ESAIM: M2AN 57 (2023) 841–864
https://doi.org/10.1051/m2an/2022084

ESAIM: Mathematical Modelling and Numerical Analysis
www.esaim-m2an.org

DISCONTINUOUS GALERKIN METHODS FOR STOCHASTIC MAXWELL EQUATIONS WITH MULTIPLICATIVE NOISE

Jiawei Sun[1], Chi-Wang Shu[2] and Yulong Xing[1,*]

Abstract. In this paper we propose and analyze finite element discontinuous Galerkin methods for the one- and two-dimensional stochastic Maxwell equations with multiplicative noise. The discrete energy law of the semi-discrete DG methods were studied. Optimal error estimate of the semi-discrete method is obtained for the one-dimensional case, and the two-dimensional case on both rectangular meshes and triangular meshes under certain mesh assumptions. Strong Taylor 2.0 scheme is used as the temporal discretization. Both one- and two-dimensional numerical results are presented to validate the theoretical analysis results.

Mathematics Subject Classification. 65M12, 65M60, 65C30, 35Q61, 60H35.

Received April 12, 2022. Accepted October 2, 2022.

1. Introduction

Stochastic Maxwell equations are commonly used to model microscopic origins of randomness in electromagnetism. The concept of stochastic Maxwell equations was firstly introduced by Rytov *et al.* [1] to describe the fluctuations of an electromagnetic field. In [2], Ord *et al.* studied the Mark Kac random walk model and modified it into the Maxwell field equations in 1+1 dimensions. Such a model describes most of the telegraph equations, and the author's modification of the Kac model constructed a strong connection between the telegraph and Maxwell equations. In [3] Horsin *et al.* applied an abstract approach and a constructive approach by generalizing the Hilbert uniqueness method to analyze the approximate controllability of the stochastic Maxwell equations. Furthermore in [4] the deterministic and stochastic integrodifferential equations in Hilbert spaces were studied and the well-posedness for the Cauchy problem of the integrodifferential equation was analyzed. Such results were motivated by mathematical modeling of electromagnetic fields in complex random media.

Numerical methods are often used to solve Maxwell equations with various forms of stochasticity. In [5], Benner *et al.* studied the time-harmonic Maxwell's equations with some uncertainty in material parameters. They compared stochastic collocation and Monte Carlo simulation, and computed a reduced model in order to lower the computational cost. In [6] Jung considered the wave and Maxwell equations with fluctuations by a random change in media parameters. The author used polynomial chaos Galerkin projections to develop

Keywords and phrases. Discontinuous Galerkin methods, stochastic Maxwell equations, multiplicative noise, energy law, optimal error estimate.

[1] Department of Mathematics, The Ohio State University, Columbus, OH 43210, USA.

[2] Division of Applied Mathematics, Brown University, Providence, RI 02912, USA.

*Corresponding author: xing.205@osu.edu.

the evolution of probability distribution function. Later in [7] Jung *et al.* studied two-dimensional transverse magnetic Maxwell equations with multiple random interfaces. They applied the polynomial chaos projection method, and computed the stochastic and deterministic parts separately. Furthermore, stochastic collocation methods for metamaterial Maxwell's equations are studied by Li *et al.* [8]. They considered the equations with random coefficient and random initial conditions. They also developed regularity analysis for stochastic metamaterial Maxwell's equations.

In the recent years, there have been many studies on various numerical methods for the stochastic Maxwell equations with either additive or multiplicative noises. In [9], Hong *et al.* studied the stochastic Maxwell equations driven by a multiplicative noise, which are given as follows:

$$\epsilon\,\mathrm{d}\mathbf{E} = \nabla\times\mathbf{H} - \lambda\mathbf{H}\circ\,\mathrm{d}W,\ \ \mu\,\mathrm{d}\mathbf{H} = -\nabla\times\mathbf{E} + \lambda\mathbf{E}\circ\,\mathrm{d}W, \tag{1.1}$$

where $\mathbf{H}$ represents the magnetic field, $\mathbf{E}$ stands for the electric field, λ is the scale of the noise, $\circ$ denotes the stochastic integral in Stratonovich sense, and W is a space time mixed Wiener process. They studied the multi-symplectic structure and the energy conservation law for (1.1), and developed a fully discrete numerical method which conserves both multi-symplecticity and energy on the discrete level. Furthermore, Cohen *et al.* [10] analyzed the general form of stochastic Maxwell equations with multiplicative noise, and constructed an exponential integrator which has a general mean square convergence order of 0.5 in time, and first order temporal rate under some assumptions. Chen *et al.* [11] applied a semi-implicit scheme for the model under a general setting and showed that the proposed method has the mean-square order of 0.5 in time. In [12], Zhang *et al.* presented a nice review article to summarize numerical methods for different kinds of stochastic Maxwell equations with both additive noise and multiplicative noise, including (1.1) studied in [9], and the model

$$\begin{aligned}\epsilon\,\mathrm{d}\mathbf{E} - \nabla\times\mathbf{H}\,\mathrm{d}t &= -\mathbf{J}_e(t,\mathbf{x},\mathbf{E},\mathbf{H})\,\mathrm{d}t - \mathbf{J}_e^r(t,\mathbf{x},\mathbf{E},\mathbf{H})\,\mathrm{d}W,\\ \mu\,\mathrm{d}\mathbf{H} + \nabla\times\mathbf{E}\,\mathrm{d}t &= -\mathbf{J}_m(t,\mathbf{x},\mathbf{E},\mathbf{H})\,\mathrm{d}t - \mathbf{J}_m^r t,\mathbf{x},\mathbf{E},\mathbf{H})\,\mathrm{d}W,\end{aligned} \tag{1.2}$$

which was studied in [11]. The properties of stochastic Maxwell equations are also provided in that paper. Hong *et al.* [13] studied the stochastic wave equation and developed numerical schemes that preserve the averaged energy evolution law. Both the compact finite difference method and the interior penalty discontinuous Galerkin (DG) finite element methods were proposed. Finite element approximations of a class of nonlinear stochastic wave equations with multiplicative noise were recently investigated in [14].

In [11], Chen *et al.* established the regularity properties of the solution of stochastic Maxwell equations with multiplicative noise (1.2). Let $\mathbb{M}$ be the differential operator defined as

$$\mathbb{M} = \begin{pmatrix} 0 & \nabla\times \\ -\nabla\times & 0 \end{pmatrix}.$$

It was shown in [11] that, under certain assumptions, the solutions are uniformly bounded in the following way:

$$\sup_{t\in[0,t_{\mathrm{end}}]}\|\mathbb{U}(t)\|_{L^p(\Omega;\mathcal{D}(\mathbb{M}^\ell))} \le C\Big(1+\|\mathbb{U}_0\|_{L^p(\Omega;\mathcal{D}(\mathbb{M}^\ell))}\Big), \tag{1.3}$$

for any given integer l, $p\ge 2$, $t\in[0,t_{\mathrm{end}}]$, where $\mathbb{U}=(\mathbf{E}^T,\mathbf{H}^T)^T$ and $\mathcal{D}(\mathbb{M}^\ell)$ stands for the domain of $\mathbb{M}^\ell$, the ℓ-th power of the operator $\mathbb{M}$, with the associated norm: $\|\mathbb{U}\|_{\mathcal{D}(\mathbb{M}^\ell)} = (\|\mathbb{U}\|_{L^2}^2 + \left\|\mathbb{M}^\ell\mathbb{U}\right\|_{L^2}^2)^{\frac{1}{2}}$. Furthermore, the Hölder continuity of the solution holds in $\mathcal{D}(\mathbb{M}^{l-1})$ norm in the expectation sense:

$$\mathbb{E}\|\mathbb{U}(t)-\mathbb{U}(s)\|_{\mathcal{D}(\mathbb{M}^{l-1})}^p \le |t-s|^{\frac{p}{2}}. \tag{1.4}$$

The high order DG finite element methods are considered in this paper. The DG method is a class of finite element methods that uses discontinuous piecewise polynomials as the basis functions, and was first introduced by Reed and Hill [15] to solve linear transport equation. In the early 1990s, Cockburn *et al.* studied the extension of DG methods for hyperbolic conservation laws in a series of papers [16–19]. DG methods adopt many advantages from both finite element and finite volume methods, including hp-adaptivity flexibility, efficient parallel implementation, the ability of handling complicated boundary conditions, etc., making them a popular choice for numerical methods of conservation laws.

In [20], Cheng *et al.* studied DG methods for the one-dimensional (1D) deterministic two-way wave equations, and investigated a family of L^2 stable high order DG methods defined through a general form of numerical fluxes. A systematic study of stability, error estimates, and dispersion analysis was carried out. In a recent paper [21], Sun and Xing extended the analysis on optimal error estimate to multi-dimensional wave equations. For the DG methods with generalized numerical fluxes, one key idea to the optimal error estimate was by constructing a global projection on unstructured meshes, which will also be useful for the analysis in this paper. Recently, DG methods have been extended for stochastic partial differential equations. Li *et al.* investigated DG methods [22] for stochastic conservation laws with multiplicative noise and provided optimal error estimate for the semilinear equations.

In this paper, we present the DG methods for both one- and two-dimensional (2D) stochastic Maxwell equations with multiplicative noise. This is an extension of our previous work [23], where we studied multi-symplectic DG methods for stochastic Maxwell equations with additive noise. We showed in [23] that the proposed methods satisfy the discrete form of the stochastic energy linear growth property and preserve the multi-symplectic structure on the discrete level. Optimal error estimate of the semi-discrete DG method was also analyzed in [23]. Another related work was studied by Chen [24], where a symplectic full discretization of multi-dimensional stochastic Maxwell equations with additive noise is provided. DG methods were used for spatial discretization and midpoint method was used for temporal discretization. A first order convergence in time and 1.5th order convergence in space was discussed in [24]. In this work, we plan to apply DG spatial discretization with generalized numerical fluxes to stochastic Maxwell equations with multiplicative noise. We will present the discrete energy growth property of the numerical solutions obtained from our semi-discrete DG scheme, following the energy law of the exact solutions. With the help of a global projection constructed in [21], we present optimal error estimate of the semi-discrete DG methods for stochastic Maxwell equations with multiplicative noise for both one- and two-dimensional cases. In the two-dimensional case, we study both rectangular meshes and triangular meshes. Strong Taylor 2.0 temporal discretization is combined with the semi-discrete DG method to derive a fully discrete method for numerical implementation. Both one- and two-dimensional numerical results are presented to validate the theoretical analysis results. Compared with stochastic Maxwell equations with additive noise studied in [23], stochastic Maxwell equations with multiplicative noise involve the nonlinear noise, which brings extra complication to the analysis. We can only show the error estimate in the expectation sense, *i.e.*, $\mathbb{E}\|u-u_h\|^2 \le Ch^{k+1}$, while in the additive noise case, the exact error estimate $\|u-u_h\| \le Ch^{k+1}$ was obtained. Multi-symplectic structure was investigated in the additive noise case, which no longer holds for the system with multiplicative noise. In addition, unstructured triangular meshes are considered in this paper.

The structure of this paper is as follows. Section 2 describes the DG method for one-dimensional stochastic Maxwell equations with multiplicative noises. Energy growth and the optimal error estimate of the proposed method are provided. Section 3 studies the DG scheme for two-dimensional stochastic Maxwell equations. Both rectangular meshes and triangular meshes are considered. The corresponding discrete energy law and optimal error estimate are also studied for both cases. The temporal discretization is briefly discussed in Section 4. Section 5 presents the numerical results to validate the theoretical results. Conclusion remarks are provided in Section 6. Throughout this paper, the spatial L^2 norm is denoted by $\|\cdot\|$, and C represents a generic positive constant independent of the spatial and temporal step size h and Δt, which can take different values in different cases. With an abuse of notation, we denote H^s as the standard Sobolev space for both scalar-valued functions and vector-valued functions. Furthermore, W_t represents the standard Brownian motion starting from 0.

2. One-dimensional stochastic Maxwell equations with multiplicative noise

In this section, we consider one-dimensional stochastic Maxwell equations with multiplicative noise:

$$\begin{cases} \mathrm{d}v = -u_x \,\mathrm{d}t + f(x,t,u,v)\,\mathrm{d}W_t, \\ \mathrm{d}u = -v_x \,\mathrm{d}t + g(x,t,u,v)\,\mathrm{d}W_t, \end{cases} \tag{2.1}$$

where $x \in I$, $t \in [0, t_{\text{end}}]$, and W_t is a standard one-dimensional Brownian motion on a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and f, g are functions that satisfy the following Lipschitz continuous and linear growth assumptions:

$$\begin{aligned} |f(x,t,u_1,v_1) - f(x,t,u_2,v_2)| + |g(x,t,u_1,v_1) - g(x,t,u_2,v_2)| &\le C(|u_1-u_2| + |v_1-v_2|), \\ |f(x,t,u,v)| + |g(x,t,u,v)| &\le C(1+|u|+|v|). \end{aligned} \tag{2.2}$$

For simplicity, the periodic boundary condition is considered in this paper.

The stochastic Maxwell equations with multiplicative noise (2.1) satisfy the following energy law. A similar result was discussed in [13] for stochastic wave equations.

Theorem 2.1 (Continuous energy law)**.** *Let u and v be the solutions to the model (2.1) under periodic boundary condition. For any t, the global stochastic energy satisfies the following energy law*

$$\mathbb{E}\left(\int_I u^2(x,t) + v^2(x,t)\,\mathrm{d}x\right) = \int_I u^2(x,0) + v^2(x,0)\,\mathrm{d}x + \int_0^t \mathbb{E}\left(\|f\|^2 + \|g\|^2\right)\mathrm{d}\tau. \tag{2.3}$$

Proof. By utilizing Itô's lemma and equations (2.1), we have

$$\begin{aligned} \mathrm{d}\left(\int_I u^2(x,t) + v^2(x,t)\,\mathrm{d}x\right) &= \int_I (2u\,\mathrm{d}u + \mathrm{d}\langle u,u\rangle_t + 2v\,\mathrm{d}v + \mathrm{d}\langle v,v\rangle_t)\,\mathrm{d}x \\ &= -\int_I (2uv_x + 2vu_x)\,\mathrm{d}t\,\mathrm{d}x + 2\int_I (gu+fv)\,\mathrm{d}W_t\,\mathrm{d}x + \int_I \mathrm{d}\langle u,u\rangle_t + \mathrm{d}\langle v,v\rangle_t\,\mathrm{d}x \\ &= 2\int_I (gu+fv)\,\mathrm{d}W_t\,\mathrm{d}x + \int_I \mathrm{d}\langle u,u\rangle_t + \mathrm{d}\langle v,v\rangle_t\,\mathrm{d}x, \end{aligned} \tag{2.4}$$

where the last equality follows from the integration by parts and periodic boundary condition. Integrating over time leads to

$$\int_I u^2(x,t) + v^2(x,t)\,\mathrm{d}x = \int_I u^2(x,0) + v^2(x,0)\,\mathrm{d}x + 2\int_0^t \int_I (gu+fv)\,\mathrm{d}x\,\mathrm{d}W_t + \int_I \langle u,u\rangle_t + \langle v,v\rangle_t\,\mathrm{d}x. \tag{2.5}$$

Integrating the second equation of (2.1) over t leads to

$$u(x,t) = u(x,0) + \int_0^t -v_x\,\mathrm{d}\tau + \int_0^t g(x,\tau,u,v)\,\mathrm{d}W_\tau, \tag{2.6}$$

therefore, we have

$$\int_I \langle u,u\rangle_t\,\mathrm{d}x = \int_I \left\langle \int_0^t g(x,\tau,u,v)\,\mathrm{d}W_\tau, \int_0^t g(x,\tau,u,v)\,\mathrm{d}W_\tau \right\rangle_t \mathrm{d}x.$$

By Itô isometry, we obtain

$$\mathbb{E}\int_I \langle u,u\rangle_t\,\mathrm{d}x = \int_0^t \mathbb{E}\|g\|^2\,\mathrm{d}\tau.$$

For the same reason, it follows that

$$\mathbb{E}\int_I \langle v, v\rangle_t \, \mathrm{d}x = \int_0^t \mathbb{E}\|f\|^2 \, \mathrm{d}\tau.$$

Note that the process $\int_0^t \int_I (gu + fv) \, \mathrm{d}x \, \mathrm{d}W_t$ is an Itô integral, thus it has zero expectation. Taking the expectation of (2.5) yields the continuous energy law (2.3). □

The one-dimensional computational domain I is partitioned into subintervals $I_j = [x_{j-1/2}, x_{j+1/2}]$, $j = 1, 2, \cdots, N$. Let us denote $x_j = (x_{j-1/2} + x_{j+1/2})/2$ to be the center of each subinterval. Denote $h_j = x_{j+1/2} - x_{j-1/2}$ to be the mesh size of each subinterval and $h = \max_j h_j$ to be the maximum mesh size. We further assume that the ratio h/h_j is bounded over all j during mesh refinement. The piecewise polynomial solution and test function space V_h^k is defined as

$$V_h^k = \left\{ w_h : w_h|_{I_j} \in P^k(I_j), \quad j = 1, 2, \cdots, N \right\},$$

where $P^k(I_j)$ stands for the space of polynomials of degree up to k on the cell I_j. Since the function in V_h^k can be discontinuous at cell interface, we use $(w_h^+)_{j+1/2}$ and $(w_h^-)_{j+1/2}$ to represent the limit of $w_h \in V_h^k$ at the interface $x_{j+1/2}$ from the right and left respectively. We denote the average and jump of the functions at the cell interfaces by $\{w_h\} = \left(w_h^+ + w_h^-\right)/2$ and $[w_h] = w_h^+ - w_h^-$.

