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A C^0 INTERIOR PENALTY METHOD FOR mTH-LAPLACE EQUATION

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Abstract. In this paper, we propose a C^0 interior penalty method for mth-Laplace equation on bounded Lipschitz polyhedral domain in $\mathbb{R}^d$, where m and d can be any positive integers. The standard H^1-conforming piecewise r-th order polynomial space is used to approximate the exact solution u, where r can be any integer greater than or equal to m. Unlike the interior penalty method in Gudi and Neilan [*IMA J. Numer. Anal.* **31** (2011) 1734–1753], we avoid computing D^m of numerical solution on each element and high order normal derivatives of numerical solution along mesh interfaces. Therefore our method can be easily implemented. After proving discrete H^m-norm bounded by the natural energy semi-norm associated with our method, we manage to obtain stability and optimal convergence with respect to discrete H^m-norm. The error estimate under the low regularity assumption of the exact solution is also obtained. Numerical experiments validate our theoretical estimate.

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1. Introduction

We consider the mth-Laplace equation

$$(-1)^m \Delta^m u = f \qquad \text{in} \quad \Omega, \tag{1.1a}$$

$$u = \frac{\partial u}{\partial \boldsymbol{\nu}} = \cdots = \frac{\partial^{m-1} u}{\partial \boldsymbol{\nu}^{m-1}} = 0 \quad \text{on} \quad \partial\Omega, \tag{1.1b}$$

where m is an arbitrary positive integer, Ω is a bounded Lipschitz polyhedral domain in $\mathbb{R}^d$ $(d = 1, 2, 3, \cdots)$, and $\boldsymbol{\nu}$ is the outward unit normal vector field along $\partial\Omega$. The source term $f \in H^{-1}(\Omega)$.

Several works have been done to solve numerically (1.1). Standard H^m conforming finite elements space requires C^{m-1} continuity and leads to complicated construction of finite element space and lots of degrees of freedom when m is large. Bramble and Zlámal [4] studied the H^m conforming finite elements space on the two dimensional triangular meshes. Meanwhile, a H^m conforming finite element space is developed by Hu and Zhang on rectangular grids for arbitrary d in [14]. Recently, Hu *et al.* introduce a H^m-conforming finite element space

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on simplicial meshes for any d in [16]. The finite element space in [16] contains piecewise r-th order polynomials with $r \geq 2^d m + 1$. Therefore, the polynomial order of finite element space in [16] is quite big. Though up to this moment they have above mentioned restrictions, conforming H^m finite element spaces are desirable in both theoretical analysis and practice. In order to simplify the construction of H^m finite element space, alternative H^m nonconforming finite element space is introduced in several works. In [22], a H^m nonconforming finite element space (named Morley–Wang–Xu elements) is introduced for $m \leq d$. Besides, Hu and Zhang also considered the H^m nonconforming finite element space in [15] on triangular meshes for $d = 2$. The finite element space in [22] is generalized for $m = d + 1$ by Wu and Xu [25]. Recently in [24], it is further generalized for arbitrary m and d but with stabilization along mesh interface in order to balance the weak continuity and the penalty terms. In order to obtain stability and optimal convergence in some discrete H^m-norm, Wang and Xu [22], Wu and Xu [24,25] propose to compute numerical approximation to $D^m u$, such that their implementation may become quite complicated as m is large. The finite element spaces in [4, 14] can be used to solve numerically (1.1) with any source term $f \in H^{-m}(\Omega)$. However, the implementation of these conforming and nonconforming finite element spaces can be quite challenging for large m. Virtual element methods have been investigated for (1.1). In [1], a conforming H^m virtual element method is introduced for convex polygonal domain in $\mathbb{R}^2$. The finite element space in [1] contains piecewise r-th order polynomials, where $r \geq 2m - 1$. The virtual element method in [1] needs strong assumption on regularity of f ($f \in H^{r-m+1}(\Omega)$) to achieve optimal convergence (see [1], Thm. 4.2). In [7], a nonconforming H^m virtual element method is developed for bounded Lipschitz polyhedral domain in $\mathbb{R}^d$, where d can be any positive integer. The design of finite element space in [7], which contains piecewise r-th ($r \geq m$) order polynomials, is based on a generalized Green's identity for H^m inner product. It is assumed that $m \leq d$ in [7]. In [17], the virtual element method in [7] is extended for $m > d$. Besides above numerical methods based on primary formulations of (1.1), a mixed formulation based on Helmholtz decomposition for tensor valued function is introduced in [19] for two dimensional domain.

We propose a C^0 interior penalty method (2.2) for (1.1) for arbitrary positive integers m and d. The finite element space of (2.2) is the standard H^1-conforming piecewise r-th order polynomials, where $r \geq m$. The design of (2.2) avoids computing D^m of numerical solution on each element and high order normal derivatives of numerical solution along mesh interfaces. In fact, equation (2.2) only gets involved with calculation of high order multiplicity of Laplace of numerical solution ($\Delta^i u_h$ for $1 \leq i \leq m$) and the gradient of high order multiplicity of Laplace of numerical solution ($\nabla \Delta^i u_h$ for $0 \leq i \leq m-1$) on both elements and mesh interfaces. Therefore our method (2.2) can be easily implemented, even when m is large and $d = 3$. After proving (Thm. 3.4) that discrete H^m-norm (see Def. 3.1) is bounded by the natural energy semi-norm associated with (2.2), we manage to show our method (2.2) has stability and optimal convergence on bounded Lipschitz polyhedral domain in $\mathbb{R}^d$ with respect to the discrete H^m-norm, for any positive integers m and d. Roughly speaking, we have

$$
\begin{aligned}
\|u_h\|_{m,h} &\leq C\|f\|_{H^{-1}(\Omega)}, \\
\|u_h - u\|_{m,h} &\leq C h^{\min(r+1-m, s-m)} \|u\|_{H^s(\Omega)},
\end{aligned}
$$

where $s \geq 2m - 1$. We refer to Theorems 3.6 and 3.7 for detailed descriptions on stability and optimal convergence. The design and analysis of our method (2.2) can be easily generalized for nonlinear partial differential equations with $(-1)^m \Delta^m u$ as their leading term. We would like to point out that our method (2.2) is not a generalization of the interior penalty method for sixth-order elliptic equation ($m = 3$) in (3.4), (3.5) of [13]. Actually, the method in [13] needs to calculate numerical approximation to $D^3 u$.

If the exact solution of (1.1) is under the low regularity assumption, Gudi *et al.* have applied the analysis technique from the *a posteriori* error analysis to derive the error estimates for the interior penalty methods for the 2nd-order, 4th-order and 6th-order elliptic equations under the low regularity assumptions in [12, 13]. In this paper, we shall extend the analysis by Gudi *et al.* for the proposed C^0 interior penalty methods for (1.1) when $m \geq 2$. Assuming $u \in H_0^m(\Omega)$ for (1.1), we have

$$
\|u - u_h\|_{m,h} \leq C \inf_{v \in V_h} (\|u - v\|_{m,h} + \operatorname{osc}_m(f)),
$$

where V_h is the C^0 conforming finite element space and $\mathrm{osc}_m(f)$ is the oscillation term defined in (4.6).

The numerical method considered in this paper works for any positive integers m and d from theoretical viewpoint. It can be applied to the practical high order equations. For instance, the modeling for plates in linear elasticity results in consideration of fourth-order partial differential equations [11]. Modeling in material science usually applies the fourth-order equation such as the Cahn–Hilliard equation [5,10] and the sixth-order equation such as the thin-film equations [3] and the phase field crystal model [9,21,23]. Recently, an eighth-order equation was considered for the nonlinear Schrödinger equation in [18]. As mentioned in [22], although there are rare practical applications for general higher order equations, the elliptic equations of order $m = d/2$ in any dimension have been used in differential geometry [6]. One can also extends the numerical methods and analysis for the solution of nonlinear Hamilton–Jacobi–Bellman equation and other phase-field models.

In the next section, we present the C^0 interior penalty method. In Section 3, we prove stability and optimal convergence with respect to discrete H^m-norm (see Def. 3.1). In Section 4, we show the error estimates under the low regularity assumption of the exact solution. In Section 5, we provide numerical experiments.

2. C^0 INTERIOR PENALTY METHOD

In this section, firstly we give notations to define the C^0 interior penalty method for (1.1). Then in Section 2.1, we derive the C^0 interior penalty method for any $m \geq 1$. Finally in Section 2.2, we provide concrete examples of the method for $m = 1, 2, 3, 4$.

Let $\mathcal{T}_h$ be a quasi-uniform conforming simplicial mesh of Ω. Here we define $h = \max_{K\in\mathcal{T}_h} h_K$ where h_K is the diameter of the element $K \in \mathcal{T}_h$. We denote by $\mathcal{F}_h$, $\mathcal{F}_h^{\mathrm{int}}$ and $\mathcal{F}_h^{\partial}$ the collections of all $(d-1)$-dimensional faces, interior faces and boundary faces of $\mathcal{T}_h$, respectively. Obviously, $\mathcal{F}_h = \mathcal{F}_h^{\mathrm{int}} \cup \mathcal{F}_h^{\partial}$. For any positive integer r, we define $V_h = H_0^1(\Omega) \cap P_r(\mathcal{T}_h)$, where $P_r(\mathcal{T}_h) = \{v_h \in L^2(\Omega) : v_h|_K \in P_r(K), \forall K \in \mathcal{T}_h\}$.

We introduce some trace operators. For any interior face $F \in \mathcal{F}_h^{\mathrm{int}}$, let $K^-, K^+ \in \mathcal{T}_h$ be two elements sharing F. We denote by $\boldsymbol{\nu}^-$ and $\boldsymbol{\nu}^+$ the outward unit normal vectors along ∂K^- and ∂K^+, respectively. For scalar function $v : \Omega \to \mathbb{R}$ and vector field $\boldsymbol{\phi} : \Omega \to \mathbb{R}^d$, which may be discontinuous across $\mathcal{F}_h^{\mathrm{int}}$, we define the following quantities. For $v^- := v|_{K^-}$, $v^+ := v|_{K^+}$, $\boldsymbol{\phi}^- := \boldsymbol{\phi}|_{K^-}$ and $\boldsymbol{\phi}^+ := \boldsymbol{\phi}|_{K^+}$, we define

$$\begin{aligned} \{\!\{v\}\!\} &= \frac{1}{2}\big(v^-|_F + v^+|_F\big), & \{\!\{\boldsymbol{\phi}\}\!\} &= \frac{1}{2}\big(\boldsymbol{\phi}^-|_F + \boldsymbol{\phi}^+|_F\big), \\ [\![v]\!] &= v^-\boldsymbol{\nu}^-|_F + v^+\boldsymbol{\nu}^+|_F, & [\![\boldsymbol{\phi}]\!] &= \boldsymbol{\phi}^- \cdot \boldsymbol{\nu}^-|_F + \boldsymbol{\phi}^+ \cdot \boldsymbol{\nu}^+|_F; \end{aligned}$$

if $F \in \partial K^+ \cap \partial\Omega$, we define

$$\{\!\{v\}\!\} = v^+|_F, \quad \{\!\{\boldsymbol{\phi}\}\!\} = \boldsymbol{\phi}^+|_F, \quad [\![v]\!] = v^+\boldsymbol{\nu}|_F, \quad [\![\boldsymbol{\phi}]\!] = \boldsymbol{\phi}^+ \cdot \boldsymbol{\nu}|_F.$$

We also define $[\![v]\!]_J|_F = v^-|_F - v^+|_F$ for $F \in \mathcal{F}_h^{\mathrm{int}}$ and $[\![v]\!]_J|_F = v^+|_F$ for $F \in \partial\Omega$.

2.1. Derivation of C^0 interior penalty method

We assume the exact solution $u \in H^{2m-1}(\Omega)$. For any $v_h \in V_h$, *via* m-times integrating by parts,

$$\begin{aligned} &((-1)^m \Delta^m u, v_h)_\Omega \\ &= \begin{cases} \big(\Delta^{\tilde{m}} u, \Delta^{\tilde{m}} v_h\big)_{\mathcal{T}_h} - \sum_{i=0}^{\tilde{m}-1} \langle \Delta^{\tilde{m}+i} u, \partial_{\boldsymbol{\nu}} \Delta^{\tilde{m}-i-1} v_h \rangle_{\partial\mathcal{T}_h} + \sum_{i=0}^{\tilde{m}-2} \big\langle \partial_{\boldsymbol{\nu}} \Delta^{\tilde{m}+i} u, \Delta^{\tilde{m}-i-1} v_h \big\rangle_{\partial\mathcal{T}_h}, \\ \qquad \text{if } m = 2\tilde{m} \quad (m \text{ is an even number}); \\ \big(\nabla\Delta^{\tilde{m}} u, \nabla\Delta^{\tilde{m}} v_h\big)_{\mathcal{T}_h} + \sum_{i=0}^{\tilde{m}-1} \langle \Delta^{\tilde{m}+i+1} u, \partial_{\boldsymbol{\nu}} \Delta^{\tilde{m}-i-1} v_h \rangle_{\partial\mathcal{T}_h} - \sum_{i=0}^{\tilde{m}-1} \big\langle \partial_{\boldsymbol{\nu}} \Delta^{\tilde{m}+i} u, \Delta^{\tilde{m}-i} v_h \big\rangle_{\partial\mathcal{T}_h}, \\ \qquad \text{if } m = 2\tilde{m}+1 \quad (m \text{ is an odd number}). \end{cases} \end{aligned}$$

Since $u \in H^{2m-1}(\Omega)$, for any $v_h \in V_h$,

$$((-1)^m \Delta^m u, v_h)_\Omega \tag{2.1}$$

$$
= \begin{cases} \left(\Delta^{\tilde{m}} u, \Delta^{\tilde{m}} v_h\right)_{\mathcal{T}_h} - \sum_{i=0}^{\tilde{m}-1} \left\langle \left\{\!\!\left\{ \Delta^{\tilde{m}+i} u \right\}\!\!\right\}, \left[\!\left[\nabla \Delta^{\tilde{m}-i-1} v_h \right]\!\right] \right\rangle_{\mathcal{F}_h} + \sum_{i=0}^{\tilde{m}-2} \left\langle \left\{\!\!\left\{ \nabla \Delta^{\tilde{m}+i} u \right\}\!\!\right\}, \left[\!\left[\Delta^{\tilde{m}-i-1} v_h \right]\!\right] \right\rangle_{\mathcal{F}_h}, \\ \qquad \text{if } m = 2\tilde{m} \quad (m \text{ is an even number}); \\ \left(\nabla \Delta^{\tilde{m}} u, \nabla \Delta^{\tilde{m}} v_h\right)_{\mathcal{T}_h} + \sum_{i=0}^{\tilde{m}-1} \left\langle \left\{\!\!\left\{ \Delta^{\tilde{m}+i+1} u \right\}\!\!\right\}, \left[\!\left[\nabla \Delta^{\tilde{m}-i-1} v_h \right]\!\right] \right\rangle_{\mathcal{F}_h} - \sum_{i=0}^{\tilde{m}-1} \left\langle \left\{\!\!\left\{ \nabla \Delta^{\tilde{m}+i} u \right\}\!\!\right\}, \left[\!\left[\Delta^{\tilde{m}-i} v_h \right]\!\right] \right\rangle_{\mathcal{F}_h}, \\ \qquad \text{if } m = 2\tilde{m}+1 \quad (m \text{ is an odd number}). \end{cases}
$$

Equation (2.1) inspires us to define the coupling term C_h in Definition 2.1.

