A FLUX GLOBALIZATION BASED WELL-BALANCED PATH-CONSERVATIVE CENTRAL-UPWIND SCHEME FOR THE SHALLOW WATER FLOWS IN CHANNELS

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Abstract. We develop a flux globalization based well-balanced (WB) path-conservative central-upwind (PCCU) scheme for the one-dimensional shallow water flows in channels. Challenges in developing numerical methods for the studied system are mainly related to the presence of nonconservative terms modeling the flow when the channel width and bottom topography are discontinuous. We use the path-conservative technique to treat these nonconservative product terms and implement this technique within the flux globalization framework, for which the friction and aforementioned nonconservative terms are incorporated into the global flux: This results in a quasi-conservative system, which is numerically solved using the Riemann-problem-solver-free central-upwind scheme. The WB property of the resulting scheme (that is, its ability to exactly preserve both still- and moving-water equilibria at the discrete level) is ensured by performing piecewise linear reconstruction for the equilibrium variables rather than the conservative variables, and then evaluating the global flux using the obtained point values of the equilibrium quantities. The robustness and excellent performance of the proposed flux globalization based WB PCCU scheme are demonstrated in several numerical examples with both continuous and discontinuous channel width and bottom topography. In these examples, we clearly demonstrate the advantage of the proposed scheme over its simpler counterparts.

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1. Introduction

This paper is concerned with the one-dimensional (1-D) hyperbolic system of balance laws modeling shallow water (SW) flows in channels. Such systems are widely used to study the environmental water system, such as rivers and canals. The studied system (see, e.g., [2, 34, 40, 42]) reads as

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\[
\begin{align*}
A_t + Q_x &= 0, \\
Q_t + \left( \frac{Q^2}{A} + \frac{g}{2} A^2 \eta \right)_x &= -\frac{g}{2} A^2 \eta_x - gAZ_x - gAS_f,
\end{align*}
\]  

(1.1)

where \( x \) is a spatial variable, \( t \) is the time, \( A(x,t) = h(x,t)b(x) \) is the wetted cross-sectional area, where \( h(x,t) \) is the water depth, \( b = b(x) \) represents channel width and \( \eta(x) := 1/b(x) \), \( Q(x,t) = q(x,t)b(x) \) is the rate of discharge, where \( q(x,t) = \frac{h(x,t)}{b(x)} \) is the discharge and \( u(x,t) \) is the velocity, \( Z = Z(x) \) is the bottom topography, \( g \) is the constant gravitational acceleration, and \( S_f \) is the Manning friction term (see, e.g., \[15,17,19,36\]) given by

\[
S_f = \frac{n^2 Q|Q|}{A^2 \rho^3}.
\]

Here, \( n \) denotes the Manning coefficient and \( \rho = \left( \frac{1}{4\eta^2} + 2\eta \right)^{-1} \) is the hydraulic radius.

Development of accurate and robust numerical method for the system (1.1) is a challenging task as this nonlinear hyperbolic system admits solutions that develop shocks, rarefactions, and contact discontinuities even when the initial data are smooth. In addition, a good numerical scheme should be able to respect a delicate balance between the flux and source terms in (1.1). We say that the scheme is well-balanced (WB) if it is capable of exactly preserving some (physically relevant) steady-state solutions of (1.1). Such solutions satisfy the following system of time-independent PDEs:

\[
\begin{align*}
Q_x &= 0, \\
\left( \frac{Q^2}{A} + \frac{g}{2} A^2 \eta \right)_x &= -\frac{g}{2} A^2 \eta_x - gAZ_x - gAS_f,
\end{align*}
\]

which can be integrated to obtain

\[
Q \equiv \hat{Q} = \text{Const}, \quad E \equiv \hat{E} = \text{Const},
\]

(1.2)

where

\[
E := \frac{u^2}{2} + g(\eta + Z) + \int_{\hat{x}}^{x} gS_f \, d\xi
\]

(1.3)

is a global quantity with \( u = Q/A \) and \( \hat{x} \) being an arbitrary number. The “lake-at-rest” (still-water) equilibria form a particular class of steady states (1.2) and (1.3) with zero discharge:

\[
Q \equiv 0, \quad w := h + Z = A\eta + Z \equiv \text{Const},
\]

(1.4)

where \( w \) represents the water surface.