The DG scheme for the one-dimensional model (2.1) takes the following formulation: for $x \in I$, $(\omega, t) \in \Omega \times [0, t_{\mathrm{end}}]$, find $v_h(\omega, x, t), u_h(\omega, x, t) \in V_h^k$, such that for any test functions $\varphi, \widetilde{\varphi} \in V_h^k$, it holds that

$$\int_{I_j} \mathrm{d}v_h \varphi(x) \, \mathrm{d}x = \left(\int_{I_j} u_h \varphi_x \, \mathrm{d}x - \left(\widehat{u}_h \varphi^-\right)_{j+\frac{1}{2}} + \left(\widehat{u}_h \varphi^+\right)_{j-\frac{1}{2}} \right) \mathrm{d}t + \int_{I_j} f \varphi \, \mathrm{d}W_t \, \mathrm{d}x, \tag{2.7}$$

$$\int_{I_j} \mathrm{d}u_h \widetilde{\varphi}(x) \, \mathrm{d}x = \left(\int_{I_j} v_h \widetilde{\varphi}_x \, \mathrm{d}x - \left(\widehat{v}_h \widetilde{\varphi}^-\right)_{j+\frac{1}{2}} + \left(\widehat{v}_h \widetilde{\varphi}^+\right)_{j-\frac{1}{2}} \right) \mathrm{d}t + \int_{I_j} g \widetilde{\varphi} \, \mathrm{d}W_t \, \mathrm{d}x, \tag{2.8}$$

where $\widehat{u}_h, \widehat{v}_h$ are the numerical fluxes defined on the cell interfaces.

In recent years, there are much attention on studying and analyzing generalized numerical fluxes with various parameters, in order to provide more flexible numerical dissipation with potential applications to complex systems. In this paper, we follow the study in [20, 21] and consider the following generalized numerical fluxes

$$\widehat{u}_h = \{u_h\} + \alpha[u_h] - \beta_1[v_h], \qquad \widehat{v}_h = \{v_h\} - \alpha[v_h] - \beta_2[u_h], \tag{2.9}$$

for some $\alpha \in \mathbb{R}$ and $\beta_1, \ \beta_2 \geq 0$. The upwind flux can be recovered with the parameters $\alpha = 0, \beta_1 = \beta_2 = 1/2$. In addition, $\alpha = 1/2, \beta_1 = \beta_2 = 0$ gives the alternating numerical which is popular in the local DG method. A subgroup of such numerical fluxes, named "$\alpha\beta$" fluxes, were considered in [20] for one-dimensional deterministic two-way wave equations, and optimal error estimate were investigated based on a specially constructed projection operator. DG method with more general numerical fluxes were studied in [21] for the one- and multi-dimensional deterministic wave equations. To provide an optimal error estimate, the key ingredient was to construct an optimal global projection on one-dimensional meshes or multi-dimensional unstructured meshes, which will also be utilized in this paper. Note that similar DG method was studied in [24] for the stochastic Maxwell equation with additive noise, using a specific choice of numerical fluxes (upwind flux) and slightly different treatment of the noise term. Linear polynomial space was considered there, and the convergence rate of $p - 1/2$ in space was obtained when the regularity H^p $(p = 1, 2)$ of the exact solution is ensured.

Next, we start by showing the following semi-discrete energy law satisfied by the numerical solutions of the proposed DG methods.

Theorem 2.2 (Semi-discrete energy law)**.** *Let v_h and u_h be the numerical solutions obtained in (2.7) and (2.8) with $\beta_1, \beta_2 \geq 0$, and let $\mathcal{P}(f)$ and $\mathcal{P}(g)$ be the L^2 projections of f and g onto V_h^k, then we have*

$$\mathbb{E}\Big(\|u_h(x,t)\|^2 + \|v_h(x,t)\|^2\Big) \leq \|u_h(x,0)\|^2 + \|v_h(x,0)\|^2 + \int_0^t \mathbb{E}\Big(\|\mathcal{P}(f)\|^2 + \|\mathcal{P}(g)\|^2\Big)\,\mathrm{d}s.$$

Moreover, the equality holds when $\beta_1 = \beta_2 = 0$ in the numerical fluxes (2.9).

Proof. Taking the test functions $\varphi = v_h$ in (2.7) and $\widetilde{\varphi} = u_h$ in (2.8), and adding the resulting two equations together, we have

$$\begin{aligned}\int_{I_j} (\mathrm{d}u_h)u_h + (\mathrm{d}v_h)v_h\,\mathrm{d}x &= \int_{I_j} gu_h + fv_h\,\mathrm{d}x\,\mathrm{d}W_t + \left(\int_{I_j} u_h(v_h)_x\,\mathrm{d}x + v_h(u_h)_x\,\mathrm{d}x\right)\mathrm{d}t\\ &\quad+ \Big(\big((\{u_h\} + \alpha[u_h] - \beta_1[v_h])v_h^+\big)_{j-\frac{1}{2}} - \big((\{u_h\} + \alpha[u_h] - \beta_1[v_h])v_h^-\big)_{j+\frac{1}{2}}\Big)\,\mathrm{d}t\\ &\quad+ \Big(\big((\{v_h\} - \alpha[v_h] - \beta_2[u_h])u_h^+\big)_{j-\frac{1}{2}} - \big((\{v_h\} - \alpha[v_h] - \beta_2[u_h])u_h^-\big)_{j+\frac{1}{2}}\Big)\,\mathrm{d}t\\ &= \int_{I_j} gu_h + fv_h\,\mathrm{d}x\,\mathrm{d}W_t + \Big(\Theta_{j-\frac{1}{2}} - \Theta_{j+\frac{1}{2}}\Big)\,\mathrm{d}t - \Big(\beta_1[v_h]^2_{j+\frac{1}{2}} + \beta_2[u_h]^2_{j+\frac{1}{2}}\Big)\,\mathrm{d}t\\ &\leq \int_{I_j} gu_h + fv_h\,\mathrm{d}x\,\mathrm{d}W_t + \Big(\Theta_{j-\frac{1}{2}} - \Theta_{j+\frac{1}{2}}\Big)\,\mathrm{d}t, \end{aligned} \tag{2.10}$$

where

$$\Theta = \left(\frac{1}{2} + \alpha\right)v_h^- u_h^+ + \left(\frac{1}{2} - \alpha\right)u_h^- v_h^+.$$

Note that the equality in (2.10) holds when $\beta_1 = \beta_2 = 0$ in the numerical fluxes (2.9).

By Itô's formula, we have

$$\mathrm{d}(u_h)^2 = 2u_h\,\mathrm{d}u_h + \mathrm{d}\langle u_h, u_h\rangle_t, \qquad \mathrm{d}(v_h)^2 = 2v_h\,\mathrm{d}v_h + \mathrm{d}\langle v_h, v_h\rangle_t. \tag{2.11}$$

It follows from (2.7) that for any $\varphi \in V_h^k$,

$$\begin{aligned}\int_{I_j} v_h(x,t)\varphi(x)\,\mathrm{d}x &= \int_{I_j} v_h(x,0)\varphi(x)\,\mathrm{d}x + \int_0^t \left(\int_{I_j} u_h\varphi_x\,\mathrm{d}x - \big(\widehat{u}_h\varphi^-\big)_{j+\frac{1}{2}} + \big(\widehat{u}_h\varphi^+\big)_{j-\frac{1}{2}}\right)\mathrm{d}\tau\\ &\quad+ \int_0^t \int_{I_j} f\varphi\,\mathrm{d}x\,\mathrm{d}W_\tau, \end{aligned} \tag{2.12}$$

therefore, for any continuous semimartingale Y,

$$\int_{I_j} \langle v_h, Y\rangle_t \varphi(x)\,\mathrm{d}x = \left\langle \int_{I_j} v_h\varphi\,\mathrm{d}x, Y\right\rangle_t = \left\langle \int_0^t \int_{I_j} f\varphi\,\mathrm{d}x\,\mathrm{d}W_\tau, Y\right\rangle_t = \left\langle \int_0^t \int_{I_j} \mathcal{P}(f)\varphi\,\mathrm{d}x\,\mathrm{d}W_\tau, Y\right\rangle_t, \tag{2.13}$$

where the second equality follows from applying (2.12), and the last equality comes from the L^2 projection property:

$$\int_{I_j} (f - \mathcal{P}(f))v\,\mathrm{d}x = 0, \quad \forall v(x) \in V_h^k.$$

Let us represent the numerical solutions v_h in the cell I_j as

$$v_h(\omega, x, t) = \sum_{l=0}^k v_j^l(\omega, t)\phi_j^l(x),$$

where $\{\phi_j^l\}$ represents the set of orthogonal Legendre basis of V_h^k over cell I_j. It can be shown that (2.12) and (2.13) lead to

$$\begin{aligned}\int_{I_j} \langle v_h, v_h\rangle_t \,\mathrm{d}x &= \int_{I_j} \left\langle v_h, \sum_{l=0}^k v_j^l \phi_j^l \right\rangle_t \mathrm{d}x = \sum_{l=0}^k \int_{I_j} \langle v_h, v_j^l\rangle_t \phi_j^l \,\mathrm{d}x = \sum_{l=0}^k \left\langle \int_0^t \int_{I_j} \mathcal{P}(f)\phi_j^l \,\mathrm{d}x \,\mathrm{d}W_\tau, v_j^l \right\rangle_t \\ &= \sum_{l=0}^k \int_{I_j} \left\langle \int_0^t \mathcal{P}(f)\,\mathrm{d}W_\tau, v_j^l\phi_j^l \right\rangle_t \mathrm{d}x = \int_{I_j} \left\langle \int_0^t \mathcal{P}(f)\,\mathrm{d}W_\tau, v_h \right\rangle_t \mathrm{d}x, \end{aligned} \tag{2.14}$$

Let us denote $\mathcal{P}(f) = \sum_{l=0}^k (P_f)_j^l \phi_j^l$. Repeating the same process as in (2.14), we can obtain

$$\int_{I_j} \left\langle \int_0^t \mathcal{P}(f)\,\mathrm{d}W_\tau, v_h \right\rangle_t \mathrm{d}x = \int_{I_j} \left\langle \int_0^t \mathcal{P}(f)\,\mathrm{d}W_\tau, \int_0^t \mathcal{P}(f)\,\mathrm{d}W_\tau \right\rangle_t \mathrm{d}x, \tag{2.15}$$

which leads to

$$\int_{I_j} \langle v_h, v_h\rangle_t \,\mathrm{d}x = \int_{I_j} \left\langle \int_0^t \mathcal{P}(f)\,\mathrm{d}W_\tau, \int_0^t \mathcal{P}(f)\,\mathrm{d}W_\tau \right\rangle_t \mathrm{d}x =: m_a(v_h, f). \tag{2.16}$$

Similarly, we have

$$\int_{I_j} \langle u_h, u_h\rangle_t \,\mathrm{d}x = \int_{I_j} \left\langle \int_0^t \mathcal{P}(g)\,\mathrm{d}W_\tau, \int_0^t \mathcal{P}(g)\,\mathrm{d}W_\tau \right\rangle_t \mathrm{d}x =: m_a(u_h, g). \tag{2.17}$$

Integrating (2.10) over t, summing up over all cells I_j and utilizing the results in (2.11), (2.16) and (2.17), we have

$$\begin{aligned}\|u_h(x,t)\|^2 + \|v_h(x,t)\|^2 \le \|u_h(x,0)\|^2 + \|v_h(x,0)\|^2 + m_a(v_h; f) + m_a(u_h; g) \\ + \int_0^t \int_I g u_h + f v_h \,\mathrm{d}x \,\mathrm{d}W_\tau,\end{aligned} \tag{2.18}$$

with periodic boundary conditions. Note that the process $\int_0^t \int_I g u_h + f v_h \,\mathrm{d}x \,\mathrm{d}W_\tau$ is an Itô integral, thus it has zero expectation. Taking expectation of equation (2.18), and applying Itô isometry onto $m_a(v_h; f)$ and $m_a(u_h; g)$, we have

$$\mathbb{E}\Big(\|u_h(x,t)\|^2 + \|v_h(x,t)\|^2\Big) \le \|u_h(x,0)\|^2 + \|v_h(x,0)\|^2 + \int_0^t \mathbb{E}\Big(\|\mathcal{P}(f)\|^2 + \|\mathcal{P}(g)\|^2\Big)\,\mathrm{d}\tau,$$

which finishes the proof. □

Next, we will provide an optimal error estimate analysis of the proposed semi-discrete DG method. We start by defining a pair of global projection operators which will be used in the error estimate analysis: on any cell I_j and for any pair of smooth functions $(q(x),\ r(x))$, define $\mathcal{P}^{\alpha,\beta_1}$ and $\mathcal{P}^{-\alpha,\beta_2}$ as

$$\int_{I_j} \big(\mathcal{P}^{\alpha,\beta_1} q - q(x)\big) w(x)\,\mathrm{d}x = 0, \qquad \forall w(x) \in P^{k-1}(I_j), \quad \forall j, \tag{2.19}$$

$$\int_{I_j} \big(\mathcal{P}^{-\alpha,\beta_2} r - r(x)\big) w(x)\,\mathrm{d}x = 0, \qquad \forall w(x) \in P^{k-1}(I_j), \quad \forall j, \tag{2.20}$$

$$\big(\{\mathcal{P}^{\alpha,\beta_1} q\} + \alpha[\mathcal{P}^{\alpha,\beta_1} q] - \beta_1[\mathcal{P}^{\alpha,\beta_1} r]\big)_{j+\frac12} = q\Big(x_{j+\frac12}\Big), \qquad \forall j, \tag{2.21}$$

$$\big(\{\mathcal{P}^{-\alpha,\beta_2} r\} - \alpha[\mathcal{P}^{-\alpha,\beta_2} r] - \beta_2[\mathcal{P}^{-\alpha,\beta_2} q]\big)_{j+\frac12} = r\Big(x_{j+\frac12}\Big), \quad \forall j. \tag{2.22}$$

This set of global projections was also introduced by Sun and Xing [21], and the following property on the projection error is studied in Lemma 2.1 of [21] and will be useful in the error estimate analysis.

