled (and extended) in Section 3.4.2. Thanks to (3.5), Goldstein's problem ((2.14), (2.15)) is rewritten as the following "modified" convected Helmholtz problem governing the only unknown φ:

$$\begin{cases} -\operatorname{div}\big(\rho_0\big(\nabla\varphi + \boldsymbol{\mathcal{S}}_0(\omega;\nabla\varphi) - c_0^{-2} D_\omega\varphi\, \boldsymbol{v}_0\big)\big) - i\omega\, \rho_0\, c_0^{-2} D_\omega\varphi = \rho_0 f, & \text{in } \Omega, \quad \text{(i)} \\ \rho_0\big(\nabla\varphi + \boldsymbol{\mathcal{S}}_0(\omega;\nabla\varphi) - c_0^{-2} D_\omega\varphi\, \boldsymbol{v}_0\big)\cdot \boldsymbol{n} - \rho_0\, c_0^{-1}(\boldsymbol{v}_0\cdot\boldsymbol{n})(\boldsymbol{\mathcal{S}}_0(\omega;\nabla\varphi)\cdot\boldsymbol{n}) = i\,\lambda\, c_0^{-1}\,\rho_0\,\varphi, & \text{on } \partial\Omega. \quad \text{(ii)} \end{cases} \tag{3.6}$$

Seeing the problem (3.6) as a perturbation of the convected Helmholtz problem (3.4) leads us to analyse (3.6) by adapting the arguments used for the (3.4) analysis. We shall also call the problem (3.6) the reduced Goldstein's problem in the sense that the only unknown is the potential φ.

As we shall recall in Section 3.3, the analysis of the convected Helmholtz equation is very close to the one of the standard Helmholtz equation. One first reduces the problem to the application of Fredholm's alternative (this is where the condition that the flow is subsonic is used), in such a way that the existence result is simply a consequence of the uniqueness result. Uniqueness is obtained by energy type boundary estimates combined with unique continuation arguments.

Adapting this analysis to the modified Helmholtz problem (3.6) is not as straightforward as one might think. The difficulty is that we do not know how to obtain the uniqueness result (this will be explained at the beginning of Sect. 3.4.3). That is why, to get around this obstacle, we shall use the analytic Fredholm theory, which requires to extend the problem to complex frequencies. This is for putting the problem in the adequate abstract framework, that we shall need the stronger admissibility condition (3.3) for the flow $\boldsymbol{v}_0$. This is also because we use this theory that we have to exclude the set of possible resonances $\mathcal{R}_{ex}$ for the existence and uniqueness result.

The rest of this section is organized as follows. Section 3.3 is devoted to a recap of the existent theory for the convected Helmholtz equation. The main section is Section 3.4 where we develop the proof of Theorem 3.11 *via* a perturbation approach explained above. Finally, in Section 3.5, we give a precise reinterpretation (when $d = 2$) of the admissibility condition as a small vorticity condition.

3.3. Resolution of the convected Helmholtz equation

As said above, the convected Helmholtz equation (3.6) is the one that must be solved when the mean flow is irrotational. However, from the mathematical point of view, this equation makes sense even for a non potential flow, $\boldsymbol{\omega}_0 := \boldsymbol{\nabla} \times \boldsymbol{v}_0 \neq \boldsymbol{0}$, even though it is physically meaningful only when $\boldsymbol{\omega}_0 = \boldsymbol{0}$.

3.3.1. Weak formulation of the problem

It is through the variational formulation that the boundary value problem will acquire a precise sense. In (3.4), in comparison with the classical Helmholtz equation, the only difference is a second order term, namely $\rho_0 D_\omega\big(c_0^{-2} D_\omega \varphi\big)$, which replaces the usual zero order term $-\omega^2 c_0^{-2}\,\rho_0\,\varphi$ ($D_\omega\varphi = -i\omega\varphi$ if $\boldsymbol{v}_0 = \boldsymbol{0}$).

Proposition 3.7. *If* $\operatorname{div}(\rho_0 \boldsymbol{v}_0) = 0$*, the weak formulation of* (3.4) *reads, setting* $V = \mathrm{H}^1(\Omega)$*,*

$$\text{Find } \varphi \in V \text{ such that} \quad \forall\, \psi \in V, \quad a(\omega;\varphi,\psi) = \ell(\psi), \tag{3.7}$$

where the sesquilinear form $a(\omega;\varphi,\psi)$ *and the antilinear form* $\ell(\psi)$ *are defined on* V *by:*

$$\left\{\begin{aligned} a(\omega;\varphi,\psi) &:= \int_\Omega \rho_0 \left[\boldsymbol{\nabla}\varphi\cdot\boldsymbol{\nabla}\overline{\psi} - (\boldsymbol{M}_0\cdot\boldsymbol{\nabla}\varphi)\big(\boldsymbol{M}_0\cdot\boldsymbol{\nabla}\overline{\psi}\big)\right] - \omega^2\int_\Omega \rho_0\, c_0^{-2}\,\varphi\,\overline{\psi} \\ &\qquad + i\omega\int_\Omega \rho_0\, c_0^{-1}\left[\varphi\big(\boldsymbol{M}_0\cdot\boldsymbol{\nabla}\overline{\psi}\big) - (\boldsymbol{M}_0\cdot\boldsymbol{\nabla}\varphi)\overline{\psi}\right] - i\omega\int_{\partial\Omega}\lambda\,\rho_0\, c_0^{-1}\varphi\,\overline{\psi}, \\ \ell(\psi) &:= \int_\Omega \rho_0\, f\,\overline{\psi}. \end{aligned}\right. \tag{3.8}$$

Proof. It is quite standard, multiplying (3.4)(i) by $\overline{\psi}$, for $\psi \in \mathrm{H}^1(\Omega)$ and integrating over Ω yields, after using Green's formula,

$$\int_\Omega \rho_0\big(\boldsymbol{\nabla}\varphi - c_0^{-2}D_\omega\varphi\,\boldsymbol{v}_0\big)\cdot\boldsymbol{\nabla}\overline{\psi} - i\omega\int_\Omega \rho_0 c_0^{-2}D_\omega\varphi\,\overline{\psi} - \big\langle\rho_0\big(\boldsymbol{\nabla}\varphi - c_0^{-2}D_\omega\varphi\,\boldsymbol{v}_0\big)\cdot\boldsymbol{n},\overline{\psi}\big\rangle_{\partial\Omega} = \int_\Omega \rho_0\, f\,\overline{\psi}.$$

Then, to obtain (3.7), it suffices to first use (3.4)(ii) in order to get

$$\big\langle\rho_0\big(\boldsymbol{\nabla}\varphi - c_0^{-2}D_\omega\varphi\,\boldsymbol{v}_0\big)\cdot\boldsymbol{n},\overline{\psi}\big\rangle_{\partial\Omega} = i\omega\int_{\partial\Omega}\lambda\,\rho_0\,c_0^{-1}\varphi\,\overline{\psi},$$

and then observe that, by definition of D_ω and $c_0^{-1}\,\boldsymbol{v}_0 = \boldsymbol{M}_0$,

$$\left\{\begin{aligned} &-c_0^{-2}D_\omega\varphi\,\big(\boldsymbol{v}_0\cdot\boldsymbol{\nabla}\overline{\psi}\big) = i\omega\, c_0^{-1}\,\varphi\,\big(\boldsymbol{M}_0\cdot\boldsymbol{\nabla}\overline{\psi}\big) - (\boldsymbol{M}_0\cdot\boldsymbol{\nabla}\varphi)\big(\boldsymbol{M}_0\cdot\boldsymbol{\nabla}\overline{\psi}\big), \\ &-i\omega\, c_0^{-2}D_\omega\varphi\,\overline{\psi} = -i\omega\, c_0^{-1}\,(\boldsymbol{M}_0\cdot\boldsymbol{\nabla}\varphi)\,\overline{\psi} - \omega^2\, c_0^{-2}\,\varphi\,\overline{\psi}, \end{aligned}\right.$$

which we substitute into (3.13). □

3.3.2. Existence and uniqueness result

In order to get tighter inequalities by avoiding factors such as ρ_+/ρ_-, we choose to introduce the weighted norms $\|\cdot\|_{\mathrm{L}^2(\Omega,\rho_0)}$ and $\|\cdot\|_{\mathrm{H}^1(\Omega,\rho_0)}$, on $\mathrm{L}^2(\Omega)$ and $\mathrm{H}^1(\Omega)$, defined for $\phi \in \mathrm{L}^2(\Omega)$, $\psi \in \mathrm{H}^1(\Omega)$ by

$$\|\phi\|_{\mathrm{L}^2(\Omega,\rho_0)} := \left(\int_\Omega \rho_0 |\phi|^2\right)^{1/2}, \quad \|\psi\|_{\mathrm{H}^1(\Omega,\rho_0)} := \left(\|\psi\|^2_{\mathrm{L}^2(\Omega,\rho_0)} + \|\boldsymbol{\nabla}\psi\|^2_{\mathrm{L}^2(\Omega,\rho_0)^d}\right)^{1/2}.$$

The assumption (2.2) on ρ_0 ensures that these norms are equivalent to the usual one used on $\mathrm{L}^2(\Omega)$ and $\mathrm{H}^1(\Omega)$.

Theorem 3.8. *The potential flow problem* (3.4) *is well-posed under the assumption* (2.13) *and the condition that the flow is strictly subsonic, namely*

$$\sup_{\boldsymbol{x}\in\Omega} |\boldsymbol{M}_0(\boldsymbol{x})|^2 < 1. \tag{3.9}$$

Proof. It is carried out in two steps: (i) the problem is of Fredholm type and (ii) and admits at most one solution.

For (i), we decompose (artificially) the sesquilinear form $a(\omega;\varphi,\psi)$ as the sum of two terms

$$a(\omega;\varphi,\psi) = a_*(\omega;\varphi,\psi) + c(\omega;\varphi,\psi),$$

where (note that we add and subtract artificially the term $\int_\Omega \rho_0\, \varphi\, \overline{\psi}$):

$$\begin{cases} a_*(\omega;\varphi,\psi) := \displaystyle\int_\Omega \rho_0 \big[\boldsymbol{\nabla}\varphi\cdot\boldsymbol{\nabla}\overline{\psi} - (\boldsymbol{M}_0\cdot\boldsymbol{\nabla}\varphi)(\boldsymbol{M}_0\cdot\boldsymbol{\nabla}\overline{\psi}) + \varphi\overline{\psi}\big] - i\omega \int_{\partial\Omega} \lambda\, \rho_0 c_0^{-1}\, \varphi\overline{\psi}, \\ c(\omega;\varphi,\psi) := i\omega \displaystyle\int_\Omega \rho_0\, c_0^{-1} \varphi(\boldsymbol{M}_0\cdot\boldsymbol{\nabla}\overline{\psi}) - i\omega \int_\Omega \rho_0\, c_0^{-1}(\boldsymbol{M}_0\cdot\boldsymbol{\nabla}\varphi)\overline{\psi} - \int_\Omega \rho_0\left(1+\omega^2 c_0^{-2}\right)\varphi\,\overline{\psi}. \end{cases}$$

Next, we observe that, thanks to (3.9), $a_*(\omega;\varphi,\psi)$ is coercive in $\mathrm{H}^1(\Omega)$. Indeed since the volume integral in the expression of $a_*(\omega;\varphi,\varphi)$ is real while the boundary integral is purely imaginary, we have

$$\begin{aligned} |a_*(\omega;\varphi,\varphi)| &\geq \int_\Omega \rho_0 \Big(|\boldsymbol{\nabla}\varphi|^2 - |(\boldsymbol{M}_0\cdot\boldsymbol{\nabla})\varphi|^2 + |\varphi|^2\Big) \\ &\geq \Big(1 - \sup_{\boldsymbol{x}\in\Omega}|\boldsymbol{M}_0(\boldsymbol{x})|^2\Big)\|\varphi\|^2_{\mathrm{H}^1(\Omega,\rho_0)}. \end{aligned} \tag{3.10}$$

It remains to show that the sesquilinear form $c(\cdot,\cdot)$ is associated with a compact operator. It is straightforward that c is a sum of three continuous sesquilinear forms, c_1, c_2 and c_3, in $\mathrm{H}^1(\Omega)$ satisfying one of these two properties:

$$\begin{aligned} &\forall\, \varphi,\psi \in \mathrm{H}^1(\Omega),\ \ |c_i(\varphi,\psi)| \leq \alpha_i \|\varphi\|_{\mathrm{L}^2(\Omega)}\|\psi\|_{\mathrm{H}^1(\Omega)} \qquad \text{(i),} \quad \text{or} \\ &\forall\, \varphi,\psi \in \mathrm{H}^1(\Omega),\ \ |c_i(\varphi,\psi)| \leq \alpha_i \|\varphi\|_{\mathrm{H}^1(\Omega)}\|\psi\|_{\mathrm{L}^2(\Omega)} \qquad \text{(ii),} \end{aligned} \tag{3.11}$$

where $\alpha_i > 0$, $i = 1,2,3$. Moreover, for $i = 1,2,3$ and denoting $C_i \in \mathcal{L}(\mathrm{H}^1(\Omega))$ the bounded operator associated with c_i, *i.e.* such that $\forall\,\varphi,\psi \in \mathrm{H}^1(\Omega), (C_i\varphi,\psi)_{\mathrm{H}^1(\Omega)} = c_i(\varphi,\psi)$, the estimation (3.11)(i) implies that C_i is compact and the estimation (3.11)(ii) implies that its adjoint C_i^* is compact and thus C_i is too. Indeed, assuming that c_i satisfies (3.11)(i), let us show that C_i is then a compact operator, the second assertion is deduced in the same fashion. Consider a sequence $(\varphi_n) \in \mathrm{H}^1(\Omega)$ bounded in $\mathrm{H}^1(\Omega)$. Up to an extraction, thanks to the compact embedding of $\mathrm{H}^1(\Omega)$ into $\mathrm{L}^2(\Omega)$ (as Ω is bounded), (φ_n) converges in $\mathrm{L}^2(\Omega)$ toward $\varphi \in \mathrm{H}^1(\Omega)$. Remarking that $\|C_i\varphi_n - C_i\varphi\|^2_{\mathrm{H}^1(\Omega)} = c_i(\varphi_n - \varphi, C_i\varphi_n - C_i\varphi)$, and applying (3.11)(i), one finds

$$\|C_i\varphi_n - C_i\varphi\|^2_{\mathrm{H}^1(\Omega)} \leq \alpha_i \|\varphi_n - \varphi\|_{\mathrm{L}^2(\Omega)}\|C_i\varphi_n - C_i\varphi\|_{\mathrm{H}^1(\Omega)}.$$

As $(C_i\varphi_n - C_i\varphi)$ is bounded in $\mathrm{H}^1(\Omega)$, this shows that $(C_i\varphi_n)$ converges in $\mathrm{H}^1(\Omega)$ (up to the extraction) and thus that C_i is compact. As a consequence, c is associated with a compact operator and the problem falls under the Fredholm alternative [41, 49].

