ESAIM: M2AN 57 (2023) 271–297
https://doi.org/10.1051/m2an/2022068

ESAIM: Mathematical Modelling and Numerical Analysis
www.esaim-m2an.org

ANALYSIS OF AN HDG METHOD FOR THE NAVIER–STOKES EQUATIONS WITH DIRAC MEASURES

Haitao Leng*

Abstract. In two dimensions, we analyze a hybridized discontinuous Galerkin (HDG) method for the Navier–Stokes equations with Dirac measures. The approximate velocity field obtained by the HDG method is shown to be pointwise divergence-free and $H(\mathrm{div})$-conforming. Under a smallness assumption on the continuous and discrete solutions, *a posteriori* error estimator, that provides an upper bound for the L^2-norm in the velocity, is proposed in the convex domain. In the polygonal domain, reliable and efficient *a posteriori* error estimator for the $W^{1,q}$-seminorm in the velocity and L^q-norm in the pressure is also proved. Finally, a Banach's fixed point iteration method and an adaptive HDG algorithm are introduced to solve the discrete system and show the performance of the obtained *a posteriori* error estimators.

Mathematics Subject Classification. 65J15, 65N50, 65N15, 76D05.

Received December 24, 2021. Accepted September 5, 2022.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be an open and bounded polygonal domain with Lipschitz boundary $\partial\Omega$, then we consider the following Navier–Stokes equations in this paper

$$-\nu\Delta\mathbf{u} + \nabla\cdot(\mathbf{u}\otimes\mathbf{u}) + \nabla p = \mathbf{F}\delta_{x_0} \quad \text{in } \Omega, \tag{1a}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in } \Omega, \tag{1b}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \partial\Omega. \tag{1c}$$

The unknowns $\mathbf{u}$ and p denote the velocity and pressure of a fluid, $\nu > 0$ is the dynamic viscosity, $\otimes$ denotes the tensor product, δ_{x_0} is the Dirac delta supported at the interior point $x_0 \in \Omega$ and $\mathbf{F} \in \mathbb{R}^2$. An instance where singular forces appear is in the modelling of the motion of active thin structures [1, 2], there the right hand side is a linear combination of Dirac deltas supported at interior points of Ω. Another instance emerges in PDE-constrained optimal control problems with point values of the velocity in the objective functional [3, 4]. The idea in the problem is that one tries to optimize the flow at a certain point. As for other applications, we mention such as the use of flagella by sessile organisms to generate feeding currents [5] and modeling the flow of a fluid through structures with singular sources [6, 7].

Keywords and phrases. Hybridized discontinuous Galerkin method, *a posteriori* error estimate, Navier–Stokes equation, divergence-free, Dirac.

School of Mathematics and information sciences, Guangzhou University, Guangzhou, Guangdong, P.R. China.
*Corresponding author: lhtdemail@163.com

The stationary Navier–Stokes system describing the motion of an incompressible fluid is one of the most important models in mathematical physics, hence the study of approximation techniques, at least in energy-type spaces, has received a tremendous attention, see [8–11] and references therein for more detail. As for the system (1), since the singular nature of δ_{x_0}, it must be understood in a completely different way. In [12, 13], *a priori* without any convergence rate and *a posteriori* error estimates for finite element methods of system (1) were obtained in weighted spaces $H_0^1(w,\Omega) \times L^2(w,\Omega)$ under a smallness assumption on the continuous and discrete solutions. In [14], Lepe *et al.* established reliable and efficient *a posteriori* error estimator for spaces $\left(W^{1,q}(\Omega)\right)^2 \times L^q(\Omega)$, where $q \in [4/3-\varepsilon, 2)$ and $\varepsilon > 0$ is a constant depending on the domain Ω. For the elliptic and Stokes versions of system (1), we refer to [15–22] to the interested readers.

In 2012, Houston and Wihler [23] studied a discontinuous Galerkin (DG) method for the elliptic version of system (1). *A priori* and *a posteriori* error estimates were proved under assumption that the point x_0 lies in the interior of an element and $\Omega \subset \mathbb{R}^2$. It is well known that DG method is very flexible when it is applied to solve the partial differential equations, but too many globally coupled degrees of freedom are always involved in the final discrete system. In order to overcome this issue, a new class of hybridizable discontinuous Galerkin (HDG) methods, which keeps the advantage of DG methods and results in a system with significantly reduced degrees of freedom, was proposed by Cockburn *et al.* [24] for the mixed formulation of elliptic equations. Currently, HDG methods have been widely used to lots of problems such as convection diffusion [25–27], linear elasticity [28, 29], Maxwell [30, 31], fluid flow [32–34] and optimal control [35], etc. Although HDG methods have been successfully used in many fields, we find that there only exist few works involving *a posteriori* error estimates, and the pioneer work was made by Cockburn and Zhang [36, 37] for second order elliptic problem. In papers [38] and [39], the authors obtained *a priori* and *a posteriori* error estimates for HDG methods for the gradient-velocity-pressure and velocity-pseudostress formulations of Brinkman equations. For the gradient-velocity-pressure formulation of Oseen and Navier–Stokes equations, efficient and reliable *a posteriori* error estimator was established in [40] and [41]. It is worth noting that *a priori* and *a posteriori* error estimates for an HDG method for elliptic problems with Dirac measures were derived by the author [42].

In [43–45], Rhebergen and Wells proposed an HDG method for the velocity-pressure formulation of Stokes and Navier–Stokes equations with divergence-free and $H(\mathrm{div})$-conforming velocity field. Based on this, we provide an HDG discretization for system (1) in this paper. Under a smallness assumption on the discrete solution, the existence and uniqueness of HDG solutions are proved by the Banach's fixed point theorem. Using duality argument, *a posteriori* error estimator, that provides an upper bound for L^2-norm in the velocity, is introduced in the convex domain. In the polygonal domain, efficient and reliable *a posteriori* error estimator for $W^{1,q}$-seminorm in the velocity and L^q-norm in the pressure is also established, where $q \in \left(\frac{4+\varepsilon}{3+\varepsilon}, 2\right)$ and $\varepsilon > 0$ is a constant depending on the domain Ω. Finally, a Banach's fixed point iteration method and an adaptive HDG algorithm are given to solve the discrete system and show the performance of the obtained *a posteriori* error estimators. To the best of our knowledge, this is the first work on pressure-robust HDG methods for the Navier–Stokes equations with Dirac measures.

Compared with [12–14], we need introducing the Oswald interpolation to prove *a posteriori* error estimator for $W^{1,q}$-seminorm in the velocity and L^q-norm in the pressure. For L^2-norm in the velocity, we also provide *a posteriori* error estimator under assumption that the domain Ω is convex. It is worth noting that *a posteriori* error estimators obtained in this paper incorporate a term $\|\mathbf{u}_h - \widehat{\mathbf{u}}_h\|_{L^q(\partial K)}$ that does not appear in [12–14] (see Lems. 4.4 and 4.5). Since the approximate velocity filed is pointwise divergence-free, the term $\|\nabla \cdot \mathbf{u}_h\|_{L^q(K)}$ included in [12–14] is not contained by this paper.

The rest of this article is arranged as follows: In Section 2, notation and the weak formulation of system (1) are introduced. In Section 3, we provide the HDG discretization of system (1) and prove the existence and uniqueness of discrete solutions. In Section 4, *a posteriori* error estimators for the errors in the velocity and pressure are obtained. In Section 5, a Banach's fixed point iteration method and an adaptive HDG algorithm are given to show the performance of the obtained *a posteriori* error estimators by two numerical examples. We end this paper by some conclusions in Section 6.

Throughout this paper, let c with or without subscript be a generic positive constant independent of the mesh size, which may be different on each occasion. For ease of exposition, we denote $A \leq cB$ by $A \lesssim B$ and $A \approx B$ by $A \lesssim B \lesssim A$.

2. Notation and preliminaries

For any bounded and open set $D \subset \mathbb{R}^d$ or $D \subset \mathbb{R}^{d-1}$, $W^{s,q}(D)$ denotes the standard Sobolev space with norm $\|\cdot\|_{s,q,D}$ and seminorm $|\cdot|_{s,q,D}$. When $q = 2$, the Sobolev space $W^{s,2}(D)$ denotes by $H^s(D)$ with norm $\|\cdot\|_{s,D}$ and seminorm $|\cdot|_{s,D}$. If we further have $s = 0$, $H^0(D)$ coincides with $L^2(D)$, and the inner product is described by $(\cdot,\cdot)_D$ for $D \subset \mathbb{R}^d$ or $\langle\cdot,\cdot\rangle_D$ for $D \subset \mathbb{R}^{d-1}$. For $q \in (1,\infty)$, let q' be the conjugate number of q such that $\frac{1}{q}+\frac{1}{q'} = 1$, then the dual space to $W_0^{s,q}(\Omega)$ is denoted by $W^{-s,q'}(\Omega)$, where $W_0^{s,q}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in the space $W^{s,q}(\Omega)$ (see [46]). If no confusion induced, we use $\langle\cdot,\cdot\rangle$ to denote the duality pairing between spaces $W^{-s,q'}(\Omega)$ and $W_0^{s,q}(\Omega)$. We define $L_0^q(\Omega) := \left\{v \in L^q(\Omega) : \int_\Omega v\,\mathrm{d}x = 0\right\}$ and $H(\mathrm{div},\Omega) := \left\{\mathbf{v} \in \left(L^2(\Omega)\right)^d : \nabla\cdot\mathbf{v} \in L^2(\Omega)\right\}$, where $\nabla\cdot$ is the divergence operator.

The weak formulation of the Navier–Stokes system (1) reads as follows: Find $(\mathbf{u},p) \in \left(W_0^{1,q}(\Omega)\right)^2 \times L_0^q(\Omega)$ $(1 < q < 2)$ such that

$$a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) + o(\mathbf{u},\mathbf{u};\mathbf{v}) = \langle \mathbf{F}\delta_{x_0},\mathbf{v}\rangle, \quad b(\mathbf{u},q) = 0, \tag{2}$$

for any $\mathbf{v} \in \left(W_0^{1,q'}(\Omega)\right)^2$ and $q \in L_0^{q'}(\Omega)$, where

$$a(\mathbf{u},\mathbf{v}) = (\nu\nabla\mathbf{u},\nabla\mathbf{v})_\Omega, \quad o(\mathbf{u},\mathbf{w};\mathbf{v}) = -(\mathbf{u}\otimes\mathbf{w},\nabla\mathbf{v})_\Omega, \quad b(\mathbf{u},q) = -(q,\nabla\cdot\mathbf{u})_\Omega.$$

From Proposition 2 of [14], we know that if the quantity $\nu^{-2}\|\mathbf{F}\delta_{x_0}\|_{-1,q,\Omega}$ is sufficiently small, the weak formulation (2) exists a unique solution $(\mathbf{u},p) \in \left(W_0^{1,q}(\Omega)\right)^2 \times L_0^q(\Omega)$ with $q \in \left(\frac{4+\varepsilon}{3+\varepsilon},2\right)$, where $\varepsilon > 0$ is a constant depending on the domain Ω. Moreover, the following estimate holds

$$\|\nabla\mathbf{u}\|_{0,q,\Omega} \lesssim \nu^{-1}\|\mathbf{F}\delta_{x_0}\|_{-1,q,\Omega}, \tag{3}$$

Remark 2.1. In the proof of the existence and uniqueness of solutions for system (1), the boundedness

$$o(\mathbf{u},\mathbf{u};\mathbf{v}) \leq \|\mathbf{u}\|_{0,2q,\Omega}^2\|\nabla\mathbf{v}\|_{0,q',\Omega} \lesssim \|\nabla\mathbf{u}\|_{0,q,\Omega}^2\|\nabla\mathbf{v}\|_{0,q',\Omega} \tag{4}$$

for the convective term is needed (see [14], Prop. 1). Since $\delta_{x_0} \in W^{-1,q}(\Omega)$ with $q \in \left(1,\frac{d}{d-1}\right)$, the Stokes version of system (1) exists a unique weak solution $(\mathbf{u},p) \in \left(W_0^{1,q}(\Omega)\right)^d \times L_0^q(\Omega)$ for $q \in \left(\frac{6-d+\varepsilon}{5-d+\varepsilon},\frac{d}{d-1}\right)$ (see [47], Thm. 1.1, Rem. 1.1). By embedding theorem ([8], Thm. II.3.2), we know that if $q < d$, $W_0^{1,q}(\Omega) \hookrightarrow L^r(\Omega)$ for $r \in \left[q,\frac{dq}{d-q}\right]$. Hence for the three-dimensional case, we can not expect $W_0^{1,q}(\Omega) \hookrightarrow L^{2q}(\Omega)$ for $q \in \left(\frac{3+\varepsilon}{2+\varepsilon},\frac{3}{2}\right)$, therefore the boundedness (4) can not be guaranteed for the convective term. This is why our analysis is restricted to the two-dimensional case.

3. The hybridized discontinuous Galerkin method

Let $\mathcal{T}_h$ be a conforming and shape-regular triangulation of the domain Ω. Denote $\mathcal{E}_h^o$ the set of all interior edges of $\mathcal{T}_h$ and $\mathcal{E}_h^\partial$ the set of all boundary edges. We define $\mathcal{E}_h = \mathcal{E}_h^o \bigcup \mathcal{E}_h^\partial$ and $\partial\mathcal{T}_h := \{\partial K : K \in \mathcal{T}_h\}$, where ∂K is the boundary of the element K. For any element $K \in \mathcal{T}_h$ and edge $F \in \mathcal{E}_h$, let h_K and h_F be the diameters of K and F. For the interior edge $F \in \mathcal{E}_h^o$, we define the jumps $[v]|_F$ and $[(\nabla\mathbf{v})\mathbf{n}]|_F$ by $[v]|_F = v^+\mathbf{n}^+ + v^-\mathbf{n}^-$ and $[(\nabla\mathbf{v})\mathbf{n}]|_F = (\nabla\mathbf{v}^+)\mathbf{n}^+ + (\nabla\mathbf{v}^-)\mathbf{n}^-$, where $v^+,\mathbf{v}^+$ and $v^-,\mathbf{v}^-$ are the traces of v and $\mathbf{v}$ on $F = \partial K^+ \cap \partial K^-$

and $\mathbf{n}^+$ and $\mathbf{n}^-$ are the outward unit normal vector to $F = \partial K^+ \cap \partial K^-$. As for the interior point $x_0 \in \Omega$, let $\mathcal{T}_{x_0} := \{K \in \mathcal{T}_h : x_0 \in \overline{K}\}$ and N be the number of elements in $\mathcal{T}_{x_0}$.

Based on the partition $\mathcal{T}_h$, we provide the following discontinuous finite element spaces:

$$\begin{aligned}
V_h &= \Big\{\mathbf{v} \in \big(L^2(\Omega)\big)^2 : \mathbf{v}\big|_K \in \big(\mathcal{P}^k(K)\big)^2,\ \forall K \in \mathcal{T}_h\Big\},\\
Q_h &= \big\{q \in L_0^2(\Omega) : q\big|_K \in \mathcal{P}^{k-1}(K),\ \forall K \in \mathcal{T}_h\big\},\\
\widehat{V}_h &= \Big\{\widehat{\mathbf{v}} \in \big(L^2(\mathcal{E}_h)\big)^2 : \widehat{\mathbf{v}}\big|_F \in \big(\mathcal{P}^k(F)\big)^2\ \forall F \in \mathcal{E}_h,\ \widehat{\mathbf{v}} = 0 \text{ on } \mathcal{E}_h^\partial\Big\},\\
\widehat{Q}_h &= \big\{\widehat{q} \in L^2(\mathcal{E}_h) : \widehat{q}\big|_F \in \mathcal{P}^k(F),\ \forall F \in \mathcal{E}_h\big\},
\end{aligned}$$

where $\mathcal{P}^s(D)$ denotes the set of polynomials of degree no larger than s on the domain D. Next, we define the mesh-dependent inner products and norms:

$$\begin{aligned}
(v_1, v_2)_{\mathfrak{D}} &= \sum_{K\in\mathfrak{D}} (v_1, v_2)_K, & \langle v_1, v_2\rangle_{\partial\mathfrak{D}} &= \sum_{K\in\mathfrak{D}} \langle v_1, v_2\rangle_{\partial K},\\
\|v\|_{0,q,\mathfrak{D}} &= \left\{\sum_{K\in\mathfrak{D}} \|v\|_{0,q,K}^q\right\}^{1/q}, & \|v\|_{0,q,\partial\mathfrak{D}} &= \left\{\sum_{K\in\mathfrak{D}} \|v\|_{0,q,\partial K}^q\right\}^{1/q},
\end{aligned}$$

for $\mathfrak{D} \subset \mathcal{T}_h$ and $1 \le q < \infty$.

Then the HDG scheme of the Navier–Stokes system (1), that is almost the same with [44,48], reads as follows: Find $(\mathbf{u}_h, \widehat{\mathbf{u}}_h, p_h, \widehat{p}_h) \in V_h \times \widehat{V}_h \times Q_h \times \widehat{Q}_h$ such that

$$a_h((\mathbf{u}_h, \widehat{\mathbf{u}}_h), (\mathbf{v}_h, \widehat{\mathbf{v}}_h)) + o_h(\mathbf{u}_h, (\mathbf{u}_h, \widehat{\mathbf{u}}_h); (\mathbf{v}_h, \widehat{\mathbf{v}}_h)) + b_h((p_h, \widehat{p}_h), (\mathbf{v}_h, \widehat{\mathbf{v}}_h)) = \frac{1}{N}\sum_{K\in\mathcal{T}_{x_0}} \mathbf{F}\cdot\mathbf{v}_h\Big|_K(x_0), \tag{5a}$$

$$b_h((q_h, \widehat{q}_h), (\mathbf{u}_h, \widehat{\mathbf{u}}_h)) = 0, \tag{5b}$$

for any $(\mathbf{v}_h, \widehat{\mathbf{v}}_h, q_h, \widehat{q}_h) \in V_h \times \widehat{V}_h \times Q_h \times \widehat{Q}_h$, where the bilinear forms a_h and b_h and the trilinear form o_h are defined by

$$\begin{aligned}
a_h((\mathbf{u}_h, \widehat{\mathbf{u}}_h), (\mathbf{v}_h, \widehat{\mathbf{v}}_h)) &= (\nu\nabla\mathbf{u}_h, \nabla\mathbf{v}_h)_{\mathcal{T}_h} + \sum_{K\in\mathcal{T}_h} \frac{\alpha\nu}{h_K}\langle \mathbf{u}_h - \widehat{\mathbf{u}}_h, \mathbf{v}_h - \widehat{\mathbf{v}}_h\rangle_{\partial K}\\
&\quad - \langle \mathbf{u}_h - \widehat{\mathbf{u}}_h, \nu(\nabla\mathbf{v}_h)\mathbf{n}\rangle_{\partial\mathcal{T}_h} - \langle \nu(\nabla\mathbf{u}_h)\mathbf{n}, \mathbf{v}_h - \widehat{\mathbf{v}}_h\rangle_{\partial\mathcal{T}_h},\\
o_h(\mathbf{w}_h, (\mathbf{u}_h, \widehat{\mathbf{u}}_h); (\mathbf{v}_h, \widehat{\mathbf{v}}_h)) &= -(\mathbf{u}_h \otimes \mathbf{w}_h, \nabla\mathbf{v}_h)_{\mathcal{T}_h} + \frac{1}{2}\langle (\mathbf{w}_h\cdot\mathbf{n})(\mathbf{u}_h + \widehat{\mathbf{u}}_h), \mathbf{v}_h - \widehat{\mathbf{v}}_h\rangle_{\partial\mathcal{T}_h}\\
&\quad + \frac{1}{2}\langle |\mathbf{w}_h\cdot\mathbf{n}|(\mathbf{u}_h - \widehat{\mathbf{u}}_h), \mathbf{v}_h - \widehat{\mathbf{v}}_h\rangle_{\partial\mathcal{T}_h},\\
b_h((p_h, \widehat{p}_h), (\mathbf{v}_h, \widehat{\mathbf{v}}_h)) &= -(p_h, \nabla\cdot\mathbf{v}_h)_{\mathcal{T}_h} + \langle (\mathbf{v}_h - \widehat{\mathbf{v}}_h)\cdot\mathbf{n}, \widehat{p}_h\rangle_{\partial\mathcal{T}_h}.
\end{aligned}$$

Here $\alpha > 0$ is the stabilization parameter that is usually chosen to be large enough to ensure the stability of the discrete system (5) (see [44,48]).