Lemma 2.3 (Projection error)**.** *If $\alpha^2+\beta_1\beta_2\neq 0$, the projections $\mathcal{P}^{\alpha,\beta_1}$, $\mathcal{P}^{-\alpha,\beta_2}$ in (2.19)–(2.22) are well defined. Furthermore, let $q,\ r\in H^{k+1}$ be smooth functions, then there exists some constant C such that*

$$\left\|\mathcal{P}^{\alpha,\beta_1}q-q\right\|^2+\left\|\mathcal{P}^{-\alpha,\beta_2}r-r\right\|^2\le Ch^{2k+2}\Big(\|q\|^2_{H^{k+1}}+\|r\|^2_{H^{k+1}}\Big).$$

Theorem 2.4 (Optimal error estimate)**.** *Suppose $\beta_1,\beta_2\ge 0$ and $\alpha^2+\beta_1\beta_2\neq 0$. Let u_h and v_h be the numerical solutions obtained by the semi-discrete DG method (2.7), (2.8), and $u,\ v\in L^2(\Omega\times[0,t_{\text{end}}],H^{k+2})$ be the strong solutions to (2.1) with $f(\cdot,\cdot,u(\cdot),v(\cdot)),g(\cdot,\cdot,u(\cdot),v(\cdot))\in L^2(\Omega\times[0,t_{\text{end}}],H^{k+1})$, then there exists some constant C such that*

$$\mathbb{E}\Big(\|u-u_h\|^2+\|v-v_h\|^2\Big)\le Ch^{2k+2},\tag{2.23}$$

where the convergence rate is optimal with respect to the polynomial degree k.

Proof. Since the exact solutions v and u also satisfy equations (2.7) and (2.8), taking the difference of the semi-discrete method and the equations satisfied by the exact solutions yields the following error equations

$$\begin{aligned}\int_{I_j}\mathrm{d}(v-v_h)\varphi(x)\,\mathrm{d}x=&\left(\int_{I_j}(u-u_h)\varphi_x\,\mathrm{d}x-\big((u-\widehat{u}_h)\varphi^-\big)_{j+\frac12}+\big((u-\widehat{u}_h)\varphi^+\big)_{j-\frac12}\right)\mathrm{d}t\\&+\int_{I_j}(f(x,t,u,v)-f(x,t,u_h,v_h))\varphi\,\mathrm{d}x\,\mathrm{d}W_t,\end{aligned}\tag{2.24}$$

$$\begin{aligned}\int_{I_j}\mathrm{d}(u-u_h)\widetilde{\varphi}(x)\,\mathrm{d}x=&\left(\int_{I_j}(v-v_h)\widetilde{\varphi}_x\,\mathrm{d}x-\big((v-\widehat{v}_h)\widetilde{\varphi}^-\big)_{j+\frac12}+\big((v-\widehat{v}_h)\widetilde{\varphi}^+\big)_{j-\frac12}\right)\mathrm{d}t\\&+\int_{I_j}(g(x,t,u,v)-g(x,t,u_h,v_h))\widetilde{\varphi}\,\mathrm{d}x\,\mathrm{d}W_t,\end{aligned}\tag{2.25}$$

for any $\varphi,\widetilde{\varphi}\in V_h^k$. Decompose the numerical error into the following two terms

$$v-v_h=\xi^v-\epsilon^v,\qquad u-u_h=\xi^u-\epsilon^u,\tag{2.26}$$

where

$$\xi^v=\mathcal{P}^{-\alpha,\beta_2}v-v_h\in V_h^k,\quad \epsilon^v=\mathcal{P}^{-\alpha,\beta_2}v-v,\quad \xi^u=\mathcal{P}^{\alpha,\beta_1}u-u_h\in V_h^k,\quad \epsilon^u=\mathcal{P}^{\alpha,\beta_1}u-u.$$

For the initial condition, we choose

$$v_h(x,0)=\mathcal{P}^{-\alpha,\beta_2}v(x,0),\qquad u_h(x,0)=\mathcal{P}^{\alpha,\beta_1}u(x,0),$$

hence $\xi^v(x,0)=\xi^u(x,0)=0$. For $q\in\{f,g\}$ and any $w\in V_h^k$, define

$$\mathcal{E}_q^j(w)=\int_{I_j}(q(x,t,u,v)-q(x,t,u_h,v_h))w\,\mathrm{d}x\,\mathrm{d}W_t,$$

which is an Itô differential, hence we have $\mathbb{E}\big(\mathcal{E}_q^j(w)\big)=0$. By choosing the test functions $\varphi=\xi^v$, $\widetilde{\varphi}=\xi^u$ in (2.24), (2.25) and summing up the equations, we obtain

$$\begin{aligned}\int_{I_j}\mathrm{d}\xi^v\xi^v+\mathrm{d}\xi^u\xi^u\,\mathrm{d}x=&\int_{I_j}\mathrm{d}\epsilon^v\xi^v+\mathrm{d}\epsilon^u\xi^u\,\mathrm{d}x+\mathcal{E}_f^j(\xi^v)+\mathcal{E}_g^j(\xi^u)\\&+\left(\int_{I_j}\xi^u\xi^v_x\,\mathrm{d}x-\big((\{\xi^u\}+\alpha[\xi^u]-\beta_1[\xi^v])(\xi^v)^-\big)_{j+\frac12}+\big((\{\xi^u\}+\alpha[\xi^u]-\beta_1[\xi^v])(\xi^v)^+\big)_{j-\frac12}\right)\mathrm{d}t\end{aligned}$$

$$
\begin{aligned}
&- \left(\int_{I_j} \epsilon^u \xi_x^v \,\mathrm{d}x - \big((\{\epsilon^u\} + \alpha[\epsilon^u] - \beta_1[\epsilon^v])(\xi^v)^-\big)_{j+\frac{1}{2}} + \big((\{\epsilon^u\} + \alpha[\epsilon^u] - \beta_1[\epsilon^v])(\xi^v)^+\big)_{j-\frac{1}{2}} \right) \mathrm{d}t \\
&+ \left(\int_{I_j} \xi^v \xi_x^u \,\mathrm{d}x - \big((\{\xi^v\} - \alpha[\xi^v] - \beta_2[\xi^u])(\xi^u)^-\big)_{j+\frac{1}{2}} + \big((\{\xi^v\} - \alpha[\xi^v] - \beta_2[\xi^u])(\xi^u)^+\big)_{j-\frac{1}{2}} \right) \mathrm{d}t \\
&- \left(\int_{I_j} \epsilon^v \xi_x^u \,\mathrm{d}x - \big((\{\epsilon^v\} - \alpha[\epsilon^v] - \beta_2[\epsilon^u])(\xi^u)^-\big)_{j+\frac{1}{2}} + \big((\{\epsilon^v\} - \alpha[\epsilon^v] - \beta_2[\epsilon^u])(\xi^u)^+\big)_{j-\frac{1}{2}} \right) \mathrm{d}t \\
&= \int_{I_j} \mathrm{d}\epsilon^v \xi^v + \mathrm{d}\epsilon^u \xi^u \,\mathrm{d}x + \left(\widetilde{\Theta}_{j-\frac{1}{2}} - \widetilde{\Theta}_{j+\frac{1}{2}} \right) \mathrm{d}t + \mathcal{E}_f^j(\xi^v) + \mathcal{E}_g^j(\xi^u) - \left(\beta_1[\xi^v]_{j+\frac{1}{2}}^2 + \beta_2[\xi^u]_{j+\frac{1}{2}}^2 \right) \mathrm{d}t, \\
&\le \int_{I_j} \mathrm{d}\epsilon^v \xi^v + \mathrm{d}\epsilon^u \xi^u \,\mathrm{d}x + \left(\widetilde{\Theta}_{j-\frac{1}{2}} - \widetilde{\Theta}_{j+\frac{1}{2}} \right) \mathrm{d}t + \mathcal{E}_f^j(\xi^v) + \mathcal{E}_g^j(\xi^u),
\end{aligned} \tag{2.27}
$$

where $\widetilde{\Theta} = \left(\frac{1}{2} + \alpha\right)(\xi^u)^+(\xi^v)^- + \left(\frac{1}{2} - \alpha\right)(\xi^v)^+(\xi^u)^-$. The last equality follows from the definition of the special projection which leads to (for the error term $\epsilon^v = \mathcal{P}^{-\alpha,\beta_2} v - v$, $\epsilon^u = \mathcal{P}^{\alpha,\beta_1} u - u$)

$$
\int_{I_j} \epsilon^u \xi_x^v \,\mathrm{d}x = \int_{I_j} \epsilon^v \xi_x^u \,\mathrm{d}x = (\{\epsilon^u\} + \alpha[\epsilon^u] - \beta_1[\epsilon^v])_{j\pm\frac{1}{2}} = (\{\epsilon^v\} - \alpha[\epsilon^v] - \beta_2[\epsilon^u])_{j\pm\frac{1}{2}} = 0,
$$

and an integration by parts which leads to

$$
\begin{aligned}
\int_{I_j} \xi^u \xi_x^v \,\mathrm{d}x + \int_{I_j} \xi^v \xi_x^u \,\mathrm{d}x &- \big((\{\xi^u\} + \alpha[\xi^u] - \beta_1[\xi^v])(\xi^v)^-\big)_{j+\frac{1}{2}} + \big((\{\xi^u\} + \alpha[\xi^u] - \beta_1[\xi^v])(\xi^v)^+\big)_{j-\frac{1}{2}} \\
&- \big((\{\xi^v\} - \alpha[\xi^v] - \beta_2[\xi^u])(\xi^u)^-\big)_{j+\frac{1}{2}} + \big((\{\xi^v\} - \alpha[\xi^v] - \beta_2[\xi^u])(\xi^u)^+\big)_{j-\frac{1}{2}} \\
&= \widetilde{\Theta}_{j-\frac{1}{2}} - \widetilde{\Theta}_{j+\frac{1}{2}} - \left(\beta_1[\xi^v]_{j+\frac{1}{2}}^2 + \beta_2[\xi^u]_{j+\frac{1}{2}}^2 \right).
\end{aligned}
$$

By Itô's lemma, we have

$$
\mathrm{d}(\xi^v)^2 = 2\,\mathrm{d}\xi^v \xi^v + \mathrm{d}\langle \xi^v, \xi^v \rangle_t, \;\; \mathrm{d}(\xi^u)^2 = 2\,\mathrm{d}\xi^u \xi^u + \mathrm{d}\langle \xi^u, \xi^u \rangle_t. \tag{2.28}
$$

Note that

$$
\mathrm{d}\big(\mathcal{P}^{\alpha,\beta_1} u\big) = \mathcal{P}^{\alpha,\beta_1}(\mathrm{d}u) = \mathcal{P}^{\alpha,\beta_1}(-v_x \,\mathrm{d}t + g \,\mathrm{d}W_t) = \mathcal{P}^{\alpha,\beta_1}(-v_x \,\mathrm{d}t) + \mathcal{P}^{\alpha,\beta_1}(g) \,\mathrm{d}W_t.
$$

We then have, for any test function $\widetilde{\varphi}$,

$$
\int_{I_j} \big(\mathrm{d}\mathcal{P}^{\alpha,\beta_1} u\big) \widetilde{\varphi} \,\mathrm{d}x = \int_{I_j} \mathcal{P}^{\alpha,\beta_1}(-v_x) \widetilde{\varphi} \,\mathrm{d}x \,\mathrm{d}t + \int_{I_j} \widetilde{\varphi} \mathcal{P}^{\alpha,\beta_1}(g) \,\mathrm{d}x \,\mathrm{d}W_t. \tag{2.29}
$$

Subtracting (2.8) from (2.29), we have

$$
\begin{aligned}
\int_{I_j} \mathrm{d}\xi^u \widetilde{\varphi} \,\mathrm{d}x = &\left(\int_{I_j} (-v_h \widetilde{\varphi_x} + \mathcal{P}^{\alpha,\beta_1}(-v_x)\widetilde{\varphi}) \,\mathrm{d}x + (\widehat{v_h} \widetilde{\varphi}^-)_{j+\frac{1}{2}} - (\widehat{v_h} \widetilde{\varphi}^+)_{j-\frac{1}{2}} \right) \mathrm{d}t \\
&+ \int_{I_j} (\mathcal{P}^{\alpha,\beta_1}(g(\cdot, u, v)) - g(\cdot, u_h, v_h)) \widetilde{\varphi} \,\mathrm{d}x \,\mathrm{d}W_t.
\end{aligned} \tag{2.30}
$$

Therefore, after integrating over t, we have

$$
\int_{I_j} \xi^u \widetilde{\varphi} \,\mathrm{d}x = \int_0^t \left(\int_{I_j} \big(-v_h \widetilde{\varphi_x} + \mathcal{P}^{\alpha,\beta_1}(-v_x)\widetilde{\varphi}\big) \,\mathrm{d}x + \big(\widehat{v_h} \widetilde{\varphi}^-\big)_{j+\frac{1}{2}} - \big(\widehat{v_h} \widetilde{\varphi}^+\big)_{j-\frac{1}{2}} \right) \mathrm{d}\tau
$$

$$+\int_0^t\int_{I_j}\big(\mathcal{P}^{\alpha,\beta_1}(g(\cdot,u,v))-g(\cdot,u_h,v_h)\big)\widetilde{\varphi}\,\mathrm{d}x\,\mathrm{d}W_\tau. \tag{2.31}$$

For any semimartingale Y, we have

$$\int_{I_j}\langle\xi^u,Y\rangle_t\widetilde{\varphi}\,\mathrm{d}x=\left\langle\int_0^t\int_{I_j}(\mathcal{P}^{\alpha,\beta_1}(g(\cdot,u,v))-g(\cdot,u_h,v_h))\widetilde{\varphi}\,\mathrm{d}x\,\mathrm{d}W_\tau,Y\right\rangle_t. \tag{2.32}$$

Let us write $\xi^u=\sum_{l=0}^k(\xi^u)_j^l\phi_j^l$. Repeating the same process as in (2.14) yields

$$\int_{I_j}\langle\xi^u,\xi^u\rangle_t\,\mathrm{d}x=\int_{I_j}\left\langle\int_0^t\mathcal{P}^{\alpha,\beta_1}(g(\cdot,u,v))-g(\cdot,u_h,v_h)\,\mathrm{d}W_\tau,\xi^u\right\rangle_t\mathrm{d}x. \tag{2.33}$$