For the point (ii) (uniqueness), let us introduce $\varphi \in \mathrm{H}^1(\Omega)$ solution of (3.4) with $f = 0$:

$$\forall\, \psi \in \mathrm{H}^1(\Omega), \quad a(\omega; \varphi, \psi) = 0.$$

Then, choosing $\psi = \varphi$ and taking the imaginary part of this equality leads to:

$$\Im m\left[a(\omega;\varphi,\varphi)\right] = 0 = \omega \int_{\partial\Omega} \lambda\, \rho_0\, c_0^{-1}\, |\varphi|^2.$$

Since $\lambda > 0$ on $\gamma \subset \partial\Omega$ (assumption (2.13)), this shows that $\varphi = 0$ on γ. From the boundary condition (2.15)(i), we deduce that $\boldsymbol{\nabla}\varphi \cdot \boldsymbol{n} = 0$ on γ. As Ω is connected, we conclude, with a classical unique continuation argument for elliptic equations [1], that $\varphi = 0$ in Ω. □

3.4. Proof of the main result

3.4.1. Preamble

Let us start with the weak formulation of the coupled problem ((2.14), (2.15)) (the natural extension of (3.7) obtained for the convected Helmholtz equation). Let us introduce the Hilbert spaces

$$V := \mathrm{H}^1(\Omega), \quad \boldsymbol{M} := \mathrm{H}^-(\boldsymbol{v}_0, \Omega), \quad \text{and} \quad \boldsymbol{L} := \mathrm{L}^2(\Omega)^d.$$

Proposition 3.9. *The weak formulation of ((2.14), (2.15)) is: find $(\varphi, \boldsymbol{\xi}) \in V \times \boldsymbol{M}$ such that*

$$\begin{cases} a(\omega;\varphi,\psi) + d(\boldsymbol{\xi},\psi) = \ell(\psi), & \forall\, \psi \in V, \quad \text{(i)} \\ b(\varphi,\boldsymbol{\zeta}) + t(\omega;\boldsymbol{\xi},\boldsymbol{\zeta}) = 0, & \forall\, \boldsymbol{\zeta} \in \boldsymbol{L}, \quad \text{(ii)} \end{cases} \tag{3.12}$$

where the sesquilinear form $a(\omega;\varphi,\psi)$ and the antilinear form $\ell(\psi)$ are defined in (3.8), while the other sesquilinear forms are given by

$$\begin{cases} t(\omega;\boldsymbol{\xi},\boldsymbol{\zeta}) := \displaystyle\int_\Omega \big(D_\omega \boldsymbol{\xi}\cdot\overline{\boldsymbol{\zeta}} + (\boldsymbol{\xi}\cdot\boldsymbol{\nabla})\boldsymbol{v}_0\cdot\overline{\boldsymbol{\zeta}}\big), \\ d(\boldsymbol{\xi},\psi) := \displaystyle\int_\Omega \rho_0\, \boldsymbol{\xi}\cdot\overline{\boldsymbol{\nabla}\psi} - \int_{\Gamma_+} \rho_0\, c_0^{-1}(\boldsymbol{v}_0\cdot\boldsymbol{n})(\boldsymbol{\xi}\cdot\boldsymbol{n})\,\overline{\psi}, \\ b(\varphi,\boldsymbol{\zeta}) := \displaystyle\int_\Omega (\boldsymbol{\omega}_0\times\boldsymbol{\nabla}\varphi)\cdot\overline{\boldsymbol{\zeta}}. \end{cases}$$

Proof. As (3.12)(ii) simply results from multiplication of (2.14)(ii) by $\overline{\boldsymbol{\zeta}}$ and integration over Ω, only (3.12)(i) deserves some comments. This equation is obtained as in the proof of Proposition 3.7, by multiplying (2.14)(i) by $\overline{\psi}$, integrating over Ω and using Green's formula, which gives

$$\int_\Omega \rho_0\big(\boldsymbol{\nabla}\varphi + \boldsymbol{\xi} - c_0^{-2} D_\omega\varphi\, \boldsymbol{v}_0\big)\cdot\boldsymbol{\nabla}\overline{\psi} - i\omega\int_\Omega \rho_0\, c_0^{-2} D_\omega\varphi\,\overline{\psi} - \big\langle \rho_0\big(\boldsymbol{\nabla}\varphi + \boldsymbol{\xi} - c_0^{-2} D_\omega\varphi\,\boldsymbol{v}_0\big)\cdot\boldsymbol{n}, \overline{\psi}\big\rangle_{\partial\Omega} = \int_\Omega \rho_0\, f\,\overline{\psi}. \tag{3.13}$$

One then concludes as in the proof of Proposition 3.7 using (2.15)(i). □

For the sequel, it is useful to rewrite (3.12) in an abstract operator form. To this purpose, *via* Riesz theorem, we introduce the linear operators associated to the sesquilinear forms in (3.12)

$$\begin{cases} A(\omega) \in \mathcal{L}(V) & \text{s.t. } (A(\omega)\varphi, \psi)_V = a(\omega; \varphi, \psi), \quad \forall\, (\varphi, \psi) \in V \times V, \\ \boldsymbol{T}(\omega) \in \mathcal{L}(\boldsymbol{M}, \boldsymbol{L}) & \text{s.t. } (\boldsymbol{T}(\omega)\boldsymbol{\xi}, \boldsymbol{\zeta})_{\boldsymbol{L}} = t(\omega; \boldsymbol{\xi}, \boldsymbol{\zeta}), \quad \forall\, (\boldsymbol{\xi}, \boldsymbol{\zeta}) \in \boldsymbol{M} \times \boldsymbol{L}, \\ D \in \mathcal{L}(\boldsymbol{M}, V) & \text{s.t. } (D\boldsymbol{\xi}, \psi)_V = d(\boldsymbol{\xi}, \psi), \quad \forall\, (\boldsymbol{\xi}, \psi) \in \boldsymbol{M} \times V, \\ \boldsymbol{B} \in \mathcal{L}(V, \boldsymbol{L}) & \text{s.t. } (\boldsymbol{B}\varphi, \boldsymbol{\zeta})_{\boldsymbol{L}} = b(\varphi, \boldsymbol{\zeta}), \quad \forall\, (\varphi, \boldsymbol{\zeta}) \in V \times \boldsymbol{L}. \end{cases} \tag{3.14}$$

Then, with $\hat{f} \in V$ such that $\ell(\varphi) = (\hat{f}, \psi)_V$ for all ψ in V, (3.12) rewrites

$$\begin{pmatrix} A(\omega) & D \\ \boldsymbol{B} & \boldsymbol{T}(\omega) \end{pmatrix} \begin{pmatrix} \varphi \\ \boldsymbol{\xi} \end{pmatrix} = \begin{pmatrix} \hat{f} \\ \boldsymbol{0} \end{pmatrix}. \tag{3.15}$$

Remark 3.10. Note that the operator $\mathbf{T}(\omega)$ is nothing but the transport differential operator $\boldsymbol{\mathcal{T}}_0(\omega)$, see (2.9), acting in the space $\mathbf{M}$ and that $\mathbf{B}\varphi = \boldsymbol{\omega}_0 \times \boldsymbol{\nabla}\varphi$, since their definition simply uses the L^2-inner product (or $\mathbf{L}$-inner product). The interpretation of $A(\omega)$ and D is less direct since they are defined through the H^1-inner product.

For solving (3.15), the most natural idea, is to try to extend the Fredholm type approach followed in Section 3.3 for the convected Helmholtz equation. However, a first obstacle is that the uniqueness proof in Theorem 3.8 is not generalisable (at least we did not succeed) to the coupled problem ((2.14), (2.15)). Indeed, let $(\varphi, \boldsymbol{\xi})$ be a solution of ((2.14), (2.15)) for $f = 0$. Taking $\psi = \varphi$ in (3.12)(i) gives, after taking the imaginary part

$$\omega \int_\Gamma \lambda\, \rho_0\, c_0^{-1}\, |\varphi|^2 = \Im m \left\{ - \int_\Omega \rho_0\, \boldsymbol{\xi} \cdot \boldsymbol{\nabla} \overline{\varphi} + \int_{\Gamma_+} \rho_0\, c_0^{-1} (\boldsymbol{v}_0 \cdot \boldsymbol{n})(\boldsymbol{\xi} \cdot \boldsymbol{n})\, \overline{\varphi}, \right\}.$$

We could conclude if, for instance, we could deduce from the transport equation (2.14)(ii) that the right hand side term is non-positive. Unfortunately, we did not succeed (even in the simple case of a laminar flow in a wave guide) which leads us to doubt that this is true in general. This is why we have chosen to use analytic Fredholm theory as explained in Section 3.2.

The idea is to extend the Goldstein's problem ((2.14), (2.15)) to *complex values* of the frequency ω, especially to

$$\omega \in \mathbb{C}^+_\beta := \{ z \in \mathbb{C} \;/\; \Im m(z) > -\beta \}$$

for $\beta > 0$ small enough: this will be made precise in the proof, see Section 3.4. As we shall see, the key properties of this domain is that it is connected, contains the real axis as well as the semi-imaginary one $\{ i\,\omega_i \;/\; \omega_i > 0 \}$.

This will permit us to apply the following abstract result from analytic Fredholm theory. For instance, from [49], Theorem 8.92:

Theorem 3.11 (Analytic Fredholm)**.** *Let V be a Hilbert space and $G \subset \mathbb{C}$ be a domain (open and connected). Let $\lambda \in G \mapsto B(\lambda) \in \mathcal{L}(V)$ an analytic map such that $B(\lambda)$ is compact for all $\lambda \in G$. Then, either,*

(1) *$(I - B(\lambda))^{-1}$ exists for no $\lambda \in G$,*
(2) *There exists $S \subset G$, with no limit point in G, such that for all $\lambda \in G \setminus S$, $(I - B(\lambda))^{-1}$ exists and the map $\lambda \in G \setminus S \mapsto (I - B(\lambda))^{-1}$, is analytic.*

From which one easily deduces the following corollary that is more directly fitted to our settings:

Corollary 3.12 (Analytic Fredholm (invertible case))**.** *Let V be a Hilbert space and $G \subset \mathbb{C}$ be a domain. Let $A^g(\omega)$, $\omega \in G$, be a family of bounded linear operators in V and assume that $A^g(\omega) = A^g_*(\omega) + C^g(\omega)$ with $A^g_*(\omega), C^g(\omega) \in \mathcal{L}(V)$, where*

(1) *$A^g_* : \omega \in G \mapsto A^g_*(\omega)$ and $C^g : \omega \in G \mapsto C^g(\omega)$ are analytic,*
(2) *$\forall\, \omega \in G$, $A^g_*(\omega)$ is invertible and $C^g(\omega)$ is compact,*
(3) *$\exists\, \omega_\diamond \in G$ such that $A^g(\omega_\diamond)$ is invertible.*

Then, there exists a set $S \subset G$ with no limit point in G such that $\forall\, \omega \in G \setminus S$, $A^g(\omega)$ is invertible.

Remark 3.13. – Here, the analyticity of $A^g : \omega \in G \mapsto A^g(\omega) \in \mathcal{L}(V)$ has to be understood in the sense of the operator norm, *i.e.*

$$\forall\, \omega \in G, \lim_{\substack{z\to 0,\\ z\in\mathbb{C}^*}} \frac{1}{z}(A^g(\omega+z) - A^g(\omega)) \quad \text{exists in} \quad \mathcal{L}(V).$$

– A Fredholm operator is a bounded linear operator such that both its kernel and a supplementary of its range are finite dimensional. Its index is the difference between these two dimensions. An invertible operator is a Fredholm operator of index 0, and because the Fredholm property and the index value are stable by adding a compact operator, assumption (2) of the Corollary 3.12 implies that $A^g(\omega)$ is a Fredholm operator of index 0.

The above result cannot be directly applied to coupled problem (3.15) in $(\varphi, \boldsymbol{\xi})$. We shall apply it to the problem in φ alone obtained formally by eliminating $\boldsymbol{\xi}$. In other words, we perform a Schur complement

$$\boldsymbol{\xi} = \boldsymbol{\xi}(\varphi) = -\,\mathbf{T}(\omega)^{-1}\,\mathbf{B}\,\varphi, \tag{3.16}$$

so that the equation governing φ is given by the "modified" convected Helmholtz equation:

$$A_r(\omega)\,\varphi = f, \quad \text{where } A_r(\omega) := A(\omega) - D\,\boldsymbol{T}(\omega)^{-1}\,\boldsymbol{B}.$$

and we aim to apply Corollary 3.12 to $V = \mathrm{H}^1(\Omega)$ and $A^g(\omega) = A_r(\omega)$.

Remark 3.14. Note that, going back to the notation (3.5) of Section 3.2,

$$-\,\mathbf{T}(\omega)^{-1}\,\mathbf{B}\,\varphi = \boldsymbol{\mathcal{S}}_0(\omega, \boldsymbol{\nabla}\varphi).$$

In Section 3.4.2, we first show that the elimination of $\boldsymbol{\xi}$ is possible, namely that (3.16) has a sense (this is of course related to the invertibility of the transport equation). In Section 3.4.3, we show that the family $A_r(\omega)$ satisfies the assumptions (1) and (2) of Corollary 3.12. Finally, in Section 3.4.4, we conclude the proof by showing that the assumption (3) is also satisfied.

3.4.2. Elimination of $\boldsymbol{\xi}$ and modified convected Helmholtz equation

As this section is dedicated to the elimination of $\boldsymbol{\xi}$ from the transport equation (2.7)(i), let us begin with the Theorem 3.16 which states the invertibility of the transport operator, introduced in (2.9), acting on $\boldsymbol{M}$, *i.e.* $\boldsymbol{\mathcal{T}}_0(\omega) : \boldsymbol{M} \subset \boldsymbol{L} \to \boldsymbol{L}$. Whereas the issue of the invertibility of this operator has been previously treated in [9], it has to be adapted here to complex frequencies ω. For completeness, let us recall the definition of a Ω-filling flow $\boldsymbol{v}_0$ which is the key assumption of the Theorem 3.16.

The characteristic field $\boldsymbol{\chi} : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ associated to $\boldsymbol{v}_0 \in \mathrm{C}^1(\mathbb{R}^d)^d$ is defined from the solutions of the following family of differential equations:

$$\forall\, (t, \boldsymbol{b}) \in \mathbb{R} \times \mathbb{R}^d, \begin{cases} \dfrac{\partial \boldsymbol{\chi}}{\partial t}(t, \boldsymbol{b}) = \boldsymbol{v}_0(\boldsymbol{\chi}(t, \boldsymbol{b})), \\ \boldsymbol{\chi}(0, \boldsymbol{b}) = \boldsymbol{b}, \end{cases}$$

(we do not use the term "flow" from differential equation theory to avoid confusion with the flow $\boldsymbol{v}_0$). Using $\boldsymbol{\chi}$, we can now precise the Definition 3.1 of a Ω-filling flow $\boldsymbol{v}_0$:

Definition 3.15 (Ω-filling flow and lifetime in Ω)**.** The flow $\boldsymbol{v}_0$ is said Ω-filling if there exists a time $T^* \in \mathbb{R}^+$ such that for all $\boldsymbol{x} \in \Omega$:

$$\exists (t, \boldsymbol{b}) \in [0, T^*] \times \Gamma_- \ \ / \ \ \boldsymbol{\chi}(t, \boldsymbol{b}) = \boldsymbol{x} \quad \text{and} \quad \forall\, \tau \in]0, t], \ \boldsymbol{\chi}(\tau, \boldsymbol{b}) \in \Omega. \tag{3.17}$$

For such a flow, its lifetime in Ω, denoted $t^*(\boldsymbol{v}_0, \Omega)$, is the infimum of all constants T^* keeping (3.17) valid.

