A PRIORI AND A POSTERIORI ERROR ESTIMATES FOR THE QUAD-CURL EIGENVALUE PROBLEM

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Abstract. In this paper, we consider a priori and a posteriori error estimates of the $H(\text{curl}^2)$-conforming finite element when solving the quad-curl eigenvalue problem. An a priori estimate of eigenvalues with convergence order $2(s-1)$ is obtained if the corresponding eigenvector $\mathbf{u} \in H^{s-1}(\Omega)$ and $\nabla \times \mathbf{u} \in H^s(\Omega)$. For the a posteriori estimate, by analyzing the associated source problem, we obtain lower and upper bounds for the errors of eigenvectors in the energy norm and upper bounds for the errors of eigenvalues. Numerical examples are presented for validation.

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1. Introduction

The quad-curl operator has important applications in the inverse electromagnetic scattering theory [8,9]. The corresponding quad-curl eigenvalue problem plays a fundamental role in the analysis and computation of the electromagnetic interior transmission eigenvalues [27,32,33]. Various numerical methods have been proposed for the quad-curl source problem, see, e.g., [6,7,13,22,35,36,38,41,42,45]. However, there exist only a few results on the numerical methods for the quad-curl eigenvalue problem. The quad-curl eigenvalue problem was first proposed in [33], where J. Sun developed a mixed finite element method by introducing an auxiliary variable $\mathbf{w} = \nabla \times \nabla \times \mathbf{u}$ and proved an a priori error estimate. In [12], H. Chen et al. designed a different mixed scheme by introducing $\delta = \nabla \times \mathbf{u}$. Two multigrid methods based on the Rayleigh quotient iteration and the inverse iteration with fixed shift were proposed and analyzed in [21].

Very recently, three of the authors and their collaborators constructed $H(\text{curl}^2)$-conforming finite elements in both two and three dimensions (2D and 3D) to solve the quad-curl source problem [23,24,39,40,44]. Based on the conforming elements, in this paper, we consider the conforming finite element method for the eigenvalue problem and derive a priori and a posteriori error estimates.

Keywords and phrases. The quad-curl problem, eigenvalue problem, a priori error estimation, a posteriori error estimation, curl-curl conforming elements.

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In the first part of the paper, we apply the classical framework of Babuška and Osborn [2, 29] to derive an a priori estimate. To this end, we prove the discrete compactness of the $H(\text{curl}^2)$-conforming finite elements with div-free condition. We can show that the conforming method is convergent and it has a convergence order of $2(s − 1)$ when the eigenvector $\mathbf{u} \in H^{s−1}(\Omega)$ and $\nabla \times \mathbf{u} \in H^s(\Omega)$.

At reentrant corners or material interfaces, the eigenvectors feature strong singularities [28]. For more efficient computation, adaptive local refinements are considered. A posteriori error estimators are essential for adaptive finite element methods. In addition, an inappropriate scheme for the quad-curl problem might lead to spurious eigenvalues. In this situation, a posteriori error estimators can be applied to test whether an eigenvalue is spurious. We refer to [3–5,15,17,25,30] for the a posteriori error estimates of electromagnetic problems and elliptic problems. In [10], S. Cao et al. developed an a posteriori error estimator for a decoupled finite element method for the quad-curl source problem. In terms of the quad-curl eigenvalue problem, to the authors’ knowledge, no work on a posteriori error estimations has been done so far. Therefore, in the second part of the paper, we consider an a posteriori error estimate for the conforming finite element method.

Due to the large kernel space of the curl operator, the Helmholtz decomposition of splitting a vector field in $H(\text{curl}^2; \Omega)$ into the irrotational and solenoidal components plays an important role in the analysis. However, in general, the irrotational component is not $H^2$-regular when $\Omega$ is non-convex. Therefore, we propose a new decomposition for $H_0(\text{curl}^2; \Omega)$, which further splits the irrotational component into a function in $H^2(\Omega)$ and a function in the kernel of curl operator. To obtain an a posteriori error estimator for the eigenvalue problem, we apply the idea of [17] to relate the eigenvalue problem to a source problem. An a posteriori error estimator for the source problem is constructed by analyzing irrotational and solenoidal components, respectively. Then an a posteriori error estimator for simple eigenvalues is obtained. The proof uses the new decomposition and makes no additional regularity assumption.

For ease of presentation, we will focus on only 3D case, the similar arguments can be used to the 2D case. The rest of this paper is organized as follows. In Section 2, we present some notation, the $H$($\text{curl}^2$)-conforming elements, the new decomposition, and an $H(\text{curl}^2)$-type Clément interpolation, which will be used in the a posteriori error analysis. In Section 3, we derive the a priori error estimate for the quad-curl eigenvalue problem. In Section 4, we prove the a posteriori error estimate. Finally, in Section 5, we show some numerical experiments.

2. Notation and basis tools

2.1. Notation

Let $\Omega \subset \mathbb{R}^3$ be a contractible Lipschitz domain. For a Lipschitz domain $D \subset \mathbb{R}^3$, $L^2(D)$ denotes the space of square integrable functions on $D$ with norm $\| \cdot \|_D$. For a positive integer $s$, $H^s(D)$ denotes the space of scalar functions in $L^2(D)$ whose derivatives up to order $s$ are also in $L^2(D)$. If $s = 0$, $H^0(D) = L^2(D)$. For vector functions, denote $L^2(D) = (L^2(D))^3$ and $H^s(D) = (H^s(D))^3$. We use $\langle \cdot, \cdot \rangle_D$ and $\langle \cdot, \cdot \rangle_{\partial D}$ to stand for the $L^2$ inner products on $D$ and $\partial D$. When $D = \Omega$, we omit the subscript $\Omega$ in the notation of norms and $L^2$ inner products.

For simplicity, denote $(\nabla \times)^2 \mathbf{u} = \nabla \times \nabla \times \mathbf{u}$. We now define a space concerning the $\nabla \times$ operator

$$H(\text{curl}^2; D) := \{ \mathbf{u} \in L^2(D) : (\nabla \times)^i \mathbf{u} \in L^2(D), \ i = 1, 2 \},$$

whose norm is given by

$$\| \mathbf{u} \|_{H(\text{curl}^2; D)} = \sqrt{\sum_{i=0}^{2} \langle (\nabla \times)^i \mathbf{u}, (\nabla \times)^i \mathbf{u} \rangle_D}.$$

We also equip the space $H(\text{curl}^2; D)$ with the following norm:

$$\| \mathbf{u} \|_D^2 = \| \mathbf{u} \|_D^2 + \| (\nabla \times)^2 \mathbf{u} \|_D^2.$$
We drop the subscript $D$ in $\| \cdot \|_D$ when $D = \Omega$. The spaces $H_0(\text{curl}^2; D)$, $H_0^1(D)$, and $H(\text{div}^0; D)$ are defined, respectively, as

\[
\begin{align*}
H_0(\text{curl}^2; D) &:= \{ u \in H(\text{curl}^2; D) : u \times n_D = 0 \text{ and } \nabla \times u \times n_D = 0 \text{ on } \partial D \}, \\
H_0^1(D) &:= \{ u \in H^1(D) : u = 0 \text{ on } \partial D \}, \\
H(\text{div}^0; D) &:= \{ u \in L^2(D) : \nabla \cdot u = 0 \},
\end{align*}
\]

where $n_D$ is the unit outward normal vector to $\partial D$.

Let $T_h$ be a shape regular simplicial triangulation of $\Omega$. Denote by $N_h$, $E_h$, and $F_h$ the sets of vertices, edges, and faces. Let $\tau_e$ and $n_f$ be the unit tangent vector of an edge $e \in E_h$ and the unit normal vector of a face $f \in F_h$, respectively. We denote $N_h^\text{int}$ and $F_h^\text{int}$ as the sets of vertices and faces in the interior of $\Omega$, respectively. Let $N_h(T)$, $E_h(T)$ and $F_h(T)$ be the sets of vertices, edges, and faces on the element $T$. Denote by $h_T$ the diameter of $T \in T_h$. Denote $h = \max_{T \in T_h} h_T$. We use $P_k$ to represent the space of polynomials with degrees at most $k$. Denote $P_k = (P_k)^3$.

### 2.2. $H(\text{curl}^2)$-conforming elements in 3D

We apply the $H(\text{curl}^2)$-conforming elements constructed in [23,24]. The shape function space for an element $T \in T_h$ is

\[ V^k_h(T) = \nabla P_{k+1}(T) \oplus pP^+_k(T), \]

where

\[ p_u = \int_0^1 u(W + t(x - W)) \times t(x - W) dt, \]

and

\[ P^+_k(T) = \begin{cases} P_1(T) \oplus B^1, & k = 1, \\
P_2(T) \oplus B^1 \oplus \hat{B}^2, & k = 2, \\
P_k(T) \oplus \hat{B}^k, & k \geq 3. \end{cases} \]

Here $B^1$ and $\hat{B}^k$ are spaces of modified bubbles defined on the Alfeld split of $T$, see [24] for more information.

**Remark 2.1.** For the bubble functions in $P^+_k(T)$, we choose the barycenter of $T$ as the base point $W$ in the Poincaré operator $p$, see [14]. For other functions, we choose $W$ to be the origin.

For $k \geq 2$, the $H(\text{curl}^2)$-conforming element with the shape function space $V^k_h(T)$ is defined by the following degrees of freedom (DOFs).

- **Vertex DOFs** $M_v(u)$ at all vertices $v_i \in N_h$:

\[
M_v(u) = \{(\nabla \times u)(v_i)\}. \tag{2.1}
\]

- **Edge DOFs** $M_e(u)$ on all edges $e_i \in E_h$:

\[
M_e(u) = \left\{ \int_{e_i} u \cdot \tau_{e_i}, q dS, \forall q \in P_k(e_i) \right\} \\
\cup \left\{ \frac{1}{\text{length}(e_i)} \int_{e_i} \nabla \times u \cdot q dS, \forall q \circ F_{T_i} \in P_{k-3}(e_i) \right\}. \tag{2.2}
\]
2.3. An \( H(\text{curl}) \)-type Clément interpolation

Let \( \omega_v \) be the union of elements sharing the vertex \( v \) and \( R_v \phi \) be the \( L^2 \) projection of \( \phi \in L^2(\omega_v) \) on \( \omega_v \), i.e., \( R_v \phi \in P_1(\omega_v) \) such that

\[
\int_{\omega_v} (\phi - R_v \phi) p dV = 0, \quad \forall p \in P_1(\omega_v).
\]
For \( \mathbf{u} \in H^{1/2+\delta}(\Omega) \) with \( \nabla \times \mathbf{u} \in H^{3/2+\delta}(\Omega) \), the lowest-order \( H(\text{curl}; \Omega) \) interpolation \( \Pi_h^2 \mathbf{u} \) defined by the DOFs (2.1)–(2.4) can be written as

\[
\Pi_h^2 \mathbf{u} = \sum_{v \in \mathcal{N}_h} \alpha_v^i(\mathbf{u}) \phi_v^i + \sum_{c \in \mathcal{E}_h} \sum_{i=1}^3 \alpha_c^i(\mathbf{u}) \phi_c^i + \sum_{f \in \mathcal{F}_h} \alpha_f(\mathbf{u}) \phi_f,
\]

where

\[
\alpha_v^i(\mathbf{u}) = \text{the } i\text{th component of } \nabla \times \mathbf{u}(v),
\]

\[
\alpha_c^i(\mathbf{u}) = \int_c \mathbf{u} \cdot \mathbf{r}_i q ds \text{ for any } q_i \in P_2(e),
\]

\[
\alpha_f(\mathbf{u}) = \int_f \mathbf{u} \cdot B_T(\hat{n}_f \times \hat{x}|_f \times \hat{n}_f) dA \text{ with } f \in \partial T,
\]

and the functions \( \phi_v^i, \phi_c^i, \) and \( \phi_f \) are the corresponding dual basis functions. Now we define an \( H(\text{curl}) \)-type Clément interpolation \( \tilde{\Pi}_h^2 \) for \( \mathbf{u} \in H^{1/2+\delta}(\Omega) \) with \( \nabla \times \mathbf{u} \in H^1(\Omega) \) by replacing \( \alpha_v^i(\mathbf{u}) \) with \( \tilde{\alpha}_v^i(\mathbf{u}) = R_v^0((\nabla \times \mathbf{u})_i)(v) \). The interpolation is well-defined and the following error estimate holds.