Let $q_h = 0$ in (5b), we have

$$\langle (\mathbf{u}_h - \widehat{\mathbf{u}}_h)\cdot\mathbf{n}, \widehat{q}_h\rangle_{\partial\mathcal{T}_h} = 0 \quad \forall \widehat{q}_h \in \widehat{Q}_h.$$

Hence $[\mathbf{u}_h\cdot\mathbf{n}] = 0$ on $\mathcal{E}_h^o$ and $\mathbf{u}_h\cdot\mathbf{n} = 0$ on $\mathcal{E}_h^\partial$, which shows that the approximate velocity field derived by the discrete system (5) is $H(\mathrm{div})$-conforming. Let $q_h = \nabla\cdot\mathbf{u}_h$ and $\widehat{q}_h = 0$ in (5b), we can yield

$$\|\nabla\cdot\mathbf{u}_h\|_{0,\mathcal{T}_h} = 0,$$

this means that the approximate velocity field obtained by the discrete system (5) is pointwise divergence-free within an element. With divergence-free and $H(\mathrm{div})$-conforming at hand, an attractive property named pressure-robustness can be obtained (see [48]).

Before proving the existence and uniqueness of discrete solutions, we introduce a mesh-dependent seminorm that will be frequently used in the subsequent proofs

$$|||(\mathbf{v},\boldsymbol{\mu})|||_v := \|\nabla\mathbf{v}\|^2_{0,\mathcal{T}_h} + \sum_{K\in\mathcal{T}_h} \alpha h_K^{-1}\|\mathbf{v}-\boldsymbol{\mu}\|^2_{0,\partial K},$$

and set

$$\|\mathbf{v}\|_{1,h} := |||(\mathbf{v},\{\{\mathbf{v}\}\})|||_v,$$

where the average of $\mathbf{v}$, $\{\{\mathbf{v}\}\}$, is defined as follows. On the interior edge $F\in\partial K^+\cap\partial K^-$, we have $\{\{\mathbf{v}\}\} = \frac{1}{2}(\mathbf{v}^+ + \mathbf{v}^-)$, where $\mathbf{v}^+$ and $\mathbf{v}^-$ are the traces of $\mathbf{v}$ on the edge F. For the boundary edge $F\in\mathcal{E}_h^\partial$, we set $\{\{\mathbf{v}\}\}=0$. Note that $\|\mathbf{v}\|_{1,h}$ is nothing but the standard discrete H^1-norm of $\mathbf{v}$.

Lemma 3.1. *For $1\le q<\infty$, we have*

$$\|\boldsymbol{v}\|_{0,q,\Omega} \lesssim \|\boldsymbol{v}\|_{1,h} \qquad \forall \boldsymbol{v}\in V(h), \tag{6}$$

$$\|\boldsymbol{v}\|_{0,q,\Omega} \le c_1\,|||(\boldsymbol{v},\boldsymbol{\mu})|||_v \qquad \forall(\boldsymbol{v},\boldsymbol{\mu})\in V(h)\times\widehat{V}_h, \tag{7}$$

where $V(h)=V_h+\left(H_0^1(\Omega)\right)^2$. In addition, the following estimate holds

$$\left(\sum_{K\in\mathcal{T}_h} h_K\|\boldsymbol{v}\|^4_{L^4(\partial K)}\right)^{1/4} \lesssim \|\boldsymbol{v}\|_{1,h} \le c_2\,|||(\boldsymbol{v},\boldsymbol{\mu})|||_v \qquad \forall(\boldsymbol{v},\boldsymbol{\mu})\in V(h)\times\widehat{V}_h. \tag{8}$$

Proof. The inequalities (6) and (7) has been proved in Proposition A.2 of [49], Theorem 2.1 of [50] and Theorem 5.3 of [51]. As for the inequality (8), it is a direct result of (6), (7) and equation (7.7) of [52]. □

Lemma 3.2 (Trace and inverse estimates)**.** *For each element $K\in\mathcal{T}_h$ and any given nonnegative integer j, the following approximations hold*

$$|v|_{i,p,K} \le c_3 h_K^{-i+\frac{2}{p}-\frac{2}{q}}\|v\|_{0,q,K}, \qquad \forall v\in\mathcal{P}^j(K),\ i=0,1,\ 1\le p,q\le\infty, \tag{9}$$

$$\|v\|_{0,q,\partial K} \lesssim h_K^{-\frac{1}{q}}\|v\|_{0,q,K} + \|v\|^{1-1/q}_{0,q,K}\|\nabla v\|^{1/q}_{0,q,K}, \qquad \forall v\in W^{1,q}(K),\ 1\le q\le\infty. \tag{10}$$

Proof. The inverse estimate (9) can be found in Lemma 4.5.3 of [53]. The trace inequality (10) can be found in Lemma 12.15 of [54]. □

Previously it was shown ([43], Lems. 4.2, 4.3) that for sufficiently large α, the bilinear form a_h is coercive and bounded, *i.e.*, there exist constants $c_a^s>0$ and $c_a^b>0$ such that for any $(\mathbf{v}_h,\widehat{\mathbf{v}}_h),(\mathbf{w}_h,\widehat{\mathbf{w}}_h)\in V_h\times\widehat{V}_h$

$$\nu c_a^s\,|||(\mathbf{v}_h,\widehat{\mathbf{v}}_h)|||_v^2 \le a_h((\mathbf{v}_h,\widehat{\mathbf{v}}_h),(\mathbf{v}_h,\widehat{\mathbf{v}}_h)), \tag{11}$$

$$|a_h((\mathbf{w}_h,\widehat{\mathbf{w}}_h),(\mathbf{v}_h,\widehat{\mathbf{v}}_h))| \le \nu c_a^b\,|||(\mathbf{w}_h,\widehat{\mathbf{w}}_h)|||_v\,|||(\mathbf{v}_h,\widehat{\mathbf{v}}_h)|||_v. \tag{12}$$

For the trilinear form o_h, it was shown [48] that for $\mathbf{w}_h\in V_h^{\mathrm{div}}$ and $\mathbf{w}_1,\mathbf{w}_2\in V_h$, there is a constant $c_c>0$ independent of mesh sizes such that

$$o_h(\mathbf{w}_h,(\mathbf{v}_h,\widehat{\mathbf{v}}_h);(\mathbf{v}_h,\widehat{\mathbf{v}}_h)) = \frac{1}{2}\langle|\mathbf{w}_h\cdot\mathbf{n}|(\mathbf{v}_h-\widehat{\mathbf{v}}_h),(\mathbf{v}_h-\widehat{\mathbf{v}}_h)\rangle_{\partial\mathcal{T}_h} \ge 0, \tag{13}$$

and

$$|o_h(\mathbf{w}_1,(\mathbf{v}_1,\widehat{\mathbf{v}}_1);(\mathbf{v}_2,\widehat{\mathbf{v}}_2)) - o_h(\mathbf{w}_2,(\mathbf{v}_1,\widehat{\mathbf{v}}_1);(\mathbf{v}_2,\widehat{\mathbf{v}}_2))| \le c_c\|\mathbf{w}_1-\mathbf{w}_2\|_{1,h}\,|||(\mathbf{v}_1,\widehat{\mathbf{v}}_1)|||_v\,|||(\mathbf{v}_2,\widehat{\mathbf{v}}_2)|||_v, \tag{14}$$

for any $(\mathbf{v}_h,\widehat{\mathbf{v}}_h),(\mathbf{v}_1,\widehat{\mathbf{v}}_1),(\mathbf{v}_2,\widehat{\mathbf{v}}_2)\in V_h\times\widehat{V}_h$, where

$$V_h^{\mathrm{div}} := \Big\{\mathbf{v}_h\in V_h : b_h((q_h,\widehat{q}_h),(\mathbf{v}_h,0))=0,\ \forall (q_h,\widehat{q}_h)\in Q_h\times\widehat{Q}_h\Big\}.$$

With all these ingredients at hand, now we are ready to prove the existence and uniqueness of discrete solutions. To this end, we define a mapping $\mathcal{F}: V_h^{\mathrm{div}}\times\widehat{V}_h\to V_h^{\mathrm{div}}\times\widehat{V}_h$ as follows: for given $(\mathbf{w}_h,\widehat{\mathbf{w}}_h)\in V_h^{\mathrm{div}}\times\widehat{V}_h$, let $(\mathbf{u}_h,\widehat{\mathbf{u}}_h)=\mathcal{F}(\mathbf{w}_h,\widehat{\mathbf{w}}_h)\in V_h^{\mathrm{div}}\times\widehat{V}_h$ be part of the solution of

$$a_h((\mathbf{u}_h,\widehat{\mathbf{u}}_h),(\mathbf{v}_h,\widehat{\mathbf{v}}_h)) + o_h(\mathbf{w}_h,(\mathbf{u}_h,\widehat{\mathbf{u}}_h);(\mathbf{v}_h,\widehat{\mathbf{v}}_h)) + b_h((p_h,\widehat{p}_h),(\mathbf{v}_h,\widehat{\mathbf{v}}_h)) = \frac{1}{N}\sum_{K\in\mathcal{T}_{x_0}}\mathbf{F}\cdot\mathbf{v}_h|_K(x_0), \tag{15}$$

for any $(\mathbf{v}_h,\widehat{\mathbf{v}}_h)\in V_h\times\widehat{V}_h$. Since the bilinear form b_h is inf-sup stable ([45], Lem. 8), the above linear discrete system (15) exists a unique solution. Obviously, $(\mathbf{u}_h,\widehat{\mathbf{u}}_h)$ is a solution of the discrete system (5) if and only if it is a fixed point of the mapping $\mathcal{F}$.

Theorem 3.1. *Assuming*

$$\nu^{-2}\frac{c_c c_1 c_2 c_3}{(c_a^s)^2}|\boldsymbol{F}| < \frac{1}{6},$$

where c_1, c_2 and c_3 are the positive constants defined in (7)–(9), then there exists a unique solution $(\mathbf{u}_h,\widehat{\mathbf{u}}_h,p_h,\widehat{p}_h)\in V_h\times\widehat{V}_h\times Q_h\times\widehat{Q}_h$ for the discrete system (5) such that

$$|||(\mathbf{u}_h,\widehat{\mathbf{u}}_h)|||_v \le \nu^{-1}\frac{2c_1c_3}{c_a^s}|\boldsymbol{F}|. \tag{16}$$

Proof. For given $(\mathbf{w}_h,\widehat{\mathbf{w}}_h)\in V_h^{\mathrm{div}}\times\widehat{V}_h$, let $(\mathbf{u}_h,\widehat{\mathbf{u}}_h)=\mathcal{F}(\mathbf{w}_h,\widehat{\mathbf{w}}_h)\in V_h^{\mathrm{div}}\times\widehat{V}_h$, then (11), (13) and (15) yield

$$\nu c_a^s\,|||(\mathbf{u}_h,\widehat{\mathbf{u}}_h)|||_v^2 \le a_h((\mathbf{u}_h,\widehat{\mathbf{u}}_h),(\mathbf{u}_h,\widehat{\mathbf{u}}_h)) \le \frac{1}{N}\sum_{K\in\mathcal{T}_{x_0}}\mathbf{F}\cdot\mathbf{u}_h|_K(x_0) \le \frac{1}{N}|\mathbf{F}|\sum_{K\in\mathcal{T}_{x_0}}\|\mathbf{u}_h\|_{L^\infty(K)}. \tag{17}$$

For a given partition $\mathcal{T}_h$, it is easily observed that there exists a real number $r>0$ such that $h_K^{-\frac{2}{r}}\le 2$ for any $K\in\mathcal{T}_{x_0}$. Hence the inverse estimate (9) and (7) yield

$$\frac{1}{N}\sum_{K\in\mathcal{T}_{x_0}}\|\mathbf{u}_h\|_{L^\infty(K)} \le \frac{c_3}{N}\sum_{K\in\mathcal{T}_{x_0}} h_K^{-\frac{2}{r}}\|\mathbf{u}_h\|_{0,r,K} \le 2c_1c_3\,|||(\mathbf{u}_h,\widehat{\mathbf{u}}_h)|||_v. \tag{18}$$

Combining (17) and (18), we obtain

$$|||(\mathbf{u}_h,\widehat{\mathbf{u}}_h)|||_v \le \nu^{-1}\frac{2c_1c_3}{c_a^s}|\mathbf{F}|,$$

which indicates that the image of the mapping $\mathcal{F}$ satisfies the bound (16) and $\mathcal{F}$ maps $\mathcal{K}_h$ to itself, where $\mathcal{K}_h := \Big\{(\mathbf{v}_h,\widehat{\mathbf{v}}_h)\in V_h^{\mathrm{div}}\times\widehat{V}_h : |||(\mathbf{v}_h,\widehat{\mathbf{v}}_h)|||_v \le \nu^{-1}\frac{2c_1c_3}{c_a^s}|\mathbf{F}|\Big\}$.

Next, we need to prove that the mapping $\mathcal{F}$ is contractive in the space $\mathcal{K}_h$. Let $(\mathbf{u}_1,\widehat{\mathbf{u}}_1)=\mathcal{F}(\mathbf{w}_1,\widehat{\mathbf{w}}_1)$ and $(\mathbf{u}_2,\widehat{\mathbf{u}}_2)=\mathcal{F}(\mathbf{w}_2,\widehat{\mathbf{w}}_2)$ for any $(\mathbf{w}_1,\widehat{\mathbf{w}}_1),(\mathbf{w}_2,\widehat{\mathbf{w}}_2)\in\mathcal{K}_h$. Then (11) and (13)–(15) result in

$$\nu c_a^s\,|||(\mathbf{u}_1-\mathbf{u}_2,\widehat{\mathbf{u}}_1-\widehat{\mathbf{u}}_2)|||_v^2 \le a_h((\mathbf{u}_1-\mathbf{u}_2,\widehat{\mathbf{u}}_1-\widehat{\mathbf{u}}_2),(\mathbf{u}_1-\mathbf{u}_2,\widehat{\mathbf{u}}_1-\widehat{\mathbf{u}}_2))$$

$$= o_h(\mathbf{w}_2, (\mathbf{u}_2, \widehat{\mathbf{u}}_2); (\mathbf{u}_1 - \mathbf{u}_2, \widehat{\mathbf{u}}_1 - \widehat{\mathbf{u}}_2)) - o_h(\mathbf{w}_1, (\mathbf{u}_1, \widehat{\mathbf{u}}_1); (\mathbf{u}_1 - \mathbf{u}_2, \widehat{\mathbf{u}}_1 - \widehat{\mathbf{u}}_2))$$
$$\leq c_c \|\mathbf{w}_1 - \mathbf{w}_2\|_{1,h} \, |||(\mathbf{u}_2, \widehat{\mathbf{u}}_2)|||_v \, |||(\mathbf{u}_1 - \mathbf{u}_2, \widehat{\mathbf{u}}_1 - \widehat{\mathbf{u}}_2)|||_v.$$

Hence

$$|||(\mathbf{u}_1 - \mathbf{u}_2, \widehat{\mathbf{u}}_1 - \widehat{\mathbf{u}}_2)|||_v \leq \nu^{-2} \frac{2c_c c_1 c_2 c_3}{(c_a^s)^2} |\mathbf{F}| \, |||(\mathbf{w}_1 - \mathbf{w}_2, \widehat{\mathbf{w}}_1 - \widehat{\mathbf{w}}_2)|||_v < \frac{1}{3} \, |||(\mathbf{w}_1 - \mathbf{w}_2, \widehat{\mathbf{w}}_1 - \widehat{\mathbf{w}}_2)|||_v,$$

from (8), which shows that the mapping $\mathcal{F}$ is contractive if $\nu^{-2} \frac{c_c c_1 c_2 c_3}{(c_a^s)^2} |\mathbf{F}| < \frac{1}{6}$. According to the Banach's fixed point theorem, we know that the mapping $\mathcal{F}$ exists a unique fixed point $(\mathbf{u}_h, \widehat{\mathbf{u}}_h) \in \mathcal{K}_h$ satisfying (16).

Furthermore, we prove the existence and uniqueness of $(p_h, \widehat{p}_h)$. Given the solution $(\mathbf{u}_h, \widehat{\mathbf{u}}_h) \in V_h^{\mathrm{div}} \times \widehat{V}_h$, we have the pressure $(p_h, \widehat{p}_h) \in Q_h \times \widehat{Q}_h$ by

$$b_h((p_h, \widehat{p}_h), (\mathbf{v}_h, \widehat{\mathbf{v}}_h)) = \frac{1}{N} \sum_{K \in \mathcal{T}_{x_0}} \mathbf{F} \cdot \mathbf{v}_h \Big|_K (x_0) - a_h((\mathbf{u}_h, \widehat{\mathbf{u}}_h), (\mathbf{v}_h, \widehat{\mathbf{v}}_h)) - o_h(\mathbf{u}_h, (\mathbf{u}_h, \widehat{\mathbf{u}}_h); (\mathbf{v}_h, \widehat{\mathbf{v}}_h)), \tag{19}$$

for any $(\mathbf{v}_h, \widehat{\mathbf{v}}_h) \in V_h \times \widehat{V}_h$. Using the approximations (12), (14) and (18), we know that the right hand side of (19) is a bounded linear functional on $V_h \times \widehat{V}_h$ with respect to the seminorm $|||(\cdot, \cdot)|||_v$. Hence the existence and uniqueness of solutions for (19) can be guaranteed by the inf-sup condition ([45], Lem. 8). □

4. *A POSTERIORI* ERROR ESTIMATES

This section is devoted to provide *a posteriori* error analysis for the errors in the velocity and pressure. Obviously, it is not an easy task because of the nonlinearity of the Navier–Stokes system and the singular nature of δ_{x_0}.

We begin this section by introducing the Lagrange and Scott-Zhang interpolations.

Lemma 4.1 (See [53], Inequality 4.8.10)**.** *Let $I_h^{sc} : L^1(\Omega) \to X_h^k$ be the Scott-Zhang interpolation, where $X_h^k := \{v \in C(\overline{\Omega}) : v|_K \in \mathcal{P}^k(K), \ \forall K \in \mathcal{T}_h\}$. Then we have*

$$\|v - I_h^{sc} v\|_{r,q,K} \lesssim h_K^{s-r} |v|_{s,q,S_K}, \tag{20}$$

for $K \in \mathcal{T}_h$ and $v \in W^{s,q}(S_K)$, where $0 \leq r \leq s \leq k+1$, $1 \leq q \leq \infty$, and $S_K :=$ interior$\big(\bigcup\{\overline{K}_i : \overline{K}_i \cap \overline{K} \neq \emptyset, \ K_i \in \mathcal{T}_h\}\big)$.

