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CONVERGENCE RESULTS OF A HETEROGENEOUS ASYNCHRONOUS NEWMARK TIME INTEGRATORS

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Abstract. This paper is concerned with the convergence analysis of the PH heterogeneous asynchronous time integrators algorithm, proposed by Prakash and Hjelmstad [*Int. J. Numer. Methods Eng.* **61** (2004) 2183–2204], and devoted to transient dynamic problems for structural analysis. According to PH method, the time discretization is performed using the well-known Newmark schemes, where the time step ratio, *i.e.*, the ratio of the macro time step to the micro time step, of two subdomains separated by an interface is a positive integer. The analysis is restricted to linear problems with two subdomains for simplification. We show that L^∞-uniform convergence of the approximated solutions is achieved taking into account damping terms and suitable regularities on load terms. We shall also give some error estimates of the method.

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1. Introduction

The introduction of the mixed time integration methods with/without different time steps (commonly known as heterogeneous asynchronous time integrators) for dynamic problems [1–16] are of considerable interest to many applications involving subdomains decomposition techniques (wave propagation [11,17,18], fluid-structure interaction problems [19], non-smooth contact problems [14]). The principle of heterogeneous asynchronous time integrators (HATI), for transient dynamic problems, consists on splitting the global problem into several subdomains, each one integrated by its own time scheme and/or its own time step. The main advantage of these methods is to be able to integrate the whole problem with several times steps (with/without heterogeneous time schemes) instead of a single time step usually controlled by the smallest element in the mesh, especially for conditionally stable schemes. Among the most popular time schemes currently used for structural dynamic problems, can be cited Newmark schemes [20], α-schemes [21–23] and the Runge–Kutta schemes [24]. In this paper, we focus on the convergence analysis of the mixed time integration method developed by Prakash and Hjelmstad in [25] for structural dynamic problems where the time discretization is performed *via* the Newmark schemes family.

To clarify the purpose of the paper, let us consider Ω as the set of nodes in $\mathbb{R}^d$ ($d = 1$, 2 or 3 in general), which represent, for instance, the nodes of a mesh or the set of discrete points like particles. Assume a partition

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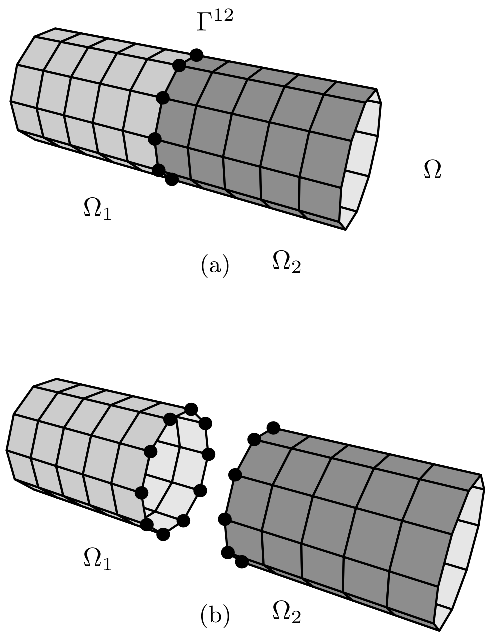

FIGURE 1. The whole domain Ω splitted into two subdomains Ω_1 and Ω_2 (a) where the nodes at the interface Γ^{12} are duplicated (b).

of Ω into two parts Ω_1 and Ω_2 such that the intersection $\Omega_1 \cap \Omega_2 = \Gamma^{12}$ represents the interface separating the two sub-domains (see Fig. 1a). In this paper, we are concerned with the following classical system of equations which describes the transient dynamic problem for the whole domain $\Omega = \Omega_1 \cup \Omega_2$ over the open interval $(0, T)$:

$$\begin{cases} M_1 \ddot{\mathcal{U}}_1(t) + C_1 \dot{\mathcal{U}}_1(t) + K_1 \mathcal{U}_1(t) + L_1^T \lambda(t) = F_1(t) \\ M_2 \ddot{\mathcal{U}}_2(t) + C_2 \dot{\mathcal{U}}_2(t) + K_2 \mathcal{U}_2(t) + L_2^T \lambda(t) = F_2(t) \\ L_1 \dot{\mathcal{U}}_1(t) + L_2 \dot{\mathcal{U}}_2(t) = V_d(t) \\ \dot{\mathcal{U}}_1(0) = V_{01} \text{ and } \mathcal{U}_1(0) = U_{01} \\ \dot{\mathcal{U}}_2(0) = V_{02} \text{ and } \mathcal{U}_2(0) = U_{02} \end{cases} \tag{1.1}$$

where the vector valued functions $\ddot{\mathcal{U}}_i$, $\dot{\mathcal{U}}_i$ and $\mathcal{U}_i$, associated with the subdomain Ω_i with $i \in \{1, 2\}$, stand for the acceleration, velocity and displacement vectors of length N_i, respectively, while the dot symbol stands for the time derivative. The vector-valued functions $t \to F_i(t)$ are the applied external forces with length N_i. The two last equations of the previous system represent the initial conditions for suitable real vectors U_{0i} and V_{0i} of length N_i. In the context of mechanical engineering, the vector-valued function λ of length N_λ play the role of the Lagrange multipliers (we conserve this definition in the paper) and ensure the continuity of the interface forces between the two subdomain. Moreover, we assume the following:

(**H1**):
(1) M_i is symmetric positive definite matrix for every $i \in \{1, 2\}$.
(2) C_i is symmetric positive definite matrix (damping matrix) for every $i \in \{1, 2\}$.
(3) K_i is symmetric positive semi-definite matrix for every $i \in \{1, 2\}$.
(4) The real matrix $\begin{pmatrix} L_1 & L_2 \end{pmatrix} \in \mathbb{R}^{N_\lambda \times (N_1 + N_2)}$ is surjective.

Since the nodes located at the interface are duplicated (Fig. 1b), the continuity of the velocity at the interface is ensured *via* the third equation of the system (1.1). Therefore, the operator $\begin{pmatrix} L_1 & L_2 \end{pmatrix}$ are constructed to ensure

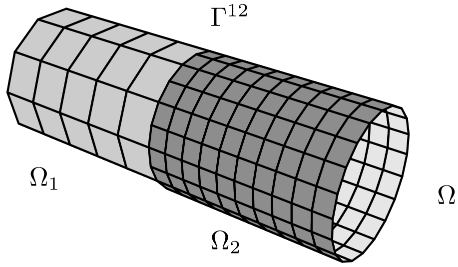

FIGURE 2. The whole domain Ω splitted into two subdomains Ω_1 and Ω_2 with non-conformal meshes.

the continuity of the velocity at the interface Γ^{12} but also the Dirichlet conditions imposed in terms of the velocity for either Ω_1 or Ω_2. We assume that the Dirichlet conditions are represented by the vector-valued function $t \to V_d(t)$ mapping $(0, T)$ to $\mathbb{R}^{N_\lambda}$ which includes the zero components corresponding to the continuities at the interface and satisfies, in addition, the following compatibility condition:

$$L_1 V_{01} + L_2 V_{02} = V_d(0). \tag{1.2}$$

In the context of this paper, the nodes at the interface does not necessary coincide (Fig. 2) which is of interest to the problems with non-conformal meshes. However, the only condition we should deal with is the surjectivity of $\big(L_1\ L_2\big)$ which is a necessary for the well-posedness (see for instance Lem. 2.2 in [26]).

The aim of this paper is the analysis of the stability and the convergence of the PH method applied to the system (1.1) taking into account the assumption (**H1**). According to PH method, the numerical solution is computed at each macro time step by solving a generalized saddle-point problem [26], where the unknowns are the kinematic quantities, related to both subdomains, as well as the Lagrange multipliers. Since the Lagrange multipliers are only computed at the macro time scales, their approximations on the fine scale is performed by mean of a linear interpolation. Under some specific regularity conditions on external loads, we shall prove that the approximated solutions converge uniformly, with respect to the norm L^∞, to the exact solution of the problem. We shall also establish some error estimates under a particular constraint on the Newmark parameters which is often satisfied in practice. To the best of the author's knowledge, a rigorous study on the convergence of the PH method under the assumption (**H1**) is not available at present. It is important to underline that the energy stability (see Lem. 3.5 below) has been established for the PH method in [25]. However, it should be stressed that the latter definition of stability, in the context of PH method, is not generally sufficient to ensure the global boundedness of the set of solutions. Indeed, the investigation of the non-singularity of the global matrix generated by the PH method in [26] has shown that the existence and uniqueness of the numerical solution may fail if the matrix C_1 is only positive semi-definite and L_2 is not onto, *i.e.*, an example of illustration has been given in Remark 3.9 in [26] with numerical validations. In this case, the set of the approximated solutions, assumed not empty, necessary contains an affine subspace. Thus, the set of solutions is unbounded despite the fact that the energy stability is satisfied. In this paper, we shall proceed differently and more rigorously to study the global stability and the convergence of the PH method under the hypothesis (**H1**).

The paper is organized as follows: a brief description of the PH algorithm is given in Section 2. In Section 3, we shall introduce the definitions of the approximated solutions and establish the boundedness results of the numerical solutions. Finally, the last section, *i.e.*, Section 4, will be devoted to the discussion of the main results on the convergence of the PH method.

2. Review of PH method

The application of the PH algorithm on the system (1.1) is performed as follows: First, we choose the micro time scale h_1 for the subdomain Ω_1 and the coarse time scale h_2 for the subdomain Ω_2 linked by $h_2 = mh_1$, where $m \in \mathbb{N}^*$ is a positive integer called the time step ratio. The equation of motion for the subdomain Ω_1 is described at the time $t_j^{(1)} = jh_1$ $(j = 0, 1, 2, \ldots)$, while the equation of motion of the subdomain Ω_2 is described at the time $t_k^{(2)} = kh_2$ $(k = 0, 1, 2, \ldots)$. It is seen that $t_k^{(2)} = kh_2 = kmh_1 = t_{km}^{(1)}$ for each $k \in \{0, 1, 2, \ldots\}$.

Now, let $\ddot{\mathcal{U}}_1^j$, $\dot{\mathcal{U}}_1^j$, $\mathcal{U}_1^j$, F_1^j and λ_1^j be the acceleration, the velocity, the displacement, the external force and the Lagrange multiplier vectors, respectively, related to the subdomain Ω_1 at the time $t_j^{(1)} = jh_1$. Similarly, let $\ddot{\mathcal{U}}_2^k$, $\dot{\mathcal{U}}_2^k$, $\mathcal{U}_2^k$, F_2^k and λ_2^k be the acceleration, the velocity, the displacement, the external force and the Lagrange multiplier vectors, respectively, related to the subdomain Ω_2 at the time $t_k^{(2)} = kh_2 = kmh_1$. The time discretization of the system using the GC method within the range $\left[t_{k-1}^{(2)}, t_k^{(2)}\right]$ $(k \in \mathbb{N}^*)$ is described as follows:

For a fixed $j \in ((k-1)m, km]$, the equilibrium equation for the subdomain Σ_1 at the time $t_j^{(1)}$ writes:

$$M_1 \ddot{\mathcal{U}}_1^j + C_1 \dot{\mathcal{U}}_1^j + K_1 \mathcal{U}_1^j + L_1^T \lambda_1^j = F_1^j. \tag{2.1}$$

The approximations of the quantities $\mathcal{U}_1^j$ and $\dot{\mathcal{U}}_1^j$ using a Newmark scheme, in terms of the parameters γ_1 and β_1, read:

$$\begin{cases} \mathcal{U}_1^j = \mathcal{U}_1^{j-1} + h_1 \dot{\mathcal{U}}_1^{j-1} + h_1^2 \left(\frac{1}{2} - \beta_1\right) \ddot{\mathcal{U}}_1^{j-1} + h_1^2 \beta_1 \ddot{\mathcal{U}}_1^j \\ \dot{\mathcal{U}}_1^j = \dot{\mathcal{U}}_1^{j-1} + h_1 (1 - \gamma_1) \ddot{\mathcal{U}}_1^{j-1} + h_1 \gamma_1 \ddot{\mathcal{U}}_1^j \\ \mathcal{U}_1^0 = U_{01} \\ \dot{\mathcal{U}}_1^0 = V_{01}. \end{cases} \tag{2.2}$$

The equilibrium equation for the subdomain Σ_2 at the time $t_k^{(2)}$ writes:

$$M_2 \ddot{\mathcal{U}}_2^k + C_2 \dot{\mathcal{U}}_2^k + K_2 \mathcal{U}_2^k + L_2^T \lambda_2^k = F_2^k. \tag{2.3}$$

Similarly, the approximations of the quantities $\mathcal{U}_2^k$ and $\dot{\mathcal{U}}_2^k$ using a Newmark scheme, characterized by the parameters γ_2 and β_2, are given by:

$$\begin{cases} \mathcal{U}_2^k = \mathcal{U}_2^{k-1} + h_2 \dot{\mathcal{U}}_2^{k-1} + h_2^2 \left(\frac{1}{2} - \beta_2\right) \ddot{\mathcal{U}}_2^{k-1} + h_2^2 \beta_2 \ddot{\mathcal{U}}_2^k \\ \dot{\mathcal{U}}_2^k = \dot{\mathcal{U}}_2^{k-1} + h_2 (1 - \gamma_2) \ddot{\mathcal{U}}_2^{k-1} + h_2 \gamma_2 \ddot{\mathcal{U}}_2^k \\ \mathcal{U}_2^0 = U_{02} \\ \dot{\mathcal{U}}_2^0 = V_{02}. \end{cases} \tag{2.4}$$

The computation of the Lagrange multiplier at the fine scale, between two coarse time instants $(k-1)h_2$ and kh_2, as proposed in the PH method is equivalent to the following interpolation:

$$\begin{cases} \lambda_1^{(k-1)m+j} = \left(1 - \frac{j}{m}\right) \lambda_1^{(k-1)m} + \frac{j}{m} \lambda_1^{km} + B_2^{(k-1)m+j} \quad 1 \le j \le m \\ \lambda_1^{km} = \lambda_2^k \quad \forall\, k \end{cases} \tag{2.5}$$

where

$$B_2^{(k-1)m+j} = L_2 F_2((k-1)h_2 + jh_1) - \left(1 - \frac{j}{m}\right) L_2 F_2((k-1)h_2) - \frac{j}{m} L_2 F_2(kh_2). \tag{2.6}$$

In this paper, the term B_2^j is assumed to be zero which occurs, for instance, when no external loads are applied to the interface Γ^{12}, *i.e.*, $L_2 F_2^j = 0$ for every index j, or when the external force F_2 are linear or constant with time. Thus, equation (2.5) implies:

$$\lambda_1^{(k-1)m+j} = \left(1 - \frac{j}{m}\right)\lambda_2^{k-1} + \frac{j}{m}\lambda_2^k. \tag{2.7}$$

Finally, the remaining equation concerns the continuity of the velocities at the interface as well as the Dirichlet conditions which are only imposed at the coarse time steps. More precisely,

$$L_1\dot{\mathcal{U}}_1^{km} + L_2\dot{\mathcal{U}}_2^k = V_d^k \quad \text{for every integer k} \geq 1 \tag{2.8}$$

where V_d^k is the value of V_d at the time $t_k^{(2)}$. The computation of the different quantities in (2.2)–(2.7) assumes the knowledge of the initial values $\lambda_1^0 = \lambda_2^0$, $\ddot{\mathcal{U}}_1^0$ and $\ddot{\mathcal{U}}_2^0$ which are arbitrary in the context of this paper. The only requirement is that the initial values satisfy the equilibrium equations as described in (2.1) and (2.3) which has a meaning from a physical point of view. In practice, if V_d is sufficiently smooth, these initial values are generally computed using the following system:

$$\begin{cases} M_1\ddot{\mathcal{U}}_1^0 + C_1\dot{\mathcal{U}}_1^0 + K_1\mathcal{U}_1^0 + L_1^T\lambda_1^0 = F_1(0) \\ M_2\ddot{\mathcal{U}}_2^0 + C_2\dot{\mathcal{U}}_2^0 + K_2\mathcal{U}_2^0 + L_2^T\lambda_2^0 = F_2(0) \\ L_1\ddot{\mathcal{U}}_1^0 + L_2\ddot{\mathcal{U}}_2^0 = \dot{V}_d(0) \\ \lambda_1^0 = \lambda_2^0 \\ \mathcal{U}_1^0 = U_{01} \quad \dot{\mathcal{U}}_1^0 = V_{01} \\ \mathcal{U}_2^0 = U_{02} \quad \dot{\mathcal{U}}_2^0 = V_{02}. \end{cases} \tag{2.9}$$