Definition 2.1. For any $w_h, v_h \in V_h$, we define the coupling term $C_h(w_h, v_h)$ along mesh interface $\mathcal{F}_h$ by

$$
\begin{aligned} & C_h(w_h, v_h) \\ & = \begin{cases} -\sum_{i=0}^{\tilde{m}-1} \left\langle \left\{\!\!\left\{ \Delta^{\tilde{m}+i} w_h \right\}\!\!\right\}, \left[\!\left[\nabla \Delta^{\tilde{m}-i-1} v_h \right]\!\right] \right\rangle_{\mathcal{F}_h} + \sum_{i=0}^{\tilde{m}-2} \left\langle \left\{\!\!\left\{ \nabla \Delta^{\tilde{m}+i} w_h \right\}\!\!\right\}, \left[\!\left[\Delta^{\tilde{m}-i-1} v_h \right]\!\right] \right\rangle_{\mathcal{F}_h}, & \text{if } m = 2\tilde{m}; \\ \sum_{i=0}^{\tilde{m}-1} \left\langle \left\{\!\!\left\{ \Delta^{\tilde{m}+i+1} w_h \right\}\!\!\right\}, \left[\!\left[\nabla \Delta^{\tilde{m}-i-1} v_h \right]\!\right] \right\rangle_{\mathcal{F}_h} - \sum_{i=0}^{\tilde{m}-1} \left\langle \left\{\!\!\left\{ \nabla \Delta^{\tilde{m}+i} w_h \right\}\!\!\right\}, \left[\!\left[\Delta^{\tilde{m}-i} v_h \right]\!\right] \right\rangle_{\mathcal{F}_h}, & \text{if } m = 2\tilde{m}+1. \end{cases} \end{aligned}
$$

In order to define C^0 interior penalty method, we need the stabilization term S_h in Definition 2.2.

Definition 2.2. For any $w_h, v_h \in V_h$, we define the stabilization term $S_h(w_h, v_h)$ along mesh interface $\mathcal{F}_h$ by

$$
\begin{aligned} & S_h(w_h, v_h) \\ & = \begin{cases} \sum_{i=0}^{\tilde{m}-1} h^{-(4i+1)} \left\langle \left[\!\left[\nabla \Delta^{\tilde{m}-i-1} w_h \right]\!\right], \left[\!\left[\nabla \Delta^{\tilde{m}-i-1} v_h \right]\!\right] \right\rangle_{\mathcal{F}_h} + \sum_{i=0}^{\tilde{m}-2} h^{-(4i+3)} \left\langle \left[\!\left[\Delta^{\tilde{m}-i-1} w_h \right]\!\right], \left[\!\left[\Delta^{\tilde{m}-i-1} v_h \right]\!\right] \right\rangle_{\mathcal{F}_h}, \\ \qquad \text{if } m = 2\tilde{m} \quad (m \text{ is an even number}); \\ \sum_{i=0}^{\tilde{m}-1} h^{-(4i+3)} \left\langle \left[\!\left[\nabla \Delta^{\tilde{m}-i-1} w_h \right]\!\right], \left[\!\left[\nabla \Delta^{\tilde{m}-i-1} v_h \right]\!\right] \right\rangle_{\mathcal{F}_h} + \sum_{i=0}^{\tilde{m}-1} h^{-(4i+1)} \left\langle \left[\!\left[\Delta^{\tilde{m}-i} w_h \right]\!\right], \left[\!\left[\Delta^{\tilde{m}-i} v_h \right]\!\right] \right\rangle_{\mathcal{F}_h}, \\ \qquad \text{if } m = 2\tilde{m}+1 \quad (m \text{ is an odd number}). \end{cases} \end{aligned}
$$

We would like to point out that $S_h(w_h, v_h) = 0$ if $m = 1$.

The C^0 interior penalty method is to find $u_h \in V_h$, such that for any v_h,

$$
a_h(u_h, v_h) = (f, v_h)_\Omega, \tag{2.2}
$$

where

$$
\begin{cases} a_h(u_h, v_h) = \left(\Delta^{\tilde{m}} u_h, \Delta^{\tilde{m}} v_h\right)_{\mathcal{T}_h} + C_h(u_h, v_h) + C_h(v_h, u_h) + \tau S_h(u_h, v_h), & \text{if } m = 2\tilde{m}; \\ a_h(u_h, v_h) = \left(\nabla \Delta^{\tilde{m}} u_h, \nabla \Delta^{\tilde{m}} v_h\right)_{\mathcal{T}_h} + C_h(u_h, v_h) + C_h(v_h, u_h) + \tau S_h(u_h, v_h), & \text{if } m = 2\tilde{m}+1. \end{cases} \tag{2.3}
$$

Here the parameter $\tau \geq 1$ shall be large enough but independent of h.

2.2. Examples of C^0 interior penalty method

- $m = 1$.
  The C^0 interior penalty method for $-\Delta u = f$ is to find $u_h \in V_h$ satisfying

$$
(\nabla u_h, \nabla v_h)_\Omega = (f, v_h)_\Omega, \qquad \forall v_h \in V_h. \tag{2.4}
$$

- $m = 2$.
  The C^0 interior penalty method for $\Delta^2 u = f$ is to find $u_h \in V_h$ satisfying

$$
\begin{aligned} (\Delta u_h, \Delta v_h)_{\mathcal{T}_h} & - \langle \{\!\{ \Delta u_h \}\!\}, [\![\nabla v_h]\!] \rangle_{\mathcal{F}_h} - \langle \{\!\{ \Delta v_h \}\!\}, [\![\nabla u_h]\!] \rangle_{\mathcal{F}_h} \\ & + \tau h^{-1} \langle [\![\nabla u_h]\!], [\![\nabla v_h]\!] \rangle_{\mathcal{F}_h} = (f, v_h)_\Omega, \qquad \forall v_h \in V_h. \end{aligned} \tag{2.5}
$$

  Actually, equation (2.5) is the C^0 interior penalty method which replaces the discontinuous finite element spaces in [20] with C^0 finite element space.

– $m = 3$.
The C^0 interior penalty method for $-\Delta^3 u = f$ is to find $u_h \in V_h$ satisfying

$$\begin{aligned}
&(\nabla\Delta u_h, \nabla\Delta v_h)_{\mathcal{T}_h} + \Big(\big\langle \big\{\!\!\big\{ \Delta^2 u_h \big\}\!\!\big\}, [\![\nabla v_h]\!]\big\rangle_{\mathcal{F}_h} - \langle \{\!\!\{\nabla\Delta u_h\}\!\!\}, [\![\Delta v_h]\!]\rangle_{\mathcal{F}_h}\Big) \\
&\quad + \Big(\big\langle \big\{\!\!\big\{ \Delta^2 v_h \big\}\!\!\big\}, [\![\nabla u_h]\!]\big\rangle_{\mathcal{F}_h} - \langle \{\!\!\{\nabla\Delta v_h\}\!\!\}, [\![\Delta u_h]\!]\rangle_{\mathcal{F}_h}\Big) \\
&\quad + \tau\big(h^{-3}\langle [\![\nabla u_h]\!], [\![\nabla v_h]\!]\rangle_{\mathcal{F}_h} + h^{-1}\langle [\![\Delta u_h]\!], [\![\Delta v_h]\!]\rangle_{\mathcal{F}_h}\big) = (f, v_h)_\Omega, \qquad \forall v_h \in V_h.
\end{aligned} \tag{2.6}$$

It is easy to see that (2.6) is quite different from the interior penalty method in [13].

– $m = 4$.
The C^0 interior penalty method for $\Delta^4 u = f$ is to find $u_h \in V_h$ satisfying

$$\begin{aligned}
&\big(\Delta^2 u_h, \Delta^2 v_h\big)_{\mathcal{T}_h} + \Big(-\big\langle \big\{\!\!\big\{ \Delta^3 u_h \big\}\!\!\big\}, [\![\nabla v_h]\!]\big\rangle_{\mathcal{F}_h} + \big\langle \big\{\!\!\big\{ \nabla\Delta^2 u_h \big\}\!\!\big\}, [\![\Delta v_h]\!]\big\rangle_{\mathcal{F}_h} - \big\langle \big\{\!\!\big\{ \Delta^2 u_h \big\}\!\!\big\}, [\![\nabla\Delta v_h]\!]\big\rangle_{\mathcal{F}_h}\Big) \\
&\quad + \Big(-\big\langle \big\{\!\!\big\{ \Delta^3 v_h \big\}\!\!\big\}, [\![\nabla u_h]\!]\big\rangle_{\mathcal{F}_h} + \big\langle \big\{\!\!\big\{ \nabla\Delta^2 v_h \big\}\!\!\big\}, [\![\Delta u_h]\!]\big\rangle_{\mathcal{F}_h} - \big\langle \big\{\!\!\big\{ \Delta^2 v_h \big\}\!\!\big\}, [\![\nabla\Delta u_h]\!]\big\rangle_{\mathcal{F}_h}\Big) \\
&\quad + \tau\big(h^{-5}\langle [\![\nabla u_h]\!], [\![\nabla v_h]\!]\rangle_{\mathcal{F}_h} + h^{-3}\langle [\![\Delta u_h]\!], [\![\Delta v_h]\!]\rangle_{\mathcal{F}_h} + h^{-1}\langle [\![\nabla\Delta u_h]\!], [\![\nabla\Delta v_h]\!]\rangle_{\mathcal{F}_h}\big) \\
&\quad = (f, v_h)_\Omega, \qquad \forall v_h \in V_h.
\end{aligned} \tag{2.7}$$

3. Analysis

In this section, firstly we prove Theorem 3.4, which states the discrete H^m-norm (see Def. 3.1) bounded by the natural energy semi-norm associated with the C^0 interior penalty method (2.2). Then we prove Theorem 3.6, which shows the energy estimate of (2.2). Finally, we prove Theorem 3.7, which gives optimal convergence of numerical approximation to u in the discrete H^m-norm. Throughout this paper, C with or without a subscript denotes a positive constant depending only on the property of Ω, the shape regularity of the meshes and the degree of polynomial spaces. The constant C can take on different values in different occurrences.

Definition 3.1. For any integers $m \geq 2$, we define the discrete H^m-norm $\|v\|_{m,h}$ by

$$\begin{aligned}
\|v\|_{m,h}^2 &= \sum_{i=0}^{m} \big\|D^i v\big\|_{L^2(\mathcal{T}_h)}^2 + \sum_{j=1}^{m-1} h^{-(2m-2j-1)} \big\| \big[\!\big[D^j v\big]\!\big]_J \big\|_{L^2(\mathcal{F}_h)}^2 \\
&:= \sum_{i=0}^{m} \sum_{K\in\mathcal{T}_h} \big\|D^i v\big\|_{L^2(K)}^2 + \sum_{j=1}^{m-1} \sum_{F\in\mathcal{F}_h} h^{-(2m-2j-1)} \big\| \big[\!\big[D^j v\big]\!\big]_J \big\|_{L^2(F)}^2, \quad \forall v \in H_0^1(\Omega) \cap H^m(\mathcal{T}_h).
\end{aligned}$$

For any $F \in \mathcal{F}_h^{\mathrm{int}}$, there are two elements $K^-, K^+ \in \mathcal{T}_h$ sharing the common face F. We denote by $v^- := v|_{K^-}$ and $v^+ := v|_{K^+}$. We define

$$\begin{aligned}
\big\| \big[\!\big[D^j v\big]\!\big]_J \big\|_{L^2(F)}^2 &= \sum_{1\leq k_1,\cdots,k_j\leq d} \Big\| \Big[\!\Big[\partial_{x_{k_1}}\partial_{x_{k_2}}\cdots\partial_{x_{k_j}} v\Big]\!\Big]_J \Big\|_{L^2(F)}^2 \\
&:= \sum_{1\leq k_1,\cdots,k_j\leq d} \Big\| \Big(\partial_{x_{k_1}}\partial_{x_{k_2}}\cdots\partial_{x_{k_j}} v^-\Big)\Big|_F - \Big(\partial_{x_{k_1}}\partial_{x_{k_2}}\cdots\partial_{x_{k_j}} v^+\Big)\Big|_F \Big\|_{L^2(F)}^2.
\end{aligned}$$

For any $F \in \mathcal{F}_h^{\partial}$, we define

$$\big\| \big[\!\big[D^j v\big]\!\big]_J \big\|_{L^2(F)}^2 = \sum_{1\leq k_1,\cdots,k_j\leq d} \Big\| \Big[\!\Big[\partial_{x_{k_1}}\partial_{x_{k_2}}\cdots\partial_{x_{k_j}} v\Big]\!\Big]_J \Big\|_{L^2(F)}^2 := \sum_{1\leq k_1,\cdots,k_j\leq d} \Big\| \Big(\partial_{x_{k_1}}\partial_{x_{k_2}}\cdots\partial_{x_{k_j}} v\Big)\Big|_F \Big\|_{L^2(F)}^2.$$

3.1. Discrete H^m-norm bounded by natural energy semi-norm

The main result of Section 3.1 is Theorem 3.4, which shows that the discrete H^m-norm (see Def. 3.1) is bounded by the natural energy semi-norm associated with the C^0 interior penalty method (2.2). The proof of Theorem 3.4 is based on Lemmas 3.2 and 3.3.

Lemma 3.2. *For any integers $r \geq m \geq 2$, there is a constant $C > 0$ such that*

$$\sum_{j=1}^{m-1} h^{-(2m-2j-1)} \left\| [\![D^j v_h]\!]_J \right\|^2_{L^2(\mathcal{F}_h)} \leq C S_h(v_h, v_h), \qquad \forall v_h \in V_h. \tag{3.1}$$

Proof. We choose $F \in \mathcal{F}_h$ arbitrarily. There is an orthonormal coordinate system $\{y_k\}_{k=1}^d$ such that the y_d-axis is parallel to normal vector along F. Therefore y_1-axis, $\cdots$, y_{d-1}-axis are all parallel to F.