**Theorem 2.3.** For any \( T \in \mathcal{T}_h \), let \( \omega_T = \cup_{v \in \mathcal{N}_h(T)\omega_v} \). Then, for \( \mathbf{u} \in H^2(\Omega) \), it holds that

\[
\| \mathbf{u} - \tilde{\Pi}_h^2 \mathbf{u} \|_T + h_T \| \nabla (\mathbf{u} - \tilde{\Pi}_h^2 \mathbf{u}) \|_T + h_T^2 \| \nabla \times (\mathbf{u} - \tilde{\Pi}_h^2 \mathbf{u}) \|_T \leq C h^2 \| \mathbf{u} \|_{2,\omega_T}.
\]

The theorem can be obtained by combining the approximation properties of \( \Pi_h^2 \) and the classic Clément interpolation.

### 2.4. A decomposition of \( H_0(\text{curl}^2; \Omega) \)

Motivated by the decomposition of \( H_0(\text{curl}; \Omega) \) in Proposition 5.1 of [19], we obtain a decomposition of the space \( H_0(\text{curl}^2; \Omega) \), which plays a critical role in the analysis.

**Lemma 2.4.** Let \( \nabla H^1_0(\Omega) \) be the set of gradients of functions in \( H^1_0(\Omega) \). Then \( \nabla H^1_0(\Omega) \) is a closed subspace of \( H_0(\text{curl}^2; \Omega) \) and

\[
H_0(\text{curl}^2; \Omega) = X \oplus \nabla H^1_0(\Omega),
\]

where \( X = \{ \mathbf{u} \in H_0(\text{curl}^2; \Omega) | (\mathbf{u}, \nabla p) = 0, \forall p \in H^1_0(\Omega) \} \). Namely, for any \( \mathbf{u} \in H_0(\text{curl}^2; \Omega) \), \( \mathbf{u} = \mathbf{u}^0 + \mathbf{u}^\perp \) with \( \mathbf{u}^0 \in \nabla H^1_0(\Omega) \) and \( \mathbf{u}^\perp \in X \). Furthermore, \( \mathbf{u}^\perp \) admits the splitting

\[
\mathbf{u}^\perp = \nabla \phi + \mathbf{v},
\]

where \( \phi \in H^1_0(\Omega) \) and \( \mathbf{v} \in H^2(\Omega) \) satisfying

\[
\| \mathbf{v} \|_2 \leq C \| \nabla \times \mathbf{u}^\perp \|_1,
\]

\[
\| \nabla \phi \| \leq C \left( \| \nabla \times \mathbf{u}^\perp \|_1 + \| \mathbf{u}^\perp \| \right).
\]

**Proof.** The proof of (2.5) can be found in [44]. To prove (2.6)–(2.8), let \( \mathcal{O} \) be a bounded, smooth, contractible open set with \( \Omega \subset \mathcal{O} \). For any \( \mathbf{u}^\perp \in X \), we can extend \( \mathbf{u}^\perp \) in the following way:

\[
\tilde{\mathbf{u}} = \begin{cases} 
\mathbf{u}^\perp, & \Omega, \\
0, & \mathcal{O}/\Omega.
\end{cases}
\]
Obviously, \( \tilde{u} \in H_0(\text{curl}; \mathcal{O}) \) and \( \nabla \times \tilde{u} \in H_0^1(\mathcal{O}) \). According to Proposition 4.1 in [16], there exists a \( w \in H^2(\mathcal{O}) \) such that
\[
\nabla \times (w - \tilde{u}) = 0 \quad \text{and} \quad \|w\|_{2,\mathcal{O}} \leq C\|\nabla \times \tilde{u}\|_{1,\mathcal{O}}.
\] (2.9)
Based on Theorem 2.9 of [20], there exists a unique function \( p \in H^1(\mathcal{O})/\mathbb{R} \) such that
\[
w - \tilde{u} = \nabla p.
\] (2.10)
Now, we restrict (2.10) to the domain \( \mathcal{O}/\hat{\Omega} \) and obtain
\[
\nabla p = w \in H^2(\mathcal{O}/\hat{\Omega}).
\] (2.11)
Using the extension theorem [11], we can extend \( p \in H^3(\mathcal{O}/\hat{\Omega}) \) to \( \tilde{p} \), which is defined on \( \mathcal{O} \) and satisfies
\[
\|\tilde{p}\|_{3,\mathcal{O}} \leq C\|p\|_{3,\mathcal{O}/\hat{\Omega}} \leq C\|\nabla p\|_{2,\mathcal{O}/\hat{\Omega}} = C\|w\|_{2,\mathcal{O}/\hat{\Omega}},
\] (2.12)
where we have used Poincaré–Friedrichs inequality for \( p \in H^3(\mathcal{O}/\hat{\Omega}) \) since we can choose \( p \) for which \( \int_{\mathcal{O}/\hat{\Omega}} p = 0 \). Restricting on \( \hat{\Omega} \), we have
\[
u^+ = w - \nabla \tilde{p} + \nabla (\tilde{p} - p) = v + \nabla \phi.
\]
Note that \( \phi = \tilde{p} - p \in H_0^1(\Omega) \) since \( \tilde{p} \) is the extension of \( p \). Therefore, (2.6) is proved. By virtue of (2.12) and (2.9), we obtain
\[
\|v\|_2 = \|w - \nabla \tilde{p}\|_2 \leq \|w - \nabla p\|_{2,\mathcal{O}} \leq C\|\nabla w\|_{2,\mathcal{O}} \leq C\|\nabla \times \tilde{u}\|_{1,\mathcal{O}} = C\|\nabla \times u^+\|_1,
\]
and
\[
\|\nabla \phi\| = \|u^+ - v\| \leq \|u^+\| + \|v\| \leq \|u^+\| + \|v\|_2 \leq C\left(\|u^+\| + \|\nabla \times u^+\|_1\right).
\]

3. An a priori error estimate for the eigenvalue problem

Following [33], the quad-curl eigenvalue problem is to seek \( \lambda \) and \( u \) such that
\[
(\nabla \times)^4 u = \lambda u \quad \text{in} \quad \Omega,
\]
\[
\nabla \cdot u = 0 \quad \text{in} \quad \Omega,
\]
\[
u \times n = 0 \quad \text{on} \quad \partial \Omega,
\]
\[
\nabla \times u \times n = 0 \quad \text{on} \quad \partial \Omega,
\]
where \( n \) is the unit outward normal to \( \partial \Omega \). The assumption that \( \Omega \) is contractible implies \( \lambda \neq 0 \). The variational form of the quad-curl eigenvalue problem is to find \( \lambda \in \mathbb{R} \) and \( u \in X \) such that
\[
((\nabla \times)^2 u, (\nabla \times)^2 v) = \lambda (u, v), \quad \forall v \in X.
\] (3.2)
We define some discrete spaces.
\[
V_h = \{v_h \in H(\text{curl}; \Omega) : v_h|_T \in V_h^k(T)\},
\]
\[
V_h^0 = \{v_h \in V_h : n \times v_h = 0 \quad \text{and} \quad n \times \nabla \times v_h = 0 \quad \text{on} \quad \partial \Omega\},
\]
\[
S_h = \{w_h \in H^1(\Omega) : w_h|_T \in P_h(T)\},
\]
\[
S_h^0 = \{w_h \in S_h : w_h|_{\partial \Omega} = 0\},
\]
\[
X_h = \{u_h \in V_h^0 \mid (u_h, \nabla q_h) = 0 \quad \text{for all} \quad q_h \in S_h^0\}.
\]
The discrete problem for (3.2) is to find \( \lambda_h \in \mathbb{R} \) and \( u_h \in X_h \) such that
\[
((\nabla \times)^2 u_h, (\nabla \times)^2 v_h) = \lambda_h (u_h, v_h), \quad \forall v \in X_h.
\] (3.3)
3.1. The source problem

We start with the associated source problem. Given \( f \in L^2(\Omega) \), find \( u \in H_0(\text{curl}^2; \Omega) \) and \( p \in H^1_0(\Omega) \) such that

\[
(\nabla \times)^2 u + u + \nabla p = f \quad \text{in } \Omega, \\
\nabla \cdot u = 0 \quad \text{in } \Omega, \\
u \times n = 0 \quad \text{on } \partial \Omega, \\
\nabla \times u \times n = 0 \quad \text{on } \partial \Omega.
\]

(3.4)

Note that \( p = 0 \) for \( f \in H(\text{div}; \Omega) \).

The weak formulation is to find \((u; p) \in H_0(\text{curl}^2; \Omega) \times H^1_0(\Omega)\) such that

\[
a(u, v) + b(v, p) = (f, v), \quad \forall v \in H_0(\text{curl}^2; \Omega), \\
b(u, q) = 0, \quad \forall q \in H^1_0(\Omega),
\]

(3.5)

where

\[
a(u, v) = ((\nabla \times)^2 u, (\nabla \times)^2 v) + (u, v), \\
b(v, p) = (v, \nabla p).
\]

Define \( Y := \{ \mathbf{w} \in H_0(\text{curl}; \Omega) : (\mathbf{w}, \nabla q) = 0, \quad \forall q \in H^1_0(\Omega) \} \), then \( \nabla \times u \in Y \). By applying Friedrichs inequality on \( \nabla \times u \), we get \( a(\cdot, \cdot) \) is coercive on \( H_0(\text{curl}^2; \Omega) \), i.e.,

\[
a(u, u) \geq \|u\|^2_{H(\text{curl}^2; \Omega)}.
\]

In addition, the following Babuška–Brezzi condition holds,

\[
\sup_{v \in H_0(\text{curl}^2; \Omega)} \frac{b(v, p)}{\|v\|_{H(\text{curl}^2; \Omega)}} \geq \frac{b(\nabla p, p)}{\|\nabla p\|_{H(\text{curl}^2; \Omega)}} = \|\nabla p\|_{H^2(\Omega)} \geq C\|p\|_1.
\]

The well-posedness of (3.5) then follows from Theorem 1.3.2 of [34]. Consequently, we can define a bounded solution operator \( A : L^2(\Omega) \to L^2(\Omega) \) such that, for \( f \in L^2(\Omega), \quad Af = u \in X \subset L^2(\Omega) \) satisfies

\[
a(Af, v) = (f, v), \quad \forall v \in X.
\]

The operator \( A \) is selfadjoint since

\[
(A\phi, \psi) = (A\psi, A\phi) = a(A\phi, A\psi) = (\phi, A\psi), \quad \forall \phi, \psi \in L^2(\Omega).
\]

\( A \) is also compact due to the following result.