This can be separated into two terms

$$\begin{aligned}\int_{I_j}\langle\xi^u,\xi^u\rangle_t\,\mathrm{d}x&=\int_{I_j}\left\langle\int_0^t\mathcal{P}^{\alpha,\beta_1}(g(\cdot,u,v))-g(\cdot,u,v)\,\mathrm{d}W_\tau,\xi^u\right\rangle_t\mathrm{d}x\\&\quad+\int_{I_j}\left\langle\int_0^t g(\cdot,u,v)-g(\cdot,u_h,v_h)\,\mathrm{d}W_\tau,\xi^u\right\rangle_t\mathrm{d}x\\&\le\frac{1}{2}\int_{I_j}\langle\xi^u,\xi^u\rangle_t\,\mathrm{d}x+C\int_{I_j}\left\langle\int_0^t\mathcal{P}^{\alpha,\beta_1}(g)-g\,\mathrm{d}W_\tau,\int_0^t\mathcal{P}^{\alpha,\beta_1}(g)-g\,\mathrm{d}W_\tau\right\rangle_t\mathrm{d}x\\&\quad+C\int_{I_j}\left\langle\int_0^t g(\cdot,u,v)-g(\cdot,u_h,v_h)\,\mathrm{d}W_\tau,\int_0^t g(\cdot,u,v)-g(\cdot,u_h,v_h)\,\mathrm{d}W_\tau\right\rangle_t\mathrm{d}x.\end{aligned} \tag{2.34}$$

where the last inequality follows from applying Cauchy inequality. Introduce the notations

$$\begin{aligned}Qu_1&=\int_{I_j}\left\langle\int_0^t\mathcal{P}^{\alpha,\beta_1}(g)-g\,\mathrm{d}W_\tau,\int_0^t\mathcal{P}^{\alpha,\beta_1}(g)-g\,\mathrm{d}W_\tau\right\rangle_t\mathrm{d}x,\\Qu_2&=\int_{I_j}\left\langle\int_0^t g(\cdot,u,v)-g(\cdot,u_h,v_h)\,\mathrm{d}W_\tau,\int_0^t g(\cdot,u,v)-g(\cdot,u_h,v_h)\,\mathrm{d}W_\tau\right\rangle_t\mathrm{d}x.\end{aligned}$$

Recall the assumptions (2.2), and by Itô isometry we have

$$\begin{aligned}\mathbb{E}(Qu_2)=\int_{I_j}\int_0^t\mathbb{E}|g(\cdot,u,v)-g(\cdot,u_h,v_h)|^2\,\mathrm{d}\tau\,\mathrm{d}x&\le C\int_{I_j}\int_0^t\mathbb{E}\big(|u-u_h|^2+|v-v_h|^2\big)\,\mathrm{d}\tau\,\mathrm{d}x\\&=C\int_{I_j}\int_0^t\mathbb{E}\Big(|\xi^u-\epsilon^u|^2+|\xi^v-\epsilon^v|^2\Big)\,\mathrm{d}\tau\,\mathrm{d}x,\end{aligned} \tag{2.35}$$

and

$$\mathbb{E}(Qu_1)=\int_{I_j}\int_0^t\mathbb{E}\big(\mathcal{P}^{\alpha,\beta_1}g-g\big)^2\,\mathrm{d}\tau\,\mathrm{d}x.$$

Therefore, after taking the expectation of equation (2.34), we conclude

$$\begin{aligned}\int_{I_j}\mathbb{E}(\langle\xi^u,\xi^u\rangle_t)\,\mathrm{d}x&\le C\mathbb{E}(Qu_1+Qu_2)\\&\le C\int_{I_j}\int_0^t\mathbb{E}\big(\mathcal{P}^{\alpha,\beta_1}g-g\big)^2\,\mathrm{d}\tau\,\mathrm{d}x+C\int_{I_j}\int_0^t\mathbb{E}\Big(|\xi^u-\epsilon^u|^2+|\xi^v-\epsilon^v|^2\Big)\,\mathrm{d}\tau\,\mathrm{d}x,\end{aligned} \tag{2.36}$$

and for the same reason,

$$\int_{I_j} \mathbb{E}(\langle \xi^v, \xi^v \rangle_t) \, \mathrm{d}x \le C \int_{I_j} \int_0^t \mathbb{E}\big(\mathcal{P}^{-\alpha,\beta_2} f - f\big)^2 \, \mathrm{d}\tau \, \mathrm{d}x + C \int_{I_j} \int_0^t \mathbb{E}\Big(|\xi^u - \epsilon^u|^2 + |\xi^v - \epsilon^v|^2\Big) \, \mathrm{d}\tau \, \mathrm{d}x. \quad (2.37)$$

After summing over all cells I_j, utilizing the periodic boundary conditions and integrating equations (2.27), (2.28) from 0 to t, we can take the expectation of the resulting equations and obtain

$$\begin{aligned}\mathbb{E}\Big(\|\xi^u(x,t)\|^2 + \|\xi^v(x,t)\|^2\Big) \le \|\xi^u(x,0)\|^2 + \|\xi^v(x,0)\|^2 + 2\mathbb{E}\bigg(\int_0^t \int_I \mathrm{d}\epsilon^v \xi^v + \mathrm{d}\epsilon^u \xi^u \, \mathrm{d}x \, \mathrm{d}\tau\bigg) \\ + \int_0^t \int_I \mathbb{E}(\langle \xi^u, \xi^u \rangle_t) \, \mathrm{d}x \, \mathrm{d}\tau + \int_0^t \int_I \mathbb{E}(\langle \xi^v, \xi^v \rangle_t) \, \mathrm{d}x \, \mathrm{d}\tau.\end{aligned}$$

Following the same derivation of equation (2.30), we have

$$\int_I \mathrm{d}\epsilon^u \widetilde{\varphi} \, \mathrm{d}x = \int_I \big(v_x - \mathcal{P}^{\alpha,\beta_1}(v_x)\big) \widetilde{\varphi} \, \mathrm{d}x \, \mathrm{d}t + \int_I \big(\mathcal{P}^{\alpha,\beta_1}(g) - g\big) \widetilde{\varphi} \, \mathrm{d}x \, \mathrm{d}W_t. \quad (2.38)$$

Based on the assumptions, we know that $\int_{I_j} (\mathcal{P}^{\alpha,\beta_1}(g) - g) \widetilde{\varphi} \, \mathrm{d}x \, \mathrm{d}W_t$ is a martingale. Therefore

$$\mathbb{E}\bigg(\int_0^t \int_I \mathrm{d}\epsilon^u \xi^u \, \mathrm{d}x\bigg) = \mathbb{E}\bigg(\int_0^t \int_I \big(v_x - \mathcal{P}^{\alpha,\beta_1}(v_x)\big) \xi^u \, \mathrm{d}x \, \mathrm{d}\tau\bigg) \le C h^{2k+2} + \int_0^t \mathbb{E}\|\xi^u(x,\tau)\|^2 \, \mathrm{d}\tau, \quad (2.39)$$

after applying the error estimate of the projection in Lemma 2.3. Similarly, we have

$$\mathbb{E}\bigg(\int_0^t \int_I \mathrm{d}\epsilon^v \xi^v \, \mathrm{d}x\bigg) \le C h^{2k+2} + \int_0^t \mathbb{E}\|\xi^v(x,\tau)\|^2 \, \mathrm{d}\tau. \quad (2.40)$$

Note that the initial condition satisfies $\|\xi^v(x,0)\| = \|\xi^u(x,0)\| = 0$, and we can utilize the results in (2.36)–(2.40) to obtain

$$\begin{aligned}\mathbb{E}\Big(\|\xi^v(x,t)\|^2 + \|\xi^v(x,t)\|^2\Big) &\le 2\mathbb{E}\bigg(\int_0^t \int_I \mathrm{d}\epsilon^v \xi^v + \mathrm{d}\epsilon^u \xi^u \, \mathrm{d}x\bigg) + C\mathbb{E}\bigg(\int_0^t \big\|\mathcal{P}^{\alpha,\beta_1} g - g\big\|^2 + \big\|\mathcal{P}^{-\alpha,\beta_2} f - f\big\|^2 \, \mathrm{d}\tau\bigg) \\ &\quad + C\mathbb{E}\bigg(\int_0^t \|\xi^v(x,\tau)\|^2 + \|\xi^u(x,\tau)\|^2 \, \mathrm{d}\tau\bigg) + C h^{2k+2} \\ &\le C \int_0^t \mathbb{E}\Big(\|\xi^v(x,\tau)\|^2 + \|\xi^u(x,\tau)\|^2\Big) \, \mathrm{d}\tau + C h^{2k+2}.\end{aligned}$$

The optimal error estimate (2.23) follows from applying Gronwall's inequality and the optimal projection error. □

Remark 2.5. In the proof, we assumed enough regularity of the exact solutions to study the "best" spatial convergence rate of the proposed method. Such convergence rate has also been observed on some numerical examples in Section 5.

3. Two-dimensional stochastic Maxwell equations with multiplicative noise

In this section, we study two-dimensional stochastic Maxwell equations in the following form

$$\begin{cases} \mathrm{d}E - T_x \, \mathrm{d}t + S_y \, \mathrm{d}t = f(\mathbf{x}, t, \boldsymbol{u}) \, \mathrm{d}W_t, \\ \mathrm{d}S + E_y \, \mathrm{d}t = g(\mathbf{x}, t, \boldsymbol{u}) \, \mathrm{d}W_t, \\ \mathrm{d}T - E_x \, \mathrm{d}t = r(\mathbf{x}, t, \boldsymbol{u}) \, \mathrm{d}W_t, \end{cases} \quad (3.1)$$

where $\mathbf{x} = (x, y)^T \in \Omega$, $t \in [0, t_{\text{end}}]$, $\boldsymbol{u} = (E, S, T)$ and $f,\ g,\ r$ are smooth functions that satisfy the following Lipschitz continuous and linear growth assumptions:

$$|f(\mathbf{x}, t, \boldsymbol{u}_1) - f(\mathbf{x}, t, \boldsymbol{u}_2)| + |g(\mathbf{x}, t, \boldsymbol{u}_1) - g(\mathbf{x}, t, \boldsymbol{u}_2)| + |r(\mathbf{x}, t, \boldsymbol{u}_1) - r(\mathbf{x}, t, \boldsymbol{u}_2)| \le C|\boldsymbol{u}_1 - \boldsymbol{u}_2|, \tag{3.2}$$

$$|f(\mathbf{x}, t, \boldsymbol{u})| + |g(\mathbf{x}, t, \boldsymbol{u})| + |r(\mathbf{x}, t, \boldsymbol{u})| \le C(1 + |E| + |S| + |T|). \tag{3.3}$$

We refer to the beginning of Section 2 on the discussion of regularity properties of the solutions. For this model, we have the following energy law satisfied by exact solutions. The proof follows almost the same analysis as that of Theorem 2.1 and is skipped here.

Theorem 3.1 (Continuous energy law)**.** *Let $E,\ S,\ T$ be solutions to the equation (3.1) on the bounded domain Ω with the periodic boundary condition, then for any t, the global stochastic energy satisfies the following energy law*

$$\begin{aligned}\mathbb{E}\left(\int_\Omega E(\mathbf{x}, t)^2 + S(\mathbf{x}, t)^2 + T(\mathbf{x}, t)^2 \,\mathrm{d}\mathbf{x}\right)& \\ = \int_\Omega E(\mathbf{x}, 0)^2 + S(\mathbf{x}, 0)^2 + T(\mathbf{x}, 0)^2 \,\mathrm{d}\mathbf{x} &+ \int_0^t \mathbb{E}\left(\|f\|^2 + \|g\|^2 + \|r\|^2\right) \mathrm{d}\tau.\end{aligned} \tag{3.4}$$

3.1. Triangular meshes

In this subsection we will consider DG methods for stochastic Maxwell equations with multiplicative noise on triangular discretization of the domain Ω, and carry out the corresponding analysis. Firstly we rewrite the equation (3.1) in the following compact form:

$$\mathrm{d}E = \nabla \cdot \mathbf{U} \,\mathrm{d}t + F(\mathbf{x}, t, E, \mathbf{U}) \,\mathrm{d}W_t, \quad \mathrm{d}\mathbf{U} = \nabla E \,\mathrm{d}t + \mathbf{G}(\mathbf{x}, t, E, \mathbf{U}) \,\mathrm{d}W_t, \tag{3.5}$$

where $\mathbf{U} = (T, -S)^T$, and F and $\mathbf{G}$ are functions satisfying (3.2) and (3.3).

Let $\Omega = \cup_{K \in \mathcal{T}_h} K$ be a quasi-uniform triangulation of the domain Ω, and Γ be the collection of triangle faces. For a triangle K, and a face $\mathscr{F} \in \partial K$, let $\mathbf{n}$ be the outer normal vector on $\mathscr{F}$. Given a face $\mathscr{F}$, let $\mathbf{n}_{\mathscr{F}}$ denote the unit vector across $\mathscr{F}$, whose direction is not essential for unspecified K. We define the finite element DG space V_h^k as

$$V_h^k = \left\{v_h : v_h|_K \in P^k(K)\right\},$$

where P^k is the space of polynomial of degree at most k. $\mathbb{V}_h$ is used to denote $(V_h^k)^2$, and denote $(P^k)^2$ as $\mathbf{P}^k$. For a given face $\mathscr{F}$, let $K^\pm$ be the two neighboring cells. For any function w or vector valued function $\mathbf{v}$, define

$$w^\pm = \lim_{\mathbf{x} \to \partial K, \mathbf{x} \in K^\pm} w(\mathbf{x}), \qquad \mathbf{v}^\pm = \lim_{\mathbf{x} \to \partial K, \mathbf{x} \in K^\pm} \mathbf{v}(\mathbf{x}).$$

The vector $\mathbf{n}^\pm$ is the outer normal vector on F with respect to $K^\pm$. Finally, we define the following notations for averages and jumps across a face $\mathscr{F}$:

$$\{w\} = \frac{1}{2}\left(w^+ + w^-\right), \qquad \{\mathbf{v}\} = \frac{1}{2}\left(\mathbf{v}^+ + \mathbf{v}^-\right), \tag{3.6}$$

$$[w\mathbf{n}] = w^+\mathbf{n}^+ + w^-\mathbf{n}^-, \qquad [\mathbf{v} \cdot \mathbf{n}] = \mathbf{v}^+ \cdot \mathbf{n}^+ + \mathbf{v}^- \cdot \mathbf{n}^-. \tag{3.7}$$

Note that over all the triangles and faces, we have the following equalities

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} w\mathbf{v} \cdot \mathbf{n} \,\mathrm{d}s = \sum_{\mathscr{F} \in \Gamma} \int_F w^+\mathbf{n}^+ \cdot \mathbf{v}^+ + w^-\mathbf{n}^- \cdot \mathbf{v}^- \,\mathrm{d}s = \sum_{\mathscr{F} \in \Gamma} \int_F \{w\}[\mathbf{n} \cdot \mathbf{v}] + [w\mathbf{n}] \cdot \{\mathbf{v}\} \,\mathrm{d}s. \tag{3.8}$$

The DG scheme for the two-dimensional system (3.5) is: find $E_h \in V_h^k,\ \mathbf{U}_h \in \mathbb{V}_h^k$, such that for all test functions $\varphi \in V_h^k,\ \ \boldsymbol{\psi} \in \mathbb{V}_h^k$, it holds that

$$\int_K \mathrm{d}E_h \varphi \,\mathrm{d}\mathbf{x} + \int_K \mathbf{U}_h \cdot \nabla\varphi \,\mathrm{d}\mathbf{x} \,\mathrm{d}t - \int_{\partial K} \boldsymbol{\mathcal{F}_1}(\mathbf{U}_h, E_h) \cdot \varphi\mathbf{n} \,\mathrm{d}s \,\mathrm{d}t = \int_K F\varphi \,\mathrm{d}\mathbf{x} \,\mathrm{d}W_t, \tag{3.9}$$

$$\int_K \mathrm{d}\mathbf{U}_h \cdot \boldsymbol{\psi}\,\mathrm{d}\mathbf{x} + \int_K E_h \nabla\cdot\boldsymbol{\psi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t - \int_{\partial K} \mathcal{F}_2(E_h,\mathbf{U}_h)\boldsymbol{\psi}\cdot\mathbf{n}\,\mathrm{d}s\,\mathrm{d}t = \int_K \mathbf{G}\cdot\boldsymbol{\psi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}W_t, \tag{3.10}$$

where the numerical fluxes are chosen to be

$$\boldsymbol{\mathcal{F}_1}(\mathbf{U}_h,E_h) = \{\mathbf{U}_h\} - \boldsymbol{\alpha}[\mathbf{U}_h\cdot\mathbf{n}] - \beta_1[E_h\mathbf{n}], \quad \mathcal{F}_2(E_h,\mathbf{U}_h) = \{E_h\} + \boldsymbol{\alpha}\cdot[E_h\mathbf{n}] - \beta_2[\mathbf{U}_h\cdot\mathbf{n}], \tag{3.11}$$

with $\boldsymbol{\alpha} = \alpha\,\mathrm{sgn}(\mathbf{r}\cdot\mathbf{n}_{\mathscr{F}})\mathbf{n}_{\mathscr{F}}$ for some number α. Note that these generalized numerical fluxes can be viewed as two-dimensional extension of the one-dimensional numerical fluxes (2.9).