Lemma 4.2 (See [53], Thm. 4.4.4, Cor. 4.4.7)**.** *Let $I_h : C(\overline{\Omega}) \to X_h^1$ be the Lagrange interpolation operator, then*

$$|v - I_h v|_{i,q,K} \lesssim h_K^{2-i} |v|_{2,q,K}, \qquad \forall v \in W^{2,q}(\Omega), \ i = 0, 1, 2, \ 1 < q < \infty, \tag{21}$$
$$|v - I_h v|_{i,q,K} \lesssim h_K^{1-i} |v|_{1,q,K}, \qquad \forall v \in W^{1,q}(\Omega), \ i = 0, 1, \ 2 < q < \infty, \tag{22}$$
$$|v - I_h v|_{0,\infty,K} \lesssim h_K^{2-\frac{2}{q}} |v|_{2,q,K}, \qquad \forall v \in W^{2,q}(\Omega), \ 1 < q < \infty, \tag{23}$$
$$|v - I_h v|_{0,\infty,K} \lesssim h_K^{1-\frac{2}{q}} |v|_{1,q,K}, \qquad \forall v \in W^{1,q}(\Omega), \ 2 < q < \infty, \tag{24}$$

for any $K \in \mathcal{T}_h$.

Next, we give an interpolation named Oswald interpolation, which shows that the function $\mathbf{v}_h \in V_h$ can be approximated by the continuous functions $\widetilde{\mathbf{v}}_h \in V_h \cap \big(H_0^1(\Omega)\big)^2$ and $\widetilde{\mathbf{v}}_h^* \in V_h \cap \big(H^1(\Omega)\big)^2$. Specifically, let $\mathcal{N}_K = \Big\{x_K^{(j)}, \ j = 1, \cdots, m\Big\}$ be the Lagrange nodes in element $K \in \mathcal{T}_h$ and $\Big\{\phi_K^{(j)}, \ j = 1, \cdots, m\Big\}$ the corresponding Lagrange basis functions. Then the function $\mathbf{v}_h$ and the Oswald interpolations $\widetilde{\mathbf{v}}_h$ and $\widetilde{\mathbf{v}}_h^*$ can be

written as $\mathbf{v}_h = \sum_{K\in\mathcal{T}_h}\sum_{j=1}^m \boldsymbol{\alpha}_K^{(j)}\phi_K^{(j)}$, $\widetilde{\mathbf{v}}_h = \sum_{v\in\mathcal{N}}\boldsymbol{\beta}^{(v)}\phi^{(v)}$, and $\widetilde{\mathbf{v}}_h^* = \sum_{v\in\mathcal{N}}\boldsymbol{\gamma}^{(v)}\phi^{(v)}$, where $\boldsymbol{\alpha}_K^{(j)} = \mathbf{v}_h\left(x_K^{(j)}\right)$, $\mathcal{N} = \bigcup_{K\in\mathcal{T}_h}\mathcal{N}_K$, $\phi^{(v)}$ is the Lagrange basis function of the node v,

$$\boldsymbol{\beta}^{(v)} = \begin{cases} \mathbf{0} & \text{If } v\in\mathcal{N}\cap\partial\Omega, \\ \frac{1}{|w_v|}\sum_{x_K^{(j)}=v}\boldsymbol{\alpha}_K^{(j)} & \text{If } v\in\mathcal{N}\backslash\partial\Omega, \end{cases} \qquad \text{and} \qquad \boldsymbol{\gamma}^{(v)} = \frac{1}{|w_v|}\sum_{x_K^{(j)}=v}\boldsymbol{\alpha}_K^{(j)}, \quad \forall v\in\mathcal{N}.$$

Here $w_v = \left\{K\in\mathcal{T}_h : v\in\overline{K}\right\}$ and $|w_v|$ denotes the number of elements in w_v.

Lemma 4.3. *For $\mathbf{v}_h\in V_h$, there exist continuous functions $\widetilde{\mathbf{v}}_h\in V_h\cap\left(H_0^1(\Omega)\right)^2$ and $\widetilde{\mathbf{v}}_h^*\in V_h\cap\left(H^1(\Omega)\right)^2$ such that*

$$\sum_{K\in\mathcal{T}_h}\|\mathbf{v}_h-\widetilde{\mathbf{v}}_h^*\|_{0,q,K}^q \lesssim \sum_{F\in\mathcal{E}_h^o} h_F\|[\mathbf{v}_h]\|_{0,q,F}^q, \tag{25}$$

$$\sum_{K\in\mathcal{T}_h} h_K^s\|\mathbf{v}_h-\widetilde{\mathbf{v}}_h\|_{0,q,K}^q \lesssim \sum_{F\in\mathcal{E}_h^o} h_F^{1+s}\|[\mathbf{v}_h]\|_{0,q,F}^q + \sum_{F\in\mathcal{E}_h^\partial} h_F^{1+s}\|\mathbf{v}_h\|_{0,q,F}^q, \tag{26}$$

$$\sum_{K\in\mathcal{T}_h} h_K^s\|\nabla(\mathbf{v}_h-\widetilde{\mathbf{v}}_h)\|_{0,q,K}^q \lesssim \sum_{F\in\mathcal{E}_h^o} h_F^{1+s-q}\|[\mathbf{v}_h]\|_{0,q,F}^q + \sum_{F\in\mathcal{E}_h^\partial} h_F^{1+s-q}\|\mathbf{v}_h\|_{0,q,F}^q, \tag{27}$$

for $1\le q<\infty$, where s is a real number.

Proof. Since the error estimates (25) and (26) can be proved similarly, we just prove the error estimate (27).

It is worth noting that the case of $q=2$ and $s=0$ has been proved in Theorem 2.2 of [55]. As for other cases, the method of proof is similar. According to the definition of $\widetilde{\mathbf{v}}_h$, we have

$$\sum_{K\in\mathcal{T}_h} h_K^s\|\nabla(\mathbf{v}_h-\widetilde{\mathbf{v}}_h)\|_{0,q,K}^q = \sum_{K\in\mathcal{T}_h} h_K^s\int_K\left|\sum_{j=1}^m\left(\boldsymbol{\alpha}_K^{(j)}-\boldsymbol{\beta}_K^{(j)}\right)\nabla\phi_K^{(j)}\right|^q \mathrm{d}x \lesssim \sum_{K\in\mathcal{T}_h} h_K^s\sum_{j=1}^m\left|\boldsymbol{\alpha}_K^{(j)}-\boldsymbol{\beta}_K^{(j)}\right|^q\left\|\nabla\phi_K^{(j)}\right\|_{0,q,K}^q,$$

where $\boldsymbol{\beta}_K^{(j)} = \boldsymbol{\beta}^{(\nu)}$ whenever $x_K^{(j)} = \nu$. From Lemma 3.2, we know that $\left\|\nabla\phi_K^{(j)}\right\|_{0,q,K} \lesssim h_K^{\frac{2}{q}-1}$. Hence the above inequality and (9) yield

$$\begin{aligned}\sum_{K\in\mathcal{T}_h} h_K^s\|\nabla(\mathbf{v}_h-\widetilde{\mathbf{v}}_h)\|_{0,q,K}^q &\lesssim \sum_{K\in\mathcal{T}_h} h_K^{s+2-q}\sum_{j=1}^m\left|\boldsymbol{\alpha}_K^{(j)}-\boldsymbol{\beta}_K^{(j)}\right|^q \\ &\lesssim \sum_{F\in\mathcal{E}_h^o} h_F^{s+2-q}\|[\mathbf{v}_h]\|_{0,\infty,F}^q + \sum_{F\in\mathcal{E}_h^\partial} h_F^{s+2-q}\|\mathbf{v}_h\|_{0,\infty,F}^q \\ &\lesssim \sum_{F\in\mathcal{E}_h^o} h_F^{s+1-q}\|[\mathbf{v}_h]\|_{0,q,F}^q + \sum_{F\in\mathcal{E}_h^\partial} h_F^{s+1-q}\|\mathbf{v}_h\|_{0,q,F}^q,\end{aligned}$$

which concludes the proof. □

4.1. Reliability

Now we are ready to prove *a posteriori* error estimate for the error $\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}$.

Lemma 4.4. *Let $(\mathbf{u},p)$ and $(\mathbf{u}_h,\widehat{\mathbf{u}}_h,p_h,\widehat{p}_h)$ be the solutions of problems (1) and (5). Under assumption that the domain Ω is convex and the quantity $\nu^{-2}|\boldsymbol{F}|$ is sufficiently small, we have*

$$\nu\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega} \lesssim \left(\sum_{K\in\mathcal{T}_h}\eta_K^2+\eta_{\partial K}^2\right)^{1/2} + \left(\sum_{F\in\mathcal{E}_h^o}\eta_F^2\right)^{1/2}, \tag{28}$$

where

$$\eta_{\partial K}^2 = \nu^2 h_K \|\mathbf{u}_h - \widehat{\mathbf{u}}_h\|_{0,\partial K}^2, \quad \eta_F^2 = \nu^2 h_F^3 \|[(\nabla \mathbf{u}_h)\boldsymbol{n}]\|_{0,F}^2 + h_F^3 \|[p_h]\|_{0,F}^2.$$

If x_0 is a node of the partition $\mathcal{T}_h$,

$$\eta_K^2 = h_K^4 \|\nu \Delta \mathbf{u}_h - \nabla \cdot (\mathbf{u}_h \otimes \mathbf{u}_h) - \nabla p_h\|_{0,K}^2,$$

otherwise,

$$\eta_K^2 = \begin{cases} |\boldsymbol{F}|^2 h_K^2 + h_K^4 \|\nu \Delta \mathbf{u}_h - \nabla \cdot (\mathbf{u}_h \otimes \mathbf{u}_h) - \nabla p_h\|_{0,K}^2, & K \in \mathcal{T}_{x_0}, \\ h_K^4 \|\nu \Delta \mathbf{u}_h - \nabla \cdot (\mathbf{u}_h \otimes \mathbf{u}_h) - \nabla p_h\|_{0,K}^2, & K \in \mathcal{T}_h \backslash \mathcal{T}_{x_0}. \end{cases}$$

Proof. In order to prove this lemma, we introduce the following auxiliary system

$$-\nu \Delta \mathbf{u}_{\mathbf{f}} + \nabla p_{\mathbf{f}} = \mathbf{f} \quad \text{in } \Omega, \qquad \nabla \cdot \mathbf{u}_{\mathbf{f}} = 0 \quad \text{in } \Omega, \qquad \mathbf{u}_{\mathbf{f}} = \mathbf{0} \quad \text{on } \partial\Omega. \tag{29}$$

It is well known ([56], Thm. 2) that the system (29) exists a unique weak solution $(\mathbf{u}_{\mathbf{f}}, p_{\mathbf{f}}) \in \left(H^2(\Omega)\right)^2 \cap \left(H_0^1(\Omega)\right)^2 \times H^1(\Omega) \cap L_0^2(\Omega)$ for any $\mathbf{f} \in \left(L^2(\Omega)\right)^2$, and

$$\nu \|\mathbf{u}_{\mathbf{f}}\|_{2,\Omega} + \|p_{\mathbf{f}}\|_{1,\Omega} \lesssim \|\mathbf{f}\|_{0,\Omega}. \tag{30}$$

Using (1), (29) and integration by parts, we infer that

$$\begin{aligned} (\mathbf{u} - \mathbf{u}_h, \mathbf{f})_\Omega &= (\nu \nabla \mathbf{u}_{\mathbf{f}}, \nabla(\mathbf{u} - \mathbf{u}_h))_{\mathcal{T}_h} + \langle \nu(\nabla \mathbf{u}_{\mathbf{f}})\mathbf{n}, \mathbf{u}_h \rangle_{\partial \mathcal{T}_h} - \langle \mathbf{u}_h \cdot \mathbf{n}, p_{\mathbf{f}} \rangle_{\partial \mathcal{T}_h} \\ &= -(\nu \Delta(\mathbf{u} - \mathbf{u}_h), \mathbf{u}_{\mathbf{f}})_{\mathcal{T}_h} - \langle \nu(\nabla \mathbf{u}_h)\mathbf{n}, \mathbf{u}_{\mathbf{f}} \rangle_{\partial \mathcal{T}_h} + \langle \nu(\nabla \mathbf{u}_{\mathbf{f}})\mathbf{n}, \mathbf{u}_h \rangle_{\partial \mathcal{T}_h} - \langle \mathbf{u}_h \cdot \mathbf{n}, p_{\mathbf{f}} \rangle_{\partial \mathcal{T}_h} \\ &= \mathbf{F} \cdot \mathbf{u}_{\mathbf{f}}(x_0) + (\nabla \cdot (\mathbf{u}_h \otimes \mathbf{u}_h - \mathbf{u} \otimes \mathbf{u}), \mathbf{u}_{\mathbf{f}})_{\mathcal{T}_h} + (\nabla(p_h - p), \mathbf{u}_{\mathbf{f}})_{\mathcal{T}_h} \\ &\quad + (\nu \Delta \mathbf{u}_h - \nabla \cdot (\mathbf{u}_h \otimes \mathbf{u}_h) - \nabla p_h, \mathbf{u}_{\mathbf{f}})_{\mathcal{T}_h} - \langle \nu(\nabla \mathbf{u}_h)\mathbf{n}, \mathbf{u}_{\mathbf{f}} \rangle_{\partial \mathcal{T}_h} \\ &\quad + \langle \nu(\nabla \mathbf{u}_{\mathbf{f}})\mathbf{n}, \mathbf{u}_h \rangle_{\partial \mathcal{T}_h} - \langle \mathbf{u}_h \cdot \mathbf{n}, p_{\mathbf{f}} \rangle_{\partial \mathcal{T}_h} \\ &= \mathbf{F} \cdot \mathbf{u}_{\mathbf{f}}(x_0) + (\nu \Delta \mathbf{u}_h - \nabla \cdot (\mathbf{u}_h \otimes \mathbf{u}_h) - \nabla p_h, \mathbf{u}_{\mathbf{f}})_{\mathcal{T}_h} + \langle (\mathbf{u}_h \otimes \mathbf{u}_h)\mathbf{n}, \mathbf{u}_{\mathbf{f}} \rangle_{\partial \mathcal{T}_h} \\ &\quad + (\mathbf{u} \otimes \mathbf{u} - \mathbf{u}_h \otimes \mathbf{u}_h, \nabla \mathbf{u}_{\mathbf{f}})_{\mathcal{T}_h} + \langle \mathbf{u}_{\mathbf{f}} \cdot \mathbf{n}, p_h \rangle_{\partial \mathcal{T}_h} - \langle \nu(\nabla \mathbf{u}_h)\mathbf{n}, \mathbf{u}_{\mathbf{f}} \rangle_{\partial \mathcal{T}_h} \\ &\quad + \langle \nu(\nabla \mathbf{u}_{\mathbf{f}})\mathbf{n}, \mathbf{u}_h \rangle_{\partial \mathcal{T}_h} - \langle \mathbf{u}_h \cdot \mathbf{n}, p_{\mathbf{f}} \rangle_{\partial \mathcal{T}_h}. \end{aligned} \tag{31}$$

Note that we have used the fact that the velocities $\mathbf{u}, \mathbf{u}_{\mathbf{f}}, \mathbf{u}_h$ are divergence-free in the derivation of equality (31).

Let $\widehat{\mathbf{v}}_h = 0$ in (5a), integration by parts results in

$$\begin{aligned} (\nu \Delta \mathbf{u}_h - \nabla \cdot (\mathbf{u}_h \otimes \mathbf{u}_h) - \nabla p_h, \mathbf{v}_h)_{\mathcal{T}_h} = &\sum_{K \in \mathcal{T}_h} \frac{\alpha \nu}{h_K} \langle \mathbf{u}_h - \widehat{\mathbf{u}}_h, \mathbf{v}_h \rangle_{\partial K} - \langle \mathbf{u}_h - \widehat{\mathbf{u}}_h, \nu(\nabla \mathbf{v}_h)\mathbf{n} \rangle_{\partial \mathcal{T}_h} \\ &- \langle (\mathbf{u}_h \otimes \mathbf{u}_h)\mathbf{n}, \mathbf{v}_h \rangle_{\partial \mathcal{T}_h} + \langle \mathbf{v}_h \cdot \mathbf{n}, \widehat{p}_h - p_h \rangle_{\partial \mathcal{T}_h} - \frac{1}{N} \sum_{K \in \mathcal{T}_{x_0}} \mathbf{F} \cdot \mathbf{v}_h \Big|_K (x_0) \\ &+ \frac{1}{2} \langle (\mathbf{u}_h \cdot \mathbf{n})(\mathbf{u}_h + \widehat{\mathbf{u}}_h), \mathbf{v}_h \rangle_{\partial \mathcal{T}_h} + \frac{1}{2} \langle |\mathbf{u}_h \cdot \mathbf{n}|(\mathbf{u}_h - \widehat{\mathbf{u}}_h), \mathbf{v}_h \rangle_{\partial \mathcal{T}_h}, \end{aligned} \tag{32}$$

for any $\mathbf{v}_h \in V_h$.