**Lemma 3.1.** \( X \) processes the continuous compactness property.

**Proof.** Since \( X \subset Y \hookrightarrow X \hookrightarrow L^2(\Omega) \) [26], then \( X \hookrightarrow L^2(\Omega) \). \( \Box \)

The \( H(\text{curl}^2) \)-conforming finite element method seeks \( u_h \in V^0_h \) and \( p_h \in S^0_h \) such that

\[
a(u_h, v_h) + b(v_h, p_h) = (f, v_h), \quad \forall v_h \in V^0_h, \\
b(u_h, q_h) = 0, \quad \forall q_h \in S^0_h.
\]

(3.6)

Since \( \nabla \times u_h \in Y \), by the Friedrichs inequality on \( \nabla \times u_h \), there exists a constant \( C \) independent of \( h \) such that

\[
a(u_h, u_h) = \|u_h\|^2 + \|(\nabla \times)^2 u_h\|^2 \geq C\|u_h\|_{H(\text{curl}^2; \Omega)}.
\]
The well-posedness of problem (3.6) is then due to the discrete Babuška–Brezzi condition,

$$
\sup_{v \in V_h^0} \frac{b(v_h, p_h)}{\|v\|_{H(\text{curl}; \Omega)}} \geq \frac{b(\nabla p_h, p_h)}{\|
abla p_h\|_{H(\text{curl}; \Omega)}} = \|
abla p_h\|_1 \geq C\|p_h\|_1.
$$

Consequently, we can define a discrete solution operator $A_h : L^2(\Omega) \rightarrow L^2(\Omega)$ such that $u_h = A_h f \in X_h$ is the solution of (3.6). We will use the standard finite element framework and the approximation property of the interpolation to obtain the approximation property of the numerical solution. To this end, we first introduce a new space:

$$
H(\text{gradcurl}; \Omega) = \{ u \in H(\text{curl}; \Omega) : \nabla \times u \in H^1(\Omega) \},
$$

and the associated space with vanishing trace

$$
H_0(\text{gradcurl}; \Omega) = \{ u \in H(\text{gradcurl}; \Omega) : n \times u = 0 \text{ and } \nabla \times u = 0 \text{ on } \partial \Omega \}.
$$

Equip the space $H(\text{gradcurl}; \Omega)$ with norm $\|u\|_{H(\text{gradcurl}; \Omega)}^2 = \|u\|^2 + \|\nabla \times u\|^2$. We can show that the space $H_0(\text{gradcurl}; \Omega)$ is equivalent to $H_0(\text{curl}; \Omega)$.

**Lemma 3.2.** The space $H_0(\text{curl}^2; \Omega)$ coincides with $H_0(\text{gradcurl}; \Omega)$. Moreover, for $u \in H_0(\text{curl}^2; \Omega)$,

$$
\|u\|_{H(\text{curl}^2; \Omega)} \leq C\|u\|_{H(\text{gradcurl}; \Omega)}. \tag{3.7}
$$

**Proof.** To prove $H_0(\text{curl}^2; \Omega) = H_0(\text{gradcurl}; \Omega)$, it suffices to show $H_0(\text{curl}^2; \Omega) \subset H_0(\text{gradcurl}; \Omega)$ since $H_0(\text{gradcurl}; \Omega) \subset H_0(\text{curl}^2; \Omega)$ is trivial. For $u \in H_0(\text{curl}^2; \Omega)$, we have $\nabla \times u \in H_0(\text{div}; \Omega) = \{ u \in L^2(\Omega) : \nabla \cdot u \in L^2(\Omega), u \cdot n = 0 \text{ on } \partial \Omega \}$ since $\nabla \cdot (\nabla \times u) = 0$ and $\nabla \times u \cdot n = \nabla \cdot (u \times n) = 0$ on $\partial \Omega$. It then follows from $H_0(\text{curl}; \Omega) \cap H_0(\text{div}; \Omega) = H_0^2(\Omega)$ ([20], Lem. 2.5) that $\nabla \times u \in H_0^2(\Omega)$, and hence $u \in H_0(\text{gradcurl}; \Omega)$. The inequality (3.7) follows from the Poincaré inequality.

**Lemma 3.3.** $C_0^S(\Omega)$ is dense in $H_0(\text{gradcurl}; \Omega)$.

**Proof.** The density of $C_0^S(\Omega)$ in $H_0(\text{gradcurl}; \Omega)$ has been proved in Theorem 3.15 of [43]. The proof uses a similar argument to the one used to prove the density of $C_0^S(\Omega)$ in $H_0(\text{curl}; \Omega)$ [26].

With the density of $C_0^S(\Omega)$ in $H_0(\text{gradcurl}; \Omega)$, we can obtain the following approximation property.

**Theorem 3.4.** For $f \in L^2(\Omega)$, it holds

$$
\|Af - A_h f\|_{H(\text{curl}^2; \Omega)} \to 0 \text{ as } h \to 0.
$$

**Proof.** The usual theory of mixed method shows that

$$
\|Af - A_h f\|_{H(\text{curl}^2; \Omega)} \leq C \inf_{v_h \in V_h^0} \|Af - v_h\|_{H(\text{curl}^2; \Omega)} + \inf_{q_h \in S_h^0} \|p - q_h\|_1. \tag{3.8}
$$

It follows from the approximation property of the canonical interopolations, Lemma 3.2, the density of $C_0^S(\Omega)$ in $H_0^1(\Omega)$, and Lemma 3.3 that

$$
\inf_{v_h \in V_h^0} \|Af - v_h\|_{H(\text{curl}^2; \Omega)} + \inf_{q_h \in S_h^0} \|p - q_h\|_1 \to 0 \text{ as } h \to 0.
$$

If $p$ and $Af$ are smoother, then from the approximation property of $\Pi_h^k$, we have the following error estimate with a convergence order.

**Theorem 3.5.** Assume that $Af \in H^{s-1}(\Omega), \nabla \times Af \in H^s(\Omega)$, and $p \in H^s(\Omega)$ ($s \geq 3/2 + \delta$ with $\delta > 0$). It holds that

$$
\|Af - A_h f\|_{H(\text{curl}^2; \Omega)} \leq Ch^{s-1} (\|Af\|_{s} + \|\nabla \times Af\|_{s} + \|p\|_{s}).
$$
3.2. An a priori error estimate of the eigenvalue problem

We first rewrite the eigenvalue problem (3.2) by adding a low-order term $u$. Find $\lambda \in \mathbb{R}$ and $(u,p) \in H_0(\text{curl}^2; \Omega) \times H_0^1(\Omega)$ such that

$$
\begin{align*}
a(u,v) + b(v,p) &= (\lambda + 1)(u,v), \quad \forall v \in H_0(\text{curl}^2; \Omega), \\
b(u,q) &= 0, \quad \forall q \in H_0^1(\Omega).
\end{align*}
$$

(3.9)

Due to the fact that $\nabla \cdot u = 0$, we have $p = 0$. Then (3.9) can be written as an operator eigenvalue problem of finding $\mu := 1/(\lambda + 1) \in \mathbb{R}$ and $u \in X$ such that

$$
Au = \mu u.
$$

(3.10)

The discrete eigenvalue problem (3.3) is equivalent to seeking $\lambda_h \in \mathbb{R}$ and $(u_h,p_h) \in V_h^0 \times S_h^0$ such that

$$
\begin{align*}
a(u_h,v_h) + b(v_h,p_h) &= (\lambda_h + 1)(u_h,v_h), \quad \forall v_h \in V_h^0, \\
b(u_h,q_h) &= 0, \quad \forall q_h \in S_h^0.
\end{align*}
$$

(3.11)

Using the operator $A_h$, the eigenvalue problem is to find $\mu_h \in \mathbb{R}$ and $u_h \in X_h$ such that

$$
A_h u_h = \mu_h u_h,
$$

(3.12)

where $\mu_h = 1/(\lambda_h + 1)$.

Let $\Lambda = \{h_n, n = 0, 1, 2, \ldots\}$ be a sequence of mesh sizes such that

$$
h_0 > h_1 > h_2 > \cdots > 0 \text{ and } h_n \to 0 \text{ as } n \to \infty.
$$

Define a collection of operators,

$$
\mathcal{A} = \{A_h : L^2(\Omega) \to L^2(\Omega), \ h \in \Lambda\}.
$$

To apply the abstract convergence theory [29], we need to verify:

1. $\mathcal{A}$ is collectively compact, i.e., for each bounded set $U \subseteq L^2(\Omega)$, $\mathcal{A}(U) = \{A_h u : u \in U, h \in \Lambda\}$ is relatively compact.
2. $\mathcal{A}$ is point-wise convergent, i.e., for $f \in L^2(\Omega)$, $A_h f \to Af$ strongly in $L^2(\Omega)$ as $h \to 0$.

Theorem 3.4 verifies (2). It remains to verify (1). We first show $\{X_h\}_{h \in \Lambda}$ has discrete compactness property.

Theorem 3.6. $\{X_h\}_{h \in \Lambda}$ processes the discrete compactness property, i.e., for every $\{w_n\}_{n=1}^\infty$ such that

- $w_n \in X_{h_n}$ for each $n$ and $h_n \to 0$ as $n \to \infty$,
- there is a constant $C$ independent of $w_n$ such that $\|w_n\|_{H(\text{curl}^2; \Omega)} \leq C$,

then there exists a subsequence, still denoted $\{w_n\}$, and a function $w \in X$ such that

$$
w_n \to w \text{ strongly in } L^2(\Omega) \text{ and weakly in } X \text{ as } n \to \infty.
$$

Proof. Let $w_n \in X_{h_n}, n = 1, 2, \ldots$ and $h_n \to 0$ as $n \to \infty$. Suppose $\|w_n\|_{H(\text{curl}^2; \Omega)} \leq C < \infty$ for all $n$. Seek $p^n \in H_0^1(\Omega)$ such that $(\nabla p^n, \nabla \xi) = (w_n, \nabla \xi)$ for all $\xi \in H_0^1(\Omega)$. Set $w^n = w_n - \nabla p^n$, clearly, $w^n$ satisfies

$$
\begin{align*}
(\nabla \times)^2 w^n &= (\nabla \times)^2 w_n, \\
\nabla \times w^n &= \nabla \times w_n, \quad \text{and} \quad \nabla \cdot w^n &= 0 \quad \text{in } \Omega, \\
\n \times w^n &= \nabla w_n \text{ on } \partial \Omega.
\end{align*}
$$

Hence $w^n \in X$ and $\|w^n\|_{H(\text{curl}^2; \Omega)} \leq C$. By Lemma 3.1, there is a subsequence, still denoted by $\{w^n\}_{n=1}^\infty$, and a function $w \in L^2(\Omega)$ such that $w^n \to w$ as $n \to \infty$ strongly in $L^2(\Omega)$. Furthermore, we can prove $w \in X$. In
fact, for any \( v \in L^2(\Omega) \), \( \lim_{n \to \infty} (\nabla \times w^n, v) \leq C \|v\| \), which implies the limit of \( (\nabla \times w^n, v) \) is a bounded linear functional on \( L^2(\Omega) \). By Riesz representation theorem, there exists a unique element \( z \in L^2(\Omega) \) such that

\[
\lim_{n \to \infty} (\nabla \times w^n, v) = (z, v).
\]