The steady states (1.2) and (1.4) are of great practical importance since many physically relevant solutions of (1.1) are, in fact, small perturbations of these steady states. We note that if a stable but non-well-balanced (NWB) scheme is applied to the studied SW system, then the numerical errors may trigger appearance of artificial waves of magnitude larger than the size of the waves to be captured. A remedy to this problem would be a mesh refinement, but in practical situations such refinement may be too computationally expensive or even unaffordable. One therefore needs to derive WB schemes.

If \( \eta(x) \equiv \text{Const} \), then the studied system (1.1) reduces to the Saint-Venant system, for which many WB schemes have been developed. For the schemes capable of preserving the “lake-at-rest” steady states (1.4), we refer the reader to the non-exhaustive list of references \[1,3,7,10,18,22,25,39\]. The WB property of these methods hinges on a special approximation of the geometric source term \(-gAZ_x\). The case of the moving-water equilibria is substantially more complicated; see, e.g., \[4–6,8,9,23,35,43\] and references therein. The difficulty is related to the fact that WB approximations of the geometric source term now need to include terms that are small for smooth solutions, but may become artificially large at discontinuities. Several WB numerical methods
for SW models in channels have been proposed. For example, schemes capable of preserving “lake-at-rest” steady states were introduced in [2,40,42] in the case of rectangular cross-section channels and in [16,20,21] in a more general case of arbitrary cross-section channels. In [34], a relaxation scheme that can exactly preserve moving-water equilibria for rectangular cross-section channels was proposed.

In this paper, we develop WB schemes for the system (1.1) using a flux globalization approach which was proposed in [12] and then applied to a variety of hyperbolic systems of balance laws in [5,9,11,13,28,29,35]. In this approach, we incorporate the source terms into the fluxes and rewrite (1.1) in the following quasi-conservative form:

\begin{align*}
A_t + Q_x &= 0, \\
Q_t + K_x &= 0,
\end{align*}

(1.5)

where \( K \) is a global flux given by

\begin{equation}
K = \frac{Q^2}{A} + \frac{g}{2}A^2\eta + R, \quad R := \int_x^x \left( \frac{g}{2}A^2\eta_x + gAZ_x + gAS_f \right) d\xi.
\end{equation}

(1.6)

We note that the steady states (1.2) and (1.4) can be rewritten in terms of the global fluxes as

\begin{equation}
Q \equiv \bar{Q} = \text{Const}, \quad K \equiv \bar{K} = \text{Const}.
\end{equation}

(1.7)

It might be challenging to numerically solve the system (1.5) and (1.6) using a Riemann-problem-solver based upwind scheme due to the global nature of the flux (1.6). Therefore, we follow [5,9,11–13,28,29] and apply a Riemann-problem-solving-free semi-discrete central-upwind (CU) scheme to the studied system (1.5) and (1.6). CU schemes were originally introduced in [24,26,27] in the context of general multidimensional hyperbolic system of conservation laws and were later extended to a variety of hyperbolic system of balance laws including several SW models; see the review paper [23] and references therein. In order to apply the CU scheme to the quasi-conservative system (1.5) and (1.6), we need to evaluate the CU numerical fluxes, which would be functions of the point values of \( A, Q, \) and \( K \) at the cell interfaces, which may be computed out of the available cell averages of \( A \) and \( Q \) using a piecewise polynomial reconstruction of \( A \) and \( Q \), and an appropriate quadrature for the integral in (1.6). This, however, will not lead to a WB scheme as \( A \) is not an equilibrium variable, that is, it does not remain constant at steady states. We may follow the lines of [9,11–13,28] and reconstruct the equilibrium quantities \( Q \) and \( K \). This will lead to a WB CU scheme, which is expected to be highly accurate and robust as long as the function \( \eta(x) \) and \( Z(x) \) are continuous.

When \( \eta \) and \( Z \) are discontinuous, the right-hand side (RHS) of (1.1) would contain nonconservative product terms, which is an additional challenge as in this case, weak solutions cannot be understood in the sense of distributions. Instead, weak solutions can be defined as the Borel measures; see, e.g., [14,30,31]. This concept of weak solutions was used to develop path-conservative (PC) finite-volume methods; see, e.g., the review papers [6,38] and references therein. The PC approach was incorporated into the CU schemes in [7], where the path-conservative central-upwind (PCCU) schemes were introduced. The drawback of the PCCU schemes, however, is that they can preserve “lake-at-rest” steady states only. In order to design moving-water equilibrium preserving PCCU schemes, the PC approach was incorporated into the flux globalization based WB CU schemes in [29]; also see [5]. This results in a flux globalization based WB PCCU scheme, which we apply to the studied SW in channel system. To this end, we reconstruct the conservative variable \( E \) given by (1.3) rather than the global flux variable \( K \) in (1.6). The resulting scheme will ensure that both \( E \) and \( K \) will remain constant at the discrete steady states. A major advantage of the new flux globalization based WB PCCU scheme is that it can preserve a much wider variety of steady states (compared to its aforementioned alternatives) including some discontinuous ones, naturally arising in the case of discontinuous \( \eta \) and/or \( Z \).