We claim that for any $1 \leq l \leq m$, there is a positive integer C' such that

$$\left\| [\![D^l \tilde{v}_h]\!]_J \right\|^2_{L^2(F)} \leq C' \left(\sum_{j=0}^{l} h^{-(2l-2j)} \left\| [\![\partial^j_{y_d} \tilde{v}_h]\!]_J \right\|^2_{L^2(F)} \right), \quad \forall \tilde{v}_h \in P_r(\mathcal{T}_h). \tag{3.2}$$

We prove (3.2) by induction. When $l = 1$, it is easy to see

$$\| [\![D\tilde{v}_h]\!]_J \|^2_{L^2(F)} = \sum_{k=1}^{d} \left\| [\![\partial_{x_k} \tilde{v}_h]\!]_J \right\|^2_{L^2(F)} = \sum_{k=1}^{d} \left\| [\![\partial_{y_k} \tilde{v}_h]\!]_J \right\|^2_{L^2(F)}.$$

By discrete inverse inequality and the fact that y_1-axis, $\cdots$, y_{d-1}-axis are all parallel to F, we have that

$$\sum_{k=1}^{d-1} \left\| [\![\partial_{y_k} \tilde{v}_h]\!]_J \right\|^2_{L^2(F)} \leq C h^{-2} \| [\![\tilde{v}_h]\!]_J \|^2_{L^2(F)}.$$

Therefore we have

$$\| [\![D\tilde{v}_h]\!]_J \|^2_{L^2(F)} \leq C \Big(h^{-2} \| [\![\tilde{v}_h]\!]_J \|^2_{L^2(F)} + \left\| [\![\partial_{y_d} \tilde{v}_h]\!]_J \right\|^2_{L^2(F)} \Big).$$

Thus (3.2) holds when $l = 1$. We assume that (3.2) holds for any $l < m$. Then by discrete inverse inequality and the fact that y_1-axis, $\cdots$, y_{d-1}-axis are all parallel to F,

$$\begin{aligned} \left\| [\![D^{l+1} \tilde{v}_h]\!]_J \right\|^2_{L^2(F)} &= \sum_{k=1}^{d} \left\| [\![\partial_{x_k} D^l \tilde{v}_h]\!]_J \right\|^2_{L^2(F)} = \sum_{k=1}^{d} \left\| [\![\partial_{y_k} D^l \tilde{v}_h]\!]_J \right\|^2_{L^2(F)} \\ &\leq C \Big(h^{-2} \left\| [\![D^l \tilde{v}_h]\!]_J \right\|^2_{L^2(F)} + \left\| [\![D^l (\partial_{y_d} \tilde{v}_h)]\!]_J \right\|^2_{L^2(F)} \Big). \end{aligned}$$

Since $\tilde{v}_h \in P_r(\mathcal{T}_h)$, then $\partial_{y_d} \tilde{v}_h \in P_r(\mathcal{T}_h)$. Since we assume (3.2) holds for l, we have

$$\begin{aligned} \left\| [\![D^l (\partial_{y_d} \tilde{v}_h)]\!]_J \right\|^2_{L^2(F)} &\leq C \left(\sum_{j=0}^{l} h^{-(2l-2j)} \left\| [\![\partial^j_{y_d} (\partial_{y_d} \tilde{v}_h)]\!]_J \right\|^2_{L^2(F)} \right), \\ h^{-2} \left\| [\![D^l \tilde{v}_h]\!]_J \right\|^2_{L^2(F)} &\leq C \left(\sum_{j=0}^{l} h^{-(2(l+1)-2j)} \left\| [\![\partial^j_{y_d} \tilde{v}_h]\!]_J \right\|^2_{L^2(F)} \right). \end{aligned}$$

Therefore (3.2) holds for $l + 1$. Thus we can conclude that the claim (3.2) is true.

Now we start to prove (3.1) by induction. Since $\|[\![\partial_{y_d} v_h]\!]_J\|_{L^2(F)} = \|[\![\nabla v_h]\!]\|_{L^2(F)}$, (3.2) and the fact $v_h \in H_0^1(\Omega)$ imply

$$\|[\![Dv_h]\!]_J\|_{L^2(F)} = \|[\![\nabla v_h]\!]\|_{L^2(F)}. \tag{3.3}$$

Since $F \in \mathfrak{F}_h$ is chosen arbitrarily, (3.3) implies that (3.1) holds when $m = 2$.

Applying (3.2) with $l = 2$, we have

$$\begin{aligned}\left\|[\![D^2 v_h]\!]_J\right\|^2_{L^2(F)} &\le C\Big(h^{-4}\|[\![v_h]\!]_J\|^2_{L^2(F)} + h^{-2}\|[\![\partial_{y_d} v_h]\!]_J\|^2_{L^2(F)} + \Big\|[\![\partial^2_{y_d} v_h]\!]_J\Big\|^2_{L^2(F)}\Big)\\ &= C\Big(h^{-2}\|[\![\nabla v_h]\!]\|^2_{L^2(F)} + \Big\|[\![\partial^2_{y_d} v_h]\!]_J\Big\|^2_{L^2(F)}\Big).\end{aligned} \tag{3.4}$$

The last equality in (3.4) holds since $v_h \in H_0^1(\Omega)$ and $\|[\![\partial_{y_d} v_h]\!]_J\|_{L^2(F)} = \|[\![\nabla v_h]\!]\|_{L^2(F)}$. We notice that

$$\Delta v_h = \Big(\partial^2_{y_1} v_h + \cdots + \partial^2_{y_{d-1}} v_h\Big) + \partial^2_{y_d} v_h. \tag{3.5}$$

Since y_1-axis, $\cdots$, y_{d-1}-axis are all parallel to F, discrete inverse inequality implies

$$\begin{aligned}\Big\|\Big[\!\!\Big[\Big(\partial^2_{y_1} v_h + \cdots + \partial^2_{y_{d-1}} v_h\Big)\Big]\!\!\Big]\Big\|^2_{L^2(F)} &= \Big\|\Big[\!\!\Big[\Big(\partial^2_{y_1} v_h + \cdots + \partial^2_{y_{d-1}} v_h\Big)\Big]\!\!\Big]_J\Big\|^2_{L^2(F)}\\ &\le Ch^{-2}\|[\![Dv_h]\!]_J\|^2_{L^2(F)} = Ch^{-2}\|[\![\nabla v_h]\!]\|^2_{L^2(F)}.\end{aligned}$$

By (3.5) and the above inequality, we have

$$\Big\|[\![\partial^2_{y_d} v_h]\!]_J\Big\|^2_{L^2(F)} \le C\Big(h^{-2}\|[\![\nabla v_h]\!]\|^2_{L^2(F)} + \|[\![\Delta v_h]\!]\|^2_{L^2(F)}\Big). \tag{3.6}$$

By (3.4), (3.6), we have

$$\left\|[\![D^2 v_h]\!]_J\right\|^2_{L^2(F)} \le C\Big(h^{-2}\|[\![\nabla v_h]\!]\|^2_{L^2(F)} + \|[\![\Delta v_h]\!]\|^2_{L^2(F)}\Big). \tag{3.7}$$

Since $F \in \mathfrak{F}_h$ is chosen arbitrarily, equations (3.3), (3.7) imply that (3.1) holds when $m = 3$.

We assume that $1 \le l < m$ is an odd number, $l = 2\tilde{l} + 1$ and

$$\left\|[\![D^l v_h]\!]_J\right\|^2_{L^2(F)} \le C\left(\sum_{i=0}^{\tilde{l}} h^{-4i}\Big\|\Big[\!\!\Big[\nabla\Delta^{\tilde{l}-i} v_h\Big]\!\!\Big]\Big\|^2_{L^2(F)} + \sum_{i=0}^{\tilde{l}-1} h^{-(4i+2)}\Big\|\Big[\!\!\Big[\Delta^{\tilde{l}-i} v_h\Big]\!\!\Big]\Big\|^2_{L^2(F)}\right). \tag{3.8}$$

Then by applying (3.8) for each $\partial_{y_k} v_h$, we have

$$\begin{aligned}\left\|[\![D^{l+1} v_h]\!]_J\right\|^2_{L^2(F)} &= \sum_{k=1}^{d}\left\|[\![D^l(\partial_{x_k} v_h)]\!]_J\right\|^2_{L^2(F)} = \sum_{k=1}^{d}\left\|[\![D^l(\partial_{y_k} v_h)]\!]_J\right\|^2_{L^2(F)}\\ &\le C\sum_{k=1}^{d}\left(\sum_{i=0}^{\tilde{l}} h^{-4i}\Big\|\Big[\!\!\Big[\nabla\Delta^{\tilde{l}-i}(\partial_{y_k} v_h)\Big]\!\!\Big]\Big\|^2_{L^2(F)} + \sum_{i=0}^{\tilde{l}-1} h^{-(4i+2)}\Big\|\Big[\!\!\Big[\Delta^{\tilde{l}-i}(\partial_{y_k} v_h)\Big]\!\!\Big]\Big\|^2_{L^2(F)}\right).\end{aligned}$$

Here $2\tilde{l} + 1 = l$. Since y_1-axis, $\cdots$, y_{d-1}-axis are all parallel to F, discrete inverse inequality implies

$$\left\|[\![D^{l+1} v_h]\!]_J\right\|^2_{L^2(F)} \le Ch^{-2}\left(\sum_{i=0}^{\tilde{l}} h^{-4i}\Big\|\Big[\!\!\Big[\nabla\Delta^{\tilde{l}-i} v_h\Big]\!\!\Big]\Big\|^2_{L^2(F)} + \sum_{i=0}^{\tilde{l}-1} h^{-(4i+2)}\Big\|\Big[\!\!\Big[\Delta^{\tilde{l}-i} v_h\Big]\!\!\Big]\Big\|^2_{L^2(F)}\right)$$

$$+ C\left(\sum_{i=0}^{\tilde{l}} h^{-4i}\left\|\left[\!\left[\nabla\Delta^{\tilde{l}-i}(\partial_{y_d}v_h)\right]\!\right]\right\|^2_{L^2(F)} + \sum_{i=0}^{\tilde{l}-1} h^{-(4i+2)}\left\|\left[\!\left[\Delta^{\tilde{l}-i}(\partial_{y_d}v_h)\right]\!\right]\right\|^2_{L^2(F)}\right).$$

Again by the fact that y_1-axis, $\cdots$, y_{d-1}-axis are all parallel to F, we have that for any $0 \le i \le \tilde{l}$,

$$\begin{aligned}
\left\|\left[\!\left[\nabla\Delta^{\tilde{l}-i}(\partial_{y_d}v_h)\right]\!\right]\right\|^2_{L^2(F)} &= \left\|\left[\!\left[\nabla\left(\partial_{y_d}\Delta^{\tilde{l}-i}v_h\right)\right]\!\right]_J\right\|^2_{L^2(F)} \\
&= \left(\left\|\left[\!\left[\partial_{y_1}\left(\partial_{y_d}\Delta^{\tilde{l}-i}v_h\right)\right]\!\right]_J\right\|^2_{L^2(F)} + \cdots + \left\|\left[\!\left[\partial_{y_{d-1}}\left(\partial_{y_d}\Delta^{\tilde{l}-i}v_h\right)\right]\!\right]_J\right\|^2_{L^2(F)}\right) \\
&\quad + \left\|\left[\!\left[\partial^2_{y_d}\left(\Delta^{\tilde{l}-i}v_h\right)\right]\!\right]_J\right\|^2_{L^2(F)} \\
&\le C\left(h^{-2}\left\|\left[\!\left[\partial_{y_d}\Delta^{\tilde{l}-i}v_h\right]\!\right]_J\right\|^2_{L^2(F)} + \left\|\left[\!\left[\partial^2_{y_d}\left(\Delta^{\tilde{l}-i}v_h\right)\right]\!\right]_J\right\|^2_{L^2(F)}\right) \\
&\le C\left(h^{-2}\left\|\left[\!\left[\nabla\Delta^{\tilde{l}-i}v_h\right]\!\right]_J\right\|^2_{L^2(F)} + \left\|\left[\!\left[\partial^2_{y_d}\left(\Delta^{\tilde{l}-i}v_h\right)\right]\!\right]_J\right\|^2_{L^2(F)}\right) \\
&\le C\left(h^{-2}\left\|\left[\!\left[\nabla\Delta^{\tilde{l}-i}v_h\right]\!\right]_J\right\|^2_{L^2(F)} + \left\|\left[\!\left[\Delta^{\tilde{l}-i+1}v_h\right]\!\right]_J\right\|^2_{L^2(F)}\right).
\end{aligned}$$

We have applied (3.6) for $\Delta^{\tilde{l}-i}v_h$ to obtain last inequality. We also notice that for any $0 \le i \le \tilde{l}-1$,

$$\left\|\left[\!\left[\Delta^{\tilde{l}-i}(\partial_{y_d}v_h)\right]\!\right]\right\|^2_{L^2(F)} = \left\|\left[\!\left[\partial_{y_d}(\Delta^{\tilde{l}-i}v_h)\right]\!\right]_J\right\|^2_{L^2(F)} = \left\|\left[\!\left[\nabla\Delta^{\tilde{l}-i}v_h\right]\!\right]\right\|^2_{L^2(F)}.$$

Therefore we have that (3.8) implies

$$\left\|\left[\!\left[D^{l+1}v_h\right]\!\right]_J\right\|^2_{L^2(F)} \le C\left(\sum_{i=0}^{\tilde{l}} h^{-4i}\left\|\left[\!\left[\Delta^{\tilde{l}-i+1}v_h\right]\!\right]\right\|^2_{L^2(F)} + \sum_{i=0}^{\tilde{l}} h^{-(4i+2)}\left\|\left[\!\left[\nabla\Delta^{\tilde{l}-i}v_h\right]\!\right]\right\|^2_{L^2(F)}\right), \tag{3.9}$$

where $l = 2\tilde{l}+1$.

Now we assume that $1 \le l < m$ is an even number, $l = 2\tilde{l}$ and

$$\left\|\left[\!\left[D^{l}v_h\right]\!\right]_J\right\|^2_{L^2(F)} \le C\left(\sum_{i=0}^{\tilde{l}-1} h^{-4i}\left\|\left[\!\left[\Delta^{\tilde{l}-i}v_h\right]\!\right]\right\|^2_{L^2(F)} + \sum_{i=0}^{\tilde{l}-1} h^{-(4i+2)}\left\|\left[\!\left[\nabla\Delta^{\tilde{l}-i-1}v_h\right]\!\right]\right\|^2_{L^2(F)}\right). \tag{3.10}$$

Then by similar argument in last paragraph, we have that (3.10) implies

$$\left\|\left[\!\left[D^{l+1}v_h\right]\!\right]_J\right\|^2_{L^2(F)} \le C\left(\sum_{i=0}^{\tilde{l}} h^{-4i}\left\|\left[\!\left[\nabla\Delta^{\tilde{l}-i}v_h\right]\!\right]\right\|^2_{L^2(F)} + \sum_{i=0}^{\tilde{l}-1} h^{-(4i+2)}\left\|\left[\!\left[\Delta^{\tilde{l}-i}v_h\right]\!\right]\right\|^2_{L^2(F)}\right), \tag{3.11}$$

where $l = 2\tilde{l}$.

