ESAIM: M2AN 57 (2023) 671–691
https://doi.org/10.1051/m2an/2022100

ESAIM: Mathematical Modelling and Numerical Analysis
www.esaim-m2an.org

SUPERCONVERGENCE AND POSTPROCESSING OF THE CONTINUOUS GALERKIN METHOD FOR NONLINEAR VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

Mingzhu Zhang, Xinyu Mao and Lijun Yi*

Abstract. We propose a novel postprocessing technique for improving the global accuracy of the continuous Galerkin (CG) method for nonlinear Volterra integro-differential equations. The key idea behind the postprocessing technique is to add a higher order Lobatto polynomial of degree $k + 1$ to the CG approximation of degree k. We first show that the CG method superconverges at the nodal points of the time partition. We further prove that the postprocessed CG approximation converges one order faster than the unprocessed CG approximation in the L^2-, H^1- and L^∞-norms. As a by-product of the postprocessed superconvergence results, we construct several a posteriori error estimators and prove that they are asymptotically exact. Numerical examples are presented to highlight the superconvergence properties of the postprocessed CG approximations and the robustness of the a posteriori error estimators.

Mathematics Subject Classification. 65L60, 65R20, 41A25.

Received April 17, 2022. Accepted December 5, 2022.

1. Introduction

In this paper, we consider the nonlinear Volterra integro-differential equation (VIDE) of the form

$$\begin{cases} u'(t) = f(t, u(t)) + \displaystyle\int_0^t K(t, s, u(s))\mathrm{d}s, \quad t \in [0, T], \\ u(0) = u_0 \in \mathbb{R}, \end{cases} \tag{1.1}$$

where the given functions $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ and $K(t, s, u) : D \times \mathbb{R} \to \mathbb{R}$ (with $D = \{(t, s) : 0 \le s \le t \le T\}$) are continuous for $t \in [0, T]$ and $(t, s) \in D$, respectively. We further assume that there exist positive constants L_1 and L_2 such that f and K satisfy the Lipschitz conditions

$$\begin{aligned} |f(t, u) - f(t, v)| &\le L_1|u - v|, \quad t \in [0, T], \\ |K(t, s, u) - K(t, s, v)| &\le L_2|u - v|, \quad (t, s) \in D \end{aligned} \tag{1.2}$$

for any $u, v \in \mathbb{R}$. Then, the problem (1.1) has a unique solution $u \in C^1([0, T])$ (see, *e.g.*, [6]).

Keywords and phrases. Volterra integro-differential equations, continuous Galerkin method, postprocessing, superconvergence.

Department of Mathematics, Shanghai Normal University, Shanghai200234, P.R. China.
*Corresponding author: ylj5152@shnu.edu.cn

VIDEs arise widely in mathematical modelling of physical, biological, engineering and other phenomena that are governed by memory effects [9, 19, 26]. During the past few decades, various numerical methods have been studied for VIDEs, such as Runge-Kutta methods [4, 20, 27, 32], collocation methods [8, 14, 24, 25, 28], continuous Galerkin (CG) methods [17, 18, 29–31] and discontinuous Galerkin (DG) methods [7, 21]. We refer the readers to the monographs [5, 6] and the literature given therein.

Among the above mentioned numerical methods, the Galerkin type methods have received considerable attention due to the (possibly) arbitrary high-order convergence. In the context of Galerkin methods, postprocessing techniques are attractive ways to improve the accuracy of an already obtained Galerkin approximation. Several postprocessing techniques have been introduced for the VIDEs. For example, defect correction methods (based on interpolation and iteration) [18, 33] and Richardson extrapolation method [34] were studied for the CG approximations of the nonlinear VIDEs with smooth kernels; superconvergence extraction technique based on Lagrange interpolation was developed in [22] for the DG approximations of the linear VIDEs with smooth and non-smooth kernels. For other types of integral equations, some postprocessing techniques for Galerkin approximations were also investigated, see, *e.g.*, [15, 16] and references therein.

The aim of this paper is to propose and analyze a novel postprocessing technique to improve the accuracy of the CG method for the VIDE (1.1) with regular solutions. The key idea of our postprocessing technique is to add a higher order Lobatto polynomial of degree $k + 1$ to the CG approximation of degree k (see 4.3), which can be regarded as a simple correction for the CG approximation. The main contributions and features of this paper can be summarized as following:

- We show that the CG method superconverges at the nodal points of the time partition with respect to the step-size.
- We prove that the proposed postprocessing technique improves the convergence rates of the CG method in the L^2-, H^1- and L^∞-norms by one order.
- Based on the postprocessed superconvergence results, we construct asymptotically exact a posteriori error estimators for the CG method as the step-size decreases.
- The postprocessing is local, in the sense that it can be done independently on each local time interval, which enables the design of parallel numerical algorithms.
- The postprocessing is very easy to implement and can achieve global superconvergence with a small cost which only requires to compute an integral on each local time interval.

This paper is organized as follows. In Section 2, we introduce the CG scheme for the VIDE 1.1 and state the *a priori* error estimates. In Section 3, we prove the nodal superconvergence estimates and obtain some superclose results. In Section 4, we describe the postprocessing technique and analyze its superconvergence properties. In Section 5, we construct several a posteriori error estimators and prove that they are asymptotically accurate. In Section 6, we present some numerical experiments to validate the theoretical results. Finally, we give some concluding remarks in Section 7.

Throughout, standard notations and conventions are followed. For an open interval $I = (a, b) \subset \mathbb{R}$, we denote by $C^m(I)$ and $C^m(\bar{I})$ the spaces of real-valued functions whose derivatives up to order m are continuous on I and $\bar{I}$, respectively. Let $L^p(I)$, $1 \le p < \infty$, be the Lebesgue spaces of p-integrable functions u, equipped with the norms $\|u\|_{L^p(I)} = (\int_I |u(t)|^p \mathrm{d}t)^{1/p}$. In addition, we denote by $L^\infty(I)$ the space of all bounded functions u on I, equipped with the norm $\|u\|_{L^\infty(I)} = \operatorname{ess\,sup}_{t\in I} |u(t)|$. Furthermore, let $W^{m,p}(I), 1 \le p \le \infty$, be the Sobolev spaces equipped with the standard norms $\|\cdot\|_{W^{m,p}(I)}$ and semi-norms $|\cdot|_{W^{m,p}(I)}$. In particular, we set $W^{m,2}(I) = H^m(I)$, and the corresponding norms and semi-norms are denoted by $\|\cdot\|_{H^m(I)}$ and $|\cdot|_{H^m(I)}$, respectively.

2. Continuous Galerkin method

Let $\mathcal{T}_h$ be a partition of $[0, T]$ into N time intervals $\{I_n := (t_{n-1}, t_n)\}_{n=1}^N$ with nodes

$$0 = t_0 < t_1 < t_2 < \cdots < t_{N-1} < t_N = T.$$

We define the length of I_n by $h_n = t_n - t_{n-1}$ and the maximum step-size by $h = \max_{1\le n\le N}\{h_n\}$. For simplicity, we assume that the time mesh is quasiuniform, *i.e.*, there exists a positive constant C_q such that

$$1 \le \frac{h}{h_n} \le C_q, \quad 1 \le n \le N,$$

although the postprocessing technique proposed in this work can be applied to an arbitrary mesh.

The trial and test function spaces are given by

$$S^{k,1}(\mathcal{T}_h) = \{u \in H^1(0,T) : u|_{I_n} \in P_k(I_n),\ 1 \le n \le N\}$$

and

$$S^{k-1,0}(\mathcal{T}_h) = \{u \in L^2(0,T) : u|_{I_n} \in P_{k-1}(I_n),\ 1 \le n \le N\},$$

respectively. Here, $P_k(I_n)$ denotes the space of polynomials of degree at most k on I_n and the space $P_{k-1}(I_n)$ is defined analogously.

To describe the CG method, we define an integral operator $K : C([0,T]) \to C([0,T])$ by

$$Ku(t) = \int_0^t K(t,s,u(s))\mathrm{d}s.$$

The CG approximation of the VIDE (1.1) can be read as: find $U \in S^{k,1}(\mathcal{T}_h)$

$$\begin{cases} \displaystyle\sum_{n=1}^N \int_{I_n} U'(t)\varphi(t)\mathrm{d}t = \sum_{n=1}^N \int_{I_n} f(t,U(t))\varphi(t)\mathrm{d}t + \sum_{n=1}^N \int_{I_n} KU(t)\varphi(t)\mathrm{d}t, \\ U(0) = u_0 \end{cases} \tag{2.1}$$

for any $\varphi \in S^{k-1,0}(\mathcal{T}_h)$.

Since the CG scheme (2.1) employs different trial and test function spaces, it can be regarded as a Petrov-Galerkin scheme. The well-posedness of the CG solution defined by (2.1) has been proved in [18, 29].

Due to the discontinuous character of the test space $S^{k-1,0}(\mathcal{T}_h)$, the CG scheme (2.1) can be decoupled into local problems on each time step, namely, if U is given on $I_m,\ 1 \le m \le n-1$, we find $U|_{I_n} \in P_k(I_n)$ such that

$$\begin{cases} \displaystyle\int_{I_n} U'(t)\varphi(t)\mathrm{d}t = \int_{I_n} f(t,U(t))\varphi(t)\mathrm{d}t + \int_{I_n} KU(t)\varphi(t)\mathrm{d}t, \\ U|_{I_n}(t_{n-1}) = U|_{I_{n-1}}(t_{n-1}) \end{cases} \tag{2.2}$$

for all $\varphi \in P_{k-1}(I_n)$. Here, we set the initial value $U|_{I_1}(t_0) = u_0$.

The following *a priori* error estimates have been established in [18, 29].

Lemma 2.1. *Let u be the solution of (1.1) and U be the CG solution of (2.1). Assume that $u \in H^{k+1}(0,T)$ with integer $k \ge 1$ and h is sufficiently small. Then*

$$\|u - U\|_{L^2(0,T)} \le Ch^{k+1}\|u\|_{H^{k+1}(0,T)}, \tag{2.3}$$

$$\|u - U\|_{H^1(0,T)} \le Ch^{k}\|u\|_{H^{k+1}(0,T)}. \tag{2.4}$$

Moreover, if $u \in W^{k+1,\infty}(0,T)$ with integer $k \ge 1$, then

$$\|u - U\|_{L^\infty(0,T)} \le Ch^{k+1}\|u\|_{W^{k+1,\infty}(0,T)}. \tag{2.5}$$

The constants $C > 0$ are independent of h.