It is more convenient to rewrite the previous time discretized equations in a block matrix representation as proposed in [25]:

$$\begin{bmatrix} \mathbb{M}_1 & & & & & & \frac{1}{m}\mathbb{L}_1^T \\ \mathbb{N}_1 & \mathbb{M}_1 & & & & & \frac{2}{m}\mathbb{L}_1^T \\ & \cdot & \cdot & & & & \cdot \\ & & \cdot & \cdot & & & \cdot \\ & & & \mathbb{N}_1 & \mathbb{M}_1 & & \frac{m}{m}\mathbb{L}_1^T \\ & & & & & \mathbb{M}_2 & \mathbb{L}_2^T \\ & & & & \mathbb{B}_1 & \mathbb{B}_2 & 0 \end{bmatrix} \begin{bmatrix} \mathbb{U}_1^{(k-1)m+1} \\ \mathbb{U}_1^{(k-1)m+2} \\ \cdot \\ \cdot \\ \mathbb{U}_1^{km} \\ \mathbb{U}_2^k \\ \lambda_2^k \end{bmatrix} = \begin{bmatrix} \mathbb{P}_1^{(k-1)m+1} - \mathbb{N}_1\mathbb{U}_1^{(k-1)m} \\ \mathbb{P}_1^{(k-1)m+2} \\ \cdot \\ \cdot \\ \mathbb{P}_1^{km} \\ \mathbb{P}_2^k - \mathbb{N}_2\mathbb{U}_2^{(k-1)} \\ V_d^k \end{bmatrix} - \begin{bmatrix} \frac{m-1}{m}\mathbb{L}_1^T\lambda_2^{(k-1)} \\ \frac{m-2}{m}\mathbb{L}_1^T\lambda_2^{(k-1)} \\ \cdot \\ \cdot \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{2.10}$$

where for each $i \in \{1, 2\}$ and each integer $j \in [(k-1)m + 1, km]$ for fixed k, we have:

$$\mathbb{M}_i = \begin{bmatrix} M_i & C_i & K_i \\ -\gamma_i h_i I_{N_i} & I_{N_i} & 0 \\ -\beta_i h_i^2 I_{N_i} & 0 & I_{N_i} \end{bmatrix}, \qquad \mathbb{N}_i = \begin{bmatrix} 0 & 0 & 0 \\ -(1-\gamma_i)h_i I_{N_i} & -I_{N_i} & 0 \\ -\left(\frac{1}{2} - \beta_i\right)h_i^2 I_{N_i} & -h_i I_{N_i} & -I_{N_i} \end{bmatrix}$$

$$\mathbb{B}_i^T = \begin{bmatrix} 0 \\ L_i^T \\ 0 \end{bmatrix}, \qquad \mathbb{L}_i^T = \begin{bmatrix} L_i^T \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbb{P}_1^j = \begin{bmatrix} F_1^j \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbb{U}_1^j = \begin{bmatrix} \ddot{\mathcal{U}}_1^j \\ \dot{\mathcal{U}}_1^j \\ \mathcal{U}_1^j \end{bmatrix}$$

$$\mathbb{P}_2^k = \begin{bmatrix} F_2^k \\ 0 \\ 0 \end{bmatrix} \qquad \text{and} \quad \mathbb{U}_2^k = \begin{bmatrix} \ddot{\mathcal{U}}_2^k \\ \dot{\mathcal{U}}_2^k \\ \mathcal{U}_2^k \end{bmatrix}. \tag{2.11}$$

Furthermore, one may notice that the system (2.10) can be cast in the following form:

$$\begin{bmatrix} \mathbb{A} & \mathbb{L}^T \\ \mathbb{B} & 0 \end{bmatrix} \begin{bmatrix} \mathbb{U}^k \\ \lambda_2^k \end{bmatrix} = \begin{bmatrix} \mathbb{F}_k \\ V_d^k. \end{bmatrix} \tag{2.12}$$

The previous system (2.12) is referred as the global problem in the paper and the involved matrix as the global operator. Comparing equations (2.12) and (2.10), the representation is unique and it is clear and the different operators $\mathbb{A}$, $\mathbb{L}$ and $\mathbb{B}$ are given by:

$$\mathbb{A} = \begin{bmatrix} \mathbb{M}_1 & & & & \\ \mathbb{N}_1 & \mathbb{M}_1 & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \mathbb{N}_1 \ \mathbb{M}_1 & \\ & & & & \mathbb{M}_2 \end{bmatrix} \quad \text{and} \quad \mathbb{L}^T = \begin{bmatrix} \frac{1}{m}\mathbb{L}_1^T \\ \frac{2}{m}\mathbb{L}_1^T \\ \cdot \\ \cdot \\ \frac{m}{m}\mathbb{L}_1^T \\ \mathbb{L}_2^T \end{bmatrix} \quad \text{and} \quad \mathbb{B}^T = \begin{bmatrix} 0 \\ 0 \\ \cdot \\ 0 \\ \mathbb{B}_1^T \\ \mathbb{B}_2^T. \end{bmatrix} \tag{2.13}$$

It is important to emphasize that the previous problem (2.12) may be viewed as a particular case of a generalized saddle-point problem and it is worth noting that similar system appears in other multi-time steps methods involving other time schemes (see for instance [10, 12]). The generalized saddle-point problems have been investigated by Nicolaides [27] in the context of Hilbert spaces using the inf-sup conditions and then by Bernardi et al. [28] (see Thm. 2.1 in [28]) by extending the study to Banach spaces using Brezzi's assumptions (see Thm. 1.1 in [29]). In the context of PH method, The non-singularity of the global matrix in (2.12) as well as the strict-positivity of the Schur complement have been studied in [26] depending on the surjectivity of the link matrix L_2 and using a special case of damping matrices (Rayleigh damping). Following the references [25, 26], we consider the following assumptions on the Newmark parameters:

(**H2**): For every $i \in \{1,2\}$, the triple (γ_i, β_i, h_i) satisfies:
(1) $\gamma_i \geq \frac{1}{2}$.
(2) $h_i > 0$.
(3) $\tilde{M}_i = M_i + h_i^2\big(\beta_i - \frac{\gamma_i}{2}\big)K_i > 0$.

3. Boundedness of the numerical solution

In this section, we introduce some definitions of the approximated numerical solutions considered in the paper. We shall also discuss the existence and the stability of the numerical solutions under the assumptions (**H1**), (**H2**) and (**HF**) (Hypothesis (**HF**) is introduced below).

Let us first briefly recall some definitions of classical functional spaces. For every open $J \subset \mathbb{R}$, we consider the following spaces:

– $\mathrm{L}^2\big(J, \mathbb{R}^l\big)$ for $1 \leq p < \infty$ is the space of square integrable Lebesgue measurable functions defined on J with values in $\mathbb{R}^l$ and equipped with the norm $\|.\|_{\mathrm{L}^2(J)} = \left(\int_J \|.\|^2\right)^{\frac{1}{2}}$, where $\|.\|$ is the Euclidian norm. We also endowed the corresponding Hilbert space with the product scalar $(\cdot,\cdot)_{2,J}$.

– $\mathrm{L}^\infty\left(J, \mathbb{R}^l\right)$ is the space of Lebesgue measurable functions f bounded almost everywhere on J and equipped with the norm $\|f\|_{\mathrm{L}^\infty(J)} = \inf\{C \geq 0 : |f(x)| \leq C \quad a.e.\}$.
– $\mathrm{W}^{k,2}\left(J, \mathbb{R}^l\right)$ the space of all Lebesgue measurable functions $v : J \mapsto \mathbb{R}^l$ such that the ith derivative $v^{(i)}$ is square integrable for every $i \in 0, \ldots, k$, $v^{(i)} \in \mathrm{L}^2\left(J, \mathbb{R}^l\right)$. The previous space is equipped with norm $\|v\|_{\mathrm{W}^{k,2}(J)} = \left(\int_J \|v\|^2 + \sum_{i=1}^k \int_J \left\|v^{(i)}\right\|^2\right)^{\frac{1}{2}}$.
– If J is either open or closed the space $\mathrm{C}^k\left(J, \mathbb{R}^l\right)$ is the set of vector-valued functions, together with their classical derivatives of order lower than or equal to k, continuous on J. The Banach space $\mathrm{C}^k\left(J, \mathbb{R}^l\right)$ is endowed with the norm $\|\phi\|_{\mathrm{C}^k(J)} = \max_{0 \leq l \leq k} \sup_{x \in J} |\phi^{(l)}(x)|$. Moreover, We denote $\mathrm{C}^\infty\left(J, \mathbb{R}^l\right) = \cap_{k \geq 0} \mathrm{C}^k\left(J, \mathbb{R}^l\right)$. If $k = 0$, the norm $\|.\|_{\mathrm{C}^0(J)}$ coincides with $\|.\|_{\mathrm{L}^\infty(J)}$.
  For the sake of conciseness, the definitions of the previous norms are independent of the dimension l.

Let $I = (0, T)$ be the open interval and h a positive reel such that $T = nh$, where $n \geq 1$ is a positive integer. Let $\left(X^k\right)_{0 \leq k \leq n}$ be a finite sequence of vectors that represent the values of the quantity X computed at the times $t_0 = 0, t_1 = h, t_2 = 2h, \ldots, t_n = nh$. To every vector $\left(X^k\right)_{0 \leq k \leq n}$ we associate the measurable functions $\bar{X}^{h,r}$ and $\tilde{X}^h$ defined on I by:

$$\begin{cases} \bar{X}^{h,r}(t) = \sum_{k=0}^{n-1} r X_{k+1} + (1-r) X_k \; \chi_{[t_k, t_{k+1})}(t) \quad r \in [0,1] \\ \tilde{X}^h(t) = \sum_{k=1}^{n} \left[\frac{t - t_{k-1}}{h}(X_k - X_{k-1}) + X_k\right] \chi_{[t_{k-1}, t_k)}(t) \end{cases} \tag{3.1}$$

with:

$$\tilde{X}^h(t_n) = X_n. \tag{3.2}$$

The definition (3.1) together with (3.2) imply that $\tilde{X}^h$ is continuous over $\bar{I}$. Moreover, if $r = 1$, $\bar{X}^{h,r}$ is simply denoted $\bar{X}^h$.

Remark 3.1. As a rule, the (weak) time derivative of $\tilde{X}^h$ is denoted $\dot{\tilde{X}}^h$ and should not be confused with the dot notation on the discrete vector $(\dot{X}^k)_k$ which refers to the discrete quantities related to the derivative of X. For instance, $\dot{\tilde{\dot{\mathcal{U}}}}{}_2^{h_2}$ is the time derivative of the interpolation $\tilde{\dot{\mathcal{U}}}{}_2^{h_2}$ constructed from the sequence $(\dot{\mathcal{U}}_2^k)_k$

Given measurable functions F_1, F_2 and V_d as described in the problem (1.1), we say that the triple $(\mathcal{U}_1, \mathcal{U}_2, \lambda)$ of measurable functions in $\bar{I}$ is a solution of the problem (1.1) if $\mathcal{U}_1$ and $\mathcal{U}_2$ are twice weakly differentiable and satisfy (1.1) almost everywhere in I together with the initial conditions. The next result discusses the existence and the uniqueness of the solution of system (1.1) in the continuous framework. We shall state it under general assumptions:

Theorem 3.2 (Existence and uniqueness)**.** *For every $i \in \{1, 2\}$, consider the hypothesis* (**H1**) *but with assumptions 2 and 3 replaced by:*

(1) *$C_i : \mathbb{R}^{N_i} \mapsto \mathbb{R}^{N_i}$ is single-valued and maximal monotone.*
(2) *$K_i : \mathbb{R}^{N_i} \mapsto \mathbb{R}^{N_i}$ is continuous (not necessary linear) and exact, i.e., there exists some Gateau-differentiable function $\psi_i : \mathbb{R}^{N_i} \mapsto \mathbb{R}$ such that $D\psi_i = K_i$. Moreover, we assume that ψ_i is nonnegative and K_i satisfies the following growth:*

$$\|K_i(u)\| \leq C_{K_i} \|u\| \quad \forall u \tag{3.3}$$

where C_{K_i} is a constant independent of u.

If the maps $t \to F_i(t)$ are measurable functions with $F_i \in \mathrm{L}^2\left(I, \mathbb{R}^{N_i}\right)$ and $t \to V_d \in W^{1,2}\left(I, \mathbb{R}^{N_\lambda}\right)$. Then, the problem (1.1) has a solution $(\mathcal{U}_1, \mathcal{U}_2, \lambda)$ in $C^1\left(\bar{I}, \mathbb{R}^{N_1}\right) \times C^1\left(\bar{I}, \mathbb{R}^{N_2}\right) \times \mathrm{L}^2\left(I, \mathbb{R}^{N_\lambda}\right) \cap W^{2,2}\left(I, \mathbb{R}^{N_1}\right) \times W^{2,2}\left(I, \mathbb{R}^{N_2}\right)$ $\times \mathrm{L}^2\left(I, \mathbb{R}^{N_\lambda}\right)$. In the special case where K_i are linear symmetric and positive semi-definite, the solution is unique.

Proof. See Appendix A. □

In the following, we consider the following assumptions on the external loads:

(**HF**):
(1) For every $i \in \{1,2\}$: $F_i \in \mathrm{W}^{1,2}\big(I, \mathbb{R}^{N_i}\big)$.
(2) $V_d \in \mathrm{W}^{2,2}\big(I, \mathbb{R}^{N_\lambda}\big)$.

In the sequel, it is not restrictive to consider $V_d \in \mathrm{C}^1\big(\bar{I}, \mathbb{R}^{N_\lambda}\big) \cap \mathrm{W}^{2,2}\big(I, \mathbb{R}^{N_\lambda}\big)$ and $F_i \in \mathrm{C}^0\big(\bar{I}, \mathbb{R}^{N_i}\big) \cap \mathrm{W}^{1,2}\big(I, \mathbb{R}^{N_i}\big)$ by the continuous representative argument. In this case, it is seen from the proof of Theorem 3.2 that (**HF**) implies that $\ddot{\mathcal{U}}_1$, $\ddot{\mathcal{U}}_2$ and λ are continuous on I. Since the initial problem (1.1) may be reduced to a problem with zero Dirichlet conditions, we will assume in the remainder of the paper that $V_d = 0$. Indeed, using the condition (**HF**) (2), we can replace $\mathcal{U}_1$ and $\mathcal{U}_2$ by $\mathcal{U}_1 - U_{1d}$ and $\mathcal{U}_2 - U_{2d}$, respectively, where the couple (U_{1d}, U_{2d}) is smooth of class C^2, *i.e.*, more precisely $U_{id} \in \mathrm{C}^2\big(\bar{I}, \mathbb{R}^{N_i}\big) \cap \mathrm{W}^{3,2}\big(I, \mathbb{R}^{N_i}\big)$ for $i = 1, 2$, and satisfies $L_1\,\dot{U}_{1d}(t) + L_2\,\dot{U}_{2d}(t) = V_d(t)$ for every $t \in \bar{I}$. In this case, the right hand sides of the two first equations in (1.1) will contains further terms, each one belongs to $\mathrm{W}^{1,2}\big(I, \mathbb{R}^{N_i}\big)$, $i = 1$ or 2 (note that the existence of (U_{1d}, U_{2d}) is guaranteed by the surjectivity of the operator $\big(L_1\ L_2\big)$). For reading convenience, the condition "$V_d = 0$" will be systematically recalled.

The two next lemma describe some basic facts on the PH method, necessary to prove the global boundedness of the numerical solutions in Lemma 3.5. The reader may also refer to [26] for more details.

Lemma 3.3. *Assume that $n = \frac{T}{h_2} \geq 1$ is a positive integer and denote $\langle \cdot, \cdot \rangle$ the duality bracket of an Euclidean space $\mathbb{R}^N$ (the notation is independent of the dimension for simplification). Under the hypothesis* (**H1**), (**H2**) *and* (**HF**) *with $V_d = 0$, assume that the system* (2.12) *has a solution for every $1 \leq k \leq n$, we claim:*

$$\begin{aligned} &\frac{1}{2}\Big[\Big\langle \ddot{\mathcal{U}}_1^{nm}, \tilde{M}_1 \ddot{\mathcal{U}}_1^{nm} \Big\rangle + \Big\langle \dot{\mathcal{U}}_1^{nm}, K_1 \dot{\mathcal{U}}_1^{nm} \Big\rangle\Big] + \frac{1}{2}\Big[\Big\langle \ddot{\mathcal{U}}_2^{n}, \tilde{M}_2 \ddot{\mathcal{U}}_2^{n} \Big\rangle + \Big\langle \dot{\mathcal{U}}_2^{n}, K_2 \dot{\mathcal{U}}_2^{n} \Big\rangle\Big] + \sum_{i=1}^{2} D_i^n \\ &\qquad \leq \sum_{i=1}^{2} \frac{1}{2}\Big[\Big\langle \ddot{\mathcal{U}}_i^{0}, \tilde{M}_i \ddot{\mathcal{U}}_i^{0} \Big\rangle + \Big\langle \dot{\mathcal{U}}_i^{0}, K_i \dot{\mathcal{U}}_i^{0} \Big\rangle\Big] + \sum_{i=1}^{2} W_i^{\mathrm{ext},n} \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} D_1^n &= \frac{1}{h_1} \sum_{j=1}^{nm} \Big\langle C_1 \dot{\mathcal{U}}_1^j - C_1 \dot{\mathcal{U}}_1^{j-1}, \dot{\mathcal{U}}_1^j - \dot{\mathcal{U}}_1^{j-1} \Big\rangle \\ D_2^n &= \frac{1}{h_2} \sum_{k=1}^{n} \Big\langle C_2 \dot{\mathcal{U}}_2^k - C_2 \dot{\mathcal{U}}_2^{(k-1)}, \dot{\mathcal{U}}_2^k - \dot{\mathcal{U}}_2^{(k-1)} \Big\rangle \end{aligned} \tag{3.5}$$

and,

$$\begin{aligned} W_1^{\mathrm{ext},n} &= \frac{1}{h_1} \sum_{j=1}^{nm} \Big\langle F_1^j - F_1^{j-1}, \dot{\mathcal{U}}_1^j - \dot{\mathcal{U}}_1^{j-1} \Big\rangle \\ W_2^{\mathrm{ext},n} &= \frac{1}{h_2} \sum_{k=1}^{n} \Big\langle F_2^k - F_2^{(k-1)}, \dot{\mathcal{U}}_2^k - \dot{\mathcal{U}}_2^{(k-1)} \Big\rangle \end{aligned} \tag{3.6}$$

Proof. This is too similar to the proof given in [25] see also [26]. □

Lemma 3.4. *The matrix $\mathbb{A}$ is non-singular and the (i,j)-th block matrix entry of the inverse of $\mathbb{A}$ is given by:*

$$\mathbb{A}_{ij}^{-1} = \begin{cases} (-1)^{i-j} \mathbb{M}_1^{-1} \big(\mathbb{N}_1 \mathbb{M}_1^{-1}\big)^{i-j} & \text{if} \quad 1 \leq j \leq i \leq m \\ \mathbb{M}_2^{-1} & \text{if} \quad i = j = m+1 \\ 0 & \text{else.} \end{cases} \tag{3.7}$$

Proof. It is enough to check that the multiplication of the block matrix (3.7) with $\mathbb{A}$ gives the identity matrix. $\square$

The "energy stability" result as a consequence of Lemma 3.3 has been originally proved in [25], in the case of zero damping $C_1 = C_2 = 0$, which implies, under the hypothesis of zero external loads, the uniform boundedness of some quantities at the macro-time steps. This result does not evidently ensures the uniform boundedness of all quantities as being highlighted in [26], where it has been proved that the global matrix in equation (2.12) may be singular when the damping matrix C_1 is only positive semi-definite. Lemma 3.5 below provides a more complete proof of stability in the case of positive definite damping matrices and under sufficiently smooth external loads.