Setting $\mathbf{v}_h = I_h \mathbf{u}_{\mathbf{f}}$ in (32) and inserting (32) in (31), we obtain

$$(\mathbf{u} - \mathbf{u}_h, \mathbf{f})_\Omega = I_1 + I_2 + I_3 + I_4 + I_5 + I_6, \tag{33}$$

where

$$
\begin{aligned}
I_1 &= \mathbf{F}\cdot(\mathbf{u_f}-I_h\mathbf{u_f})(x_0),\\
I_2 &= (\nu\Delta\mathbf{u}_h-\nabla\cdot(\mathbf{u}_h\otimes\mathbf{u}_h)-\nabla p_h,\mathbf{u_f}-I_h\mathbf{u_f})_{\mathcal{T}_h},\\
I_3 &= (\mathbf{u}\otimes\mathbf{u}-\mathbf{u}_h\otimes\mathbf{u}_h,\nabla\mathbf{u_f})_{\mathcal{T}_h}+\langle(\mathbf{u}_h\otimes\mathbf{u}_h)\mathbf{n},\mathbf{u_f}-I_h\mathbf{u_f}\rangle_{\partial\mathcal{T}_h},\\
I_4 &= \langle(\nu\nabla\mathbf{u_f}-p_\mathbf{f}\mathbf{I})\mathbf{n},\mathbf{u}_h\rangle_{\partial\mathcal{T}_h}-\langle\nu(\nabla I_h\mathbf{u_f})\mathbf{n},\mathbf{u}_h-\widehat{\mathbf{u}}_h\rangle_{\partial\mathcal{T}_h},\\
I_5 &= \langle p_h,\mathbf{u_f}\cdot\mathbf{n}\rangle_{\partial\mathcal{T}_h}+\langle I_h\mathbf{u_f}\cdot\mathbf{n},\widehat{p}_h-p_h\rangle_{\partial\mathcal{T}_h},\\
I_6 &= \sum_{K\in\mathcal{T}_h}\frac{\alpha\nu}{h_K}\langle\mathbf{u}_h-\widehat{\mathbf{u}}_h,I_h\mathbf{u_f}\rangle_{\partial K}-\langle\nu(\nabla\mathbf{u}_h)\mathbf{n},\mathbf{u_f}\rangle_{\partial\mathcal{T}_h}\\
&\quad+\frac{1}{2}\langle(\mathbf{u}_h\cdot\mathbf{n})(\mathbf{u}_h+\widehat{\mathbf{u}}_h),I_h\mathbf{u_f}\rangle_{\partial\mathcal{T}_h}+\frac{1}{2}\langle|\mathbf{u}_h\cdot\mathbf{n}|(\mathbf{u}_h-\widehat{\mathbf{u}}_h),I_h\mathbf{u_f}\rangle_{\partial\mathcal{T}_h}.
\end{aligned}
$$

Here $\mathbf{I}$ is the identity tensor. Next we are going to estimate I_1, I_2, I_3, I_4, I_5 and I_6 separately. If x_0 is a node of the partition $\mathcal{T}_h$, we have $I_1 = 0$, otherwise,

$$
I_1 \le |\mathbf{F}|\frac{1}{N}\sum_{K\in\mathcal{T}_{x_0}}\|\mathbf{u_f}-I_h\mathbf{u_f}\|_{0,\infty,K}\lesssim|\mathbf{F}|\frac{1}{N}\sum_{K\in\mathcal{T}_{x_0}}h_K|\mathbf{u_f}|_{2,K},\tag{34}
$$

by using (23). Moreover, the Lagrange interpolation error estimate (21) can yield

$$
\begin{aligned}
I_2 &\lesssim \sum_{K\in\mathcal{T}_h}h_K^2\|\nu\Delta\mathbf{u}_h-\nabla\cdot(\mathbf{u}_h\otimes\mathbf{u}_h)-\nabla p_h\|_{0,K}|\mathbf{u_f}|_{2,K}\\
&\lesssim\left(\sum_{K\in\mathcal{T}_h}h_K^4\|\nu\Delta\mathbf{u}_h-\nabla\cdot(\mathbf{u}_h\otimes\mathbf{u}_h)-\nabla p_h\|_{0,K}^2\right)^{1/2}|\mathbf{u_f}|_{2,\Omega}.
\end{aligned}\tag{35}
$$

As for the term I_3, according to the embedding theorem ([8], Thm. II.3.2), we know that $W_0^{1,q}(\Omega)\hookrightarrow L^4(\Omega)$ for $q\in\left(\frac{4+\varepsilon}{3+\varepsilon},2\right)$. Hence inequalities (3), (7) and (16) deduce that

$$
\begin{aligned}
(\mathbf{u}\otimes\mathbf{u}-\mathbf{u}_h\otimes\mathbf{u}_h,\nabla\mathbf{u_f})_{\mathcal{T}_h} &\le\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}\big(\|\mathbf{u}\|_{L^4(\Omega)}+\|\mathbf{u}_h\|_{L^4(\Omega)}\big)\|\nabla\mathbf{u_f}\|_{L^4(\Omega)}\\
&\lesssim\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}\big(\|\nabla\mathbf{u}\|_{0,q,\Omega}+|||(\mathbf{u}_h,\widehat{\mathbf{u}}_h)|||_v\big)\|\mathbf{u_f}\|_{2,\Omega}\\
&\lesssim\nu^{-1}|\mathbf{F}|\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega}\|\mathbf{u_f}\|_{2,\Omega}.
\end{aligned}\tag{36}
$$

On the other hand, in virtue of the normal continuity of $\mathbf{u}_h$, we have from (10), (8), (16) and (21)

$$
\begin{aligned}
\langle(\mathbf{u}_h\otimes\mathbf{u}_h)\mathbf{n},\mathbf{u_f}-I_h\mathbf{u_f}\rangle_{\partial\mathcal{T}_h} &= \langle((\mathbf{u}_h-\widehat{\mathbf{u}}_h)\otimes\mathbf{u}_h)\mathbf{n},\mathbf{u_f}-I_h\mathbf{u_f}\rangle_{\partial\mathcal{T}_h}\\
&\le\sum_{K\in\mathcal{T}_h}\|\mathbf{u}_h-\widehat{\mathbf{u}}_h\|_{L^4(\partial K)}\|\mathbf{u}_h\|_{L^4(\partial K)}\|\mathbf{u_f}-I_h\mathbf{u_f}\|_{0,\partial K}\\
&\lesssim\sum_{K\in\mathcal{T}_h}h_K\|\mathbf{u}_h-\widehat{\mathbf{u}}_h\|_{0,\partial K}h_K^{1/4}\|\mathbf{u}_h\|_{L^4(\partial K)}|\mathbf{u_f}|_{2,K}\\
&\lesssim\nu^{-1}|\mathbf{F}|\left(\sum_{K\in\mathcal{T}_h}h_K^2\|\mathbf{u}_h-\widehat{\mathbf{u}}_h\|_{0,\partial K}^2\right)^{1/2}|\mathbf{u_f}|_{2,\Omega}.
\end{aligned}\tag{37}
$$

Since $\nu\nabla\mathbf{u_f}-p_\mathbf{f}\mathbf{I}$ and $\mathbf{u}_h$ are normal continuous, from (10), (21) and (20) we arrive at

$$
I_4=\langle(\nu\nabla\mathbf{u_f}-p_\mathbf{f}\mathbf{I})\mathbf{n},\mathbf{u}_h-\widehat{\mathbf{u}}_h\rangle_{\partial\mathcal{T}_h}-\langle(\nu\nabla I_h\mathbf{u_f}-I_h^{sc}p_\mathbf{f}\mathbf{I})\mathbf{n},\mathbf{u}_h-\widehat{\mathbf{u}}_h\rangle_{\partial\mathcal{T}_h}
$$

$$\leq \sum_{K\in\mathcal{T}_h} \left(\nu\|\nabla(\mathbf{u_f}-I_h\mathbf{u_f})\|_{0,\partial K} + \|p_{\mathbf{f}} - I_h^{sc}p_{\mathbf{f}}\|_{0,\partial K}\right)\|\mathbf{u}_h-\widehat{\mathbf{u}}_h\|_{0,\partial K} \tag{38}$$

$$\lesssim \left(\sum_{K\in\mathcal{T}_h} h_K\|\mathbf{u}_h-\widehat{\mathbf{u}}_h\|_{0,\partial K}^2\right)^{1/2} (\nu|\mathbf{u_f}|_{2,\Omega} + |p_{\mathbf{f}}|_{1,\Omega}),$$

and

$$I_5 = \langle(\mathbf{u_f}-I_h\mathbf{u_f})\cdot\mathbf{n}, p_h\rangle_{\partial\mathcal{T}_h} \lesssim \left(\sum_{F\in\mathcal{E}_h^o} h_F^3\|[p_h]\|_{0,F}^2\right)^{1/2} |\mathbf{u_f}|_{2,\Omega}. \tag{39}$$

Finally, we turn to deal with the term I_6. Let $\mathbf{v}_h = 0$ in (5a), we have

$$\begin{aligned} 0 = &\sum_{K\in\mathcal{T}_h} \frac{\alpha\nu}{h_K}\langle\mathbf{u}_h-\widehat{\mathbf{u}}_h, \widehat{\mathbf{v}}_h\rangle_{\partial K} - \langle\nu(\nabla\mathbf{u}_h)\mathbf{n}, \widehat{\mathbf{v}}_h\rangle_{\partial\mathcal{T}_h} \\ &+ \frac{1}{2}\langle(\mathbf{u}_h\cdot\mathbf{n})(\mathbf{u}_h+\widehat{\mathbf{u}}_h), \widehat{\mathbf{v}}_h\rangle_{\partial\mathcal{T}_h} + \frac{1}{2}\langle|\mathbf{u}_h\cdot\mathbf{n}|(\mathbf{u}_h-\widehat{\mathbf{u}}_h), \widehat{\mathbf{v}}_h\rangle_{\partial\mathcal{T}_h}, \end{aligned} \tag{40}$$

for any $\widehat{\mathbf{v}}_h \in \widehat{V}_h$, thus

$$\begin{aligned} I_6 &= \langle\nu(\nabla\mathbf{u}_h)\mathbf{n}, I_h\mathbf{u_f}-\mathbf{u_f}\rangle_{\partial\mathcal{T}_h} \\ &\leq \sum_{F\in\mathcal{E}_h^o} \nu\|[(\nabla\mathbf{u}_h)\mathbf{n}]\|_{0,F}\|I_h\mathbf{u_f}-\mathbf{u_f}\|_{0,F} \lesssim \left(\sum_{F\in\mathcal{E}_h^o} h_F^3\|[(\nabla\mathbf{u}_h)\mathbf{n}]\|_{0,F}^2\right)^{1/2} \nu|\mathbf{u_f}|_{2,\Omega}. \end{aligned} \tag{41}$$

Combining (30), (33)–(39) and (41), we conclude the proof by

$$\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega} = \sup_{\mathbf{f}\in(L^2(\Omega))^2\setminus\mathbf{0}} \frac{(\mathbf{u}-\mathbf{u}_h,\mathbf{f})_\Omega}{\|\mathbf{f}\|_{0,\Omega}} \lesssim \nu^{-1}\left(\sum_{K\in\mathcal{T}_h}\eta_K^2+\eta_{\partial K}^2\right)^{1/2} + \nu^{-1}\left(\sum_{F\in\mathcal{E}_h^o}\eta_F^2\right)^{1/2}.$$

□

Remark 4.1. From the estimate (36) we know that the assumption that the quantity $\nu^{-2}|\mathbf{F}|$ is sufficiently small is mainly used to control the nonlinear term $(\mathbf{u}\otimes\mathbf{u}-\mathbf{u}_h\otimes\mathbf{u}_h, \nabla\mathbf{u_f})_{\mathcal{T}_h}$.

Remark 4.2. Since $\langle(\mathbf{u}_h\cdot\mathbf{n})\widehat{\mathbf{u}}_h, \widehat{\mathbf{v}}_h\rangle_{\partial\mathcal{T}_h} = 0$ for any $\widehat{\mathbf{v}}_h \in \widehat{V}_h$, the equality (40) yields

$$\nu[(\nabla\mathbf{u}_h)\mathbf{n}] = 2\left\{\left\{\tau(\mathbf{u}_h-\widehat{\mathbf{u}}_h) + \frac{1}{2}\Pi_k^\partial((\mathbf{u}_h\cdot\mathbf{n}+|\mathbf{u}_h\cdot\mathbf{n}|)(\mathbf{u}_h-\widehat{\mathbf{u}}_h))\right\}\right\}, \tag{42}$$

on each interior face F, where $\tau = \frac{\alpha\nu}{h_K}$ and Π_k^∂ is the L^2-projection onto $\mathcal{P}^k(F)$. In virtue of (42), a simple calculation yields

$$\nu h_F^{3/2}\|[(\nabla\mathbf{u}_h)\mathbf{n}]\|_{0,F} \lesssim \nu\sum_{K\in P_F} h_K^{1/2}\|\mathbf{u}_h-\widehat{\mathbf{u}}_h\|_{0,F\cap\partial K} + \sum_{K\in P_F} h_K h_K^{1/4}\|\mathbf{u}_h\|_{L^4(F\cap\partial K)}\|\mathbf{u}_h-\widehat{\mathbf{u}}_h\|_{0,F\cap\partial K}$$

where $P_F = \{K\in\mathcal{T}_h : F\in\partial K\}$. Summing the above inequality over all interior edges, we obtain by (8) and (16)

$$\left(\sum_{F\in\mathcal{E}_h^o}\nu^2 h_F^3\|[(\nabla\mathbf{u}_h)\mathbf{n}]\|_{0,F}^2\right)^{1/2} \lesssim \left(\sum_{K\in\mathcal{T}_h}\nu^2 h_K\|\mathbf{u}_h-\widehat{\mathbf{u}}_h\|_{0,\partial K}^2\right)^{1/2}$$

$$+\left(\sum_{K\in\mathcal{T}_h} h_K\|\mathbf{u}_h\|^4_{L^4(\partial K)}\right)^{1/4}\left(\sum_{K\in\mathcal{T}_h} h_K^4\|\mathbf{u}_h-\widehat{\mathbf{u}}_h\|^4_{0,\partial K}\right)^{1/4}$$
$$\lesssim\left(\sum_{K\in\mathcal{T}_h}\nu^2 h_K\|\mathbf{u}_h-\widehat{\mathbf{u}}_h\|^2_{0,\partial K}\right)^{1/2}+\nu^{-1}|\mathbf{F}|\left(\sum_{K\in\mathcal{T}_h} h_K^2\|\mathbf{u}_h-\widehat{\mathbf{u}}_h\|^2_{0,\partial K}\right)^{1/2},$$

which indicates that the estimator $\left(\sum_{F\in\mathcal{E}_h^o}\nu^2 h_F^3\|[(\nabla\mathbf{u}_h)\mathbf{n}]\|^2_{0,F}\right)^{1/2}$ can be controlled by the estimator $\left(\sum_{K\in\mathcal{T}_h}\nu^2 h_K\|\mathbf{u}_h-\widehat{\mathbf{u}}_h\|^2_{0,\partial K}\right)^2$ under assumption that the quantity $\nu^{-2}|\mathbf{F}|$ is sufficiently small.

Since the velocity $\mathbf{u}$ and pressure p of the Navier–Stokes system (1) belong to $\left(W_0^{1,q}(\Omega)\right)^2$ and $L_0^q(\Omega)$ with $q\in\left(\frac{4+\varepsilon}{3+\varepsilon},2\right)$, it is reasonable to consider *a posteriori* error estimates for $\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,q,\mathcal{T}_h}$ and $\|p-p_h\|_{0,q,\Omega}$.

From [46], we know that the space $W^{1,q}(\Omega)$ $(1<q<\infty)$ is reflexive, hence

$$\|\nabla\mathbf{v}\|_{0,q,\Omega}=\sup_{\mathbf{f}\in\left(W^{-1,q'}(\Omega)\right)^2/\mathbf{0}}\frac{\langle\mathbf{v},\mathbf{f}\rangle}{\|\mathbf{f}\|_{-1,q',\Omega}},\quad\forall\mathbf{v}\in\left(W_0^{1,q}(\Omega)\right)^2,\tag{43}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality paring between the spaces $\left(W^{-1,q'}(\Omega)\right)^2$ and $\left(W_0^{1,q}(\Omega)\right)^2$.

Lemma 4.5. *Let $(\mathbf{u},p)$ and $(\mathbf{u}_h,\widehat{\mathbf{u}}_h,p_h,\widehat{p}_h)$ be the solutions of problems (1) and (5). If the quantity $\nu^{-2}|\boldsymbol{F}|$ is sufficiently small, we have*

$$\nu\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,q,\mathcal{T}_h}\lesssim\left(\sum_{K\in\mathcal{T}_h}\zeta^q_{q,K}+\zeta^q_{q,\partial K}\right)^{1/q}+\left(\sum_{F\in\mathcal{E}_h^o}\zeta^q_{q,F}\right)^{1/q},\tag{44}$$

for $q\in\left(\frac{4+\varepsilon}{3+\varepsilon},2\right)$, where

$$\zeta^q_{q,\partial K}=\nu^q h_K^{1-q}\|\mathbf{u}_h-\widehat{\mathbf{u}}_h\|^q_{0,q,\partial K},\quad \zeta^q_{q,F}=\nu^q h_F\|[(\nabla\mathbf{u}_h)\boldsymbol{n}]\|^q_{0,q,F}+h_F\|[p_h]\|^q_{0,q,F}.$$

If x_0 is a node of the partition $\mathcal{T}_h$,

$$\zeta^q_{q,K}=h_K^q\|\nu\Delta\mathbf{u}_h-\nabla\cdot(\mathbf{u}_h\otimes\mathbf{u}_h)-\nabla p_h\|^q_{0,q,K},$$

otherwise,

$$\zeta^q_{q,K}=\begin{cases}|\boldsymbol{F}|^q h_K^{2-q}+h_K^q\|\nu\Delta\mathbf{u}_h-\nabla\cdot(\mathbf{u}_h\otimes\mathbf{u}_h)-\nabla p_h\|^q_{0,q,K}, & K\in\mathcal{T}_{x_0},\\ h_K^q\|\nu\Delta\mathbf{u}_h-\nabla\cdot(\mathbf{u}_h\otimes\mathbf{u}_h)-\nabla p_h\|^q_{0,q,K}, & K\in\mathcal{T}_h\backslash\mathcal{T}_{x_0}.\end{cases}$$

Proof. We know from Theorem 1.1, Remark 1.1 of [47] that the auxiliary system (29) exists a unique weak solution $(\mathbf{u}_\mathbf{f},p_\mathbf{f})\in\left(W_0^{1,q'}(\Omega)\right)^2\times L_0^{q'}(\Omega)$ for $\mathbf{f}\in\left(W^{-1,q'}(\Omega)\right)^2$ with $q'\in(2,4+\varepsilon)$, and

$$\nu\|\mathbf{u}_\mathbf{f}\|_{1,q',\Omega}+\|p_\mathbf{f}\|_{0,q',\Omega}\lesssim\|\mathbf{f}\|_{-1,q',\Omega}.\tag{45}$$

Since the discrete velocity $\mathbf{u}_h$ does not belong to $\left(W_0^{1,q}(\Omega)\right)^2$, we can not use the duality argument directly for $\mathbf{u}-\mathbf{u}_h$. Here let $\widetilde{\mathbf{u}}_h\in V_h\cap\left(H_0^1(\Omega)\right)^2$ be the Oswald interpolation of $\mathbf{u}_h$, then by using (2) and integration by parts we have

$$\langle\mathbf{u}-\widetilde{\mathbf{u}}_h,\mathbf{f}\rangle=(\nabla(\mathbf{u}-\widetilde{\mathbf{u}}_h),\nu\nabla\mathbf{u}_\mathbf{f})_{\mathcal{T}_h}-(p_\mathbf{f},\nabla\cdot(\mathbf{u}-\widetilde{\mathbf{u}}_h))_{\mathcal{T}_h}$$

$$
\begin{aligned}
&= \mathbf{F}\cdot\mathbf{u}_{\mathbf{f}}(x_0) + (\mathbf{u}\otimes\mathbf{u}, \nabla\mathbf{u}_{\mathbf{f}})_\Omega + (\nabla(\mathbf{u}_h - \widetilde{\mathbf{u}}_h), \nu\nabla\mathbf{u}_{\mathbf{f}})_{\mathcal{T}_h} \\
&\quad + (\nu\Delta\mathbf{u}_h, \mathbf{u}_{\mathbf{f}})_{\mathcal{T}_h} - \langle \nu(\nabla\mathbf{u}_h)\mathbf{n}, \mathbf{u}_{\mathbf{f}}\rangle_{\partial\mathcal{T}_h} - (p_{\mathbf{f}}, \nabla\cdot(\mathbf{u}_h - \widetilde{\mathbf{u}}_h))_{\mathcal{T}_h} \\
&= \mathbf{F}\cdot\mathbf{u}_{\mathbf{f}}(x_0) + (\nu\Delta\mathbf{u}_h - \nabla\cdot(\mathbf{u}_h\otimes\mathbf{u}_h) - \nabla p_h, \mathbf{u}_{\mathbf{f}})_{\mathcal{T}_h} \\
&\quad + (\nabla(\mathbf{u}_h - \widetilde{\mathbf{u}}_h), \nu\nabla\mathbf{u}_{\mathbf{f}})_{\mathcal{T}_h} - (p_{\mathbf{f}}, \nabla\cdot(\mathbf{u}_h - \widetilde{\mathbf{u}}_h))_{\mathcal{T}_h} \\
&\quad + (\mathbf{u}\otimes\mathbf{u} - \mathbf{u}_h\otimes\mathbf{u}_h, \nabla\mathbf{u}_{\mathbf{f}})_\Omega + \langle(\mathbf{u}_h\otimes\mathbf{u}_h)\mathbf{n}, \mathbf{u}_{\mathbf{f}}\rangle_{\partial\mathcal{T}_h} \\
&\quad + \langle \mathbf{u}_{\mathbf{f}}\cdot\mathbf{n}, p_h\rangle_{\partial\mathcal{T}_h} - \langle \nu(\nabla\mathbf{u}_h)\mathbf{n}, \mathbf{u}_{\mathbf{f}}\rangle_{\partial\mathcal{T}_h}.
\end{aligned} \tag{46}
$$