3. Natural superconvergence of the CG method

In this section, we show that the CG method for the VIDE (1.1) superconverges at the nodal points of the time partition. We further prove that the projection $\Pi^k u$ (see (3.12)) is superclose to the CG solution U.

3.1. Preliminaries

Let L_k be the Legendre polynomial of degree k on $[-1,1]$. It is well-known that there holds the orthogonality

$$\int_{-1}^{1} L_k(x)L_l(x)dx = \frac{2}{2k+1}\delta_{l,k}, \quad \forall k,l \geq 0, \tag{3.1}$$

where $\delta_{l,k}$ is the Kronecker symbol. Let ϕ_k be the Lobatto polynomial of degree k on $[-1,1]$, namely (see, *e.g.*, [11])

$$\phi_0(x) = \frac{1-x}{2}, \quad \phi_1(x) = \frac{1+x}{2}, \quad \phi_k(x) = \int_{-1}^{x} L_{k-1}(s)\mathrm{d}s, \quad k \geq 2. \tag{3.2}$$

It is easy to verify that

$$\phi_k(x) = \frac{1}{2k-1}\left(L_k(x) - L_{k-2}(x)\right), \quad \phi_k(\pm 1) = 0, \quad k \geq 2. \tag{3.3}$$

For our purpose, we also define the shifted Legendre and Lobatto polynomials on I_n by

$$L_{k,n}(t) = L_k\left(\frac{2t - t_n - t_{n-1}}{h_n}\right), \quad k \geq 0 \tag{3.4}$$

and

$$\phi_{k,n}(t) = \phi_k\left(\frac{2t - t_n - t_{n-1}}{h_n}\right), \quad k \geq 0 \tag{3.5}$$

for $t \in I_n$, respectively.

Combining (3.1)–(3.5), we can obtain the following properties of the shifted Legendre and Lobatto polynomials $L_{k,n}$ and $\phi_{k,n}$ in a straightforward way.

Lemma 3.1. *For the polynomials $L_{k,n}(x)$ and $\phi_{k,n}(x)$, there hold*

$$\|L_{k,n}\|^2_{L^2(I_n)} = \frac{h_n}{2k+1}, \quad k \geq 0, \tag{3.6a}$$

$$\|\phi_{k,n}\|^2_{L^2(I_n)} = \frac{2h_n}{(2k-3)(2k-1)(2k+1)}, \quad k \geq 2, \tag{3.6b}$$

$$\|\phi'_{k,n}\|^2_{L^2(I_n)} = \frac{4}{(2k-1)h_n}, \quad k \geq 2, \tag{3.6c}$$

$$\|\phi_{k,n}\|_{L^\infty(I_n)} \leq \frac{2}{2k-1}, \quad k \geq 2. \tag{3.6d}$$

Let $\mathcal{T}_h$ be a given partition of $[0,T]$ with N subintervals $\{I_n\}_{n=1}^N$. For any $u \in H^1(I_n)$, we have $u' \in L^2(I_n)$. Since the Legendre polynomials form a complete orthogonal basis of the L^2 space, by using the Riesz-Fischer Theorem (see Theorem 3 in Chapter 7.3 of [23]), we can expand $u' \in L^2(I_n)$ into the Fourier-Legendre series

$$u' = \sum_{i=0}^{\infty} \hat{u}_{i,n} L_{i,n}, \tag{3.7}$$

with $\hat{u}_{i,n} = \frac{2i+1}{h_n}\int_{I_n} u' L_{i,n}\mathrm{d}t$ be the Fourier coefficients of u', here the "=" means that the partial sums of the Fourier-Legendre series of the function u' converge to u' in the sense of the metric in $L^2(I_n)$, namely,

$$\lim_{k\to\infty} \|u' - \sum_{i=0}^{k} \hat{u}_{i,n} L_{i,n}\|_{L^2(I_n)} = 0.$$

Moreover, due to the generalized Parseval's identity (see Corollary of Theorem 1 in Chapter 7.3 of [23]), the Fourier-Legendre series (3.7) can be integrated term by term over the interval $(t_{n-1}, t) \subset I_n$, which implies that

$$u(t) = u(t_{n-1}) + \sum_{i=0}^{\infty} \hat{u}_{i,n} \int_{t_{n-1}}^{t} L_{i,n}(s)\mathrm{d}s := \sum_{i=0}^{\infty} \alpha_{i,n}\phi_{i,n}(t), \quad t \in I_n, \tag{3.8}$$

where $\alpha_{0,n} = u(t_{n-1}),\ \alpha_{1,n} = u(t_n)$ and

$$\alpha_{i,n} = \frac{h_n}{2}\hat{u}_{i-1,n} = \frac{2i-1}{2}\int_{I_n} u' L_{i-1,n}\mathrm{d}t, \quad i \geq 2. \tag{3.9}$$

We now define an projector $\pi^k_{I_n} : H^1(I_n) \to P_k(I_n)$ with $k \geq 1$ by

$$\pi^k_{I_n} u(t) = \sum_{i=0}^{k} \alpha_{i,n}\phi_{i,n}(t). \tag{3.10}$$

It is worth noting that the projection $\pi^k_{I_n} u$ has been frequently used for the superconvergence analysis of the finite element methods and finite volume methods for various partial differential equations; see, *e.g.*, [11,12] and the references therein. In this paper, we shall also use this projection for superconvergence analysis of the CG method for VIDEs.

Due to (3.2), (3.4) and (3.5), there holds $\phi'_{k,n}(t) = \frac{2}{h_n} L_{k-1,n}(t)$ for $k \geq 2$. Then, using (3.8), (3.10) and the orthogonality property of the Legendre polynomials, gives

$$\int_{I_n} (u - \pi^k_{I_n} u)'\varphi\mathrm{d}t = \int_{I_n} \Big(\sum_{i=k+1}^{\infty} \alpha_{i,n}\phi_{i,n}\Big)'\varphi\mathrm{d}t = \frac{2}{h_n}\sum_{i=k+1}^{\infty} \alpha_{i,n}\int_{I_n} L_{i-1,n}\varphi\mathrm{d}t = 0 \tag{3.11}$$

for any $\varphi \in P_{k-1}(I_n)$. Moreover, due to (3.3) and (3.5), there holds $\phi_{k,n}(t_{n-1}) = \phi_{k,n}(t_n) = 0$ for $k \geq 2$. Then, using (3.10), (3.2) and (3.5), we deduce that

$$\pi^k_{I_n} u(t_{n-1}) = \alpha_{0,n}\phi_{0,n}(t_{n-1}) + \alpha_{1,n}\phi_{1,n}(t_{n-1}) = \alpha_{0,n} = u(t_{n-1}),$$

and similarly, we have $\pi^k_{I_n} u(t_n) = \alpha_{1,n} = u(t_n)$.

For any $u \in H^1(0,T)$, we further define a global projection $\Pi^k u$ interval-wise by

$$\Pi^k u|_{I_n} = \pi^k_{I_n} u, \quad 1 \leq n \leq N. \tag{3.12}$$

Since $\pi^k_{I_n} u(t_{n-1}) = u(t_{n-1})$ and $\pi^k_{I_n} u(t_n) = u(t_n)$, there holds $\Pi^k u \in S^{k,1}(\mathcal{T}_h)$.

We recall the following approximation properties of $\pi^k_{I_n}$ from [31].

Lemma 3.2. *Assume that $u \in H^{k+1}(I_n)$ with integer $k \geq 1$. Then*

$$\|u - \pi^k_{I_n} u\|_{L^2(I_n)} \leq C h_n^{k+1}\|u\|_{H^{k+1}(I_n)}, \tag{3.13}$$

$$\|u - \pi^k_{I_n} u\|_{H^1(I_n)} \leq C h_n^{k}\|u\|_{H^{k+1}(I_n)}. \tag{3.14}$$

Moreover, if $u \in W^{k+1,\infty}(I_n)$ with interger $k \geq 1$, then

$$\|u - \pi^k_{I_n} u\|_{L^\infty(I_n)} \leq C h_n^{k+1}\|u\|_{W^{k+1,\infty}(I_n)}. \tag{3.15}$$

The constants $C > 0$ are independent of h_n.

3.2. Superconvergence at the nodes

In this section, we will prove that the CG method superconverges at the nodes of the time partition $\mathcal{T}_h$ with respect to the step-size h.

Theorem 3.3. *Let $u \in H^{k+1}(0,T)$ with integer $k \geq 1$ be the solution of (1.1) and U be the CG solution of (2.1). Assume that $\theta_1(t) := f_u(t,u(t)) \in C^{k-1}([0,T])$ and $\theta_2(t,s) := K_u(t,s,u(s)) \in C^{k-1}(D)$ with $D = \{(t,s) : 0 \leq s \leq t \leq T\}$. We further assume that there exist positive constants M_1 and M_2 such that $|f_{uu}(t,u)| \leq M_1$ on $[0,T]\times\mathbb{R}$ and $|K_{uu}(t,s,u)| \leq M_2$ on $D\times\mathbb{R}$. Then for h sufficiently small, there holds*

$$|(u-U)(t_n)| \leq Ch^{2k}\|u\|_{H^{k+1}(0,T)}, \quad 1 \leq n \leq N, \tag{3.16}$$

where the constant $C > 0$ is independent of h.

Proof. Let $e := u - U$. By (1.1) and (2.2), there holds

$$\int_{I_n} e'\varphi \mathrm{d}t = \int_{I_n} \left(f(t,u(t)) - f(t,U(t))\right)\varphi \mathrm{d}t + \int_{I_n} (Ku(t) - KU(t))\varphi \mathrm{d}t \tag{3.17}$$

for any $\varphi \in P_{k-1}(I_n)$.