The rest of the paper is organized as follows. In Section 2, we introduce the flux globalization based WB PCCU scheme for the SW flows in channels. In Section 3, we present the numerical results, which demonstrate the WB property, high resolution and robustness of the proposed scheme.
2. Flux globalization based WB PCCU scheme

In this section, we develop a semi-discrete second-order flux globalization based WB PCCU scheme for the SW in channels equations (1.1). We rewrite (1.1) in the following vector form:

\[ U_t + F(U, V)_x = B(U)V_x + S(U, V), \quad U = (A, Q)^T, \quad V = (\eta, Z)^T, \]
\[ F(U, V) = \begin{pmatrix} Q^2 \\ A^2 + g^2 A^2 \eta \end{pmatrix}, \quad B(U) = \begin{pmatrix} 0 \\ -g A^2 - g A \end{pmatrix}, \quad S(U, V) = \begin{pmatrix} 0 \\ -g AS_f \end{pmatrix}. \] (2.1)

In order to apply a finite-volume method to (2.1), we first introduce a finite-volume mesh consisting of uniform cells \( C_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}] \) of size \( \Delta x = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}} \) centered at \( x_j = \left(x_{j-\frac{1}{2}} + x_{j+\frac{1}{2}}\right)/2, \ j = 1, \ldots, N \). We denote by \( \overline{U}_j \) the computed cell averages

\[ \overline{U}_j(t) \approx \frac{1}{\Delta x} \int_{C_j} U(x, t) \, dx, \]

which are assumed to be available at a certain time level \( t \). In the framework of semi-discrete schemes, the cell averages are evolved in time by using an appropriate time discretization, which results in a system of ODEs, which is then numerically integrated using an appropriate ODE solver. In Sections 2.1 and 2.2, we will present two semi-discrete PCCU schemes for the system (2.1).

2.1. PCCU scheme

In this section, we apply the PCCU scheme introduced in [7] to the studied system (2.1). In the PCCU scheme, the ODE system for the cell averages reads as (from now on we will omit the time dependence of the indexed quantities for the sake of brevity):

\[ \frac{d\overline{U}_j}{dt} = -\frac{1}{\Delta x} \left\{ \mathcal{F}_{j+\frac{1}{2}} - \mathcal{F}_{j-\frac{1}{2}} - B_j - \frac{a_{j-\frac{1}{2}}^-}{a_{j-\frac{1}{2}}^+ - a_{j-\frac{1}{2}}^-} B_{\Psi, j-\frac{1}{2}} + \frac{a_{j+\frac{1}{2}}^-}{a_{j+\frac{1}{2}}^- - a_{j+\frac{1}{2}}^+} B_{\Psi, j+\frac{1}{2}} - S_j \right\}, \] (2.2)

where

\[ \mathcal{F}_{j+\frac{1}{2}} = \frac{a_{j+\frac{1}{2}}^+}{a_{j+\frac{1}{2}}^- - a_{j+\frac{1}{2}}^+} \left( \frac{F(U_{j+\frac{1}{2}}, V^-_{j+\frac{1}{2}})}{a_{j+\frac{1}{2}}^- - a_{j+\frac{1}{2}}^+} - \frac{F(U_{j+\frac{1}{2}}, V^+_{j+\frac{1}{2}})}{a_{j+\frac{1}{2}}^- - a_{j+\frac{1}{2}}^+} \right) + \frac{a_{j+\frac{1}{2}}^-}{a_{j+\frac{1}{2}}^- - a_{j+\frac{1}{2}}^+} \left( U_{j+\frac{1}{2}}^+ - U_{j+\frac{1}{2}}^- \right). \] (2.3)

Here, \( U_{j+\frac{1}{2}}^\pm \) and \( V_{j+\frac{1}{2}}^\pm \) are the left and right point values of \( U \) and \( V \), respectively, at the cell interfaces \( x = x_{j+\frac{1}{2}} \). We obtain these values using a piecewise linear reconstruction based on the generalized minmod limiter; see Appendix A.