Next, we start by showing the following semi-discrete energy law satisfied by numerical solutions of the proposed DG methods.

Theorem 3.2 (Semi-discrete energy law)**.** *Let E_h and $\mathbf{U}_h$ be the numerical solutions obtained in* (3.9) *and* (3.10) *with $\beta_1 \geq 0$, $\beta_2 \geq 0$, then we have*

$$\mathbb{E}\Big(\|E_h(\mathbf{x},t)\|^2 + \|\mathbf{U}_h(\mathbf{x},t)\|^2\Big) \leq \|E_h(\mathbf{x},0)\|^2 + \|\mathbf{U}_h(\mathbf{x},0)\|^2 + \int_0^t \mathbb{E}\Big(\|\mathcal{P}(F)\|^2 + \|\mathcal{P}(\mathbf{G})\|^2\Big)\,\mathrm{d}\tau. \tag{3.12}$$

Moreover, the equality holds when $\beta_1 = \beta_2 = 0$ in the numerical fluxes (3.11)*.*

Proof. By taking the test function $\varphi = E_h$ in (3.9) and $\boldsymbol{\psi} = \mathbf{U}_h$ in (3.10), and summing the resulting equations over all cells K, we obtain

$$\begin{aligned}\int_\Omega \mathrm{d}E_h E_h + \mathrm{d}\mathbf{U}_h\cdot\mathbf{U}_h\,\mathrm{d}\mathbf{x} + \int_\Omega \mathbf{U}_h\cdot\nabla E_h + E_h\nabla\cdot\mathbf{U}_h\,\mathrm{d}\mathbf{x}\,\mathrm{d}t \\ - \sum_{K\in\mathcal{T}_h}\int_{\partial K} \boldsymbol{\mathcal{F}_1}(\mathbf{U}_h,E_h)\cdot E_h\mathbf{n} + \mathcal{F}_2(E_h,\mathbf{U}_h)\mathbf{U}_h\cdot\mathbf{n}\,\mathrm{d}s\,\mathrm{d}t = \int_\Omega F E_h + \mathbf{G}\cdot\mathbf{U}_h\,\mathrm{d}\mathbf{x}\,\mathrm{d}W_t.\end{aligned} \tag{3.13}$$

By an integration by parts and applying the equality (3.8), we can have

$$\begin{aligned}&\int_\Omega \mathbf{U}_h\cdot\nabla E_h + E_h\nabla\cdot\mathbf{U}_h\,\mathrm{d}\mathbf{x}\,\mathrm{d}t - \sum_{K\in\mathcal{T}_h}\int_{\partial K}\boldsymbol{\mathcal{F}_1}(\mathbf{U}_h,E_h)\cdot E_h\mathbf{n} + \mathcal{F}_2(E_h,\mathbf{U}_h)\mathbf{U}_h\cdot\mathbf{n}\,\mathrm{d}s\,\mathrm{d}t \\ &\quad= \sum_{K\in\mathcal{T}_h}\int_{\partial K} E_h\mathbf{U}_h\cdot\mathbf{n}\,\mathrm{d}s\,\mathrm{d}t - \sum_{\mathscr{F}\in\Gamma}\int_F \boldsymbol{\mathcal{F}_1}(\mathbf{U}_h,E_h)\cdot[E_h\mathbf{n}] + \mathcal{F}_2(E_h,\mathbf{U}_h)[\mathbf{U}_h\cdot\mathbf{n}]\,\mathrm{d}s\,\mathrm{d}t \\ &\quad= \sum_{K\in\mathcal{T}_h}\int_{\partial K} E_h\mathbf{U}_h\cdot\mathbf{n}\,\mathrm{d}s\,\mathrm{d}t - \sum_{\mathscr{F}\in\Gamma}\int_F (\{\mathbf{U}_h\} - \boldsymbol{\alpha}[\mathbf{U}_h\cdot\mathbf{n}])\cdot[E_h\mathbf{n}] + (\{E_h\} + \boldsymbol{\alpha}\cdot[E_h\mathbf{n}])[\mathbf{U}_h\cdot\mathbf{n}]\,\mathrm{d}s\,\mathrm{d}t \\ &\qquad+ \sum_{\mathscr{F}\in\Gamma}\int_F \beta_1|[E_h\mathbf{n}]|^2 + \beta_2[\mathbf{U}_h\cdot\mathbf{n}]^2\,\mathrm{d}s\,\mathrm{d}t = \sum_{\mathscr{F}\in\Gamma}\int_F \beta_1|[E_h\mathbf{n}]|^2 + \beta_2[\mathbf{U}_h\cdot\mathbf{n}]^2\,\mathrm{d}s\,\mathrm{d}t.\end{aligned} \tag{3.14}$$

Therefore, equation (3.13) becomes

$$\int_\Omega \mathrm{d}E_h E_h + \mathrm{d}\mathbf{U}_h\cdot\mathbf{U}_h\,\mathrm{d}\mathbf{x} + \sum_{\mathscr{F}\in\Gamma}\int_F \beta_1|[E_h\mathbf{n}]|^2 + \beta_2[\mathbf{U}_h\cdot\mathbf{n}]^2\,\mathrm{d}s\,\mathrm{d}t = \int_\Omega F E_h + \mathbf{G}\cdot\mathbf{U}_h\,\mathrm{d}\mathbf{x}\,\mathrm{d}W_t. \tag{3.15}$$

By Itô's lemma, we have

$$\mathrm{d}E_h E_h = \frac{1}{2}(\mathrm{d}(E_h)^2 - \mathrm{d}\langle E_h,E_h\rangle_t), \quad \mathrm{d}\mathbf{U}_h\cdot\mathbf{U}_h = \frac{1}{2}(d|\mathbf{U}_h|^2 - \mathrm{d}\langle\mathbf{U}_h,\mathbf{U}_h\rangle_t).$$

Following an exact same process as in the derivation of (2.16), and applying the Itô isometry, we have

$$\int_K \mathbb{E}\langle E_h,E_h\rangle_t\,\mathrm{d}\mathbf{x} = \int_K\int_0^t \mathbb{E}(\mathcal{P}(F))^2\,\mathrm{d}\tau\,\mathrm{d}\mathbf{x}, \qquad \int_K \mathbb{E}\langle\mathbf{U}_h,\mathbf{U}_h\rangle_t\,\mathrm{d}\mathbf{x} = \int_K\int_0^t \mathbb{E}(\mathcal{P}(\mathbf{G}))^2\,\mathrm{d}\tau\,\mathrm{d}\mathbf{x}.$$

By plugging these into (3.15), integrating over time t, summing over all K, and taking expectation, we obtain

$$\mathbb{E}\Big(\|E(\mathbf{x},t)\|^2+\|\mathbf{U}(\mathbf{x},t)\|^2\Big)\le\|E(\mathbf{x},0)\|^2+\|\mathbf{U}(\mathbf{x},0)\|^2+\int_0^t\mathbb{E}\Big(\|\mathcal{P}(F)\|^2+\|\mathcal{P}(\mathbf{G})\|^2\Big)\,\mathrm{d}\tau, \tag{3.16}$$

where we use the fact that the right-hand side of (3.13) $\int_\Omega FE_h+\mathbf{G}\cdot\mathbf{U}_h\,\mathrm{d}\mathbf{x}\,\mathrm{d}W_t$ is a martingale, hence its expectation equals to zero. It is easy to observe that the equality holds when $\beta_1=\beta_2=0$. ☐

Similar to the one-dimensional case, to provide the optimal error estimate, we need to introduce the following pair of projections $\mathbb{P}^{\mathbf{U}}\mathbf{U}$ and $\mathcal{P}^E E$ [21]: for any $K\in\mathcal{T}_h$,

$$\int_K\big(\mathbb{P}^{\mathbf{U}}\mathbf{U}-\mathbf{U}\big)\cdot\mathbf{v}\,\mathrm{d}\mathbf{x}=0,\qquad\forall\mathbf{v}\in\mathbf{P}^{k-1}(K),\forall K\in\mathcal{T}_h, \tag{3.17}$$

$$\int_K\big(\mathcal{P}^E E-E\big)w\,\mathrm{d}\mathbf{x}=0,\qquad\forall w\in P^{k-1}(K),\forall K\in\mathcal{T}_h, \tag{3.18}$$

$$\int_{\partial K}\boldsymbol{\mathcal{F}_1}\big(\mathbb{P}^{\mathbf{U}}\mathbf{U},\mathcal{P}^E E\big)\cdot\mu\mathbf{n}\,\mathrm{d}s=\int_{\partial K}\mathbf{U}\cdot\mu\mathbf{n}\,\mathrm{d}s,\quad\forall\mu\in P^k(\mathscr{F}),\forall\mathscr{F}\in\Gamma, \tag{3.19}$$

$$\int_{\partial K}\mathcal{F}_2\big(\mathcal{P}^E E,\mathbb{P}^{\mathbf{U}}\mathbf{U}\big)\nu\,\mathrm{d}s=\int_{\partial K}E\nu\,\mathrm{d}s,\qquad\forall\nu\in P^k(\mathscr{F}),\forall\mathscr{F}\in\Gamma. \tag{3.20}$$

The following lemma on the projection is studied in Lemma 3.1 of [21] and will be useful in the analysis of error estimate. The detailed proof is skipped here, but we remark that when $\beta_1=0$, an additional assumption on the unit normal direction is needed, namely, there exists a unit vector $\mathbf{r}$ such that $|\mathbf{r}\cdot\mathbf{n}_{\mathscr{F}}|\ge\kappa>0$ for all $\mathbf{n}_{\mathscr{F}}$. Further discussion can be found in [21].

Lemma 3.3. *Suppose $\beta_1\ge0,\ \beta_2>0$, and $|\boldsymbol{\alpha}|^2+\beta_1\beta_2\ne0$, then the projection pair defined in (3.17)–(3.20) is well defined, and there exists some constant C independent of mesh size h, such that*

$$\big\|\mathbb{P}^{\mathbf{U}}\mathbf{U}-\mathbf{U}\big\|^2+\big\|\mathcal{P}^E E-E\big\|^2\le Ch^{2k+2}\Big(\|\mathbf{U}\|^2_{H^{k+1}}+\|E\|^2_{H^{k+1}}\Big).$$

Theorem 3.4 (Optimal error estimate)**.** *Suppose $\beta_1\ge0,\ \beta_2>0$, and $|\boldsymbol{\alpha}|^2+\beta_1\beta_2\ne0$. Let E_h and $\mathbf{U}_h$ be the numerical solutions obtained by semi-discrete DG method (3.9), (3.10), and $(E,\mathbf{U}^T)\in L^2(\Omega\times[0,t_{\mathrm{end}}];H^{k+2})$ are strong solutions, and $(F,\mathbf{G}^T)\in L^2(\Omega\times[0,t_{\mathrm{end}}],H^{k+1})$, then there exists some constant C such that*

$$\mathbb{E}\Big(\|E-E_h\|^2+\|\mathbf{U}-\mathbf{U}_h\|^2\Big)\le Ch^{2k+2}.$$

Proof. For simplicity, for $f,\ \varphi\in V_h^k$, and $\mathbf{g},\ \boldsymbol{\psi}\in\mathbb{V}_h^k$, introduce the notation

$$\mathcal{A}_K(f,\mathbf{g};\varphi,\boldsymbol{\psi})=\int_K\mathbf{g}\cdot\nabla\varphi+f\nabla\cdot\boldsymbol{\psi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t-\int_{\partial K}\boldsymbol{\mathcal{F}_1}(\mathbf{g},f)\cdot\varphi\mathbf{n}+\mathcal{F}_2(f,\mathbf{g})\boldsymbol{\psi}\cdot\mathbf{n}\,\mathrm{d}s\,\mathrm{d}t.$$

We further define

$$\xi^{\mathbf{U}}=\mathbb{P}^{\mathbf{U}}\mathbf{U}-\mathbf{U}_h,\quad\xi^E=\mathcal{P}^E E-E_h,\quad\epsilon^{\mathbf{U}}=\mathbb{P}^{\mathbf{U}}\mathbf{U}-\mathbf{U},\quad\epsilon^E=\mathcal{P}^E E-E,$$

and choose the initial condition

$$\mathbf{U}_h(\mathbf{x},0)=\mathbb{P}^{\mathbf{U}}\mathbf{U}(\mathbf{x},0),\qquad E_h(\mathbf{x},0)=\mathcal{P}^E E(\mathbf{x},0),$$

hence $\xi^E(\mathbf{x},0)=0,\ \xi^{\mathbf{U}}(\mathbf{x},0)=\mathbf{0}$.

It can be observed that the following error equations hold

$$\begin{aligned}\int_K \mathrm{d}\big(\xi^E-\epsilon^E\big)\varphi\,\mathrm{d}\mathbf{x}+\int_K\big(\xi^{\mathbf{U}}-\epsilon^{\mathbf{U}}\big)\cdot\nabla\varphi\,\mathrm{d}\mathbf{x}\,\mathrm{d}t-\int_{\partial K}\boldsymbol{\mathcal{F}_1}\big(\xi^{\mathbf{U}}-\epsilon^{\mathbf{U}},\xi^E-\epsilon^E\big)\cdot\varphi\mathbf{n}\,\mathrm{d}s\,\mathrm{d}t\\ =\int_K(F(E,\mathbf{U})-F(E_h,\mathbf{U}_h))\varphi\,\mathrm{d}\mathbf{x}\,\mathrm{d}W_t,\\ \int_K \mathrm{d}\big(\xi^{\mathbf{U}}-\epsilon^{\mathbf{U}}\big)\cdot\boldsymbol{\psi}\,\mathrm{d}\mathbf{x}+\int_K\big(\xi^E-\epsilon^E\big)\nabla\cdot\boldsymbol{\psi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}t-\int_{\partial K}\mathcal{F}_2\big(\xi^E-\epsilon^E,\xi^{\mathbf{U}}-\epsilon^{\mathbf{U}}\big)\boldsymbol{\psi}\cdot\mathbf{n}\,\mathrm{d}s\,\mathrm{d}t\\ =\int_K(\mathbf{G}(E,\mathbf{U})-\mathbf{G}(E_h,\mathbf{U}_h))\cdot\boldsymbol{\psi}\,\mathrm{d}\mathbf{x}\,\mathrm{d}W_t,\end{aligned}$$

where we dropped the dependence of F, $\mathbf{G}$ on $\mathbf{x},t$ for simplicity. Taking the test functions $\varphi=\xi^E$ and $\boldsymbol{\psi}=\xi^{\mathbf{U}}$, and summing up two equations, we obtain

$$\begin{aligned}\int_K \mathrm{d}\xi^E\xi^E+\,\mathrm{d}\xi^{\mathbf{U}}\cdot\xi^{\mathbf{U}}\,\mathrm{d}\mathbf{x}-\int_K \mathrm{d}\epsilon^E\xi^E+\,\mathrm{d}\epsilon^{\mathbf{U}}\cdot\xi^{\mathbf{U}}\,\mathrm{d}\mathbf{x}+\mathcal{A}_K\big(\xi^E,\xi^{\mathbf{U}};\xi^E,\xi^{\mathbf{U}}\big)-\mathcal{A}_K\big(\epsilon^E,\epsilon^{\mathbf{U}};\xi^E,\xi^{\mathbf{U}}\big)\\ =\int_K(F(E,\mathbf{U})-F(E_h,\mathbf{U}_h))\xi^E+(\mathbf{G}(E,\mathbf{U})-\mathbf{G}(E_h,\mathbf{U}_h))\cdot\xi^{\mathbf{U}}\,\mathrm{d}\mathbf{x}\mathrm{d}W_t.\end{aligned}\tag{3.21}$$