Lemma 3.5 (Boundedness result)**.** *Under the hypothesis* (**H1**)*,* (**H2**)-(1) *and* (**HF**) *with* $V_d = 0$*, there exist some constants* $h_{10}, h_{20} > 0$ *for fixed* m *(with* $h_{20} = mh_{10}$*), such that if we define the set* S_h *by*

$$S_h = \left\{(h_1, h_2) \in (0, h_{10}) \times (0, h_{20}) | h_2 = mh_1, n_1 = \frac{T}{h_1} \in \mathbb{N}^*, n_2 = \frac{T}{h_2} \in \mathbb{N}^*\right\}. \tag{3.8}$$

Then, for fixed $(h_1, h_2) \in S_h$*, the problem* (2.10) *has one unique solution* $((\mathbb{U}_1^j)_{1\le j\le n_1}$*,* $(\mathbb{U}_2^k)_{1\le k\le n_2}$*,* $(\lambda_2^k)_{1\le k\le n_2})$*. Moreover,*

$$\sup_{(h_1,h_2)\in S_h} \sum_{i=1,2} \left(\left\|\bar{\ddot{\mathcal{U}}}_i^{h_i}\right\|_{\mathrm{L}^\infty(I)} + \left\|\bar{\dot{\mathcal{U}}}_i^{h_i}\right\|_{\mathrm{L}^\infty(I)} + \left\|\bar{\mathcal{U}}_i^{h_i}\right\|_{\mathrm{L}^\infty(I)} \right) \le c_T \sqrt{B_T} \tag{3.9}$$

$$\sup_{(h_1,h_2)\in S_h} \sum_{i=1,2} \left\|\bar{\lambda}_i^{h_i}\right\|_{\mathrm{L}^\infty(I)} \le c_T \sqrt{B_T}. \tag{3.10}$$

In particular:

$$\sup_{(h_1,h_2)\in S_h} \sum_{i=1,2} \left(\left\|\bar{\ddot{\mathcal{U}}}_i^{h_i}\right\|_{L^2(I)} + \left\|\bar{\dot{\mathcal{U}}}_i^{h_i}\right\|_{L^2(I)} + \left\|\bar{\mathcal{U}}_i^{h_i}\right\|_{L^2(I)} \right) \le c_T \sqrt{B_T} \tag{3.11}$$

$$\sup_{(h_1,h_2)\in S_h} \sum_{i=1,2} \left\|\bar{\lambda}_i^{h_i}\right\|_{L^2(I)} \le c_T \sqrt{B_T} \tag{3.12}$$

where $B_T = \sum_{i=1}^2 [\|\mathcal{U}_i^0\|^2 + \|\dot{\mathcal{U}}_i^0\|^2 + \|\ddot{\mathcal{U}}_i^0\|^2 + \|\lambda_i^0\|^2 + \|F_i\|^2_{W^{1,2}(I)}]$ *and the coefficient* $c_T > 0$ *only depends on the constants* γ_i*,* $\beta_i, h_{10}, h_{20}, m, T$ *and the matrices* K_i*,* C_i*,* M_i *and* L_i $(i \in \{1, 2\})$*. The same results hold also for* $\tilde{\ddot{\mathcal{U}}}_i^{h_i}$*,* $\tilde{\dot{\mathcal{U}}}_i^{h_i}$*,* $\tilde{\mathcal{U}}_i^{h_i}$ *and* $\tilde{\lambda}_i^{h_i}$*.*

Proof of Lemma 3.5. First let us fix h_{10} and h_{20} to be sufficiently small such that the triples $(\gamma_1, \beta_1, h_{10})$ and $(\gamma_2, \beta_2, h_{20})$ satisfy the assumption (**H2**) and we choose arbitrary $(h_1, h_2) \in S_h$. Moreover, we keep the definitions of n_1 and n_2 as in the definition of S_h (3.8). When no confusion should arise and in order to avoid useless repetitions, we shall denote $c_T > 0$ as an arbitrary constant, independent of h_1, h_2, and only depends on the constants γ_i, $\beta_i, h_{10}, h_{20}, m, T$ and the matrices K_i, C_i, M_i and L_i $(i \in \{1, 2\})$.

First, assume that the system (2.12) has a solution for every $1 \le k \le n_2$ and for every h_2 such that $(h_1, h_2) \in S_h$. For every $i \in \{1, 2\}$, let $0 < l_i \le n_i$ be an integer and choose a constant $\epsilon > 0$. By virtue of Young's, Jensen's and Cauchy's inequalities, the terms $W_i^{\mathrm{ext},l}$ are bounded by:

$$\begin{aligned} W_i^{\mathrm{ext},l} &= \frac{1}{h_l} \sum_{j=1}^{l_i} \left\langle F_i^j - F_i^{j-1}, \dot{\mathcal{U}}_i^j - \dot{\mathcal{U}}_i^{j-1} \right\rangle \\ &\le \frac{1}{h_1} \sum_{j=1}^{l_i} \frac{1}{\epsilon} \left\| F_i^j - F_i^{j-1} \right\|^2 + \frac{1}{h_1} \sum_{j=1}^{l_i} \epsilon \left\| \dot{\mathcal{U}}_i^j - \dot{\mathcal{U}}_i^{j-1} \right\|^2 \end{aligned}$$

$$\le \frac{1}{h_1}\sum_{j=1}^{l_i}\frac{1}{\epsilon}h_1\int_{t_{j-1}^{(i)}}^{t_j^{(i)}}\left(\dot{F}_i\right)^2\mathrm{d}t+\frac{\epsilon}{h_1}\sum_{j=1}^{l_i}\left\|\dot{\mathcal{U}}_i^j-\dot{\mathcal{U}}_i^{j-1}\right\|^2$$
$$=\frac{1}{\epsilon}\int_I\left(\dot{F}_i\right)^2\mathrm{d}t+\frac{\epsilon}{h_1}\sum_{j=1}^{l_i}\left\|\dot{\mathcal{U}}_i^j-\dot{\mathcal{U}}_i^{j-1}\right\|^2. \tag{3.13}$$

In the view of equation (3.4), the previous equation (3.13) and the definitions in equation (3.5), if $\epsilon > 0$ is strictly lower that the smallest eigenvalues of both C_1 and C_2 (these eigenvalues are strictly positive by assumption (**H1**)), we infer, for every $1 \le l \le n_2$, that:

$$\frac{1}{2}\left[\left\langle\ddot{\mathcal{U}}_1^{lm},\tilde{M}_1\ddot{\mathcal{U}}_1^{lm}\right\rangle+\left\langle\dot{\mathcal{U}}_1^{lm},K_1\dot{\mathcal{U}}_1^{lm}\right\rangle\right]+\frac{1}{h_1}\sum_{j=1}^{lm}\left\|\dot{\mathcal{U}}_1^j-\dot{\mathcal{U}}_1^{j-1}\right\|^2\le c_TB_T \tag{3.14}$$

and

$$\frac{1}{2}\left[\left\langle\ddot{\mathcal{U}}_2^l,\tilde{M}_2\ddot{\mathcal{U}}_2^l\right\rangle+\left\langle\dot{\mathcal{U}}_2^l,K_2\dot{\mathcal{U}}_2^l\right\rangle\right]+\frac{1}{h_2}\sum_{j=1}^{l}\left\|\dot{\mathcal{U}}_2^j-\dot{\mathcal{U}}_2^{j-1}\right\|^2\le c_TB_T \tag{3.15}$$

which easily yields $\sup_{(h_1,h_2)\in S_h}\|\bar{\ddot{\mathcal{U}}}_2^{h_2}\|_{\mathrm{L}^\infty(I)} \le c_T\sqrt{B_T}$. Using the Newmark approximation relations (2.4), we have again $\sup_{(h_1,h_2)\in S_h}\|\bar{\dot{\mathcal{U}}}_2^{h_2}\|_{\mathrm{L}^\infty(I)} \le c_T\sqrt{B_T}$. Indeed, using equation (2.4), it is seen that the velocities $(\dot{\mathcal{U}}_2^k)_{1\le k\le n_2}$ are bounded by:

$$\left\|\dot{\mathcal{U}}_2^k\right\|=\left\|\dot{\mathcal{U}}_2^0+\sum_{l=1}^{k}\Big(h_2\left(1-\gamma_2\right)\ddot{\mathcal{U}}_2^{l-1}+h_2\gamma_2\ddot{\mathcal{U}}_2^l\Big)\right\|\quad\forall\,1\le k\le n_2$$
$$\le\left\|\dot{\mathcal{U}}_2^0\right\|+kh_2\max_{0\le l\le n_2}\left\|\ddot{\mathcal{U}}_2^l\right\|\le\left\|\dot{\mathcal{U}}_2^0\right\|+T\max_{0\le l\le n_2}\left\|\ddot{\mathcal{U}}_2^l\right\|\le c_T\sqrt{B_T}. \tag{3.16}$$

By analogous arguments, the same result may be obtained for the displacement vector $\mathcal{U}_2$.

Now, it remains to prove the boundedness of the quantities at the micro time steps, associated with the subdomain Ω_1, as well as the Lagrange multipliers. This can be achieved in several steps:

Step 1. We claim $\sup_{(h_1,h_2)\in S_h}\|\bar{\dot{\mathcal{U}}}_1^{h_1}\|_{\mathrm{L}^\infty(I)} \le c_T\sqrt{B_T}$.

Indeed, for every $1 \le j \le n_1$, combining equation (3.14) and the Cauchy–Schwarz inequality gives:

$$\sup_{(h_1,h_2)\in S_h}\max_j\left\|\dot{\mathcal{U}}_1^j\right\|=\sup_{(h_1,h_2)\in S_h}\max_j\left\|\dot{\mathcal{U}}_1^0+\sum_{l=1}^{j}\dot{\mathcal{U}}_1^l-\dot{\mathcal{U}}_1^{l-1}\right\|$$
$$\le\sup_{(h_1,h_2)\in S_h}\max_j\left\|\dot{\mathcal{U}}_1^0\right\|+\sqrt{\frac{T}{h_1}\sum_{l=1}^{j}\left\|\dot{\mathcal{U}}_1^j-\dot{\mathcal{U}}_1^{j-1}\right\|^2}\le c_T\sqrt{B_T}. \tag{3.17}$$

Step 2. $\left(\left\|\mathbb{U}_1^{lm}\right\|\right)_l$ is uniformly bounded, *i.e.*, the displacement, the velocity and the acceleration vectors of the subdomain Ω_1, computed at the macro time steps, are uniformly bounded.

Using assumption (**H2**) and equation (3.14), it is clear that the acceleration components computed at the macro time steps are uniformly bounded:

$$\sup_{(h_1,h_2)\in S_h}\max_{0\le l\le n_2}\left\|\ddot{\mathcal{U}}_1^{lm}\right\|\le c_T\sqrt{B_T}. \tag{3.18}$$

To prove the same result for the displacement vector, pick up an integer $0 \leq l \leq n_2$. By virtue of equation (2.2), we have:

$$\begin{aligned}
\mathcal{U}_1^{lm} &= \mathcal{U}_1^0 + h_1 \sum_{j=0}^{lm-1} \dot{\mathcal{U}}_1^j + h_1^2 \sum_{j=1}^{lm} \left[\left(\frac{1}{2} - \beta_1 \right) \ddot{\mathcal{U}}_1^{j-1} + \beta_1 \ddot{\mathcal{U}}_1^j \right] \\
&= \mathcal{U}_1^0 + h_1 \sum_{j=0}^{lm-1} \dot{\mathcal{U}}_1^j + \frac{h_1^2}{2} \sum_{j=1}^{lm} \left[(1-\gamma_1) \ddot{\mathcal{U}}_1^{j-1} + \gamma_1 \ddot{\mathcal{U}}_1^j \right] + \beta_1 h_1^2 \left(\ddot{\mathcal{U}}_1^{lm} - \ddot{\mathcal{U}}_1^0 \right) - \frac{1}{2} \gamma_1 h_1^2 \left(\ddot{\mathcal{U}}_1^{lm} - \ddot{\mathcal{U}}_1^0 \right) \\
&= \mathcal{U}_1^0 + h_1 \sum_{j=0}^{lm-1} \dot{\mathcal{U}}_1^j + \frac{h_1}{2} \sum_{j=1}^{lm} \left[\dot{\mathcal{U}}_1^j - \dot{\mathcal{U}}_1^{j-1} \right] + \beta_1 h_1^2 \left(\ddot{\mathcal{U}}_1^{lm} - \ddot{\mathcal{U}}_1^0 \right) - \frac{1}{2} \gamma_1 h_1^2 \left(\ddot{\mathcal{U}}_1^{lm} - \ddot{\mathcal{U}}_1^0 \right). \qquad (3.19)
\end{aligned}$$

Thus, combining step 1 and equation (3.18), we end up with:

$$\begin{aligned}
\sup_{(h_1,h_2)\in S_h} \max_{1\leq l\leq n_2} \left\| \mathcal{U}_1^{lm} \right\| &\leq \left\| \mathcal{U}_1^0 \right\| + \frac{3T}{2} \sup_{(h_1,h_2)\in S_h} \left\| \bar{\dot{\mathcal{U}}}_1^{h_1} \right\|_{\mathrm{L}^\infty(I)} + 2\left(\beta_1 + \frac{\gamma_1}{2} \right) \sup_{(h_1,h_2)\in S_h} \max_{0\leq l\leq n_2} h_1^2 \left\| \ddot{\mathcal{U}}_1^{lm} \right\| \\
&\leq c_T \sqrt{B_T}. \qquad (3.20)
\end{aligned}$$

As a consequence, we conclude:

$$\sup_{(h_1,h_2)\in S_h} \max_{0\leq l\leq n_2} \left\| \mathbb{U}_1^{lm} \right\| \leq c_T \sqrt{B_T}. \qquad (3.21)$$

Step 3. Boundedness of the Lagrange multipliers:

At each macro time step kh_2, we have:

$$\begin{aligned}
M_1 \ddot{\mathcal{U}}_1^{km} + C_1 \dot{\mathcal{U}}_1^{km} + K_1 \mathcal{U}_1^{km} + L_1^T \lambda_2^k &= F_1^{km} \\
M_2 \ddot{\mathcal{U}}_2^k + C_2 \dot{\mathcal{U}}_2^k + K_2 \mathcal{U}_2^k + L_2^T \lambda_2^k &= F_2^k \qquad (3.22)
\end{aligned}$$

where the equality $\lambda_1^{km} = \lambda_2^k$ has been used (see Eq. (2.5)). Since $\left(L_1 \ L_2 \right)$ is surjective and $F_i \in \mathrm{W}^{1,2}\left(I, \mathbb{R}^{N_i}\right)$ is bounded on I, we have for every $1 \leq k \leq n_2$:

$$\begin{aligned}
\left\| \lambda_2^k \right\| &\leq \left\| \left(L_2 L_2^T + L_1 L_1^T \right)^{-1} \left(L_2 M_2 \ddot{\mathcal{U}}_2^k + L_2 C_2 \dot{\mathcal{U}}_2^k + L_2 K_2 \mathcal{U}_2^k - L_2 F_2^k \right) \right\| \\
&\quad + \left\| \left(L_2 L_2^T + L_1 L_1^T \right)^{-1} \left(L_1 M_1 \ddot{\mathcal{U}}_1^{km} + L_1 C_1 \dot{\mathcal{U}}_1^{km} + L_1 K_1 \mathcal{U}_1^{km} - L_1 F_1^{km} \right) \right\| \\
&\lesssim \max_{0\leq k\leq n_2} \left(\left\| \ddot{\mathcal{U}}_2^k \right\| + \left\| \dot{\mathcal{U}}_2^k \right\| + \left\| \mathcal{U}_2^k \right\| + \left\| F_2^k \right\| \right) + \max_{0\leq k\leq n_2} \left(\left\| \ddot{\mathcal{U}}_1^{km} \right\| + \left\| \dot{\mathcal{U}}_1^{km} \right\| + \left\| \mathcal{U}_1^{km} \right\| + \left\| F_1^{km} \right\| \right) \\
&\leq c_T \sqrt{B_T} \qquad (3.23)
\end{aligned}$$

where the uniform boundedness of the quantities at the macro time steps has been used as well as the continuous embedding $\mathrm{W}^{1,2}(I, \mathbb{R}^{N_i}) \subset \mathrm{L}^\infty(I, \mathbb{R}^{N_i})$ for $i = 1, 2$. Hence, $\sup_{(h_1,h_2)\in S_h} \|\bar{\lambda}_2^{h_2}\|_{\mathrm{L}^\infty(I)} \leq c_T \sqrt{B_T}$. By the interpolation relation (2.7), it is clear that $(\lambda_1^i)_i$ is uniformly bounded, *i.e.*, $\sup_{(h_1,h_2)\in S_h} \|\bar{\lambda}_1^{h_1}\|_{\mathrm{L}^\infty(I)} \leq c_T \sqrt{B_T}$.