The values \( a_{j+\frac{1}{2}}^\pm \) in (2.2) and (2.3) are one-sided local speeds of propagation, which can be estimated using the eigenvalues of the Jacobian \( \partial F/\partial U \) as follows:

\[ a_{j+\frac{1}{2}}^+ = \max \left\{ u_{j+\frac{1}{2}}^- + \sqrt{g h_{j+\frac{1}{2}}^-}, u_{j+\frac{1}{2}}^+ + \sqrt{g h_{j+\frac{1}{2}}^+}, 0 \right\}, \]
\[ a_{j+\frac{1}{2}}^- = \min \left\{ u_{j+\frac{1}{2}}^- - \sqrt{g h_{j+\frac{1}{2}}^-}, u_{j+\frac{1}{2}}^+ - \sqrt{g h_{j+\frac{1}{2}}^+}, 0 \right\}, \]

where \( u_{j+\frac{1}{2}}^\pm = Q_{j+\frac{1}{2}}^\pm \sqrt{A_{j+\frac{1}{2}}^\pm} \) and \( h_{j+\frac{1}{2}}^\pm = A_{j+\frac{1}{2}}^\pm \). 

Next, the terms \( B_j \) and \( B_{\Psi, j+\frac{1}{2}} \) in (2.2) reflect the contribution of the term \( B(U)V_x \) on the RHS of the studied system (2.1). Their first components are obviously 0 and the second components are computed as in
In particular, $B_j^{(2)}$ are evaluated by the following second-order quadrature:

$$B_j^{(2)} = \int_{C_j} [B(U) V_x]^{(2)} \, dx = - \frac{q}{2} A^2 \eta_x + g A Z_x \, dx$$

$$\approx - \frac{q}{2} A_{j+\frac{1}{2}}^+ A_{j-\frac{1}{2}}^- \left( \eta_{j+\frac{1}{2}}^- - \eta_{j-\frac{1}{2}}^+ \right) - \frac{q}{2} \left( A_{j+\frac{1}{2}}^- + A_{j-\frac{1}{2}}^+ \right) \left( Z_{j+\frac{1}{2}}^- - Z_{j-\frac{1}{2}}^+ \right).$$

In order to compute $B_{\Psi,j+\frac{1}{2}}^{(2)}$, we use the linear paths

$$\Psi_{j+\frac{1}{2}}(s) = \left( \begin{array}{c} A_{j+\frac{1}{2}}(s) \\ Q_{j+\frac{1}{2}}(s) \end{array} \right) = \frac{1}{2} \left( A_j^+ + s \left( A_{j+\frac{1}{2}}^+ - A_{j-\frac{1}{2}}^- \right) \right),$$

$$\Phi_{j+\frac{1}{2}}(s) = \frac{1}{2} \left( \begin{array}{c} \eta_{j+\frac{1}{2}}(s) \\ Z_{j+\frac{1}{2}}(s) \end{array} \right) = \frac{1}{2} \left( \eta_j^- + s \left( \eta_{j+\frac{1}{2}}^+ - \eta_{j+\frac{1}{2}}^- \right) \right),$$

so that

$$B_{\Psi,j+\frac{1}{2}}^{(2)} = \frac{1}{2} \left[ B \left( \Psi_{j+\frac{1}{2}}(s) \right) \Phi'_{j+\frac{1}{2}}(s) \right]^{(2)} \, ds = - \frac{1}{2} \left[ \frac{q}{2} A_{j+\frac{1}{2}}^2 \eta_{j+\frac{1}{2}}'(s) + g A_{j+\frac{1}{2}}(s) Z'_{j+\frac{1}{2}}(s) \right] \, ds$$

$$= - \frac{q}{2} A_{j+\frac{1}{2}}^+ A_{j-\frac{1}{2}}^- \left( \eta_{j+\frac{1}{2}}^+ - \eta_{j+\frac{1}{2}}^- \right) - \frac{q}{2} \left( A_{j+\frac{1}{2}}^- + A_{j-\frac{1}{2}}^+ \right) \left( Z_{j+\frac{1}{2}}^+ - Z_{j+\frac{1}{2}}^- \right).$$