From our definition of projections (3.17)–(3.20), we can conclude that $\mathcal{A}_K(\epsilon^E,\epsilon^{\mathbf{U}};\xi^E,\xi^{\mathbf{U}})=0$. Following the exact same analysis as in (3.14), we can have

$$\sum_{K\in\mathcal{T}_h}\mathcal{A}_K\big(\xi^E,\xi^{\mathbf{U}};\xi^E,\xi^{\mathbf{U}}\big)=\sum_{\mathscr{F}\in\Gamma}\int_F\beta_1^2\big|\big[\xi^E\mathbf{n}\big]\big|^2+\beta_2\big[\xi^{\mathbf{U}}\cdot\mathbf{n}\big]^2\,\mathrm{d}s\,\mathrm{d}t\geq 0.$$

In addition, the term $M_1:=\int_K(F(E,\mathbf{U})-F(E_h,\mathbf{U}_h))\xi^E+(\mathbf{G}(E,\mathbf{U})-\mathbf{G}(E_h,\mathbf{U}_h))\cdot\xi^{\mathbf{U}}\,\mathrm{d}\mathbf{x}\,\mathrm{d}W_t$ on the right-hand side is an martingale, and its expectation equals to zero. Therefore, summing the equation (3.21) over all the cells K leads to

$$\int_\Omega \mathrm{d}\xi^E\xi^E+\,\mathrm{d}\xi^{\mathbf{U}}\cdot\xi^{\mathbf{U}}\,\mathrm{d}\mathbf{x}\leq\int_\Omega \mathrm{d}\epsilon^E\xi^E+\,\mathrm{d}\epsilon^{\mathbf{U}}\cdot\xi^{\mathbf{U}}\,\mathrm{d}\mathbf{x}+M_1.\tag{3.22}$$

Following the same derivation of equations (2.39) and (2.40), we have

$$\mathbb{E}\bigg(\int_\Omega\int_0^t \mathrm{d}\epsilon^E\xi^E+\,\mathrm{d}\epsilon^{\mathbf{U}}\cdot\xi^{\mathbf{U}}\,\mathrm{d}\mathbf{x}\bigg)\leq Ch^{2k+2}+C\int_0^t\mathbb{E}\Big(\big\|\xi^E(\mathbf{x},\tau)\big\|^2+\big\|\xi^{\mathbf{U}}(\mathbf{x},\tau)\big\|^2\Big)\,\mathrm{d}\tau.\tag{3.23}$$

By Itô lemma, we have

$$\mathrm{d}\xi^E\xi^E=\frac{1}{2}\Big(\mathrm{d}\big(\xi^E\big)^2-\,\mathrm{d}\big\langle\xi^E,\xi^E\big\rangle_t\Big),\quad \mathrm{d}\xi^{\mathbf{U}}\cdot\xi^{\mathbf{U}}=\frac{1}{2}\big(\mathrm{d}|\xi^{\mathbf{U}}|^2-\,\mathrm{d}\big\langle\xi^{\mathbf{U}},\xi^{\mathbf{U}}\big\rangle_t\big).$$

Follow the same process as the steps in (2.29)–(2.36) in one-dimensional case, we have

$$\begin{aligned}\int_K\mathbb{E}\big(\big\langle\xi^E,\xi^E\big\rangle_t+\big\langle\xi^{\mathbf{U}},\xi^{\mathbf{U}}\big\rangle_t\big)\,\mathrm{d}\mathbf{x}\leq C\int_K\int_0^t\mathbb{E}\Big(\big|\mathcal{P}^EF-F\big|^2+|\mathbb{P}^{\mathbf{U}}\mathbf{G}-\mathbf{G}|^2\Big)\,\mathrm{d}\tau\,\mathrm{d}\mathbf{x}\\ +C\int_K\int_0^t\mathbb{E}\big(|\xi^E-\epsilon^E|^2+|\xi^{\mathbf{U}}-\epsilon^{\mathbf{U}}|^2\big)\,\mathrm{d}\tau\,\mathrm{d}\mathbf{x}.\end{aligned}\tag{3.24}$$

Summing over all the cells K, combining these results with (3.22), integrating over time t, and taking expectation, we will have

$$\mathbb{E}\Big(\big\|\xi^E(\mathbf{x},t)\big\|^2+\big\|\xi^{\mathbf{U}}(\mathbf{x},t)\big\|^2\Big)\leq\big\|\xi^E(\mathbf{x},0)\big\|^2+\big\|\xi^{\mathbf{U}}(\mathbf{x},0)\big\|^2+C\int_0^t\mathbb{E}\Big(\big\|\xi^E(\mathbf{x},\tau)\big\|^2+\big\|\xi^{\mathbf{U}}(\mathbf{x},\tau)\big\|^2\Big)\,\mathrm{d}\tau+Ch^{2k+2},$$

after utilizing the optimal projection error in Lemma 3.3. Note that the chosen initial condition satisfies $\xi^E(x,0) = \xi^{\mathbf{U}}(x,0) = 0$, we have

$$\mathbb{E}\Big(\|E - E_h\|^2 + \|\mathbf{U} - \mathbf{U}_h\|^2\Big) \le Ch^{2k+2},$$

after applying Gronwall's inequality and the optimal projection error. □

3.2. Rectangular meshes

In this subsection, we will investigate DG methods on cartesian meshes, and study the stability and error estimate of the proposed methods. The stability result and its proof is similar to the case of triangular meshes, but different technique is needed for the proof of optimal error estimate, which will be discussed in details.

The two-dimensional rectangular computational domain Ω is set to be $I \times J$, and we consider the rectangular partition with the cells denoted by $I_i \times J_j = [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}]$ for $i = 1, 2, \cdots, N_x$ and $j = 1, 2, \cdots, N_y$. Let $x_i = \frac{1}{2}(x_{i-\frac{1}{2}} + x_{i+\frac{1}{2}})$, and $y_j = \frac{1}{2}(y_{j-\frac{1}{2}} + y_{j+\frac{1}{2}})$. Furthermore, we define the mesh size in both directions as $h_{x,i} = x_{i+\frac{1}{2}} - x_{i-\frac{1}{2}}$, $h_{y,j} = y_{j+\frac{1}{2}} - y_{j-\frac{1}{2}}$, with $h_x = \max_i h_{x,i}$, $h_y = \max_j h_{y,j}$ and $h = \max(h_x, h_y)$ being the maximum mesh size. Similar to the one-dimensional case, we define the two dimensional piecewise polynomial space $\mathbb{V}_h^k$ as follows:

$$\mathbb{V}_h^k = \big\{w(x,y) : w|_{I_i \times J_j} \in Q^k(I_i \times J_j) = P^k(I_i) \otimes P^k(J_j), \quad i = 1, 2, \cdots, N_x; j = 1, 2, \cdots, N_y\big\}.$$

The DG scheme for (3.1) is as follows: find $E_h,\ S_h,\ T_h \in \mathbb{V}_h^k$, such that for all $\varphi,\ \psi,\ \phi \in \mathbb{V}_h^k$,

$$\begin{aligned}\int_{J_j}\int_{I_i} \mathrm{d}E_h \varphi \,\mathrm{d}x\,\mathrm{d}y = &-\int_{J_j} \mathcal{A}_{I_i}(T_h, \varphi; \alpha_1)\,\mathrm{d}y\,\mathrm{d}t + \int_{I_i} \mathcal{A}_{J_j}(S_h, \varphi; -\alpha_2)\,\mathrm{d}x\,\mathrm{d}t \\ &+ \int_{J_j}\int_{I_i} f\varphi\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}W_t, \end{aligned} \tag{3.25}$$

$$\int_{J_j}\int_{I_i} \mathrm{d}S_h \psi\,\mathrm{d}x\,\mathrm{d}y = \int_{I_i} \mathcal{A}_{J_j}(E_h, \psi; \alpha_2)\,\mathrm{d}x\,\mathrm{d}t + \int_{J_j}\int_{I_i} g\psi\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}W_t, \tag{3.26}$$

$$\int_{J_j}\int_{I_i} \mathrm{d}T_h \phi\,\mathrm{d}x\,\mathrm{d}y = -\int_{J_j} \mathcal{A}_{I_i}(E_h, \phi; -\alpha_1)\,\mathrm{d}y\,\mathrm{d}t + \int_{J_j}\int_{I_i} r\phi\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}W_t, \tag{3.27}$$

where, for simplicity, the following operators are defined: for $\alpha \in \mathbb{R}$, $p,\ q \in \mathbb{V}_h^k$,

$$\begin{aligned}\mathcal{A}_{I_i}(p,q;\alpha) &= \int_{I_i} p q_x\,\mathrm{d}x - \big(\widehat{p_\alpha} q^-\big)_{i+\frac{1}{2},y} + \big(\widehat{p_\alpha} q^+\big)_{i-\frac{1}{2},y}, \\ \mathcal{A}_{J_j}(p,q;\alpha) &= \int_{J_j} p q_y\,\mathrm{d}y - \big(\widehat{p_\alpha} q^-\big)_{x,j+\frac{1}{2}} + \big(\widehat{p_\alpha} q^+\big)_{x,j-\frac{1}{2}}\end{aligned}$$

with the numerical fluxes defined as follows:

$$\text{for } q \in \mathbb{V}_h^k \text{ and } \alpha \in \{\pm\alpha_1, \pm\alpha_2\} \subset \mathbb{R},\ \widehat{q_\alpha} = \{q\} + \alpha[q]. \tag{3.28}$$

Note that the numerical fluxes (3.28) can be viewed as a special case of (2.9) with $\beta_1 = \beta_2 = 0$. They are chosen such that the optimal error estimate can be easily analyzed.

We first provide the following semi-discrete energy law satisfied by numerical solutions of the proposed DG methods on rectangular meshes. The proof is identical to that of Theorem 3.2, and is skipped here.

Theorem 3.5 (Semi-discrete energy law)**.** *Let $E_h,\ S_h,\ T_h \in \mathbb{V}_h^k$ be the numerical solutions of the semi-discrete DG methods (3.25)–(3.27), then we have*

$$\mathbb{E}\Big(\|E_h(x,y,t)\|^2 + \|S_h(x,y,t)\|^2 + \|T_h(x,y,t)\|^2\Big) \tag{3.29}$$
$$= \|E_h(x,y,0)\|^2 + \|S_h(x,y,0)\|^2 + \|T_h(x,y,0)\|^2 + \int_0^t \mathbb{E}(\|\mathcal{P}(f)\|^2 + \|\mathcal{P}(g)\|^2 + \|\mathcal{P}(r)\|^2)\,\mathrm{d}\tau.$$

Before we study the error estimate, some preparations on projections are provided. Let us define the generalized Radau projection as

$$\mathbb{P}_x^{\alpha} = \mathcal{P}_x^{\alpha,0} \otimes \mathcal{P}_y,\ \mathbb{P}_y^{\alpha} = \mathcal{P}_x \otimes \mathcal{P}_y^{\beta,0},\ \mathbb{P}^{\alpha,\beta} = \mathcal{P}_x^{\alpha,0} \otimes \mathcal{P}_y^{\beta,0}, \tag{3.30}$$

where $\mathcal{P}$ is the standard one-dimensional L^2 projection, and $\mathcal{P}_x^{\alpha,0}$, $\mathcal{P}_y^{\beta,0}$ are the one-dimensional generalized Radau projections (2.19)–(2.22) defined as follows: On the cell I_i, J_j and for any function $q(x), r(y)$, the projections $\mathcal{P}_x^{\alpha,0}, \mathcal{P}_y^{\beta,0}$ into the space V_h^k are given by

$$\int_{I_i} \big(\mathcal{P}_x^{\alpha,0} q - q(x)\big)\phi(x)\,\mathrm{d}x = 0,\ \forall \phi(x) \in P^{k-1}(I_i),\quad \text{and } \big(\{\mathcal{P}_x^{\alpha,0} q\} + \alpha[\mathcal{P}^{\alpha,0} q]\big)_{i+\frac{1}{2}} = q\Big(x_{i+\frac{1}{2}}\Big),$$
$$\int_{J_j} \big(\mathcal{P}_y^{\beta,0} r - r(y)\big)\psi(y)\,\mathrm{d}y = 0,\ \forall \psi(y) \in P^{k-1}(J_j),\quad \text{and } \big(\{\mathcal{P}_y^{\beta,0} r\} + \alpha[\mathcal{P}^{\beta,0} r]\big)_{j+\frac{1}{2}} = r\Big(y_{j+\frac{1}{2}}\Big).$$

The following lemmas are studied in [25] and will be useful in our error estimate analysis.

Lemma 3.6 (Superconvergence property)**.** *Let $\mathbb{P}^{\alpha,\beta}$ be the projection defined in (3.30) with $\alpha, \beta \neq 0$. For any function $w(x,y) \in H^{k+1}$, denote $\epsilon = \mathbb{P}^{\alpha,\beta} w - w$. For any $\phi \in \mathbb{V}_h^k$, there exists some constant C such that*

$$\left|\sum_{i,j} \int_{J_j} \mathcal{A}_{I_i}(\epsilon,\phi,\alpha)\,\mathrm{d}y\right| \le Ch^{k+1}\|\phi\|, \quad \left|\sum_{i,j} \int_{I_i} \mathcal{A}_{J_j}(\epsilon,\phi,\beta)\,\mathrm{d}x\right| \le Ch^{k+1}\|\phi\|.$$

Lemma 3.7 (Projection error)**.** *Let Π be any projection defined in (3.30) with $\alpha, \beta \neq 0$. For any function $w(x,y) \in H^{k+1}$, there exists some constant C such that*

$$\|\Pi w - w\| \le Ch^{k+1}\|w\|_{H^{k+1}}.$$

Now we turn to the optimal error estimate.

Theorem 3.8 (Optimal error estimate)**.** *Suppose $\alpha_1, \alpha_2 \neq 0$, and let $E_h, S_h, T_h \in \mathbb{V}_h^k$ be the numerical solutions given by the DG scheme (3.25)–(3.27), and $E, T, S \in L^2(\Omega \times [0, t_{\text{end}}]; H^{k+2})$ are exact solutions to (3.1), and $f, g, r \in H^{k+1}$, then there exists some constant C such that*

$$\mathbb{E}\Big(\|E - E_h\|^2 + \|S - S_h\|^2 + \|T - T_h\|^2\Big) \le Ch^{2k+2}. \tag{3.31}$$

Proof. We start by introducing

$$\xi^E = \mathbb{P}^{-\alpha_1,\alpha_2} E - E_h, \qquad \xi^S = \mathbb{P}_y^{-\alpha_2} S - S_h, \qquad \xi^T = \mathbb{P}_x^{\alpha_1} T - T_h,$$
$$\epsilon^E = \mathbb{P}^{-\alpha_1,\alpha_2} E - E, \qquad \epsilon^S = \mathbb{P}_y^{-\alpha_2} S - S, \qquad \epsilon^T = \mathbb{P}_x^{\alpha_1} T - T,$$

which leads to

$$E - E_h = \xi^E - \epsilon^E, \qquad S - S_h = \xi^S - \epsilon^S, \qquad T - T_h = \xi^T - \epsilon^T.$$

We choose the initial conditions as follows:

$$E_h(x,y,0) = \mathbb{P}^{-\alpha_1,\alpha_2} E(x,y,0), \qquad S_h(x,y,0) = \mathbb{P}_y^{-\alpha_2} S(x,y,0), \qquad T_h(x,y,0) = \mathbb{P}_x^{\alpha_1} T(x,y,0),$$

hence, $\xi^E(x,y,0)=\xi^S(x,y,0)=\xi^T(x,y,0)=0$.