Picking \( v \in H^1(\Omega) \), we have

\[
(w, \nabla \times v) = \lim_{n \to \infty} (w^n, \nabla \times v) = \lim_{n \to \infty} (\nabla \times w^n, v) = (z, v),
\]

which implies \( z = \nabla \times w \in L^2(\Omega) \). Moreover, it holds that

\[
(w, \nabla \times v) = (\nabla \times w, v) + (w \times n, v) = (z, v) + (w \times n, v),
\]

which leads to \( \langle w \times n, v \rangle = 0 \) for all \( v \in H^{1/2}(\partial \Omega) \) because of the surjectivity from \( H^1(\Omega) \) to \( H^{1/2}(\partial \Omega) \). Therefore, we arrive at \( w \times n = 0 \) on \( \partial \Omega \), and hence \( w \in H_0(\text{curl}; \Omega) \). We can then prove \( \nabla \times w \in H_0(\text{curl}; \Omega) \) by replacing \( w \) with \( \nabla \times w \). Finally, \( w \in X \) since

\[
(w, \nabla q) = \lim_{n \to \infty} (w^n, \nabla q) = 0, \quad \forall q \in H_0^1(\Omega).
\]

The weak convergence of \( w^n \to w \) in \( X \) then follows. By Lemma 7.15 of [26], \( w^n \in H^{1/2+s}(\Omega) \) with \( s > 0 \), and it holds

\[
\|w^n\|_{1/2+s} \leq C \|\nabla \times w_n\|. \tag{3.13}
\]

Since \( w^n \in H^{1/2+s}(\Omega) \) and \( \nabla \times w^n = \nabla \times w_n \in C^0(\Omega) \), we know by Lemma 4.1 of [23] that the interpolation \( \Pi_{h_n} w^n \) is well-defined. Since \( \Pi_{h_n} w_n = w_n, \Pi_{h_n} \nabla p^n = \nabla \pi_{h_n} \nabla p^n \) with \( \pi_{h_n} \) the Lagrange interpolation (see [24], Lem. 5.3 and [23], Lem. 4.5). Hence, using the fact that \( w \in H(\text{curl}^2; \Omega) \) and \( w_n \in X_{h_n} \),

\[
\|w - w_n\|^2 = (w - w_n, w - \Pi_{h_n} w^n) + (w - w_n, \Pi_{h_n} w^n - w_n)
= (w - w_n, w - \Pi_{h_n} w^n) + (w - w_n, -\nabla \pi_{h_n} p^n)
= (w - w_n, w - \Pi_{h_n} w^n) \leq \|w - w_n\| \|w - \Pi_{h_n} w^n\|,
\]

which implies

\[
\|w - w_n\| \leq \|w - \Pi_{h_n} w^n\| \leq \|w - w^n\| + \|w^n - \Pi_{h_n} w^n\|. \tag{3.14}
\]

To estimate the second term on the right-hand side, we apply Theorem 2.2 and (3.13) to obtain

\[
\|w^n - \Pi_{h_n} w^n\| \leq C h_n^{1/2+s}(\|w^n\|_{1/2+s} + \|\nabla \times w^n\|) \leq C h_n^{1/2+s} \|\nabla \times w^n\|. \tag{3.15}
\]

Combining (3.14) and (3.15) leads to

\[
\|w - w_n\| \leq \|w - w^n\| + C h_n^{1/2+s} \|w_n\|_{H(\text{curl}^2; \Omega)}.
\]

Since the right-hand side converges to zero, we have proved that \( w_n \to w \) in \( L^2(\Omega) \) as \( n \to \infty \). The weak convergence \( w_n \to w \) in \( X \) follows from the strong convergence \( w_n \to w \) in \( L^2(\Omega) \), the weak convergence \( w^n \to w \) in \( X \), and the fact that \( \nabla \times w_n = \nabla \times w^n \).

\( \Box \)

**Theorem 3.7.** \( A \) is collectively compact.
Proof. Suppose $U \subset L^2(\Omega)$ is a bounded set. For any $u \in U$, according to the wellposedness of (3.6), $A_h u \in X_h$ satisfies
\[
\| (\nabla \times)^2 A_h u \| + \| A_h u \| \leq C \| u \|.
\]
By the Friedrichs inequality, we have
\[
\| A_h u \|_{H(\text{curl}^2; \Omega)} \leq C \| u \|,
\]
which implies $\{ A_h u : u \in U, h \in \Lambda \} \subset W := \cup_{h \in \Lambda} X_h$ is bounded in $H(\text{curl}^2; \Omega)$.

To prove that the set $\{ A_h u : u \in U, h \in \Lambda \}$ is relatively compact, it suffices to show that $W \hookrightarrow L^2(\Omega)$. Suppose $\{ w_n \}_{n=1}^\infty \subset W$ is bounded in $H(\text{curl}^2; \Omega)$. Then $w_n \in X_{h_n}$ for some $h_n$. If $h_n \to 0$ as $n \to \infty$, there exists a convergent subspace in $L^2(\Omega)$ according to the discrete compactness of $X_h$ (Thm. 3.6). If $h_n \geq \delta > 0$, then $\{ w_n \}_{n=1}^\infty$ is contained in a finite dimensional space, and hence there exists a convergent subsequence. \hfill \Box

Theorem 3.8. Let $\mu$ be an eigenvalue of $A$ with multiplicity $m$ and $E(\mu)$ be the associated eigenspace. Let $\{ \phi_j \}_{j=1}^m$ be an orthonormal basis for $E(\mu)$. There exist exactly $m$ discrete eigenvalues $\mu_{j,h}$ and the associated eigenfunctions $\phi_{j,h}, j = 1, 2, \ldots, m$, of $A_h$ such that
\[
|\mu - \mu_{j,h}| \to 0, \text{ as } h \to 0, \tag{3.16}
\]
and
\[
|\mu - \mu_{j,h}| \leq C \max_{1 \leq i \leq m} \text{a}(\phi_i - \phi_{i,h}, \phi_i - \phi_{i,h}). \tag{3.17}
\]
Moreover, if $\phi \in H^{s-1}(\Omega)$ and $\nabla \times \phi \in H^s(\Omega)$ for any $\phi \in E(\mu)$, then, for $h$ small enough,
\[
|\mu - \mu_{j,h}| = O(h^{2(s-1)}). \tag{3.18}
\]
Proof. According to Theorem 4 of [29], it holds that
\[
|\mu - \mu_{j,h}| \leq C \left\{ \sum_{i,k=1}^m |(A - A_h)\phi_i, \phi_k| + \|(A - A_h)E(\mu)\|_2^2 \right\}.
\]
Since $\phi_i, A\phi_i \in X$, we have
\[
((A - A_h)\phi_i, \phi_k) = (\nabla \times \nabla \times (A - A_h)\phi_i, \nabla \times \nabla \times A\phi_k) + ((A - A_h)\phi_i, A\phi_k)
\]
\[
= (\nabla \times \nabla \times (A - A_h)\phi_i, \nabla \times \nabla \times (A - A_h)\phi_k) + ((A - A_h)\phi_i, (A - A_h)\phi_k)
\]
\[
\leq \|(A - A_h)\phi_i\| \|(A - A_h)\phi_k\|,
\]
which together with the fact that $E(\mu)$ is finite dimensional leads to
\[
|\mu - \mu_{j,h}| \leq C \max_{1 \leq i \leq m} \left\{ \|(A - A_h)\phi_i\|^2 + \|(A - A_h)E(\mu)\|_2^2 \right\}
\]
\[
\leq C \max_{1 \leq i \leq m} \|(A - A_h)\phi_i\|^2_{H(\text{curl}^2; \Omega)}.
\]
Then (3.16) follows from the pointwise convergence of $A_h$ to $A$ in $H(\text{curl}^2; \Omega)$ (Thm. 3.4). Since $\nabla \cdot \phi_i = 0$, it follows from (3.8) that
\[
\|(A - A_h)\phi_i\|_{H(\text{curl}^2; \Omega)} \leq \inf_{v_h \in X_h} \| A\phi_i - v_h \|_{H(\text{curl}^2; \Omega)}
\]
\[
= \inf_{v_h \in X_h} \mu \| \phi_i - (1/\mu)v_h \|_{H(\text{curl}^2; \Omega)} \leq \mu \| \phi_i - \phi_{i,h} \|_{H(\text{curl}^2; \Omega)}
\]
Furthermore, for the next section. The reason that we consider the quad-curl problem \((3.4)\) with the low-order term \(\mathcal{E}\) is to make \(\phi \in E(\mu)\). Furthermore, if \(\phi \in H^{s-1}(\Omega)\) and \(\nabla \times \phi \in H^s(\Omega)\), according to Theorem 3.5, we have that
\[
\| (A - A_h) \phi \|_{H(\text{curl}, \Omega)} \leq C h^{s-1} (\| A \phi \|_{s-1} + \| \nabla \times A \phi \|_s) \leq C \mu h^{s-1} (\| \phi \|_{s-1} + \| \nabla \times \phi \|_s).
\]
Since \(E(\mu)\) is finite dimensional, we obtain (3.18).

\[\square\]

Remark 3.9. The estimate \(|\mu - \mu_{i,h}| \leq C \max_{1 \leq i \leq m} a(\phi_i - \phi_{i,h}, \phi_i - \phi_{i,h})\) will be applied to obtain (4.2) in the next section. The reason that we consider the quad-curl problem \((3.4)\) with the low-order term \(u\) is to make \(\| \phi - \phi_h \|_{H(\text{curl}^2, \Omega)} \leq C \sqrt{a(\phi - \phi_h, \phi - \phi_h)}\) hold.

4. A posteriori error estimates for the eigenvalue problem

Assume that \((\lambda; u; p) \in \mathbb{R} \times H_0(\text{curl}^2; \Omega) \times H_0^1(\Omega)\) is a simple eigenpair of \((3.9)\) with \(|u|_0 = 1\) and \((\lambda_h; u_h; p_h) \in \mathbb{R} \times V_h^0 \times S_h^0\) is the associated finite element eigenpair of \((3.11)\) with \(|u_h|_0 = 1\). According to Theorem 3.8 and (3.28a) of [1], the following inequalities hold:

\[
\| u - u_h \| \leq C \rho_{\Omega}(h) \| u - u_h \|, \quad |\lambda_h - \lambda| \leq C \| u - u_h \|^2,
\]

where
\[
\rho_{\Omega}(h) = \sup_{f \in L^2(\Omega), \| f \|_1 \neq 0} \inf_{\nabla v \in V_h^0} \| Af - v \|_{H(\text{curl}^2; \Omega)}.
\]

It is obvious that \(\rho_{\Omega}(h) \to 0\) as \(h \to 0\).