We then have the following theorem:

Theorem 3.16 (Invertibility of $\boldsymbol{\mathcal{T}}_0(\omega)$ for $\omega \in \mathbb{C}$)**.** *If the flow $\boldsymbol{v}_0$ is Ω-filling and satisfies $\operatorname{div}(\rho_0 \boldsymbol{v}_0) = 0$, with the lifetime $t^*(\boldsymbol{v}_0, \Omega)$ in Ω, then, for all $\omega \in \mathbb{C}$, the time-harmonic transport operator $\boldsymbol{\mathcal{T}}_0(\omega) : \boldsymbol{M} \subset \boldsymbol{L} \to \boldsymbol{L}$ defined by (2.9) is invertible and the following estimates hold: denoting $\omega_i = \Im m\, \omega$, for all $\boldsymbol{g} \in \boldsymbol{L}$,*

$$\begin{aligned} &\left\| \boldsymbol{\mathcal{T}}_0^{-1}(\omega)\, \boldsymbol{g} \right\|_{\mathrm{L}^2(\Omega, \rho_0)^d} &&\leq C_0(\boldsymbol{v}_0, \Omega, \omega_i)\, \|\boldsymbol{g}\|_{\mathrm{L}^2(\Omega, \rho_0)^d} && \text{(i)}, \\ &\left\| \boldsymbol{\mathcal{T}}_0^{-1}(\omega)\, \boldsymbol{g} \right\|_{\mathrm{L}^2(\Gamma_+,\, \rho_0(\boldsymbol{v}_0 \cdot \boldsymbol{n}))} &&\leq C_+(\boldsymbol{v}_0, \Omega, \omega_i)\, \|\boldsymbol{g}\|_{\mathrm{L}^2(\Omega, \rho_0)^d} && \text{(ii)}, \end{aligned} \tag{3.18}$$

where the weighted L^2*-norm along* Γ_+ *is given by, remembering that* $\boldsymbol{v}_0 \cdot \boldsymbol{n} > 0$ *on* Γ_+,

$$\|w\|^2_{\mathrm{L}^2(\Gamma_+,\, \rho_0(\boldsymbol{v}_0 \cdot \boldsymbol{n}))} := \int_{\Gamma_+} |w|^2\, \rho_0 (\boldsymbol{v}_0 \cdot \boldsymbol{n})\, \mathrm{d}\sigma,$$

and $C_0(\boldsymbol{v}_0, \Omega, \omega_i)$ and $C_+(\boldsymbol{v}_0, \Omega, \omega_i)$ are respectively given, with (Φ, Ψ) defined in (3.2), by

$$\begin{cases} C_0(\boldsymbol{v}_0, \Omega, \omega_i) := t^*(\boldsymbol{v}_0, \Omega)\, \Phi(2\, t^*(\boldsymbol{v}_0, \Omega)(\|\boldsymbol{\nabla} \boldsymbol{v}_0\|_{\mathrm{L}^\infty} - \omega_i)), & \text{(i)} \\ C_+(\boldsymbol{v}_0, \Omega, \omega_i) := \sqrt{t^*(\boldsymbol{v}_0, \Omega)}\; \Psi(2\, t^*(\boldsymbol{v}_0, \Omega)(\|\boldsymbol{\nabla} \boldsymbol{v}_0\|_{\mathrm{L}^\infty} - \omega_i)). & \text{(ii)} \end{cases} \tag{3.19}$$

Remark 3.17. As the map $\Phi : \mathbb{R} \to \mathbb{R}$ is increasing (Lem. 3.4), equation (3.19)(i) implies that the function $\omega_i \in \mathbb{R} \mapsto C_0(\boldsymbol{v}_0, \Omega, \omega_i)$ is (strictly) decreasing. In particular

$$\forall\, \omega_i \geq 0, \quad C_0(\boldsymbol{v}_0, \Omega, \omega_i) \leq C_0(\boldsymbol{v}_0, \Omega, 0) = t^*(\boldsymbol{v}_0, \Omega)\, \Phi(2\, t^*(\boldsymbol{v}_0, \Omega) \|\boldsymbol{\nabla} \boldsymbol{v}_0\|_{\mathrm{L}^\infty}).$$

Moreover, as $\Phi(s)$ and $\Psi(s)$ tend to 0 when $s \to -\infty$ (Lem. 3.4 again), it is also clear from formulas (3.19) that

$$C_0(\boldsymbol{v}_0, \Omega, \omega_i) \to 0 \ \text{ and,} \quad C_+(\boldsymbol{v}_0, \Omega, \omega_i) \to 0 \quad \text{ when } \omega_i \to +\infty.$$

Remark 3.18. The estimate (3.18)(ii) is similar to a trace theorem in the space $\mathrm{H}(\Omega, \boldsymbol{v}_0)$, where a special care is given to explicit the trace constant.

Proof. The proof of Theorem 3.16 is a straightforward adaptation of the same result for real frequencies detailed in [9]. The key point consists in obtaining the following two *a priori* estimates (obviously related to (3.18))

$$\begin{aligned} &\forall\, \boldsymbol{\xi} \in \boldsymbol{M}, \quad \|\boldsymbol{\xi}\|_{\mathrm{L}^2(\Omega, \rho_0)^d} &&\leq C_0(\boldsymbol{v}_0, \Omega, \omega_i)\, \|\boldsymbol{\mathcal{T}}_0(\omega) \boldsymbol{\xi}\|_{\mathrm{L}^2(\Omega, \rho_0)^d}, && \text{(i)} \\ &\forall\, \boldsymbol{\xi} \in \boldsymbol{M}, \quad \|\boldsymbol{\xi}\|_{\mathrm{L}^2(\Gamma_+,\, \rho_0(\boldsymbol{v}_0 \cdot \boldsymbol{n}))} &&\leq C_+(\boldsymbol{v}_0, \Omega, \omega_i)\, \|\boldsymbol{\mathcal{T}}_0(\omega) \boldsymbol{\xi}\|_{\mathrm{L}^2(\Omega, \rho_0)^d}. && \text{(ii)} \end{aligned} \tag{3.20}$$

Once the estimates (3.20) are established, the proof of the invertibility of $\boldsymbol{\mathcal{T}}_0(\omega)$ essentially relies on functional analytic arguments and (3.20) leads to (3.18). We divide the proof into three steps.

Step 1. Preliminary material. We use the same technique of "characteristic" change of variables as in [9]. Thanks to the Ω-filling property of $\boldsymbol{v}_0$ and the fact that the boundary Γ_- is piecewise C^1, the function $\boldsymbol{\chi}$ induces a piecewise-C^1-diffeomorphism $\boldsymbol{\chi} : \Theta \mapsto \Omega$ on the fiber bundle

$$\Theta := \big\{ (t, \boldsymbol{b}) \in \mathbb{R}_+ \times \Gamma_- \ : \ \forall\, \tau \in (0, t], \ \ \boldsymbol{\chi}(\tau, \boldsymbol{b}) \in \Omega \big\}.$$

In what follows, we endow Θ with the push-forward measure $\mu = (\rho_0 \mathrm{d}\boldsymbol{x}) \circ \boldsymbol{\chi}$ such that for any Lebesgue measurable set $B \subset \Theta$,

$$\mu(B) := (\rho_0 \mathrm{d}\boldsymbol{x})\big[\boldsymbol{\chi}(B)\big].$$

This leads to the change of variables formula: given $f : \Omega \to \mathbb{C}$,

$$\int_\Theta f \circ \boldsymbol{\chi}(t, \boldsymbol{b})\, \mathrm{d}\mu(t, \boldsymbol{b}) = \int_\Omega f(\boldsymbol{x}) \rho_0(\boldsymbol{x})\, \mathrm{d}\boldsymbol{x}. \tag{3.21}$$

A classical consequence of $\mathrm{div}(\rho_0 \boldsymbol{v}_0) = 0$ [5, 18], is that the measure $\mathrm{d}\mu(t, \boldsymbol{b})$ can be written as the product measure of $\mathrm{d}t$ and the surfacic measure $\mathrm{d}\sigma$ on Γ_- (which means nothing else, but the fact that the volume is preserved along the streamlines):

$$\mathrm{d}\mu(t, \boldsymbol{b}) = \mathrm{d}t\, \mathrm{d}\sigma(\boldsymbol{b}),$$

and thus the integration over Θ reads, given $g : \Theta \to \mathbb{C}$,

$$\int_\Theta g(t, \boldsymbol{b})\, \mathrm{d}\mu(t, \boldsymbol{b}) = \int_\Theta g(t, \boldsymbol{b})\, \mathrm{d}t\, \mathrm{d}\sigma(\boldsymbol{b}) = \int_{\Gamma_-} \left(\int_0^{t(\boldsymbol{b})} g(t, \boldsymbol{b})\, \mathrm{d}t \right) \mathrm{d}\sigma(\boldsymbol{b}), \tag{3.22}$$

where $t(\boldsymbol{b}) := \sup\{t \geq 0 \,:\, \forall\, \tau \in (0, t],\ \boldsymbol{\chi}(t, \boldsymbol{b}) \in \Omega\}$ is by definition the lifetime in Ω of $\boldsymbol{b} \in \Gamma_-$. Note that this notion is related to the global lifetime in Ω *via*

$$t^*(\boldsymbol{v}_0, \Omega) = \sup_{\boldsymbol{b} \in \Gamma_-} t(b). \tag{3.23}$$

As a consequence of formula (3.21), the operator $\boldsymbol{\mathcal{S}} \in \mathcal{L}\big(\mathrm{L}^2(\Omega, \rho_0)^d, \mathrm{L}^2(\Theta, \mu)^d\big)$ defined by

$$\forall\, \boldsymbol{\xi} \in \mathrm{L}^2(\Omega)^d, \quad \forall\, (t, \boldsymbol{b}) \in \Theta, \quad \boldsymbol{\mathcal{S}}\boldsymbol{\xi}(t, \boldsymbol{b}) := \boldsymbol{\xi}(\boldsymbol{\chi}(t, \boldsymbol{b}))$$

is an isometry, namely:

$$\forall\, \boldsymbol{\xi} \in \mathrm{L}^2(\Omega, \rho_0)^d, \quad \|\boldsymbol{\mathcal{S}}\boldsymbol{\xi}\|_{\mathrm{L}^2(\Theta)^d} = \|\boldsymbol{\xi}\|_{\mathrm{L}^2(\Omega, \rho_0)^d}. \tag{3.24}$$

The main interest of the change of variables $\boldsymbol{x} = \boldsymbol{\chi}(t, \boldsymbol{b})$ is that it simplifies the expression of the transport operator in Θ variables. More precisely, one observes that setting

$$\widehat{\boldsymbol{M}} := \left\{ \widehat{\boldsymbol{\zeta}} \in \mathrm{L}^2(\Theta, \mu)^d \ \Big/\ \frac{\partial \widehat{\boldsymbol{\zeta}}}{\partial t} \in \mathrm{L}^2(\Theta, \mu)^d,\ \forall\, \boldsymbol{b} \in \Gamma_-,\ \widehat{\boldsymbol{\zeta}}(0, \boldsymbol{b}) = \boldsymbol{0} \right\},$$

then $\boldsymbol{\xi} \in \boldsymbol{M}$ if and only if $\widehat{\boldsymbol{\xi}} := \boldsymbol{\mathcal{S}}\boldsymbol{\xi} \in \widehat{\boldsymbol{M}}$. Moreover, one has the formula ($\boldsymbol{v}_0 \cdot \boldsymbol{\nabla}$ simply becomes the t-derivative, this is pure computation, see [9] for $d = 2$)

$$\forall\, \boldsymbol{\zeta} \in \boldsymbol{M}, \quad \boldsymbol{\mathcal{S}}((\boldsymbol{v}_0 \cdot \boldsymbol{\nabla})\boldsymbol{\zeta}) = \frac{\partial}{\partial t}(\boldsymbol{\mathcal{S}}\boldsymbol{\zeta}). \tag{3.25}$$

As a consequence if we introduce the operator $\boldsymbol{\mathcal{Q}}_0(\omega)$ defined by

$$\forall\, \widehat{\boldsymbol{\zeta}} \in \widehat{\boldsymbol{M}}, \quad \boldsymbol{\mathcal{Q}}_0(\omega)\, \widehat{\boldsymbol{\zeta}} := \frac{\partial \widehat{\boldsymbol{\zeta}}}{\partial t} - i\omega\, \widehat{\boldsymbol{\zeta}} + \widehat{\boldsymbol{J}}_0\, \widehat{\boldsymbol{\zeta}},$$

where $\widehat{\boldsymbol{J}}_0(t, \boldsymbol{b}) \in \mathcal{L}(\mathbb{R}^d)$ is defined as

$$\widehat{\boldsymbol{J}}_0(t, \boldsymbol{b}) := \boldsymbol{J}_0(\chi(t, \boldsymbol{b})), \quad \text{where } \boldsymbol{J}_0(\boldsymbol{x}) := D\boldsymbol{v}_0(\boldsymbol{x}),\ \textit{i.e.}\ (\boldsymbol{J}_0)_{ij}(\boldsymbol{x}) = \frac{\partial \boldsymbol{v}_{0,i}}{\partial x_j}(\boldsymbol{x}),$$

one has the commutation property (this is pure computation, see [9] for $d = 2$)

$$\forall\, \boldsymbol{\xi} \in \boldsymbol{M}, \quad \boldsymbol{\mathcal{S}}\, \boldsymbol{\mathcal{T}}_0(\omega)\boldsymbol{\xi} = \boldsymbol{\mathcal{Q}}_0(\omega)\, \boldsymbol{\mathcal{S}}\boldsymbol{\xi}. \tag{3.26}$$

Step 2. Proof of the volume estimate (3.20)(i).