Set $\mathbf{v}_h = I_h\mathbf{u}_{\mathbf{f}}$ in (32) and insert (32) in (46) to infer that

$$
\langle \mathbf{u} - \widetilde{\mathbf{u}}_h, \mathbf{f}\rangle = I_1 + I_2 + I_3 + I_6 + J_1 + J_2, \tag{47}
$$

where I_1, I_2, I_3 and I_6 are defined in the proof of Lemma 4.4, and

$$
\begin{aligned}
J_1 &= (\nabla(\mathbf{u}_h - \widetilde{\mathbf{u}}_h), \nu\nabla\mathbf{u}_{\mathbf{f}})_{\mathcal{T}_h} - (p_{\mathbf{f}}, \nabla\cdot(\mathbf{u}_h - \widetilde{\mathbf{u}}_h))_{\mathcal{T}_h}, \\
J_2 &= -\langle \mathbf{u}_h - \widehat{\mathbf{u}}_h, \nu(\nabla I_h\mathbf{u}_{\mathbf{f}})\mathbf{n}\rangle_{\partial\mathcal{T}_h} + \langle I_h\mathbf{u}_{\mathbf{f}}\cdot\mathbf{n}, \widehat{p}_h - p_h\rangle_{\partial\mathcal{T}_h} + \langle \mathbf{u}_{\mathbf{f}}\cdot\mathbf{n}, p_h\rangle_{\partial\mathcal{T}_h}.
\end{aligned}
$$

If x_0 is a node of the partition $\mathcal{T}_h$, $I_1 = 0$, otherwise, we have from (24)

$$
I_1 \le \frac{1}{N}|\mathbf{F}| \sum_{K\in\mathcal{T}_{x_0}} \|\mathbf{u}_{\mathbf{f}} - I_h\mathbf{u}_{\mathbf{f}}\|_{0,\infty,K} \lesssim \frac{1}{N}|\mathbf{F}| \sum_{K\in\mathcal{T}_{x_0}} h_K^{1-\frac{2}{q'}} |\mathbf{u}_{\mathbf{f}}|_{1,q',K}. \tag{48}
$$

On the other hand, the Lagrange interpolation error estimate (22) and Lemma 4.3 can yield

$$
I_2 \lesssim \left(\sum_{K\in\mathcal{T}_h} h_K^q \|\nu\Delta\mathbf{u}_h - \nabla\cdot(\mathbf{u}_h\otimes\mathbf{u}_h) - \nabla p_h\|_{0,q,K}^q\right)^{1/q} |\mathbf{u}_{\mathbf{f}}|_{1,q',\Omega}, \tag{49}
$$

and

$$
\begin{aligned}
J_1 &\lesssim \left(\sum_{F\in\mathcal{E}_h^o} h_F^{1-q}\|[\mathbf{u}_h]\|_{0,q,F}^q + \sum_{F\in\mathcal{E}_h^\partial} h_F^{1-q}\|\mathbf{u}_h\|_{0,q,F}^q\right)^{1/q} \left(\nu\|\nabla\mathbf{u}_{\mathbf{f}}\|_{0,q',\Omega} + \|p_{\mathbf{f}}\|_{0,q',\Omega}\right) \\
&\lesssim \left(\sum_{K\in\mathcal{T}_h} h_K^{1-q}\|\mathbf{u}_h - \widehat{\mathbf{u}}_h\|_{0,q,\partial K}^q\right)^{1/q} \left(\nu\|\nabla\mathbf{u}_{\mathbf{f}}\|_{0,q',\Omega} + \|p_{\mathbf{f}}\|_{0,q',\Omega}\right).
\end{aligned} \tag{50}
$$

According to the embedding theorem ([8], Thm. II.3.2) we know that $W_0^{1,q}(\Omega) \hookrightarrow L^{2q}(\Omega)$ for $q \in \left(\frac{4+\varepsilon}{3+\varepsilon}, 2\right)$. Hence we deduce by (3), (7), (16), inverse estimate and Lemma 4.3 that

$$
\begin{aligned}
(\mathbf{u}\otimes\mathbf{u} - \mathbf{u}_h\otimes\mathbf{u}_h, \nabla\mathbf{u}_{\mathbf{f}})_\Omega &\le \|\mathbf{u} - \mathbf{u}_h\|_{0,2q,\Omega}\left(\|\mathbf{u}\|_{0,2q,\Omega} + \|\mathbf{u}_h\|_{0,2q,\Omega}\right)\|\nabla\mathbf{u}_{\mathbf{f}}\|_{0,q',\Omega} \\
&\lesssim \|\mathbf{u} - \mathbf{u}_h\|_{0,2q,\Omega}\left(\|\nabla\mathbf{u}\|_{0,q,\Omega} + |||(\mathbf{u}_h, \widehat{\mathbf{u}}_h)|||_v\right)\|\nabla\mathbf{u}_{\mathbf{f}}\|_{0,q',\Omega} \\
&\lesssim \nu^{-1}|\mathbf{F}|\left(\|\nabla(\mathbf{u} - \widetilde{\mathbf{u}}_h)\|_{0,q,\Omega} + \|\mathbf{u}_h - \widetilde{\mathbf{u}}_h\|_{0,2q,\Omega}\right)\|\nabla\mathbf{u}_{\mathbf{f}}\|_{0,q',\Omega},
\end{aligned} \tag{51}
$$

and

$$
\|\mathbf{u}_h - \widetilde{\mathbf{u}}_h\|_{0,2q,\Omega} \lesssim \left(\sum_{K\in\mathcal{T}_h} h_K\|\mathbf{u}_h - \widehat{\mathbf{u}}_h\|_{0,2q,\partial K}^{2q}\right)^{1/2q} \lesssim \left(\sum_{K\in\mathcal{T}_h} \|\mathbf{u}_h - \widehat{\mathbf{u}}_h\|_{0,q,\partial K}^q\right)^{1/q}. \tag{52}
$$

As for another part of I_3, we have from (9), (10), (7), (16) and (22)

$$\begin{aligned}\langle(\mathbf{u}_h\otimes\mathbf{u}_h)\mathbf{n},\mathbf{u_f}-I_h\mathbf{u_f}\rangle_{\partial\mathcal{T}_h}&=\langle((\mathbf{u}_h-\widehat{\mathbf{u}}_h)\otimes\mathbf{u}_h)\mathbf{n},\mathbf{u_f}-I_h\mathbf{u_f}\rangle_{\partial\mathcal{T}_h}\\&\le\sum_{K\in\mathcal{T}_h}\|\mathbf{u}_h-\widehat{\mathbf{u}}_h\|_{0,2q,\partial K}\|\mathbf{u}_h\|_{0,2q,\partial K}\|\mathbf{u_f}-I_h\mathbf{u_f}\|_{0,q',\partial K}\\&\lesssim\sum_{K\in\mathcal{T}_h}h_K^{-1/q}\|\mathbf{u}_h-\widehat{\mathbf{u}}_h\|_{0,q,\partial K}\|\mathbf{u}_h\|_{0,2q,K}h_K^{1/q}|\mathbf{u_f}|_{1,q',K}\\&\lesssim\nu^{-1}|\mathbf{F}|\left(\sum_{K\in\mathcal{T}_h}\|\mathbf{u}_h-\widehat{\mathbf{u}}_h\|_{0,q,\partial K}^q\right)^{1/q}|\mathbf{u_f}|_{1,q',\Omega}.\end{aligned}\tag{53}$$

Since $\mathbf{u_f}\in\left(W_0^{1,q'}(\Omega)\right)^2$ with $q'\in(2,4+\varepsilon)$ is continuous, by using (9), (10), (22) and (40) we can get

$$\begin{aligned}J_2&=-\langle\mathbf{u}_h-\widehat{\mathbf{u}}_h,\nu(\nabla I_h\mathbf{u_f})\mathbf{n}\rangle_{\partial\mathcal{T}_h}-\langle(I_h\mathbf{u_f}-\mathbf{u_f})\cdot\mathbf{n},p_h\rangle_{\partial\mathcal{T}_h}\\&\lesssim\sum_{K\in\mathcal{T}_h}\nu h_K^{1/q-1}\|\mathbf{u}_h-\widehat{\mathbf{u}}_h\|_{0,q,\partial K}\|\nabla\mathbf{u_f}\|_{0,q',K}+\left(\sum_{F\in\mathcal{E}_h^o}h_F\|[p_h]\|_{0,q,F}^q\right)^{1/q}|\mathbf{u_f}|_{1,q',\Omega},\end{aligned}\tag{54}$$

and

$$\begin{aligned}I_6=\langle\nu(\nabla\mathbf{u}_h)\mathbf{n},I_h\mathbf{u_f}-\mathbf{u_f}\rangle_{\partial\mathcal{T}_h}&=\sum_{F\in\mathcal{E}_h^o}\langle\nu[(\nabla\mathbf{u}_h)\mathbf{n}],I_h\mathbf{u_f}-\mathbf{u_f}\rangle_F\\&\lesssim\nu\left(\sum_{F\in\mathcal{E}_h^o}h_F\|[(\nabla\mathbf{u}_h)\mathbf{n}]\|_{0,q,F}^q\right)^{1/q}|\mathbf{u_f}|_{1,q',\Omega}.\end{aligned}\tag{55}$$

Combining (43), (45) and (47)–(55), we obtain

$$\|\nabla(\mathbf{u}-\widetilde{\mathbf{u}}_h)\|_{0,q,\Omega}\lesssim\nu^{-1}\left(\sum_{K\in\mathcal{T}_h}\zeta_{q,K}^q+\zeta_{q,\partial K}^q\right)^{1/q}+\nu^{-1}\left(\sum_{F\in\mathcal{E}_h^o}\zeta_{q,F}^q\right)^{1/q},\tag{56}$$

under assumption that the quantity $\nu^{-2}|\mathbf{F}|$ is sufficiently small. Finally, the error estimate (44) can be derived by (56), Lemma 4.3 and the triangle inequality. □

Remark 4.3. According to (42), we also have from Lemma 3.2

$$\nu h_F^{1/q}\|[(\nabla\mathbf{u}_h)\mathbf{n}]\|_{0,q,F}\lesssim\nu\sum_{K\in P_F}h_K^{1/q-1}\|\mathbf{u}_h-\widehat{\mathbf{u}}_h\|_{0,F\cap\partial K}+\sum_{K\in P_F}\|\mathbf{u}_h\|_{0,2q,K}\|\mathbf{u}_h-\widehat{\mathbf{u}}_h\|_{0,q,F\cap\partial K}.$$

Sum the above inequality over all interior edges to yield

$$\left(\sum_{F\in\mathcal{E}_h^o}\nu^q h_F\|[(\nabla\mathbf{u}_h)\mathbf{n}]\|_{0,q,F}^q\right)^{1/q}\lesssim\left(\sum_{K\in\mathcal{T}_h}\nu^q h_K^{1-q}\|\mathbf{u}_h-\widehat{\mathbf{u}}_h\|_{0,q,\partial K}^q\right)^{1/q}+\nu^{-1}|\mathbf{F}|\left(\sum_{K\in\mathcal{T}_h}\|\mathbf{u}_h-\widehat{\mathbf{u}}_h\|_{0,q,\partial K}^q\right)^{1/q},$$

which shows that the estimator $\left(\sum_{F\in\mathcal{E}_h^o}\nu^q h_F\|[(\nabla\mathbf{u}_h)\mathbf{n}]\|_{0,q,F}^q\right)^{1/q}$ is also can be controlled by $\left(\sum_{K\in\mathcal{T}_h}\nu^q h_K^{1-q}\|\mathbf{u}_h-\widehat{\mathbf{u}}_h\|_{0,q,\partial K}^q\right)^{1/q}$ under assumption that the quantity $\nu^{-2}|\mathbf{F}|$ is sufficiently small.

Lemma 4.6. *Let* $(\mathbf{u}, p)$ *and* $(\mathbf{u}_h, \widehat{\mathbf{u}}_h, p_h, \widehat{p}_h)$ *be the solutions of problems* (1) *and* (5). *If the quantity* $\nu^{-2}|\boldsymbol{F}|$ *is sufficiently small, we have*

$$\|p - p_h\|_{0,q,\Omega} \lesssim \left(\sum_{K\in\mathcal{T}_h} \zeta_{q,K}^q + \zeta_{q,\partial K}^q\right)^{1/q} + \left(\sum_{F\in\mathcal{E}_h^o} \zeta_{q,F}^q\right)^{1/q}, \tag{57}$$

for $q \in \left(\frac{4+\varepsilon}{3+\varepsilon}, 2\right)$, *where* $\zeta_{q,K}$, $\zeta_{q,\partial K}$ *and* $\zeta_{q,F}$ *are defined in Lemma 4.5.*

Proof. From Theorem 2.6 of [57] and equation (3.8) of [14], we know that the following inf-sup condition holds

$$\|r\|_{0,q,\Omega} \lesssim \sup_{\mathbf{v}\in\left(W_0^{1,q'}(\Omega)\right)^2/\mathbf{0}} \frac{(r, \nabla\cdot\mathbf{v})_\Omega}{\|\nabla\mathbf{v}\|_{0,q',\Omega}}, \quad \forall r \in L_0^q(\Omega). \tag{58}$$

Since $p - p_h \in L_0^q(\Omega)$, this theorem can be proved by the above inf-sup condition. For any $\mathbf{v} \in \left(W_0^{1,q'}(\Omega)\right)^2$, (2) and integration by parts yield

$$\begin{aligned}(p - p_h, \nabla\cdot\mathbf{v})_\Omega &= (\nu\nabla\mathbf{u}, \nabla\mathbf{v})_\Omega - (\mathbf{u}\otimes\mathbf{u}, \nabla\mathbf{v})_\Omega - \mathbf{F}\cdot\mathbf{v}(x_0) - (p_h, \nabla\cdot\mathbf{v})_\Omega \\ &= (\nu\nabla(\mathbf{u}-\mathbf{u}_h), \nabla\mathbf{v})_{\mathcal{T}_h} + (\mathbf{u}_h\otimes\mathbf{u}_h - \mathbf{u}\otimes\mathbf{u}, \nabla\mathbf{v})_\Omega \\ &\quad - \mathbf{F}\cdot\mathbf{v}(x_0) - (\nu\Delta\mathbf{u}_h - \nabla\cdot(\mathbf{u}_h\otimes\mathbf{u}_h) - \nabla p_h, \mathbf{v})_{\mathcal{T}_h} \\ &\quad - \langle\mathbf{v}\cdot\mathbf{n}, p_h\rangle_{\partial\mathcal{T}_h} + \langle\nu(\nabla\mathbf{u}_h)\mathbf{n}, \mathbf{v}\rangle_{\partial\mathcal{T}_h} - \langle(\mathbf{u}_h\otimes\mathbf{u}_h)\mathbf{n}, \mathbf{v}\rangle_{\partial\mathcal{T}_h}.\end{aligned} \tag{59}$$

Set $\mathbf{v}_h = I_h\mathbf{v}$ in (32) and insert (32) in (59) to infer that

$$(p - p_h, \nabla\cdot\mathbf{v})_\Omega = G - I_1 - I_2 - I_3 - J_2 - I_6, \tag{60}$$

where

$$G = (\nu\nabla(\mathbf{u}-\mathbf{u}_h), \nabla\mathbf{v})_{\mathcal{T}_h},$$

and I_1, I_2, I_3, I_6 and J_2 are defined in the proof of Lemma 4.4 and Lemma 4.5 with $\mathbf{u_f}$ replaced by $\mathbf{v}$. A simple application of Cauchy-Schwartz inequality derives

$$G \le \nu\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,q,\mathcal{T}_h}\|\nabla\mathbf{v}\|_{0,q',\Omega}. \tag{61}$$

Therefore combining (48), (49), (51)–(55), (58), (60) and (61) we have

$$\|p - p_h\|_{0,q,\Omega} \lesssim \nu\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,q,\mathcal{T}_h} + \left(\sum_{K\in\mathcal{T}_h} \zeta_{q,K}^q + \zeta_{q,\partial K}^q\right)^{1/q} + \left(\sum_{F\in\mathcal{E}_h^o} \zeta_{q,F}^q\right)^{1/q} + \nu\|\nabla(\mathbf{u}-\widetilde{\mathbf{u}}_h)\|_{0,q,\Omega},$$

under assumption that the quantity $\nu^{-2}|\mathbf{F}|$ is sufficiently small. We conclude the proof by using (44), (56) and the above inequality. □

4.2. Efficiency

Let element and edge bubble functions B_K and B_F be defined as that in [58, 59]. Then by a similar method with [14, 15], we define

$$\psi_K(x) = \begin{cases} B_K(x)\frac{|x-x_0|^2}{h_K^2} & \text{if } x_0 \in K, \\ B_K(x) & \text{otherwise,}\end{cases}$$

for any $K \in \mathcal{T}_h$, and

$$\psi_F(x) = \begin{cases} B_F(x)\frac{|x-x_0|^2}{h_F^2} & \text{if } x_0 \in S_F, \\ B_F(x) & \text{otherwise,} \end{cases}$$

for any $F \in \mathcal{E}_h^o$, where $S_F := \text{interior}\big(\bigcup\{\overline{K} : F \in \partial K,\ K \in \mathcal{T}_h\}\big)$. Moreover the following properties hold for the defined functions ψ_K and ψ_F.

Lemma 4.7. *For each $K \in \mathcal{T}_h$ and $F \in \mathcal{E}_h^o$, we have*

$$\|v\|_{0,q,K} \lesssim \left\|v\psi_K^{\frac{1}{q}}\right\|_{0,q,K}, \quad \|w\|_{0,q,F} \lesssim \left\|w\psi_F^{\frac{1}{q}}\right\|_{0,q,F},$$

for $1 < q < \infty$, $v|_K \in \mathcal{P}^j(K)$, and $w|_F \in \mathcal{P}^j(F)$, where j is a nonnegative integer.

Lemma 4.8. *Let $(\mathbf{u}, p)$ and $(\mathbf{u}_h, \widehat{\mathbf{u}}_h, p_h, \widehat{p}_h)$ be the solutions of problems (1) and (5). If x_0 is a node of the partition $\mathcal{T}_h$, we have*

$$\left\{\sum_{K\in\mathcal{T}_h} \zeta_{q,K}^q\right\}^{1/q} \lesssim \nu\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,q,\mathcal{T}_h} + \|p-p_h\|_{0,q,\Omega} + \nu^{-2}|\boldsymbol{F}|\Big(\nu\|\nabla(\mathbf{u}-\widetilde{\mathbf{u}}_h)\|_{0,q,\Omega} + \nu\|\mathbf{u}_h-\widetilde{\mathbf{u}}_h\|_{0,2q,\mathcal{T}_h}\Big),$$

for $q \in \left(\frac{4+\varepsilon}{3+\varepsilon}, 2\right)$, where $\widetilde{\mathbf{u}}_h \in V_h \cap \big(H_0^1(\Omega)\big)^2$ is the Oswald interpolation of $\mathbf{u}_h$.