Step 4. The kinematic quantities related to Ω_1, computed at the micro time steps are uniformly bounded.

In the view of the explicit expression of $\mathbb{A}^{-1}$ in Lemma 3.4 and equation (2.10), the expression of the quantities at the fine time step related to the subdomain Ω_1 can be written, for every $(k-1)m+1 \leq i \leq km$, as:

$$\mathbb{U}_1^i = \sum_{l=(k-1)m+1}^{i} (-1)^{i-l} \left(\mathbb{M}_1^{-1} \mathbb{N}_1 \right)^{i-l} \mathbb{M}_1^{-1} \left(\mathbb{P}_1^l - \mathbb{L}_1^T \lambda_1^l \right) + (-1)^{i-(k-1)m} \left(\mathbb{M}_1^{-1} \mathbb{N}_1 \right)^{i-(k-1)m} \mathbb{U}_1^{(k-1)m}. \qquad (3.24)$$

Recall that $\lambda_1^{(k-1)m+l} = \frac{l}{m}\lambda_2^k + \frac{m-l}{m}\lambda_2^{(k-1)}$ for $1 \leq l \leq m$. Moreover, it is not difficult to prove that $\mathbb{M}_1^{-1}$ and $\mathbb{N}_1$ are uniformly bounded when the step h_1 varies in the compact $[0, h_{10}]$. Thus, combining steps 1–3 and again the continuous embedding $\mathrm{W}^{1,2}\big(I, \mathbb{R}^{N_1}\big) \subset \mathrm{L}^\infty\big(I, \mathbb{R}^{N_1}\big)$, one may conclude that:

$$\sup_{(h_1,h_2)\in S_h} \max_{0\leq j\leq n_1} \left\|\mathbb{U}_1^j\right\| \leq c_T\sqrt{B_T}. \tag{3.25}$$

The uniform boundedness related to the L^2-norm is simply a consequence of the continuous embedding $\mathrm{L}^\infty\big(I, \mathbb{R}^l\big) \subset L^2\big(I, \mathbb{R}^l\big)$ for every integer l.

Now it remains to prove the existence and the uniqueness of the solution. For this purpose, it is sufficient to show that the null-space of the matrix $\begin{bmatrix} \mathbb{A} & \mathbb{L}^T \\ \mathbb{B} & 0 \end{bmatrix}$ in equation (2.12) is zero. Indeed, taking into consideration the preceding boundedness results, one may infer that the null-space of $\mathbb{A}$ is necessary bounded, hence it is reduced to the single element $\{0\}$. This completes the proof. □

Remark 3.6. If $V_d \neq 0$, we can still obtain the uniform boundedness of the numerical solutions, without transforming the problem as described in the paragraph before Lemma 3.5, but with the additional assumption: $\ddot{V}_d \in BV\big(\bar{I}, \mathbb{R}^{N_\lambda}\big)$ (see Appendix B).

4. Convergence results

This section is focused on the main results of the paper, namely Theorems 4.1, 4.3 and 4.4, which discuss the uniform convergence of the approximated numerical solutions to the solution of the main problem in equation (1.1) as well as the error estimates. For this purpose, we assume, throughout this section, that the hypotheses of Lemma 3.5 are satisfied with $V_d = 0$. Moreover, we choose a sequence $(h_{1,l}, h_{2,l})_{l\geq 0} \in S_h^{\mathbb{N}}$ converging to 0 (the set S_h is the same defined in Lem. 3.5 by (3.8)) where the time step ratio m being constant, and we define, for $i = 1, 2$, $n_{i,l} = \frac{T}{h_{i,l}} \in \mathbb{N}^*$. Moreover, for sake of simplification, the discrete times are still denoted by $t_p^{(i)} = ph_{i,l}$ for every $i = 1, 2$ (see the first paragraph of Sect. 2). According to Lemma 3.5 and by a compactness argument, we may assume the following weak convergences in $\mathrm{L}^2\big(I, \mathbb{R}^{N_i}\big)$ and $\mathrm{L}^2\big(I, \mathbb{R}^{N_\lambda}\big)$ for every $i \in \{1, 2\}$:

$$\begin{aligned} &\bar{\ddot{\mathcal{U}}}_i^{h_{i,l}} \overset{w}{\rightharpoonup} A_i \quad \tilde{\dot{\mathcal{U}}}_i^{h_{i,l}} \overset{w}{\rightharpoonup} V_i \\ &\tilde{\mathcal{U}}_i^{h_{i,l}} \overset{w}{\rightharpoonup} U_i \quad \bar{\lambda}_i^{h_{i,l}} \overset{w}{\rightharpoonup} \lambda_i. \end{aligned} \tag{4.1}$$

Theorem 4.1. *Define $\lambda = \lambda_2$. The triple (U_1, U_2, λ) is uniquely represented by a triple in $\mathbf{S} = C^1\big(\bar{I}, \mathbb{R}^{N_1}\big) \times C^1\big(\bar{I}, \mathbb{R}^{N_2}\big) \times \mathrm{L}^2\big(I, \mathbb{R}^{N_\lambda}\big) \cap W^{2,2}\big(I, \mathbb{R}^{N_1}\big) \times W^{2,2}\big(I, \mathbb{R}^{N_2}\big) \times \mathrm{L}^2\big(I, \mathbb{R}^{N_\lambda}\big)$ which is the unique solution of the problem (1.1) in $\mathbf{S}$. Moreover, we have:*

(1) *$\lambda_1 = \lambda_2$ a.e.*
(2) *$\dot{U}_i = V_i$ and $\dot{V}_i = A_i$ a.e.*
(3) *The following uniform convergences hold for $i = 1, 2$:*

$$\begin{aligned} &\lim_{h_{i,l}\to 0} \left\|\tilde{\mathcal{U}}_i^{h_{i,l}} - U_i\right\|_{\mathrm{L}^\infty(\bar{I})} = 0 \\ &\lim_{h_{i,l}\to 0} \left\|\tilde{\dot{\mathcal{U}}}_i^{h_{i,l}} - \dot{U}_i\right\|_{\mathrm{L}^\infty(\bar{I})} = 0 \end{aligned} \tag{4.2}$$

with $U_i(0) = U_{0i}$ and $\dot{U}_i(0) = V_{0i}$. Moreover, the convergences in (4.2) and (4.1) hold for every sequence $(h_{1,l}, h_{2,l})_{l\geq 0} \in S_h^{\mathbb{N}}$ converging to 0.

Remark 4.2. Henceforth, we assume, without loss of generality, that $(U_1, U_2) \in C^1(\bar{I}, \mathbb{R}^{N_1}) \times C^1(\bar{I}, \mathbb{R}^{N_2})$. Moreover, to be consistent with the notations in equation (1.1), the solution (U_1, U_2, λ) in Theorem 4.1 is henceforth denoted $(\mathcal{U}_1, \mathcal{U}_2, \lambda)$.

Theorem 4.1 only shows that the convergence of the acceleration and the Lagrange multipliers are in the weak sense. However, one may notice that the L^∞ strong convergence of either the acceleration field or the Lagrange multipliers necessary imply the strong convergence of the second quantity with the same norm. Given the assumption of Theorem 4.1, we can still obtain a strong convergence using an alternative definition of the approximate solution, piecewise constant on each macro time interval of length $h_{2,l}$ as follows: If X_1 and X_2 stand for one of the following fields: displacement, velocity, acceleration and the Lagrange multipliers, related to the subdomains Ω_1 and Ω_2, respectively, we define the sequence of the piecewise constant functions $(\hat{X}_1^{h_{2,l}})_l$ and $(\hat{X}_2^{h_{2,l}})_l$ by:

$$\begin{cases} \hat{X}_1^{h_{2,l}}(t) = \sum_{k=1}^{n_{2,l}} \hat{X}_1^k \chi_{\left[t_{k-1}^{(2)}, t_k^{(2)}\right)}(t) \\ \hat{X}_2^{h_{2,l}}(t) = \sum_{k=1}^{n_{2,l}} \hat{X}_2^k \chi_{\left[t_{k-1}^{(2)}, t_k^{(2)}\right)}(t) \end{cases} \tag{4.3}$$

where, for every $1 \le k \le n_{2,l}$, the averages $\hat{X}_1^k$ and $\hat{X}_2^k$ are defined by:

$$\begin{cases} \hat{X}_1^k = \frac{1}{m}\left[\gamma_1 X_1^{km} + (1-\gamma_1) X_1^{(k-1)m} + \sum_{l=1}^{m-1} X_1^{(k-1)m+l}\right] \\ \hat{X}_2^k = \gamma_2 X_2^k + (1-\gamma_2) X_2^{(k-1)}. \end{cases} \tag{4.4}$$

It is clear from the definition (4.3), that $\hat{X}_2^{h_{2,l}} = \bar{X}_2^{h_{2,l}, r=\gamma_2}$ (see the definitions in Eq. (3.1)). The following Theorem states a strong convergence of the acceleration and the Lagrange multipliers vectors,

Theorem 4.3. *If $\gamma_1 = m\gamma_2 - \frac{m-1}{2}$ then:*

$$\begin{aligned} \lim_{h_{2,l}\to 0} \left\| \hat{\lambda}_i^{h_{2,l}} - \lambda \right\|_{L^\infty(I)} &= 0 \\ \lim_{h_{2,l}\to 0} \left\| \hat{\ddot{\mathcal{U}}}_i^{h_{2,l}} - \ddot{\mathcal{U}}_i \right\|_{L^\infty(I)} &= 0. \end{aligned} \tag{4.5}$$

For every sequence $(h_{1,l}, h_{2,l})_{l\ge 0} \in S_h^{\mathbb{N}}$ converging to 0. In particular the previous result holds for every $m \ge 1$ when $\gamma_1 = \gamma_2 = \frac{1}{2}$.

Note that the particular choice $\gamma_1 = \gamma_2 = \frac{1}{2}$ is usually used in practice since it does not introduce artificial damping to the solution. We end this part by the following theorem which establishes some error estimates of the PH method with respect to the macro time step.

Theorem 4.4. *Let $(h_{1,l}, h_{2,l}) \in S_h$. If $\gamma_1 = m\gamma_2 - \frac{m-1}{2}$ and $\dot{F}_i \in L^\infty\left(I, \mathbb{R}^{N_i}\right)$ for $i = 1, 2$, then:*

$$\begin{aligned} \left\| \tilde{\mathcal{U}}_i^{h_{i,l}} - \mathcal{U}_i \right\|_{L^\infty(\bar{I})} &\le c_T h_{2,l} \\ \left\| \tilde{\dot{\mathcal{U}}}_i^{h_{i,l}} - \dot{\mathcal{U}}_i \right\|_{L^\infty(\bar{I})} &\le c_T h_{2,l} \\ \left\| \hat{\ddot{\mathcal{U}}}_i^{h_{2,l}} - \ddot{\mathcal{U}}_i \right\|_{L^\infty(I)} &\le c_T h_{2,l} \\ \left\| \hat{\lambda}_i^{h_{2,l}} - \lambda \right\|_{L^\infty(I)} &\le c_T h_{2,l}. \end{aligned} \tag{4.6}$$

For every $i \in \{1, 2\}$, where c_T is a constant independent of $h_{2,l}$.

Remark 4.5. It is interesting to point out that numerical investigations of the convergence of PH method have been conducted on split-oscillators (without damping) in [12], which show that the order of convergence is nearly preserved even with time step ratios close to 20.

Now, we shall give seperatly the proofs of the previous results.

4.1. Proof of Theorem 4.1

The proof is divided into several steps:

Claim 1. We claim:

$$\dot{\bar{\dot{\mathcal{U}}}}_i^{h_{i,l}} \overset{w}{\rightharpoonup} A_i \quad \dot{\bar{\mathcal{U}}}_i^{h_{i,l}} \overset{w}{\rightharpoonup} V_i \quad \text{in } \mathrm{L}^2. \tag{4.7}$$

In particular, we have $\dot{U}_i = V_i$ a.e. and $\dot{V}_i = A_i$ a.e.

Proof of Claim 1. We restrict ourselves with the case $\dot{\bar{\dot{\mathcal{U}}}}_i^{h_{i,l}} \overset{w}{\rightharpoonup} A_i$ in L^2, the remaining case may be proved by analogous arguments.

Let $i \in \{1,2\}$ and $\phi \in \mathrm{C}^\infty(\bar{I}, \mathbb{R}^{N_i})$. Using the definitions of the approximate solutions introduced in equation (3.1) and the Newmark relations equations (2.2) and (2.4), we have:

$$\begin{aligned}
\int_I \left\langle \dot{\bar{\dot{\mathcal{U}}}}_i^{h_{i,l}}, \phi \right\rangle \mathrm{d}t &= \sum_{k=0}^{n_{i,l}-1} \int_{kh_{i,l}}^{(k+1)h_{i,l}} \left\langle \left(\gamma_i \ddot{\mathcal{U}}_i^{k+1} + (1-\gamma_i)\ddot{\mathcal{U}}_i^k \right), \phi \right\rangle \mathrm{d}t \\
&= (1-\gamma_i) \int_0^{h_{i,l}} \left\langle \ddot{\mathcal{U}}_i^0, \phi \right\rangle + \gamma_i \int_{h_{i,l}(n_{i,l}-1)}^{n_{i,l}h_{i,l}} \left\langle \ddot{\mathcal{U}}_i^{n_{i,l}}, \phi \right\rangle \\
&\quad + \int_0^{(n_{i,l}-1)h_{i,l}} \left\langle \bar{\ddot{\mathcal{U}}}_i^{h_{i,l}}, \gamma_i \phi + (1-\gamma_i)\phi(.+h_{i,l}) \right\rangle \mathrm{d}t.
\end{aligned} \tag{4.8}$$

Using the Boundedness result in Lemma 3.5 and the continuity of the function ϕ, it is not difficult to see:

$$\lim_l \int_I \left\langle \dot{\bar{\dot{\mathcal{U}}}}_i^{h_{i,l}}, \phi \right\rangle \mathrm{d}t = \int_I \langle A_i, \phi \rangle \, \mathrm{d}t. \tag{4.9}$$

According to Lemma 3.5, we have $(\dot{\bar{\dot{\mathcal{U}}}}_i^{h_{i,l}})_l$ is uniformly bounded in L^2 norm. By density of $\mathrm{C}^\infty(\bar{I}, \mathbb{R}^{N_i})$ in $\mathrm{L}^2(I, \mathbb{R}^{N_i})$, we deduce $\dot{\bar{\dot{\mathcal{U}}}}_i^{h_{i,l}} \overset{w}{\rightharpoonup} A_i$ in L^2. $\square$

Claim 2. We claim:

$$\lambda_1 = \lambda_2 \quad a.e. \tag{4.10}$$

Proof of Claim 2. The arguments are similar to those of Claim 1. Indeed, let $\phi \in \mathrm{C}^\infty(\bar{I}, \mathbb{R}^{N_\lambda})$ and define:

$$\bar{\phi}^{h_{2,l}}(t) = \sum_{k=1}^{n_{2,l}} \phi(kh_{2,l})\, \chi_{((k-1)h_{2,l}, kh_{2,l}]}(t) \tag{4.11}$$

using the interpolation relation in equation (2.7), we obtain:

$$\begin{aligned}
\int_I \left\langle \bar{\lambda}_1^{h_{1,l}}, \bar{\phi}^{h_{2,l}} \right\rangle \mathrm{d}t &= \sum_{k=0}^{n_{2,l}-1} \phi((k+1)h_{2,l}) \left(\frac{m-1}{2}\lambda_2^k + \frac{m+1}{2}\lambda_2^{k+1} \right) h_{1,l} \\
&= \frac{m-1}{2}\phi(h_{2,l})\lambda_2^0 h_{1,l} + \frac{m+1}{2}\phi(T)\lambda_2^{n_{2,l}} h_{1,l} \\
&\quad + \int_0^{T-h_{2,l}} \left\langle \frac{m+1}{2m}\bar{\phi}^{h_{2,l}} + \frac{m-1}{2m}\bar{\phi}^{h_{2,l}}(\cdot + h_{2,l}), \bar{\lambda}_2^{h_{2,l}} \right\rangle.
\end{aligned} \tag{4.12}$$

By the continuity of ϕ and the uniform boundedness of $(\bar{\lambda}_1^{h_{1,l}})_l$ and $(\bar{\lambda}_2^{h_{2,l}})_l$, it is not difficult to observe that:

$$\int_I \langle \lambda_1, \phi \rangle \, \mathrm{d}t = \lim_l \int_I \left\langle \bar{\lambda}_1^{h_{1,l}}, \bar{\phi}^{h_{2,l}} \right\rangle \mathrm{d}t = \int_I \langle \lambda_2, \phi \rangle \, \mathrm{d}t. \tag{4.13}$$

Hence, $\lambda_1 = \lambda_2$ $(a.e.)$. $\square$

Claim 3. For every $i \in \{1,2\}$, U_i and $\dot{U}_i$ may be selected to be continuous over $\bar{I}$ with $U_i(0) = U_{0i}$ and $\dot{U}_i(0) = V_{0i}$ and up to a subsequence, we claim:

$$\lim_{h_{i,l}\to 0}\left\|\tilde{\mathcal{U}}_i^{h_{i,l}} - U_i\right\|_{\mathrm{L}^\infty(\bar{I})} = 0 \quad \text{and} \quad \lim_{h_{i,l}\to 0}\left\|\dot{\tilde{\mathcal{U}}}_i^{h_{i,l}} - \dot{U}_i\right\|_{\mathrm{L}^\infty(\bar{I})} = 0. \tag{4.14}$$

Furthermore, we have:

$$\lim_{h_{i,l}\to 0}\left\|\bar{\mathcal{U}}_i^{h_{i,l}} - U_i\right\|_{\mathrm{L}^\infty(I)} = 0 \quad \text{and} \quad \lim_{h_{i,l}\to 0}\left\|\dot{\bar{\mathcal{U}}}_i^{h_{i,l}} - \dot{U}_i\right\|_{\mathrm{L}^\infty(I)} = 0. \tag{4.15}$$

Henceforth, U_i and $\dot{U}_i$ may be assumed to be continuous on $\bar{I}$.