From (3.26), the isometry result (3.24) and $\widehat{\boldsymbol{\xi}} = \boldsymbol{\mathcal{S}}\boldsymbol{\xi}$, it is clear that proving (3.20)(i) amounts to proving

$$\forall\, \widehat{\boldsymbol{\xi}} \in \widehat{\boldsymbol{M}}, \quad \left\|\widehat{\boldsymbol{\xi}}\right\|_{\mathrm{L}^2(\Theta,\mu)^d} \leq C_0(\boldsymbol{v}_0, \Omega, \omega) \left\|\boldsymbol{\mathcal{Q}}_0(\omega)\widehat{\boldsymbol{\xi}}\right\|_{\mathrm{L}^2(\Theta,\mu)^d}. \tag{3.27}$$

Of course, by a density-continuity argument, it suffices to establish (3.27) when

$$\widehat{\boldsymbol{\xi}} \in D_-(\Theta) := \Big\{\boldsymbol{\zeta} \in \widehat{\boldsymbol{M}} \;/\; a.e.\ \boldsymbol{b} \in \Gamma_-,\ t \to \boldsymbol{\zeta}(t,\boldsymbol{b}) \in \mathrm{C}^1([0,t(\boldsymbol{b})])\Big\}.$$

which is a dense subset of $\widehat{\boldsymbol{M}}$. For such a $\widehat{\boldsymbol{\xi}}$, we write

$$\begin{aligned}\widehat{\boldsymbol{\xi}}(t,\boldsymbol{b})\, e^{-i\omega t} &= \int_0^t \frac{\partial}{\partial\tau}\Big(\widehat{\boldsymbol{\xi}}(\tau,\boldsymbol{b})e^{-i\omega\tau}\Big)\mathrm{d}\tau = \int_0^t e^{-i\omega\tau}\left(\frac{\partial\widehat{\boldsymbol{\xi}}}{\partial\tau}(\tau,\boldsymbol{b}) - i\omega\,\widehat{\boldsymbol{\xi}}(\tau,\boldsymbol{b})\right)\mathrm{d}\tau \\ &= \int_0^t e^{-i\omega\tau}\,\boldsymbol{\mathcal{Q}}_0(\omega)\,\widehat{\boldsymbol{\xi}}(\tau,\boldsymbol{b})\,\mathrm{d}\tau - \int_0^t e^{-i\omega\tau}\,\widehat{\boldsymbol{J}}_0(\tau,\boldsymbol{b})\,\widehat{\boldsymbol{\xi}}(\tau,\boldsymbol{b})\,\mathrm{d}\tau.\end{aligned}$$

Then, using $\left|e^{-i\omega t}\,\widehat{\boldsymbol{\xi}}(t,\boldsymbol{b})\right| = e^{\,\omega_i t}\left|\widehat{\boldsymbol{\xi}}(t,\boldsymbol{b})\right|$ we get

$$e^{\omega_i t}\left|\widehat{\boldsymbol{\xi}}(t,\boldsymbol{b})\right| \leq \int_0^t e^{\,\omega_i\tau}\left|\boldsymbol{\mathcal{Q}}_0(\omega)\widehat{\boldsymbol{\xi}}(\tau,\boldsymbol{b})\right|\mathrm{d}\tau + \int_0^t e^{\,\omega_i\tau}\left|\widehat{\boldsymbol{\xi}}(\tau,\boldsymbol{b})\right|\left|\widehat{\boldsymbol{J}}_0(\tau,\boldsymbol{b})\right|\mathrm{d}\tau. \tag{3.28}$$

Let us give the particular Gronwall's lemma which will provide the estimate of $e^{\omega_i t}\left|\widehat{\boldsymbol{\xi}}(t,\boldsymbol{b})\right|$ from (3.28).

Lemma 3.19. *Let $\alpha,\beta : \mathbb{R}_+ \to \mathbb{R}_+$ be two continuous functions. If a continuous function $u : \mathbb{R}_+ \to \mathbb{R}$ satisfies*

$$\forall\, t \geq 0, \quad u(t) \leq \int_0^t \alpha(\tau)\mathrm{d}\tau + \int_0^t \beta(\tau)u(\tau)\mathrm{d}\tau \quad \text{and} \quad u(0) = 0, \tag{3.29}$$

then,

$$\forall\, t \geq 0, \quad u(t) \leq \int_0^t \alpha(\tau)\exp\Big(\int_\tau^t \beta(\tau')\mathrm{d}\tau'\Big)\mathrm{d}\tau. \tag{3.30}$$

A proof of the Lemma 3.19 has been given in the article [9]. Denoting $\alpha : \tau \mapsto e^{\omega_i\tau}\left|\boldsymbol{\mathcal{Q}}_0(\omega)\widehat{\boldsymbol{\xi}}(\tau,\boldsymbol{b})\right|$ and $\beta : \tau \mapsto \left|\widehat{\boldsymbol{J}}_0(\tau,\boldsymbol{b})\right|$, the inequality (3.28) reads as (3.29) where $u : t \mapsto e^{\omega_i t}\left|\widehat{\boldsymbol{\xi}}(t,\boldsymbol{b})\right|$. It then follows from (3.30):

$$e^{\,\omega_i t}\left|\widehat{\boldsymbol{\xi}}(t,\boldsymbol{b})\right| \leq \int_0^t \exp\Big(\int_\tau^t \left|\widehat{\boldsymbol{J}}_0(\tau',\boldsymbol{b})\right|\mathrm{d}\tau'\Big)\, e^{\,\omega_i\tau}\left|\boldsymbol{\mathcal{Q}}_0(\omega)\widehat{\boldsymbol{\xi}}(\tau,\boldsymbol{b})\right|\mathrm{d}\tau,$$

and Cauchy–Schwartz inequality leads to

$$\left|\widehat{\boldsymbol{\xi}}(t,\boldsymbol{b})\right|^2 \leq \mathcal{I}(\omega_i;t,\boldsymbol{b})\int_0^t \left|\boldsymbol{\mathcal{Q}}_0\widehat{\boldsymbol{\xi}}(\tau,\boldsymbol{b})\right|^2\mathrm{d}\tau, \tag{3.31}$$

where $\mathcal{I}(\omega_i;t,\boldsymbol{b}) := \int_0^t \exp\Big(2\int_\tau^t\left|\widehat{\boldsymbol{J}}_0(\tau',\boldsymbol{b})\right|\mathrm{d}\tau'\Big)\, e^{-2\omega_i(t-\tau)}\,\mathrm{d}\tau$. Since $\left|\widehat{\boldsymbol{J}}_0(\tau',\boldsymbol{b})\right| \leq \|\boldsymbol{\nabla}\boldsymbol{v}_0\|_{\mathrm{L}^\infty}$, we get

$$\mathcal{I}(\omega_i;t,\boldsymbol{b}) \leq \mathcal{I}_*(\omega_i;t) := \int_0^t e^{\,2\,(\|\boldsymbol{\nabla}\boldsymbol{v}_0\|_{\mathrm{L}^\infty}-\omega_i)(t-\tau)}\,\mathrm{d}\tau = t\,\Psi(2t(\|\boldsymbol{\nabla}\boldsymbol{v}_0\|_{\mathrm{L}^\infty} - \omega_i))^2, \tag{3.32}$$

where the last equality results from an explicit computation and definition (3.2) of Ψ. Note that by definition (3.19) of $C_+(\boldsymbol{v}_0,\Omega,\omega_i)$,

$$\mathcal{I}_*(\omega_i; t^*(\boldsymbol{v}_0,\Omega)) = C_+(\boldsymbol{v}_0,\Omega,\omega_i)^2. \tag{3.33}$$

Finally, using (3.32) in (3.31) and integrating the resulting inequality over Θ gives

$$\begin{aligned}
\left\|\widehat{\boldsymbol{\xi}}\right\|^2_{\mathrm{L}^2(\Theta,\mu)^d} &= \int_{\Gamma_-}\int_0^{t(\boldsymbol{b})}\left|\widehat{\boldsymbol{\xi}}(t,\boldsymbol{b})\right|^2 \mathrm{d}t\,\mathrm{d}\sigma(\boldsymbol{b})\\
&\le \int_{\Gamma_-}\int_0^{t(\boldsymbol{b})}\left(\mathcal{I}_*(\omega_i;t)\int_0^t\left|\boldsymbol{\mathcal{Q}}_0\widehat{\boldsymbol{\xi}}(\tau,\boldsymbol{b})\right|^2\mathrm{d}\tau\right)\mathrm{d}t\,\mathrm{d}\sigma(\boldsymbol{b})\\
&\le \int_{\Gamma_-}\left(\int_0^{t(\boldsymbol{b})}\left|\boldsymbol{\mathcal{Q}}_0\widehat{\boldsymbol{\xi}}(\tau,\boldsymbol{b})\right|^2\mathrm{d}\tau\right)\left(\int_0^{t(\boldsymbol{b})}\mathcal{I}_*(\omega_i;t)\,\mathrm{d}t\right)\mathrm{d}\sigma(\boldsymbol{b})\\
&\le \left(\int_0^{t^*(\boldsymbol{v}_0,\Omega)}\mathcal{I}_*(\omega_i;t)\,\mathrm{d}t\right)\left(\int_{\Gamma_-}\int_0^{t(\boldsymbol{b})}\left|\boldsymbol{\mathcal{Q}}_0\widehat{\boldsymbol{\xi}}(\tau,\boldsymbol{b})\right|^2\mathrm{d}\tau\mathrm{d}\sigma(\boldsymbol{b})\right)\quad(\text{by } (3.23))\\
&\le \left(\int_0^{t^*(\boldsymbol{v}_0,\Omega)}\mathcal{I}_*(\omega_i;t)\,\mathrm{d}t\right)\ \left\|\boldsymbol{\mathcal{Q}}_0(\omega)\widehat{\boldsymbol{\xi}}\right\|^2_{\mathrm{L}^2(\Theta,\mu)^d},
\end{aligned}$$

and an exact computation, with the expressions (3.2), (3.19)(i), (3.32) of respectively Φ, C_0, $\mathcal{I}_*(\omega_i;t)$, gives:

$$\begin{aligned}
\int_0^{t^*(\boldsymbol{v}_0,\Omega)}\mathcal{I}_*(\omega_i;t)\mathrm{d}t &= \int_0^{t^*(\boldsymbol{v}_0,\Omega)} t\Psi(2t(\|\boldsymbol{\nabla}\boldsymbol{v}_0\|_{\mathrm{L}^\infty}-\omega_i))^2\mathrm{d}t\\
&= \int_0^{t^*(\boldsymbol{v}_0,\Omega)}\frac{e^{2t(\|\boldsymbol{\nabla}\boldsymbol{v}_0\|_{\mathrm{L}^\infty}-\omega_i)}-1}{2(\|\boldsymbol{\nabla}\boldsymbol{v}_0\|_{\mathrm{L}^\infty}-\omega_i)}\mathrm{d}t\\
&= t^*(\boldsymbol{v}_0,\Omega)^2\,\Phi(2t^*(\boldsymbol{v}_0,\Omega)(\|\boldsymbol{\nabla}\boldsymbol{v}_0\|_{\mathrm{L}^\infty}-\omega_i))^2\\
&= C_0(\boldsymbol{v}_0,\Omega,\omega_i)^2,
\end{aligned}$$

which leads to the desired estimate (3.27).

Step 3. Proof of the boundary estimate (3.20)(ii).

We first observe that, by Green's formula,

$$\|\boldsymbol{\xi}\|^2_{\mathrm{L}^2(\Gamma_+,\rho_0(\boldsymbol{v}_0\cdot\boldsymbol{n}))} := \int_{\Gamma_+}\rho_0\,|\boldsymbol{\xi}|^2\,(\boldsymbol{v}_0\cdot\boldsymbol{n})\,\mathrm{d}\sigma = 2\,\Re e\int_\Omega \rho_0(\boldsymbol{v}_0\cdot\boldsymbol{\nabla})\boldsymbol{\xi}\cdot\overline{\boldsymbol{\xi}}.$$

Thus using the change of variables formula (3.21), (3.22) and (3.25)

$$\|\boldsymbol{\xi}\|^2_{\mathrm{L}^2(\Gamma_+,\rho_0(\boldsymbol{v}_0\cdot\boldsymbol{n}))} = 2\,\Re e\int_\Theta\frac{\partial\widehat{\boldsymbol{\xi}}}{\partial t}\cdot\overline{\widehat{\boldsymbol{\xi}}}\,\mathrm{d}\mu(t,\boldsymbol{b}) = 2\,\Re e\int_{\Gamma_-}\int_0^{t(\boldsymbol{b})}\frac{\partial\widehat{\boldsymbol{\xi}}}{\partial t}\cdot\overline{\widehat{\boldsymbol{\xi}}}\,\mathrm{d}t\,\mathrm{d}\sigma(\boldsymbol{b}).$$

Thus, using again (3.31) and the inequality $\mathcal{I}(\omega_i;t(\boldsymbol{b}),\boldsymbol{b})\le\mathcal{I}_*(\omega_i;t(\boldsymbol{b}))\le\mathcal{I}_*(\omega_i;t^*(\boldsymbol{v}_0,\Omega))$,

$$\begin{aligned}
\|\boldsymbol{\xi}\|^2_{\mathrm{L}^2(\Gamma_+,\rho_0(\boldsymbol{v}_0\cdot\boldsymbol{n}))} &= \int_{\Gamma_-}\left|\widehat{\boldsymbol{\xi}}(t(\boldsymbol{b}),\boldsymbol{b})\right|^2\mathrm{d}\sigma(\boldsymbol{b})\\
&\le \int_{\Gamma_-}\mathcal{I}(\omega_i;t(\boldsymbol{b}),\boldsymbol{b}))\left(\int_0^{t(\boldsymbol{b})}\left|\boldsymbol{\mathcal{Q}}_0\widehat{\boldsymbol{\xi}}(\tau,\boldsymbol{b})\right|^2\mathrm{d}\tau\right)\mathrm{d}\sigma(\boldsymbol{b})\\
&\le \mathcal{I}_*(\omega_i;t^*(\boldsymbol{v}_0,\Omega))\int_{\Gamma_-}\int_0^{t(\boldsymbol{b})}\left|\boldsymbol{\mathcal{Q}}_0\widehat{\boldsymbol{\xi}}(\tau,\boldsymbol{b})\right|^2\mathrm{d}\tau\mathrm{d}\sigma(\boldsymbol{b})
\end{aligned}$$

$$= \mathcal{I}_*(\omega_i; t^*(\boldsymbol{v}_0, \Omega)) \left\| \boldsymbol{\mathcal{Q}}_0 \widehat{\boldsymbol{\xi}} \right\|^2_{\mathrm{L}^2(\Theta,\mu)^d}.$$

Finally, using (3.33) and $\left\| \boldsymbol{\mathcal{Q}}_0 \widehat{\boldsymbol{\xi}} \right\|_{\mathrm{L}^2(\Theta,\mu)^d} = \|\boldsymbol{\mathcal{T}}_0(\omega)\boldsymbol{\xi}\|_{\mathrm{L}^2(\Omega,\rho_0)^d}$ (again by (3.26), (3.24) and $\widehat{\boldsymbol{\xi}} = \boldsymbol{\mathcal{S}}\boldsymbol{\xi}$):

$$\|\boldsymbol{\xi}\|^2_{\mathrm{L}^2(\Gamma_+,\rho_0(\boldsymbol{v}_0\cdot\boldsymbol{n}))} \leq C_+(\boldsymbol{v}_0, \Omega, \omega_i)^2 \, \|\boldsymbol{\mathcal{T}}_0(\omega)\boldsymbol{\xi}\|^2_{\mathrm{L}^2(\Omega,\rho_0)^d}.$$

□

We can now apply Theorem 3.16 to reduce the Goldstein problem ((2.14), (2.15)) to the "modified" convected Helmholtz problem (3.6) with unknown $\varphi \in V$ by giving a rigorous sense to

$$\boldsymbol{\mathcal{S}}_0(\omega; \boldsymbol{\nabla}\varphi) := \boldsymbol{\mathcal{T}}_0^{-1}(\boldsymbol{\omega}_0 \times \boldsymbol{\nabla}\varphi) \in \boldsymbol{H}, \quad \forall\, \varphi \in V, \tag{3.34}$$

since $\boldsymbol{\omega}_0 \times \boldsymbol{\nabla}\varphi \in \boldsymbol{L}$. We deduce in particular, from the estimates (3.18), the following continuity estimates

$$\begin{aligned}
&\forall\, \varphi \in V, \quad \|\boldsymbol{\mathcal{S}}_0(\omega; \boldsymbol{\nabla}\varphi)\|_{\mathrm{L}^2(\Omega,\rho_0)^d} \leq C_0(\boldsymbol{v}_0, \Omega, \omega_i)\, \|\boldsymbol{\omega}_0\|_{\mathrm{L}^\infty} \|\boldsymbol{\nabla}\varphi\|_{\mathrm{L}^2(\Omega,\rho_0)^d}, && \text{(i)}\\
&\forall\, \varphi \in V, \quad \|\boldsymbol{\mathcal{S}}_0(\omega; \boldsymbol{\nabla}\varphi)\|_{\mathrm{L}^2(\Gamma_+,\,\rho_0(\boldsymbol{v}_0\cdot\boldsymbol{n}))} \leq C_+(\boldsymbol{v}_0, \Omega, \omega_i)\, \|\boldsymbol{\omega}_0\|_{\mathrm{L}^\infty} \|\boldsymbol{\nabla}\varphi\|_{\mathrm{L}^2(\Omega,\rho_0)^d}. && \text{(ii)}
\end{aligned} \tag{3.35}$$

The next proposition gives the weak formulation of the reduced Goldstein's problem (3.6).