Proof. Let $\mathbf{v}_h = \psi_K(\nu\Delta\mathbf{u}_h - \nabla\cdot(\mathbf{u}_h\otimes\mathbf{u}_h) - \nabla p_h)$, then Lemma 4.7 and the inverse estimate (9) yield

$$\begin{aligned} \|\nu\Delta\mathbf{u}_h - \nabla\cdot(\mathbf{u}_h\otimes\mathbf{u}_h) - \nabla p_h\|_{0,K}^2 &\lesssim (\nu\Delta\mathbf{u}_h - \nabla\cdot(\mathbf{u}_h\otimes\mathbf{u}_h) - \nabla p_h, \mathbf{v}_h)_K \\ &= (\nu\Delta(\mathbf{u}_h-\mathbf{u}) - \nabla(p_h-p), \mathbf{v}_h)_K + (\nabla\cdot(\mathbf{u}\otimes\mathbf{u}-\mathbf{u}_h\otimes\mathbf{u}_h), \mathbf{v}_h)_K \\ &= (\nu\nabla(\mathbf{u}-\mathbf{u}_h), \nabla\mathbf{v}_h)_K - (p-p_h, \nabla\cdot\mathbf{v}_h)_K - (\mathbf{u}\otimes\mathbf{u}-\mathbf{u}_h\otimes\mathbf{u}_h, \nabla\mathbf{v}_h)_K. \end{aligned} \tag{62}$$

Again, the inverse estimate (9) derives

$$\begin{aligned} (\nu\nabla(\mathbf{u}-\mathbf{u}_h), \nabla\mathbf{v}_h)_K - (p-p_h, \nabla\cdot\mathbf{v}_h)_K &\le \Big(\nu\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,q,K} + \|p-p_h\|_{0,q,K}\Big)\|\nabla\mathbf{v}_h\|_{0,q',K} \\ &\lesssim h_K^{-\frac{2}{q}}\Big(\nu\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,q,K} + \|p-p_h\|_{0,q,K}\Big)\|\mathbf{v}_h\|_{0,K}, \end{aligned} \tag{63}$$

and

$$(\mathbf{u}\otimes\mathbf{u}-\mathbf{u}_h\otimes\mathbf{u}_h, \nabla\mathbf{v}_h)_K \le h_K^{-\frac{2}{q}}\|\mathbf{u}-\mathbf{u}_h\|_{0,2q,K}\Big(\|\mathbf{u}\|_{0,2q,K} + \|\mathbf{u}_h\|_{0,2q,K}\Big)\|\mathbf{v}_h\|_{0,K}. \tag{64}$$

Combining (62)–(64), we arrive at

$$\begin{aligned} &h_K\|\nu\Delta\mathbf{u}_h - \nabla\cdot(\mathbf{u}_h\otimes\mathbf{u}_h) - \nabla p_h\|_{0,q,K} \\ &\quad\lesssim h_K^{\frac{2}{q}}\|\nu\Delta\mathbf{u}_h - \nabla\cdot(\mathbf{u}_h\otimes\mathbf{u}_h) - \nabla p_h\|_{0,K} \\ &\quad\lesssim \nu\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,q,K} + \|p-p_h\|_{0,q,K} + \|\mathbf{u}-\mathbf{u}_h\|_{0,2q,K}\Big(\|\mathbf{u}\|_{0,2q,K} + \|\mathbf{u}_h\|_{0,2q,K}\Big). \end{aligned} \tag{65}$$

Summing (65) over all elements in $\mathcal{T}_h$, we obtain by (3), Lemma 3.1, (16) and embedding theorem

$$\left\{\sum_{K\in\mathcal{T}_h} \zeta_{q,K}^q\right\}^{1/q} \lesssim \nu\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,q,\mathcal{T}_h} + \|p-p_h\|_{0,q,\Omega}$$

$$
\begin{aligned}
&+ \|\mathbf{u}-\mathbf{u}_h\|_{0,2q,\mathcal{T}_h}\Big(\|\mathbf{u}\|_{0,2q,\Omega}+\|\mathbf{u}_h\|_{0,2q,\Omega}\Big)\\
\lesssim\ & \nu\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,q,\mathcal{T}_h}+\|p-p_h\|_{0,q,\Omega}\\
&+\Big(\|\nabla(\mathbf{u}-\widetilde{\mathbf{u}}_h)\|_{0,q,\Omega}+\|\mathbf{u}_h-\widetilde{\mathbf{u}}_h\|_{0,2q,\mathcal{T}_h}\Big)\Big(\|\nabla\mathbf{u}\|_{0,q,\Omega}+|||(\mathbf{u}_h,\widehat{\mathbf{u}}_h)|||_v\Big)\\
\lesssim\ & \nu\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,q,\mathcal{T}_h}+\|p-p_h\|_{0,q,\Omega}\\
&+\nu^{-2}|\mathbf{F}|\Big(\nu\|\nabla(\mathbf{u}-\widetilde{\mathbf{u}}_h)\|_{0,q,\Omega}+\nu\|\mathbf{u}_h-\widetilde{\mathbf{u}}_h\|_{0,2q,\mathcal{T}_h}\Big),
\end{aligned}
$$

which concludes the proof. □

If x_0 is not a node of the partition $\mathcal{T}_h$, from the definition of $\zeta_{q,K}$ we know that the efficiency of $|\mathbf{F}|h_K^{\frac{2}{q}-1}$ needs to be proved for any $K\in\mathcal{T}_{x_0}$. To this end, we introduce a smooth bubble function B_{K,x_0} for each $K\in\mathcal{T}_{x_0}$ ([15], Sect. 3) such that

$$
\operatorname{supp}(B_{K,x_0})\subset S_K,\quad B_{K,x_0}(x_0)=1,\quad \|B_{K,x_0}\|_{0,\infty,S_K}=1,\quad \|\nabla B_{K,x_0}\|_{0,\infty,S_K}\lesssim h_K^{-1},
$$

where S_K is defined in Lemma 4.1. Moreover, the following estimates hold

$$
|B_{K,x_0}|_{r,q',S_K}\lesssim h_K^{2/q'-r},\quad r=0,1,\ 1\le q'\le\infty. \tag{66}
$$

Lemma 4.9. *Let $(\mathbf{u},p)$ and $(\mathbf{u}_h,\widehat{\mathbf{u}}_h,p_h,\widehat{p}_h)$ be the solutions of problems (1) and (5), then*

$$
\begin{aligned}
\Bigg\{\sum_{K\in\mathcal{T}_{x_0}}|\boldsymbol{F}|^q h_K^{2-q}\Bigg\}^{1/q}\lesssim\ & \nu\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,q,\mathcal{T}_h}+\|p-p_h\|_{0,q,\Omega}\\
&+\nu^{-2}|\boldsymbol{F}|\Big(\nu\|\nabla(\mathbf{u}-\widetilde{\mathbf{u}}_h)\|_{0,q,\mathcal{T}_h}+\nu\|\mathbf{u}_h-\widetilde{\mathbf{u}}_h\|_{0,2q,\Omega}\Big)\\
&+\Bigg\{\sum_{F\in\mathcal{E}_h^o}\zeta_{q,F}^q\Bigg\}^{1/q}+\nu^{-2}|\boldsymbol{F}|\Bigg\{\sum_{K\in\mathcal{T}_h}\nu^q\|\mathbf{u}_h-\widehat{\mathbf{u}}_h\|_{0,q,\partial K}^q\Bigg\}^{1/q},
\end{aligned}
$$

for $q\in\Big(\frac{4+\varepsilon}{3+\varepsilon},2\Big)$, where $\widetilde{\mathbf{u}}_h\in V_h\cap\big(H_0^1(\Omega)\big)^2$ is the Oswald interpolation of $\mathbf{u}_h$.

Proof. Let $\mathbf{v}=B_{K,x_0}|\mathbf{F}|^{q-1}\operatorname{sign}(\mathbf{F})$, then by using (9) and (66) we have

$$
\begin{aligned}
|\mathbf{F}|^q &= \langle\mathbf{F}\delta_{x_0},\mathbf{v}\rangle\\
&= -(\nu\Delta(\mathbf{u}-\mathbf{u}_h)-\nabla(p-p_h),\mathbf{v})_{S_K}+(\nabla\cdot(\mathbf{u}\otimes\mathbf{u}-\mathbf{u}_h\otimes\mathbf{u}_h),\mathbf{v})_{S_K}\\
&\quad-(\nu\Delta\mathbf{u}_h-\nabla\cdot(\mathbf{u}_h\otimes\mathbf{u}_h)-\nabla p_h,\mathbf{v})_{S_K}\\
&= (\nu\nabla(\mathbf{u}-\mathbf{u}_h),\nabla\mathbf{v})_{S_K}-(p-p_h,\nabla\cdot\mathbf{v})_{S_K}+\langle\nu(\nabla\mathbf{u}_h)\mathbf{n},\mathbf{v}\rangle_{\partial S_K}\\
&\quad-\langle p_h,\mathbf{v}\cdot\mathbf{n}\rangle_{\partial S_K}-(\mathbf{u}\otimes\mathbf{u}-\mathbf{u}_h\otimes\mathbf{u}_h,\nabla\mathbf{v})_{S_K}-\langle(\mathbf{u}_h\otimes\mathbf{u}_h)\mathbf{n},\mathbf{v}\rangle_{\partial S_K}\\
&\quad-(\nu\Delta\mathbf{u}_h-\nabla\cdot(\mathbf{u}_h\otimes\mathbf{u}_h)-\nabla p_h,\mathbf{v})_{S_K}\\
&\lesssim |\mathbf{F}|^{q-1}h_K^{\frac{2}{q'}-1}\Big(\nu\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,q,S_K}+\|p-p_h\|_{0,q,S_K}\Big)\\
&\quad+\sum_{F\in\mathcal{E}_h^o(S_K)}\Big(\nu\|[(\nabla\mathbf{u}_h)\mathbf{n}]\|_{0,q,F}+\|[p_h]\|_{0,q,F}\Big)\|\mathbf{v}\|_{0,q',F}\\
&\quad+|\mathbf{F}|^{q-1}h_K^{\frac{2}{q'}-1}\|\mathbf{u}-\mathbf{u}_h\|_{0,2q,S_K}\Big(\|\mathbf{u}\|_{0,2q,S_K}+\|\mathbf{u}_h\|_{0,2q,S_K}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{K'\in S_K} \|\mathbf{u}_h - \widehat{\mathbf{u}}_h\|_{0,2q,\partial K'} \|\mathbf{u}_h\|_{0,2q,\partial K'} \|\mathbf{v}\|_{0,q',\partial K'} \\
&+ |\mathbf{F}|^{q-1} h_K^{\frac{2}{q'}} \sum_{K'\in S_K} \|\nu\Delta\mathbf{u}_h - \nabla\cdot(\mathbf{u}_h\otimes\mathbf{u}_h) - \nabla p_h\|_{0,q,K'} \\
\lesssim\ & |\mathbf{F}|^{q-1} h_K^{\frac{2}{q'}-1} \Big(\nu\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,q,S_K} + \|p-p_h\|_{0,q,S_K} \\
&+ \left\{ \sum_{F\in\mathcal{E}_h^o(S_K)} h_F \Big(\nu^q \|[(\nabla\mathbf{u}_h)\mathbf{n}]\|_{0,q,F}^q + \|[p_h]\|_{0,q,F}^q \Big) \right\}^{1/q} \\
&+ \|\mathbf{u}-\mathbf{u}_h\|_{0,2q,S_K} \Big(\|\mathbf{u}\|_{0,2q,S_K} + \|\mathbf{u}_h\|_{0,2q,S_K}\Big) + \\
&+ \|\mathbf{u}_h\|_{0,2q,S_K} \left\{ \sum_{K'\in S_K} \|\mathbf{u}_h - \widehat{\mathbf{u}}_h\|_{0,q,\partial K'}^q \right\}^{1/q} + \left\{ \sum_{K'\in S_K} \zeta_{q,K'}^q \right\}^{1/q} \Bigg),
\end{aligned}
$$

where $\mathcal{E}_h^o(S_K)$ denotes the set of all interior edges of S_K. Hence

$$
\begin{aligned}
|\mathbf{F}| h_K^{\frac{2}{q}-1} \lesssim\ & \nu\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,q,S_K} + \|p-p_h\|_{0,q,S_K} + \|\mathbf{u}-\mathbf{u}_h\|_{0,2q,S_K} \Big(\|\mathbf{u}\|_{0,2q,S_K} + \|\mathbf{u}_h\|_{0,2q,S_K}\Big) \\
&+ \left\{ \sum_{F\in\mathcal{E}_h^o(S_K)} h_F \Big(\nu^q \|[(\nabla\mathbf{u}_h)\mathbf{n}]\|_{0,q,F}^q + \|[p_h]\|_{0,q,F}^q \Big) \right\}^{1/q} \\
&+ \|\mathbf{u}_h\|_{0,2q,S_K} \left\{ \sum_{K'\in S_K} \|\mathbf{u}_h - \widehat{\mathbf{u}}_h\|_{0,q,\partial K'}^q \right\}^{1/q} + \left\{ \sum_{K'\in S_K} \zeta_{q,K'}^q \right\}^{1/q}.
\end{aligned}
$$

Summing the above inequality over all elements in $\mathcal{T}_{x_0}$, we can obtain by Lemma 4.8

$$
\begin{aligned}
\left\{ \sum_{K\in\mathcal{T}_{x_0}} |\mathbf{F}|^q h_K^{2-q} \right\}^{1/q} \lesssim\ & \nu\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,q,\mathcal{T}_h} + \|p-p_h\|_{0,q,\Omega} \\
&+ \nu^{-2}|\mathbf{F}| \Big(\nu\|\nabla(\mathbf{u}-\widetilde{\mathbf{u}}_h)\|_{0,q,\mathcal{T}_h} + \nu\|\mathbf{u}_h - \widetilde{\mathbf{u}}_h\|_{0,2q,\Omega}\Big) \\
&+ \left\{ \sum_{F\in\mathcal{E}_h^o} \zeta_{q,F}^q \right\}^{1/q} + \nu^{-2}|\mathbf{F}| \left\{ \sum_{K\in\mathcal{T}_h} \nu^q \|\mathbf{u}_h - \widehat{\mathbf{u}}_h\|_{0,q,\partial K}^q \right\}^{1/q},
\end{aligned}
$$

which concludes the proof. □

Next, we are going to prove the efficiency of $\zeta_{q,F}$.

Lemma 4.10. *Let $(\mathbf{u},p)$ and $(\mathbf{u}_h,\widehat{\mathbf{u}}_h,p_h,\widehat{p}_h)$ be the solutions of problems (1) and (5), then we have*

$$
\begin{aligned}
\left\{ \sum_{F\in\mathcal{E}_h^o} \zeta_{q,F}^q \right\}^{1/q} \lesssim\ & \nu\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,q,\mathcal{T}_h} + \|p-p_h\|_{0,q,\Omega} \\
&+ \nu^{-2}|\boldsymbol{F}| \Big(\nu\|\nabla(\mathbf{u}-\widetilde{\mathbf{u}}_h)\|_{0,q,\Omega} + \nu\|\mathbf{u}_h - \widetilde{\mathbf{u}}_h\|_{0,2q,\Omega}\Big) \\
&+ \|p-\widetilde{p}_h^*\|_{0,q,\Omega} + \nu^{-2}|\boldsymbol{F}| \left\{ \sum_{K\in\mathcal{T}_h} \nu^q \|\mathbf{u}_h - \widehat{\mathbf{u}}_h\|_{0,q,\partial K}^q \right\}^{1/q},
\end{aligned}
$$

for $q \in \left(\frac{4+\varepsilon}{3+\varepsilon}, 2\right)$, *where* $\widetilde{\mathbf{u}}_h \in V_h \cap \left(H_0^1(\Omega)\right)^2$ *and* $\widetilde{p}_h^* \in Q_h \cap H^1(\Omega)$ *are the Oswald interpolations of* $\mathbf{u}_h$ *and* p_h.

Proof. Let $P_F : L^\infty(F) \to L^\infty(S_F)$ be the continuation operator ([59], Sect. 3) such that

$$\|P_F v\|_{0,S_F} \lesssim h_F^{1/2} \|v\|_{0,F}, \quad \forall v \in \mathcal{P}^j(F),\ F \in \mathcal{E}_h^o, \tag{67}$$

where S_F is defined in the beginning of Subsection 4.2 and j is a nonnegative integer. Here we set $\mathbf{v}_h = \psi_F(P_F([(\nabla \mathbf{u}_h)\mathbf{n}]))$. Then by using (9), (10), Lemma 4.7 and (67), we obtain that

$$\begin{aligned}
\|[(\nabla \mathbf{u}_h)\mathbf{n}]\|_{0,F}^2 &\lesssim \langle [(\nabla \mathbf{u}_h)\mathbf{n}], \mathbf{v}_h \rangle_F = \int_{S_F} \nabla \cdot ((\nabla \mathbf{u}_h)\mathbf{v}_h) \\
&= (\Delta \mathbf{u}_h, \mathbf{v}_h)_{S_F} + (\nabla \mathbf{u}_h, \nabla \mathbf{v})_{S_F} \\
&= \left(\nu \Delta \mathbf{u}_h - \nabla \cdot (\mathbf{u}_h \otimes \mathbf{u}_h) - \nabla p_h, \nu^{-1} \mathbf{v}_h\right)_{S_F} \\
&\quad + \left(\nabla \cdot (\mathbf{u}_h \otimes \mathbf{u}_h - \mathbf{u} \otimes \mathbf{u}), \nu^{-1} \mathbf{v}_h\right)_{S_F} + \left(\nabla (p_h - p), \nu^{-1} \mathbf{v}_h\right)_{S_F} \\
&\quad + \left(\nu \nabla (\mathbf{u}_h - \mathbf{u}), \nu^{-1} \nabla \mathbf{v}_h\right)_{S_F} \\
&= \left(\nu \Delta \mathbf{u}_h - \nabla \cdot (\mathbf{u}_h \otimes \mathbf{u}_h) - \nabla p_h, \nu^{-1} \mathbf{v}_h\right)_{S_F} \\
&\quad + \left(\nu \nabla (\mathbf{u}_h - \mathbf{u}), \nu^{-1} \nabla \mathbf{v}_h\right)_{S_F} + \left(p - p_h, \nu^{-1} \nabla \cdot \mathbf{v}_h\right)_{S_F} + \nu^{-1} \langle p_h, \mathbf{v}_h \cdot \mathbf{n} \rangle_{\partial S_F} \\
&\quad + \left(\mathbf{u} \otimes \mathbf{u} - \mathbf{u}_h \otimes \mathbf{u}_h, \nu^{-1} \nabla \mathbf{v}_h\right)_{S_F} + \nu^{-1} \langle (\mathbf{u}_h \otimes \mathbf{u}_h)\mathbf{n}, \mathbf{v}_h \rangle_{\partial S_F} \\
&\lesssim \nu^{-1} h_F^{\frac{2}{q'} - \frac{3}{2}} \Bigg(\left\{ \sum_{K \in S_F} \zeta_{q,K}^q \right\}^{1/q} + \nu \|\nabla (\mathbf{u} - \mathbf{u}_h)\|_{0,q,S_F} + \|p - p_h\|_{0,q,S_F} \\
&\quad + h_F^{\frac{1}{q}} \|[p_h]\|_{0,q,F} + \|\mathbf{u}_h\|_{0,2q,S_F} \left\{ \sum_{K \in S_F} \|\mathbf{u}_h - \widehat{\mathbf{u}}_h\|_{0,q,\partial K}^q \right\}^{1/q} \\
&\quad + \|\mathbf{u} - \mathbf{u}_h\|_{0,2q,S_F} \Big(\|\mathbf{u}\|_{0,2q,S_F} + \|\mathbf{u}_h\|_{0,2q,S_F} \Big) \Bigg) \|[(\nabla \mathbf{u}_h)\mathbf{n}]\|_{0,F}.
\end{aligned}$$