Define two projection operators \(R_h, Q_h\) as follows. For \(u \in H_0(\text{curl}^2; \Omega)\) and \(p \in H_0^1(\Omega)\), find \(R_h u \in V_h^0, Q_h p \in S_h^0\), such that
\[
a(u - R_h u, v_h) + b(v_h, p - Q_h p) = 0, \quad \forall v_h \in V_h^0,
\]
\[
b(u - R_h u, q_h) = 0, \quad \forall q_h \in S_h^0.
\]

According to the orthogonality and the uniqueness of the discrete eigenvalue problem,
\[
u_h = (\lambda_h + 1) R_h A u_h.
\]

Let \((u_h^h; p_h^h)\) be the solution of \((3.5)\) with \(f = (\lambda_h + 1) u_h\). Then
\[
u_h = (\lambda_h + 1) A u_h \text{ and } u_h = R_h u^h.
\]

The following theorem relates the eigenvalue problem to the source problem \((3.5)\) with \(f = (\lambda_h + 1) u_h\).

Theorem 4.1. Let \(\rho_{\Omega}(h) = \rho_{\Omega}(h) + \| u - u_h \|\). It holds that
\[
\| u^h - R_h u_h \| - C r(h) \| u - u_h \| \leq \| u - u_h \| \leq \| u^h - R_h u_h \| + C r(h) \| u - u_h \|.
\]

Furthermore, for \(h\) small enough, there exist two constants \(c\) and \(C\) such that
\[
c \| u^h - R_h u_h \| \leq \| u - u_h \| \leq C \| u^h - R_h u_h \|.
\]
Proof. Since $u_h = R_h u^h$, by the triangle inequality, we have that

$$-\| u - u^h \| + \| u^h - R_h u^h \| \leq \| u - u_h \| \leq \| u - u^h \| + \| u^h - R_h u^h \|.$$  

Using the fact $u = (\lambda + 1)A u$ and (4.3), we obtain that

$$\| u - u^h \| = \| (\lambda + 1)A u - (\lambda_h + 1)A u_h \| \leq |\lambda + 1|\| A(u - u_h) \| + |\lambda - \lambda_h|\| A u_h \|. \quad (4.6)$$

Due to the well-posedness of (3.5), it holds that

$$\| A(u - u_h) \| \leq C \| u - u_h \|,$$

which, together with (4.1) and (4.2), leads to

$$\| u - u^h \| \leq Cr(h)\| u - u_h \|. \quad (4.7)$$

Then (4.4) follows immediately. Note that $r(h) \to 0$ as $h \to 0$. For $h$ small enough, (4.4) implies (4.5). □

Now to obtain an a posteriori error estimate for the eigenvalue problem, it suffices to derive an a posteriori error estimate for the source problem with $f = (\lambda_h + 1)u_h$. The exact solution and numerical solution are $(u^h, p^h)$ and $(u_h, 0)$, respectively.

Denote the total errors by $e := u^h - u_h$ and $\epsilon := p_h - 0 = p^h$. Then $e \in H_0(\text{curl}^2; \Omega)$ and $\epsilon \in H_0^1(\Omega)$ satisfy the defect equations

$$a(e, v) + b(v, \epsilon) = r_1(v), \quad \forall v \in H_0(\text{curl}^2; \Omega), \quad (4.8)$$

$$b(e, q) = r_2(\nabla q), \quad \forall q \in H_0^1(\Omega), \quad (4.9)$$

where

$$r_1(v) = (f, v) - ((\nabla \times)^2 u_h, (\nabla \times)^2 v) - (u_h, v),$$
$$r_2(\nabla q) = -(u_h, \nabla q).$$

We have the following Galerkin orthogonality

$$r_1(v_h) = 0, \quad \forall v_h \in V_h^0, \quad (4.10)$$
$$r_2(\nabla q_h) = 0, \quad \forall q_h \in S_h^0. \quad (4.11)$$

The error estimator will be constructed by employing Lemma 2.4. Writing $e = e^0 + e^\perp$ and $v = v^0 + v^\perp$ with $e^0, v^0 \in \nabla H_0^1(\Omega)$ and $e^\perp, v^\perp \in X$, we obtain that

$$\begin{align*}
(a(e^0, v^0) + (v^0, \nabla \epsilon) = r_1(v^0), & \quad \forall v^0 \in \nabla H_0^1(\Omega), \quad (4.12) \\
((\nabla \times)^2 e^\perp, (\nabla \times)^2 v^\perp) + (e^\perp, v^\perp) = r_1(v^\perp), & \quad \forall v^\perp \in X, \quad (4.13) \\
(e^0, \nabla q) = r_2(\nabla q), & \quad \forall q \in H_0^1(\Omega). \quad (4.14)
\end{align*}$$

The estimators for the irrotational part $e^0$, the solenoidal part $e^\perp$, and $\nabla \epsilon$ will be derived separately. Firstly, we consider the irrotational part $e^0$ and $\nabla \epsilon$. For a $\vartheta \in H_0^1(\Omega)$, we have

$$r_1(\nabla \vartheta) = \sum_{T \in T_h} (f - u_h, \nabla \vartheta)_T = \sum_{T \in T_h} - (\nabla \cdot f, \vartheta)_T + \sum_{f \in F_h^{\text{int}}} \left\langle \| n_f \cdot f \|, \vartheta \right\rangle_f$$
$$+ \sum_{T \in T_h} (\nabla \cdot u_h, \vartheta)_T - \sum_{f \in F_h^{\text{int}}} \left\langle \| n_f \cdot u_h \|, \vartheta \right\rangle_f,$$
where \( \mathcal{F}_h^{\text{int}} \) is the common face of two adjacent elements \( T_1, T_2 \in \mathcal{T}_h \), \( n_f \) is the unit normal vector of \( f \) directed towards the interior of \( T_1 \), and the jump

\[
\| n_f \cdot u_h \|_f = n_f \cdot u_h |_{T_2} - n_f \cdot u_h |_{T_1}.
\]

We also have

\[
(r_2(\nabla \vartheta) = \sum_{T \in \mathcal{T}_h} -\langle u_h, \nabla \vartheta \rangle_T \sum_{T \in T_h} \langle (\nabla \cdot u_h, \vartheta) \rangle_T - \sum_{f \in \mathcal{F}_h^{\text{int}}} \left\langle \| n_f \cdot u_h \|_f, \vartheta \right \rangle_f.
\]

We introduce the error terms which are related to the upper and lower bounds for \( e^0 \) and \( \nabla \varepsilon \):

\[
\eta_0^2 := \sum_{T \in \mathcal{T}_h} (\eta_0^T)^2 + \sum_{f \in \mathcal{F}_h^{\text{int}}} (\eta_0^f)^2,
\]

\[
\eta_3^2 := \sum_{T \in \mathcal{T}_h} (\eta_3^T)^2 + \sum_{f \in \mathcal{F}_h^{\text{int}}} (\eta_3^f)^2,
\]

where

\[
\eta_0^T := h_T \| \nabla \cdot f \|_T, \quad T \in \mathcal{T}_h,
\]

\[
\eta_0^f := h_f^{1/2} \| n_f \cdot f \|_f, \quad f \in \mathcal{F}_h^{\text{int}},
\]

\[
\eta_3^T := h_T \| \nabla \cdot u_h \|_T, \quad T \in \mathcal{T}_h,
\]

\[
\eta_3^f := h_f^{1/2} \| n_f \cdot u_h \|_f, \quad f \in \mathcal{F}_h^{\text{int}}.
\]

Next, we consider the bounds for \( e^\perp \). For \( w \in X \), the residual \( r_1(w) \) can be expressed as

\[
r_1(w) = \sum_{T \in \mathcal{T}_h} (f - u_h, w)_T - \left( (\nabla \times)^2 u_h, (\nabla \times)^2 w \right)_T
\]

\[
= \sum_{T \in \mathcal{T}_h} (f - (\nabla \times)^4 u_h - u_h, w)_T + \sum_{f \in \mathcal{F}_h^{\text{int}}} \left\langle \| (\nabla \times)^2 u_h \times n_f \|_f, \nabla \times w \right \rangle_f
\]

\[
+ \sum_{f \in \mathcal{F}_h^{\text{int}}} \left\langle \| (\nabla \times)^3 u_h \times n_f \|_f, w \right \rangle_f,
\]

where \( \| (\nabla \times)^2 u_h \times n_f \|_f \) and \( \| (\nabla \times)^3 u_h \times n_f \|_f \) stand for the jump of the tangential component of \( (\nabla \times)^2 u_h \) and \( (\nabla \times)^3 u_h \), respectively. The bounds for \( \| e^\perp \| \) contain the error terms

\[
\eta_1^2 := \sum_{T \in \mathcal{T}_h} (\eta_1^T)^2 + \sum_{f \in \mathcal{F}_h^{\text{int}}} \left( \eta_1^f \right)^2 + \sum_{f \in \mathcal{F}_h^{\text{int}}} \left( \eta_1^f \right)^2,
\]

\[
\eta_2^2 := \sum_{T \in \mathcal{T}_h} (\eta_2^T)^2,
\]

where

\[
\eta_1^T := h_T^2 \| \pi_h f - (\nabla \times)^4 u_h - u_h \|_T, \quad T \in \mathcal{T}_h,
\]

\[
\eta_1^f := h_f^2 \| f - \pi_h f \|_f, \quad T \in \mathcal{T}_h,
\]

\[
\eta_1^f := h_f^{1/2} \| n_f \cdot (\nabla \times)^2 u_h \|_f, \quad f \in \mathcal{F}_h^{\text{int}},
\]

\[
\eta_1^f := h_f^{3/2} \| n_f \cdot (\nabla \times)^3 u_h \|_f, \quad f \in \mathcal{F}_h^{\text{int}}.
\]
and \( \pi_h f \) denotes the \( L^2 \)-projection of \( f \) onto \( \{ v_h \in L^2(\Omega) : v_h|_T \in P_h(T), \forall T \in \mathcal{T}_h \} \). For \( T \in \mathcal{T}_h \), we define a local error indicator \( \eta_h(u_h, T) \) by

\[
\eta_h^2(u_h, T) = (\eta_0^T)^2 + (\eta_1^T)^2 + (\eta_2^T)^2 + \sum_{f \in T} ((\eta_0^f)^2 + (\eta_1^f)^2 + (\eta_2^f)^2 + (\eta_3^f)^2),
\]

and a global a posteriori error estimator by

\[
\eta_h^2(u_h, \Omega) = \eta_0^2 + \eta_1^2 + \eta_2^2.
\]

Now we state the a posteriori estimate for \( e \) and \( \varepsilon \) in the energy norm.

**Theorem 4.2.** Let \( \eta_0, \eta_1, \eta_2, \) and \( \eta_3 \) be defined in (4.15), (4.17), (4.18), and (4.16), respectively. Then if \( h < 1 \),

\[
\gamma_1(\eta_0 + \eta_1 + \eta_3) - \gamma_2 \eta_2 \leq \| e \| + \| \nabla \varepsilon \| \leq \Gamma_1(\eta_0 + \eta_1 + \eta_3) + \Gamma_2 \eta_2,
\]

and, if \( h \) is small enough,

\[
\gamma_3(\eta_1 + \eta_3) - \gamma_4(\eta_2 + C h^2 \eta_0) \leq \| e \| \leq \Gamma_3(\eta_0 + \eta_1 + \eta_3) + \Gamma_4 \eta_4,
\]

where \( \gamma_1, \gamma_2, \gamma_3, \gamma_4, \Gamma_1, \Gamma_2, \Gamma_3, \) and \( \Gamma_4 \) are some constants independent of \( h \).