According to (3.3), (3.7)–(3.11) and the fact that $F \in \mathcal{F}_h$ is chosen arbitrarily, we can conclude that the proof is complete. □

According to (3.1c) in Theorem 3.1 of [8], there is a constant $C > 0$ such that

$$\|\nabla\tilde{v}_h\|^2_{L^2(\mathcal{T}_h)} + \left\|D^2\tilde{v}_h\right\|^2_{L^2(\mathcal{T}_h)} \le C\left(\|\Delta\tilde{v}_h\|^2_{L^2(\mathcal{T}_h)} + h^{-1}\|[\![\nabla\tilde{v}_h]\!]\|^2_{L^2(\mathcal{F}_h)} + h^{-3}\|[\![\tilde{v}_h]\!]\|^2_{L^2(\mathcal{F}_h)}\right), \quad \forall\tilde{v}_h \in P_{\tilde{r}}(\mathcal{T}_h), \tag{3.12}$$

where $\tilde{r} \ge 2$ is a positive integer.

Lemma 3.3. *We define $2\tilde{m}+1=m$ if m is an odd number, while $2\tilde{m}=m$ if m is an even number. Then there is a positive constant C such that*

$$\|D^m v_h\|^2_{L^2(\mathcal{T}_h)} \le \begin{cases} C\Big(\big\|\Delta^{\tilde{m}} v_h\big\|^2_{L^2(\mathcal{T}_h)} + \sum_{j=1}^{m-1} h^{-(2m-2j-1)}\big\|\big[\!\big[D^j v_h\big]\!\big]_J\big\|^2_{L^2(\mathcal{F}_h)}\Big), & \text{if } m=2\tilde{m};\\ C\Big(\big\|\nabla\Delta^{\tilde{m}} v_h\big\|^2_{L^2(\mathcal{T}_h)} + \sum_{j=1}^{m-1} h^{-(2m-2j-1)}\big\|\big[\!\big[D^j v_h\big]\!\big]_J\big\|^2_{L^2(\mathcal{F}_h)}\Big), & \text{if } m=2\tilde{m}+1,\end{cases} \tag{3.13}$$

for any $v_h \in V_h$.

Proof. It is easy to see that (3.13) holds when $m=1$. By (3.12), (3.13) holds when $m=2$.

It is easy to see

$$\big\|D^3 v_h\big\|^2_{L^2(\mathcal{T}_h)} = \sum_{k=1}^{d}\big\|D^2(\partial_{x_k} v_h)\big\|^2_{L^2(\mathcal{T}_h)}.$$

Applying (3.12) to each $\partial_{x_k} v_h$, we have

$$\begin{aligned}\big\|D^3 v_h\big\|^2_{L^2(\mathcal{T}_h)} &\le C\sum_{k=1}^{d}\Big(\|\Delta(\partial_{x_k} v_h)\|^2_{L^2(\mathcal{T}_h)} + h^{-3}\big\|[\![\partial_{x_k} v_h]\!]_J\big\|^2_{L^2(\mathcal{F}_h)} + h^{-1}\big\|[\![D(\partial_{x_k} v_h)]\!]_J\big\|^2_{L^2(\mathcal{F}_h)}\Big)\\ &= C\Big(\|\nabla\Delta v_h\|^2_{L^2(\mathcal{T}_h)} + h^{-3}\|[\![D v_h]\!]_J\|^2_{L^2(\mathcal{F}_h)} + h^{-1}\big\|\big[\!\big[D^2 v_h\big]\!\big]_J\big\|^2_{L^2(\mathcal{F}_h)}\Big).\end{aligned}$$

Thus (3.13) holds when $m=3$.

For any $2<l\le m$, we have

$$\big\|D^l v_h\big\|^2_{L^2(\mathcal{T}_h)} = \big\|D^2\big(D^{l-2} v_h\big)\big\|^2_{\mathcal{T}_h}.$$

Applying (3.12) to each component of $D^{l-2}v_h$, we have

$$\big\|D^l v_h\big\|^2_{L^2(\mathcal{T}_h)} \le C\Big(\big\|D^{l-2}\Delta v_h\big\|^2_{L^2(\mathcal{T}_h)} + h^{-3}\big\|\big[\!\big[D^{l-2} v_h\big]\!\big]_J\big\|^2_{L^2(\mathcal{F}_h)} + h^{-1}\big\|\big[\!\big[D^{l-1} v_h\big]\!\big]_J\big\|^2_{L^2(\mathcal{F}_h)}\Big). \tag{3.14}$$

According to (3.14) and the fact (3.13) holds for $m=1,2,3$, we can conclude that the proof is complete. $\square$

According to Lemmas 3.2, 3.3 and the discrete Poincaré inequality, we immediately have the following Theorem 3.4.

Theorem 3.4. *For any integers $r\ge m\ge 1$, there is a constant $C>0$ such that*

$$\|v_h\|^2_{m,h} \le \begin{cases} C\Big(\big\|\Delta^{\tilde{m}} v_h\big\|^2_{L^2(\mathcal{T}_h)} + S_h(v_h,v_h)\Big), & \text{if } m=2\tilde{m};\\ C\Big(\big\|\nabla\Delta^{\tilde{m}} v_h\big\|^2_{L^2(\mathcal{T}_h)} + S_h(v_h,v_h)\Big), & \text{if } m=2\tilde{m}+1,\end{cases}$$

for any $v_h\in V_h$. $\|v_h\|_{m,h}$ is introduced in Definition 3.1. We point out that the right hand side of the above inequality is the natural energy semi-norm associated with the method (2.2).

3.2. Energy estimate of C^0 interior penalty method

We provide Theorem 3.6, which shows energy estimate of C^0 interior penalty method (2.2) with respect to the discrete H^m-norm (see Def. 3.1). Before we prove Theorem 3.6, we introduce Lemma 3.5.

Lemma 3.5. *For any integers $r \geq m \geq 2$ and any spatial dimension $d \geq 1$, there is a positive number $\tau_0 \geq 1$ such that for any $v_h \in V_h$,*

$$4|C_h(v_h, v_h)| \leq \begin{cases} \left\|\Delta^{\tilde{m}} v_h\right\|^2_{L^2(\mathcal{T}_h)} + \tau_0 S_h(v_h, v_h), & \text{if } m = 2\tilde{m} \quad (m \text{ is an even number}); \\ \left\|\nabla\Delta^{\tilde{m}} v_h\right\|^2_{L^2(\mathcal{T}_h)} + \tau_0 S_h(v_h, v_h), & \text{if } m = 2\tilde{m}+1 \quad (m \text{ is an odd number}). \end{cases} \tag{3.15}$$

Proof. We prove (3.15) for $m = 2\tilde{m}$ (m is an even number) in the following. It is similar to prove (3.15) for m which is an odd number.

According to Definition 2.1, discrete trace inequality and inverse inequality,

$$\begin{aligned}
|C_h(v_h, v_h)| &= \left| -\sum_{i=0}^{\tilde{m}-1} \left\langle \left\{\!\!\left\{ \Delta^{\tilde{m}+i} v_h \right\}\!\!\right\}, \left[\!\left[\nabla\Delta^{\tilde{m}-i-1} v_h \right]\!\right] \right\rangle_{\mathcal{F}_h} + \sum_{i=0}^{\tilde{m}-2} \left\langle \left\{\!\!\left\{ \nabla\Delta^{\tilde{m}+i} v_h \right\}\!\!\right\}, \left[\!\left[\Delta^{\tilde{m}-i-1} v_h \right]\!\right] \right\rangle_{\mathcal{F}_h} \right| \\
&\leq \sum_{i=0}^{\tilde{m}-1} \left\| \left\{\!\!\left\{ \Delta^{\tilde{m}+i} v_h \right\}\!\!\right\} \right\|_{L^2(\mathcal{F}_h)} \left\| \left[\!\left[\nabla\Delta^{\tilde{m}-i-1} v_h \right]\!\right] \right\|_{L^2(\mathcal{F}_h)} \\
&\quad + \sum_{i=0}^{\tilde{m}-2} \left\| \left\{\!\!\left\{ \nabla\Delta^{\tilde{m}+i} v_h \right\}\!\!\right\} \right\|_{L^2(\mathcal{F}_h)} \left\| \left[\!\left[\Delta^{\tilde{m}-i-1} v_h \right]\!\right] \right\|_{L^2(\mathcal{F}_h)} \\
&\leq C \sum_{i=0}^{\tilde{m}-1} h^{-\frac{1}{2}} \left\| \Delta^{\tilde{m}+i} v_h \right\|_{L^2(\mathcal{T}_h)} \left\| \left[\!\left[\nabla\Delta^{\tilde{m}-i-1} v_h \right]\!\right] \right\|_{L^2(\mathcal{F}_h)} \\
&\quad + C \sum_{i=0}^{\tilde{m}-2} h^{-\frac{1}{2}} \left\| \nabla\Delta^{\tilde{m}+i} v_h \right\|_{L^2(\mathcal{T}_h)} \left\| \left[\!\left[\Delta^{\tilde{m}-i-1} v_h \right]\!\right] \right\|_{L^2(\mathcal{F}_h)} \\
&\leq C \sum_{i=0}^{\tilde{m}-1} h^{-(2i+\frac{1}{2})} \left\| \Delta^{\tilde{m}} v_h \right\|_{L^2(\mathcal{T}_h)} \left\| \left[\!\left[\nabla\Delta^{\tilde{m}-i-1} v_h \right]\!\right] \right\|_{L^2(\mathcal{F}_h)} \\
&\quad + C \sum_{i=0}^{\tilde{m}-2} h^{-\left(2i+\frac{3}{2}\right)} \left\| \Delta^{\tilde{m}} v_h \right\|_{L^2(\mathcal{T}_h)} \left\| \left[\!\left[\Delta^{\tilde{m}-i-1} v_h \right]\!\right] \right\|_{L^2(\mathcal{F}_h)} \\
&\leq \frac{1}{4} \left\| \Delta^{\tilde{m}} v_h \right\|^2_{L^2(\mathcal{T}_h)} + C \left(\sum_{i=0}^{\tilde{m}-1} h^{-(4i+1)} \left\| \left[\!\left[\nabla\Delta^{\tilde{m}-i-1} v_h \right]\!\right] \right\|^2_{L^2(\mathcal{F}_h)} + \sum_{i=0}^{\tilde{m}-2} h^{-(4i+3)} \left\| \left[\!\left[\Delta^{\tilde{m}-i-1} v_h \right]\!\right] \right\|^2_{L^2(\mathcal{F}_h)} \right).
\end{aligned}$$

By Definition 2.2, it is easy to see that (3.15) holds. Therefore the proof is complete. □

Theorem 3.6. *When $m = 1$, the method (2.2) is well-posed such that*

$$\|u_h\|_{H^1(\Omega)} \leq C\|f\|_{H^{-1}(\Omega)}. \tag{3.16}$$

For any $m \geq 2$, there is a positive number $\tau_0 \geq 1$ which is the same as Lemma 3.5, such that if $\tau \geq \tau_0$, then the method (2.2) is well-posed such that

$$\|u_h\|_{m,h} \leq C\|f\|_{H^{-1}(\Omega)}. \tag{3.17}$$

Here $u_h \in V_h$ is the numerical solution of the method (2.2).

Proof. By (2.4), the method is the standard finite element method for Poisson equation when $m = 1$. Therefore, the method (2.2) is well-posed and (3.16) holds, when $m = 1$.

Now we consider $m \geq 2$. By the definition of the bilinear form $a_h(\cdot,\cdot)$, Theorem 3.4 and Lemma 3.5, the coercivity and the continuity of $a_h(\cdot,\cdot)$ are obtained which imply the well-posedness of the method (2.2). We assume $m = 2\tilde{m}$ to be an even number. By taking $v_h = u_h$ in the method (2.2), we have

$$\|\Delta^{\tilde{m}} u_h\|^2_{L^2(\mathcal{T}_h)} + 2C_h(u_h, u_h) + \tau S_h(u_h, u_h) = (f, u_h)_\Omega.$$

We choose τ_0 the same as Lemma 3.5. Then Lemma 3.5 implies

$$\frac{1}{2}\|\Delta^{\tilde{m}} u_h\|^2_{L^2(\mathcal{T}_h)} + \frac{\tau}{2} S_h(u_h, u_h) \leq (f, u_h)_\Omega \leq \|f\|_{H^{-1}(\Omega)} \|u_h\|_{H^1(\Omega)} \leq \|f\|_{H^{-1}(\Omega)} \|u_h\|_{m,h},$$

if $\tau \geq \tau_0$. Then by Theorem 3.4 and the above inequality, we obtain (3.17) when m is an even number.

It is similar to show that (3.17) holds when m is an odd number. Thus we can conclude that the proof is complete. □

3.3. Error analysis of C^0 interior penalty method

We provide Theorem 3.7, which gives error analysis of C^0 interior penalty method (2.2) with respect to the discrete H^m-norm (see Def. 3.1).

Theorem 3.7. *We assume that the exact solution $u \in H_0^m(\Omega) \cap H^s(\Omega)$ where $s \geq 2m - 1$. When $m = 1$, we have*

$$\|u - u_h\|_{H^1(\Omega)} \leq Ch^{\min(r, s-1)} \|u\|_{H^s(\Omega)}. \tag{3.18}$$

For $m \geq 2$, we assume that $\tau \geq \tau_0 \geq 1$ where τ_0 is the same as Theorem 3.6. Then we have

$$\|u - u_h\|_{m,h} \leq Ch^{\min(r+1-m, s-m)} \|u\|_{H^s(\Omega)}. \tag{3.19}$$

Here $u_h \in V_h$ is the numerical solution of the method (2.2).

Proof. When $m = 1$, the method (2.2) is the standard finite element method (2.4) for Poisson equation with homogeneous Dirichlet boundary condition. So it is easy to see that (3.18) holds. In the following, we assume $m \geq 2$.

By Theorem 3.6, the method (2.2) has the unique numerical solution $u_h \in V_h$.