Since both numerical solutions and exact solutions satisfy equations (3.25)–(3.27), we have the following error equations

$$\begin{aligned}\int_{J_j}\int_{I_i}\mathrm{d}(E-E_h)\varphi\,\mathrm{d}x\,\mathrm{d}y &= -\int_{J_j}\mathcal{A}_{I_i}(T-T_h,\varphi;\alpha_1)\,\mathrm{d}y\,\mathrm{d}t+\int_{I_i}\mathcal{A}_{J_j}(S-S_h,\varphi;-\alpha_2)\,\mathrm{d}x\,\mathrm{d}t\\ &\quad+\int_{J_j}\int_{I_i}(f(x,t,\boldsymbol{u})-f(x,t,\boldsymbol{u}_h))\varphi\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}W_t, \qquad (3.32)\\ \int_{J_j}\int_{I_i}\mathrm{d}(S-S_h)\psi\,\mathrm{d}x\,\mathrm{d}y &= \int_{I_i}\mathcal{A}_{J_j}(E-E_h,\psi;\alpha_2)\,\mathrm{d}x\,\mathrm{d}t\\ &\quad+\int_{J_j}\int_{I_i}(g(x,t,\boldsymbol{u})-g(x,t,\boldsymbol{u}_h))\psi\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}W_t, \qquad (3.33)\\ \int_{J_j}\int_{I_i}\mathrm{d}(T-T_h)\phi\,\mathrm{d}x\,\mathrm{d}y &= -\int_{J_j}\mathcal{A}_{I_i}(E-E_h,\phi;-\alpha_1)\,\mathrm{d}y\,\mathrm{d}t\\ &\quad+\int_{J_j}\int_{I_i}(r(x,t,\boldsymbol{u})-r(x,t,\boldsymbol{u}_h))\phi\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}W_t, \qquad (3.34)\end{aligned}$$

for all $\varphi,\psi,\phi\in\mathbb{V}_h^k$. Choose the test functions $\varphi=\xi^E,\quad\psi=\xi^S,\quad\phi=\xi^T$, and notice that

$$\int_{J_j}\mathcal{A}_{I_i}(\epsilon^T,\xi^E;\alpha_1)\,\mathrm{d}y\,\mathrm{d}t=\int_{I_i}\mathcal{A}_{J_j}(\epsilon^S,\xi^E;-\alpha_2)\,\mathrm{d}x\,\mathrm{d}t=0,$$

following the definition of the projections. For a function $q\in\{f,g,r\}$ and $w\in\mathbb{V}_h^k$, define

$$\mathbf{E}_q^{i,j}(w)=\int_{J_j}\int_{I_i}(q(x,t,\boldsymbol{u})-q(x,t,\boldsymbol{u}_h))w\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}W_t,$$

which is an Itô integral, hence we have $\mathbb{E}\big(\mathbf{E}_q^{i,j}(w)\big)=0$ for any w. Therefore, equations (3.32)-(3.34) become

$$\begin{aligned}&\int_{J_j}\int_{I_i}\mathrm{d}\xi^E\xi^E\,\mathrm{d}x\,\mathrm{d}y-\int_{J_j}\int_{I_i}\mathrm{d}\epsilon^E\xi^E\,\mathrm{d}x\,\mathrm{d}y=-\int_{J_j}\mathcal{A}_{I_i}\Big(\xi^T,\xi^E;\alpha_1\Big)\,\mathrm{d}y\,\mathrm{d}t+\int_{I_i}\mathcal{A}_{J_j}\Big(\xi^S,\xi^E;-\alpha_2\Big)\,\mathrm{d}x\,\mathrm{d}t+\mathbf{E}_f^{i,j}\Big(\xi^E\Big),\\ &\int_{J_j}\int_{I_i}\mathrm{d}\xi^S\xi^S\,\mathrm{d}x\,\mathrm{d}y-\int_{J_j}\int_{I_i}\mathrm{d}\epsilon^S\xi^S\,\mathrm{d}x\,\mathrm{d}y=\int_{I_i}\mathcal{A}_{J_j}\Big(\xi^E,\xi^S;\alpha_2\Big)\,\mathrm{d}x\,\mathrm{d}t-\int_{I_i}\mathcal{A}_{J_j}\Big(\epsilon^E,\xi^S;\alpha_2\Big)\,\mathrm{d}x\,\mathrm{d}t+\mathbf{E}_g^{i,j}\Big(\xi^S\Big),\\ &\int_{J_j}\int_{I_i}\mathrm{d}\xi^T\xi^T\,\mathrm{d}x\,\mathrm{d}y-\int_{J_j}\int_{I_i}\mathrm{d}\epsilon^T\xi^T\,\mathrm{d}x\,\mathrm{d}y=-\int_{J_j}\mathcal{A}_{I_i}\Big(\xi^E,\xi^T;-\alpha_1\Big)\,\mathrm{d}y\,\mathrm{d}t+\int_{J_j}\mathcal{A}_{I_i}\Big(\epsilon^E,\xi^T;-\alpha_1\Big)\,\mathrm{d}y\,\mathrm{d}t+\mathbf{E}_r^{i,j}\Big(\xi^T\Big).\end{aligned}$$

Summing up these equations and applying integration by parts, we obtain

$$\begin{aligned}&\int_{J_j}\int_{I_i}\mathrm{d}\xi^E\xi^E+\mathrm{d}\xi^S\xi^S+\mathrm{d}\xi^T\xi^T\,\mathrm{d}x\,\mathrm{d}y\\ &\quad=\int_{J_j}\int_{I_i}\mathrm{d}\epsilon^E\xi^E+\mathrm{d}\epsilon^S\xi^S+\mathrm{d}\epsilon^T\xi^T\,\mathrm{d}x\,\mathrm{d}y-\int_{J_j}\Big(\Pi_{i-\frac12,y}-\Pi_{i+\frac12,y}\Big)\,\mathrm{d}y\,\mathrm{d}t+\int_{I_i}\Big(\bar\Pi_{x,j-\frac12}-\bar\Pi_{x,j+\frac12}\Big)\,\mathrm{d}x\,\mathrm{d}t\\ &\quad\quad-\int_{I_i}\mathcal{A}_{J_j}(\epsilon^E,\xi^S;\alpha_2)\,\mathrm{d}x\,\mathrm{d}t+\int_{J_j}\mathcal{A}_{I_i}(\epsilon^E,\xi^T;-\alpha_1)\,\mathrm{d}y\,\mathrm{d}t+\mathbf{E}_f^{i,j}(\xi^E)+\mathbf{E}_g^{i,j}(\xi^S)+\mathbf{E}_r^{i,j}(\xi^T), \qquad (3.35)\end{aligned}$$

where

$$\Pi=\Big(\frac12+\alpha_1\Big)\big(\xi^T\big)^+\big(\xi^E\big)^-+\Big(\frac12-\alpha_1\Big)\big(\xi^T\big)^-\big(\xi^E\big)^+,\quad \bar\Pi=\Big(\frac12+\alpha_2\Big)\big(\xi^S\big)^-\big(\xi^E\big)^++\Big(\frac12-\alpha_2\Big)\big(\xi^S\big)^+\big(\xi^E\big)^-.$$

By Itô's lemma, we have

$$\mathrm{d}\big(\xi^E\big)^2 = 2\,\mathrm{d}\xi^E\xi^E + \mathrm{d}\big\langle \xi^E, \xi^E\big\rangle_t, \quad \mathrm{d}\big(\xi^S\big)^2 = 2\,\mathrm{d}\xi^S\xi^S + \mathrm{d}\big\langle \xi^S, \xi^S\big\rangle_t, \quad \mathrm{d}\big(\xi^T\big)^2 = 2\,\mathrm{d}\xi^T\xi^T + \mathrm{d}\big\langle \xi^T, \xi^T\big\rangle_t.$$

Following the same process as in the derivation of (2.36), we have

$$\begin{aligned}
\int_{J_j}\int_{I_i} \mathbb{E}\big\langle \xi^E, \xi^E\big\rangle_t \,\mathrm{d}x\,\mathrm{d}y &\le C\int_{J_j}\int_{I_i}\int_0^t \mathbb{E}\big(\mathbb{P}^{-\alpha_1,\alpha_2}(f) - f\big)^2\,\mathrm{d}\tau\,\mathrm{d}x\,\mathrm{d}y + C\mathbb{E}\mathbf{e}_{i,j},\\
\int_{J_j}\int_{I_i} \mathbb{E}\big\langle \xi^S, \big\rangle_t \,\mathrm{d}x\,\mathrm{d}y &\le C\int_{J_j}\int_{I_i}\int_0^t \mathbb{E}\big(\mathbb{P}_y^{-\alpha_2}(g) - g\big)^2\,\mathrm{d}\tau\,\mathrm{d}x\,\mathrm{d}y + C\mathbb{E}\mathbf{e}_{i,j},\\
\int_{J_j}\int_{I_i} \mathbb{E}\big\langle \xi^T, \xi^T\big\rangle_t \,\mathrm{d}x\,\mathrm{d}y &\le C\int_{J_j}\int_{I_i}\int_0^t \mathbb{E}(\mathbb{P}_x^{\alpha_1}(r) - r)^2\,\mathrm{d}\tau\,\mathrm{d}x\,\mathrm{d}y + C\mathbb{E}\mathbf{e}_{i,j},
\end{aligned}$$

where

$$\mathbf{e}_{i,j} := \int_{J_j}\int_{I_i}\int_0^t \big|\xi^E - \epsilon^E\big|^2 + \big|\xi^S - \epsilon^S\big|^2 + \big|\xi^T - \epsilon^T\big|^2\,\mathrm{d}\tau\,\mathrm{d}x\,\mathrm{d}y.$$

After taking expectation on equation (3.35), summing over all the cells, and integrating over t, we have

$$\begin{aligned}
&\frac{1}{2}\int_J\int_I \mathbb{E}\Big(\big(\xi^E\big)^2 + \big(\xi^S\big)^2 + \big(\xi^T\big)^2\Big)\,\mathrm{d}x\,\mathrm{d}y\\
&= \frac{1}{2}\int_J\int_I \mathbb{E}\big(\big\langle \xi^E, \xi^E\big\rangle_t + \big\langle \xi^S, \xi^S\big\rangle_t + \big\langle \xi^T, \xi^T\big\rangle_t\big)\,\mathrm{d}x\,\mathrm{d}y + \int_J\int_I\int_0^t \mathbb{E}(\mathrm{d}\xi^E\xi^E + \mathrm{d}\xi^S\xi^S + \mathrm{d}\xi^T\xi^T)\,\mathrm{d}x\,\mathrm{d}y\\
&\le C\int_0^t \mathbb{E}\Big(\big\|\mathbb{P}^{-\alpha_1,\alpha_2}(f) - f\big\|^2 + \|\mathbb{P}_x^{\alpha_1}(r) - r\|^2 + \big\|\mathbb{P}_y^{-\alpha_2}(g) - g\big\|^2\Big)\,\mathrm{d}\tau\\
&+ \int_J\int_I\int_0^t \mathbb{E}\big(\mathrm{d}\epsilon^E\xi^E + \mathrm{d}\epsilon^S\xi^S + \mathrm{d}\epsilon^T\xi^T\big)\,\mathrm{d}x\,\mathrm{d}y + Ch^{k+1}\int_0^t \mathbb{E}\big(\big\|\xi^T\big\| + \big\|\xi^S\big\|\big)\,\mathrm{d}\tau + C\sum_{i,j}\mathbb{E}\mathbf{e}_{i,j}.
\end{aligned}$$

Following the same derivation of equations (2.39) and (2.40), we have

$$\begin{aligned}
\int_J\int_I\int_0^t \mathbb{E}\big(\mathrm{d}\epsilon^E\xi^E + \mathrm{d}\epsilon^S\xi^S + \mathrm{d}\epsilon^T\xi^T\big)\,\mathrm{d}x\,\mathrm{d}y &\le Ch^{2k+2}\\
&+ \int_0^t \mathbb{E}\Big(\big\|\xi^E(x,y,\tau)\big\|^2 + \big\|\xi^S(x,y,\tau)\big\|^2 + \big\|\xi^T(x,y,\tau)\big\|^2\Big)\,\mathrm{d}\tau,
\end{aligned}$$

after applying the superconvergence property in Lemma 3.6. We utilize the projection error property and Young's inequality to obtain

$$\mathbb{E}\Big(\big\|\xi^E\big\|^2 + \big\|\xi^S\big\|^2 + \big\|\xi^T\big\|^2\Big) \le C\int_0^t \mathbb{E}\Big(\big\|\xi^E(x,y,\tau)\big\|^2 + \big\|\xi^S(x,y,\tau)\big\|^2 + \big\|\xi^T(x,y,\tau)\big\|^2\Big)\,\mathrm{d}\tau + Ch^{2k+2}.$$

The optimal error estimate (3.31) follows from applying Gronwall's inequality and the optimal projection error. □

4. Temporal discretization

After DG spatial discretization, the semi-discrete DG methods (2.7) and (2.8), or (3.9), (3.10) can be obtained. To solve the resulting stochastic differential equations, we present Taylor 2.0 strong scheme [22] as the temporal discretization in this section.