Proof of Claim 3. We choose $i = 1$ or 2 and we restrict ourselves with the first convergences in equations (4.14) and (4.15) since the same argument may be applied to the second case. By Claim 1, it is clear that $\tilde{\mathcal{U}}_i^{h_{i,l}} \overset{w}{\rightharpoonup} U_i$ in $\mathrm{W}^{1,2}(I,\mathbb{R}^{N_i})$. The first equation in (4.14) is then an immediate consequence of the compact embedding $\mathrm{W}^{1,2}\left(I,\mathbb{R}^{N_i}\right) \hookrightarrow \mathrm{C}^0\left(\bar{I},\mathbb{R}^{N_i}\right)$ [30] and noticing the following continuity at 0: $\lim_{t\to 0^+}\tilde{\mathcal{U}}_i^{h_{i,l}}(t) = \tilde{\mathcal{U}}_i^{h_{i,l}}(0) = U_{0i}$ and $\lim_{t\to 0^+}\dot{\tilde{\mathcal{U}}}_i^{h_{i,l}}(t) = \dot{\tilde{\mathcal{U}}}_i^{h_{i,l}}(0) = V_{0i}$.

The first equation in (4.15) is immediately obtained from the following triangle inequality:

$$\begin{aligned}\left\|\bar{\mathcal{U}}_i^{h_{i,l}} - U_i\right\|_{\mathrm{L}^\infty(I)} &\le \left\|\bar{\mathcal{U}}_i^{h_{i,l}} - \tilde{\mathcal{U}}_i^{h_{i,l}}\right\|_{\mathrm{L}^\infty(I)} + \left\|\tilde{\mathcal{U}}_i^{h_{i,l}} - U_i\right\|_{\mathrm{L}^\infty(\bar{I})} \\ &\le Ch_{i,l} + \left\|\tilde{\mathcal{U}}_i^{h_{i,l}} - U_i\right\|_{\mathrm{L}^\infty(\bar{I})}\end{aligned} \tag{4.16}$$

where C is independent of $h_{i,l}$. In equation (4.16), the inequality $\|\bar{\mathcal{U}}_i^{h_{i,l}} - \tilde{\mathcal{U}}_i^{h_{i,l}}\|_{\mathrm{L}^\infty(I)} \le Ch_{i,l}$ is justified by Lemma 3.5, equations (2.2) and (2.4). □

Next, we prove that the triple (U_1, U_2, λ) is the solution of the problem and the constraint relation in (1.1) is satisfied by $(\dot{U}_1, \dot{U}_2)$.

As mentioned before, the constraint relation in equation (1.1) is only enforced at the coarse time step *via* equation (2.8). Taking into account the compatibility condition (1.2), it follows:

$$L_1\dot{\tilde{\mathcal{U}}}_1^{h_{2,l}} + L_2\dot{\tilde{\mathcal{U}}}_2^{h_{2,l}} = 0 \tag{4.17}$$

or

$$L_1\dot{\bar{\mathcal{U}}}_1^{h_{2,l}} + L_2\dot{\bar{\mathcal{U}}}_2^{h_{2,l}} = 0 \tag{4.18}$$

By Claim 3, and using the same subsequence, we also have:

$$\lim_{h_{2,l}\to 0}\left\|\dot{\bar{\mathcal{U}}}_1^{h_{2,l}} - \dot{U}_1\right\|_{\mathrm{L}^\infty(I)} = 0. \tag{4.19}$$

Using Claims 1 3, equations (4.19) and (4.1), we infer (for the same subsequence in Eq. (4.19)):

$$\begin{array}{ll}\ddot{\bar{\mathcal{U}}}_i^{h_{i,l}} \overset{w}{\rightharpoonup} \ddot{U}_i & \dot{\bar{\mathcal{U}}}_i^{h_{i,l}} \overset{w}{\rightharpoonup} \dot{U}_i \\ \bar{\mathcal{U}}_i^{h_{i,l}} \overset{w}{\rightharpoonup} U_i & \bar{\lambda}_i^{h_{i,l}} \overset{w}{\rightharpoonup} \lambda\end{array} \tag{4.20}$$

where the weak convergences are in the L^2 sense. Moreover, one may notice that:

$$\begin{cases} M_1\ddot{\bar{\mathcal{U}}}_1^{h_{1,l}} + C_1\dot{\bar{\mathcal{U}}}_1^{h_{1,l}} + K_1\bar{\mathcal{U}}_1^{h_{1,l}}(t) + L_1^T\bar{\lambda}_1^{h_{1,l}} = \bar{F}_1^{h_{1,l}} \\ M_2\ddot{\bar{\mathcal{U}}}_2^{h_{2,l}} + C_2\dot{\bar{\mathcal{U}}}_2^{h_{2,l}} + K_2\bar{\mathcal{U}}_2^{h_{2,l}} + L_2^T\bar{\lambda}_2^{h_{2,l}} = \bar{F}_1^{h_{2,l}} \\ L_1\dot{\bar{\mathcal{U}}}_1^{h_{2,l}} + L_2\dot{\bar{\mathcal{U}}}_2^{h_{2,l}} = 0 \end{cases} \tag{4.21}$$

where the discrete terms related to the external forces $\bar{F}_i^{h_{i,l}}$ are defined by:

$$\bar{F}_i^{h_{i,l}}(t) = \sum_{k=0}^{n_{i,l}-1} F_i^{k+1}\ \chi_{[t_k,t_{k+1})}(t). \tag{4.22}$$

First, using the linearity of the problem and equation (4.20), it is seen that the terms $M_i\bar{\ddot{\mathcal{U}}}_i^{h_{i,l}}$, $C_i\bar{\dot{\mathcal{U}}}_i^{h_{i,l}}$, $K_i\bar{\mathcal{U}}_i^{h_{i,l}}$ and $L_i^T\bar{\lambda}_i^{h_{i,l}}$ converge (weakly) to $M_i\ddot{U}_i$, $C_i\dot{U}_i$, K_iU_i and $L_i^T\lambda$, respectively. The uniform convergence in equation (4.19) shows that the constraint relation in equation (1.1) is also satisfied, namely, $L_1\dot{U}_1+L_2\dot{U}_2=0$. The convergence of the terms related to the external forces is a direct consequence of the regularity assumed in (**HF**).

As a consequence, it follows that the triple $(U_1,U_2,\lambda)\in C^1(\bar{I},\mathbb{R}^{N_1})\times C^1(\bar{I},\mathbb{R}^{N_2})\times \mathrm{L}^2(I,\mathbb{R}^{N_\lambda})\cap \mathrm{W}^{2,2}(I,\mathbb{R}^{N_1})\times \mathrm{W}^{2,2}(I,\mathbb{R}^{N_2})\times \mathrm{L}^2(I,\mathbb{R}^{N_\lambda})$ satisfies the system (1.1).

Finally, by the uniqueness of the solution according to Theorem 3.2, one may infer that the claims of Theorem 4.1 are satisfied for every sequence $(h_{1,l},h_{2,l})_{l\geq 0}\in S_h^{\mathbb{N}}$ converging to 0, in particular, the convergences in (4.2) and (4.1). This completes the proof. □

4.2. Proof of Theorem 4.3

Using the second relation in equation (2.2), we have for every integer $0\leq k<n_{2,l}$:

$$\begin{aligned}\dot{\mathcal{U}}_1^{(k+1)m} &= \dot{\mathcal{U}}_1^{km} + \sum_{l=0}^{m-1} h_1\,(1-\gamma_1)\,\ddot{\mathcal{U}}_1^{km+l} + h_1\gamma_1\,\ddot{\mathcal{U}}_1^{km+l+1}\\ &= \dot{\mathcal{U}}_1^{km} + h_2\hat{\ddot{U}}_1^{k+1}\end{aligned} \tag{4.23}$$

and

$$\dot{\mathcal{U}}_2^{(k+1)} = \dot{\mathcal{U}}_2^{k} + h_2\,(1-\gamma_2)\,\ddot{\mathcal{U}}_2^{k} + h_2\gamma_2\,\ddot{\mathcal{U}}_1^{k+1} = \dot{\mathcal{U}}_2^{k} + h_2\hat{\ddot{U}}_2^{k+1}. \tag{4.24}$$

Taking into account the compatibility condition (1.2) and the relation (2.8), we have for every integer $0\leq k\leq n_{2,l}$:

$$L_1\dot{\mathcal{U}}_1^{km} + L_2\dot{\mathcal{U}}_2^{k} = 0. \tag{4.25}$$

Thus, for every integer $0\leq k<n_{2,l}$:

$$L_1\hat{\ddot{\mathcal{U}}}_1^{k+1} + L_2\hat{\ddot{\mathcal{U}}}_2^{k+1} = 0. \tag{4.26}$$

Using the sequence $(h_{2,l})_l$, we infer:

$$L_1\hat{\ddot{\mathcal{U}}}_1^{h_{2,l}}(t) + L_2\hat{\ddot{\mathcal{U}}}_2^{h_{2,l}}(t) = 0 \quad \forall t\in I. \tag{4.27}$$

Moreover, using the linearity of the problem (1.1), we end up with:

$$\begin{cases} M_1\hat{\ddot{\mathcal{U}}}_1^{h_{2,l}}(t) + C_1\hat{\dot{\mathcal{U}}}_1^{h_{2,l}}(t) + K_1\hat{\mathcal{U}}_1^{h_{2,l}}(t) + L_1^T\hat{\lambda}_1^{h_{2,l}}(t) = \hat{F}_1^{h_{2,l}}(t)\\ M_2\hat{\ddot{\mathcal{U}}}_2^{h_{2,l}}(t) + C_2\hat{\dot{\mathcal{U}}}_2^{h_{2,l}}(t) + K_2\hat{\mathcal{U}}_2^{h_{2,l}}(t) + L_2^T\hat{\lambda}_2^{h_{2,l}}(t) = \hat{F}_2^{h_{2,l}}(t)\\ L_1\hat{\ddot{\mathcal{U}}}_1^{h_{2,l}}(t) + L_2\hat{\ddot{\mathcal{U}}}_2^{h_{2,l}}(t) = 0.\end{cases} \tag{4.28}$$

The key of the proof is the following equality:

$$\hat{\lambda}_1^{h_{2,l}}(t) = \hat{\lambda}_2^{h_{2,l}}(t) \quad \forall t\in I. \tag{4.29}$$

Equation (4.29) is a direct consequence of elementary computations taking into account the assumption $\gamma_1 = m\gamma_2 - \frac{m-1}{2}$ as stated in Theorem 4.3. By the uniform convergence in Theorem 4.1 and the regularity of external loads in assumption (**HF**), we have:

$$\lim_{h_{2,l}\to 0}\left\|\hat{\mathcal{U}}_i^{h_{2,l}} - \mathcal{U}_i\right\|_{\mathrm{L}^\infty(I)} = 0$$
$$\lim_{h_{2,l}\to 0}\left\|\hat{\dot{\mathcal{U}}}_i^{h_{2,l}} - \dot{\mathcal{U}}_i\right\|_{\mathrm{L}^\infty(I)} = 0$$
$$\lim_{h_{2,l}\to 0}\left\|\hat{F}_i^{h_{2,l}} - F_i\right\|_{\mathrm{L}^\infty(I)} = 0. \tag{4.30}$$

Indeed, restricting our focus on the displacements, the triangular inequality entails:

$$\left\|\hat{\mathcal{U}}_i^{h_{2,l}} - \mathcal{U}_i\right\|_{\mathrm{L}^\infty(I)} \leq \left\|\hat{\mathcal{U}}_i^{h_{2,l}} - \tilde{\mathcal{U}}_i^{h_{i,l}}\right\|_{\mathrm{L}^\infty(I)} + \left\|\tilde{\mathcal{U}}_i^{h_{i,l}} - \mathcal{U}_i\right\|_{\mathrm{L}^\infty(I)}. \tag{4.31}$$

Moreover, using Lemma 3.5 and equation (2.2), we write for $i = 1$ and on each micro interval $J_{k,p} = [t_{km+p}, t_{km+p+1})$ $(0 \leq p < m$ and $k < n_{1,l})$:

$$\begin{aligned}\left(\hat{\mathcal{U}}_1^{h_{2,l}} - \tilde{\mathcal{U}}_1^{h_{1,l}}\right)_{|J_{k,p}} &= \frac{1}{m}\sum_{l=0}^{m-1}\gamma_1\left(\mathcal{U}_1^{km+l+1} - \mathcal{U}_1^{km+l}\right) + \frac{1}{m}\sum_{l=0}^{m-1}\left(\mathcal{U}_1^{km+l} - \mathcal{U}_1^{km+p+1}\right)\\ &\lesssim \gamma_1 h_{1,l} + \frac{1}{m}\sum_{l=0}^{m-1}|l-p-1|h_{1,l} \leq c_T h_{2,l}\end{aligned} \tag{4.32}$$

where c_T is independent of k, p and $h_{2,l}$. Following the same argument, the same may be obtained for $i = 2$. Thus:

$$\lim_{h_{2,l}\to 0}\left\|\hat{\mathcal{U}}_i^{h_{2,l}} - \mathcal{U}_i\right\|_{\mathrm{L}^\infty(I)} = 0. \tag{4.33}$$

The same reasoning may be used to prove the remaining equations in (4.30).

Now, by virtue of equations (4.28) and (4.29), we have for every $t \in I$:

$$\begin{aligned}&\left\langle M_1\hat{\ddot{\mathcal{U}}}_1^{h_{2,l}} - M_1\ddot{\mathcal{U}}_1, \hat{\ddot{\mathcal{U}}}_1^{h_{2,l}} - \ddot{\mathcal{U}}_1\right\rangle + \left\langle M_2\hat{\ddot{\mathcal{U}}}_2^{h_{2,l}} - M_2\ddot{\mathcal{U}}_2, \hat{\ddot{\mathcal{U}}}_2^{h_{2,l}} - \ddot{\mathcal{U}}_2\right\rangle\\ &+ \left\langle C_1\hat{\dot{\mathcal{U}}}_1^{h_{2,l}} - C_1\dot{\mathcal{U}}_1, \hat{\ddot{\mathcal{U}}}_1^{h_{2,l}} - \ddot{\mathcal{U}}_1\right\rangle + \left\langle C_2\hat{\dot{\mathcal{U}}}_2^{h_{2,l}} - C_2\dot{\mathcal{U}}_2, \hat{\ddot{\mathcal{U}}}_2^{h_{2,l}} - \ddot{\mathcal{U}}_2\right\rangle\\ &+ \left\langle K_1\hat{\mathcal{U}}_1^{h_{2,l}} - K_1\mathcal{U}_1, \hat{\ddot{\mathcal{U}}}_1^{h_{2,l}} - \ddot{\mathcal{U}}_1\right\rangle + \left\langle K_2\hat{\mathcal{U}}_2^{h_{2,l}} - K_2\mathcal{U}_2, \hat{\ddot{\mathcal{U}}}_2^{h_{2,l}} - \ddot{\mathcal{U}}_2\right\rangle\\ &= \left\langle \hat{F}_1^{h_{2,l}} - F_1, \hat{\ddot{\mathcal{U}}}_1^{h_{2,l}} - \ddot{\mathcal{U}}_1\right\rangle + \left\langle \hat{F}_2^{h_{2,l}} - F_2, \hat{\ddot{\mathcal{U}}}_2^{h_{2,l}} - \ddot{\mathcal{U}}_2\right\rangle.\end{aligned} \tag{4.34}$$

Combining the strict non-negativity of the mass matrices and the Cauchy–Schwartz inequality, we get:

$$\begin{aligned}\left\|\hat{\ddot{\mathcal{U}}}_1^{h_{2,l}} - \ddot{\mathcal{U}}_1\right\|_{\mathrm{L}^\infty(I)} + \left\|\hat{\ddot{\mathcal{U}}}_2^{h_{2,l}} - \ddot{\mathcal{U}}_2\right\|_{\mathrm{L}^\infty(I)} \leq c_1\Bigg(&\left\|\hat{\dot{\mathcal{U}}}_1^{h_{2,l}} - \dot{\mathcal{U}}_1\right\|_{\mathrm{L}^\infty(I)} + \left\|\hat{\dot{\mathcal{U}}}_2^{h_{2,l}} - \dot{\mathcal{U}}_2\right\|_{\mathrm{L}^\infty(I)}\\ &+ \left\|\hat{\mathcal{U}}_1^{h_{2,l}} - \mathcal{U}_1\right\|_{\mathrm{L}^\infty(I)} + \left\|\hat{\mathcal{U}}_2^{h_{2,l}} - \mathcal{U}_2\right\|_{\mathrm{L}^\infty(I)}\\ &+ \left\|\hat{F}_1^{h_{2,l}} - F_1\right\|_{\mathrm{L}^\infty(I)} + \left\|\hat{F}_2^{h_{2,l}} - F_2\right\|_{\mathrm{L}^\infty(I)}\Bigg)\end{aligned} \tag{4.35}$$

where c_1 is independent of $h_{2,l}$. As a consequence, using (4.30), we deduce:

$$\lim_{h_{2,l}\to 0}\left\|\hat{\ddot{\mathcal{U}}}_1^{h_{2,l}} - \ddot{\mathcal{U}}_1\right\|_{\mathrm{L}^\infty(I)} = 0 \quad \text{and} \quad \lim_{h_{2,l}\to 0}\left\|\hat{\ddot{\mathcal{U}}}_2^{h_{2,l}} - \ddot{\mathcal{U}}_2\right\|_{\mathrm{L}^\infty(I)} = 0. \tag{4.36}$$