Using the Taylor's theorem with Lagrange remainder in the variable u, there exist functions χ_1 and χ_2, whose values $\chi_1(t)$ and $\chi_2(t)$ at t are between $u(t)$ and $U(t)$, such that

$$f(t,u(t)) - f(t,U(t)) = \theta_1(t)e(t) + R_1(t)e^2(t), \tag{3.18}$$

$$K(t,s,u(s)) - K(t,s,U(s)) = \theta_2(t,s)e(s) + R_2(t,s)e^2(s), \tag{3.19}$$

where

$$\theta_1(t) := f_u(t,u(t)), \quad \theta_2(t,s) := K_u(t,s,u(s)),$$
$$R_1(t) := -\frac{f_{uu}(t,\chi_1(t))}{2}, \quad R_2(t,s) := -\frac{K_{uu}(t,s,\chi_2(t))}{2}\cdot$$

Assume that there are positive constants M_1 and M_2 such that $|f_{uu}(t,u)| \leq M_1$ on $[0,T]\times\mathbb{R}$ and $|K_{uu}(t,s,u)| \leq M_2$ on $D\times\mathbb{R}$. Then we have

$$|R_1| \leq \frac{M_1}{2} \quad \text{and} \quad |R_2| \leq \frac{M_2}{2}\cdot \tag{3.20}$$

In view of (3.18) and (3.19), we can rewrite (3.17) as

$$\int_{I_n} \left(e' - \theta_1 e - \int_0^t \theta_2(t,s)e(s)\mathrm{d}s\right)\varphi \mathrm{d}t = \int_{I_n} R_1 e^2\varphi \mathrm{d}t + \int_{I_n}\left(\int_0^t R_2(t,s)e^2(s)\mathrm{d}s\right)\varphi \mathrm{d}t \tag{3.21}$$

for any $\varphi \in P_{k-1}(I_n)$.

We next construct a linear auxiliary problem: find w such that

$$\begin{cases} w'(t) + \theta_1(t)w(t) + \displaystyle\int_t^{t_n} \theta_2(s,t)w(s)\mathrm{d}s = 0, \quad t \in [0,t_n], \\ w(t_n) = 1. \end{cases} \tag{3.22}$$

Assume that $\theta_1(t) = f_u(t,u(t)) \in C^{k-1}([0,T])$ and $\theta_2(t,s) = K_u(t,s,u(s)) \in C^{k-1}(D)$. We assert that there exists a positive constant C (which is independent of h) such that

$$\|w\|_{H^k(0,t_n)} \leq C, \quad k \geq 1. \tag{3.23}$$

In order to verify (3.23), we introduce the function $\widetilde{w}(t) = w(t_n - t)$, then the backward problem (3.22) can be rewritten as a forward VIDE

$$\begin{cases} -\widetilde{w}'(t) + \tilde{\theta}_1(t)\widetilde{w}(t) + \displaystyle\int_0^t \tilde{\theta}_2(s,t)\widetilde{w}(s)\mathrm{d}s = 0, \quad t \in [0, t_n], \\ \widetilde{w}(0) = 1, \end{cases}$$

where $\tilde{\theta}_1(t) := \theta_1(t_n - t)$ and $\tilde{\theta}_2(s,t) = \theta_2(t_n - s, t_n - t)$. Noting that $\tilde{\theta}_1 \in C^{k-1}([0,T])$ and $\tilde{\theta}_2 \in C^{k-1}(D)$, applying Theorem 3.1.4 of [5] to the above forward VIDE, we have $\widetilde{w} \in C^k([0,t_n])$, and thus $w \in C^k([0,t_n])$, which also implies (3.23).

Using integration by parts, the fact $e(0) = 0$ and (3.22), we obtain

$$\begin{aligned} &\int_0^{t_n} \left(e' - \theta_1 e - \int_0^t \theta_2(t,s)e(s)\mathrm{d}s\right) w\mathrm{d}t \\ =&\int_0^{t_n} e'w\mathrm{d}t - \int_0^{t_n} \theta_1 ew dt - \int_0^{t_n}\int_0^t \theta_2(t,s)e(s)w(t)\mathrm{d}s\mathrm{d}t \\ =&e(t_n)w(t_n) - e(0)w(0) - \int_0^{t_n} w'e\mathrm{d}t - \int_0^{t_n} \theta_1 we dt - \int_0^{t_n}\int_t^{t_n} \theta_2(s,t)w(s)e(t)\mathrm{d}s\mathrm{d}t \\ =&e(t_n) - \int_0^{t_n}\left(w' + \theta_1 w + \int_t^{t_n} \theta_2(s,t)w(s)\mathrm{d}s\right)e(t)\mathrm{d}t \\ =&e(t_n), \end{aligned} \tag{3.24}$$

where we have used the fact that

$$\int_0^{t_n}\int_0^t \theta_2(t,s)e(s)w(t)\mathrm{d}s\mathrm{d}t = \int_0^{t_n}\int_0^s \theta_2(s,t)e(t)w(s)\mathrm{d}t\mathrm{d}s = \int_0^{t_n}\int_t^{t_n} \theta_2(s,t)w(s)e(t)\mathrm{d}s\mathrm{d}t$$

by exchanging the order of integration.

For any function $v \in L^2(I_n)$, we denote by $\bar{v} \in P_{k-1}(I_n)$ the usual L^2 projection of v onto $P_{k-1}(I_n)$, *i.e.*,

$$\int_{I_n} (v - \bar{v})\varphi\mathrm{d}t = 0, \quad \forall \varphi \in P_{k-1}(I_n).$$

Due to (5.8.27) of [10], there holds

$$\|v - \bar{v}\|_{L^2(I_n)} \le Ch_n^k \|v\|_{H^k(I_n)}. \tag{3.25}$$

Noting that

$$\left\|\int_0^t |e(s)|\mathrm{d}s\right\|_{L^2(0,t_n)} \le \left\{\int_0^{t_n}\left(\int_0^t \mathrm{d}s \int_0^t e^2(s)\mathrm{d}s\right)\mathrm{d}t\right\}^{\frac{1}{2}} \le \frac{\sqrt{2}}{2}T\|e\|_{L^2(0,t_n)}, \tag{3.26}$$

then using the assumptions $\theta_1(t) \in C^{k-1}([0,T])$ and $\theta_2(t,s) \in C^{k-1}(D)$, we have

$$\left\|e' - \theta_1 e - \int_0^t \theta_2(t,s)e(s)\mathrm{d}s\right\|_{L^2(0,t_n)} \le \|e'\|_{L^2(0,t_n)} + C\|e\|_{L^2(0,t_n)} + C\left\|\int_0^t |e(s)|\mathrm{d}s\right\|_{L^2(0,t_n)} \le C\|e\|_{H^1(0,t_n)}. \tag{3.27}$$

Similarly to (3.26), there holds

$$\left\|\int_0^t e^2(s)\mathrm{d}s\right\|_{L^2(0,t_n)} \le \left\{\int_0^{t_n}\left(\int_0^t \mathrm{d}s\int_0^t e^4(s)\mathrm{d}s\right)\mathrm{d}t\right\}^{\frac{1}{2}} \le \frac{\sqrt{2}}{2}T\|e^2\|_{L^2(0,t_n)},$$

which together with (3.20) implies

$$\left\|R_1e^2+\int_0^t R_2(t,s)e^2(s)\mathrm{d}s\right\|_{L^2(0,t_n)} \le C\|e^2\|_{L^2(0,t_n)}+C\left\|\int_0^t e^2(s)\mathrm{d}s\right\|_{L^2(0,t_n)} \le C\|e^2\|_{L^2(0,t_n)}. \tag{3.28}$$

Combining (3.21), (3.25), (3.27), (3.28), (3.23), and the L^2-stability of the L^2-projection $\bar{w}$ of w, gives

$$\begin{aligned}
&\int_0^{t_n}\left(e'-\theta_1 e-\int_0^t\theta_2(t,s)e(s)\mathrm{d}s\right)w\mathrm{d}t\\
=&\sum_{i=1}^n\int_{I_i}\left(e'-\theta_1 e-\int_0^t\theta_2(t,s)e(s)\mathrm{d}s\right)(w-\bar{w})\mathrm{d}t+\sum_{i=1}^n\int_{I_i}\left(R_1e^2+\int_0^t R_2(t,s)e^2(s)\mathrm{d}s\right)\bar{w}\mathrm{d}t\\
\le&\left\|e'-\theta_1 e-\int_0^t\theta_2(t,s)e(s)\mathrm{d}s\right\|_{L^2(0,t_n)}\left(\sum_{i=1}^n\|w-\bar{w}\|^2_{L^2(I_i)}\right)^{\frac12}\\
&+\left\|R_1e^2+\int_0^t R_2(t,s)e^2(s)\mathrm{d}s\right\|_{L^2(0,t_n)}\left(\sum_{i=1}^N\|\bar{w}\|^2_{L^2(I_i)}\right)^{\frac12}\\
\le& Ch^k\|e\|_{H^1(0,t_n)}\|w\|_{H^k(0,t_n)}+C\|e^2\|_{L^2(0,t_n)}\|w\|_{L^2(0,t_n)}\\
\le& Ch^k\|e\|_{H^1(0,t_n)}+C\|e^2\|_{L^2(0,t_n)}.
\end{aligned}\tag{3.29}$$

Moreover, noticing the embedding inequality $\|e\|_{L^\infty(0,t_n)}\le C\|e\|_{H^1(0,t_n)}$, we find

$$\|e^2\|_{L^2(0,t_n)}=\left(\int_0^{t_n}e^4\mathrm{d}t\right)^{\frac12}\le\|e\|_{L^\infty(0,t_n)}\left(\int_0^{t_n}e^2\mathrm{d}t\right)^{\frac12}\le C\|e\|_{H^1(0,t_n)}\|e\|_{L^2(0,t_n)}. \tag{3.30}$$

Finally, combining (3.24), (3.29) and (3.30), then using (2.4) and (2.3), gives

$$\begin{aligned}
|e(t_n)|&\le Ch^k\|e\|_{H^1(0,t_n)}+C\|e\|_{H^1(0,t_n)}\|e\|_{L^2(0,t_n)}\\
&\le Ch^{2k}\|u\|_{H^{k+1}(0,T)}+Ch^{2k+1}\|u\|^2_{H^{k+1}(0,T)}\\
&\le Ch^{2k}\|u\|_{H^{k+1}(0,T)}.
\end{aligned}$$

This ends the proof of (3.16). □

Remark 3.4. It is worth noting that the nodal superconvergence rate $O(h^{2k})$ has been proved in Theorem 3.2 of [18] provided that $u\in W^{k+1,\infty}(0,T)$. In Theorem 3.3, we prove the same nodal superconvergence rate $O(h^{2k})$ by using an alternative approach and a weaker assumption on u (*i.e.*, $u\in H^{k+1}(0,T)$).