Again, the inverse estimate (9) yields

$$\begin{aligned}
\nu h_F^{\frac{1}{q}} \|[(\nabla \mathbf{u}_h)\mathbf{n}]\|_{0,q,F} &\lesssim \nu h_F^{\frac{2}{q} - \frac{1}{2}} \|[(\nabla \mathbf{u}_h)\mathbf{n}]\|_{0,F} \\
&\lesssim \left\{ \sum_{K \in S_F} \zeta_{q,K}^q \right\}^{1/q} + \nu \|\nabla (\mathbf{u} - \mathbf{u}_h)\|_{0,q,S_F} + \|p - p_h\|_{0,q,S_F} \\
&\quad + h_F^{\frac{1}{q}} \|[p_h]\|_{0,q,F} + \|\mathbf{u}_h\|_{0,2q,S_F} \left\{ \sum_{K \in S_F} \|\mathbf{u}_h - \widehat{\mathbf{u}}_h\|_{0,q,\partial K}^q \right\}^{1/q} \\
&\quad + \|\mathbf{u} - \mathbf{u}_h\|_{0,2q,S_F} \Big(\|\mathbf{u}\|_{0,2q,S_F} + \|\mathbf{u}_h\|_{0,2q,S_F} \Big).
\end{aligned}$$

Summing the above estimate over all interior edges, we arrive at

$$\begin{aligned}
\left\{ \sum_{F \in \mathcal{E}_h^o} \nu^q h_F \|[(\nabla \mathbf{u}_h)\mathbf{n}]\|_{0,q,F}^q \right\}^{1/q} &\lesssim \nu \|\nabla (\mathbf{u} - \mathbf{u}_h)\|_{0,q,\mathcal{T}_h} + \|p - p_h\|_{0,q,\Omega} \\
&\quad + \nu^{-2} |\mathbf{F}| \Big(\nu \|\nabla (\mathbf{u} - \widetilde{\mathbf{u}}_h)\|_{0,q,\Omega} + \nu \|\mathbf{u}_h - \widetilde{\mathbf{u}}_h\|_{0,2q,\Omega} \Big)
\end{aligned} \tag{68}$$

$$+\left\{\sum_{F\in\mathcal{E}_h^o} h_F\|[p_h]\|_{0,q,F}^q\right\}^{1/q}+\nu^{-2}|\mathbf{F}|\left\{\sum_{K\in\mathcal{T}_h}\nu^q\|\mathbf{u}_h-\widehat{\mathbf{u}}_h\|_{0,q,\partial K}^q\right\}^{1/q},$$

by (3), Lemma 3.1, (16) and Lemma 4.8.

On the other hand, the Oswald interpolation error estimate (25) and Lemma 3.2 can yield

$$\|[p_h]\|_{0,q,F}=\|[p_h-\widetilde{p}_h^*]\|_{0,q,F}\lesssim h_F^{-\frac{1}{q}}\|p_h-\widetilde{p}_h^*\|_{0,q,S_F}\le h_F^{-\frac{1}{q}}\Big(\|p-p_h\|_{0,q,S_F}+\|p-\widetilde{p}_h^*\|_{0,q,S_F}\Big). \tag{69}$$

We conclude the proof by combining (68) and (69). □

Obviously, the remaining task of this subsection is to prove the efficiency of $\zeta_{q,\partial K}$.

Lemma 4.11. *Let $(\mathbf{u},p)$ and $(\mathbf{u}_h,\widehat{\mathbf{u}}_h,p_h,\widehat{p}_h)$ be the solutions of problems (1) and (5). If the quantity $\nu^{-2}|\boldsymbol{F}|$ is sufficiently small, we have*

$$\left\{\sum_{K\in\mathcal{T}_h}\zeta_{q,\partial K}^q\right\}^{1/q}\lesssim\nu\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,q,\mathcal{T}_h}+\|p-p_h\|_{0,q,\Omega}+\|p-\widetilde{p}_h^*\|_{0,q,\Omega}+\left\{\sum_{K\in\mathcal{T}_h}\nu^q h_K^{-q}\|\mathbf{u}_h-\widetilde{\mathbf{u}}_h\|_{0,q,K}^q\right\}^{1/q},$$

for $q\in\left(\frac{4+\varepsilon}{3+\varepsilon},2\right)$, where $\widetilde{\mathbf{u}}_h\in V_h\cap\left(H_0^1(\Omega)\right)^2$ and $\widetilde{p}_h^*\in Q_h\cap H^1(\Omega)$ are the Oswald interpolations of $\mathbf{u}_h$ and p_h.

Proof. For each interior edge $F=\partial K^+\cap\partial K^-$, a simple calculation yields

$$\mathbf{u}_h^+-\widehat{\mathbf{u}}_h=\frac{1}{2}[\mathbf{u}_h]+\{\{\mathbf{u}_h-\widehat{\mathbf{u}}_h\}\},\quad \mathbf{u}_h^--\widehat{\mathbf{u}}_h=-\frac{1}{2}[\mathbf{u}_h]+\{\{\mathbf{u}_h-\widehat{\mathbf{u}}_h\}\},$$

where $\mathbf{u}_h^+$ and $\mathbf{u}_h^-$ are the traces of $\mathbf{u}_h$ on F. Hence Hölder inequality derives

$$|\mathbf{u}_h^+-\widehat{\mathbf{u}}_h|^q+|\mathbf{u}_h^--\widehat{\mathbf{u}}_h|^q\le 2^q(|[\mathbf{u}_h]|^q+|\{\{\mathbf{u}_h-\widehat{\mathbf{u}}_h\}\}|^q).$$

Sum the above inequality over all interior edges to infer that

$$\sum_{K\in\mathcal{T}_h}h_K^{1-q}\|\mathbf{u}_h-\widehat{\mathbf{u}}_h\|_{0,q,\partial K}^q\lesssim\sum_{F\in\mathcal{E}_h^o}h_F^{1-q}\Big(\|[\mathbf{u}_h]\|_{0,q,F}^q+\|\{\{\mathbf{u}_h-\widehat{\mathbf{u}}_h\}\}\|_{0,q,F}^q\Big)+\sum_{F\in\mathcal{E}_h^\partial}h_F^{1-q}\|\mathbf{u}_h\|_{0,q,F}^q. \tag{70}$$

Inserting (42) in (70), we obtain

$$\begin{aligned}\left\{\sum_{K\in\mathcal{T}_h}h_K^{1-q}\|\mathbf{u}_h-\widehat{\mathbf{u}}_h\|_{0,q,\partial K}^q\right\}^{1/q}&\lesssim\left\{\sum_{F\in\mathcal{E}_h^o}h_F^{1-q}\|[\mathbf{u}_h]\|_{0,q,F}^q+\sum_{F\in\mathcal{E}_h^\partial}h_F^{1-q}\|\mathbf{u}_h\|_{0,q,F}^q\right\}^{1/q}\\&\quad+\left\{\sum_{F\in\mathcal{E}_h^o}h_F\Big(\|[(\nabla\mathbf{u}_h)\mathbf{n}]\|_{0,q,F}^q\right.\\&\quad\left.+\nu^{-q}\|\{\{\Pi_k^\partial((\mathbf{u}_h\cdot\mathbf{n}+|\mathbf{u}_h\cdot\mathbf{n}|)(\mathbf{u}_h-\widehat{\mathbf{u}}_h))\}\}\|_{0,q,F}^q\Big)\right\}^{1/q}\\&\lesssim\left\{\sum_{F\in\mathcal{E}_h^o}h_F^{1-q}\|[\mathbf{u}_h]\|_{0,q,F}^q+\sum_{F\in\mathcal{E}_h^\partial}h_F^{1-q}\|\mathbf{u}_h\|_{0,q,F}^q\right\}^{1/q}\\&\quad+\left\{\sum_{F\in\mathcal{E}_h^o}h_F\|[(\nabla\mathbf{u}_h)\mathbf{n}]\|_{0,q,F}^q\right\}^{1/q}+\nu^{-2}|\mathbf{F}|\left\{\sum_{K\in\mathcal{T}_h}\|\mathbf{u}_h-\widehat{\mathbf{u}}_h\|_{0,q,\partial K}^q\right\}^{1/q}.\end{aligned} \tag{71}$$

Then Lemma 4.3, 4.10 and (71), together with the estimate (52), get

$$\left\{\sum_{K\in\mathcal{T}_h}\zeta_{q,\partial K}^q\right\}^{1/q} \lesssim \left\{\sum_{F\in\mathcal{E}_h^o}\nu^q h_F^{1-q}\|[\mathbf{u}_h]\|_{0,q,F}^q + \sum_{F\in\mathcal{E}_h^\partial}\nu^q h_F^{1-q}\|\mathbf{u}_h\|_{0,q,F}^q\right\}^{1/q} + \nu\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,q,\mathcal{T}_h} + \|p-p_h\|_{0,q,\Omega} + \|p-\widetilde{p}_h^*\|_{0,q,\Omega}, \tag{72}$$

under assumption that the quantity $\nu^{-2}|\mathbf{F}|$ is sufficiently small.

On the other hand, the trace inequality (10) and the inverse estimate (9) arrive at

$$h_F^{\frac{1}{q}-1}\|[\mathbf{u}_h]\|_{0,q,F} = h_F^{\frac{1}{q}-1}\|[\mathbf{u}_h-\widetilde{\mathbf{u}}_h]\|_{0,q,F} \lesssim h_F^{-1}\|\mathbf{u}_h-\widetilde{\mathbf{u}}_h\|_{0,q,S_F}. \tag{73}$$

Combining (72) and (73) we conclude the proof. □

4.3. Main result

With all these ingredients at hand, we are in position to prove the main result of this section.

Theorem 4.1. *Let $(\mathbf{u},p)$ and $(\mathbf{u}_h,\widehat{\mathbf{u}}_h,p_h,\widehat{p}_h)$ be the solutions of problems (1) and (5). Let $\zeta_{q,K}$, $\zeta_{q,\partial K}$ and $\zeta_{q,F}$ be defined in Lemma 4.5. If the quantity $\nu^{-2}|\boldsymbol{F}|$ is sufficiently small, then we have*

$$E_q \approx \left\{\sum_{K\in\mathcal{T}_h}\zeta_{q,K}^q + \zeta_{q,\partial K}^q\right\}^{1/q} + \left\{\sum_{F\in\mathcal{E}_h^o}\zeta_{q,F}^q\right\}^{1/q},$$

for $q\in\left(\frac{4+\varepsilon}{3+\varepsilon},2\right)$, where

$$E_q = \nu\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,q,\mathcal{T}_h} + \|p-p_h\|_{0,q,\Omega} + \|p-\widetilde{p}_h^*\|_{0,q,\Omega} + \left\{\sum_{K\in\mathcal{T}_h}\nu^q h_K^{-q}\|\mathbf{u}_h-\widetilde{\mathbf{u}}_h\|_{0,q,K}^q\right\}^{1/q},$$

$\widetilde{\mathbf{u}}_h\in V_h\cap\left(H_0^1(\Omega)\right)^2$ and $\widetilde{p}_h^*\in Q_h\cap H^1(\Omega)$ are the Oswald interpolations of $\mathbf{u}_h$ and p_h.

Proof. Lemma 4.5 and Lemma 4.6 yield

$$\nu\|\nabla(\mathbf{u}-\mathbf{u}_h)\|_{0,q,\mathcal{T}_h} + \|p-p_h\|_{0,q,\Omega} \lesssim \left\{\sum_{K\in\mathcal{T}_h}\zeta_{q,K}^q + \zeta_{q,\partial K}^q\right\}^{1/q} + \left\{\sum_{F\in\mathcal{E}_h^o}\zeta_{q,F}^q\right\}^{1/q},$$

which, together with Lemma 4.3, arrives at

$$\begin{aligned}\|p-\widetilde{p}_h^*\|_{0,q,\Omega} + \left\{\sum_{K\in\mathcal{T}_h}\nu^q h_K^{-q}\|\mathbf{u}_h-\widetilde{\mathbf{u}}_h\|_{0,q,K}^q\right\}^{1/q} &\lesssim \|p-p_h\|_{0,q,\Omega} + \|p_h-\widetilde{p}_h^*\|_{0,q,\Omega} + \left\{\sum_{K\in\mathcal{T}_h}\zeta_{q,\partial K}^q\right\}^{1/q}\\ &\lesssim \left\{\sum_{K\in\mathcal{T}_h}\zeta_{q,K}^q + \zeta_{q,\partial K}^q\right\}^{1/q} + \left\{\sum_{F\in\mathcal{E}_h^o}\zeta_{q,F}^q\right\}^{1/q}.\end{aligned}$$

Hence we have

$$E_q \lesssim \left\{\sum_{K\in\mathcal{T}_h}\zeta_{q,K}^q + \zeta_{q,\partial K}^q\right\}^{1/q} + \left\{\sum_{F\in\mathcal{E}_h^o}\zeta_{q,F}^q\right\}^{1/q}. \tag{74}$$

On the other hand, combine Lemmas 4.8–4.11 and (52) to deduce that

$$\left\{\sum_{K\in\mathcal{T}_h}\zeta_{q,K}^q+\zeta_{q,\partial K}^q\right\}^{1/q}+\left\{\sum_{F\in\mathcal{E}_h^o}\zeta_{q,F}^q\right\}^{1/q}\lesssim E_q, \tag{75}$$

which, together with (74), concludes the proof. □

5. Numerical experiments

In this section, two numerical examples will be provided to verify the performance of the obtained *a posteriori* error estimators.

Here we set

$$\eta^2=\sum_{K\in\mathcal{T}_h}\left(\eta_K^2+\eta_{\partial K}^2+\sum_{F\in\partial K\setminus\partial\Omega}\eta_F^2\right),$$

$$\zeta_q^q=\sum_{K\in\mathcal{T}_h}\left(\zeta_{q,K}^q+\zeta_{q,\partial K}^q+\sum_{F\in\partial K\setminus\partial\Omega}\zeta_{q,F}^q\right)\quad\forall q\in\left(\frac{4+\varepsilon}{3+\varepsilon},2\right).$$

The following algorithm will be utilized for the implementation of adaptive HDG methods.

Adaptive HDG algorithm

Step 1. Given the initial mesh $\mathcal{T}_h^0$, the iteration number $\mathbf{M}$ and $\theta\in(0,1]$.
Set $i=0$
Step 2. Solve the HDG system (5) on the mesh $\mathcal{T}_h^i$
Step 3. Compute the error estimator η or ζ_q
Step 4. Select the marking set $\mathcal{M}_h\subset\mathcal{T}_h^i$ such that
$\sum_{K\in\mathcal{M}_h}\eta_{*,K}\geq\theta\eta_*$,
where $\eta_*=\eta^2$ or ζ_q^q, and $\eta_{*,K}$ is the restriction of η_* on the element K
Step 5. **if** $i\geq\mathbf{M}$
Stop
else
Refine the marking set $\mathcal{M}_h$ by bisection to generate a new mesh $\mathcal{T}_h^{i+1}$. Set $i=i+1$ and go to **Step** 2
end

In Step 2, the following Banach's fixed point iteration method is used to solve the HDG system (5).

Banach′s fixed point iteration

Step 1. Given a initial guess $(\mathbf{u}_0,\widehat{\mathbf{u}}_0)$ and a tolerance **tol**. Set $i=0$
Step 2. Solve $(\mathbf{u}_{i+1},\widehat{\mathbf{u}}_{i+1})=\mathcal{F}(\mathbf{u}_i,\widehat{\mathbf{u}}_i)$ from the system (15)
Step 3. **if** $|||\mathcal{F}(\mathbf{u}_i,\widehat{\mathbf{u}}_i)-(\mathbf{u}_i,\widehat{\mathbf{u}}_i)|||_v<\mathbf{tol}$
Stop
else
Set $i=i+1$ and go to **Step** 2
end

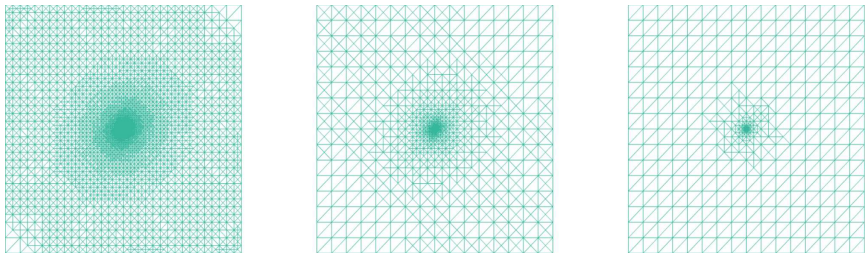

FIGURE 1. *Left–right*: adaptive meshes generated by $\zeta_{1.2}$, $\zeta_{1.4}$ and $\zeta_{1.6}$ after 20 adaptive iterations. Here $\theta = 0.4$.

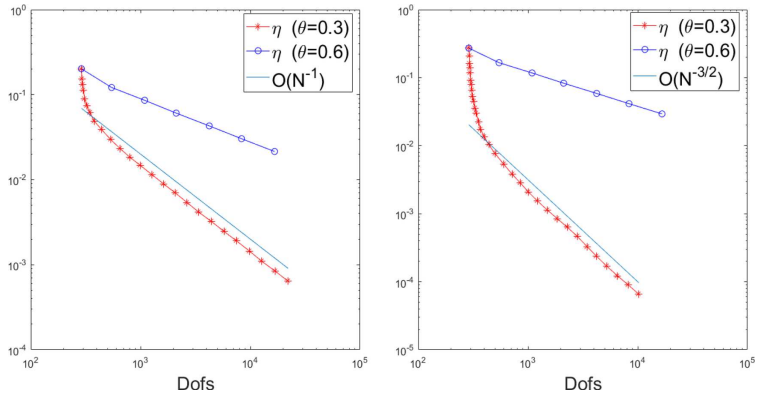

FIGURE 2. *Left–right*: convergence histories of η for $k = 1$ and $k = 2$.

Let $\gamma > 0$ be the contractive ratio of the mapping $\mathcal{F}$, then it is easily to obtain that

$$|||\mathcal{F}(\mathbf{u}_i, \widehat{\mathbf{u}}_i) - (\mathbf{u}_i, \widehat{\mathbf{u}}_i)|||_v \leq \gamma^i \, |||\mathcal{F}(\mathbf{u}_0, \widehat{\mathbf{u}}_0) - (\mathbf{u}_0, \widehat{\mathbf{u}}_0)|||_v, \tag{76}$$

$$|||(\mathbf{u}_h, \widehat{\mathbf{u}}_h) - (\mathbf{u}_i, \widehat{\mathbf{u}}_i)|||_v \leq \gamma^i \, |||(\mathbf{u}_h, \widehat{\mathbf{u}}_h) - (\mathbf{u}_0, \widehat{\mathbf{u}}_0)|||_v. \tag{77}$$

Under assumption that the quantity $\nu^{-2}|\mathbf{F}|$ is sufficiently small, the inequality (76) indicates that the above Banach's iteration method will stop after a limited number of iterations for any initial guesses. On the other hand, the inequality (77) shows that the solution obtained by the above Banach's iteration method is a good enough approximation for the HDG solution.