Proposition 3.20. *If the flow $\boldsymbol{v}_0$ is admissible and if $\operatorname{div}(\rho_0 \boldsymbol{v}_0) = 0$, the weak formulation of (3.6) reads, with $\ell(\psi)$ defined as in (3.8):*

$$\text{For } f \in \mathrm{L}^2(\Omega), \text{ find } \varphi \in V \text{ s.t. } \forall\, \psi \in V, \;\; a^g(\omega; \varphi, \psi) = \ell(\psi), \tag{3.36}$$

where the sesquilinear form $a^g(\omega, \varphi, \psi)$ is defined on V by:

$$a^g(\omega; \varphi, \psi) := a(\omega; \varphi, \psi) + d(\boldsymbol{\mathcal{S}}_0(\omega; \boldsymbol{\nabla}\varphi), \psi) \tag{3.37}$$

where $a(\omega; \varphi, \psi)$ and $d(\omega; \varphi, \psi)$ are the sesquilinear forms defined in (3.8) and (3.14).

Proof. It is a straightforward consequence of weak formulation ((2.14), (2.15)) of the Goldstein's problem. Indeed, (2.15) means, *via* Theorem 3.16, that $\boldsymbol{\xi} = \boldsymbol{\mathcal{S}}_0(\omega; \boldsymbol{\nabla}\varphi) \in \boldsymbol{H}$ that we substitute into (2.14) to get (3.36). □

3.4.3. Fredholm analytic property

This section is dedicated to show that the assumptions (1) and (2) of the Corollary 3.12 are satisfied for the operator $A^g(\omega) := A_r(\omega)$ defined in the preamble 3.4.1. The check of the assumption (1) is easy as the sesquilinear forms involved depend only polynomially on ω. We have the following proposition:

Proposition 3.21 (Analyticity)**.** *The map $\omega \in \mathbb{C} \mapsto A^g(\omega) \in \mathcal{L}(V)$ is analytic.*

Proof. Using the notations of the preamble, as the sesquilinear forms a and t defined respectively on $V \times V$ and $\boldsymbol{H} \times \boldsymbol{L}$ have a polynomial dependence in the variable $\omega \in \mathbb{C}$, the associated operators $A(\omega)$ and $\boldsymbol{T}(\omega)$ depends also polynomially, thus analytically, on ω. Then, by standard results, if $\omega \mapsto \boldsymbol{T}(\omega)$ is analytic and $\boldsymbol{T}(\omega)$ is invertible for all ω, the maps $\omega \mapsto \boldsymbol{T}(\omega)^{-1}$ is also analytic. We finally deduce the analyticity of the mapping $\omega \mapsto A^g(\omega) := A(\omega) - D\,\boldsymbol{T}(\omega)^{-1}\boldsymbol{B}$ as a sum and composition of analytic maps. □

We now investigate the Fredholmness of $A^g(\omega)$ to verify assumption (2). In the proof of Theorem 3.8, we proved the Fredholmness of the convected Helmholtz problem by artificially decomposing its sesquilinear form a as $a = a_* + c$ and showed that a_* is coercive and c is associated with a compact operator. To use a similar approach for a^g, it worths first to mention that the first part of the sesquilinear form $d(\boldsymbol{\mathcal{S}}_0(\omega;\boldsymbol{\nabla}\varphi),\psi)$ namely

$$\int_\Omega \rho_0\,\boldsymbol{\mathcal{S}}_0(\omega;\boldsymbol{\nabla}\varphi)\cdot\boldsymbol{\nabla}\overline{\psi}, \tag{3.38}$$

is not compact in $V = \mathrm{H}^1(\Omega)$. This is due to the fact that the operator $\mathcal{T}_0$ is regularizing along the streamlines (this is estimate (3.18)) but not along the transverse directions. As a consequence, equation (3.38) needs to be incorporated in the "coercive part" of the decomposition of a^g. This leads us to the following decomposition

$$\forall\,(\varphi,\psi)\in V\times V,\quad a^g(\omega;\varphi,\psi) = a^g_*(\omega;\varphi,\psi) + c^g(\omega;\varphi,\psi),$$
$$\text{where}\quad \begin{cases} a^g_*(\omega;\varphi,\psi) := a_*(\omega;\varphi,\psi) + \displaystyle\int_\Omega \rho_0\,\boldsymbol{\mathcal{S}}_0(\omega;\boldsymbol{\nabla}\varphi)\cdot\boldsymbol{\nabla}\overline{\psi},\\ c^g(\omega;\varphi,\psi) := c(\omega;\varphi,\psi) - \displaystyle\int_{\Gamma_+} \rho_0(\boldsymbol{M}_0\cdot\boldsymbol{n})\,\boldsymbol{\mathcal{S}}_0(\omega;\boldsymbol{\nabla}\varphi)\cdot\boldsymbol{n}\,\overline{\psi}. \end{cases}$$

Let $A_*(\omega), C(\omega), A^g_*(\omega), C^g(\omega)$ in $\mathcal{L}(V)$ be the operators defined, *via* Riesz theorem, by

$$\forall\,\varphi,\psi\in V,\quad \begin{cases} a_*(\omega;\varphi,\psi) = (A_*(\omega)\varphi,\psi)_V,\ a^g_*(\omega;\varphi,\psi) = (A^g_*(\omega)\varphi,\psi)_V,\\ c(\omega;\varphi,\psi) = (C(\omega)\varphi,\psi)_V,\quad c^g(\omega;\varphi,\psi) = (C^g(\omega)\varphi,\psi)_V. \end{cases}$$

Proposition 3.22 (Fredholmness)**.** *If the flow* $\boldsymbol{v}_0$ *is admissible* (Def. 3.5*, which includes in particular* (3.3)), *and satisfies* $\operatorname{div}(\rho_0\boldsymbol{v}_0) = 0$*, there exists* $\beta > 0$ *such that for all* $\omega\in\mathbb{C}^+_\beta$*:*

(i) $A^g_*(\omega)$ *is invertible,*
(ii) $C^g(\omega)$ *is a compact operator,*

so that $A^g(\omega)$ *is a Fredholm operator of index* 0*.*

Proof. (i) We have proved that $a_*(\omega;\cdot,\cdot)$ is coercive for real frequency ω, see (3.10). If $\omega := \omega_r + i\omega_i$ is a complex number (where $\omega_r,\omega_i\in\mathbb{R}$), we have, for $\varphi\in\mathrm{H}^1(\Omega)$,

$$|a_*(\omega,\varphi,\varphi)| \geq \Re e(a_*(\omega,\varphi,\varphi)) = \int_\Omega \rho_0\Big(|\boldsymbol{\nabla}\varphi|^2 - |\boldsymbol{M}_0\cdot\boldsymbol{\nabla}\varphi|^2 + |\varphi|^2\Big) + \omega_i\int_{\partial\Omega}\lambda\rho_0 c_0^{-1}|\varphi|^2.$$

As the additional term $a^g_*(\omega;\varphi,\varphi) - a_*(\omega,\varphi,\varphi)$ has *a priori* no sign, we simply bound it from below by minus its absolute value. Using Cauchy–Schwarz inequality, we get

$$\begin{aligned} |a^g_*(\omega;\varphi,\varphi)| &= \left|a_*(\omega;\varphi,\varphi) + \int_\Omega \rho_0\,\boldsymbol{\mathcal{S}}_0(\omega;\boldsymbol{\nabla}\varphi)\cdot\boldsymbol{\nabla}\overline{\varphi}\right|\\ &\geq |a_*(\omega;\varphi,\varphi)| - \|\boldsymbol{\mathcal{S}}_0(\omega;\boldsymbol{\nabla}\varphi)\|_{\mathrm{L}^2(\Omega,\rho_0)^d}\|\boldsymbol{\nabla}\varphi\|_{\mathrm{L}^2(\Omega,\rho_0)^d}, \end{aligned}$$

and thus,

$$|a^g_*(\omega;\varphi,\varphi)| \geq g(\omega_i)\,\|\varphi\|^2_{\mathrm{H}^1(\Omega,\rho_0)} + \omega_i\int_{\partial\Omega}\lambda\,\rho_0 c_0^{-1}|\varphi|^2,$$

where we have set

$$g(\omega_i) := 1 - \|\boldsymbol{M}_0\|^2_{\mathrm{L}^\infty} - \|\boldsymbol{\omega}_0\|_{\mathrm{L}^\infty}\,C_0(\boldsymbol{v}_0,\Omega,\omega_i). \tag{3.39}$$

As $\omega_i \mapsto C_0(\boldsymbol{v}_0, \Omega, \omega_i)$ is decreasing (Rem. 3.17), $\omega_i \mapsto g(\omega_i)$ is increasing and we can bound by below as follow:

$$|a_*^g(\omega; \varphi, \varphi)| \geq g(-\beta)\, \|\varphi\|^2_{\mathrm{H}^1(\Omega,\rho_0)} - \beta \int_{\partial\Omega} \lambda\, \rho_0 c_0^{-1} |\varphi|^2.$$

By boundedness of λ and c_0^{-1} and trace theorem, there exists $C > 0$ such that

$$\int_{\partial\Omega} \lambda\, \rho_0 c_0^{-1} |\varphi|^2 \leq C\, \|\varphi\|^2_{\mathrm{H}^1(\Omega,\rho_0)}.$$

It follows that

$$\forall\, \omega \in \mathbb{C}^+_\beta, \quad |a_*^g(\omega; \varphi, \varphi)| \geq [g(-\beta) - C\beta]\, \|\varphi\|^2_{\mathrm{H}^1(\Omega,\rho_0)}. \tag{3.40}$$

By a continuity argument (3.19), we see that

$$\lim_{\beta \to 0} \big[g(-\beta) - C\beta \big] = g(0),$$

where, by definition of $g(\omega_i)$, (3.39), and $C_0(\boldsymbol{v}_0, \Omega, \omega_i)$, (3.19),

$$g(0) = 1 - \|\boldsymbol{M}_0\|^2_{\mathrm{L}^\infty} - \|\boldsymbol{\omega}_0\|_{\mathrm{L}^\infty}\, t^*(\boldsymbol{v}_0, \Omega)\, \Phi(2t^*(\boldsymbol{v}_0, \Omega) \|\boldsymbol{\nabla} \boldsymbol{v}_0\|_{\mathrm{L}^\infty}) > 0,$$

where the strict inequality is nothing but (3.3). By continuity, we can find $\beta > 0$ (small enough) such that $g(-\beta) - C\beta > 0$. Thus, for such a β, equation (3.40) provides the coercivity of $a_*^g(\omega; \cdot, \cdot)$ for any $\omega \in \mathbb{C}^+_\beta$, uniformly with respect to $\omega \in \mathbb{C}^+_\beta$.

As a consequence, $A_*^g(\omega)$ is invertible for any $\omega \in \mathbb{C}^+_\beta$.

(ii) The compactness of the operator $C(\omega)$ associated to the sesquilinear form $c(\omega, \cdot, \cdot)$ has already been proved in the proof of Theorem 3.8. It remains to show the compactness of the operator $C^d(\omega) = C^g(\omega) - C(\omega)$ associated with the sesquilinear form

$$\varphi, \psi \in V, \quad c^d(\omega, \varphi, \psi) := \int_{\Gamma_+} \rho_0\, (\boldsymbol{M}_0 \cdot \boldsymbol{n})\, (\boldsymbol{\mathcal{S}}_0(\omega; \boldsymbol{\nabla}\varphi) \cdot \boldsymbol{n})\, \overline{\psi}.$$

This will rely on a sharp continuity estimate for $c^d(\omega, \varphi, \psi)$ which will provide more than the simple continuity of $C^d(\omega)$.

First, by Cauchy–Schwarz inequality, $\boldsymbol{M}_0 = c_0^{-1}\, \boldsymbol{v}_0$ and boundedness of various coefficients, we have, for some $C > 0$,

$$\big| c^d(\omega, \varphi, \psi) \big| \leq \big\| c_0^{-1} \big\|_{\mathrm{L}^\infty} \|(\boldsymbol{\mathcal{S}}_0(\omega; \boldsymbol{\nabla}\varphi) \cdot \boldsymbol{n})\|_{\mathrm{L}^2(\Gamma_+, \rho_0(\boldsymbol{v}_0 \cdot \boldsymbol{n}))}\, \|\psi\|_{\mathrm{L}^2(\Gamma_+,\, \rho_0(\boldsymbol{v}_0 \cdot \boldsymbol{n}))}.$$

Therefore, using the trace inequality (3.35)(ii), we have, with $\omega_i := \Im m\, \omega$,

$$\big| c^d(\omega, \varphi, \psi) \big| \leq C_+(\boldsymbol{v}_0, \Omega, \omega_i)\, \|\boldsymbol{\omega}_0\|_{\mathrm{L}^\infty} \|\boldsymbol{\nabla}\varphi\|_{\mathrm{L}^2(\Omega,\rho_0)^d} \|\psi\|_{\mathrm{L}^2(\Gamma_+,\, \rho_0(\boldsymbol{v}_0 \cdot \boldsymbol{n}))}$$

and the compactness of $C^d(\omega)$ is a consequence of the compactness of the map $\psi \in V \mapsto \psi_{|\Gamma_+} \in \mathrm{L}^2(\Gamma_+)$ (or the compact embedding of $\mathrm{H}^{\frac{1}{2}}(\Gamma_+)$ in $\mathrm{L}^2(\Gamma_+)$), in a same fashion as the compactness of the operator C associated with c in the proof of Theroem 3.8.

□

3.4.4. Invertibility at one frequency $\omega_\diamond$

To fulfill the conditions of the Corollary 3.12, it remains to verify that the operator $A^g(\omega)$, is invertible at one frequency $\omega_\diamond$ in the domain $\mathbb{C}^*_\beta$ where the problem has been shown to be of Fredholm type in the previous section. This value will be found on the positive imaginary semi-axis provided that the imaginary part of $\omega_\diamond$ is large enough.

Remark 3.23. The idea of looking at frequencies with a positive imaginary part $\omega + i\varepsilon$, $\varepsilon > 0$ is classical in particular for the standard Helmholtz equation: it corresponds to the limiting absorption procedure. Looking at $\omega + i\varepsilon$ can be interpreted physically as adding absorption in the original model. For the standard Helmholtz equation, or for the convected Helmholtz equation in a subsonic flow, coercivity is recovered for any $\varepsilon > 0$, even arbitrarily small. This will not be the case here for the Goldstein's problem, due to the coupling with the transport equation: we shall need the absorption ε to be large enough. This will appear as a technical necessity but this is also physically meaningful: in aeroacoustics the possible presence of hydrodynamic instabilities require a large enough absorption to be counterbalanced.