3.3. Superclose between the projection $\Pi^k u$ and the CG solution

In this section, we prove that the CG solution U is closer to the projection $\Pi^k u$ than to the solution u, and we call such phenomena as superclose. To this end, we divide the error into two parts, *i.e.*,

$$e=u-U=(u-\Pi^k u)+(\Pi^k u-U):=\eta+\xi.$$

Here, $\eta=u-\Pi^k u$ and $\xi=\Pi^k u-U$.

Theorem 3.5. *Suppose that the assumptions of Theorem 3.3 hold. Then*

$$\|(\Pi^k u-U)'\|_{L^2(0,T)}\le Ch^{k+1}\|u\|_{H^{k+1}(0,T)}, \tag{3.31}$$

$$\|\Pi^k u - U\|_{L^2(0,T)} \le C h^{\min\{2k,k+2\}} \|u\|_{H^{k+1}(0,T)}. \tag{3.32}$$

Moreover, if $u \in W^{k+1,\infty}(0,T)$ *with integer* $k \ge 1$*, then we have*

$$\|\Pi^k u - U\|_{L^\infty(0,T)} \le C h^{\min\{2k,k+2\}} \|u\|_{W^{k+1,\infty}(0,T)}. \tag{3.33}$$

The constants $C > 0$ *are independent of* h*.*

Proof. Due to (3.7) of [29], there holds

$$\|\xi'\|^2_{L^2(I_n)} \le C\|e\|^2_{L^2(I_n)} + C h_n \|e\|^2_{L^2(0,t_n)}. \tag{3.34}$$

Summing up (3.34) over all element $I_n,\ 1 \le n \le N$, then using (2.3) leads to

$$\|\xi'\|^2_{L^2(0,T)} \le C\sum_{n=1}^{N} \|e\|^2_{L^2(I_n)} + C\sum_{n=1}^{N} h_n \|e\|^2_{L^2(0,t_n)} \le C\|e\|^2_{L^2(0,T)} \le C h^{2(k+1)} \|u\|^2_{H^{k+1}(0,T)}.$$

This proves (3.31).

We next prove (3.32). Due to the definition of $\Pi^k u$ and (3.16), we have

$$|\xi(t_n)| = |\Pi^k u(t_n) - U(t_n)| = |u(t_n) - U(t_n)| \le C h^{2k} \|u\|_{H^{k+1}(0,T)}. \tag{3.35}$$

Noting the fact

$$\xi(t) = \xi(t_n) - \int_t^{t_n} \xi'(s)\mathrm{d}s, \quad t \in [t_{n-1}, t_n],$$

and then using the Cauchy-Schwarz inequality, gives

$$|\xi(t)|^2 = \left(\xi(t_n) - \int_t^{t_n} \xi'(s)\mathrm{d}s\right)^2 \le 2\xi^2(t_n) + 2h_n \|\xi'\|^2_{L^2(I_n)}. \tag{3.36}$$

Integrating (3.36) with respect to the variable t on I_n, we obtain

$$\|\xi\|^2_{L^2(I_n)} \le \int_{I_n} 2|\xi(t_n)|^2 \mathrm{d}t + \int_{I_n} 2h_n \|\xi'\|^2_{L^2(I_n)} \mathrm{d}t \le 2h_n |\xi(t_n)|^2 + 2h_n^2 \|\xi'\|^2_{L^2(I_n)}. \tag{3.37}$$

Summing up (3.37) over all elements $I_n,\ 1 \le n \le N$, then using (3.35) and (3.31), gives

$$\|\xi\|^2_{L^2(0,T)} \le \sum_{n=1}^{N} 2h_n |\xi(t_n)|^2 + \sum_{n=1}^{N} 2h_n^2 \|\xi'\|^2_{L^2(I_n)} \le C h^{4k} \|u\|^2_{H^{k+1}(0,T)} + C h^{2k+4} \|u\|^2_{H^{k+1}(0,T)}, \tag{3.38}$$

which implies (3.32).

It remains to show (3.33). According to the Sobolev inequality, there holds

$$\|\xi\|^2_{L^\infty(I_n)} \le \frac{2}{h_n} \|\xi\|^2_{L^2(I_n)} + 2h_n \|\xi'\|^2_{L^2(I_n)}. \tag{3.39}$$

Combining (3.39), (3.37), (3.35) and (3.34), we get

$$\|\xi\|^2_{L^\infty(I_n)} \le C|\xi(t_n)|^2 + C h_n \|\xi'\|^2_{L^2(I_n)} \le C h^{4k} \|u\|^2_{H^{k+1}(0,T)} + C h_n \|e\|^2_{L^2(I_n)} + C h_n^2 \|e\|^2_{L^2(0,t_n)}. \tag{3.40}$$

Using the triangle inequality, (3.37) and (3.34), gives

$$\begin{aligned} \|e\|^2_{L^2(I_n)} &\le 2\|\xi\|^2_{L^2(I_n)} + 2\|\eta\|^2_{L^2(I_n)} \\ &\le 4h_n |\xi(t_n)|^2 + 4h_n^2 \|\xi'\|^2_{L^2(I_n)} + 2\|\eta\|^2_{L^2(I_n)} \\ &\le 4h_n |\xi(t_n)|^2 + C h_n^2 \|e\|^2_{L^2(I_n)} + C h_n^3 \|e\|^2_{L^2(0,t_n)} + 2\|\eta\|^2_{L^2(I_n)}, \end{aligned}$$

which implies that

$$\left(1 - Ch_n^2 - Ch_n^3\right)\|e\|_{L^2(I_n)}^2 \le 4h_n|\xi(t_n)|^2 + Ch_n^3\|e\|_{L^2(0,t_{n-1})}^2 + 2\|\eta\|_{L^2(I_n)}^2. \tag{3.41}$$

Suppose that there are two positive constants h_0 and c_0 such that $1 - Ch_0^2 - Ch_0^3 \ge c_0$. Then for $h \le h_0$, we can rewrite (3.41) as

$$\|e\|_{L^2(I_n)}^2 \le \frac{4}{c_0}h_n|\xi(t_n)|^2 + \frac{C}{c_0}h_n^3\|e\|_{L^2(0,t_{n-1})}^2 + \frac{2}{c_0}\|\eta\|_{L^2(I_n)}^2,$$

which together with (3.35), (2.3) and (3.13), gives

$$\|e\|_{L^2(I_n)}^2 \le Ch^{4k+1}\|u\|_{H^{k+1}(0,T)}^2 + Ch^{2k+5}\|u\|_{H^{k+1}(0,T)}^2 + Ch_n^{2k+2}\|u\|_{H^{k+1}(I_n)}^2, \tag{3.42}$$

where the constants $C > 0$ are independent of h but depending on c_0.

Inserting (3.42) and (2.3) into (3.40), we have

$$\|\xi\|_{L^\infty(I_n)}^2 \le Ch^{4k}\|u\|_{H^{k+1}(0,T)}^2 + Ch^{2k+4}\|u\|_{H^{k+1}(0,T)}^2 + Ch_n^{2k+3}\|u\|_{H^{k+1}(I_n)}^2. \tag{3.43}$$

Noting that $\|u\|_{H^{k+1}(0,T)} \le C\|u\|_{W^{k+1,\infty}(0,T)}$ and $\|u\|_{H^{k+1}(I_n)} \le Ch_n^{\frac{1}{2}}\|u\|_{W^{k+1,\infty}(I_n)}$, then by (3.43) we get

$$\begin{aligned}\|\xi\|_{L^\infty(0,T)}^2 &= \max_{1\le n\le N}\left\{\|\xi\|_{L^\infty(I_n)}^2\right\}\\ &\le Ch^{4k}\|u\|_{H^{k+1}(0,T)}^2 + Ch^{2k+4}\|u\|_{H^{k+1}(0,T)}^2 + C\max_{1\le n\le N}\left\{h_n^{2k+4}\|u\|_{W^{k+1,\infty}(I_n)}^2\right\}\\ &\le Ch^{\min\{4k,2k+4\}}\|u\|_{W^{k+1,\infty}(0,T)},\end{aligned}$$

which implies (3.33). □

4. Postprocessed superconvergence analysis of the CG method

In this section, we propose a postprocessing technique for the CG method. Based on the superclose results established in Theorem 3.5, we prove that the postprocessed CG solution U^* is superconvergent to the exact solution u.

4.1. Postprocessed CG solution

Let $\mathcal{T}_h$ be a given partition of $[0,T]$ with subintervals $\{I_n\}_{n=1}^N$. Recall that for any $u \in H^1(I_n)$ there holds (see (3.8))

$$u(t) = \sum_{i=0}^{\infty}\alpha_{i,n}\phi_{i,n}(t), \quad t \in I_n.$$

In view of (3.9) and (1.1), we have

$$\alpha_{k+1,n} = \frac{2k+1}{2}\int_{I_n} u'(t)L_{k,n}(t)\mathrm{d}t = \frac{2k+1}{2}\int_{I_n}\left(f(t,u(t)) + Ku(t)\right)L_{k,n}(t)\mathrm{d}t \tag{4.1}$$

for $k \ge 1$. By replacing u with U in (4.1), we get a computable term

$$\alpha_{k+1,n}^* := \frac{2k+1}{2}\int_{I_n}(f(t,U(t)) + KU(t))L_{k,n}(t)\mathrm{d}t, \quad k \ge 1. \tag{4.2}$$

Then, we define the postprocessed CG solution U^* on each time step I_n by

$$U^*(t) = U(t) + \alpha_{k+1,n}^*\phi_{k+1,n}(t), \quad t \in I_n \tag{4.3}$$

for $1 \le n \le N$, where $\phi_{k+1,n}$ is the Lobatto polynomial of degree $k+1$ as defined in (3.2). Clearly, U^* can be regarded as a simple correction for the CG solution u.

4.2. Postprocessed superconvergence results

The main results of this section are stated in the following theorem.