Since $u \in H_0^m(\Omega)$, it is easy to see that for any $0 \leq j \leq m-1$, every component of $D^j u$ is continuous across any face $F \in \mathcal{F}_h^{\text{int}}$ and is equal to zero along $\partial\Omega$. Therefore by Definitions 2.1 and 2.2, we have

$$C_h(v_h, u) = S_h(u, v_h) = 0, \qquad \forall v_h \in V_h. \tag{3.20}$$

We denote by $\Pi_h u \in V_h$ the standard L^2-orthogonal projection of u into V_h. We define $e_u = \Pi_h u - u_h$ and $\delta_u = \Pi_h u - u$. Since $u \in H^s(\Omega)$ and $s \geq 2m-1$, we have

$$\left\|D^j \delta_u\right\|_{L^2(\mathcal{T}_h)} \leq Ch^{\max(\min(r+1-j, s-j), 0)} \|u\|_{H^s(\Omega)}, \quad \forall 0 \leq j \leq 2m-1. \tag{3.21}$$

We assume $m = 2\tilde{m}$ to be an even number. By (2.1), (3.20) and the method (2.2), we have

$$\left\|\Delta^{\tilde{m}} e_u\right\|^2_{L^2(\mathcal{T}_h)} + 2C_h(e_u, e_u) + \tau S_h(e_u, e_u) = \left(\Delta^{\tilde{m}} \delta_u, \Delta^{\tilde{m}} e_u\right)_{\mathcal{T}_h} + C_h(\delta_u, e_u) + C_h(e_u, \delta_u) + \tau S_h(\delta_u, e_u). \tag{3.22}$$

By Lemma 3.5 and (3.22), we have

$$\frac{1}{2}\left\|\Delta^{\tilde{m}} e_u\right\|^2_{L^2(\mathcal{T}_h)} + \frac{\tau}{2} S_h(e_u, e_u) \leq \left(\Delta^{\tilde{m}} \delta_u, \Delta^{\tilde{m}} e_u\right)_{\mathcal{T}_h} + C_h(\delta_u, e_u) + C_h(e_u, \delta_u) + \tau S_h(\delta_u, e_u). \tag{3.23}$$

It is easy to see that

$$
\begin{aligned}
\left(\Delta^{\tilde m}\delta_u, \Delta^{\tilde m} e_u\right)_{\mathcal{T}_h} \le \left\|\Delta^{\tilde m}\delta_u\right\|_{L^2(\mathcal{T}_h)}\left\|\Delta^{\tilde m} e_u\right\|_{\mathcal{T}_h} \le C h^{\min(r+1-m,s-m)}\left\|\Delta^{\tilde m} e_u\right\|_{\mathcal{T}_h}\|u\|_{H^s(\Omega)}, \tau S_h(\delta_u, e_u)\\
&= \tau \sum_{i=0}^{\tilde m-1} h^{-(4i+1)}\left\langle \left[\!\left[\nabla\Delta^{\tilde m-i-1}\delta_u\right]\!\right], \left[\!\left[\nabla\Delta^{\tilde m-i-1}e_u\right]\!\right]\right\rangle_{\mathcal{F}_h}\\
&\quad + \tau \sum_{i=0}^{\tilde m-2} h^{-(4i+3)}\left\langle \left[\!\left[\Delta^{\tilde m-i-1}\delta_u\right]\!\right], \left[\!\left[\Delta^{\tilde m-i-1}e_u\right]\!\right]\right\rangle_{\mathcal{F}_h}\\
&\le \tau \sum_{i=0}^{\tilde m-1}\left(h^{-(4i+1)}\left\|\left[\!\left[\nabla\Delta^{\tilde m-i-1}\delta_u\right]\!\right]\right\|^2_{L^2(\mathcal{F}_h)}\right)^{\frac12}\left(h^{-(4i+1)}\left\|\left[\!\left[\nabla\Delta^{\tilde m-i-1}e_u\right]\!\right]\right\|^2_{L^2(\mathcal{F}_h)}\right)^{\frac12}\\
&\quad + \tau \sum_{i=0}^{\tilde m-2}\left(h^{-(4i+3)}\left\|\left[\!\left[\Delta^{\tilde m-i-1}\delta_u\right]\!\right]\right\|^2_{L^2(\mathcal{F}_h)}\right)^{\frac12}\left(h^{-(4i+3)}\left\|\left[\!\left[\Delta^{\tilde m-i-1}e_u\right]\!\right]\right\|^2_{L^2(\mathcal{F}_h)}\right)^{\frac12}\\
&\le C\tau h^{\min(r+1-m,s-m)}\left(S_h(e_u,e_u)\right)^{\frac12}\|u\|_{H^s(\Omega)}.
\end{aligned}
$$

We have used trace inequality and (3.21) to obtain the last inequality above.

By trace inequality and (3.21) again, we have

$$
\begin{aligned}
C_h(\delta_u, e_u) &= -\sum_{i=0}^{\tilde m-1}\left\langle \left\{\!\!\left\{\Delta^{\tilde m+i}\delta_u\right\}\!\!\right\}, \left[\!\left[\nabla\Delta^{\tilde m-i-1}e_u\right]\!\right]\right\rangle_{\mathcal{F}_h} + \sum_{i=0}^{\tilde m-2}\left\langle \left\{\!\!\left\{\nabla\Delta^{\tilde m+i}\delta_u\right\}\!\!\right\}, \left[\!\left[\Delta^{\tilde m-i-1}e_u\right]\!\right]\right\rangle_{\mathcal{F}_h}\\
&\le \sum_{i=0}^{\tilde m-1}\left(h^{4i+1}\left\|\left\{\!\!\left\{\Delta^{\tilde m+i}\delta_u\right\}\!\!\right\}\right\|^2_{L^2(\mathcal{F}_h)}\right)^{\frac12}\left(h^{-(4i+1)}\left\|\left[\!\left[\nabla\Delta^{\tilde m-i-1}e_u\right]\!\right]\right\|^2_{L^2(\mathcal{F}_h)}\right)^{\frac12}\\
&\quad + \sum_{i=0}^{\tilde m-2}\left(h^{4i+3}\left\|\left\{\!\!\left\{\nabla\Delta^{\tilde m+i}\delta_u\right\}\!\!\right\}\right\|^2_{L^2(\mathcal{F}_h)}\right)^{\frac12}\left(h^{-(4i+3)}\left\|\left[\!\left[\Delta^{\tilde m-i-1}e_u\right]\!\right]\right\|^2_{L^2(\mathcal{F}_h)}\right)^{\frac12}\\
&\le C\sum_{i=0}^{\tilde m-1}\left(h^{4i+1}\left(h^{-1}\left\|\Delta^{\tilde m+i}\delta_u\right\|^2_{L^2(\mathcal{T}_h)} + h\left\|\Delta^{\tilde m+i}\delta_u\right\|^2_{H^1(\mathcal{T}_h)}\right)\right)^{\frac12}\left(h^{-(4i+1)}\left\|\left[\!\left[\nabla\Delta^{\tilde m-i-1}e_u\right]\!\right]\right\|^2_{L^2(\mathcal{F}_h)}\right)^{\frac12}\\
&\quad + C\sum_{i=0}^{\tilde m-2}\left(h^{4i+3}\left(h^{-1}\left\|\nabla\Delta^{\tilde m+i}\delta_u\right\|^2_{L^2(\mathcal{T}_h)} + h\left\|\nabla\Delta^{\tilde m+i}\delta_u\right\|^2_{H^1(\mathcal{T}_h)}\right)\right)^{\frac12}\\
&\quad \times\left(h^{-(4i+3)}\left\|\left[\!\left[\Delta^{\tilde m-i-1}e_u\right]\!\right]\right\|^2_{L^2(\mathcal{F}_h)}\right)^{\frac12}\\
&\le C h^{\min(r+1-m,s-m)}\left(S_h(e_u,e_u)\right)^{\frac12}\|u\|_{H^s(\Omega)}.
\end{aligned}
$$

By inverse inequality and discrete trace inequality,

$$
\begin{aligned}
C_h(e_u, \delta_u) &= -\sum_{i=0}^{\tilde m-1}\left\langle \left\{\!\!\left\{\Delta^{\tilde m+i}e_u\right\}\!\!\right\}, \left[\!\left[\nabla\Delta^{\tilde m-i-1}\delta_u\right]\!\right]\right\rangle_{\mathcal{F}_h} + \sum_{i=0}^{\tilde m-2}\left\langle \left\{\!\!\left\{\nabla\Delta^{\tilde m+i}e_u\right\}\!\!\right\}, \left[\!\left[\Delta^{\tilde m-i-1}\delta_u\right]\!\right]\right\rangle_{\mathcal{F}_h}\\
&\le \sum_{i=0}^{\tilde m-1}\left(h^{4i+1}\left\|\left\{\!\!\left\{\Delta^{\tilde m+i}e_u\right\}\!\!\right\}\right\|^2_{L^2(\mathcal{F}_h)}\right)^{\frac12}\left(h^{-(4i+1)}\left\|\left[\!\left[\nabla\Delta^{\tilde m-i-1}\delta_u\right]\!\right]\right\|^2_{L^2(\mathcal{F}_h)}\right)^{\frac12}\\
&\quad + \sum_{i=0}^{\tilde m-2}\left(h^{4i+3}\left\|\left\{\!\!\left\{\nabla\Delta^{\tilde m+i}e_u\right\}\!\!\right\}\right\|^2_{L^2(\mathcal{F}_h)}\right)^{\frac12}\left(h^{-(4i+3)}\left\|\left[\!\left[\Delta^{\tilde m-i-1}\delta_u\right]\!\right]\right\|^2_{L^2(\mathcal{F}_h)}\right)^{\frac12}\\
&\le C\sum_{i=0}^{\tilde m-1}\left(h^{4i}\left\|\Delta^{\tilde m+i}e_u\right\|^2_{L^2(\mathcal{T}_h)}\right)^{\frac12}\left(h^{-(4i+1)}\left\|\left[\!\left[\nabla\Delta^{\tilde m-i-1}\delta_u\right]\!\right]\right\|^2_{L^2(\mathcal{F}_h)}\right)^{\frac12}
\end{aligned}
$$

$$
\begin{aligned}
&+ C\sum_{i=0}^{\tilde m-2}\Big(h^{4i+2}\big\|\nabla\Delta^{\tilde m+i}e_u\big\|^2_{L^2(\mathcal{T}_h)}\Big)^{\frac12}\Big(h^{-(4i+3)}\big\|[\![\Delta^{\tilde m-i-1}\delta_u]\!]\big\|^2_{L^2(\mathcal{F}_h)}\Big)^{\frac12}\\
\le{}& C\sum_{i=0}^{\tilde m-1}\Big(\big\|\Delta^{\tilde m}e_u\big\|^2_{L^2(\mathcal{T}_h)}\Big)^{\frac12}\Big(h^{-(4i+1)}\big\|[\![\nabla\Delta^{\tilde m-i-1}\delta_u]\!]\big\|^2_{L^2(\mathcal{F}_h)}\Big)^{\frac12}\\
&+ C\sum_{i=0}^{\tilde m-2}\Big(\big\|\Delta^{\tilde m}e_u\big\|^2_{L^2(\mathcal{T}_h)}\Big)^{\frac12}\Big(h^{-(4i+3)}\big\|[\![\Delta^{\tilde m-i-1}\delta_u]\!]\big\|^2_{L^2(\mathcal{F}_h)}\Big)^{\frac12}\\
\le{}& Ch^{\min(r+1-m,s-m)}\big\|\Delta^{\tilde m}e_u\big\|^2_{L^2(\mathcal{T}_h)}\|u\|_{H^s(\Omega)}.
\end{aligned}
$$

Combing (3.23) with above estimates, we have

$$\|\Delta^{\tilde m}e_u\|^2_{L^2(\mathcal{T}_h)} + \tau S_h(e_u,e_u) \le Ch^{2\min(r+1-m,s-m)}\|u\|^2_{H^s(\Omega)}.$$

We have used the fact that τ is independent of h to obtain the above inequality. Then by Theorem 3.4, we have

$$\|e_u\|^2_{m,h} \le C\Big(\|\Delta^{\tilde m}e_u\|^2_{L^2(\mathcal{T}_h)} + \tau S_h(e_u,e_u)\Big) \le Ch^{2\min(r+1-m,s-m)}\|u\|^2_{H^s(\Omega)}.$$

Now we obtain the error estimate (3.19) when $m\ge 2$ is an even number. It is similar to show that (3.19) holds when $m \ge 2$ is an odd number. Therefore we can conclude that the proof is complete. □

4. Error analysis under the low regularity assumption

In the above section, we assume the exact solution $u \in H_0^m(\Omega)\cap H^s(\Omega)$ with $s \ge 2m-1$. Since this regularity assumption may be high for the realistic problems, we further deduce the error analysis in this section for the exact solution under the low regularity assumption, *i.e.*, $u \in H_0^m(\Omega)$. In this case, the Galerkin orthogonality does not hold true for the C^0 interior penalty method if $m \ge 2$. We derive the error analysis by the technique developed by Gudi in [12] which utilizes the analysis idea from the *a posteriori* error analysis.

Let $V := H_0^m(\Omega)$ and V_h^c be the H^m-conforming finite element space in V. One can refer to the construction of H^m-conforming finite element space in V in any dimension according to a recent work in [16]. For any $v, w \in V$, let $a(v,w) = (\Delta^{\tilde m}v, \Delta^{\tilde m}w)_\Omega$ if $m = 2\tilde m$ and $a(v,w) = (\nabla\Delta^{\tilde m}v, \nabla\Delta^{\tilde m}w)_\Omega$ if $m = 2\tilde m+1$. As the three abstract assumptions in [12], firstly we assume there exists an enriching operator $E_h : V_h \to V_h^c$ such that

$$\sum_{K\in\mathcal{T}_h} h_K^{-2m}\|v - E_h v\|^2_{L^2(K)} + \|E_h v\|^2_V \le C\|v\|^2_{m,h}, \quad \forall v \in V_h. \tag{4.1}$$

Actually, for the cases of $m = 2$ and 3, this enriching operator E_h has been constructed by averaging technique and the above estimate has been derived in [12, 13].