Proposition 3.24. *For a strictly subsonic* (3.9)*, a* Ω*-filling flow* $\boldsymbol{v}_0$ *such that* $\operatorname{div}(\rho_0\boldsymbol{v}_0) = 0$*, and for* $\omega_\diamond = i\,\omega_i$ *with* $\omega_i > 0$ *large enough (obviously* $\omega^* \in \mathbb{C}^+_\beta$*),* $A^g(\omega_\diamond)$ *is invertible.*

Proof. Of course, it suffices to prove the coercivity of $a^g(i\omega_i, \cdot, \cdot)$ for ω_i large enough. Let us begin with $a(i\omega; \varphi, \varphi)$ (associated to the convected Helmholtz's problem). Noticing that

$$\Re e(\,\varphi(\boldsymbol{M}_0 \cdot \boldsymbol{\nabla}\overline{\varphi}) - (\boldsymbol{M}_0 \cdot \boldsymbol{\nabla}\varphi)\overline{\varphi}\,) = 0,$$

we see that

$$\Re e\,(a(i\omega_i; \varphi, \varphi)) = \int_\Omega \rho_0 \left(|\boldsymbol{\nabla}\varphi|^2 - |\boldsymbol{M}_0 \cdot \boldsymbol{\nabla}\varphi|^2 + \frac{\omega_i^2}{c_0^2}\,|\varphi|^2 \right) + \omega_i \int_{\partial\Omega} \rho_0\, c_0^{-1} \lambda\, |\varphi|^2.$$

Then, using $\lambda \geq 0$ on $\partial\Omega$, one gets:

$$\Re e\,(a(i\omega_i; \varphi, \varphi)) \geq \alpha_*(\omega_i) \|\varphi\|^2_{\mathrm{H}^1(\Omega,\rho_0)}, \quad \alpha_*(\omega_i) := \min\Big(1 - \|\boldsymbol{M}_0\|^2_{\mathrm{L}^\infty}, \omega_i^2\, \|c_0\|^{-2}_{\mathrm{L}^\infty}\Big). \tag{3.41}$$

Next, we treat the remaining part (3.37) of a^g, namely (see (3.37))

$$\Re e\,(d(\boldsymbol{\mathcal{S}}_0(i\omega_i; \boldsymbol{\nabla}\varphi), \psi)) = \Re e\bigg(\int_\Omega \rho_0\, \boldsymbol{\mathcal{S}}_0(i\omega_i; \boldsymbol{\nabla}\varphi) \cdot \boldsymbol{\nabla}\overline{\varphi} \bigg) - \Re e\bigg(\int_{\Gamma_+} \rho_0 (\boldsymbol{M}_0 \cdot \boldsymbol{n})(\boldsymbol{\mathcal{S}}_0(i\omega_i; \boldsymbol{\nabla}\varphi) \cdot \boldsymbol{n})\, \overline{\varphi} \bigg).$$

First, by Cauchy–Schwarz,

$$\left| \int_\Omega \rho_0\, \boldsymbol{\mathcal{S}}_0(i\omega_i; \boldsymbol{\nabla}\varphi) \cdot \boldsymbol{\nabla}\overline{\varphi} \right| \;\leq \|\boldsymbol{\mathcal{S}}_0(i\omega_i; \boldsymbol{\nabla}\varphi)\|_{\mathrm{L}^2(\Omega,\rho_0)^d} \|\boldsymbol{\nabla}\varphi\|_{\mathrm{L}^2(\Omega,\rho_0)^d},$$

so that, using (3.35)(i),

$$\left| \int_\Omega \rho_0\, \boldsymbol{\mathcal{S}}_0(i\omega_i; \boldsymbol{\nabla}\varphi) \cdot \boldsymbol{\nabla}\overline{\varphi} \right| \;\leq\; C_0(\boldsymbol{v}_0, \Omega, i\omega_i)\, \|\boldsymbol{\omega}_0\|_{\mathrm{L}^\infty} \|\boldsymbol{\nabla}\varphi\|^2_{\mathrm{L}^2(\Omega,\rho_0)^d}. \tag{3.42}$$

In a similar way, using again Cauchy–Schwarz, $\boldsymbol{M}_0 = c_0^{-1}\, \boldsymbol{v}_0$ and the boundedness of ρ_0 and c_0^{-1}, we get:

$$\left| \int_{\Gamma_+} \rho_0 (\boldsymbol{M}_0 \cdot \boldsymbol{n})(\boldsymbol{\mathcal{S}}_0(i\omega_i; \boldsymbol{\nabla}\varphi) \cdot \boldsymbol{n})\, \overline{\varphi} \right| \leq \left\|c_0^{-1}\right\|_{\mathrm{L}^\infty} \|(\boldsymbol{\mathcal{S}}_0(i\omega_i; \boldsymbol{\nabla}\varphi) \cdot \boldsymbol{n})\|_{\mathrm{L}^2(\Gamma_+, \rho_0(\boldsymbol{v}_0 \cdot \boldsymbol{n}))}\, \|\varphi\|_{\mathrm{L}^2(\Gamma_+, \rho_0(\boldsymbol{v}_0 \cdot \boldsymbol{n}))}.$$

Thus, introducing the non dimensionless constant

$$C_{tr}(\rho_0, \boldsymbol{v}_0, \Omega) := \sup_{\varphi \in \mathrm{H}^1(\Omega)} \frac{\|\varphi\|_{\mathrm{L}^2(\Gamma_+,\, \rho_0(\boldsymbol{v}_0 \cdot \boldsymbol{n}))}}{\|\varphi\|_{\mathrm{H}^1(\Omega,\rho_0)}} < +\infty \quad \text{(by trace theorem)},$$

and using (3.35)(ii), we conclude that

$$\left| \int_{\Gamma_+} \rho_0 (\boldsymbol{M}_0 \cdot \boldsymbol{n})(\boldsymbol{\mathcal{S}}_0(\omega; \boldsymbol{\nabla}\varphi) \cdot \boldsymbol{n}) \overline{\varphi} \right| \leq \left\|c_0^{-1}\right\|_{\mathrm{L}^\infty} \|\boldsymbol{\omega}_0\|_{\mathrm{L}^\infty}\, C_{tr}(\rho_0, \boldsymbol{v}_0)\; C_+(\boldsymbol{v}_0, \Omega, \omega_i) \|\varphi\|^2_{\mathrm{H}^1(\Omega,\rho_0)}. \tag{3.43}$$

Finally, regrouping (3.41)–(3.43) in (3.37) and setting

$$\alpha(\omega_i) := \alpha_*(\omega_i) - \left[C_0(\boldsymbol{v}_0, \Omega, \omega_i) + \left\|c_0^{-1}\right\|_{\mathrm{L}^\infty} C_{tr}(\rho_0, \boldsymbol{v}_0)\, C_+(\boldsymbol{v}_0, \Omega, \omega_i)\right] \|\boldsymbol{\omega}_0\|_{\mathrm{L}^\infty},$$

we obtain the lower bound

$$\Re e\left(a^g(i\omega_i; \varphi, \varphi)\right) \geq \alpha(\omega_i) \|\varphi\|^2_{\mathrm{H}^1(\Omega,\rho_0)}.$$

As $C_0(\boldsymbol{v}_0, \Omega, i\omega_i)$ and $C_+(\boldsymbol{v}_0, \Omega, i\omega_i)$ tends to 0 when $\omega_i \to +\infty$, *cf.* Remark 3.17, and since $\alpha_*(\omega_i) = 1 - \|\boldsymbol{M}_0\|^2_{\mathrm{L}^\infty}$, for ω_i large enough (see (3.41)), we have

$$\lim_{\omega_i \to +\infty} \alpha(i\omega_i) = 1 - \|\boldsymbol{M}_0\|^2_{\mathrm{L}^\infty} > 0. \tag{3.44}$$

It is then easy to conclude. □

Remark 3.25. It is worthwhile noticing that Proposition 3.24 does not require that the flow is admissible. Only the Ω-filling property, for defining $\boldsymbol{\mathcal{S}}_0(i\omega_i; \boldsymbol{\nabla}\varphi)$, and the fact that the flow is subsonic, for (3.44), are needed. This is explained by the fact that the hydrodynamic effects, due to the unknown $\boldsymbol{\xi}$, are killed at high absorption as shown by Theorem 3.16.

3.4.5. End of the main result (Thm. 3.6) proof

Since we have gathered all the Corollary 3.12 conditions, there exists a subset $\mathcal{R}^{\mathbb{C}}_{ex} \subset \mathbb{C}^+_\beta$, with no limit point in $\mathbb{C}^+_\beta$, such that for all frequencies $\omega \in \mathbb{C}^+_\beta \setminus \mathcal{R}^{\mathbb{C}}_{ex}$, the reduced Goldstein's problem (3.6) is well-posed and so is the Goldstein's problem ((2.14), (2.15)). We deduce that the same is true for all frequencies $\omega \in \mathbb{R} \setminus \mathcal{R}_{ex}$, where the set $\mathcal{R}_{ex} := \mathcal{R}^{\mathbb{C}}_{ex} \cap \mathbb{R}$ has no limit point in $\mathbb{R}$, which ends the proof of the main result.

3.5. On the admissibility condition as a low vorticity condition

In this section, essentially for technical reasons, we suppose that $d = 2$ that allows us to use the simple characterization (3.1) of Ω-filling flows. However, we conjecture that the content of this section remains valid for $d = 3$.

We wish to reinterpret the condition (3.3) as imposing on the mean flow to be subsonic, as in the potential case, but also to be of low vorticity and not more. In this aim, we consider a family of flows parameterized by a real $\eta \in \mathbb{R}$, $(\rho_0^\eta, p_0^\eta, \boldsymbol{v}_0^\eta)$ whose velocity flows are perturbation of a (strictly) subsonic potential flow. More precisely, these are of the form:

$$\boldsymbol{v}_0^\eta := \boldsymbol{v}_0 + \eta\, \boldsymbol{w}_0, \quad \boldsymbol{v}_0 = \boldsymbol{\nabla}\varphi_0, \quad \|\boldsymbol{\nabla} \times \boldsymbol{w}_0\|_{\mathrm{L}^\infty} = 1, \tag{3.45}$$

where $\boldsymbol{w}_0$ is $\mathrm{C}^1\left(\mathbb{R}^d, \mathbb{R}^d\right)$ and we assume that the acoustic velocity field c_0^η satisfies

$$\lim_{\eta \to 0} \|c_0^\eta - c_0\|_{\mathrm{L}^\infty} = 0.$$

The vorticity of these flows is proportional to η (in such a way that, for these flows "small vorticity" is equivalent to "small η"):

$$\boldsymbol{\omega}_0^\eta := \boldsymbol{\nabla} \times (\boldsymbol{\nabla}\varphi_0 + \eta\, \boldsymbol{w}_0) = \eta\, \boldsymbol{\nabla} \times \boldsymbol{w}_0 \quad \Longrightarrow \quad \|\boldsymbol{\omega}_0^\eta\|_{\mathrm{L}^\infty} = |\eta|.$$

We also assume that $\boldsymbol{w}_0$ satisfies

$$\boldsymbol{w}_0 \cdot \boldsymbol{n} = 0 \quad \text{on } \partial\Omega, \tag{3.46}$$

(which is for instance the case if $\operatorname{supp}(\boldsymbol{w}_0) \subset \Omega$), such that the inflow and outflow boundaries (*cf.* definitions (2.3)) are independent of η, *i.e.* with the notations of Remark 2.2:

$$\forall\, \eta, \quad \Gamma_\pm(\boldsymbol{v}_0^\eta) = \Gamma_\pm(\boldsymbol{v}_0) \equiv \Gamma_\pm.$$

Remark 3.26. We think that the assumption is not really essential but it suppresses many tedious difficulties in the proof of our results.

Theorem 3.27. *Let $\boldsymbol{v}_0$ strictly be subsonic and Ω filling and $\boldsymbol{v}_0^\eta$ be defined by (3.45) where the vector field $\boldsymbol{w}_0$ satisfies (3.46). Then, there is a threshold $\eta^* > 0$ such that for all $\eta \in]-\eta^*, \eta^*[$, the flow $\boldsymbol{v}_0^\eta$ is admissible in the sense of Definition 3.5 and, therefore, so that the Goldstein's problem ((2.14), (2.15)) associated with the flow $(\rho_0^\eta, p_0^\eta, \boldsymbol{v}_0^\eta)$ is well posed in the sense of* Theorem 3.6.

The proof relies on the following technical lemma (whose proof is given in appendix) that provides a uniform control (in η) of the lifetimes of the flows $\boldsymbol{v}_0^\eta$

Lemma 3.28. *There exists $\eta^* > 0$ and $T^* > 0$ such that:*

$$\forall\, \eta \in (-\eta^*, \eta^*), \quad t^*(\boldsymbol{v}_0^\eta, \Omega) \leq T^*.$$

Proof of Theorem 3.27. One first remarks that, using the 2D characterization (3.1) of the Ω-filling property, for η small enough, as $\inf_\Omega |\boldsymbol{v}_0^\eta| > 0$, $\boldsymbol{v}_0^\eta$ is Ω-filling.

Then, if we denote $\boldsymbol{M}_0^\eta := \boldsymbol{v}_0^\eta / c_0^\eta$ the Mach flow of the perturbed flow, with our assumptions, $\boldsymbol{M}_0^\eta$ converges uniformly in Ω towards $\boldsymbol{M}_0 = \boldsymbol{v}_0/c_0$ and in particular $\|\boldsymbol{M}_0^\eta\|_{\mathrm{L}^\infty} \to \|\boldsymbol{M}_0\|_{\mathrm{L}^\infty}$ when $\eta \to 0$. In a same fashion, one easily sees that $\|\boldsymbol{\nabla} \boldsymbol{v}_0^\eta\|_{\mathrm{L}^\infty} \to \|\boldsymbol{\nabla} \boldsymbol{v}_0\|_{\mathrm{L}^\infty}$ when $\eta \to 0$. Finally, as a consequence of Lemma 3.28, as $t^*(\boldsymbol{v}_0^\eta, \Omega)\Phi(2t^*(\boldsymbol{v}_0^\eta, \Omega)\|\boldsymbol{\nabla} \boldsymbol{v}_0^\eta\|_{\mathrm{L}^\infty})$ remains bounded when $\eta \to 0$ and as $\|\boldsymbol{\omega}_0^\eta\|_{\mathrm{L}^\infty} = |\eta| \to 0$, one gets:

$$\lim_{\eta \to 0} \Big[1 - \|\boldsymbol{M}_0^\eta\|_{\mathrm{L}^\infty}^2 - \|\boldsymbol{\omega}_0^\eta\|_{\mathrm{L}^\infty}\, t^*(\boldsymbol{v}_0^\eta, \Omega)\, \Phi\big(2\, t^*(\boldsymbol{v}_0^\eta, \Omega) \|\boldsymbol{\nabla} \boldsymbol{v}_0^\eta\|_{\mathrm{L}^\infty}\big) \Big] = 1 - \|\boldsymbol{M}_0^\eta\|_{\mathrm{L}^\infty}^2 > 0.$$

Thus $\boldsymbol{v}_0^\eta$ is admissible for η small enough. One concludes with Theorem 3.6. □

Appendix A. A first order absorbing boundary condition

In this appendix, we explain the construction of the absorbing boundary condition (2.11) of Section 2.3.1, as an approximate transparent condition.