Since \( f = (\lambda_h + 1)u_h \), according to the definition of \( \eta_0, \eta_2, \) and \( \eta_3 \), we have that \( \eta_0 = \lambda_h \eta_3 \) and \( \eta_2 = 0 \). Then by Theorems 4.1, 4.2, and (4.2), we can obtain the following a posteriori error estimates for the eigenvectors and eigenvalues.

**Theorem 4.3.** For \( h \) small enough, there exist constants \( c_1, C_1, \) and \( C_2 \) such that

\[
c_1(\eta_1 + \eta_3) \leq \| u - u_h \| \leq C_1(\eta_1 + (\lambda_h + 1)\eta_3),
\]

and

\[
|\lambda - \lambda_h| \leq C_2(\eta_1 + (\lambda_h + 1)\eta_3)^2,
\]

where \( \eta_1 \) and \( \eta_3 \) are respectively defined in (4.17) and (4.16) with \( f = (\lambda_h + 1)u_h \).

**The proof of Theorem 4.2.** Since \( e = e^0 + e^\perp \), the proof is split into three parts corresponding to \( e^0, e^\perp, \) and \( \varepsilon \), respectively.

(i) **Estimation of the irrotational part** \( e^0 \). Based on (4.14), we can rewrite \( e^0 = \nabla \varphi \) with \( \varphi \) solving the following uniformly positive definite variational problem on \( H^1_0(\Omega) \). Seek \( \varphi \in H^1_0(\Omega) \) such that

\[
(\nabla \varphi, \nabla q) = r_2(\nabla q), \quad \forall q \in H^1_0(\Omega).
\]

Note that \( r_2(\nabla q_h) = 0, \forall q_h \in S^0_h \). Define a projection operator \( P^k_h : H^1_0(\Omega) \rightarrow S^k_h \) such that (see, e.g., [3,29,31])

\[
P^k_h \phi = \phi, \quad \forall \phi \in S^0_h, \quad (4.20)
\]

\[
\| \phi - P^k_h \phi \|_T \leq Ch_T \| \nabla \phi \|_{\omega_T}, \quad (4.21)
\]

\[
\| \phi - P^k_h \phi \|_f \leq C \sqrt{h_T} \| \nabla \phi \|_{\omega_f}, \quad (4.22)
\]

\[
\| \nabla P^k_h \phi \|_T \leq C \| \nabla \phi \|_{\omega_T}, \quad (4.23)
\]
where \( \omega_T \) is defined in Theorem 2.3 and \( \omega_f \) is the union of elements sharing at least one vertex with \( f \in \mathcal{F}_h \). Due to (4.14) and the orthogonal property (4.11), we have that

\[
\|e^0\|^2 = r_2(e^0) = r_2(\nabla \varphi - \nabla P_h^k \varphi) = -(u_h, \nabla (\varphi - P_h^k \varphi)).
\]  

(4.24)

Using integration by parts, (4.21), and (4.22), we obtain that

\[
\left( u_h, \nabla (\varphi - P_h^k \varphi) \right) = \sum_{T \in \mathcal{T}_h} \left( \nabla \cdot u_h, \varphi - P_h^k \varphi \right)_T + \sum_{f \in \mathcal{F}_h^\text{int}} \langle \|n_f \cdot u_h\|_f, \varphi - P_h^k \varphi \rangle_f
\]

\[
\leq C \sum_{T \in \mathcal{T}_h} \|\nabla \cdot u_h\|_T h_T \|\nabla \varphi\|_{\omega_T} + C \sum_{f \in \mathcal{F}_h^\text{int}} \|n_f \cdot u_h\|_f \sqrt{h_f} \|\nabla \varphi\|_{\omega_f}
\]

\[
\leq C \left( \sum_{T \in \mathcal{T}_h} \|\nabla \cdot u_h\|_T^2 h_T^2 \right)^{1/2} \|e^0\| + C \left( \sum_{f \in \mathcal{F}_h^\text{int}} \|n_f \cdot u_h\|_f^2 h_f \right)^{1/2} \|e^0\|.
\]

Therefore, we have

\[
\|e^0\| \leq C \eta_3.
\]  

(4.25)

We now derive lower bounds for \( e^0 \) using the bubble functions.

Denote by \( \lambda_1^T, \lambda_2^T, \lambda_3^T, \lambda_4^T \) the barycentric coordinates of \( T \in \mathcal{T}_h \) and define the bubble function \( b_T \) by

\[
b_T = \begin{cases} 
256 \lambda_1^T \lambda_2^T \lambda_3^T \lambda_4^T, & \text{on } T, \\
0, & \text{on } \Omega \setminus T.
\end{cases}
\]

Given \( f \in \mathcal{F}_h \), a common edge of \( T_1 \) and \( T_2 \), let \( \varpi_f = T_1 \cup T_2 \) and enumerate the vertices of \( T_1 \) and \( T_2 \) such that the vertices of \( f \) are numbered first. Define the face-bubble function \( b_f \) by

\[
b_f = \begin{cases} 
27 \lambda_1^T \lambda_2^T \lambda_3^T, & \text{on } T_i, \ i = 1, 2, \\
0, & \text{on } \Omega \setminus \varpi_f.
\end{cases}
\]

Using the technique of Lemma 3.3 in [37], we have the following norm equivalences.

\[
\|b_T \phi_h\|_T \leq \|\phi_h\|_T \leq C \|b_T^{1/2} \phi_h\|_T, \quad \forall \phi_h \in P_h(T),
\]

(4.26)

\[
\|b_f \phi_h\|_f \leq \|\phi_h\|_f \leq C \|b_f^{1/2} \phi_h\|_f, \quad \forall \phi_h \in P_f(f).
\]

(4.27)

Using (4.26), integration by parts, the inverse inequality, and the fact that \( b_T \nabla \cdot u_h \in H_0^1(T) \subset H_0^1(\Omega) \), we have that

\[
\frac{(\eta_3^T)^2}{h_T^2} = \|\nabla \cdot u_h\|_T^2 \leq C \|b_T^{1/2} \nabla \cdot u_h\|_T^2 = C (\nabla \cdot u_h, b_T \nabla \cdot u_h)_T
\]

\[
= -C (u_h, \nabla (b_T \nabla \cdot u_h))_T = C r_2 (\nabla (b_T \nabla \cdot u_h))
\]

\[
= C \left( e^0, \nabla (b_T \nabla \cdot u_h) \right)_T \leq C \|e^0\|_T \|\nabla (b_T \nabla \cdot u_h)\|_T
\]

\[
\leq \frac{C}{h_T} \|e^0\|_T \|\nabla \cdot u_h\|_T \leq \frac{C}{h_T^2} \|e^0\|_T \|\eta_3^T\|_T,
\]

which implies that

\[
\eta_3^T \leq C \|e^0\|_T.
\]

(4.28)
We define a continuous operator $P_T : L^p(f) \rightarrow L^p(T)$ as in [37]. According to Lemma 3.3 of [37],
\[
\|P_T \sigma\|_T \leq C h_T^{1/2} \|\sigma\|_f.
\]

Denote $\|n_f \cdot u_h\|_{f; T_i} = P_T \|n_f \cdot u_h\|_f$, then
\[
\|\|n_f \cdot u_h\|_{f; T_i}\|_{T_i} \leq C h_T^{1/2} \|\|n_f \cdot u_h\|_f\|_f.
\]

The estimate of the local upper bound for $\eta^f_3$ can be obtained similarly:

\[
\frac{(\eta^f_3)^2}{h_f} = \|\|n_f \cdot u_h\|_f\|_f^2 \leq C \|\|n_f \cdot u_h\|_{f; T_i}\|_f^2
\]
\[
= C \sum_{i=1}^2 \left( \left( u_h, \nabla (b_f \|n_f \cdot u_h\|_{f; T_i}) \right)_{T_i} + \left( \nabla \cdot u_h, b_f \|n_f \cdot u_h\|_{f; T_i} \right)_{T_i} \right)
\]
\[
= C \sum_{i=1}^2 \left( \left( e^0, \nabla (b_f \|n_f \cdot u_h\|_{f; T_i}) \right)_{T_i} + \left( \nabla \cdot u_h, b_f \|n_f \cdot u_h\|_{f; T_i} \right)_{T_i} \right)
\]
\[
\leq C \sum_{i=1}^2 (h_T^{-1} \|e^0\|_{T_i} + \|\nabla \cdot u_h\|_{T_i}) \eta^f_3,
\]

where we have used

\[
\|\nabla (b_f \|n_f \cdot u_h\|_{f; T_i})\|_{T_i} \leq C h_T^{-1} \|b_f \|n_f \cdot u_h\|_{f; T_i}\|_{T_i} \leq C h_T^{-1/2} \|\|n_f \cdot u_h\|_f\|_f.
\]

Consequently,

\[
\eta^f_3 \leq C (\|e^0\|_{\omega_f} + \eta^T_3 + \eta^T_2) \leq C \|e^0\|_{\omega_f}.
\]

Now collecting (4.25), (4.28), and (4.30), we have that

\[
e \eta_3 \leq \|e^0\| \leq C \eta_3.
\]

(ii) **Estimation of $\nabla \varepsilon$.** Similar to the upper estimate of $e^0$, we can obtain an upper bound for $\|\nabla \varepsilon\|$. Due to (4.12) and (4.14), we have

\[
\|\nabla \varepsilon\|^2 = r_1(\nabla \varepsilon) - r_2(\nabla \varepsilon) = r_1(\nabla (\varepsilon - P_h^k \varepsilon)) - r_2(\nabla (\varepsilon - P_h^k \varepsilon)) = (f, \nabla (\varepsilon - P_h^k \varepsilon)).
\]

From integration by parts, (4.21), and (4.22),

\[
\|\nabla \varepsilon\|^2 = \sum_{T \in T_h} -(\nabla \cdot f, \varepsilon - P_h^k \varepsilon)_{T} + \sum_{f \in F_{int}^h} \langle \|n_f \cdot f\|_f, \varepsilon - P_h^k \varepsilon \rangle_f
\]
\[
\leq \sum_{T \in T_h} \|\nabla \cdot f\|_{h_T T} \|\varepsilon\|_{\omega_T} + \|\|n_f \cdot f\|_f \|_{\sqrt{h_f} \|\nabla \varepsilon\|_{\omega_f}}
\]
\[
\leq C \|\nabla \varepsilon\| \left( \left( \sum_{T \in T_h} \|\nabla \cdot f\|^2_{h_T^2} \right)^{1/2} + \left( \sum_{f \in F_{int}^h} \|\|n_f \cdot f\|_f^2 \right)^{1/2} \right)^{1/2}.
\]

Therefore, we have that

\[
\|\nabla \varepsilon\| \leq C \eta_0.
\]
By a similar argument to the lower estimate of $\varepsilon^0$, we have the lower bounds for $\nabla \varepsilon$:

$$\eta_0^T \leq \|\nabla \varepsilon\|_T, \quad (4.33)$$

$$\eta_0^{T_f} \leq C(\|\nabla \varepsilon\|_{\pi_f} + \eta_0^{T_1} + \eta_0^{T_2}) \leq C\|\nabla \varepsilon\|_{\pi_f}. \quad (4.34)$$