By the linearity of the system (4.28), we get the same result for the Lagrange multipliers and this completes the proof. □

4.3. Proof of Theorem 4.4:

If $i=1$ or 2, we define the piecewise twice continuously differentiable function $\underset{\sim}{\boldsymbol{U}}_i^{h_{i,l}}$ by:

$$\begin{cases} \underset{\sim}{\boldsymbol{U}}_i^{h_{i,l}}(t) = \sum_{p=0}^{n_{i,l}-1}\left[\mathcal{U}_i^p + \left(t-t_p^{(i)}\right)\dot{\mathcal{U}}_i^p + \frac{(t-t_p^{(i)})^2}{2}\left(\gamma_i\ddot{\mathcal{U}}_i^{p+1} + (1-\gamma_i)\ddot{\mathcal{U}}_i^p\right)\right]\chi_{\left[t_p^{(i)},t_{p+1}^{(i)}\right)}(t) \\ \underset{\sim}{\boldsymbol{U}}_i^{h_{i,l}}(T) = \mathcal{U}_i^{n_{i,l}-1} + h_{i,l}\dot{\mathcal{U}}_i^{n_{i,l}-1} + \frac{h_{i,l}^2}{2}\left(\gamma_i\ddot{\mathcal{U}}_i^{n_{i,l}} + (1-\gamma_i)\ddot{\mathcal{U}}_i^{n_{i,l}-1}\right). \end{cases} \tag{4.37}$$

In this proof, we shall frequently use the notation $\mathcal{O}$ to mean that the equality $A=\mathcal{O}(B(h_{2,l}))$, where A is a vector possibly depends on the time $t\in I$ and the scalar B depends on $h_{2,l}$, is equivalent to the existence of a constant C independent of the time step $h_{2,l}$ and $t\in I$ such that $\|A\|\le CB(h_{2,l})$ for all t. In order to establish the estimates (4.6), it is sufficient to prove, for every $0\le t\le T$, that:

$$\begin{aligned}
\underset{\sim}{\boldsymbol{U}}_1^{h_{1,l}}(t) - \mathcal{U}_1(t) &= A_1(h_{2,l}) + \sum_{i=1}^{2}\int_0^t \phi_{1i}(s)\left(\underset{\sim}{\boldsymbol{U}}_i^{h_{i,l}} - \mathcal{U}_i\right) + \sum_{i=1}^{2}\int_0^t \psi_{1i}(s)\left(\dot{\underset{\sim}{\boldsymbol{U}}}_i^{h_{i,l}} - \dot{\mathcal{U}}_i\right)\mathrm{d}s \\
\dot{\underset{\sim}{\boldsymbol{U}}}_1^{h_{1,l}}(t) - \dot{\mathcal{U}}_1(t) &= A_2(h_{2,l}) + \sum_{i=1}^{2}\int_0^t \phi_{2i}(s)\left(\underset{\sim}{\boldsymbol{U}}_i^{h_{i,l}} - \mathcal{U}_i\right) + \sum_{i=1}^{2}\int_0^t \psi_{2i}(s)\left(\dot{\underset{\sim}{\boldsymbol{U}}}_i^{h_{i,l}} - \dot{\mathcal{U}}_i\right)\mathrm{d}s \\
\underset{\sim}{\boldsymbol{U}}_2^{h_{2,l}}(t) - \mathcal{U}_2(t) &= A_3(h_{2,l}) + \sum_{i=1}^{2}\int_0^t \phi_{3i}(s)\left(\underset{\sim}{\boldsymbol{U}}_i^{h_{i,l}} - \mathcal{U}_i\right) + \sum_{i=1}^{2}\int_0^t \psi_{3i}(s)\left(\dot{\underset{\sim}{\boldsymbol{U}}}_i^{h_{i,l}} - \dot{\mathcal{U}}_i\right)\mathrm{d}s \\
\dot{\underset{\sim}{\boldsymbol{U}}}_2^{h_{2,l}}(t) - \dot{\mathcal{U}}_2(t) &= A_4(h_{2,l}) + \sum_{i=1}^{2}\int_0^t \phi_{4i}(s)\left(\underset{\sim}{\boldsymbol{U}}_i^{h_{i,l}} - \mathcal{U}_i\right) + \sum_{i=1}^{2}\int_0^t \psi_{4i}(s)\left(\dot{\underset{\sim}{\boldsymbol{U}}}_i^{h_{i,l}} - \dot{\mathcal{U}}_i\right)\mathrm{d}s
\end{aligned} \tag{4.38}$$

where the time dependant quantities A_i satisfy $A_i(h_{2,l})=\mathcal{O}(h_{2,l})$. The maps ϕ_{ij} and ψ_{ij} are continuous, mapping $\bar{I}$ to the space of square matrices and independent of the time step $h_{2,l}$. Notice that the conclusions of the Theorem follow if equation (4.38) holds. Indeed, a direct application of Jones inequality [31] (see also Thm. 1.2.1 in [32]), a generalized form of the Gronwall inequality for piecewise continuous functions, shows that for $i=1,2$:

$$\left\|\underset{\sim}{\boldsymbol{U}}_i^{h_{i,l}} - \mathcal{U}_i\right\|_{\mathrm{L}^\infty(\bar{I})} = \mathcal{O}(h_{2,l}) \quad \text{and} \quad \left\|\dot{\underset{\sim}{\boldsymbol{U}}}_i^{h_{i,l}} - \dot{\mathcal{U}}_i\right\|_{\mathrm{L}^\infty(\bar{I})} = \mathcal{O}(h_{2,l}). \tag{4.39}$$

Moreover, using equations (2.2), (2.4) and Lemma 3.5, it is not difficult to prove:

$$\left\|\tilde{\mathcal{U}}_i^{h_{i,l}} - \underset{\sim}{\boldsymbol{U}}_i^{h_{i,l}}\right\|_{\mathrm{L}^\infty(\bar{I})} = \mathcal{O}(h_{2,l}). \tag{4.40}$$

Thus, taking into account the equality $\dot{\tilde{\mathcal{U}}}_i^{h_{i,l}} = \dot{\underset{\sim}{\boldsymbol{U}}}_i^{h_{i,l}}$, we get the desired result for the displacement and the velocities vectors in equation (4.6). Moreover, using equations (4.35) and (4.39) together with the following estimates:

$$\begin{aligned}
\left\|\hat{\mathcal{U}}_i^{h_{2,l}} - \underset{\sim}{\boldsymbol{U}}_i^{h_{i,l}}\right\|_{\mathrm{L}^\infty(I)} &= \mathcal{O}(h_{2,l}) \\
\left\|\hat{\dot{\mathcal{U}}}_i^{h_{2,l}} - \dot{\underset{\sim}{\boldsymbol{U}}}_i^{h_{i,l}}\right\|_{\mathrm{L}^\infty(I)} &= \mathcal{O}(h_{2,l}) \\
\left\|\hat{F}_i^{h_{2,l}} - F_i\right\|_{\mathrm{L}^\infty(I)} &= \mathcal{O}(h_{2,l}).
\end{aligned} \tag{4.41}$$

The estimates for the acceleration and the Lagrange multipliers vectors follow immediately, where the Lagrange multipliers are expressed directly using equation (4.28). The proof of the first equality in (4.41) is simple. Indeed, restricting to $i=1$, $\hat{\mathcal{U}}_1^{h_{2,l}} - \underset{\sim}{\boldsymbol{U}}_1^{h_{1,l}}$ can be written, on each fine time interval $J_{k,p}=[t_{km+p}^{(1)}, t_{km+p+1}^{(1)})$, as:

$$\left(\hat{\mathcal{U}}_1^{h_{2,l}} - \underset{\sim}{\boldsymbol{U}}_1^{h_{1,l}}\right)_{|J_{k,p}} = \frac{1}{m}\left(\gamma_1 \mathcal{U}_1^{(k+1)m} + (1-\gamma_1)\mathcal{U}_1^{km} + \sum_{l=1}^{m-1} \mathcal{U}_1^{km+l}\right) - \mathcal{U}_1^{km+p} + \mathcal{O}(h_{1,l})$$
$$= \frac{1}{m}\sum_{l=0}^{m-1} \gamma_1\left(\mathcal{U}_1^{km+l+1} - \mathcal{U}_1^{km+l}\right) + \frac{1}{m}\sum_{l=0}^{m-1}\left(\mathcal{U}_1^{km+l} - \mathcal{U}_1^{km+p}\right) + \mathcal{O}(h_{1,l}). \tag{4.42}$$

Using the Newmark relations (2.2) and (2.4) and Lemma 3.5, the first sum in (4.42) is $\mathcal{O}(h_{1,l})$ and the second sum is $\mathcal{O}(\frac{1}{m}\sum_{l=0}^{m-1}|p-l|h_{1,l}) = \mathcal{O}(h_{2,l})$, we deduce that $\hat{\mathcal{U}}_1^{h_{2,l}} - \underset{\sim}{\boldsymbol{U}}_1^{h_{1,l}} = \mathcal{O}(h_{2,l})$. The case $i = 2$ and the second equality in (4.41) follows from similar arguments. The third equality is a consequence of the regularity assumption on external loads.

Now, we take only the two first formula in (4.38), the proof of the remaining ones being analogous. In the case of the displacement vector, it is sufficient to observe:

$$\underset{\sim}{\boldsymbol{U}}_1^{h_{1,l}}(t) - \mathcal{U}_1(t) = A_1(h_{2,l}) + \int_0^t \left(\dot{\underset{\sim}{\boldsymbol{U}}}_1^{h_{1,l}} - \dot{\mathcal{U}}_1\right)\mathrm{d}s \tag{4.43}$$

where

$$A_1(h_{2,l}) = \sum_{p \le \left\lfloor \frac{t}{h_1}\right\rfloor} \frac{(h_{1,l})^2}{2}(\gamma_1 - 2\beta_1)\left[\ddot{\mathcal{U}}_1^{p+1} - \ddot{\mathcal{U}}_1^{p}\right] = \mathcal{O}(h_{2,l}) \tag{4.44}$$

where $\lfloor x \rfloor$ is the greatest integer less than or equal to x. This completes the proof for the displacements.

Now, focusing on the velocities vectors and using the fact that $\ddot{\underset{\sim}{\boldsymbol{U}}}_1^{h_{1,l}} = \bar{\ddot{\mathcal{U}}}_1^{h_{1,l},r=\gamma_1}$, we get:

$$\begin{aligned}
\dot{\underset{\sim}{\boldsymbol{U}}}_1^{h_{1,l}}(t) - \dot{\mathcal{U}}_1(t) &= \int_0^t \left(\ddot{\underset{\sim}{\boldsymbol{U}}}_1^{h_{1,l}} - \ddot{\mathcal{U}}_1\right)\mathrm{d}s \\
&= -\int_0^t M_1^{-1}K_1\left(\bar{\mathcal{U}}_1^{h_{1,l},r=\gamma_1} - \underset{\sim}{\boldsymbol{U}}_1^{h_{1,l}}\right)\mathrm{d}s - \int_0^t M_1^{-1}C_1\left(\bar{\dot{\mathcal{U}}}_1^{h_{1,l},r=\gamma_1} - \dot{\underset{\sim}{\boldsymbol{U}}}_1^{h_{1,l}}\right)\mathrm{d}s \\
&\quad - \int_0^t M_1^{-1}L_1^T\left(\bar{\lambda}_1^{h_{1,l},r=\gamma_1} - \hat{\lambda}_1^{h_{2,l}}\right) + \int_0^t M_1^{-1}\left(\bar{F}_1^{h_{1,l},r=\gamma_1} - F_1\right)\mathrm{d}s \\
&\quad - \int_0^t M_1^{-1}K_1\left(\underset{\sim}{\boldsymbol{U}}_1^{h_{1,l}} - \mathcal{U}_1\right)\mathrm{d}s - \int_0^t M_1^{-1}C_1\left(\dot{\underset{\sim}{\boldsymbol{U}}}_1^{h_{1,l}} - \dot{\mathcal{U}}_1\right)\mathrm{d}s - \int_0^t M_1^{-1}L_1^T\left(\hat{\lambda}_1^{h_{2,l}} - \lambda\right) \\
&= I_1 + I_2 + I_3 + I_4 - \int_0^t M_1^{-1}K_1\left(\underset{\sim}{\boldsymbol{U}}_1^{h_{1,l}} - \mathcal{U}_1\right)\mathrm{d}s - \int_0^t M_1^{-1}C_1\left(\dot{\underset{\sim}{\boldsymbol{U}}}_1^{h_{1,l}} - \dot{\mathcal{U}}_1\right)\mathrm{d}s \\
&\quad - \int_0^t M_1^{-1}L_1^T\left(\hat{\lambda}_1^{h_{2,l}} - \lambda\right)\mathrm{d}s.
\end{aligned} \tag{4.45}$$

In the second equality of equation (4.45), only the equilibrium equation in (1.1) have been used as well as the definitions of the approximate solutions, *i.e.*, equations (3.1) and (4.3), noticing that:

$$M_1\bar{\ddot{\mathcal{U}}}_1^{h_{1,l},r=\gamma_1} + C_1\bar{\dot{\mathcal{U}}}_1^{h_{1,l},r=\gamma_1} + K_1\bar{\mathcal{U}}_1^{h_{1,l},r=\gamma_1} + L_1^T\bar{\lambda}_1^{h_{1,l},r=\gamma_1} = \bar{F}_1^{h_{1,l},r=\gamma_1}. \tag{4.46}$$

Now, we shall estimate I_1, I_2, I_3 and I_4. We start I_1 by omitting the constant multiplicative terms:

$$\int_0^t \left(\bar{\mathcal{U}}_1^{h_{1,l},r=\gamma_1} - \underset{\sim}{\boldsymbol{U}}_1^{h_{1,l}}\right)\mathrm{d}s = \sum_{p\le\left\lfloor\frac{t}{h_{1,l}}\right\rfloor}\int_{t_p^{(1)}}^{t\wedge t_{p+1}^{(1)}}\Bigg[\gamma_1\mathcal{U}_1^{p+1} + (1-\gamma_1)\mathcal{U}_1^p - \mathcal{U}_1^p - \left(s - t_p^{(1)}\right)\dot{\mathcal{U}}_1^p$$

$$- \frac{\left(s - t_p^{(1)}\right)^2}{2}\left(\gamma_1 \ddot{\mathcal{U}}_1^{p+1} + (1-\gamma_1)\ddot{\mathcal{U}}_1^{p}\right)\Bigg] \mathrm{d}s. \tag{4.47}$$

Moreover, using the boundedness results in Lemma 3.5, one may notice that for every p:

$$\gamma_1 \mathcal{U}_1^{p+1} + (1-\gamma_1)\mathcal{U}_1^{p} - \mathcal{U}_1^{p} = \gamma_1 h_{1,l} \dot{\mathcal{U}}_1^{p} + \mathcal{O}\Big((h_{1,l})^2\Big). \tag{4.48}$$

Combining equations (4.47) and (4.48), we conclude that $I_1 = \mathcal{O}(h_{2,l})$. Using the same approach, we prove that $I_2 = \mathcal{O}(h_{2,l})$.