In what follows, we denote Ω_{ext} the exterior domain as the unbounded connected component of $\mathbb{R}^d \setminus \Gamma$, so that in particular $\Gamma = \partial\Omega_{\mathrm{ext}}$. Ideally, a transparent boundary condition on $\partial\Omega$ would result from a generalized Dirichlet-to-Neumann (DtN) operator acting on a couple of boundary data

$$T : \big(\varphi^0, \boldsymbol{\xi}^0\big) : \Gamma \times \Gamma_+ \to \mathbb{C} \times \mathbb{C}^d,$$

and is related to the notion of generalized Neumann trace defined in (2.16):

$$T\big(\varphi^0, \boldsymbol{\xi}^0\big) := -\big(\boldsymbol{\nabla}\phi\big(\varphi^0, \boldsymbol{\xi}^0\big) + \boldsymbol{\zeta}\big(\varphi^0, \boldsymbol{\xi}^0\big)\big) \cdot \boldsymbol{n} + c_0^{-1}(\boldsymbol{M}_0 \cdot \boldsymbol{n}) D_\omega \phi\big(\varphi^0, \boldsymbol{\xi}^0\big), \tag{A.1}$$

where $\phi := \phi\big(\varphi^0, \boldsymbol{\xi}^0\big)$ and $\boldsymbol{\zeta} := \zeta\big(\varphi^0, \boldsymbol{\xi}^0\big)$ are solution of the exterior Dirichlet boundary value problem (posed in Ω_{ext}) associated to the convected Helmholtz equation with constant coefficients $(c_0, \boldsymbol{M}_0)$ (the exterior medium is homogeneous with a uniform flow):

$$\begin{cases} \left(-i\dfrac{\omega}{c_0} + \boldsymbol{M}_0 \cdot \boldsymbol{\nabla}\right)^2 \phi - \Delta\phi - \boldsymbol{\nabla} \cdot \boldsymbol{\zeta} = 0, & \text{in } \Omega_{\mathrm{ext}} \\ \left(-i\dfrac{\omega}{c_0} + \boldsymbol{M}_0 \cdot \boldsymbol{\nabla}\right)\boldsymbol{\zeta} = \boldsymbol{0}, & \text{in } \Omega_{\mathrm{ext}} \\ \phi = \varphi_0 \quad \text{on } \Gamma, \quad \boldsymbol{\zeta} = \boldsymbol{\xi}_0 & \text{on } \Gamma_+. \end{cases} \tag{A.2}$$

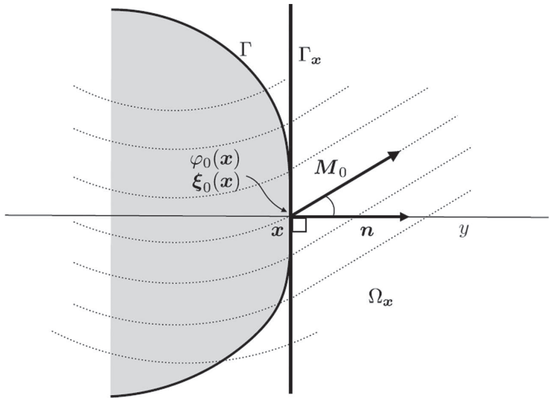

FIGURE A.1. Illustration of the local 1D problem geometry around $\boldsymbol{x} \in \Gamma_+$.

Note that the outflow boundary Γ_+ becomes the inflow boundary for the exterior domain reason why we need to prescribe $\boldsymbol{\zeta}$ on Γ_+ (instead on Γ_+ for the interior problem, *cf.* Sect. 2.3). Also note that the transport equation for $\boldsymbol{\zeta}$ is completely decoupled from ϕ because the vorticity $\boldsymbol{\omega}_0$ is $\mathbf{0}$ in the exterior domain Ω_{ext}.

Once T is known, a transparent boundary condition for the interior problem in $(\varphi, \boldsymbol{\xi})$ is

$$\rho_0(\nabla\varphi + \boldsymbol{\xi}) \cdot \boldsymbol{n} - \rho_0 c_0^{-1}(\boldsymbol{M}_0 \cdot \boldsymbol{n}) D_\omega \varphi + \rho_0\, T(\varphi, \boldsymbol{\xi}) = 0 \quad \text{on } \Gamma. \tag{A.3}$$

The above operator T in non local along Γ and cannot be computed in practice. That is why or goal in this section will be to construct a "local approximation" based on the usual idea that works for the Dirichlet-to-Neumann map which occurs in Helmholtz equation boundary conditions study. More precisely, this consists in considering the problem in a small neighborhood of a point $\boldsymbol{x} \in \Gamma$ with outgoing normal vector $\boldsymbol{n}$ and

– assimilate locally the boundary Γ to the tangent plane $\Gamma_{\boldsymbol{x}}$ and the exterior domain to the half-space $\Omega_{\boldsymbol{x}} := \{\boldsymbol{y}/(\boldsymbol{y} - \boldsymbol{x}) \cdot \boldsymbol{n} > 0\}$,
– consider that around Γ, the boundary data are constant (in other words their lateral variations are neglected).

This leads to consider the ($\boldsymbol{x}$-dependent) half-space problem (A.2) in which Ω_{ext} is replaced by $\Omega_{\boldsymbol{x}}$ and the boundary data are constant along $\Gamma_{\boldsymbol{x}}$ as shown in Figure A.1. *i.e.* $\varphi_0 \equiv \varphi_0(\boldsymbol{x})$ and $\boldsymbol{\xi}_0 \equiv \boldsymbol{\xi}_0(\boldsymbol{x})$. By translational invariance along $\Gamma_{\boldsymbol{x}}$ we deduce that the solution $(\phi_{\boldsymbol{x}}, \boldsymbol{\zeta}_{\boldsymbol{x}})$ is a 1D function of the space coordinate normal to $\Gamma_{\boldsymbol{x}}$. If we call y this variable, so that $\Omega_{\boldsymbol{x}}$ corresponds to $y > 0$ and $\Gamma_{\boldsymbol{x}}$ corresponds to $y = 0$, the 1D problems rewrites

$$\begin{cases} \left(-i\dfrac{\omega}{c_0} + (\boldsymbol{M}_0 \cdot \boldsymbol{n})\dfrac{\mathrm{d}}{\mathrm{d}y}\right)^2 \phi_{\boldsymbol{x}} - \dfrac{\mathrm{d}^2\phi_{\boldsymbol{x}}}{\mathrm{d}y^2} - \dfrac{\mathrm{d}}{\mathrm{d}y}(\boldsymbol{\zeta}_{\boldsymbol{x}} \cdot \boldsymbol{n}) = 0, & y > 0, \\ \left(-i\dfrac{\omega}{c_0} + (\boldsymbol{M}_0 \cdot \boldsymbol{n})\dfrac{\mathrm{d}}{\mathrm{d}y}\right)\boldsymbol{\zeta}_{\boldsymbol{x}} = \mathbf{0}, & y > 0, \\ \phi_{\boldsymbol{x}}(0) = \varphi_0(\boldsymbol{x}) \quad \text{on } \Gamma, & \boldsymbol{\zeta}_{\boldsymbol{x}}(0) = \boldsymbol{\xi}_0(\boldsymbol{x}) \quad \text{on } \Gamma_+. \end{cases} \tag{A.4}$$

Note that the above is valid for any point $\boldsymbol{x}$ along the boundary Γ with the particularity that $\boldsymbol{\xi}_0(\boldsymbol{x}) = 0$ if $\boldsymbol{x} \notin \Gamma_+$. From the transport equation we deduce that

$$\boldsymbol{\zeta}_{\boldsymbol{x}}(y) = \boldsymbol{\xi}_0(\boldsymbol{x})\, e^{i\,\frac{\omega}{c_0}\,\frac{y}{\boldsymbol{M}_0\cdot\boldsymbol{n}}} \tag{A.5}$$

that we substitute into the equation for $\phi_{\boldsymbol{x}}$, which gives

$$\left(-i\frac{\omega}{c_0} + (\boldsymbol{M}_0 \cdot \boldsymbol{n})\frac{\mathrm{d}}{\mathrm{d}y}\right)^2 \phi_{\boldsymbol{x}} - \frac{\mathrm{d}^2\phi_{\boldsymbol{x}}}{\mathrm{d}y^2} = i\,\frac{\omega}{c_0}\left(\frac{\boldsymbol{\xi}_0(\boldsymbol{x}) \cdot \boldsymbol{n}}{\boldsymbol{M}_0 \cdot \boldsymbol{n}}\right) e^{i\,\frac{\omega}{c_0}\,\frac{y}{\boldsymbol{M}_0\cdot\boldsymbol{n}}}. \tag{A.6}$$

The solution is the sum of a particular solution $\phi_p + \phi_h$ where ϕ_h is a solution of the homogeneous equation. Of course ϕ_p is of the form

$$\phi_p(y) = \phi_{0,p}\, e^{i\,\frac{\omega}{c_0}\,\frac{y}{\boldsymbol{M}_0\cdot\boldsymbol{n}}},$$

where, noticing that the first term of the left hand side of the equation (A.6) vanishes for $\phi_{\boldsymbol{x}} = \phi_p$, one computes that

$$\phi_{0,p} = i\,\frac{c_0}{\omega}\,(\boldsymbol{M}_0\cdot\boldsymbol{n})\,(\boldsymbol{\xi}_0(\boldsymbol{x})\cdot\boldsymbol{n}).$$

On the other hand, we know that ϕ_h is of the form

$$\phi_h(y) = A^+\,e^{i\,k^+(\omega)\,y} + A^-\,e^{i\,k^-(\omega)\,y},$$

where $(k^+(\omega), k^-(\omega))$ are the two solutions of the quadratic dispersion relation in k:

$$k^2 = \left(\frac{\omega}{c_0} - (\boldsymbol{M}_0\cdot\boldsymbol{n})\,k\right)^2,$$

that is to say, taking into account that the the flow is subsonic ($|\boldsymbol{M}_0| < 1$),

$$k^+(\omega) = \frac{\omega}{c_0}\,(1+\boldsymbol{M}_0\cdot\boldsymbol{n})^{-1} > 0, \quad k^-(\omega) = -\frac{\omega}{c_0}\,(1-\boldsymbol{M}_0\cdot\boldsymbol{n})^{-1} < 0.$$

Moreover, we shall retain only in the homogeneous solution, the one that is going at infinity (this can be fully justified by limiting absorption, we omit the details) which yields $A^- = 0$. Therefore $A^+ = \varphi_0(\boldsymbol{x}) - \phi_{0,p}$ in order that $\phi_{\boldsymbol{x}}(0) = \varphi_0(\boldsymbol{x})$. This leads to

$$\phi_{\boldsymbol{x}}(y) = \varphi_0(\boldsymbol{x})\,e^{i\,\frac{\omega}{c_0}\,\frac{x}{1+\boldsymbol{M}_0\cdot\boldsymbol{n}}} + i\,\frac{c_0}{\omega}(\boldsymbol{M}_0\cdot\boldsymbol{n})(\boldsymbol{\xi}_0(\boldsymbol{x})\cdot\boldsymbol{n})\Big[e^{i\,\frac{\omega}{c_0}\,\frac{y}{\boldsymbol{M}_0\cdot\boldsymbol{n}}} - e^{i\,\frac{\omega}{c_0}\,\frac{y}{1+\boldsymbol{M}_0\cdot\boldsymbol{n}}}\Big]. \tag{A.7}$$

We then define the approximate operator, consistently with the 1D approximate solution $(\phi_{\boldsymbol{x}}, \boldsymbol{\zeta}^{\boldsymbol{x}})$ (A.1), as

$$\begin{aligned} -T_{ap}\,(\varphi_0,\boldsymbol{\xi}_0)(\boldsymbol{x}) &= \phi'_{\boldsymbol{x}}(0) + \boldsymbol{\zeta}^{\boldsymbol{x}}(0)\cdot\boldsymbol{n} + (\boldsymbol{M}_0\cdot\boldsymbol{n})\big(i\omega\,c_0^{-1}\,\phi_{\boldsymbol{x}} - (\boldsymbol{M}_0\cdot\boldsymbol{n})\phi'_{\boldsymbol{x}}\big)(0) \\ &= \big[1-(\boldsymbol{M}_0\cdot\boldsymbol{n})^2\big]\phi'_{\boldsymbol{x}}(0) + i\omega\,c_0^{-1}(\boldsymbol{M}_0\cdot\boldsymbol{n})\phi_{\boldsymbol{x}}(0) + \boldsymbol{\zeta}^{\boldsymbol{x}}(0)\cdot\boldsymbol{n}. \end{aligned} \tag{A.8}$$

By (A.4), $\phi_{\boldsymbol{x}}(0) = \varphi_0(\boldsymbol{x})$ and $\boldsymbol{\zeta}^{\boldsymbol{x}}(0) = \boldsymbol{\xi}_0(\boldsymbol{x})$, thus

$$i\omega\,c_0^{-1}(\boldsymbol{M}_0\cdot\boldsymbol{n})\,\phi_{\boldsymbol{x}}(0) + \boldsymbol{\zeta}^{\boldsymbol{x}}(0)\cdot\boldsymbol{n} = i\omega\,c_0^{-1}(\boldsymbol{M}_0\cdot\boldsymbol{n})\,\varphi_0(\boldsymbol{x}) + \boldsymbol{\xi}_0(\boldsymbol{x})\cdot\boldsymbol{n}, \tag{A.9}$$

while, from (A.7), one gets $(\phi_{\boldsymbol{x}})'(0) = (1+\boldsymbol{M}_0\cdot\boldsymbol{n})^{-1}\big(i\,\omega\,c_0^{-1}\varphi_0(\boldsymbol{x}) - \boldsymbol{\xi}_0(\boldsymbol{x})\cdot\boldsymbol{n}\big)$, thus

$$\big[1-(\boldsymbol{M}_0\cdot\boldsymbol{n})^2\big](\phi_{\boldsymbol{x}})'(0) = (1-\boldsymbol{M}_0\cdot\boldsymbol{n})\,\big(i\,\omega\,c_0^{-1}\varphi_0(\boldsymbol{x}) - \boldsymbol{\xi}_0(\boldsymbol{x})\cdot\boldsymbol{n}\big). \tag{A.10}$$

Then, substituting (A.9) and (A.10) into (A.8) gives which, using ((A.5), (A.7)), gives

$$T_{ap}\,(\varphi_0,\boldsymbol{\xi}_0)(\boldsymbol{x}) = -\,i\,\frac{\omega}{c_0}\varphi_0(\boldsymbol{x}) - \boldsymbol{\xi}_0(\boldsymbol{x})\cdot\boldsymbol{n}.$$

Finally, changing T into T_{ap} in the transparent condition (A.3) leads to the desired absorbing condition (2.11).