Secondly, by the definition of $a_h(\cdot,\cdot)$ in (2.3) and Lemma 3.5, choosing τ as in Theorem 3.6, we easily have that

$$\|v_h\|^2_{m,h} \le Ca_h(v_h,v_h), \quad \forall v_h \in V_h. \tag{4.2}$$

Thirdly, we have the following estimate: for any $v \in V$, $w \in V_h^c$ and $v_h \in V_h$, it holds that

$$|a(v,w) - a_h(v_h,w)| \le C\|v - v_h\|_{m,h}\|w\|_V. \tag{4.3}$$

Actually, for $m = 2\tilde m$, due to the fact that $w \in V_h^c$ and $v \in V$, we can derive

$$
\begin{aligned}
a(v,w) - a_h(v_h,w) &= (\Delta^{\tilde m}v, \Delta^{\tilde m}w) - (\Delta^{\tilde m}v_h, \Delta^{\tilde m}w) - C_h(v_h,w) - C_h(w,v_h) - \tau S_h(v_h,w)\\
&= (\Delta^{\tilde m}(v - v_h), \Delta^{\tilde m}w) - C_h(w,v_h)
\end{aligned}
$$

$$= (\Delta^{\tilde{m}}(v - v_h), \Delta^{\tilde{m}} w) - C_h(w, v - v_h). \tag{4.4}$$

By the trace inequality and inverse estimate, we have

$$\begin{aligned} C_h(w, v - v_h) &= -\sum_{i=0}^{\tilde{m}-1} \langle \{\!\{ \Delta^{\tilde{m}+i} w \}\!\}, [\![\nabla \Delta^{\tilde{m}-i-1}(v - v_h)]\!] \rangle_{\mathcal{F}_h} + \sum_{i=0}^{\tilde{m}-2} \langle \{\!\{ \nabla \Delta^{\tilde{m}+i} w \}\!\}, [\![\Delta^{\tilde{m}-i-1}(v - v_h)]\!] \rangle_{\mathcal{F}_h} \\ &\leq C \sum_{i=0}^{\tilde{m}-1} \|\Delta^{\tilde{m}} w\|_{\mathcal{T}_h} h^{-\frac{1}{2}-2i} \| [\![\nabla \Delta^{\tilde{m}-i-1}(v - v_h)]\!] \|_{\mathcal{F}_h} \\ &\quad + C \sum_{i=0}^{\tilde{m}-2} \|\Delta^{\tilde{m}} w\|_{\mathcal{T}_h} h^{-\frac{3}{2}-2i} \| [\![\Delta^{\tilde{m}-i-1}(v - v_h)]\!] \|_{\mathcal{F}_h}, \end{aligned}$$

which, together with (4.4), yields the estimate (4.3). For the case of $m = 2\tilde{m} + 1$, one can similarly derive (4.3) and we omit the details here.

By the estimates (4.1)–(4.3) and following Lemma 2.1 in [12], we have

$$\|u - u_h\|_{m,h} \leq C \inf_{v \in V_h} \left(\|u - v\|_{m,h} + \sup_{\phi \in V_h \setminus \{0\}} \frac{(f, \phi - E_h \phi)_\Omega - a_h(v, \phi - E_h \phi)}{\|\phi\|_{m,h}} \right). \tag{4.5}$$

In order to get the upper bound for the second term on the right-hand side of (4.5), we first provide two lemmas.

Lemma 4.1. *Let $v \in V_h$. There exists a positive constant C independent of mesh size such that*

$$\sum_{K \in \mathcal{T}_h} h^{2m} \|f - (-1)^m \Delta^m v\|_{L^2(K)}^2 \leq \begin{cases} C\big(\|\Delta^{\tilde{m}}(u - v)\|_{\mathcal{T}_h}^2 + \operatorname{osc}_m^2(f)\big), & \text{if } m = 2\tilde{m}, \\ C\big(\|\nabla \Delta^{\tilde{m}}(u - v)\|_{\mathcal{T}_h}^2 + \operatorname{osc}_m^2(f)\big), & \text{if } m = 2\tilde{m} + 1, \end{cases}$$

where

$$\operatorname{osc}_m(f) = \left(\sum_{K \in \mathcal{T}_h} h_K^{2m} \inf_{\overline{f} \in P_{r-m}(K)} \|f - \overline{f}\|_{L^2(K)}^2 \right)^{\frac{1}{2}}. \tag{4.6}$$

Proof. We provide the proof for the case of $m = 2\tilde{m}$, and the case of $m = 2\tilde{m} + 1$ can be similarly deduced. Let $b_K \in P_{m(d+1)}(K) \cap H_0^m(K)$ be the bubble function defined on K such that $b_K(x_K) = 1$, where x_K is the barycenter of the element K. Let $\psi = b_K(\overline{f} - (-1)^m \Delta^m v)$ on $K \in \mathcal{T}_h$ and $\psi = 0$ on $\Omega \setminus K$. We easily have that

$$C_1 \|\overline{f} - (-1)^m \Delta^m v\|_{L^2(K)} \leq \|\psi\|_{L^2(K)} \leq C_2 \|\overline{f} - (-1)^m \Delta^m v\|_{L^2(K)}. \tag{4.7}$$

It follows integration by parts that

$$(f - (-1)^m \Delta^m v, \psi)_K = ((-1)^m \Delta^m u - (-1)^m \Delta^m v, \psi)_K = \big(\Delta^{\tilde{m}}(u - v), \Delta^{\tilde{m}} \psi\big)_K.$$

By the inverse estimate, we further have

$$\begin{aligned} C \|\overline{f} - (-1)^m \Delta^m v\|_{L^2(K)}^2 &\leq \big(\overline{f} - (-1)^m \Delta^m v, \psi\big)_K \\ &= (\overline{f} - f, \psi)_K + (f - (-1)^m \Delta^m v, \psi)_K \\ &= (\overline{f} - f, \psi)_K + \big(\Delta^{\tilde{m}}(u - v), \Delta^{\tilde{m}} \psi\big)_K \\ &\leq C \Big(\|f - \overline{f}\|_{L^2(K)} + h_K^{-m} \|\Delta^{\tilde{m}}(u - v)\|_{L^2(K)} \Big) \|\psi\|_{L^2(K)}, \end{aligned}$$

which, together with (4.7), yields that

$$h_K^m\|\overline{f}-(-1)^m\Delta^m v\|_{L^2(K)} \le C\Big(h_K^m\|f-\overline{f}\|_{L^2(K)} + \|\Delta^{\tilde{m}}(u-v)\|_{L^2(K)}\Big).$$

By the above estimate and the triangular inequality, we directly obtain

$$h^{2m}\|f-(-1)^m\Delta^m v\|_{L^2(K)}^2 \le C\Big(h_K^{2m}\|f-\overline{f}\|_{L^2(K)}^2 + \|\Delta^{\tilde{m}}(u-v)\|_{L^2(K)}^2\Big),$$

which yields the desired estimate. □

Lemma 4.2. *Let $v \in V_h$. For $m = 2\tilde{m}$, there exists a positive constant C independent of mesh size such that, for $i = 0, \cdots, \tilde{m}-1$,*

$$\sum_{F\in\mathcal{F}_h} h_F^{4i+1}\big\|\,[\![\Delta^{\tilde{m}+i}v]\!]\,\big\|_{L^2(F)}^2 \le C\Big(\|\Delta^{\tilde{m}}(u-v)\|_{\mathcal{T}_h}^2 + \mathrm{osc}_m^2(f)\Big), \tag{4.8}$$

$$\sum_{F\in\mathcal{F}_h} h_F^{4i+3}\big\|\,[\![\nabla\Delta^{\tilde{m}+i}v]\!]\,\big\|_{L^2(F)}^2 \le C\Big(\|\Delta^{\tilde{m}}(u-v)\|_{\mathcal{T}_h}^2 + \mathrm{osc}_m^2(f)\Big). \tag{4.9}$$

For $m = 2\tilde{m}+1$, there exists a positive constant C independent of mesh size such that

$$\sum_{F\in\mathcal{F}_h} h_F^{4i+3}\big\|\,[\![\Delta^{\tilde{m}+i+1}v]\!]\,\big\|_{L^2(F)}^2 \le C\Big(\|\nabla\Delta^{\tilde{m}}(u-v)\|_{\mathcal{T}_h}^2 + \mathrm{osc}_m^2(f)\Big), \quad i = 0, \cdots, \tilde{m}-1, \tag{4.10}$$

$$\sum_{F\in\mathcal{F}_h} h_F^{4i+1}\big\|\,[\![\nabla\Delta^{\tilde{m}+i}v]\!]\,\big\|_{L^2(F)}^2 \le C\Big(\|\nabla\Delta^{\tilde{m}}(u-v)\|_{\mathcal{T}_h}^2 + \mathrm{osc}_m^2(f)\Big), \quad i = 0, \cdots, \tilde{m}. \tag{4.11}$$

Proof. For brevity, we only provide the proof for the case of $m = 2\tilde{m}$. The estimates (4.10) and (4.11) for the case of $m = 2\tilde{m}+1$ can be similarly deduced. The proof is based on the induction approach.

Now we prove (4.8) with $i = 0$. For any $F \in \mathcal{F}_h^{\mathrm{int}}$, we denote $\omega_F = K^- \cup K^+$ where $\partial K^- \cap \partial K^+ = F$. Let $\boldsymbol{\nu}_F$ be the unit normal vector along F pointing from K^- to K^+. Let $\xi_1 \in P_{r-1}(\omega_F)$ be defined by

$$\Delta^{\tilde{m}-i}\xi_1|_F = 0, \qquad i = 1, \cdots, \tilde{m}, \tag{4.12}$$

$$\nabla\Delta^{\tilde{m}-i}\xi_1\cdot\boldsymbol{\nu}_F|_F = 0, \qquad i = 2, \cdots, \tilde{m}, \tag{4.13}$$

$$\nabla\Delta^{\tilde{m}-1}\xi_1\cdot\boldsymbol{\nu}_F|_F = [\![\Delta^{\tilde{m}}v]\!]_J|_F. \tag{4.14}$$

For the construction of ξ_1, we can firstly assume $r = m$. Let λ_{d+1}^+ and λ_{d+1}^- be the linear basis functions at the nodes opposite to the face F on K^+ and K^- respectively. We choose $\xi_1|_{K^+} = C^+(\lambda_{d+1}^+)^{m-1}$ and $\xi_1|_{K^-} = C^-(\lambda_{d+1}^-)^{m-1}$, where C^+ and C^- are constants. It is obviously that ξ_1 satisfies (4.12) and (4.13). One can easily choose C^+ and C^- such that

$$C^+\nabla\Delta^{\tilde{m}-1}(\lambda_{d+1}^+)^{m-1}\cdot\boldsymbol{\nu}_F|_{\partial K^+\cap F} = C^-\nabla\Delta^{\tilde{m}-1}(\lambda_{d+1}^-)^{m-1}\cdot\boldsymbol{\nu}_F|_{\partial K^-\cap F} = [\![\Delta^{\tilde{m}}v]\!]_J|_F.$$

For $r > m$, one can similarly construct $\xi_1|_{K^+} = (\lambda_{d+1}^+)^{m-1}\xi_1^+$ and $\xi_1|_{K^-} = (\lambda_{d+1}^-)^{m-1}\xi_1^-$, where $\xi_1^+ \in P_{r-m}(K^+), \xi_1^- \in P_{r-m}(K^-)$ such that ξ_1 satisfies (4.12)–(4.14).

Let $\xi_2 \in H_0^m(\omega_F)$ be a piecewise polynomial bubble function such that $\xi_2(m_F) = 1$, where m_F is the barycenter of F. Denote $\phi = \xi_1\xi_2$ on ω_F and extend it by zero on $\Omega\setminus\omega_F$. It follows from the definitions of ξ_1, ξ_2 and integration by parts that

$$\begin{aligned} C\big\|\,[\![\Delta^{\tilde{m}}v]\!]\,\big\|_{L^2(F)}^2 &\le \big\langle [\![\Delta^{\tilde{m}}v]\!]_J, \xi_2\nabla\Delta^{\tilde{m}-1}\xi_1\cdot\boldsymbol{\nu}_F\big\rangle_F \\ &= \big\langle [\![\Delta^{\tilde{m}}v]\!]_J, \nabla\Delta^{\tilde{m}-1}\phi\cdot\boldsymbol{\nu}_F\big\rangle_F \end{aligned}$$

$$
\begin{aligned}
&= \left(\Delta^{\tilde{m}} v, \Delta^{\tilde{m}}\phi\right)_{\omega_F} + \left(\nabla\Delta^{\tilde{m}} v, \nabla\Delta^{\tilde{m}-1}\phi\right)_{\omega_F} \\
&= \left(\Delta^{\tilde{m}} v, \Delta^{\tilde{m}}\phi\right)_{\omega_F} - \left(\Delta^{2\tilde{m}} v, \phi\right)_{\omega_F} \\
&= \left(\Delta^{\tilde{m}} (v-u), \Delta^{\tilde{m}}\phi\right)_{\omega_F} + \left(f - \Delta^{m} v, \phi\right)_{\omega_F}.
\end{aligned}
$$

By the scaling argument, for $K \in \omega_F$, we have

$$
\|\xi_1\|_{L^\infty(K)} \le C h_F^{m-\frac{1}{2}-\frac{d}{2}} \big\| \llbracket \Delta^{\tilde{m}} v \rrbracket_J \big\|_{L^2(F)} = C h_F^{m-\frac{1}{2}-\frac{d}{2}} \big\| \llbracket \Delta^{\tilde{m}} v \rrbracket \big\|_{L^2(F)},
$$

which directly yields that

$$
\|\phi\|_{L^2(K)} \le C \|\xi_1\|_{L^\infty(K)} \|\xi_2\|_{L^2(K)} \le C h_F^{m-\frac{1}{2}} \big\| \llbracket \Delta^{\tilde{m}} v \rrbracket \big\|_{L^2(F)}. \tag{4.15}
$$

By the inverse estimate, we have

$$
\big\| \llbracket \Delta^{\tilde{m}} v \rrbracket \big\|^2_{L^2(F)} \le C \sum_{K\in\omega_F} \Big(h_K^{-m} \big\| \Delta^{\tilde{m}}(u-v) \big\|_{L^2(K)} + \|f - \Delta^m v\|_{L^2(K)} \Big) \|\phi\|_{L^2(K)}. \tag{4.16}
$$

Combining (4.15) and (4.16) yields

$$
h_F \big\| \llbracket \Delta^{\tilde{m}} v \rrbracket \big\|^2_{L^2(F)} \le C \sum_{K\in\omega_F} \Big(\big\| \Delta^{\tilde{m}}(u-v) \big\|^2_{L^2(K)} + h_F^{2m} \|f - \Delta^m v\|_{L^2(K)} \Big). \tag{4.17}
$$

The above estimate (4.17) can be similarly deduced for the case of $F \in \partial\Omega$. Now combining (4.17), Lemma 4.1 and summing over all $F \in \mathfrak{F}_h$, we get the estimate (4.8) with $i = 0$.