Theorem 4.1. *Suppose that the assumptions of Theorem 3.3 hold. Let u be the solution of (1.1) and U^* be the postprocessed CG solution as defined in (4.3). We further assume that $u \in H^{k+2}(0,T)$ with integer $k \geq 1$. Then*

$$\|u - U^*\|_{L^2(0,T)} \leq Ch^{\min\{2k,k+2\}}\|u\|_{H^{k+2}(0,T)}, \tag{4.4}$$

$$\|u - U^*\|_{H^1(0,T)} \leq Ch^{k+1}\|u\|_{H^{k+2}(0,T)}. \tag{4.5}$$

Moreover, if $u \in W^{k+2,\infty}(0,T)$ with integer $k \geq 1$, then

$$\|u - U^*\|_{L^\infty(0,T)} \leq Ch^{\min\{2k,k+2\}}\|u\|_{W^{k+2,\infty}(0,T)}. \tag{4.6}$$

The constants $C > 0$ are independent of h.

Proof. By (3.10) and (4.3), we have

$$\begin{aligned} u - U^* &= \left(u - \pi_{I_n}^{k+1}u\right) + \left(\pi_{I_n}^{k+1}u - \pi_{I_n}^{k}u - \alpha_{k+1,n}^*\phi_{k+1,n}\right) + \left(\pi_{I_n}^{k}u - U\right) &(4.7)\\ &= \left(u - \pi_{I_n}^{k+1}u\right) + \left(\alpha_{k+1,n} - \alpha_{k+1,n}^*\right)\phi_{k+1,n} + \left(\pi_{I_n}^{k}u - U\right). &(4.8)\end{aligned}$$

We fist show (4.4). Using (1.2), the Cauchy-Schwarz inequality and (3.6a), we get

$$\begin{aligned} |\alpha_{k+1,n} - \alpha_{k+1,n}^*| &= \frac{2k+1}{2}\left|\int_{I_n}\left(f(t,u) - f(t,U) + Ku(t) - KU(t)\right)L_{k,n}\mathrm{d}t\right| \\ &\leq \frac{2k+1}{2}\int_{I_n}\left(L_1|e| + L_2\int_0^t |e|\mathrm{d}s\right)|L_{k,n}|\mathrm{d}t \\ &\leq \frac{2k+1}{2}\left(L_1\|e\|_{L^2(I_n)} + Ch_n^{\frac{1}{2}}\|e\|_{L^2(0,T)}\right)\|L_{k,n}\|_{L^2(I_n)} \\ &\leq Ch_n^{\frac{1}{2}}\|e\|_{L^2(I_n)} + Ch_n\|e\|_{L^2(0,T)}. \end{aligned} \tag{4.9}$$

which together with (3.6b), yields

$$\|(\alpha_{k+1,n} - \alpha_{k+1,n}^*)\phi_{k+1,n}\|_{L^2(I_n)} \leq |\alpha_{k+1,n} - \alpha_{k+1,n}^*| \cdot \|\phi_{k+1,n}\|_{L^2(I_n)} \leq Ch_n\|e\|_{L^2(I_n)} + Ch_n^{\frac{3}{2}}\|e\|_{L^2(0,T)}. \tag{4.10}$$

Combining (4.7) and (4.10), gives

$$\|u - U^*\|_{L^2(I_n)} \leq Ch_n\|e\|_{L^2(I_n)} + Ch_n^{\frac{3}{2}}\|e\|_{L^2(0,T)} + \|u - \pi_{I_n}^{k+1}u\|_{L^2(I_n)} + \|\pi_{I_n}^{k}u - U\|_{L^2(I_n)}. \tag{4.11}$$

Squaring (4.11) and summing it up over I_n for $1 \leq n \leq N$, then using (3.12), (3.13), (2.3) and (3.32), we get

$$\begin{aligned} \|u - U^*\|_{L^2(0,T)}^2 &\leq Ch^2\|e\|_{L^2(0,T)}^2 + C\|u - \Pi^{k+1}u\|_{L^2(0,T)}^2 + C\|\Pi^k u - U\|_{L^2(0,T)}^2 \\ &\leq Ch^{2k+4}\|u\|_{H^{k+1}(0,T)}^2 + Ch^{2k+4}\|u\|_{H^{k+2}(0,T)}^2 + Ch^{\min\{4k,2k+4\}}\|u\|_{H^{k+1}(0,T)}^2 \\ &\leq Ch^{\min\{4k,2k+4\}}\|u\|_{H^{k+2}(0,T)}^2, \end{aligned} \tag{4.12}$$

which implies (4.4).

We next prove (4.5). Using (4.9), (3.6b) and (3.6c), we get

$$\|(\alpha_{k+1,n} - \alpha_{k+1,n}^*)\phi_{k+1,n}\|_{H^1(I_n)} \leq |\alpha_{k+1,n} - \alpha_{k+1,n}^*| \cdot \|\phi_{k+1,n}\|_{H^1(I_n)} \leq C\|e\|_{L^2(I_n)} + Ch_n^{\frac{1}{2}}\|e\|_{L^2(0,T)}. \tag{4.13}$$

Using (4.7) and (4.13), gives

$$\|u-U^*\|_{H^1(I_n)} \le C\|e\|_{L^2(I_n)} + Ch_n^{\frac{1}{2}}\|e\|_{L^2(0,T)} + \|u-\pi_{I_n}^{k+1}u\|_{H^1(I_n)} + \|\pi_{I_n}^k u - U\|_{H^1(I_n)}. \tag{4.14}$$

Similar to the derivation of (4.12), combing (4.14), (3.12), (3.14), (2.3), (3.31) and (3.32), yields

$$\begin{aligned}\|u-U^*\|^2_{H^1(0,T)} &\le C\|e\|^2_{L^2(0,T)} + C\|u-\Pi^{k+1}u\|^2_{H^1(0,T)} + C\|\Pi^k u - U\|^2_{H^1(0,T)}\\ &\le Ch^{2k+2}\|u\|^2_{H^{k+1}(0,T)} + Ch^{2k+2}\|u\|^2_{H^{k+2}(0,T)} + Ch^{\min\{4k,2k+4\}}\|u\|^2_{H^{k+1}(0,T)}\\ &\le Ch^{2k+2}\|u\|^2_{H^{k+2}(0,T)}.\end{aligned}$$

which implies (4.5).

It remains to show (4.6). According to (4.9) and (3.6d), we have

$$\|(\alpha_{k+1,n}-\alpha^*_{k+1,n})\phi_{k+1,n}\|_{L^\infty(I_n)} \le |\alpha_{k+1,n}-\alpha^*_{k+1,n}|\cdot\|\phi_{k+1,n}\|_{L^\infty(I_n)} \le Ch_n^{\frac{1}{2}}\|e\|_{L^2(I_n)} + Ch_n\|e\|_{L^2(0,T)}. \tag{4.15}$$

Using (4.7) and (4.15), gives

$$\|u-U^*\|^2_{L^\infty(I_n)} \le Ch_n\|e\|^2_{L^2(I_n)} + Ch_n^2\|e\|^2_{L^2(0,T)} + C\|u-\pi_{I_n}^{k+1}u\|^2_{L^\infty(I_n)} + C\|\pi_{I_n}^k u - U\|^2_{L^\infty(I_n)}. \tag{4.16}$$

Using (4.16) and (3.42), then combining (3.12), (2.3), (3.15), and (3.33), we get

$$\begin{aligned}\|u-U^*\|^2_{L^\infty(0,T)} \le& C\max_{1\le n\le N}\left\{h_n\|e\|^2_{L^2(I_n)}\right\} + Ch^2\|e\|^2_{L^2(0,T)} + C\|u-\Pi^{k+1}u\|^2_{L^\infty(0,T)} + C\|\Pi^k u - U\|^2_{L^\infty(0,T)}\\ \le& C\max_{1\le n\le N}\left\{h^{4k+2}\|u\|^2_{H^{k+1}(0,T)} + h^{2k+6}\|u\|^2_{H^{k+1}(0,T)} + h_n^{2k+3}\|u\|^2_{H^{k+1}(I_n)}\right\}\\ &+ Ch^{2k+4}\|u\|^2_{H^{k+1}(0,T)} + C\max_{1\le n\le N}\left\{h_n^{2k+4}\|u\|^2_{W^{k+2,\infty}(I_n)}\right\} + Ch^{\min\{4k,2k+4\}}\|u\|^2_{W^{k+1,\infty}(0,T)}\\ \le& Ch^{\min\{4k,2k+4\}}\|u\|^2_{W^{k+2,\infty}(0,T)}.\end{aligned}$$

Here, we have used the estimates $\|u\|_{H^{k+1}(0,T)} \le C\|u\|_{W^{k+1,\infty}(0,T)} \le C\|u\|_{W^{k+2,\infty}(0,T)}$ and $\|u\|_{H^{k+1}(I_n)} \le Ch_n^{\frac{1}{2}}\|u\|_{W^{k+1,\infty}(I_n)}$. This completes the proof of (4.6). □

Remark 4.2. According to Lemma 2.1 and Theorem 4.1, we observe that the convergence rates of the L^2-, H^1- and L^∞-error estimates are improved by one order, namely,

$$\|u-U^*\|_{L^2(0,T)} = O(h^{k+2}), \quad \|u-U^*\|_{L^\infty(0,T)} = O(h^{k+2}), \quad k\ge 2,$$

and

$$\|u-U^*\|_{H^1(0,T)} = O(h^{k+1}), \quad k \ge 1.$$

Moreover, using (4.3), the fact $\phi_{k+1,n}(t_n) = 0$ and (3.16), we have

$$|(u-U^*)(t_n)| = |(u-U)(t_n)| \le Ch^{2k}\|u\|_{H^{k+1}(0,T)}, \quad 1\le n\le N, \quad k\ge 1,$$

which implies that the postprocessed CG solution U^* keeps the same nodal superconvergence as the CG solution.

5. Asymptoticlly exact a posteriori error estimators

In this section, we construct several asymptotically exact a posteriori error estimators for the CG method based on the postprocessed superconvergence results.