Combining (4.32), (4.33), and (4.34), we arrive at

$$c\eta_0 \leq \|\nabla \varepsilon\| \leq C\eta_0. \quad (4.35)$$

(iii) Estimation of the solenoidal part $\varepsilon^1$. We start with proving the upper bound for $\eta_0^T$ by using $b_T$ again. Employing the similar technique of Lemma 3.3 in [37], we have the following estimates for any $v$ in finite dimensional spaces:

$$\|v\|_T^2 \leq C\|b_T v\|_T^2, \quad (4.36)$$

$$\|b_T^* v\|_T^2 \leq C\|v\|_T^2. \quad (4.37)$$

Let $\phi_h = \pi_h f - (\nabla \times)^4 u_h - u_h$, we then have

$$\left(\frac{\eta_1^T}{h_T^2}\right)^2 \leq \|\pi_h f - (\nabla \times)^4 u_h - u_h\|_T^2 \leq C \left(f - (\nabla \times)^4 u_h - u_h, b_T^2 \phi_h\right)_T + C \left(\pi_h f - f, b_T^2 \phi_h\right)_T \quad \text{(by (4.36))}$$

$$= Cr_1(b_T^2 \phi_h) + C \left(\pi_h f - f, b_T^2 \phi_h\right)_T \quad \text{(by (4.36))}$$

$$\leq C\|v\|_T\|b_T^2 \phi_h\|_T + C\|\nabla \varepsilon\|_T \|b_T^2 \phi_h\|_T + C\eta_2^T h_T^{-2} \|b_T^2 \phi_h\|_T. \quad (4.38)$$

Due to the inverse inequality and (4.37), it holds that

$$\|b_T^2 \phi_h\|_T^2 = \|b_T^2 \phi_h\|_T^2 \|\nabla \varepsilon\|_T \leq C h_T^{-4} \|b_T^2 \phi_h\|_T^2 \leq C h_T^{-4} \|\phi_h\|_T^2. \quad (4.39)$$

Thus we obtain that

$$\left(\frac{\eta_1^T}{h_T^2}\right)^2 \leq C \frac{\eta_1^T}{h_T^2} \left(h_T^{-2}\|v\|_T + \|\nabla \varepsilon\|_T + h_T^{-2} \eta_2^T\right).$$

Dividing the above inequality by $\frac{\eta_1^T}{h_T^2}$ and multiplying by $h_T^2$, we obtain

$$\eta_1^T \leq C \left(\|v\|_T + h_T^2 \|\nabla \varepsilon\|_T + \eta_2^T\right). \quad (4.38)$$

Next we estimate the upper bound for $\eta_{1,1}^T$ by using the bubble functions $b_T$ and $b_f$. Let $T_1$ and $T_2$ be two elements sharing the face $f$. Denote $\psi_h |_{T_i} = P_{T_i} \|n_f \times (\nabla \times)^2 u_h\|_{T_i}$ for $i = 1, 2$, then

$$\|\psi_h |_{T_i}\|_{T_i} \leq Ch_{T_i}^{1/2} \|n_f \times (\nabla \times)^2 u_h\|_{T_i}. \quad (4.39)$$

Denote $\omega_{f,1} = (b_{T_1} - b_{T_2}) b_f n_f \times \psi_h$. A simple calculation shows that

$$(\nabla \times \omega_{f,1}) |_{T_i} = \frac{256}{27} \left(\frac{S_f}{3|T_1|} + \frac{S_f}{3|T_2|}\right) b_T^2 \psi_h,$$
where \( S_f \) stands for the area of the face \( f \). Similar to (4.36) and (4.37), the following inequalities hold

\[
\|v\|_f \leq C\|b_f v\|_f, \tag{4.40}
\]
\[
\|(b_{T_1} - b_{T_2})b_f v\|_{\partial \Omega} \leq C\|v\|_{\partial \Omega}. \tag{4.41}
\]

Now we are ready to construct the upper bound for \( \eta_{11}^f \):

\[
h_f^{-1} \|\nabla \times (\nabla \times)^2 u_h\|_f 
\leq C\langle \|\nabla \times (\nabla \times)^2 u_h\|_f, \nabla \times \omega_{f,1} \rangle_f \quad \text{(by \ (4.40))}
\]
\[
= C((\nabla \times)^2 u_h, \omega_{f,1}) - C((\nabla \times)^2 u_h, (\nabla \times)^2 \omega_{f,1})
\]
\[
= C\|\omega_{f,1}\| - C((f - u_h - (\nabla \times)^4 u_h, \omega_{f,1}) \quad (\omega_{f,1} \in H_0(\text{curl}^2; \Omega))
\]
\[
\leq C\|e\|_{\partial \Omega} \|\omega_{f,1}\|_f + C\|\omega_{f,1}\|_f \left(\|\nabla \varepsilon\|_f + \sum_{i=1}^{2} h_{T_i}^{-2}(\eta_{11}^{T_i} + \eta_{22}^{T_i})\right).
\]

By applying the inverse inequality, (4.39), and (4.41), we get

\[
\|\omega_{f,1}\|_f \leq h_f^{-2}\|\omega_{f,1}\|_f \leq h_f^{-3/2}\|\nabla \times (\nabla \times)^2 u_h\|_f,
\]

which, together with (4.38), leads to

\[
\eta_{11}^f \leq C\left(\|e\|_{\partial \Omega} + \eta_{22}^{T_1} + \eta_{22}^{T_2} + h_{T_1}^2 \|\nabla \varepsilon\|_{T_1} + h_{T_2}^2 \|\nabla \varepsilon\|_{T_2}\right). \tag{4.42}
\]

The upper bound for \( \eta_{12}^f \) can be constructed in a similar way. Extend \( \|\nabla \times (\nabla \times)\nabla u_h\|_f \) to \( \|\nabla \times (\nabla \times)^3 u_h\|_{f:T_i} \) on \( T_i \) such that

\[
\|\nabla \times (\nabla \times)^3 u_h\|_{f:T_i} \leq C\!h_i^{1/2}\|\nabla \times (\nabla \times)^3 u_h\|_f. \tag{4.43}
\]

Denote \( \omega_{f,2|T_i} = b_f^2\|\nabla \times (\nabla \times)^3 u_h\|_{f:T_i} \), then

\[
\|\nabla \times (\nabla \times)^3 u_h\|_f \leq C\langle \|\nabla \times (\nabla \times)^3 u_h\|_f, \omega_{f,2|T_i} \rangle_f 
\leq C(\nabla \times)^3 u_h, \omega_{f,2|T_i} \rangle_f
\leq (\nabla \times)^3 u_h, \omega_{f,2|T_i} \rangle_f
\leq C\|\nabla \times (\nabla \times)^3 u_h\|_f \|\omega_{f,2|T_i}\|_f
\leq C\|\nabla \times (\nabla \times)^3 u_h\|_f
\leq C\|\nabla \times (\nabla \times)^3 u_h\|_f.
\]

Dividing the above inequality by \( \|\nabla \times (\nabla \times)^3 u_h\|_f \) and applying (4.38) and (4.42), we obtain

\[
\eta_{12}^f \leq C \left(\eta_{22}^{T_1} + \eta_{22}^{T_2} + \|e\|_{T_1 \omega T_2} + h_{T_1}^2 \|\nabla \varepsilon\|_{T_1} + h_{T_2}^2 \|\nabla \varepsilon\|_{T_2}\right). \tag{4.44}
\]

Collecting (4.38), (4.42), and (4.44), we have that

\[
\eta_1 \leq C \left(\eta_2 + \|e\| + h^2 \|\nabla \varepsilon\|\right). \tag{4.45}
\]
It remains to construct the upper bound of \( e^+ \). For \( e^+ \in X \subset H_0(\text{curl}^2; \Omega) \), according to Lemma 2.4, \( e^+ = w + \nabla \psi \) with \( w \in H^2(\Omega) \) and \( \psi \in H^1_0(\Omega) \). Then we have
\[
\|e^+\|^2 = r_1(e^+) = r_1(w) + r_1(\nabla \psi).
\]
Due to the Galerkin orthogonality (4.10), for any \( w_h \in V_h^0 \),
\[
r_1(w) = r_1(w - w_h)
\]
\[
= \sum_{T \in T_h} \left( (f - u_h - (\nabla \times)^4 u_h, w - w_h)_T + \sum_{f \in F_h(T)} \langle n_f \times (\nabla \times)^3 u_h, w - w_h \rangle_f \right)
\]
\[
- \sum_{f \in F_h(T)} \langle n_f \times (\nabla \times)^3 u_h, \nabla \times (w - w_h) \rangle_f \right) \]
\[
\leq \sum_{T \in T_h} \left( \|\pi_h f - u_h - (\nabla \times)^4 u_h\|_T \|w - w_h\|_T + \|\pi_h f - f\|_T \|w - w_h\|_T \right)
\]
\[
+ \sum_{f \in F_h(T)} \|h_T^{-1}\|_f \|n_f \times (\nabla \times)^3 u_h\|_f \|\nabla \times (w - w_h)\|_f
\]
\[
+ \sum_{f \in F_h(T)} \|h_T^{-1}\|_f \|n_f \times (\nabla \times)^3 u_h\|_f \|w - w_h\|_f
\]
\[
\leq C(\eta_1 + \eta_2) \left( \sum_{T \in T_h} \left( h_T^{-4}\|w - w_h\|_T^2 + \sum_{f \in F_h(T)} h_f^{-1}\|\nabla \times (w - w_h)\|_f^2 \right) \right)^{1/2}
\]
\[
+ \sum_{f \in F_h(T)} h_f^{-3}\|w - w_h\|_f \right)
\]
Let \( w_h = \Pi_h^2 w \). According to the trace inequality and Theorem 2.3, we obtain
\[
\sum_{T \in T_h} \left( h_T^{-4}\|w - w_h\|_T^2 + \sum_{f \in F_h(T)} h_f^{-1}\|\nabla \times (w - w_h)\|_f^2 + \sum_{f \in F_h(T)} h_f^{-3}\|w - w_h\|_f \right) \leq C\|w\|_2^2.
\]
Furthermore, we use (2.6), (2.7), and the Poincaré inequality to obtain
\[
r_1(w) \leq C(\eta_1 + \eta_2)\|w\|_2 \leq C(\eta_1 + \eta_2)\|\nabla \times e^+\|_1 \leq C(\eta_1 + \eta_2)\|e^+\|.
\]
Similar to the proof of (4.32), using (2.8), it holds that
\[
r_1(\nabla \psi) \leq C(\eta_0 + \eta_3)\|\nabla \psi\| \leq C(\eta_0 + \eta_3)\|e^+\|.
\]
Hence,
\[
\|e^+\| \leq C(\eta_0 + \eta_1 + \eta_2 + \eta_3).
\]
Combining (4.32), (4.35), (4.45), and (4.46), we obtain Theorem 4.2.