Next we prove (4.9) with $i = 0$. For any $F \in \mathfrak{F}_h^{\text{int}}$, let $\eta_1 \in P_{r-3}(\omega_F)$ be defined by $\Delta^{\tilde{m}-i}\eta_1|_F = 0$, $\nabla\Delta^{\tilde{m}-i}\eta_1 \cdot \boldsymbol{\nu}_F|_F = 0$ with $i = 2, \cdots, \tilde{m}$, and $\Delta^{\tilde{m}-1}\eta_1|_F = \llbracket \nabla\Delta^{\tilde{m}} v \rrbracket|_F$. Here η_1 can be similarly constructed as ξ_1. Let $\eta_2 \in H_0^m(\omega_F)$ be a piecewise polynomial bubble function such that $\eta_2(m_F) = 1$, where m_F is the barycenter of F. Denote $\psi = \eta_1\eta_2$ on ω_F and extend it by zero on $\Omega \setminus \omega_F$. By integration by parts, we have

$$
\begin{aligned}
C\big\| \llbracket \nabla\Delta^{\tilde{m}} v \rrbracket \big\|^2_{L^2(F)} &\le \left\langle \llbracket \nabla\Delta^{\tilde{m}} v \rrbracket, \eta_2 \Delta^{\tilde{m}-1}\eta_1 \right\rangle_F \\
&= \left\langle \llbracket \nabla\Delta^{\tilde{m}} v \rrbracket, \Delta^{\tilde{m}-1}\psi \right\rangle_F \\
&= \left(\nabla\Delta^{\tilde{m}} v, \nabla\Delta^{\tilde{m}-1}\psi\right)_{\omega_F} + \left(\Delta^{\tilde{m}+1} v, \Delta^{\tilde{m}-1}\psi\right)_{\omega_F} \\
&= -\left(\Delta^{\tilde{m}} v, \Delta^{\tilde{m}}\psi\right)_{\omega_F} + \sum_{K\in\omega_F} \int_{\partial K} \llbracket \Delta^{\tilde{m}} v \rrbracket \left\{\!\!\left\{ \nabla\Delta^{\tilde{m}-1}\psi \right\}\!\!\right\} + \left(\Delta^{2\tilde{m}} v, \psi\right)_{\omega_F} \\
&= \left(\Delta^{\tilde{m}}(u-v), \Delta^{\tilde{m}}\psi\right)_{\omega_F} + \sum_{K\in\omega_F} \int_{\partial K} \llbracket \Delta^{\tilde{m}} v \rrbracket \left\{\!\!\left\{ \nabla\Delta^{\tilde{m}-1}\psi \right\}\!\!\right\} + \left(\Delta^{2\tilde{m}} v - f, \psi\right)_{\omega_F}.
\end{aligned}
$$

By the scaling argument, for $K \in \omega_F$, we have

$$
\|\eta_1\|_{L^\infty(K)} \le C h_F^{m-\frac{3}{2}-\frac{d}{2}} \big\| \llbracket \nabla\Delta^{\tilde{m}} v \rrbracket \big\|_{L^2(F)},
$$

which yields

$$
\|\psi\|_{L^2(K)} \le C h_F^{m-\frac{3}{2}} \big\| \llbracket \nabla\Delta^{\tilde{m}} v \rrbracket \big\|_{L^2(F)}. \tag{4.18}
$$

By the inverse estimate and the trace inequality, we get

$$
h_F^3 \big\| \llbracket \nabla\Delta^{\tilde{m}} v \rrbracket \big\|^2_{L^2(F)} \le C \sum_{K\in\omega_F} \Big(\big\| \Delta^{\tilde{m}}(u-v) \big\|_{L^2(K)} + h_F^m \|f - \Delta^m v\|_{L^2(K)} + h_F^{\frac{1}{2}} \big\| \llbracket \Delta^{\tilde{m}} v \rrbracket \big\|_{L^2(\partial K)} \Big) h_F^{3-m} \|\psi\|_{L^2(K)}.
$$

The above estimate can be similarly deduced for the case of $F \in \partial\Omega$. Combining the above estimate with (4.18), (4.17), Lemma 4.1 and summing over all $F \in \mathcal{F}_h$, we obtain the estimate (4.9) with $i = 0$.

We assume (4.8) and (4.9) hold true for $0 < i = k \leq \tilde{m} - 2$, we would like to prove that (4.8) and (4.9) hold true with $i = k + 1$. Since the derivations are similar, we only show the proof for (4.8) with $i = k + 1$.

For any $F \in \mathcal{F}_h^{\text{int}}$, let $\gamma_1 \in P_{r-4k-5}(\omega_F)$ be defined by $\Delta^{\tilde{m}-l}\gamma_1|_F = 0$ with $l = k+2, \cdots, \tilde{m}$, $\nabla\Delta^{\tilde{m}-l}\gamma_1 \cdot \boldsymbol{\nu}_F|_F = 0$ with $l = k+3, \cdots, \tilde{m}$, $\nabla\Delta^{\tilde{m}-k-2}\gamma_1 \cdot \boldsymbol{\nu}_F|_F = \llbracket \Delta^{\tilde{m}+k+1}v \rrbracket_J|_F$. Here γ_1 can be similarly constructed as ξ_1. Let $\gamma_2 \in H_0^m(\omega_F)$ be a piecewise polynomial bubble function such that $\gamma_2(m_F) = 1$, where m_F is the barycenter of F. Denote $\varphi = \gamma_1\gamma_2$ on ω_F and extend it by zero on $\Omega \setminus \omega_F$. By integration by parts, we have

$$\begin{aligned} C\big\|\llbracket \Delta^{\tilde{m}+k+1}v \rrbracket\big\|_{L^2(F)}^2 &\leq \big\langle \llbracket \Delta^{\tilde{m}+k+1}v \rrbracket_J, \gamma_2\nabla\Delta^{\tilde{m}-k-2}\gamma_1 \cdot \boldsymbol{\nu}_F \big\rangle_F \\ &= \big\langle \llbracket \Delta^{\tilde{m}+k+1}v \rrbracket_J, \nabla\Delta^{\tilde{m}-k-2}\varphi \cdot \boldsymbol{\nu}_F \big\rangle_F \\ &= \big(\Delta^{\tilde{m}+k+1}v, \Delta^{\tilde{m}-k-1}\varphi\big)_{\omega_F} + \big(\nabla\Delta^{\tilde{m}+k+1}v, \nabla\Delta^{\tilde{m}-k-2}\varphi\big)_{\omega_F}. \end{aligned} \tag{4.19}$$

By the definition of φ, integration by parts yields

$$\big(\nabla\Delta^{\tilde{m}+k+1}v, \nabla\Delta^{\tilde{m}-k-2}\varphi\big)_{\omega_F} = -\big(\Delta^{2\tilde{m}}v, \varphi\big)_{\omega_F}. \tag{4.20}$$

We also have

$$\begin{aligned} \big(\Delta^{\tilde{m}+k+1}v, \Delta^{\tilde{m}-k-1}\varphi\big)_{\omega_F} &= \big(\Delta^{\tilde{m}+k}v, \Delta^{\tilde{m}-k}\varphi\big)_{\omega_F} - \int_F \llbracket \Delta^{\tilde{m}+k}v \rrbracket_J \nabla\Delta^{\tilde{m}-k-1}\varphi \cdot \boldsymbol{\nu}_F \\ &\quad + \int_F \llbracket \nabla\Delta^{\tilde{m}+k}v \rrbracket \Delta^{\tilde{m}-k-1}\varphi \\ &:= T_0 + T_1 + T_2. \end{aligned} \tag{4.21}$$

By the scaling argument, for $K \in \omega_F$, we have

$$\|\gamma_1\|_{L^\infty(K)} \leq Ch_F^{2\tilde{m}-2k-\frac{5}{2}-\frac{d}{2}}\big\|\llbracket \Delta^{\tilde{m}+k+1}v \rrbracket\big\|_{L^2(F)},$$

which yields

$$\|\varphi\|_{L^2(K)} \leq Ch_F^{2\tilde{m}-2k-\frac{5}{2}}\big\|\llbracket \Delta^{\tilde{m}+k+1}v \rrbracket\big\|_{L^2(F)}. \tag{4.22}$$

By the trace inequality, inverse estimate and (4.22), we obtain

$$\begin{aligned} T_1 &\leq C\big\|\llbracket \Delta^{\tilde{m}+k}v \rrbracket\big\|_{L^2(F)} h_F^{-\frac{1}{2}} \big\|\nabla\Delta^{\tilde{m}-k-1}\varphi\big\|_{L^2(\omega_F)} \\ &\leq C\big\|\llbracket \Delta^{\tilde{m}+k}v \rrbracket\big\|_{L^2(F)} h_F^{-\frac{1}{2}} h_F^{-(2\tilde{m}-2k-1)}\|\varphi\|_{L^2(\omega_F)} \\ &\leq Ch_F^{-2}\big\|\llbracket \Delta^{\tilde{m}+k}v \rrbracket\big\|_{L^2(F)}\big\|\llbracket \Delta^{\tilde{m}+k+1}v \rrbracket\big\|_{L^2(F)}. \end{aligned}$$

Similarly, by the trace inequality, inverse estimate and (4.22), we have

$$T_2 \leq Ch_F^{-1}\big\|\llbracket \nabla\Delta^{\tilde{m}+k}v \rrbracket\big\|_{L^2(F)}\big\|\llbracket \Delta^{\tilde{m}+k+1}v \rrbracket\big\|_{L^2(F)}.$$

For the estimate of T_0, following integration by parts, the trace inequality, inverse estimate and (4.22) yields

$$T_0 = \big(\Delta^{\tilde{m}}v, \Delta^{\tilde{m}}\varphi\big)_{\omega_F} - \sum_{l=0}^{k-1}\int_F \llbracket \Delta^{\tilde{m}+l}v \rrbracket_J \nabla\Delta^{\tilde{m}-l-1}\varphi \cdot \boldsymbol{\nu}_F + \sum_{l=0}^{k-1}\int_F \llbracket \nabla\Delta^{\tilde{m}+l}v \rrbracket \Delta^{\tilde{m}-l-1}\varphi,$$

$$\begin{aligned}
&\leq \left(\Delta^{\tilde{m}} v, \Delta^{\tilde{m}} \varphi\right)_{\omega_F} + C\sum_{l=0}^{k-1} \left\| [\![\Delta^{\tilde{m}+l} v]\!] \right\|_{L^2(F)} h_F^{-\frac{1}{2}} \left\|\nabla\Delta^{\tilde{m}-l-1}\varphi\right\|_{L^2(\omega_F)} \\
&\quad + C\sum_{l=0}^{k-1} \left\| [\![\nabla\Delta^{\tilde{m}+l} v]\!] \right\|_{L^2(F)} h_F^{-\frac{1}{2}} \left\|\Delta^{\tilde{m}-l-1}\varphi\right\|_{L^2(\omega_F)} \\
&\leq \left(\Delta^{\tilde{m}} v, \Delta^{\tilde{m}} \varphi\right)_{\omega_F} \\
&\quad + C\sum_{l=0}^{k-1} \left(h_F^{2l+\frac{1}{2}} \left\| [\![\nabla\Delta^{\tilde{m}+l} v]\!] \right\|_{L^2(F)} + h_F^{2l+\frac{3}{2}} \left\| [\![\nabla\Delta^{\tilde{m}+l} v]\!] \right\|_{L^2(F)} \right) h_F^{-2k-\frac{5}{2}} \left\| [\![\Delta^{\tilde{m}+k+1} v]\!] \right\|_{L^2(F)}.
\end{aligned}$$

Combining (4.19)–(4.21) and the upper bounds for T_0, T_1 and T_2, we have

$$\begin{aligned}
C\left\| [\![\Delta^{\tilde{m}+k+1} v]\!] \right\|_{L^2(F)}^2 &\leq -(\Delta^{2\tilde{m}} v, \varphi)_{\omega_F} + (\Delta^{\tilde{m}} v, \Delta^{\tilde{m}}\varphi)_{\omega_F} \\
&+ C\sum_{l=0}^{k-1} \left(h_F^{2l+\frac{1}{2}} \left\| [\![\nabla\Delta^{\tilde{m}+l} v]\!] \right\|_{L^2(F)} + h_F^{2l+\frac{3}{2}} \left\| [\![\nabla\Delta^{\tilde{m}+l} v]\!] \right\|_{L^2(F)} \right) h_F^{-2k-\frac{5}{2}} \left\| [\![\Delta^{\tilde{m}+k+1} v]\!] \right\|_{L^2(F)} \\
&+ Ch_F^{-2} \left\| [\![\Delta^{\tilde{m}+k} v]\!] \right\|_{L^2(F)} \left\| [\![\Delta^{\tilde{m}+k+1} v]\!] \right\|_{L^2(F)} + Ch_F^{-1} \left\| [\![\nabla\Delta^{\tilde{m}+k} v]\!] \right\|_{L^2(F)} \left\| [\![\Delta^{\tilde{m}+k+1} v]\!] \right\|_{L^2(F)}.
\end{aligned} \tag{4.23}$$

We note that

$$-\left(\Delta^{2\tilde{m}} v, \varphi\right)_{\omega_F} + \left(\Delta^{\tilde{m}} v, \Delta^{\tilde{m}}\varphi\right)_{\omega_F} = \left(f - \Delta^{2\tilde{m}} v, \varphi\right)_{\omega_F} + \left(\Delta^{\tilde{m}} v - \Delta^{\tilde{m}} u, \Delta^{\tilde{m}}\varphi\right)_{\omega_F}.$$

For any $F \in \partial\Omega$, we can derive the similar estimate as (4.23). Now combining (4.23), the Cauchy–Schwarz inequality, the estimates of (4.8) and (4.9) with $i \leq k$, the inverse estimate and (4.22) and summing over all $F \in \mathcal{F}_h$, we can finally obtain (4.8) with $i = k+1$. □

Now we can start to derive the upper bound for the second term on the right-hand side of (4.5). We provide the estimate for the case of $m = 2\tilde{m}$, and the estimate for the case of $m = 2\tilde{m}+1$ can be similarly obtained. For any $\phi \in V_h \setminus \{0\}$, let $\zeta = \phi - E_h\phi$. By the definition of $a_h(\cdot,\cdot)$ in (2.3) and integration by parts, we have

$$\begin{aligned}
(f,\zeta)_\Omega - a_h(v,\zeta) &= (f,\zeta)_\Omega - \left(\Delta^{\tilde{m}} v, \Delta^{\tilde{m}}\zeta\right)_{\mathcal{T}_h} - C_h(v,\zeta) - C_h(\zeta,v) - \tau S_h(v,\zeta) \\
&= (f - (-1)^m \Delta^m v, \zeta)_\Omega - \sum_{i=0}^{\tilde{m}-1} \left\langle \{\!\{\Delta^{\tilde{m}+i} v\}\!\}, [\![\nabla\Delta^{\tilde{m}-i-1}\zeta]\!] \right\rangle_{\mathcal{F}_h} \\
&\quad - \sum_{i=0}^{\tilde{m}-1} \left\langle [\![\Delta^{\tilde{m}+i} v]\!], \{\!\{\nabla\Delta^{\tilde{m}-i-1}\zeta\}\!\} \right\rangle_{\mathcal{F}_h} \\
&\quad + \sum_{i=0}^{\tilde{m}-2} \left\langle \{\!\{\nabla\Delta^{\tilde{m}+i} v\}\!\}, [\![\Delta^{\tilde{m}-i-1}\zeta]\!] \right\rangle_{\mathcal{F}_h} + \sum_{i=0}^{\tilde{m}-1} \left\langle [\![\nabla\Delta^{\tilde{m}+i} v]\!], \{\!\{\Delta^{\tilde{m}-i-1}\zeta\}\!\} \right\rangle_{\mathcal{F}_h} \\
&\quad - C_h(v,\zeta) - C_h(\zeta,v) - \tau S_h(v,\zeta).
\end{aligned}$$