In view of the definition (4.3) of the postprocessed solution U^*, we define the a posteriori error estimators by

$$\begin{aligned}\eta_0 &:= \|U^* - U\|_{L^2(0,T)} = \|\alpha^*_{k+1,n}\phi_{k+1,n}\|_{L^2(0,T)},\\ \eta_1 &:= \|U^* - U\|_{H^1(0,T)} = \|\alpha^*_{k+1,n}\phi_{k+1,n}\|_{H^1(0,T)},\\ \eta_\infty &:= \|U^* - U\|_{L^\infty(0,T)} = \|\alpha^*_{k+1,n}\phi_{k+1,n}\|_{L^\infty(0,T)}.\end{aligned} \tag{5.1}$$

For convenience, we also define the unprocessed and postprocessed L^2-, H^1- and L^∞-errors

$$\begin{aligned}&E_0 := \|u - U\|_{L^2(0,T)}, \quad E_1 := \|u - U\|_{H^1(0,T)}, \quad E_\infty := \|u - U\|_{L^\infty(0,T)},\\ &E_0^* := \|u - U^*\|_{L^2(0,T)}, \quad E_1^* := \|u - U^*\|_{H^1(0,T)}, \quad E_\infty^* := \|u - U^*\|_{L^\infty(0,T)}.\end{aligned} \tag{5.2}$$

In the following theorem, we show that the a posteriori error estimators $\eta_m,\ m = 0, 1, \infty$, are asymptotically exact as $h \to 0$ under reasonable assumptions.

Theorem 5.1. *Assume the hypotheses of Theorem 4.1. We further assume that there exist positive constants $C_m(u),\ m = 0, 1, \infty$, independent of h but depending on u such that*

$$E_1 \geq C_1(u)h^k, \quad k \geq 1 \quad \text{and} \quad E_m \geq C_m(u)h^{k+1}, \quad m = 0, \infty, \quad k \geq 2, \tag{5.3}$$

where the errors $E_m,\ m = 0, 1, \infty$, are defined by (5.2). Then, the a posteriori error estimators η_m as defined in (5.1) are asymptotically exact, i.e.,

$$\lim_{h\to 0} \frac{\eta_m}{E_m} = 1, \quad m = 0, 1, \infty. \tag{5.4}$$

Proof. According to (5.1) and the triangle inequality, there hold

$$\eta_m = \|(U^* - u) + (u - U)\|_m \geq E_m - E_m^* \tag{5.5}$$

and

$$\eta_m = \|(U^* - u) + (u - U)\|_m \leq E_m + E_m^* \tag{5.6}$$

for $m = 0, 1, \infty$. Here, we denote by $\|\cdot\|_m$ the L^2-, H^1- and L^∞-norms for $m = 0, 1, \infty$, respectively. In view of (5.5) and (5.6), then using Theorem 4.1 and (5.3), we obtain

$$\left|\frac{\eta_m}{E_m} - 1\right| \leq \frac{E_m^*}{E_m} = O(h) \to 0 \quad \text{as} \quad h \to 0,$$

which implies (5.4). ☐

Remark 5.2. Proving the assumption (5.3) remains an open issue in our situation, which is beyond the scope of the present paper. However, let us make some comments. First, for one-dimensional elliptic problems, similar estimate as (5.3) has been proved in [2]. Second, in order to prove the asymptotic exactness of a posterior error estimators, such assumption has been frequently used in the existing literature (see, *e.g.*, [1] and the reference therein). Third, numerical results show that the convergence rates as predicted by Lemma 2.1 are sharp, *i.e.*, $E_m = O(h^{k+1})$ with $m = 0, \infty$ and $E_1 = O(h^k)$, which also imply the lower bounds as stated in (5.3). In that sense, the assumption (5.3) is reasonable.

6. Numerical experiments

In this section, we present some numerical results to verify the theoretical findings. The test problems are solved by the CG method with uniform degree k on uniform or nonuniform time partitions. Let U be the CG solution and U^* be the postprocessed CG solution. We denote by $e := u - U$ and $e^* := u - U^*$ the error

TABLE 1. Example 1: maximum nodal errors and convergence orders.

k	N	$\max\|e(t_n)\|$	order	k	N	$\max\|e(t_n)\|$	Order
	4	1.71 E-03	2.16		4	6.70 E-09	4.93
1	8	4.83 E-04	1.83	3	8	9.21 E-11	6.19
	16	1.23 E-04	1.98		16	1.53 E-12	5.91
	4	1.58 E-05	4.54		4	1.17 E-11	9.14
2	8	9.32 E-07	4.08	4	8	3.25 E-14	8.49
	16	4.53 E-08	4.36		16	1.11 E-16	8.19

functions, and by 'order' the actual rate of convergence of the unprocessed or postprocessed CG method with respect to N, where N is the number of elements in the time partition. Clearly, $N \simeq 1/h$ for the uniform and quasi-uniform time partitions.

Throughout, all integrals (including the L^2- and H^1-error norms) are numerically evaluated by the Gauss-Legendre quadrature formula with $k+1$-points unless otherwise specified. Let $\{t_n\}_{n=0}^N$ be the nodes of a given time partition of $[0,T]$. We calculate the L^∞-errors by

$$\|e\|_{L^\infty(0,T)} \approx \max_{1\le n\le N,\ 0\le j\le 20} \{|e(t_{n,j})|\}, \quad \|e^*\|_{L^\infty(0,T)} \approx \max_{1\le n\le N,\ 0\le j\le 20} \{|e^*(t_{n,j})|\},$$

where $t_{n,j} = t_{n-1} + jh_n/20$ with $0 \le j \le 20$.

For measuring the efficiency of the a posteriori error estimators, we define the global effectivity indices by

$$\zeta_m := \frac{\eta_m}{E_m}, \quad m = 0, 1, \infty,$$

where η_m and E_m are defined by (5.1) and (5.2), respectively.

6.1. Example 1

We consider the nonlinear VIDE (*cf.* [29]):

$$\begin{cases} u'(t) = f(t,u(t)) + \displaystyle\int_0^t t\cos(s)e^{tu(s)}\mathrm{d}s, & t \in [0,T], \\ u(0) = 0, \end{cases} \tag{6.1}$$

where $f(t,u(t)) = 1+\cos(t) - e^{t\sin(t)} - \cos(t+2\sin(t)) + \cos(t+2u(t))$ and the solution of (6.1) is $u(t) = \sin(t)$.

Consider the problem (6.1) with $T = 1$. Following [13], we use the nonuniform time partition of $[0,T]$ with nodal points given by

$$t_n = \frac{n}{N} + 0.3\frac{1}{N}\sin\left(\frac{n\pi}{N}\right) rand(), \quad 0 \le n \le N,$$

where $rand()$ returns a uniformly distributed random number in $(0,1)$. Obviously, this mesh is generated based on a random perturbation on the uniform mesh.

We first consider the maximum nodal errors. In Table 1, we list the maximum nodal errors and convergence rates for $k = 1,2,3$ and 4. Here, we denote by $\max|e(t_n)|$ the maximum absolute errors at nodal points $\{t_n\}_{n=1}^N$ of the time partition. It can be seen that the CG method superconvergences at the nodal points with order $2k$, which confirms the theoretical result in Theorem 3.3.

We next consider the performance of the postprocessing technique as defined by (4.3). In Tables 2–4, we list the unprocessed and postprocessed L^2-, H^1-, L^∞-errors and their convergence orders for $k = 1,2,3$ and 4. Clearly, the convergence of the L^2- and L^∞-errors are improved by one order for $k \ge 2$, while the convergence of the H^1-errors is improved by one order for $k \ge 1$, which coincides well with Theorem 4.1. We also observe

TABLE 2. Example 1: L^2-errors, convergence orders, and global effectivity indices ζ_0.

k	N	$\|e\|_{L^2(0,T)}$	Order	$\|e^*\|_{L^2(0,T)}$	Order	ζ_0
	8	4.15 E-04	1.82	2.90 E-04	1.94	1.64
1	16	1.05 E-04	1.98	7.20 E-05	2.01	1.63
	32	2.64 E-05	1.99	1.79 E-05	2.01	1.63
	8	9.45 E-06	3.20	4.34 E-07	4.30	1.00
2	16	1.10 E-06	3.11	2.29 E-08	4.25	1.00
	32	1.34 E-07	3.03	1.34 E-09	4.10	1.00
	8	3.55 E-08	4.07	1.20 E-09	5.13	1.00
3	16	2.28 E-09	3.96	3.55 E-11	5.08	1.00
	32	1.44 E-10	3.99	1.10 E-12	5.01	1.00
	8	4.34 E-10	4.82	1.04 E-11	5.94	1.00
4	16	1.19 E-11	5.18	1.51 E-13	6.11	1.00
	32	3.68 E-13	5.02	2.26 E-15	6.06	1.00

TABLE 3. Example 1: H^1-errors, convergence orders, and global effectivity indices ζ_1.

k	N	$\|e\|_{H^1(0,T)}$	Order	$\|e^*\|_{H^1(0,T)}$	Order	ζ_1
	8	1.86 E-02	0.92	6.91 E-04	1.84	1.00
1	16	9.39 E-03	0.99	1.71 E-04	2.01	1.00
	32	4.71 E-03	1.00	4.19 E-05	2.03	1.00
	8	5.46 E-04	2.13	1.44 E-05	3.11	1.00
2	16	1.30 E-04	2.07	1.64 E-06	3.14	1.00
	32	3.21 E-05	2.02	2.00 E-07	3.03	1.00
	8	3.21 E-06	3.04	6.17 E-08	4.09	1.00
3	16	4.16 E-07	2.95	3.78 E-09	4.03	1.00
	32	5.22 E-08	3.00	2.37 E-10	4.00	1.00
	8	5.08 E-08	3.88	6.39 E-10	4.95	1.00
4	16	2.88 E-09	4.14	1.83 E-11	5.13	1.00
	32	1.78 E-10	4.01	5.56 E-13	5.04	1.00

that the the global effectivity indices ζ_0 and ζ_∞ always approach 1 for $k \geq 2$ and the index ζ_1 approaches 1 for $k \geq 1$, which implies that the error estimators are asymptotically exact and confirms the results in Theorem 5.1. Moreover, we find that although the convergence orders of the L^2- and L^∞-errors for $k = 1$ are not improved after postprocessing, the errors are slightly improved, and the effectivity indices ζ_0 and ζ_∞ are around 1.63 and 1.33, respectively, which are within a reasonable range.

6.2. Example 2

We consider the nonlinear VIDE (*cf.* [3]):

$$\begin{cases} u'(t) = f(t, u(t)) + \displaystyle\int_0^t (1 - t + s)u^2(s)\mathrm{d}s, \quad t \in [0, T], \\ u(0) = 1, \end{cases} \tag{6.2}$$

where $f(t, u(t)) = t - \ln(1 + t) - \frac{1}{(1+t)^2} - tu(t)$ and the solution of (6.2) is $u(t) = \frac{1}{1+t}$.