It remains to establish that $I_3 = \mathcal{O}(h_{2,l})$. Using equations (2.7) and (4.3), we compute $\bar{\lambda}_1^{h_{1,l},r=\gamma_1} - \hat{\lambda}_1^{h_{2,l}}$ on each micro interval $J_{k,p} = [t^{(1)}_{km+p}, t^{(1)}_{km+p+1})$ $(0 \le p < m$ and $k < n_{2,l})$ as follows:

$$\begin{aligned}
\left(\bar{\lambda}_1^{h_{1,l},r=\gamma_1} - \hat{\lambda}_1^{h_{2,l}}\right)\Big|_{J_{k,p}} &= \gamma_1 \lambda_1^{km+p+1} + (1-\gamma_1)\lambda_1^{km+p} - \frac{1}{m}\left[\gamma_1 \lambda_1^{(k+1)m} + (1-\gamma_1)\lambda_1^{km} + \sum_{l=1}^{m-1} \lambda_1^{km+l}\right]\\
&= \frac{2p-m+1}{2m}\Big[\lambda_1^{(k+1)m} - \lambda_1^{km}\Big].
\end{aligned} \tag{4.49}$$

Taking into account the previous equation (4.49), we write:

$$\begin{aligned}
\int_0^t \left(\bar{\lambda}_1^{h_{1,l},r=\gamma_1} - \hat{\lambda}_1^{h_{2,l}}\right) &= \sum_{k<\left\lfloor \frac{t}{h_{2,l}}\right\rfloor} \left(\lambda_1^{(k+1)m} - \lambda_1^{km}\right)\left(\sum_{p=0}^{m-1} \int_{t^{(1)}_{km+p}}^{t^{(1)}_{km+p+1}} \frac{2p-m+1}{2m}\,\mathrm{d}s\right)\\
&\quad + \int_{\left\lfloor \frac{t}{h_{2,l}}\right\rfloor h_{2,l}}^{t} \left(\bar{\lambda}_1^{h_{1,l},r=\gamma_1} - \hat{\lambda}_1^{h_{2,l}}\right)\mathrm{d}s.
\end{aligned} \tag{4.50}$$

By the boundedness results in Lemma 3.5, we have:

$$\left|\int_{\left\lfloor \frac{t}{h_{2,l}}\right\rfloor h_2}^{t} \left(\bar{\lambda}_1^{h_{1,l},r=\gamma_1} - \hat{\lambda}_1^{h_{2,l}}\right)\mathrm{d}s\right| \lesssim \left(t - \left\lfloor \frac{t}{h_{2,l}}\right\rfloor h_{2,l}\right) = \mathcal{O}(h_{2,l}). \tag{4.51}$$

Moreover, using the equality $t^{(1)}_{km+p+1} - t^{(1)}_{km+p} = h_{1,l}$ and the fact that $\sum_{p=0}^{m-1} \frac{2p-m+1}{2m} = 0$, we have:

$$\sum_{p=0}^{m-1} \frac{2p-m+1}{2m} \int_{t^{(1)}_{km+p}}^{t^{(1)}_{km+p+1}} 1\mathrm{d}s = 0. \tag{4.52}$$

Combining equations (4.49) to (4.52), we deduce that $I_3 = \mathcal{O}(h_{2,l})$. With the focus on the remaining term, *i.e.*, I_4, it is easy to see that $I_4 = \mathcal{O}(h_{2,l})$ using the strong assumptions on the regularity of external loads in Theorem 4.4. In fact, for every $t \in [t_p^{(1)}, t_{p+1}^{(1)}](p \le n_{1,l})$:

$$\begin{aligned}
\bar{F}_1^{h_{1,l},r=\gamma_1}(t) - F_1(t) &= \gamma_1 \int_t^{t^{(1)}_{p+1}} \dot{F}_1(s)\,\mathrm{d}s + (1-\gamma_1)\int_{t_p^{(1)}}^{t} \dot{F}_1(s)\,\mathrm{d}s\\
&\lesssim \gamma_1 h_{1,l} + (1-\gamma_1) h_{1,l} = \mathcal{O}(h_{2,l}).
\end{aligned} \tag{4.53}$$

Now, using equation (4.28), let us rewrite the last term in equation (4.45) as follows:

$$\int_0^t M_1^{-1} L_1^T \left(\hat{\lambda}_1^{h_{2,l}} - \lambda\right)\mathrm{d}s = -\, M_1^{-1} L_1^T \mathbb{H}^{-1}\bigg(\int_0^t L_1 M_1^{-1} K_1 \Big(\hat{\mathcal{U}}_1^{h_{2,l}} - \underset{\sim}{\boldsymbol{U}}_1^{h_{1,l}}\Big)\,\mathrm{d}s$$

$$
\begin{aligned}
&+ \int_0^t L_2 M_2^{-1} K_2 \Big(\hat{\mathcal{U}}_2^{h_{2,l}} - \underset{\sim}{\boldsymbol{U}}_2^{h_{2,l}}\Big)\,\mathrm{d}s + \int_0^t L_1 M_1^{-1} C_1 \Big(\dot{\hat{\mathcal{U}}}_1^{h_{2,l}} - \dot{\underset{\sim}{\boldsymbol{U}}}_1^{h_{1,l}}\Big)\,\mathrm{d}s \\
&+ \int_0^t L_2 M_2^{-1} C_2 \Big(\dot{\hat{\mathcal{U}}}_2^{h_{2,l}} - \dot{\underset{\sim}{\boldsymbol{U}}}_2^{h_{2,l}}\Big)\,\mathrm{d}s - \int_0^t L_1 M_1^{-1} \Big(\hat{F}_1^{h_{2,l}} - F_1\Big)\,\mathrm{d}s \\
&- \int_0^t L_2 M_2^{-1} \Big(\hat{F}_2^{h_{2,l}} - F_2\Big)\,\mathrm{d}s + \int_0^t L_1 M_1^{-1} K_1 \Big(\underset{\sim}{\boldsymbol{U}}_1^{h_{1,l}} - \mathcal{U}_1\Big)\,\mathrm{d}s \\
&+ \int_0^t L_1 M_1^{-1} C_1 \Big(\dot{\underset{\sim}{\boldsymbol{U}}}_1^{h_{1,l}} - \dot{\mathcal{U}}_1\Big)\,\mathrm{d}s + \int_0^t L_2 M_2^{-1} K_2 \Big(\underset{\sim}{\boldsymbol{U}}_2^{h_{2,l}} - \mathcal{U}_2\Big)\,\mathrm{d}s \\
&+ \int_0^t L_2 M_2^{-1} C_2 \Big(\dot{\underset{\sim}{\boldsymbol{U}}}_2^{h_{2,l}} - \dot{\mathcal{U}}_2\Big)\,\mathrm{d}s\Bigg) \\
={}& - M_1^{-1} L_1^T \mathbb{H}^{-1} \Bigg(I_5 + I_6 + I_7 + I_8 + I_9 + I_{10} \\
&+ \int_0^t L_1 M_1^{-1} K_1 \Big(\underset{\sim}{\boldsymbol{U}}_1^{h_{1,l}} - \mathcal{U}_1\Big)\,\mathrm{d}s + \int_0^t L_1 M_1^{-1} C_1 \Big(\dot{\underset{\sim}{\boldsymbol{U}}}_1^{h_{1,l}} - \dot{\mathcal{U}}_1\Big) \\
&+ \int_0^t L_2 M_2^{-1} K_2 \Big(\underset{\sim}{\boldsymbol{U}}_2^{h_{2,l}} - \mathcal{U}_2\Big)\,\mathrm{d}s + \int_0^t L_2 M_2^{-1} C_2 \Big(\dot{\underset{\sim}{\boldsymbol{U}}}_2^{h_{2,l}} - \dot{\mathcal{U}}_2\Big)\,\mathrm{d}s\Bigg)
\end{aligned} \tag{4.54}
$$

where $\mathbb{H} = L_1 M_1^{-1} L_1^T + L_2 M_2^{-1} L_2^T$. As previously, we show that $I_i = \mathcal{O}(h_{2,l})$ ($i \in \{5,6,7,8,9,10\}$). We start with I_5 and we write by omitting the useless constant terms:

$$
\begin{aligned}
\int_0^t \Big(\hat{\mathcal{U}}_1^{h_{2,l}} - \underset{\sim}{\boldsymbol{U}}_1^{h_{1,l}}\Big)\mathrm{d}s ={}& \sum_{k<\left\lfloor \frac{t}{h_{2,l}} \right\rfloor} \Bigg[\int_{t_{km}^{(1)}}^{t_{(k+1)m}^{(1)}} \frac{1}{m}\Bigg(\gamma_1 \mathcal{U}_1^{(k+1)m} + (1-\gamma_1)\mathcal{U}_1^{km} + \sum_{l=1}^{m-1} \mathcal{U}_1^{km+l}\Bigg)\,\mathrm{d}s \\
&- \sum_{p=0}^{m-1} \int_{t_{km+p}^{(1)}}^{t_{km+p+1}^{(1)}} \Bigg(\mathcal{U}_1^{km+p} + \Big(s - t_{km+p}^{(1)}\Big)\dot{\mathcal{U}}_1^{km+p} + \frac{\Big(s - t_{km+p}^{(1)}\Big)^2}{2} \\
&\times \Big(\gamma_1 \ddot{\mathcal{U}}_1^{km+p+1} + (1-\gamma_1)\ddot{\mathcal{U}}_1^{km+p}\Big)\Bigg)\,\mathrm{d}s\Bigg] + \int_{\left\lfloor \frac{t}{h_{2,l}} \right\rfloor h_{2,l}}^t \Big(\hat{\mathcal{U}}_1^{h_{2,l}} - \underset{\sim}{\boldsymbol{U}}_1^{h_{1,l}}\Big)\,\mathrm{d}s \\
={}& \sum_{k<\left\lfloor \frac{t}{h_{2,l}} \right\rfloor} \Bigg[\sum_{p=0}^{m-1} \Bigg(\int_{t_{km}^{(1)}}^{t_{(k+1)m}^{(1)}} \frac{1}{m}\Big(\gamma_1 \mathcal{U}_1^{km+p+1} + (1-\gamma_1)\mathcal{U}_1^{km+p}\Big)\,\mathrm{d}s \\
&- \int_{t_{km+p}^{(1)}}^{t_{km+p+1}^{(1)}} \Bigg(\mathcal{U}_1^{km+p} + \Big(s - t_{km+p}^{(1)}\Big)\dot{\mathcal{U}}_1^{km+p} + \frac{\Big(s - t_{km+p}^{(1)}\Big)^2}{2} \\
&\times \Big(\gamma_1 \ddot{\mathcal{U}}_1^{km+p+1} + (1-\gamma_1)\ddot{\mathcal{U}}_1^{km+p}\Big)\Bigg)\,\mathrm{d}s\Bigg)\Bigg] + \int_{\left\lfloor \frac{t}{h_{2,l}} \right\rfloor h_{2,l}}^t \Big(\hat{\mathcal{U}}_1^{h_{2,l}} - \underset{\sim}{\boldsymbol{U}}_1^{h_{1,l}}\Big)\,\mathrm{d}s.
\end{aligned} \tag{4.55}
$$

Again, it is not difficult to see that $\int_{\lfloor \frac{t}{h_{2,l}} \rfloor h_{2,l}}^{t} \Big(\hat{\mathcal{U}}_1^{h_{2,l}} - \underset{\sim}{\boldsymbol{U}}_1^{h_{1,l}}\Big) \mathrm{d}s = \mathcal{O}(h_{2,l})$. Moreover, for $k < \lfloor \frac{t}{h_{2,l}} \rfloor$, we have:

$$\begin{aligned}\int_{t_{km}^{(1)}}^{t_{(k+1)m}^{(1)}} \frac{1}{m}\Big(\gamma_1 \mathcal{U}_1^{km+p+1} + (1-\gamma_1)\mathcal{U}_1^{km+p}\Big) - \int_{t_{km+p}^{(1)}}^{t_{km+p+1}^{(1)}} \mathcal{U}_1^{km+p} &= \frac{h_{2,l}}{m}\mathcal{U}_1^{km+p} - h_{1,l}\mathcal{U}_1^{km+p} + \mathcal{O}\big(h_{1,l}^2\big) \\ &= \mathcal{O}\big(h_{1,l}^2\big). \end{aligned} \tag{4.56}$$

Since the remaining terms in (4.55) are of order at least $h_{1,l}^2$, we infer that $\int_0^t \Big(\hat{\mathcal{U}}_1^{h_{2,l}} - \underset{\sim}{\boldsymbol{U}}_1^{h_{1,l}}\Big) \mathrm{d}s = \mathcal{O}(h_{2,l})$ or $I_5 = \mathcal{O}(h_{2,l})$. Similarly, elementary calculations by combining Lemma 3.5, the regularity assumption on external loads and the Newmark relations (2.2) and (2.4), show that $I_i = \mathcal{O}(h_{2,l})$ $(i \in \{6, 7, 8, 9, 10\})$.

Combining (4.45) and (4.54), we can finally write:

$$\underset{\sim}{\boldsymbol{U}}_1^{h_{1,l}}(t) - \mathcal{U}_1(t) = A_2(h_{2,l}) + \sum_{i=1}^{2} \int_0^t \phi_{2i}(s)\Big(\underset{\sim}{\boldsymbol{U}}_i^{h_{i,l}}(t) - \mathcal{U}_i\Big) + \sum_{i=1}^{2} \int_0^t \psi_{2i}(s)\Big(\dot{\underset{\sim}{\boldsymbol{U}}}_i^{h_{i,l}} - \dot{\mathcal{U}}_i\Big)\,\mathrm{d}s \tag{4.57}$$

where $A_2(h_2) = \mathcal{O}(h_{2,l})$. The functions ϕ_{2i} and ψ_{2i} are continuous which map $\bar{I}$ to the space of square matrices.

The proof is complete. □

5. Conclusions

In this paper, the convergence analysis of the PH heterogeneous asynchronous Newark time integrators has been studied, *i.e.*, more precisely in Theorems 4.1, 4.3 and 4.4, using suitable definitions of the approximate solutions. The analysis is mainly based on the uniform boundedness results of the numerical solutions established Lemma 3.5. It turns out that for sufficiently regular external loads, uniform convergence with respect to the norm L^∞ is achieved. Moreover, the estimates furnished in Theorem 4.4, under particular conditions on Newmark parameters and external loads, show that the error convergence is at least of the first order with respect to the macro time step. All the results in this work are obtained under the assumptions of damped domains and linear framework; it would be interesting to generalize these results by considering non-linearities or weakening the damping assumptions.

Appendix A. Proof of Theorem 3.2

It is not restrictive to replace the problem in Theorem 3.2 by:

$$\begin{cases} M\ddot{\mathcal{U}}(t) + C\dot{\mathcal{U}}(t) + K\,\mathcal{U}(t) + L^T\lambda(t) = F(t) \quad \forall t \in I \\ L\dot{\mathcal{U}}(t) = V_d(t) \quad \forall t \in I \\ \dot{\mathcal{U}}(0) = V_0 \text{ and } \mathcal{U}(0) = U_0 \quad \text{with} \quad L\,V_0 = V_d(0) \end{cases} \tag{A.1}$$

where the operators M, C and K fulfills the same requirements as M_i, C_i and K_i by taking $M = \begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}$, $K = \begin{bmatrix} K_1 & 0 \\ 0 & K_2 \end{bmatrix}$, $C = \begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}$ and $L = \begin{bmatrix} L_1 & L_2 \end{bmatrix}$. K will therefore satisfies the estimate:

$$\|K(u)\| \leq C_K \|u\|. \tag{A.2}$$

The operator L is onto and we may assume, up to a change of coordinates, that all entries of L are zeros except the diagonal entries which are equal to 1. Instead of considering the unknown $\mathcal{U}$, we consider $\mathcal{V}(t) = \dot{\mathcal{U}}(t) - \dot{\mathcal{U}}_d(t)$, where $\mathcal{U}_d \in \mathrm{W}^{2,2}\big(I, \mathbb{R}^N\big)$ (N is the length of the vector $\mathcal{U}$) satisfying $L\dot{\mathcal{U}}_d = V_d(t)$ $\forall t \in \bar{I}$ with $\dot{\mathcal{U}}_d(0) = V_0$. The system (A.1) becomes:

$$\begin{cases} M\dot{\mathcal{V}}(t) + \tilde{C}\,\mathcal{V}(t) + \tilde{K}\Big(t, \int_0^t \mathcal{V}\,\mathrm{d}u\Big) + L^T\lambda(t) = \tilde{F}(t) \quad \forall t \in I \\ L\,\mathcal{V}(t) = 0 \quad \forall t \in I \\ \mathcal{V}(0) = 0 \end{cases} \tag{A.3}$$

where $\tilde{C}$, $\tilde{K}$ and $\tilde{F}$ are defined by:

$$\begin{cases} \tilde{C}(t,u) = C\Big(u + \dot{\mathcal{U}}_d\Big) \\ \tilde{K}(t,u) = K\Big(u + \int_0^t \dot{\mathcal{U}}_d + U_0\Big) \\ \tilde{F} = F - M\ddot{\mathcal{U}}_d. \end{cases} \tag{A.4}$$

Since $\ddot{\mathcal{U}}_d$ is square integrable, it follows that $\tilde{F}$ is square integrable. Let $\mathbf{P}_L$ be the projection operator on the vector space $E_L = \ker(L)$. The system (A.1) shows that we need to prove the existence of a solution $\mathcal{V} \in \mathrm{W}^{1,2}(I, E_L)$ of the problem:

$$\begin{cases} \mathbf{P}_L M \mathbf{P}_L^T \dot{\mathcal{V}}(t) + \mathbf{P}_L \tilde{C}\big(t, \mathbf{P}_L^T \mathcal{V}(t)\big) + \mathbf{P}_L \tilde{K}\Big(t, \int_0^t \mathbf{P}_L^T \mathcal{V}\,\mathrm{d}u\Big) = \mathbf{P}_L \tilde{F}(t) \quad \forall t \in I \\ \mathcal{V}(0) = 0. \end{cases} \tag{A.5}$$

Observe that $\mathbf{P}_L M \mathbf{P}_L^T$ is linear and non-singular. Moreover, the operators $\tilde{C}$ and $\tilde{K}$ are both Carathéodory since C is single valued and maximal monotone on a finite dimensional space and K is continuous by hypothesis. Applying the Carathéodory theorem (see (A61) in [33]), the problem (A.5) admits a continuous solution $\mathcal{V}$ in some interval $[0,\delta) \subset I$ where $\dot{\mathcal{V}}$ exists a.e. in $J = (0,\delta)$. Now, we prove that we may choose $\delta = T$.