5. NUMERICAL EXAMPLES

In this section, we will present some numerical results in 2D. The quad-curl problem in 2D is
\[
(\nabla \times)^2 u = \lambda u \quad \text{in} \ \Omega,
\]
\[
\nabla \cdot u = 0 \quad \text{in} \ \Omega,
\]
\[
u \times n = 0 \quad \text{on} \ \partial \Omega,
\]
\[
\nabla \times u = 0 \quad \text{on} \ \partial \Omega,
\]
(5.1)
Figure 1. Sample meshes for $\Omega_1$ (left), $\Omega_2$ (middle), and $\Omega_3$ (right).

where $\nabla \times u = \hat{\nabla}u_{x_2} - \hat{\nabla}u_{x_1}$ for $u = (u_1, u_2)$ and $\nabla \times u = (\hat{\nabla}u_{x_2}, -\hat{\nabla}u_{x_1})$ for a scalar $u$. Since the outermost $\nabla \times$ is acting on a scalar, the function $\sigma = \nabla \times u$ satisfies

$$\Delta \sigma \in H^1(\Omega) \quad \text{in } \Omega,$$

$$\sigma = 0 \quad \text{on } \partial \Omega.$$ (5.2)

When $\Omega$ is a polygon, according to Theorem 14.6 of [18], $\sigma \in H^{1+\pi/\omega-\epsilon}(\Omega)$ for any $\epsilon > 0$. Here $\omega$ is the largest interior angle at the corners of $\Omega$.

Since $u \in X$, according to Lemma 2.4, $u = \nabla \phi + v$ with $\phi \in H^1_0(\Omega)$ and $v \in H^2(\Omega)$. In addition, according to the proof of Lemma 2.4, $\phi$ actually satisfies $\Delta \phi = H^1(\Omega)$, and hence $\phi \in H^{1+\pi/\omega-\epsilon}(\Omega)$. Therefore $u \in H^{\min(\pi/\omega-\epsilon,2)}(\Omega)$.

5.1. A priori error estimates

We consider three different domains:

- $\Omega_1$: the unit square given by $(0,1) \times (0,1)$,
- $\Omega_2$: the L-shaped domain given by $(0,1) \times (0,1)/([1/2,1] \times (0,1/2])$,
- $\Omega_3$: given by $(0,1) \times (0,1)/([1/4,3/4] \times [1/4,3/4])$.

The eigenvectors on $\Omega_1$ are in $\{u \in H^{2-\epsilon}(\Omega) : \nabla \times u \in H^3-\epsilon(\Omega)\}$. The eigenvectors on $\Omega_2$ are in $\{u \in H^{2/3-\epsilon}(\Omega) : \nabla \times u \in H^{5/3-\epsilon(\Omega)}\}$. According to Theorem 3.8, the convergence orders for $\Omega_1$ and $\Omega_2$ are 4 and $4/3$, respectively.

The initial meshes of the domains are shown in Figure 1. In Tables 1–3, we list the first five eigenvalues. Since the exact eigenvalues are unknown, the relative error is adopted:

$$\text{Error} = \left| \frac{\lambda_i^h - \lambda_i^{h/2}}{\lambda_i^{h/2}} \right| .$$

Table 1. The first 5 eigenvalues of $\Omega_1$ with the fourth-order elements.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$\lambda_1^h$</th>
<th>$\lambda_2^h$</th>
<th>$\lambda_3^h$</th>
<th>$\lambda_4^h$</th>
<th>$\lambda_5^h$</th>
</tr>
</thead>
<tbody>
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<td>1/4</td>
<td>7.08101988e+02</td>
<td>7.08102390e+02</td>
<td>2.35145718e+03</td>
<td>4.25922492e+03</td>
<td>5.02522026e+03</td>
</tr>
<tr>
<td>1/8</td>
<td>7.07978763e+02</td>
<td>7.07978786e+02</td>
<td>2.35006082e+03</td>
<td>4.25597055e+03</td>
<td>5.02401495e+03</td>
</tr>
<tr>
<td>1/16</td>
<td>7.07971973e+02</td>
<td>7.07971975e+02</td>
<td>2.3499027e+03</td>
<td>4.25582307e+03</td>
<td>5.02399272e+03</td>
</tr>
<tr>
<td>1/32</td>
<td>7.07971564e+02</td>
<td>7.07971564e+02</td>
<td>2.34988613e+03</td>
<td>4.25581473e+03</td>
<td>5.02399235e+03</td>
</tr>
<tr>
<td>1/64</td>
<td>7.07971528e+02</td>
<td>7.07971555e+02</td>
<td>2.34985878e+03</td>
<td>4.25581421e+03</td>
<td>5.02399235e+03</td>
</tr>
</tbody>
</table>
Table 2. The first 5 eigenvalues of $\Omega_2$ with the fourth-order elements.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$\lambda_1^h$</th>
<th>$\lambda_2^h$</th>
<th>$\lambda_3^h$</th>
<th>$\lambda_4^h$</th>
<th>$\lambda_5^h$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/4</td>
<td>5.34885649e+02</td>
<td>1.57586875e+03</td>
<td>6.10288551e+03</td>
<td>6.40711482e+03</td>
<td>1.09459861e+04</td>
</tr>
<tr>
<td>1/8</td>
<td>5.35061810e+02</td>
<td>1.57477474e+03</td>
<td>6.09565399e+03</td>
<td>6.39162468e+03</td>
<td>1.09184358e+04</td>
</tr>
<tr>
<td>1/16</td>
<td>5.3522062e+02</td>
<td>1.57468312e+03</td>
<td>6.09528577e+03</td>
<td>6.37101466e+03</td>
<td>1.09152964e+04</td>
</tr>
<tr>
<td>1/32</td>
<td>5.3529267e+02</td>
<td>1.57467206e+03</td>
<td>6.09528045e+03</td>
<td>6.36787675e+03</td>
<td>1.09143027e+04</td>
</tr>
<tr>
<td>1/64</td>
<td>5.35320748e+02</td>
<td>1.57466664e+03</td>
<td>6.09528434e+03</td>
<td>6.36661570e+03</td>
<td>1.09139180e+04</td>
</tr>
</tbody>
</table>

Table 3. The first 5 non-zero eigenvalues of $\Omega_3$ with the fourth-order elements.

<table>
<thead>
<tr>
<th>$h$</th>
<th>$\lambda_1^h$</th>
<th>$\lambda_2^h$</th>
<th>$\lambda_3^h$</th>
<th>$\lambda_4^h$</th>
<th>$\lambda_5^h$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/4</td>
<td>9.43570924e+02</td>
<td>9.43570924e+02</td>
<td>3.35118080e+03</td>
<td>5.10757870e+03</td>
<td>1.03672699e+04</td>
</tr>
<tr>
<td>1/8</td>
<td>9.40543704e+02</td>
<td>9.40543704e+02</td>
<td>3.33230800e+03</td>
<td>5.11255084e+03</td>
<td>1.03470233e+04</td>
</tr>
<tr>
<td>1/16</td>
<td>9.39507116e+02</td>
<td>9.39507116e+02</td>
<td>3.32612997e+03</td>
<td>5.11519580e+03</td>
<td>1.03445476e+04</td>
</tr>
<tr>
<td>1/32</td>
<td>9.39103168e+02</td>
<td>9.39103168e+02</td>
<td>3.32373447e+03</td>
<td>5.11630255e+03</td>
<td>1.03438189e+04</td>
</tr>
<tr>
<td>1/64</td>
<td>9.38943028e+02</td>
<td>9.38943036e+02</td>
<td>3.32278551e+03</td>
<td>5.11679450e+03</td>
<td>1.03435487e+04</td>
</tr>
</tbody>
</table>

Table 4. Convergence rate for $\Omega_1$ with the fourth-order elements (relative error).

<table>
<thead>
<tr>
<th>$h$</th>
<th>$\lambda_1^h$</th>
<th>Error</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/4</td>
<td>7.08101988e+02</td>
<td>1.74021691e-04</td>
<td>–</td>
</tr>
<tr>
<td>1/8</td>
<td>7.07978763e+02</td>
<td>9.59045415e-06</td>
<td>4.1815</td>
</tr>
<tr>
<td>1/16</td>
<td>7.07979173e+02</td>
<td>5.77928313e-07</td>
<td>4.0527</td>
</tr>
<tr>
<td>1/32</td>
<td>7.07976564e+02</td>
<td>5.08588883e-08</td>
<td>3.5063</td>
</tr>
<tr>
<td>1/64</td>
<td>7.07976682e+02</td>
<td>–</td>
<td>–</td>
</tr>
</tbody>
</table>

Table 5. Convergence rate for $\Omega_2$ with the fourth-order elements (relative error).

<table>
<thead>
<tr>
<th>$h$</th>
<th>$\lambda_1^h$</th>
<th>Error</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/4</td>
<td>5.34885649e+02</td>
<td>3.29341761e-04</td>
<td>–</td>
</tr>
<tr>
<td>1/8</td>
<td>5.35061810e+02</td>
<td>2.99502830e-04</td>
<td>0.1370</td>
</tr>
<tr>
<td>1/16</td>
<td>5.3522062e+02</td>
<td>1.31169871e-04</td>
<td>1.1911</td>
</tr>
<tr>
<td>1/32</td>
<td>5.3529267e+02</td>
<td>5.32057764e-05</td>
<td>1.3018</td>
</tr>
<tr>
<td>1/64</td>
<td>5.35320748e+02</td>
<td>–</td>
<td>–</td>
</tr>
</tbody>
</table>

Table 6. Convergence rate for $\Omega_3$ with the fourth-order elements (relative error).

<table>
<thead>
<tr>
<th>$h$</th>
<th>$\lambda_1^h$</th>
<th>Error</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/4</td>
<td>9.43570924e+02</td>
<td>3.29341761e-04</td>
<td>–</td>
</tr>
<tr>
<td>1/8</td>
<td>9.40543704e+02</td>
<td>2.99502830e-04</td>
<td>0.1370</td>
</tr>
<tr>
<td>1/16</td>
<td>9.39507116e+02</td>
<td>1.31169871e-04</td>
<td>1.1911</td>
</tr>
<tr>
<td>1/32</td>
<td>9.39103168e+02</td>
<td>5.32057764e-05</td>
<td>1.3018</td>
</tr>
<tr>
<td>1/64</td>
<td>9.38943028e+02</td>
<td>–</td>
<td>–</td>
</tr>
</tbody>
</table>

Tables 4-6 show the convergence rates of the relative errors for the first eigenvalues. We can observe a convergence order 4 for $\Omega_1$ and 4/3 for $\Omega_2$, which agrees with the theoretical results. We can also observe a convergence order 4/3 for $\Omega_3$ even if this case is not covered in the theoretical analysis.
5.2. *A posteriori* error estimates

Figure 2 shows global error estimators $\eta_h^2(u_h, \Omega)$ and the relative errors of some simple eigenvalues for the three domains. It can be observed that the relative errors and the estimators have the same convergence rates, which confirms the upper bound estimate for the simple eigenvalues. Figure 3 shows the distribution of the local indicators $\eta_h(u_h, T)$. The estimators are large at corners and catch the singularities effectively.
6. Conclusion

We proved a priori and robust a posteriori error estimates for the $H(\text{curl}^2)$-conforming finite element method when solving the quad-curl eigenvalue problem. Due to a new decomposition of the function in $H(\text{curl}^2; \Omega)$, the theory assumes no extra regularity of the eigenfunctions. The a posteriori error estimator is essential for the adaptive finite element method. It can also be applied to test spurious eigenvalues.

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References


A PRIORI AND A POSTERIORI ERROR ESTIMATES