Let us consider the general matrix-valued stochastic differential equations of the form

$$\begin{cases} dX_t^{i,j} = a^{i,j}(X_t)\,\mathrm{d}t + b^{i,j}(X_t)\,\mathrm{d}W_t, \quad t > 0 \\ X_0^{i,j} = x_0^{i,j}, \end{cases} \tag{4.1}$$

with $i = 0, 1, \cdots, m$ and $j = 0, 1, \cdots, M+1$. Suppose that $Y_n^{i,j}$ is a numerical approximation of $X_{t_n}^{i,j}$ at the time t_n, and $Y_0^{i,j} = x_0^{i,j}$. The recurrent equation between $Y_n^{i,j}$ and $Y_{n+1}^{i,j}$ is derived below. Define the following operators:

$$\mathcal{L}^0 V = \sum_{j=0}^{M+1} \sum_{i=0}^{m} a^{i,j} \frac{\partial V}{\partial x_{i,j}} + \frac{1}{2} \sum_{l,j=0}^{M+1} \sum_{m,i=0}^{m} b^{i,j} b^{m,l} \frac{\partial^2 V}{\partial x_{i,j} \partial x_{m,l}},$$

and

$$\mathcal{L}^1 V = \sum_{j=0}^{M} \sum_{i=0}^{m} b^{i,j} \frac{\partial V}{\partial x_{i,j}}$$

where $V : \mathbb{R}^{(m+1)\times(M+2)} \to \mathbb{R}$ is a twice differentiable function. As studied in Section 10.5 of [26] and Appendix A of [22], the following Taylor scheme has 2.0 order of convergence:

$$\begin{aligned} Y_{n+1}^{i,j} = {} & Y_n^{i,j} + a^{i,j}(Y_n)\tau + b^{i,j}(Y_n)\Delta W + \frac{1}{2}\mathcal{L}^1 b^{i,j}(Y_n)\big((\Delta W)^2 - \tau\big) \\ & + \frac{1}{2}\mathcal{L}^0 a^{i,j}(Y_n)\tau^2 + \mathcal{L}^0 b^{i,j}(Y_n)(\Delta W \tau - \Delta Z) + \mathcal{L}^1 a^{i,j}(Y_n)\Delta Z \\ & + \frac{1}{6}\mathcal{L}^1\mathcal{L}^1 b^{i,j}(Y_n)\Big((\Delta W)^2 - 3\tau\Big)\Delta W + \mathcal{L}^1\mathcal{L}^0 b^{i,j}(Y_n)(-\Delta U + \Delta W \Delta Z) \\ & + \mathcal{L}^1\mathcal{L}^1 a^{i,j}(Y_n)\left(\frac{1}{2}\Delta U - \frac{1}{4}\tau^2\right) + \mathcal{L}^0\mathcal{L}^1 b^{i,j}(Y_n)\left(\frac{1}{2}\Delta U - \Delta W \Delta Z + \frac{1}{2}(\Delta W)^2\tau - \frac{1}{4}\tau^2\right) \\ & + \frac{1}{24}\mathcal{L}^1\mathcal{L}^1\mathcal{L}^1 b^{i,j}(Y_n)\big((\Delta W)^4 - 6(\Delta W)^2\tau + 3\tau^2\big), \end{aligned} \tag{4.2}$$

where

$$\tau = t_{n+1} - t_n, \quad \Delta W = W_{t_{n+1}} - W_{t_n},$$

and

$$\Delta Z = \int_{t_n}^{t_{n+1}} W_s - W_{t_n}\,\mathrm{d}s, \quad \Delta U = \int_{t_n}^{t_{n+1}} (W_s - W_{t_n})^2\,\mathrm{d}s.$$

To numerically compute the stochastic variables ΔW, ΔZ and ΔU, we define a new process

$$v(s) = \frac{W_{t_n + \tau s} - W_{t_n}}{\sqrt{\tau}}, \qquad s \in [0, 1],$$

and have

$$\Delta W = \tau^{\frac{1}{2}} v(1), \ \Delta Z = \tau^{\frac{3}{2}} \int_0^1 v(s)\,\mathrm{d}s, \ \Delta U = \tau^2 \int_0^1 v^2(s)\,\mathrm{d}s,$$

which can be evaluated by solving the following system of equations

$$\begin{cases} \mathrm{d}x = \mathrm{d}v(s), & x(0) = 0, \\ \mathrm{d}y = x ds, & y(0) = 0, \\ \mathrm{d}z = x^2\,\mathrm{d}s, & z(0) = 0. \end{cases} \tag{4.3}$$

at the moment $s = 1$. The system (4.3) can be solved numerically to obtain the approximation of ΔW, ΔZ and ΔU. We refer to Appendix A.2.2 of [22] for the details of implementation.

TABLE 1. Numerical error and convergence rates of 1D case when $k = 1$.

Nx	Nt	$\left(\mathbb{E}\|e_u\|^2\right)^{1/2}$	Rate	$\left(\mathbb{E}\|e_v\|^2\right)^{1/2}$	Rate
20	200	0.02537	–	7.855E−3	–
40	400	6.407E−3	1.9851	1.978E−3	1.9893
80	800	1.567E−3	2.0313	4.837E−4	2.0320
160	1600	3.913E−4	2.0022	1.206E−4	2.0046

TABLE 2. Numerical error and convergence rates of 1D case when $k = 2$.

Nx	Nt	$\left(\mathbb{E}\|e_u\|^2\right)^{1/2}$	Rate	$\left(\mathbb{E}\|e_v\|^2\right)^{1/2}$	Rate
20	200	6.254E−4	−	2.073E−4	−
40	400	7.970E−5	2.9722	2.383E−5	3.1204
80	800	9.808E−6	3.0225	3.127E−6	2.9302
160	1600	1.269E−6	2.9500	4.071E−7	2.9412

5. NUMERICAL EXPERIMENT

In this section we present numerical results of the proposed scheme for the one-dimensional and two-dimensional stochastic Maxwell equations with multiplicative noise. We use polynomials of degree k in our proposed DG methods for spatial discretization, and Taylor 2.0 strong scheme for the temporal discretization. The accuracy tests are provided for both 1D case and 2D case to demonstrate the convergence rate. The time history of the discrete energy is also presented for both examples. The Monte-Carlo method is used to compute the stochastic term, and the expectation is computed by averaging over all the samples.

We first consider the following one-dimensional equations

$$\begin{cases} \mathrm{d}v = -u_x\,\mathrm{d}t + v\,\mathrm{d}W_t, \\ \mathrm{d}u = -v_x\,\mathrm{d}t + u\,\mathrm{d}W_t, \end{cases}$$

with periodic boundary conditions and the exact solutions given by

$$\begin{cases} v = (\sin(x-t) + \cos(x+t))e^{W_t - \frac{1}{2}t}, \\ u = (\sin(x-t) - \cos(x+t))e^{W_t - \frac{1}{2}t}. \end{cases} \tag{5.1}$$

The computational domain is set to be $[0, 2\pi]$, and the final time $T = 0.5$. Nx and Nt are used to denote the number of space steps and time steps, and we use uniform meshes as spatial discretization. The numerical initial condition is taken as the projection of the exact solutions (5.1) at $t = 0$. We apply Monte Carlo simulation with 3000 samples to approximate the expectation. Table 1 and Table 2 show the convergence rate of numerical errors $e_u = u - u_h$ and $e_v = v - v_h$, when $k = 1$ and $k = 2$ respectively. In both examples Δt is chosen small enough to ensure the spatial error dominates, and we can observe the optimal error estimate (the expected $(k+1)$-th order of convergence), which is consistent with the result in Theorem 2.4 for the semi-discrete DG method.

Next we consider the two-dimensional stochastic Maxwell equations

$$\begin{cases} \mathrm{d}E - T_x\,\mathrm{d}t + S_y\,\mathrm{d}t = E\,\mathrm{d}W_t, \\ \mathrm{d}S + E_y\,\mathrm{d}t = S\,\mathrm{d}W_t, \\ \mathrm{d}T - E_x\,\mathrm{d}t = T\,\mathrm{d}W_t, \end{cases}$$

TABLE 3. Numerical error and convergence rates of 2D case when $k = 1$.

Nx	Ny	Nt	$(\mathbb{E}\|e_E\|^2)^{1/2}$	Rate	$(\mathbb{E}\|e_S\|^2)^{1/2}$	Rate	$(\mathbb{E}\|e_T\|^2)^{1/2}$	rate
20	20	20	0.02582	–	0.01807	–	0.01807	–
40	40	40	6.690E−3	1.9484	4.674E−3	1.9510	4.674E−3	1.9510
80	80	80	1.648E−3	2.0215	1.160E−3	2.0106	1.160E−3	2.0106
160	160	160	3.984E−4	2.0482	2.810E−4	2.0453	2.810E−4	2.0453

TABLE 4. Numerical error and convergence rates of 2D case when $k = 2$.

Nx	Ny	Nt	$(\mathbb{E}\|e_E\|^2)^{1/2}$	Rate	$(\mathbb{E}\|e_S\|^2)^{1/2}$	Rate	$(\mathbb{E}\|e_T\|^2)^{1/2}$	rate
20	20	20	8.029E−4	–	5.741E−4	–	5.741E−4	–
40	40	40	1.052E−4	2.9317	7.509E−5	2.9347	7.509E−5	2.9347
80	80	80	1.254E−5	3.0695	9.113E−6	3.0425	9.113E−6	3.0425
160	160	160	1.572E−6	2.9952	1.179E−6	2.9507	1.179E−6	2.9507

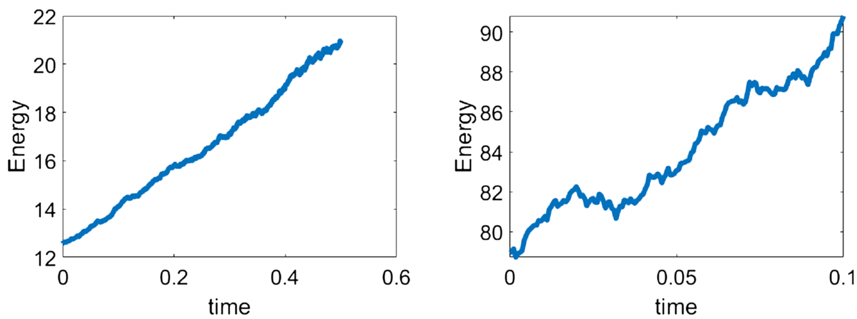

FIGURE 1. The time history of the averaged energy. Left: 1D result; Right: 2D result.

with periodic boundary conditions. The exact solutions take the form

$$\begin{cases} E = (\sin(x+t) - \cos(y+t))e^{W_t - \frac{1}{2}t}, \\ S = \sin(x+t)e^{W_t - \frac{1}{2}t}, \\ T = \cos(y+t)e^{W_t - \frac{1}{2}t}. \end{cases} \tag{5.2}$$

The space domain is $[0, 2\pi]^2$. The numerical initial condition is taken as the projection of the exact solutions (5.2) at $t = 0$. Nx, Ny and Nt are used to denote the number of space cells in x, y directions, and the number of time steps, and uniform rectangular meshes are considered as spatial discretization. We run the simulation until final time $T = 0.1$. Monte Carlo simulation with 500 samples is used to approximate the expectation. Table 3 and Table 4 present the convergence rate of numerical errors $e_w = w - w_h$, $w = E, S, T$, for the cases of $k = 1$ and $k = 2$ respectively. We can observe that in both cases, the optimal convergence rate is achieved.

The discrete energy law satisfied by the numerical solutions was studied in Theorem 2.2 for the one-dimensional system, and in Theorems 3.2 and 3.5 for the two-dimensional system. In Figure 1, the time history of averaged energy is shown for two cases.

6. Conclusion remarks

In this paper we applied high order DG scheme for one- and two-dimensional stochastic Maxwell equations with multiplicative noise. We provide the semi-discrete energy law for both cases. Optimal error estimate of the semi-discrete method is obtained for one-dimensional case, and two-dimensional case on both rectangular meshes and triangular meshes under certain mesh assumptions. The semi-discrete method is combined with strong Taylor 2.0 temporal discretization, and numerical results are presented to validate the optimal error estimates and the growth of energy.

Acknowledgements. Chi-Wang Shu's work is partially supported with NSF grant DMS-2010107, AFOSR grant FA9550-20-1-0055 and Yulong Xing's work is partially supported with NSF grant DMS-1753581.

References

[1] S.M. Rytov, Y.A. Kravtsov and V.I. Tatarskii, Principles of Statistical Radiophysics: Elements and Random Fields 3. Springer, Berlin (1989).

[2] G.N. Ord, A stochastic model of Maxwell's equations in 1+1 dimensions. *Int. J. Theor. Phys.* **35** (1996) 263–266.

[3] T. Horsin, I. Stratis and A. Yannacopoulos, On the approximate controllability of the stochastic Maxwell equations. *IMA J. Math. Control Inf.* **27** (2010) 103–118.

[4] K.B. Liaskos, I.G. Stratis and A.N. Yannacopoulos, Stochastic integrodifferential equations in Hilbert spaces with applications in electromagnetics. *J. Integral Equ. App.* **22** (2010) 559–590.

[5] P. Benner and J. Schneider, Uncertainty quantification for Maxwell's equations using stochastic collocation and model order reduction. *Int. J. Uncertainty Quantif.* **5** (2015) 195–208.

[6] C. Jung, Evolution of probability distribution in time for solutions of hyperbolic equations. *J. Sci. Comput.* **41** (2009) 13–48.

[7] C. Jung, B. kwon, A. Mahalov and T. Nguyen, Maxwell solutions in media with multiple random interfaces. *Int. J. Numer. Anal. Model.* **11** (2014) 194–213.

[8] J. Li, Z. Fang and G. Lin, Regularity analysis of metamaterial Maxwell's equations with random coefficients and initial conditions. *Comput. Methods Appl. Mech. Eng.* **335** (2018) 24–51.

[9] J. Hong, L. Ji, L. Zhang and J. Cai, An energy-conserving method for stochastic Maxwell equations with multiplicative noise. *J. Comput. Phys.* **351** (2017) 216–229.

[10] D. Cohen, J. Cui, J. Hong and L. Sun, Exponential integrators for stochastic Maxwell's equations driven by Itô noise. *J. Comput. Phys.* **410** (2020) 109382.

[11] C. Chen, J. Hong and J. Ji, Mean-square convergence of a semidiscrete scheme for stochastic Maxwell equations. *SIAM J. Numer. Anal.* **57** (2019) 728–750.

[12] L. Zhang, C. Chen, J. Hong and L. Ji, A review on stochastic multi-symplectic methods for stochastic Maxwell equations. *Commun. Appl. Math. Comput.* **1** (2019) 467–501.

[13] J. Hong, B. Hou and L. Sun, Energy-preserving fully-discrete schemes for nonlinear stochastic wave equations with multiplicative noise. *J. Comput. Phys.* **451** (2022) 110829.

[14] Y. Li, S. Wu and Y. Xing, Finite element approximations of a class of nonlinear stochastic wave equation with multiplicative noise. *J. Sci. Comput.* **91** (2022) 53.

[15] W.H. Reed and T. Hill, Triangular mesh methods for the neutron transport equation. Technical report, Los Alamos Scientific Lab., N. Mex. (USA) (1973).

[16] B. Cockburn, S. Hou and C.-W. Shu, The Runge–Kutta local projection discontinuous Galerkin finite element method for conservation laws IV: the multidimensional case. *Math. Comput.* **54** (1990) 545–581.

[17] B. Cockburn, G. Karniadakis and C.-W. Shu, The development of discontinuous galerkin methods, in Discontinuous Galerkin Methods: Theory, Computation and Applications. *Lecture Notes in Computational Science and Engineering, Part I: Overview*, edited by B. Cockburn, G. Karniadakis and C.-W. Shu. Vol. 11, Springer (2000) 3–50.

[18] B. Cockburn, S.-Y. Lin and C.-W. Shu, TVB Runge–Kutta local projection discontinuous Galerkin finite element method for conservation laws III: one dimensional systems. *J. Comput. Phys.* **84** (1989) 90–113.

[19] B. Cockburn and C.-W. Shu, TVB Runge–Kutta local projection discontinuous Galerkin finite element method for conservation laws II: general framework. *Math. Comput.* **52** (1989) 411–435.

[20] Y. Cheng, C.-S. Chou, F. Li and Y. Xing, L^2 stable discontinuous Galerkin methods for one-dimensional two-way wave equations. *Math. Comput.* **86** (2017) 121–155.

[21] Z. Sun and Y. Xing, Optimal error estimates of discontinuous Galerkin methods with generalized fluxes for wave equations on unstructured meshes. *Math. Comput.* **90** (2021) 1741–1772.

[22] Y. Li, C.-W. Shu and S. Tang, A discontinuous Galerkin method for stochastic conservation laws. *SIAM J. Sci. Comput.* **42** (2020) A54–A86.

[23] J. Sun, Y. Xing and C.-W. Shu, Multi-symplectic discontinuous Galerkin methods for the stochastic Maxwell equations with additive noise. *J. Comput. Phys.* **461** (2022) 111199.

[24] C. Chen, A symplectic discontinuous Galerkin full discretization for stochastic Maxwell equations. *SIAM J. Numer. Anal.* **59** (2021) 2197–2217.

[25] X. Meng, C.-W. Shu and B. Wu, Optimal error estimates for discontinuous Galerkin methods based on upwind-biased fluxes for linear hyperbolic equations. *Math. Comput.* **85** (2016) 1225–1261.

[26] P. Kloeden and E. Platen, in Numerical Solution of Stochastic Differential Equations. *Applications in Mathematics, Stochastic Modelling and Applied Probability*, 3rd edition. Vol. 23. Springer-Verlag, Berlin (1999).