Consider the problem (6.2) with $T = 10$. The uniform time partitions which consist of N elements are used for this example.

TABLE 4. Example 1: L^∞-errors, convergence orders, and global effectivity indices ζ_∞.

k	N	$\|e\|_{L^\infty(0,T)}$	Order	$\|e^*\|_{L^\infty(0,T)}$	Order	ζ_∞
1	8	1.11 E-03	1.64	4.88 E-04	1.83	1.33
	16	3.22 E-04	1.79	1.24 E-04	1.98	1.23
	32	8.12 E-05	1.99	3.08 E-05	2.00	1.33
2	8	2.20 E-05	3.18	9.32 E-07	4.08	1.00
	16	2.77 E-06	2.99	6.06 E-08	3.94	1.00
	32	3.58 E-07	2.95	3.91 E-09	3.95	1.00
3	8	1.26 E-07	3.94	4.08 E-09	4.81	1.00
	16	8.73 E-09	3.85	1.32 E-10	4.95	1.00
	32	5.47 E-10	4.00	4.59 E-12	4.84	1.00
4	8	1.45 E-09	4.28	3.26 E-11	5.49	0.99
	16	3.77 E-11	5.27	5.26 E-13	5.95	0.99
	32	1.20 E-12	4.98	8.33 E-15	5.98	1.00

TABLE 5. Example 2: maximum nodal errors and convergence orders.

k	N	$\max\|e(t_n)\|$	Order	k	N	$\max\|e(t_n)\|$	Order
1	32	4.37 E-03	2.10	3	32	2.82 E-07	5.76
	64	1.03 E-03	2.08		64	4.38 E-09	6.01
	128	2.59 E-04	1.99		128	6.84 E-11	6.00
2	32	4.66 E-05	3.87	4	32	1.13 E-09	7.57
	64	2.85 E-06	4.03		64	4.36 E-12	8.01
	128	1.77 E-07	4.01		128	1.70 E-14	8.00

TABLE 6. Example 2: L^2-errors, convergence orders, and global effectivity indices ζ_0.

k	N	$\|e\|_{L^2(0,T)}$	Order	$\|e^*\|_{L^2(0,T)}$	Order	ζ_0
1	128	5.26 E-04	2.00	5.04 E-04	2.00	0.86
	256	1.31 E-04	2.00	1.26 E-04	2.00	0.87
	512	3.29 E-05	2.00	3.15 E-05	2.00	0.87
2	128	5.18 E-06	2.98	3.04 E-07	3.96	1.00
	256	6.50 E-07	2.99	1.91 E-08	3.99	1.00
	512	8.13 E-08	3.00	1.20 E-09	4.00	1.00
3	128	7.87 E-08	3.97	3.66 E-09	4.95	1.00
	256	4.95 E-09	3.99	1.16 E-10	4.99	1.00
	512	3.10 E-10	4.00	3.62 E-12	5.00	1.00
4	128	1.30 E-09	4.95	5.57 E-11	5.92	1.00
	256	4.10 E-11	4.99	8.83 E-13	5.98	1.00
	512	1.28 E-12	5.00	1.39 E-14	5.99	1.00

In Table 5, we show the numerical convergence rates of the maximum nodal errors. It can be seen that the convergence rates of order $2k$ are observed in Table 5, thereby confirming the theoretical result in Theorem 3.3.

In Tables 6–8, we list the unprocessed and postprocessed L^2-, H^1-, L^∞-errors and their convergence orders, as well as the global effectivity indices for different k. We observe the same superconvergence results of the postprocessed CG approximations as those presented in Example 1. Additionally, we note that, the global

TABLE 7. Example 2: H^1-errors, convergence orders, and global effectivity indices ζ_1.

k	N	$\|e\|_{H^1(0,T)}$	Order	$\|e^*\|_{H^1(0,T)}$	Order	ζ_1
	128	2.01 E-02	0.99	7.55 E-04	2.00	1.00
1	256	1.01 E-02	1.00	1.89 E-04	2.00	1.00
	512	5.04 E-03	1.00	4.72 E-05	2.00	1.00
	128	5.13 E-04	1.98	9.93 E-06	2.98	1.00
2	256	1.29 E-04	1.99	1.25 E-06	3.00	1.00
	512	3.22 E-05	2.00	1.56 E-07	3.00	1.00
	128	1.19 E-05	2.96	1.51 E-07	3.96	1.00
3	256	1.50 E-06	2.99	9.53 E-09	3.99	1.00
	512	1.88 E-07	3.00	5.96 E-10	4.00	1.00
	128	2.64 E-07	3.94	2.51 E-09	4.94	1.00
4	256	1.67 E-08	3.98	7.93 E-11	4.99	1.00
	512	1.04 E-09	4.00	2.48 E-12	5.00	1.00

TABLE 8. Example 2: L^∞-errors, convergence orders, and global effectivity indices ζ_∞.

k	N	$\|e\|_{L^\infty(0,T)}$	Order	$\|e^*\|_{L^\infty(0,T)}$	Order	ζ_∞
	128	1.30 E-03	1.80	2.65 E-04	2.03	1.05
1	256	3.51 E-04	1.89	6.54 E-05	2.02	1.03
	512	9.15 E-05	1.94	1.62 E-05	2.01	1.01
	128	1.99 E-05	2.80	1.12 E-06	3.70	0.99
2	256	2.67 E-06	2.90	7.81 E-08	3.84	0.99
	512	3.45 E-07	2.95	5.16 E-09	3.92	1.00
	128	3.85 E-07	3.74	1.24 E-08	4.72	1.00
3	256	2.64 E-08	3.86	4.28 E-10	4.85	1.00
	512	1.73 E-09	3.93	1.41 E-11	4.93	1.00
	128	6.73 E-09	4.70	2.18 E-10	5.65	1.00
4	256	2.35 E-10	4.84	3.87 E-12	5.82	1.00
	512	7.76 E-12	4.92	6.45 E-14	5.91	1.00

TABLE 9. Example 2: CPU time (in seconds) of the unprocessed and postprocessed CG approximations.

k	N	CPUT	CPUT-P	k	N	CPUT	CPUT-P
	128	0.57	0.05		128	6.66	0.18
1	256	1.48	0.15	3	256	24.14	0.76
	512	4.17	0.57		512	68.81	3.18
	128	2.17	0.09		128	10.24	0.23
2	256	7.22	0.38	4	256	40.15	0.93
	512	25.62	1.70		512	107.87	3.70

effectivity indices ζ_m with $m = 0, 1, \infty$ always approach 1 (except the case of $k = 1$ for ζ_0), which implies that the error estimators are asymptotically exact.

In Table 9, we list the CPU time (in seconds) for obtaining the unprocessed and postprocessed CG approximations, where CPUT and CPUT-P denote the CPU time cost of the unprocessed and postprocessed CG

TABLE 10. Example 3: L^2-errors, convergence orders, and global effectivity indices ζ_0.

k	N	$\|e\|_{L^2(0,T)}$	Order	$\|e^*\|_{L^2(0,T)}$	Order	ζ_0
	16	2.17 E-04	1.99	1.78 E-04	1.99	1.65
1	32	5.44 E-05	2.00	4.45 E-05	2.00	1.65
	64	1.36 E-05	2.00	1.11 E-05	2.00	1.65
	16	6.13 E-06	2.93	6.75 E-07	3.87	0.99
2	32	7.74 E-07	2.99	4.28 E-08	3.98	1.00
	64	9.70 E-08	3.00	2.68 E-09	4.00	1.00
	16	6.73 E-07	4.03	7.02 E-08	5.07	0.98
3	32	4.17 E-08	4.01	2.11 E-09	5.06	0.99
	64	2.60 E-09	4.00	6.40 E-11	5.04	1.00
	16	7.62 E-08	4.95	8.16 E-09	5.94	0.98
4	32	2.39 E-09	4.99	1.25 E-10	6.02	0.99
	64	7.49 E-11	5.00	1.92 E-12	6.03	1.00

approximations, respectively. It can be seen that, the CPU time cost of the postprocessing process is far less than the cost of the original CG approximation, which is almost negligible.

6.3. Example 3

Although the present theory does not apply to weakly singular VIDEs with nonsmooth solutions, we expect good results for the postprocessed CG approximation when (nonuniform) graded meshes are used. Thus, we consider the linear VIDE with weakly singular kernel:

$$\begin{cases} u'(t) + u(t) + \displaystyle\int_0^t (t-s)^{-\frac{1}{2}} e^s u(s) \mathrm{d}s = f(t), \quad t \in [0,T], \\ u(0) = 0. \end{cases} \tag{6.3}$$

We choose the right-hand side f such that the solution of (6.3) is given by $u(t) = t^{\frac{3}{2}} e^{-t}$. Obviously, $u \in H^{2-\varepsilon}(0,T)$(with arbitrary $\varepsilon > 0$) and the second-order derivative of u is unbounded near $t = 0$.

Consider the problem (6.3) with $T = 1$. We use the hp-version of the composite Gauss-Legendre quadrature developed in [35] to evaluate the involved weakly singular integrals. Moreover, in order to capture the initial singularity of the solution at $t = 0$, we use the graded mesh with nodes given by

$$t_n = \left(\frac{n}{N}\right)^\gamma T, \quad 0 \le n \le N.$$

Throughout this example, we set the grading parameter $\gamma = k + 1$. It is worth pointing out that, the selection of the optimal grading parameter γ has been studied for the collocation method [8] and DG method [21] for weakly singular VIDEs. However, in our situation of the CG method, it still needs further investigation.

In Tables 10–12, we list the unprocessed and postprocessed L^2-, H^1-, L^∞-errors and their convergence orders, as well as the global effectivity indices for different k. From these tables, we observe the same superconvergence results of the postprocessed CG approximations as those reported in Examples 1 and 2 for smooth solutions. In addition, we see again that the global effectivity indices ζ_m with $m = 0, 1, \infty$ always approach 1 (except the case of $k = 1$ for ζ_0 and ζ_∞).