By the definition of $C_h(\cdot,\cdot)$, the trace inequality, inverse estimate and (4.1), we further have

$$\begin{aligned}
&(f,\zeta)_\Omega - a_h(v,\zeta) \\
&\quad = (f - (-1)^m \Delta^m v, \zeta)_\Omega - \sum_{i=0}^{\tilde{m}-1} \left\langle [\![\Delta^{\tilde{m}+i} v]\!], \{\!\{\nabla\Delta^{\tilde{m}-i-1}\zeta\}\!\} \right\rangle_{\mathcal{F}_h} + \sum_{i=0}^{\tilde{m}-1} \left\langle [\![\nabla\Delta^{\tilde{m}+i} v]\!], \{\!\{\Delta^{\tilde{m}-i-1}\zeta\}\!\} \right\rangle_{\mathcal{F}_h} \\
&\qquad + \sum_{i=0}^{\tilde{m}-1} \left\langle \{\!\{\Delta^{\tilde{m}+i}\zeta\}\!\}, [\![\nabla\Delta^{\tilde{m}-i-1} v]\!] \right\rangle_{\mathcal{F}_h} - \sum_{i=0}^{\tilde{m}-2} \left\langle \{\!\{\nabla\Delta^{\tilde{m}+i}\zeta\}\!\}, [\![\Delta^{\tilde{m}-i-1} v]\!] \right\rangle_{\mathcal{F}_h}
\end{aligned}$$

$$
\begin{aligned}
&- \tau \sum_{i=0}^{\tilde{m}-1} h^{-(4i+1)} \langle [\![\nabla \Delta^{\tilde{m}-i-1} v]\!], [\![\nabla \Delta^{\tilde{m}-i-1} \zeta]\!] \rangle_{\mathcal{F}_h} - \tau \sum_{i=0}^{\tilde{m}-2} h^{-(4i+3)} \langle [\![\Delta^{\tilde{m}-i-1} v]\!], [\![\Delta^{\tilde{m}-i-1} \zeta]\!] \rangle_{\mathcal{F}_h} \\
\leq{}& \|f - (-1)^m \Delta^m v\|_{L^2(\mathcal{T}_h)} \|\zeta\|_{L^2(\mathcal{T}_h)} + C \sum_{i=0}^{\tilde{m}-1} \big\| [\![\Delta^{\tilde{m}+i} v]\!] \big\|_{L^2(\mathcal{F}_h)} h^{-2\tilde{m}+2i+\frac{1}{2}} \|\zeta\|_{L^2(\mathcal{T}_h)} \\
&+ C \sum_{i=0}^{\tilde{m}-1} \big\| [\![\nabla \Delta^{\tilde{m}+i} v]\!] \big\|_{L^2(\mathcal{F}_h)} h^{-2\tilde{m}+2i+\frac{3}{2}} \|\zeta\|_{L^2(\mathcal{T}_h)} \\
&+ C \sum_{i=0}^{\tilde{m}-1} \big\| [\![\nabla \Delta^{\tilde{m}-i-1} v]\!] \big\|_{L^2(\mathcal{F}_h)} h^{-2\tilde{m}-2i-\frac{1}{2}} \|\zeta\|_{L^2(\mathcal{T}_h)} \\
&+ C \sum_{i=0}^{\tilde{m}-2} \big\| [\![\Delta^{\tilde{m}-i-1} v]\!] \big\|_{L^2(\mathcal{F}_h)} h^{-2\tilde{m}-2i-\frac{3}{2}} \|\zeta\|_{L^2(\mathcal{T}_h)} \\
&+ C \sum_{i=0}^{\tilde{m}-1} h^{-(4i+1)} \big\| [\![\nabla \Delta^{\tilde{m}-i-1} v]\!] \big\|_{L^2(\mathcal{F}_h)} \big\| [\![\nabla \Delta^{\tilde{m}-i-1} \zeta]\!] \big\|_{L^2(\mathcal{F}_h)} \\
&+ C \sum_{i=0}^{\tilde{m}-2} h^{-(4i+3)} \big\| [\![\Delta^{\tilde{m}-i-1} v]\!] \big\|_{L^2(\mathcal{F}_h)} \big\| [\![\Delta^{\tilde{m}-i-1} \zeta]\!] \big\|_{L^2(\mathcal{F}_h)} \\
\leq{}& C h^m \|f - (-1)^m \Delta^m v\|_{L^2(\mathcal{T}_h)} \|\phi\|_{m,h} + C \sum_{i=0}^{\tilde{m}-1} h^{2i+\frac{1}{2}} \big\| [\![\Delta^{\tilde{m}+i} v]\!] \big\|_{L^2(\mathcal{F}_h)} \|\phi\|_{m,h} \\
&+ C \sum_{i=0}^{\tilde{m}-1} h^{2i+\frac{3}{2}} \big\| [\![\nabla \Delta^{\tilde{m}+i} v]\!] \big\|_{L^2(\mathcal{F}_h)} \|\phi\|_{m,h} + C \sum_{i=0}^{\tilde{m}-1} h^{-2i-\frac{1}{2}} \big\| [\![\nabla \Delta^{\tilde{m}-i-1} v]\!] \big\|_{L^2(\mathcal{F}_h)} \|\phi\|_{m,h} \\
&+ C \sum_{i=0}^{\tilde{m}-2} h^{-2i-\frac{3}{2}} \big\| [\![\Delta^{\tilde{m}-i-1} v]\!] \big\|_{L^2(\mathcal{F}_h)} \|\phi\|_{m,h} \\
&+ C \sum_{i=0}^{\tilde{m}-1} h^{-(4i+1)} \big\| [\![\nabla \Delta^{\tilde{m}-i-1} v]\!] \big\|_{L^2(\mathcal{F}_h)} \big\| [\![\nabla \Delta^{\tilde{m}-i-1} \phi]\!] \big\|_{L^2(\mathcal{F}_h)} \\
&+ C \sum_{i=0}^{\tilde{m}-2} h^{-(4i+3)} \big\| [\![\Delta^{\tilde{m}-i-1} v]\!] \big\|_{L^2(\mathcal{F}_h)} \big\| [\![\Delta^{\tilde{m}-i-1} \phi]\!] \big\|_{L^2(\mathcal{F}_h)}.
\end{aligned}
$$

Combining the above estimate, Lemmas 4.1 and 4.2, we directly have

$$
(f, \zeta)_\Omega - a_h(v, \zeta) \leq C \Big(\big\| \Delta^{\tilde{m}} (u - v) \big\|_{\mathcal{T}_h} + \mathrm{osc}_m(f) \Big) \|\phi\|_{m,h},
$$

which yields

$$
\sup_{\phi \in V_h \setminus \{0\}} \frac{(f, \phi - E_h \phi)_\Omega - a_h(v, \phi - E_h \phi)}{\|\phi\|_{m,h}} \leq C(\|u - v\|_{m,h} + \mathrm{osc}_m(f)). \tag{4.24}
$$

The above estimate (4.24) can be similarly deduced for the case of $m = 2\tilde{m} + 1$ and we omit the details here.

Now by the estimates (4.5) and (4.24), we obtain the following convergence result for the C^0 interior penalty method for the mth-Laplace equation (1.1) with $m \geq 2$.

Theorem 4.3. *For $m \geq 2$, we assume that the exact solution $u \in H_0^m(\Omega)$ for (1.1), $\tau \geq \tau_0 \geq 1$ where τ_0 is the same as Theorem 3.6, and there exists an enriching operator $E_h : V_h \to V_h^c$ such that (4.1) holds true. Then*

Table 1. Example 5.1: errors with estimated rates of convergence when $r = m$ and $m = 2$, 3 and 4, respectively.

$1/h$	$\|u-u_h\|_{2,h}$	Order	$\|u-u_h\|_{3,h}$	Order	$\|u-u_h\|_{4,h}$	Order
8	1.1095e − 1	–	4.5054e − 1	–	6.3198e − 1	–
16	5.9870e − 2	0.89	2.4651e − 1	0.87	3.5797e − 1	0.82
32	3.0564e − 2	0.97	1.2672e − 1	0.96	1.9051e − 1	0.91
64	1.5388e − 2	0.99	6.4245e − 2	0.98	9.7934e − 2	0.96

there is a positive constant C independent of h such that

$$\|u-u_h\|_{m,h} \leq C \inf_{v\in V_h} (\|u-v\|_{m,h} + \mathrm{osc}_m(f)). \tag{4.25}$$

Here $u_h \in V_h$ is the numerical solution of the method (2.2).

Now we immediately have the following estimates. Assuming the exact solution $u \in H_0^m(\Omega)\cap H^s(\Omega)$ for (1.1), $s > m$, we have

$$\|u-u_h\|_{m,h} \leq C\Big(h^{\min(r+1-m,s-m)}\|u\|_{H^s(\Omega)} + \mathrm{osc}_m(f)\Big).$$

In particular, if the oscillation term $\mathrm{osc}_m(f)$ is zero, we have

$$\|u-u_h\|_{m,h} \leq Ch^{\min(r+1-m,s-m)}\|u\|_{H^s(\Omega)}.$$

5. Numerical experiments and discussions

In this section, we provide several numerical experiments to verify the theoretical prediction of the C^0 interior penalty finite element method proposed in the previous sections in two and three dimensions. We calculate the rate of convergence of $\|u-u_h\|_{m,h}$ in various discrete H^m norms and compare each computed rate with its theoretical estimate. It is pointed out that the estimated convergence rates have very little dependency on the particular value when $\tau = O(1)$, so we choose $\tau = 1$ in the following tests. All the numerical experiments are carried out in C, and the resulting linear algebraic systems are solved using GMRES solvers from the PETSc package [2].

Example 5.1. For this test, we solve (2.5)–(2.7), namely $m = 2$, 3 and 4, respectively, using the standard r-th order piecewise continuous H^1-conforming finite element space V_h defined in Section 2 with $\Omega = (0,1)^2$. We use the following data:

$$f(x,y) = 2^m\pi^{2m}\sin(\pi x)\sin(\pi y),$$

so that the exact solution is

$$u(x,y) = \sin(\pi x)\sin(\pi y),$$

which satisfies the mth-Laplace equation (1.1a) and homogeneous boundary conditions (1.1b).

We list the errors along with their estimated rates of convergence in Tables 1 and 2 when $r = m$ and $r = m+1$, respectively. It is remarked that long double in C99 standard is used to represent extended precision floating point value for the 4th-order Laplacian operator, which is accurate up to 10^{-20}. The tables indicate the following rates of convergence:

$$\begin{aligned} \|u-u_h\|_{m,h} &= O(h), \quad \text{when} \quad r = m, \\ \|u-u_h\|_{m,h} &= O(h^2), \text{ when} \quad r = m+1. \end{aligned}$$

TABLE 2. Example 5.1: errors with estimated rates of convergence when $r = m+1$ and $m = 2$, 3 and 4, respectively.

$1/h$	$\|u-u_h\|_{2,h}$	Order	$\|u-u_h\|_{3,h}$	Order	$\|u-u_h\|_{4,h}$	Order
8	2.5510e − 2	–	4.8538e − 2	–	9.0579e − 2	–
16	6.7880e − 3	1.91	1.3371e − 2	1.86	2.5477e − 2	1.83
32	1.7207e − 3	1.98	3.5090e − 3	1.93	6.8738e − 3	1.89
64	4.3317e − 4	1.99	8.9566e − 4	1.97	1.8039e − 3	1.93

TABLE 3. Example 5.2: errors with estimated rates of convergence when $r = m = 3$.

$1/h$	$\|u_1-u_{1,h}\|_{3,h,\Omega_1}$	Order	$\|u_2-u_{2,h}\|_{3,h,\Omega_2}$	Order
8	7.7412e − 2	–	5.2910e − 1	–
16	4.1198e − 2	0.91	4.1801e − 1	0.34
32	2.1623e − 2	0.93	3.0600e − 2	0.45
64	1.0962e − 2	0.98	2.1488e − 2	0.51

Example 5.2. In the second example, we test the proposed method in which the solutions have partial regularity on a convex domain [13] and a non-convex one [25], respectively. To this end, we solve the third-Laplace equation

$$(-\Delta)^3 u = f.$$

The first solution is defined on the square domain $\Omega_1 = (0,1)^2$ with homogeneous Dirichlet boundary conditions. The data f is chosen such that the exact solution is given by

$$u_1(x,y) = \big(x^2+y^2\big)^{7.1/4}\big(x-x^2\big)^3\big(y-y^2\big)^3.$$

Here $u_1 \in H^s(\Omega_1)$ and $4 \le s < 4.1$.

While the second solution is on the 2D L-shaped domain $\Omega_2 = (-1,1)^2 \backslash [0,1) \times (-1,0]$ with Dirichlet boundary conditions given explicitly by

$$u_2(r,\theta) = r^{2.5}\sin(2.5\theta).$$

where (r,θ) are polar coordinates. Here $f = 0$ and $u_2 \in H^{3+1/2}(\Omega_2)$ due to the singularity at the origin.

In both cases, the observed errors of the proposed method converge asymptotically with the optimal order h and $h^{1/2}$, respectively, in the discrete H^3 norm, as shown in Table 3.

Example 5.3. Our last example is a three-dimensional problem. We take the cubic domain $(0,1)^3$ as the computational domain and the exact solution u is given by

$$u(x,y,z) = \sin(\pi x)\sin(\pi y)\sin(\pi z),$$

which satisfies the third-Laplace equation (1.1a) $(m = 3)$ and homogeneous boundary conditions (1.1b).

We list the errors and rates of convergence in Table 4, which indicates that the computed solution converges asymptotically linearly to the exact solution in the discrete H^3 norm. The observed rate is in agreement with Theorem 3.7.

TABLE 4. Example 5.3: with estimated rates of convergence when $r = m = 3$.

$1/h$	$\|u - u_h\|_{3,h}$	Order
8	3.1836e − 1	–
16	1.6594e − 1	0.94
32	8.5896e − 2	0.95
64	4.3247e − 2	0.99

6. Conclusion

A C^0 interior penalty method is considered for mth-Laplace equation on bounded Lipschitz polyhedral domain in $\mathbb{R}^d$ in this paper. In order to avoid computing D^m of numerical solution on each element, we reformulate the C^0 interior penalty method for the odd and even m respectively, and only the gradient and Laplace operators are used in the new method. A rigorous and detailed analysis is given for the key estimate that the discrete H^m-norm of the solution can be bounded by the natural energy semi-norm associated with our method. Then the stability estimate and the optimal error estimates with respect to discrete H^m-norm are achieved. The error estimate under the low regularity assumption of the exact solution is also provided. We believe that the proposed C^0 interior penalty method for mth-Laplace equation can be applied for the nonlinear high order partial differential equations which will be our consideration in future.

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