Note that throughout this section the figure of convergence history is plotted in log-log coordinates and the stabilization parameter is set as $\alpha = 6k^2$, where k is the polynomial order.

Example 5.1. On the convex domain $\Omega = (0,1)^2$, we set $x_0 = (0.5, 0.5)^T$, $\mathbf{F} = (1,1)^T$ and $\nu = 1$. Notice that the exact solution for this problem is unknown.

In this example, let x_0 be a vertex of the initial mesh. In Figure 1, the adaptive meshes generated by $\zeta_{1.2}$, $\zeta_{1.4}$ and $\zeta_{1.6}$ after 20 adaptive iterations are provided. We find that the mesh nodes are concentrated around the interior point x_0, which indicates that the obtained *a posteriori* error estimator ζ_q can efficiently grab the singularity of δ_{x_0}. In Figure 2, we give the convergence history of η for different k and θ. We find that the rate of convergence is $O\big(N^{-(k+1)/2}\big)$ for $\theta = 0.3$, but not for $\theta = 0.6$. In Figure 3, the convergence history of ζ_q $(q = 1.2, 1.4)$ and the corresponding effectivity index $\mathrm{eff}_q = \frac{\zeta_q}{E_q}$ are performed. Since the example has no

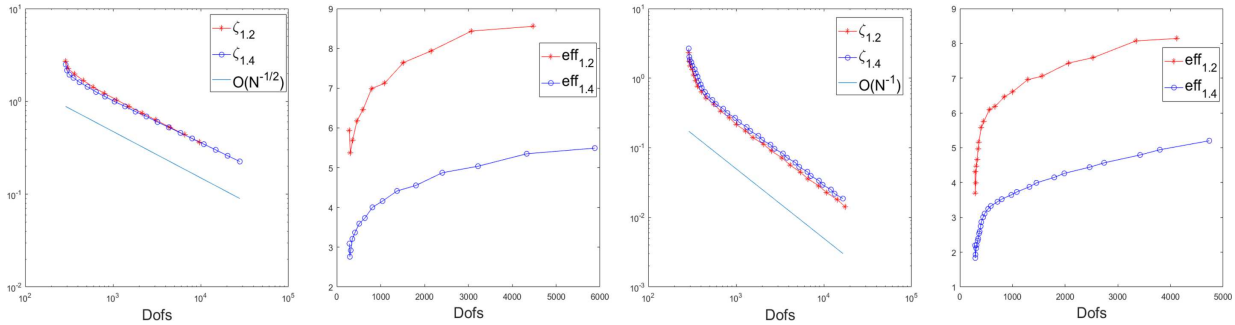

FIGURE 3. *Left 1–left 2*: convergence histories and the corresponding effectivity indexes for $k = 1$ and $\theta = 0.4$. *Right 2–right 1*: convergence histories and the corresponding effectivity indexes for $k = 2$ and $\theta = 0.4$.

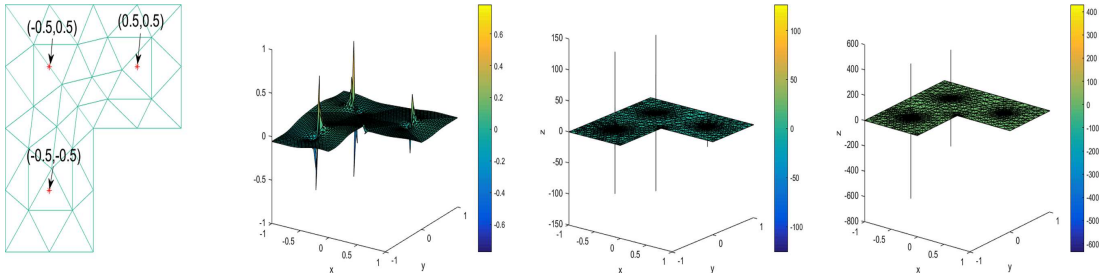

FIGURE 4. *Left 1*: initial mesh. *Left 2*: surface of p_h on the uniform mesh with 1601 nodes. *Right 2*: surface of p_h on the adaptive mesh generated by $\zeta_{1.2}$ with 1554 nodes. *Right 1*: surface of p_h on the adaptive mesh generated by $\zeta_{1.4}$ with 1655 nodes. Here $\nu = 1$, $k = 2$ and $\theta = 0.3$.

exact solutions, we use the discrete solution on the adaptive mesh with more than 20 000 nodes as the reference solution. We can see that the error estimator ζ_q can get the convergence rate $O\big(N^{-k/2}\big)$. Moreover, the effectivity indexes $\mathrm{eff}_{1.2}$ and $\mathrm{eff}_{1.4}$ are probably in the intervals $[6, 9]$ and $[4, 6]$, which indicates that ζ_q is a good enough approximation of E_q.

Example 5.2. In this example, the right hand side of momentum equation is a linear combination of Dirac deltas. Specifically, we will replace (1a) with

$$-\nu\Delta\mathbf{u} + \nabla\cdot(\mathbf{u}\otimes\mathbf{u}) + \nabla p = \sum_{v\in\mathcal{V}} \mathbf{F}_v\delta_v \quad \text{in } \Omega.$$

where $\mathcal{V} \subset \Omega$ is a finite set of interior points and $\mathbf{F}_v \in \mathbb{R}^2$.

Here we set $\Omega = (-1, 1)^2\backslash[0, 1) \times (-1, 0]$,

$$\mathcal{V} := \Big\{(0.5, 0.5)^T, (-0.5, 0.5)^T, (-0.5, -0.5)^T\Big\},$$

and

$$\mathbf{F}_{(0,5,0.5)^T} = [0.02, 0.02]^T, \quad \mathbf{F}_{(-0.5,0.5)^T} = [-0.03, -0.03]^T, \quad \mathbf{F}_{(-0.5,-0.5)^T} = [0.04, 0.04]^T.$$

The initial mesh of this example is shown in the left graph of Figure 4. Moreover, the surface of the discrete pressure p_h on the uniform and adaptive meshes is also provided in Figure 4. It is easy to find that the mesh nodes

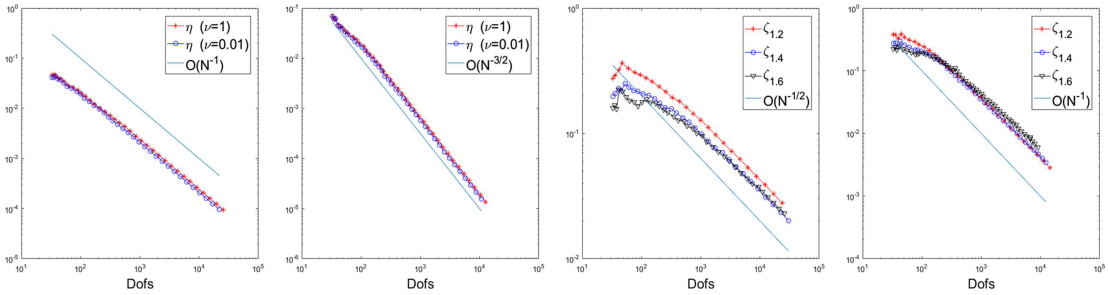

FIGURE 5. *Left 1–left 2*: convergence histories of η with different ν for $k = 1$ and $k = 2$. *Right 2–right 1*: The convergence history of ζ_q $(q = 1.2, 1.4, 1.6)$ with $\nu = 1$ for $k = 1$ and $k = 2$. Here we set $\theta = 0.3$.

are concentrated around the interior point $v \in \mathcal{V}$ and the reentrant corner, and the pressure is better represented with adaptive refinement. For $\nu = 1$ and $\nu = 0.01$, the convergence history of η is presented in Figure 5, and the convergence rate $O\big(N^{-(k+1)/2}\big)$ can be obtained. We find that the *a posteriori* error estimator η almost has no difference for $\nu = 1$ and $\nu = 0.01$. Furthermore, we give the convergence history of ζ_q $(q = 1.2, 1.4, 1.6)$ in Figure 5, and the convergence rate $O\big(N^{-k/2}\big)$ can be derived.

6. Conclusion

In this paper, we study a pressure-robust HDG method for the Navier–Stokes equations with Dirac measures. *A posteriori* error estimator, that provides an upper bound for L^2-norm in the velocity, is introduced. Moreover, *a posteriori* error estimator, that provides an upper and a lower bounds for $W^{1,q}$-seminorm in the velocity and L^q-norm in the pressure, is also proposed. Finally, a Banach's fixed point iteration method and an adaptive HDG algorithm are provided to solve the discrete system (5) and show the performance of the obtained *a posteriori* error estimators.

Acknowledgements. The work was supported by the NSF of China (Grant No. 12001209).

References

[1] G. Fulford and J. Blake, Muco-ciliary transport in the lung. *J. Theor. Biol.* **121** (1986) 381–402.
[2] L. Lacouture, A numerical method to solve the Stokes problem with punctual force in source term. *C. R. Mécanique* **343** (2015) 187–191.
[3] A. Allendes, F. Fuica, E. Otárola and D. Quero, An adaptive FEM for the pointwise tracking optimal control problem of the Stokes equations. *SIAM J. Sci. Comput.* **41** (2019) A2967–A2998.
[4] F. Fuica, E. Otárola and D. Quero, Error estimates for optimal control problems involving the Stokes system and Dirac measures. *Appl. Math. Optim.* **84** (2021) 1717–1750.
[5] J.J.L. Higdon, The generation of feeding currents by flagellar motions. *J. Fluid Mech.* **94** (1979) 305–330.
[6] H. Hasimoto, On the periodic fundamental solutions of the Stokes equations and their application to viscous flow past a cubic array of spheres. *J. Fluid Mech.* **5** (1959) 317–328.
[7] N. Liron and R. Shahar, Stokes flow due to a Stokeslet in pipe. *J. Fluid Mech.* **86** (1978) 727–744.
[8] G.P. Galdi, An Introduction to the Mathematical Theory of the Navier–Stokes Equations: Steady-State Problems. Springer, New York (2011).
[9] V. Girault and P.A. Raviart, Finite Element Methods for Navier–Stokes Equations. *Springer Series in Computational Mathematics.* Vol. 5. Springer, Berlin (1986).
[10] V. John, Finite Element Methods for Incompressible Flow Problems. *Springer Series in Computational Mathematics.* Vol. 51. Springer (2016).
[11] R. Teman, Navier–Stokes Equations, 3rd edition. Elsevier Science Publishers B. V, Amsterdam (1984).
[12] A. Allendes, E. Otárola and A.J. Salgado, A posteriori error estimates for the stationary Navier–Stokes equations with Dirac measures. *SIAM J. Sci. Comput.* **42** (2020) A1860–A1884.

[13] E. Otárola and A.J. Salgado, A weighted setting for the stationary Navier–Stokes equations under singular forcing. *Appl. Math. Lett.* **90** (2020) 105933.

[14] F. Lepe, E. Otárola and D. Quero, Error estimates for FEM discretizations of the Navier–Stokes equations with Dirac measures. *J. Sci. Comput.* **87** (2021) 97.

[15] R. Araya, E. Behrens and R. Rodriguez, A posteriori error estimates for elliptic problems with Dirac delta source terms. *Numer. Math.* **105** (2006) 193–216.

[16] A. Allendes, E. Otárola and A.J. Salgado, A posteriori error estimates for the Stokes problem with singular sources. *Comput. Methods Appl. Mech. Eng.* **345** (2019) 1007–1032.

[17] T. Apel, O. Benedix, D. Sirch and B. Vexler, A priori mesh grading for an elliptic problem with Dirac right-hand side. *SIAM J. Numer. Anal.* **49** (2011) 992–1005.

[18] I. Babuška, Error bounds for the finite element method. *Numer. Anal.* **16** (1971) 322–333.

[19] F. Fuica, F. Lepe, E. Otárola and D. Quero, A posteriori error estimates in $W^{1,p} \times L^p$ spaces for the Stokes systems with Dirac measures. *Comput. Math. Appl.* **94** (2021) 47–59.

[20] R.H. Nochetto, E. Otárola and A.J. Salgado, Piecewise polynomial interpolation in Muckenhoupt weighted Sobolev spacea and applications. *Numer. Math.* **132** (2016) 85–130.

[21] E. Otárola and A.J. Salgado, On the analysis and approximation of some models of fluids over weighted spaces on convex polyhedra. *Numer. Math.* **151** (2022) 185–218.

[22] R. Scott, Finite element convergence for singular data. *Numer. Math.* **21** (1973) 317–327.

[23] P. Houston and T.P. Wihler, Discontinuous Galerkin methods for problems with Dirace delta source. *ESAIM: M2AN* **46** (2012) 1467–1483.

[24] B. Cockburn, J. Gopalakrishnan and R. Lazarov, Unified hybridization of discontinuous Galerkin, mixed, and continuous Galerkin methods for second order elliptic problems. *SIAM J. Numer. Anal.* **47** (2009) 1319–1365.

[25] H. Chen, J. Li and W. Qiu, Robust a posteriori error estimates for HDG method for convection diffusion equations. *IMA J. Numer. Anal.* **36** (2016) 437–462.

[26] G. Fu, W. Qiu and W. Zhang, An analysis of HDG methods for convection-dominated diffusion problems. *ESAIM: M2AN* **49** (2015) 225–256.

[27] H. Leng and Y. Chen, Adaptive hybridizable discontinuous methods for nonstationary convection diffusion problems. *Adv. Comput. Math.* **46** (2020) 50.

[28] W. Qiu, J. Shen and K. Shi, An HDG method for linear elasticity with strong symmetric stresses. *Math. Comput.* **87** (2018) 69–93.

[29] S.-C. Soon, B. Cockburn and H.K. Stolarski, A hybridizable discontinuous Galerkin method for linear elasticity. *Int. J. Numer. Methods Eng.* **80** (2009) 1058–1092.

[30] H. Chen, W. Qiu and K. Shi, A priori and computable a posteriori error estimates for an HDG method for the coercive Maxwell equations. *Comput. Methods Appl. Mech. Eng.* **333** (2018) 287–310.

[31] N.C. Nguyen, J. Peraire and B. Cockburn, Hybridizable discontinuous Galerkin methods for the time harmonic Maxwell's equations. *J. Comput. Phys.* **230** (2011) 7151–7175.

[32] B. Cockburn and J. Gopalakrishnan, The derivation of hybridizable discontinuous Galerkin methods for Stokes flow. *SIAM J. Numer. Anal.* **47** (2009) 1092–1125.

[33] B. Cockburn, J. Gopalakrishnan, N.C. Nguyen, J. Peraire and F.-J. Sayas, Analysis of HDG methods for Stokes flow. *Math. Comput.* **80** (2011) 723–760.

[34] G. Fu, Y. Jin and W. Qiu, Parameter-free superconvergence $H(\mathrm{div})$-conforming HDG methods for the Brinkman equations. *IMA J. Numer. Anal.* **39** (2019) 957–982.

[35] H. Leng and H. Chen, Adaptive HDG methods for the Brinkman equations with application to optimal control. *J. Sci. Comput.* **87** (2021) 46.

[36] B. Cockburn and W. Zhang, A posteriori error estimates for HDG methods. *J. Sci. Comput.* **51** (2012) 582–607.

[37] B. Cockburn and W. Zhang, A posteriori error analysis for hybridizable discontinuous Galerkin methods for second order elliptic problems. *SIAM J. Numer. Anal.* **51** (2013) 676–693.

[38] R. Araya, M. Solano and P. Vega, Analysis of an adaptive HDG method for the Brinkman problem. *IMA J. Numer. Anal.* **39** (2019) 1502–1528.

[39] L.F. Gatica and F.A. Sequeira, A priori and a posteriori error analyses of an HDG method for the Brinkman problem. *Comput. Math. Appl.* **75** (2018) 1191–1212.

[40] R. Araya, M. Solano and P. Vega, A posteriori error analysis of an HDG method for the Oseen problem. *Appl. Numer. Math.* **146** (2019) 291–308.

[41] H. Leng, Adaptive HDG methods for the steady-state incompressible Navier–Stokes equations. *J. Sci. Comput.* **87** (2021) 37.

[42] H. Leng and Y. Chen, A hybridizable discontinuous Galerkin method for second order elliptic equations with Dirac delta source. *ESAIM: M2AN* **56** (2022) 385–406.

[43] S. Rhebergen and G.N. Wells, Analysis of a hybridized/interface stabilized finite element method for the Stokes equations. *SIAM J. Numer. Anal.* **55** (2017) 1982–2003.

[44] S. Rhebergen and G.N. Wells, A hybridizable discontinuous Galerkin method for the Navier–Stokes equations with pointwise divergence-free velocity field. *J. Sci. Comput.* **76** (2018) 1484–1501.

[45] S. Rhebergen and G.N. Wells, An embedded-hybridized discontinuous Galerkin finite element method for the Stokes equations. *Comput. Methods Appl. Mech. Eng.* **358** (2020) 112619.

[46] R.A. Adams, Sobolev Spaces. Academic Press (1975).
[47] E.D. Svensson and S. Larsson, Pointwise a posteriori error estimates for the Stokes equations in polyhedral domains. Preprint (2006).
[48] K.L.A. Kirk and S. Rhebergen, Analysis of a pressure-robost hybridized discontinuous Galerkin method for the stationary Navier–Stokes equations. *J. Sci. Comput.* **81** (2019) 881–897.
[49] A. Cesmelioglu, B. Cockburn and W. Qiu, Analysis of a hybridizable discontinuous Galerkin method for the steady-state incompressible Navier–Stokes equations. *Math. Comput.* **86** (2017) 1643–1670.
[50] D.A. Di Pietro and A. Ern, Discrete functional analysis tools for discontinuous Galerkin methods with application to the incompressible Navier–Stokes equations. *Math. Comput.* **79** (2010) 1303–1330.
[51] D.A. Di Pietro and A. Ern, Mathematical Aspects of Discontinuous Galerkin Methods. *Mathématiques and Applications (Berlin) [Mathematics and Applications]*. Vol. 69. Springer, Heidelberg (2012) MR2882148.
[52] O.A. Karakashian and W.N. Jureidini, A nonconforming finite element method for the stationary Navier–Stokes equations. *SIAM J. Numer. Anal.* **35** (1998) 93–120.
[53] S.C. Brenner and L.R. Scott, The Mathematical Theory of Finite Element Methods. Springer-Verlag, New York (2008).
[54] A. Ern and J.L. Guermond, Finite Elements I: Approximation and Interpolation. Vol. 72 of *Texts in Applied Mathematics*. Springer (2021).
[55] O.A. Karakashian and F. Pascal, A posteriori error estimates for a discontinuous Galerkin approximation of second-order elliptic problems. *SIAM J. Numer. Anal.* **41** (2003) 2374–2399.
[56] R.B. Kellogg and J.E. Osborn, A regularity result for the Stokes problem in a convex polygon. *J. Funct. Anal.* **21** (1976) 397–431.
[57] G. Acosta and R.G. Durán, Divergence Operator and Related Inequalities. *SpringerBriefs in Mathematics*. Springer, New York (2017).
[58] R. Verfürth, A Review of A Posteriori Error Estimation and Adaptive Mesh-Refinement Techniques. Wiley and Teubner, New York (1996).
[59] R. Verfürth, A posteriori error estimators for convection-diffusion equations. *Numer. Math.* **80** (1998) 641–663.