Multiplying each side of the first equation (A.5) by $\mathcal{V}$. Using the monotonicity of C, we obtain for $t < \delta$:

$$\begin{aligned} c_1\|\mathcal{V}(t)\|^2 + \psi\Big(\int_0^t \mathbf{P}_L^T \mathcal{V}\,\mathrm{d}u + \int_0^t \dot{\mathcal{U}}_d + U_0\Big) &\le \psi(U_0) + \Big(\mathbf{P}_L M \mathbf{P}_L^T \dot{\mathcal{V}}, \mathcal{V}\Big)_{2,(0,t)} \\ &\quad + \Big(\mathbf{P}_L \tilde{C}(\cdot, \mathbf{P}_L^T \mathcal{V}) - \mathbf{P}_L \tilde{C}(\cdot, \mathbf{P}_L^T 0), \mathcal{V}\Big)_{2,(0,t)} \\ &\quad + \Big(\mathbf{P}_L \tilde{K}\Big(\cdot, \int_0^\cdot \mathbf{P}_L^T \mathcal{V}\,\mathrm{d}u\Big), \mathcal{V} + \dot{\mathcal{U}}_d\Big)_{2,(0,t)} \\ &= \Big(\mathbf{P}_L \tilde{F}, \mathcal{V}\Big)_{2,(0,t)} + \psi(U_0) + \Big(\mathbf{P}_L \tilde{K}\Big(\cdot, \int_0^\cdot \mathbf{P}_L^T \mathcal{V}\,\mathrm{d}u\Big), \dot{\mathcal{U}}_d\Big)_{2,(0,t)} \\ &\quad - \Big(\mathbf{P}_L \tilde{C}(\cdot, \mathbf{P}_L^T 0), \mathcal{V}\Big)_{2,(0,t)} \end{aligned} \tag{A.6}$$

where c_1 only depends on M. Define the continuous function ϕ by $\phi(t) = \max_{[0,t]}\|\mathcal{V}(s)\|^2$. Since ψ is nonnegative, combining the estimate equation (3.3) and the previous inequality equation (A.6) and using Cauchy's and Young's inequalities it is not difficult to prove:

$$c_1\phi(t) \le c_2 T + \Big\|\tilde{F}\Big\|^2_{\mathrm{L}^2(I)} + \psi(U_0) + c_3 T \int_0^t \phi(u)\,\mathrm{d}u \tag{A.7}$$

where $c_2 > 0$ and $c_3 > 0$ constants independent of t in J and depends on $\mathcal{U}_d$, C_K and the vector $\mathbf{P}_L \tilde{C} \mathbf{P}_L^T 0$. By Gronwall lemma, a constant c_4 (independent of $t \in J$ and δ) exists such that:

$$\sup_{s \in (0,t)} \|\mathcal{V}(s)\| \le c_4. \tag{A.8}$$

Combining equation (A.8) and the properties of the operators involved in equation (A.5), we have $\dot{\mathcal{V}} \in \mathrm{L}^2((0,\delta), E_L)$ and $\mathcal{V}$ has a limit at δ^- (the solution may be extended to δ), hence, the parameter δ may be chosen equal to T. Moreover, $\mathcal{V}$ may be extended by continuity to T with $\mathcal{V}(T) = \lim_{t\to T^-} \mathcal{V}(t)$ and $\dot{\mathcal{V}} \in \mathrm{L}^2(I, E_L)$ by equation (A.5).

Now, let $\mathbf{L}_L$ be the space defined by:

$$\mathbf{L}_L = \big\{u \in \mathrm{L}^2\big(J, \mathbb{R}^N\big) \,|\, Lu = 0 \quad a.e.\big\}. \tag{A.9}$$

It is clear that $\mathbf{L}_L^\perp$ coincides with $\left\{L^T v \,|\, v \in \mathrm{L}^2\big(J, \mathbb{R}^{N_\lambda}\big)\right\}$. Moreover, we have:

$$M\mathbf{P}_L^T\,\dot{\mathcal{V}} + \tilde{C}\big(\cdot, \mathbf{P}_L^T\,\mathcal{V}\big) + \tilde{K}\left(\cdot, \int_0^\cdot \mathbf{P}_L^T \mathcal{V}\,\mathrm{d}u\right) - F \in \mathbf{L}_L^\perp. \tag{A.10}$$

Thus, there exists $\lambda \in \mathrm{L}^2\big(J, \mathbb{R}^{N_\lambda}\big)$ such that:

$$M\mathbf{P}_L^T\,\dot{\mathcal{V}} + \tilde{C}\big(\cdot, \mathbf{P}_L^T\,\mathcal{V}\big) + \tilde{K}\left(\cdot, \int_0^\cdot \mathbf{P}_L^T \mathcal{V}\,\mathrm{d}u\right) - F = -L^T\lambda. \tag{A.11}$$

Finally, if K_1 and K_2 are linear symmetric and positive semi-definite, then K satisfies the same properties. The uniqueness of the solution is ensured. Indeed, if (U_1, λ_1) and (U_2, λ_2) are two solutions. Using the notations of (A.5), we obtain:

$$\begin{aligned}\Big(M\dot{\mathcal{V}}_1 - M\dot{\mathcal{V}}_2, \mathcal{V}_1 - \mathcal{V}_2\Big)_{2,I} + \Big(\tilde{C}(\cdot, \mathcal{V}_1) - \tilde{C}(\cdot, \mathcal{V}_2), \mathcal{V}_1 - \mathcal{V}_2\Big)_{2,I}& \\ + \left(K\int_0^\cdot \mathcal{V}_1\,\mathrm{d}u - K\int_0^\cdot \mathcal{V}_2\,\mathrm{d}u, \mathcal{V}_1 - \mathcal{V}_2\right)_{2,I} = 0.&\end{aligned} \tag{A.12}$$

By the monotoniciy of $\tilde{C}$ and the fact that M is positive definite, we have:

$$\|\mathcal{V}_1 - \mathcal{V}_2\|_{\mathrm{L}^2(I)} \leq 0. \tag{A.13}$$

As consequence, by the continuity on $\bar{I}$, we have $\mathcal{V}_1 = \mathcal{V}_2$. Since L^T is one-to-one, we conclude $\lambda_1 = \lambda_2$ $(a.e.)$.

Appendix B. Uniform boundedness when $V_d \neq 0$

The purpose is to give some elements of the proof of the boundedness results as described in Lemma 3.5 under the hypothesis $V_d \in \mathrm{W}^{2,2}(I, \mathbb{R}^{N_\lambda})$ and $\ddot{V}_d \in BV(\bar{I}, \mathbb{R}^{N_\lambda})$. First we consider (U_{1d}, U_{2d}) as defined in the paragraph before Lemma 3.5. Notice that we can choose (U_{1d}, U_{2d}) such that $\ddot{U}_{id} \in BV(I, \mathbb{R}^{N_i})$ for each $i = 1, 2$. Now, we define the vectors $(U_{id}^j)_{0\leq j\leq n_i}$, $(\dot{U}_{id}^j)_{0\leq j\leq n_i}$ and $(\ddot{U}_{id}^j)_{0\leq j\leq n_i}$, with $i = 1, 2$, by:

$$\begin{cases} U_{id}^j = U_{id}^{j-1} + h_1\,\dot{U}_{id}^{j-1} + h_i^2\,\big(\frac{1}{2} - \beta_i\big)\,\ddot{U}_{id}^{j-1} + h_i^2\,\beta_i\,\ddot{U}_{id}^j \\ \dot{U}_{id}^j = \dot{U}_{id}^{j-1} + h_i\,(1-\gamma_i)\,\ddot{U}_{id}^{j-1} + h_i\gamma_i\,\ddot{U}_{id}^j \\ \dot{U}_{id}^j = \dot{U}_{id}\Big(t_j^{(i)}\Big) \quad \forall\, 0 \leq j \leq n_i \\ U_{id}^0 = U_{id}(0) \\ \ddot{U}_{id}^0 = \ddot{U}_{id}(0). \end{cases} \tag{B.1}$$

We also define for every $0 \leq j \leq n_i$:

$$\begin{cases} \mathcal{V}_i^j = \mathcal{U}_i^j - U_{id}^j \\ \dot{\mathcal{V}}_i^j = \dot{\mathcal{U}}_i^j - \dot{U}_{id}^j \\ \ddot{\mathcal{V}}_i^j = \ddot{\mathcal{U}}_i^j - \ddot{U}_{id}^j \end{cases} \tag{B.2}$$

with $i \in \{1, 2\}$. Hence, we end up with the following system:

$$\begin{cases} M_i\,\ddot{\mathcal{V}}_i^j + C_i\,\dot{\mathcal{V}}_i^j + K_i\,\mathcal{V}_i^j + L_i^T\lambda_i^j = F_i^j - M_i\ddot{U}_{id}^j - C_i\dot{U}_{id}^j - K_iU_{id}^j \\ \mathcal{V}_i^j = \mathcal{V}_i^{j-1} + h_i\,\dot{\mathcal{V}}_i^{j-1} + h_i^2\,\big(\frac{1}{2} - \beta_i\big)\,\ddot{\mathcal{V}}_1^{j-1} + h_i^2\,\beta_i\,\ddot{\mathcal{V}}_i^j \\ \dot{\mathcal{V}}_i^j = \dot{\mathcal{V}}_i^{j-1} + h_i\,(1-\gamma_i)\,\ddot{\mathcal{V}}_i^{j-1} + h_i\gamma_i\,\ddot{\mathcal{V}}_i^j \\ \mathcal{V}_i^0 = U_{0i} - U_{id}(0) \\ \dot{\mathcal{V}}_i^0 = V_{0i} - \dot{U}_{id}(0). \end{cases} \tag{B.3}$$

The constraint relation (2.8) becomes:

$$L_1\dot{\mathcal{V}}_1^{km} + L_2\dot{\mathcal{V}}_2^{k} = 0 \quad \text{for every integer k} \geq 0. \tag{B.4}$$

Taking the right hand side of the first equation (B.3) as $\tilde{F}_i^j$, *i.e.*, $\tilde{F}_i^j = F_i^j - M_i\ddot{U}_{id}^j - C_i\dot{U}_{id}^j - K_iU_{id}^j$ and taking into account the proof of Lemma 3.5, it is sufficient to establish, for every $i = 1, 2$, that:

$$\sup_{(h_1,h_2)\in S_h} \frac{1}{h_1}\sum_{j=1}^{n_i}\left\|\tilde{F}_i^j - \tilde{F}_i^{j-1}\right\|^2 \leq C(F_i, U_{id}) \tag{B.5}$$

where $C(F_i, U_{id})$ is a constant depending only on F_i and U_{id}. For this purpose, we have to estimate the error of $\ddot{U}_{id}^j - \ddot{U}_{id}(t_j^{(i)})$.

Using the second equation in (B.1) and taking into account $\ddot{U}_{id}^0 = \ddot{U}_{id}(0)$, we obtain for $i \in \{1, 2\}$ and every integer $j \in [\![1, n_i]\!]$:

$$\ddot{U}_{id}^j - \ddot{U}_{id}\left(t_j^{(i)}\right) = \frac{1}{\gamma_i}\sum_{l=0}^{j-1}(-1)^l\left(\frac{1-\gamma_i}{\gamma_i}\right)^l \eta_{i(j-l)} \tag{B.6}$$

with:

$$\eta_{ij} = \frac{1}{h_i}\left(\dot{U}_{id}\left(t_j^{(i)}\right) - \dot{U}_{id}\left(t_{j-1}^{(i)}\right)\right) - \gamma_i\ddot{U}_{id}\left(t_j^{(i)}\right) - (1-\gamma_i)\ddot{U}_{id}\left(t_{j-1}^{(i)}\right). \tag{B.7}$$

Next, we have to look for an upper bound of η_{ij}. The term η_{ij} can be written, for every $j \in [\![1, n_i]\!]$, as:

$$\begin{aligned}
\eta_{ij} &= \frac{1}{h_i}\int_{t_{j-1}^{(i)}}^{t_j^{(i)}} \ddot{U}_{id}(t)\,\mathrm{d}t - \gamma_i\ddot{U}_{id}\left(t_j^{(i)}\right) - (1-\gamma_i)\ddot{U}_{id}\left(t_{j-1}^{(i)}\right) \\
&= \frac{\gamma_i}{h_i}\int_{t_{j-1}^{(i)}}^{t_j^{(i)}}\int_{t_j^{(i)}}^{t} \dddot{U}_{id}(s)\,\mathrm{d}s\,\mathrm{d}t + \frac{1-\gamma_i}{h_i}\int_{t_{j-1}^{(i)}}^{t_{j-1}^{(i)}}\int_{t_{j-1}^{(i)}}^{t} \dddot{U}_{id}(s)\,\mathrm{d}s\,\mathrm{d}t \\
&= -\frac{\gamma_i}{h_i}\int_{t_{j-1}^{(i)}}^{t_j^{(i)}}\left(t - t_{j-1}^{(i)}\right)\dddot{U}_{id}\,\mathrm{d}t - \frac{1-\gamma_i}{h_i}\int_{t_{j-1}^{(i)}}^{t_j^{(i)}}\left(t - t_j^{(i)}\right)\dddot{U}_{id}\,\mathrm{d}t.
\end{aligned} \tag{B.8}$$

We have to distinguish two cases: $\gamma_i > \frac{1}{2}$ and $\gamma_i = \frac{1}{2}$. In the first case, it is seen that $0 \leq \frac{|1-\gamma_i|}{\gamma_i} < 1$ and $\dddot{U}_{id}$ is bounded (since $\dddot{U}_{id} \in BV(\bar{I}, \mathbb{R}^{N_i})$). Thus:

$$\|\eta_{ij}\| \leq \max(\gamma_i, |1-\gamma_i|)\|\dddot{U}_{id}\|_{\mathrm{L}^\infty(I)}h_i \quad \forall 1 \leq j \leq n_i. \tag{B.9}$$

Combining with (B.6), we obtain:

$$\left\|\ddot{U}_{id}^j - \ddot{U}_{id}\left(t_j^{(i)}\right)\right\| \leq \frac{\max(\gamma_i, |1-\gamma_i|)}{\gamma_i - |1-\gamma_i|}\|\dddot{U}_{id}\|_{\mathrm{L}^\infty(I)}h_i \quad \forall 1 \leq j \leq n_i. \tag{B.10}$$

Now, if $\gamma_i = \frac{1}{2}$, using (B.6) and the Riemann–Stieltjes integral we have:

$$\begin{aligned}
\|\eta_{ij}\| &= \left\|-\frac{1}{2h_i}\int_{t_{j-1}^{(i)}}^{t_j^{(i)}}\left(t - t_{j-1}^{(i)}\right)\dddot{U}_{id}\,\mathrm{d}t - \frac{1}{2h_i}\int_{t_{j-1}^{(i)}}^{t_j^{(i)}}\left(t - t_j^{(i)}\right)\dddot{U}_{id}\,\mathrm{d}t\right\| \\
&= \left\|\frac{1}{2h_i}\int_{t_{j-1}^{(i)}}^{t_j^{(i)}}\left(t - t_{j-1}^{(i)}\right)\left(t - t_j^{(i)}\right)\mathrm{d}\dddot{U}_{id}\right\|
\end{aligned}$$

$$\leq \frac{1}{2} V_{[t_{j-1}^{(i)}, t_j^{(i)}]}(\dddot{U}_{id}) h_i \tag{B.11}$$

where $V_J(f)$ is the total variation of a function f defined on the interval J. Combining once again with (B.6), we obtain:

$$\left\| \ddot{U}_{id}^j - \ddot{U}_{id}\left(t_j^{(i)}\right) \right\| \leq V_{\bar{I}}(\dddot{U}_{id}) h_i \quad \forall 1 \leq j \leq n_i. \tag{B.12}$$

Moreover, by virtue of (B.10) and (B.12):

$$\begin{aligned} \sup_{(h_1,h_2)\in S_h} \frac{1}{h_1} \sum_{j=1}^{n_i} \left\| \ddot{U}_{id}^j - \ddot{U}_{id}^{j-1} \right\|^2 \leq \sup_{(h_1,h_2)\in S_h} \frac{3}{h_1} &\left[\sum_{j=1}^{n_i} \left\| \ddot{U}_{id}^j - \ddot{U}_{id}\left(t_j^{(i)}\right) \right\|^2 + \sum_{j=1}^{n_i} \left\| \ddot{U}_{id}\left(t_j^{(i)}\right) - \ddot{U}_{id}\left(t_{j-1}^{(i)}\right) \right\|^2 \right. \\ &\left. + \sum_{j=1}^{n_i} \left\| \ddot{U}_{id}^{j-1} - \ddot{U}_{id}\left(t_{j-1}^{(i)}\right) \right\|^2 \right] \\ &\leq c_T \left[\|\ddot{U}_{id}\|^2_{\mathrm{L}^\infty(I)} + V_{\bar{I}}(\dddot{U}_{id})^2 + \left(\int_I (\dddot{U}_{id})^2 \,\mathrm{d}t \right) \right] \end{aligned} \tag{B.13}$$

where c_T is a constant depending only on γ_i and T. In addition, since $(\dot{U}_{id}^j)_{0\leq j\leq n_i}$ and $(\ddot{U}_{id}^j)_{0\leq j\leq n_i}$ are uniformly bounded using the regularity of U_{id} and the equations (B.1), (B.10) and (B.12), we obtain similar estimates for $(U_{id}^j)_{0\leq j\leq n_i}$ and $(\dot{U}_{id}^j)_{0\leq j\leq n_i}$.

As a consequence, the proof of Lemma 3.5 shows that $\|\bar{\ddot{\mathcal{V}}}_i^{h_i}\|_{\mathrm{L}^\infty(I)}$, $\|\bar{\dot{\mathcal{V}}}_i^{h_i}\|_{\mathrm{L}^\infty(I)}$, $\|\bar{\mathcal{V}}_i^{h_i}\|_{\mathrm{L}^\infty(I)}$ and $\|\bar{\lambda}_i^{h_i}\|_{\mathrm{L}^\infty(I)}$ are uniformly bounded. Again, from the uniform boundedness of $(U_{id}^j)_{0\leq j\leq n_i}$, $(\dot{U}_{id}^j)_{0\leq j\leq n_i}$ and $(\ddot{U}_{id}^j)_{0\leq j\leq n_i}$ using the regularity of U_{id} and the equations (B.1), (B.10) and (B.12), we conclude that $\|\bar{\ddot{\mathcal{U}}}_i^{h_i}\|_{\mathrm{L}^\infty(I)}$, $\|\bar{\dot{\mathcal{U}}}_i^{h_i}\|_{\mathrm{L}^\infty(I)}$, $\|\bar{\mathcal{U}}_i^{h_i}\|_{\mathrm{L}^\infty(I)}$ and $\|\bar{\lambda}_i^{h_i}\|_{\mathrm{L}^\infty(I)}$ are uniformly bounded. This completes the proof.

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