7. Concluding remarks

In this paper, we propose and analyze a simple but efficient postprocessing technique for the CG method of nonlinear VIDEs with smooth kernels. We prove that the postprocessing improves the convergence of the CG

TABLE 11. Example 3: H^1-errors, convergence orders, and global effectivity indices ζ_1.

k	N	$\|e\|_{H^1(0,T)}$	Order	$\|e^*\|_{H^1(0,T)}$	Order	ζ_1
	16	1.35 E-02	0.98	6.90 E-04	2.00	1.00
1	32	6.78 E-03	0.99	1.72 E-04	2.00	1.00
	64	3.39 E-03	1.00	4.30 E-05	2.00	1.00
	16	6.10 E-04	1.94	1.08 E-05	3.08	1.00
2	32	1.54 E-04	1.98	1.27 E-06	3.09	1.00
	64	3.87 E-05	2.00	1.52 E-07	3.06	1.00
	16	5.65 E-05	2.95	1.22 E-06	4.17	1.00
3	32	7.14 E-06	2.98	7.07 E-08	4.11	1.00
	64	8.95 E-07	3.00	4.20 E-09	4.07	1.00
	16	6.18 E-06	3.91	1.54 E-07	5.07	1.00
4	32	3.94 E-07	3.97	4.55 E-09	5.08	1.00
	64	2.48 E-08	3.99	1.36 E-10	5.06	1.00

TABLE 12. Example 3: L^∞-errors, convergence orders, and global effectivity indices ζ_∞.

k	N	$\|e\|_{L^\infty(0,T)}$	Order	$\|e^*\|_{L^\infty(0,T)}$	Order	ζ_∞
	16	5.39 E-04	1.94	4.51 E-04	2.00	1.62
1	32	1.36 E-04	1.99	1.13 E-04	2.00	1.63
	64	3.41 E-05	2.00	2.82 E-05	2.00	1.64
	16	1.42 E-05	3.01	1.37 E-06	3.58	1.00
2	32	1.84 E-06	2.95	9.65 E-08	3.83	0.98
	64	2.48 E-07	2.89	6.25 E-09	3.95	0.97
	16	1.68 E-06	3.91	1.31 E-07	5.01	0.99
3	32	1.05 E-07	4.00	3.89 E-09	5.07	1.00
	64	6.60 E-09	3.99	1.17 E-10	5.06	1.00
	16	2.19 E-07	4.79	1.86 E-08	5.73	0.97
4	32	7.18 E-09	4.93	3.03 E-10	5.94	0.99
	64	2.28 E-10	4.98	4.75 E-12	6.00	0.99

method in the L^2-, H^1- and L^∞-norms by one order for regular solutions. As a result, we construct several asymptotically exact a posteriori error estimators as the step-size approaches zero.

Numerical results show that, for weakly singular VIDEs with nonsmooth solutions, after postprocessing the convergence rates of the CG method with graded meshes can be also improved by one order, but this is not covered by our theoretical results. Thus, the superconvergence analysis of the postprocessed CG method for weakly singular VIDEs will be a topic of our future research.

The a posterior error estimators developed in this paper can be used for adaptive implementation of the CG time stepping method for VIDEs, although the efficiency of the local error estimators still need further study.

Acknowledgements. The work is supported by the National Natural Science Foundation of China (Grant Nos. 12171322, 11771298 and 12271366), the Natural Science Foundation of Shanghai (Grant Nos. 21ZR1447200 and 22ZR1445500), and the Science and Technology Innovation Plan of Shanghai (Grant No. 20JC1414200).

References

[1] M. Ainsworth and J.T. Oden, A Posteriori Error Estimation in Finite Element Analysis. John Wiley & Sons, New York (2000).

[2] I. Babuška and W.C. Rheinboldt, A posteriori error analysis of finite element solutions for one-dimensional problems. *SIAM J. Numer. Anal.* **18** (1981) 565–589.

[3] H. Brunner, On the numerical solution of nonlinear Volterra integro-differential equations. *BIT Numer. Math.* **13** (1973) 381–390.

[4] H. Brunner, Implicit Runge-Kutta methods of optimal order for Volterra integro-differential equations. *Math. Comput.* **42** (1984) 95-109.

[5] H. Brunner, Collocation Methods for Volterra Integral and Related Functional Equations. Cambridge University Press, Cambridge (2004).

[6] H. Brunner and P.J. van der Houwen, The Numerical Solution of Volterra Equations. North-Holland, Amsterdam (1986).

[7] H. Brunner and D. Schötzau, *hp*-discontinuous Galerkin time-stepping for Volterra integrodifferential equations. *SIAM J. Numer. Anal.* **44** (2006) 22–245.

[8] H. Brunner, A. Pedas and G. Vainikko, Piecewise polynomial collocation methods for linear Volterra integro-differential equations with weakly singular kernels. *SIAM J. Numer. Anal.* **39** (2001) 957–982.

[9] T.A. Burton, Volterra Integral and Differential Equations, 2nd edition. Vol. 202, Mathematics in Science and Engineering, Elsevier B.V. Amsterdam (2005).

[10] C. Canuto, M.Y. Hussaini, A. Quarteroni and T.A. Zang, Spectral Methods: Fundamentals in Single Domains, Scientific Computation. Springer, Berlin, Heidelberg (2006).

[11] W.X. Cao, Z.M. Zhang and Q.S. Zou, Is $2k$-conjecture valid for finite volume methods? *SIAM J. Numer. Anal.* **53** (2015) 942–962.

[12] W.X. Cao, H.L. Liu and Z.M. Zhang, Superconvergence of the direct discontinuous Galerkin method for convection-diffusion equations. *Numer. Methods Partial Differ. Equ.* **33** (2017) 290–317.

[13] W.X. Cao, L.L. Jia and Z.M. Zhang, A C^1 Petrov-Galerkin method and Gauss collocation method for 1D general elliptic problems and superconvergence. *Discrete Contin. Dyn. Syst. Ser. B* **26** (2021) 81–105.

[14] Q. Hu, Stieltjes derivatives and β-polynomial spline collocation for Volterra integrodifferential equations with singularities. *SIAM J. Numer. Anal.* **33** (1996) 208–220.

[15] Q.M. Huang and H.H. Xie, Superconvergence of Galerkin solutions for Hammerstein equations. *Int. J. Numer. Anal. Model.* **6** (2009) 696–710.

[16] H. Kaneko and Y. Xu, Superconvergence of the iterated Galerkin methods for Hammerstein equations. *SIAM J. Numer. Anal.* **33** (1996) 1048–1064.

[17] T. Lin, Y.P. Lin, M. Rao and S.H. Zhang, Petrov-Galerkin methods for linear Volterra integro-differential equations. *SIAM J. Numer. Anal.* **38** (2000) 937–963.

[18] T. Lin, Y.P. Lin, P. Luo, M. Rao and S.H. Zhang, Petrov-Galerkin methods for nonlinear Volterra integro-differential equations. *Dyn. Contin. Discrete Impuls. Syst. Ser. B* **8** (2001) 405–426.

[19] P. Linz, Analytical and Numerical Methods for Volterra Equations. SIAM, Philadelphia, PA (1985).

[20] Ch. Lubich, Runge-Kutta theory for Volterra integro-differential equations. *Numer. Math.* **40** (1982) 119–135.

[21] K. Mustapha, A superconvergent discontinuous Galerkin method for Volterra integro-differential equations, smooth and nonsmooth kernels. *Math. Comput.* **82** (2013) 1987–2005.

[22] K. Mustapha and J.K. Ryan, Post-processing discontinuous Galerkin solutions to Volterra integro-differential equations: analysis and simulations. *J. Comput. Appl. Math.* **253** (2013) 89–103.

[23] I.P. Natanson, Theory of Functions of a Real Variable, Translated from the Russian by Leo F. Boron with the collaboration of Edwin Hewitt. Frederick Ungar Publishing Co. New York (1955).

[24] T. Tang, A note on collocation methods for Volterra integro-differential equations with weakly singular kernels. *IMA J. Numer. Anal.* **13** (1993) 93–99.

[25] Z.Q. Wang, Y.L. Guo and L.J. Yi, An *hp*-version Legendre-Jacobi spectral collocation method for Volterra integro-differential equations with smooth and weakly singular kernels. *Math. Comput.* **86** (2017) 2285–2324.

[26] A.M. Wazwaz, Linear and Nonlinear Integral Equations, Methods and Applications. Higher Education Press, Beijing, Springer, Heidelberg (2011).

[27] J. Wen, C.M. Huang and M. Li, Stability analysis of Runge-Kutta methods for Volterra integro-differential equations. *Appl. Numer. Math.* **146** (2019) 73–88.

[28] Y.X. Wei and Y.P. Chen, Convergence analysis of the Legendre spectral collocation methods for second order Volterra integro-differential equations. *Numer. Math. Theory Methods Appl.* **4** (2011) 419–438.

[29] L.J. Yi, An *h-p* version of the continuous Petrov-Galerkin finite element method for nonlinear Volterra integro-differential equations. *J. Sci. Comput.* **65** (2015) 715–734.

[30] L.J. Yi and B.Q. Guo, An *h-p* Petrov-Galerkin finite element method for linear Volterra integro-differential equations. *Sci. China Math.* **57** (2014) 2285–2300.

[31] L.J. Yi and B.Q. Guo, An *h-p* version of the continuous Petrov-Galerkin finite element method for Volterra integro-differential equations with smooth and nonsmooth kernels. *SIAM J. Numer. Anal.* **53** (2015) 2677–2704.

[32] W. Yuan and T. Tang, The numerical analysis of implicit Runge-Kutta methods for a certain nonlinear integro-differential equation. *Math. Comput.* **54** (1990) 155–168.

[33] S.H. Zhang, T. Lin, Y.P. Lin and M. Rao, Defect correction and a posteriori error estimation of Petrov-Galerkin methods for nonlinear Volterra integro-differential equations. *Appl. Math.* **45** (2000) 241–263.

[34] S.H. Zhang, T. Lin, Y.P. Lin and M. Rao, Extrapolation and a-posteriori error estimators of Petrov-Galerkin methods for non-linear Volterra integro-differential equations. *J. Comput. Math.* **19** (2001) 407–422.

[35] M.Z. Zhang, X.Y. Mao and L.J. Yi, Exponential convergence of the *hp*-version of the composite Gauss-Legendre quadrature for integrals with endpoint singularities. *Appl. Numer. Math.* **170** (2021) 340–352.