Appendix B. Proof of the technical Lemma 3.28

Let us first recall the statement of this lemma:

Lemma B.1. *There exists $\eta^* > 0$ and $T^* > 0$ such that:*

$$\forall\, \eta \in (-\eta^*, \eta^*), \quad t^*(\boldsymbol{v}_0^\eta, \Omega) \le T^*.$$

Proof. In the same fashion that $\boldsymbol{\chi}$ denotes the characteristics of the flow $\boldsymbol{v}_0$, let us introduce $\boldsymbol{\chi}^\eta$ the characteristics of the perturbed flow $\boldsymbol{v}_0^\eta$, *i.e.* $\boldsymbol{\chi}^\eta : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$ solution of the following system of ordinary differential equations:

$$\begin{cases} \dfrac{\partial_t \boldsymbol{\chi}^\eta}{\partial t}(t, \boldsymbol{x}) = \boldsymbol{v}_0^\eta(\boldsymbol{\chi}^\eta(t, \boldsymbol{x})), \\ \boldsymbol{\chi}^\eta(0, \boldsymbol{x}) = \boldsymbol{x}, \end{cases} \quad \text{for all} \quad (t, \boldsymbol{x}) \in \mathbb{R} \times \mathbb{R}^d.$$

The proof consists of the following steps for which a justification will be given:

(1) General theorems of ODE theory provide that the mappings $(t, \boldsymbol{x}) \mapsto \boldsymbol{\chi}^\eta(t, \boldsymbol{x})$ converge uniformly on any compact of $\mathbb{R} \times \mathbb{R}^d$ when $\eta \to 0$. However, in our case, we can merely get a stronger result by establishing the following estimate:

$$\forall\, \eta \in \mathbb{R},\ \forall\, (t, \boldsymbol{x}) \in \mathbb{R} \times \mathbb{R}^d, \quad |\boldsymbol{\chi}^\eta(t, \boldsymbol{x}) - \boldsymbol{\chi}(t, \boldsymbol{x})| \le |\eta| \big(t + t^2 \|\boldsymbol{\nabla} \boldsymbol{v}_0\|_{\mathrm{L}^\infty} e^{t\, \|\boldsymbol{\nabla} \boldsymbol{v}_0\|_{\mathrm{L}^\infty}}\big) \|\boldsymbol{w}_0\|_{\mathrm{L}^\infty}.$$

(2) There exists a small enough $\varepsilon > 0$ such that the flow $\boldsymbol{v}_0$ still fills the extended domain $\Omega^\varepsilon := \Omega + B(0, \varepsilon)$ (*i.e.* $\boldsymbol{v}_0$ is Ω^ε-filling). Thus, the exit time of Ω^ε for the flow $\boldsymbol{v}_0$ is well-defined by:

$$t^*(\boldsymbol{v}_0, \Omega^\varepsilon) := \sup_{\boldsymbol{x} \in \Omega^\varepsilon} t^*(\boldsymbol{x}, \boldsymbol{v}_0, \Omega^\varepsilon) < +\infty,$$

where the exit time starting from $\boldsymbol{x} \in \Omega^\varepsilon$ is defined by:

$$t^*(\boldsymbol{x}, \boldsymbol{v}_0, \Omega^\varepsilon) := \sup\{t \ge 0 \ /\ \forall\, \tau \in [0, t),\ \ \boldsymbol{\chi}(t, \boldsymbol{x}) \in \Omega^\varepsilon\}.$$

(3) There exists $\eta^* > 0$ such that:

$$\forall\, \eta \in (-\eta^*, \eta^*), \quad t^*(\boldsymbol{v}_0^\eta, \Omega) \le t^*(\boldsymbol{v}_0, \Omega^\varepsilon),$$

which provides the expected bound with $T^* := t^*(\boldsymbol{v}_0, \Omega^\varepsilon)$.

The geometry of the problem is represented on the Figure B.1 where the streamlines of the flow, its characteristics and the perturbation area are shown. Let us justify each of the steps (1), (2) and (3).

(1) The inequality follows a Gronwall estimation of the difference $\boldsymbol{\chi}^\eta(t, \boldsymbol{x}) - \boldsymbol{\chi}(t, \boldsymbol{x})$ by using the ODE and the fact that $\boldsymbol{v}_0^\eta := \boldsymbol{v}_0 + \eta\, \boldsymbol{w}_0$. More precisely, for $(t, \boldsymbol{x}) \in \mathbb{R}^+ \times \mathbb{R}^d$, we have:

$$\begin{aligned} \boldsymbol{\chi}^\eta(t, \boldsymbol{x}) - \boldsymbol{\chi}(t, \boldsymbol{x}) &= \boldsymbol{\chi}^\eta(0, \boldsymbol{x}) - \boldsymbol{\chi}(0, \boldsymbol{x}) + \int_0^t \partial_t \boldsymbol{\chi}^\eta(s, \boldsymbol{x}) - \partial_t \boldsymbol{\chi}(s, \boldsymbol{x})\, \mathrm{d}s \\ &= \int_0^t \boldsymbol{v}_0^\eta(\boldsymbol{\chi}^\eta(s, \boldsymbol{x})) - \boldsymbol{v}_0(\boldsymbol{\chi}(s, \boldsymbol{x}))\, \mathrm{d}s \\ &= \int_0^t \boldsymbol{v}_0^\eta(\boldsymbol{\chi}^\eta(s, \boldsymbol{x})) - \boldsymbol{v}_0(\boldsymbol{\chi}^\eta(s, \boldsymbol{x})) + \boldsymbol{v}_0(\boldsymbol{\chi}^\eta(s, \boldsymbol{x})) - \boldsymbol{v}_0(\boldsymbol{\chi}(s, \boldsymbol{x}))\, \mathrm{d}s \\ &= \int_0^t \eta\, \boldsymbol{w}_0(\boldsymbol{\chi}(s, \boldsymbol{x})) + \boldsymbol{v}_0(\boldsymbol{\chi}^\eta(s, \boldsymbol{x})) - \boldsymbol{v}_0(\boldsymbol{\chi}(s, \boldsymbol{x}))\, \mathrm{d}s. \end{aligned}$$

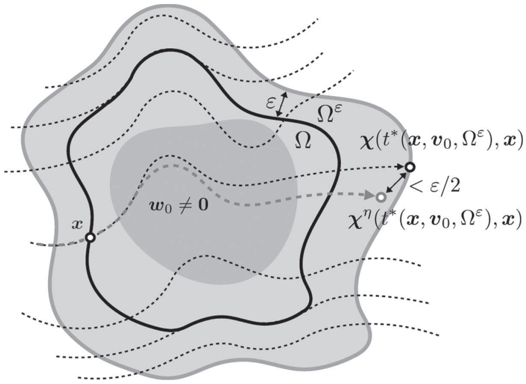

FIGURE B.1. The control of the distance between the reference and the perturbed characteristics from $\boldsymbol{x}$ ensures that if the reference characteristic is, at time t, outside of Ω^ε, then, the perturbed characteristic is outside of Ω at the same time t.

Then, by mean value inequality,

$$|\boldsymbol{\chi}^\eta(t,\boldsymbol{x}) - \boldsymbol{\chi}(t,\boldsymbol{x})| \leq t\,|\eta|\,\|\boldsymbol{w}_0\|_{\mathrm{L}^\infty} + \|\boldsymbol{\nabla}\boldsymbol{v}_0\|_{\mathrm{L}^\infty} \int_0^t |\boldsymbol{\chi}^\eta(s,\boldsymbol{x}) - \boldsymbol{\chi}(s,\boldsymbol{x})|\,\mathrm{d}s,$$

and by Gronwall's lemma we deduce

$$\begin{aligned}|\boldsymbol{\chi}^\eta(t,\boldsymbol{x}) - \boldsymbol{\chi}(t,\boldsymbol{x})| &\leq t\,|\eta|\,\|\boldsymbol{w}_0\|_{\mathrm{L}^\infty} + \|\boldsymbol{\nabla}\boldsymbol{v}_0\|_{\mathrm{L}^\infty} \int_0^t e^{(t-s)\|\boldsymbol{\nabla}\boldsymbol{v}_0\|_{\mathrm{L}^\infty}}\, s|\eta|\|\boldsymbol{w}_0\|_{\mathrm{L}^\infty}\mathrm{d}s,\\ &\leq t\,|\eta|\,\|\boldsymbol{w}_0\|_{\mathrm{L}^\infty} + \|\boldsymbol{\nabla}\boldsymbol{v}_0\|_{\mathrm{L}^\infty}\, e^{t\|\boldsymbol{\nabla}\boldsymbol{v}_0\|_{\mathrm{L}^\infty}}\, t^2|\eta|\|\boldsymbol{w}_0\|_{\mathrm{L}^\infty},\end{aligned}$$

which is the expected inequality.

(2) The 2D characterization (3.1) of Ω-filling states that $\boldsymbol{v}_0$ is Ω-filling if and only if $\inf_\Omega |\boldsymbol{v}_0| > 0$. Thus, it is straightforward that the continuity of $\boldsymbol{v}_0$ and the fact that $\inf_\Omega |\boldsymbol{v}_0| > 0$ imply that there exists $\varepsilon > 0$ such that $\inf_{\Omega^\varepsilon} |\boldsymbol{v}_0| > 0$.

(3) Thanks to (1), given $\eta^* > 0$ such that

$$\forall\, \eta \in (-\eta^*, \eta^*),\ \ \forall\,(t,\boldsymbol{x}) \in [0, t^*(\boldsymbol{v}_0, \Omega^\varepsilon)] \times \Omega^\varepsilon, \quad |\boldsymbol{\chi}^\eta(t,\boldsymbol{x}) - \boldsymbol{\chi}(t,\boldsymbol{x})| \leq \varepsilon/2.$$

Then, let $\boldsymbol{x} \in \Omega$. One has $\boldsymbol{\chi}(t^*(\boldsymbol{x},\boldsymbol{v}_0,\Omega^\varepsilon),\boldsymbol{x}) \in \partial\Omega^\varepsilon$ and thus is at a distance ε to Ω:

$$\mathrm{dist}(\boldsymbol{\chi}(t^*(\boldsymbol{x},\boldsymbol{v}_0,\Omega^\varepsilon)),\Omega) = \varepsilon.$$

One also has that for $\eta \in (-\eta^*, \eta^*)$, $(t^*(\boldsymbol{x},\boldsymbol{v}_0,\Omega^\varepsilon),\boldsymbol{x}) \in [0, t^*(\boldsymbol{v}_0,\Omega^\varepsilon)] \times \Omega^\varepsilon$, leading to $|\boldsymbol{\chi}^\eta(t^*(\boldsymbol{x},\boldsymbol{v}_0,\Omega^\varepsilon),\boldsymbol{x}) - \boldsymbol{\chi}(t^*(\boldsymbol{x},\boldsymbol{v}_0,\Omega^\varepsilon),\boldsymbol{x})| \leq \varepsilon/2$, which implies that

$$\boldsymbol{\chi}^\eta(t^*(\boldsymbol{x},\boldsymbol{v}_0,\Omega^\varepsilon),\boldsymbol{x}) \notin \Omega.$$

By definition of $t^*(\boldsymbol{x},\boldsymbol{v}_0^\eta,\Omega)$, we have $\forall\,\tau \in [0, t^*(\boldsymbol{x},\boldsymbol{v}_0^\eta,\Omega))$, $\boldsymbol{\chi}^\eta(\tau,\boldsymbol{x}) \in \Omega$, from which we deduce:

$$t^*(\boldsymbol{x},\boldsymbol{v}_0^\eta,\Omega) \leq t^*(\boldsymbol{x},\boldsymbol{v}_0,\Omega^\varepsilon) \leq t^*(\boldsymbol{v}_0,\Omega^\varepsilon).$$

Passing to the sup over Ω in this inequality, one finally gets, for any $\eta \in (-\eta^*, \eta^*)$:

$$t^*(\boldsymbol{v}_0^\eta,\Omega) \leq t^*(\boldsymbol{v}_0,\Omega^\varepsilon),$$

which is the announced result. □

Appendix C. On an alternative approach to the existence result

Let us mention a possible alternative to the approach of Section 3 to the analysis of Goldstein's coupling. We have not chosen to follow this approach in a first step for reasons that we shall mention later, but this could be the object of a companion paper.

Let us restart from the abstract block form (3.15) of the Goldstein's problem:

$$\begin{pmatrix} A(\omega) & D \\ \boldsymbol{B} & \boldsymbol{T}(\omega) \end{pmatrix} \begin{pmatrix} \varphi \\ \boldsymbol{\xi} \end{pmatrix} = \begin{pmatrix} \hat{f} \\ \boldsymbol{0} \end{pmatrix}.$$

In our approach in Section 3.1, we proceed *via* the elimination of $\boldsymbol{\xi}$. An natural alternative would be to eliminate φ *via* (formally)

$$\varphi = A(\omega)^{-1}\Big(\hat{f} - D\,\boldsymbol{\xi}\Big),$$

so that, we are led to a "reduced" transport equation on $\boldsymbol{\xi}$

$$\boldsymbol{T}(\omega) - \boldsymbol{B}\,A(\omega)^{-1}\,D = -\,\boldsymbol{B}\,A(\omega)^{-1}\,\hat{f}. \tag{C.1}$$

To do so, we first have to check that the first step is possible, which is related to the invertibility of $A(\omega)$, that is to say the resolution of the convected Helmholtz equation. We know, see Section 3.3, that $A(\omega)$ is invertible under the only assumption that the flow is strictly subsonic. Then it remains to treat the reduced transport equation which is far from standard since the operator $\boldsymbol{B}\,A(\omega)^{-1}\,D$ is a fully non local perturbation of the transport operator.

For the perturbation analysis, we cannot use, as in Section 3.4, the Fredholm approach which is not adapted (at least to our knowledge) to the transport equation. However, we can try to use the Banach fixed point theorem. More precisely, assuming that the transport operator $\mathbf{T}(\omega)$ is invertible, which is guaranteed (Sect. 3.4.2) under the condition that the flow is Ω-filling, we rewrite (C.1) as

$$\boldsymbol{I} - \boldsymbol{T}(\omega)^{-1}\mathbf{B}\,A(\omega)^{-1}\,D = -\,\boldsymbol{T}(\omega)^{-1}\;\boldsymbol{B}\,A(\omega)^{-1}\,\hat{f}.$$

We can then conclude to the solvability of (C.1) under the formal condition

$$\big\|\boldsymbol{T}(\omega)^{-1}\boldsymbol{B}\,A(\omega)^{-1}\,D\big\| < 1, \tag{C.2}$$

where $\boldsymbol{T}(\omega)^{-1}\boldsymbol{B}\,A(\omega)^{-1}\,D$ is seen as an operator of $\mathcal{L}(\boldsymbol{M})$. Of course a sufficient condition for (C.2) is

$$\|\boldsymbol{B}\|\,\|D\|\,\big\|\boldsymbol{T}(\omega)^{-1}\big\|\,\big\|A(\omega)^{-1}\big\| < 1. \tag{C.3}$$

If, for a while, we forget about the presence of $\big\|A(\omega)^{-1}\big\|$, considering the estimate (3.35)(i) (remember that $\boldsymbol{T}(\omega)^{-1} = \boldsymbol{\mathcal{S}}_0(\omega; \boldsymbol{\nabla}\varphi)$, one sees that the condition (C.3) is at least qualitatively very similar to our admissibility condition (3.5), the norm $\|\boldsymbol{\omega}_0\|_{\mathrm{L}^\infty}$ of the vorticity being hidden in the estimate (3.35)(i) which is frequency independent for real frequencies. However, the presence of $\big\|A(\omega)^{-1}\big\|$ makes the condition (C.3) much less explicit than (3.5), especially because it is hard to get an explicit upper bound for $\big\|A(\omega)^{-1}\big\|$. This quantity does depend on the frequency and may become large if the presence of resonances of the convected Helmholtz equation (*i.e.* poles of the resolvent $A(\omega)^{-1}$) that could be close to the real axis. This is one of the reasons that led us to privilege the approach adopted in this paper.

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