Finally, the term $S_j^{(2)}$ is approximated using the trapezoidal quadrature rule:

$$S_j^{(2)} = \int_{C_j} [S(U, V)]^{(2)} \, dx = - \int_{C_j} g A S_f \, dx \approx - \frac{q}{2} \left( A_{j+\frac{1}{2}}^+ (S_f)_{j+\frac{1}{2}}^- + A_{j-\frac{1}{2}}^- (S_f)_{j-\frac{1}{2}}^+ \right).$$

### 2.1.1. Still-water equilibria preserving PCCU scheme

We note that the PCCU scheme (2.2) and (2.3) is not WB as it cannot preserve even the simplest still-water equilibria (1.4). We therefore follow [7] and modify the numerical diffusion by replacing the CU numerical flux (2.3) with

$$F_{j+\frac{1}{2}} = \frac{a_j^+}{a_j^-} \left( U_{j+\frac{1}{2}}^+, V_{j+\frac{1}{2}}^- \right) - \frac{a_j^-}{a_j^+} \left( U_{j+\frac{1}{2}}^-, V_{j+\frac{1}{2}}^+ \right)$$

$$\left[ \begin{array}{c} F_{j+\frac{1}{2}}^+ \\ F_{j+\frac{1}{2}}^- \end{array} \right] = \left[ \begin{array}{c} \eta_{j+\frac{1}{2}}^+ \\ Z_{j+\frac{1}{2}}^+ \end{array} \right]$$

where

$$J_{j+\frac{1}{2}}^* = \left( \begin{array}{cc} 0 & 1 \\ \frac{g}{2} & 0 \end{array} \right)$$

and

$$\tilde{B}_{\Psi,j+\frac{1}{2}} = \left( \begin{array}{c} 0 \\ \frac{g}{2} \left( A_{j+\frac{1}{2}}^+ + A_{j-\frac{1}{2}}^- \right) \left( Z_{j+\frac{1}{2}}^+ - Z_{j-\frac{1}{2}}^- \right) - g A_{j+\frac{1}{2}}^+ A_{j-\frac{1}{2}}^- \left( \eta_{j+\frac{1}{2}}^+ - \eta_{j+\frac{1}{2}}^- \right) \right)$$
are obtained in a similar way as in equations (4.20), (4.22) of [7]. It is straightforward to verify that the term
\[
U^+_{j+\frac{1}{2}} - U^-_{j+\frac{1}{2}} - \left( J^+_j \right)_{j+\frac{1}{2}}^{-1} \mathbf{B}^*_{j+1} = \left( A^+_{j+\frac{1}{2}} - A^-_{j+\frac{1}{2}} \right) \left( 1 + \frac{Z^+_{j+\frac{1}{2}} - Z^-_{j+\frac{1}{2}}}{A^+_{j+\frac{1}{2}} \eta^+_j + A^-_{j+\frac{1}{2}} \eta^-_{j+\frac{1}{2}}} \right),
\] (2.9)
and thus both (2.9) and (2.8) vanish as long as \( A^+_{j+\frac{1}{2}} \eta^+_j + A^-_{j+\frac{1}{2}} \eta^-_{j+\frac{1}{2}} = A^+_{j+\frac{1}{2}} \eta^+_j + A^-_{j+\frac{1}{2}} \eta^-_{j+\frac{1}{2}} \), that is, when the data are at a “lake-at-rest” steady state (1.4).

The PCCU scheme (2.2), (2.8), (2.9) is capable of preserving still-water equilibria, but it cannot, in general, preserve moving-water steady states (1.2) and (1.3). In the next section, we develop a flux globalization based PCCU scheme, which is capable of exactly preserve discrete steady states given by either (1.2), (1.3) or (1.7).

### 2.2. Flux globalization based PCCU scheme

In this section, we follow the approach recently introduced in [5,29] and introduce the flux globalization based semi-discrete WB PCCU scheme, respectively.

The cell averages \( \bar{U}_j \) are evolved in time by solving the following system of ODEs:
\[
\frac{d \bar{U}_j}{dt} = -\frac{\mathcal{K}_{j+\frac{1}{2}} - \mathcal{K}_{j-\frac{1}{2}}}{\Delta x},
\] (2.10)
which is a semi-discrete approximation of the quasi-conservative system (1.5). Here,
\[
\mathcal{K}_{j+\frac{1}{2}} = \frac{a^+_{j+\frac{1}{2}} K^-_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}} K^+_{j+\frac{1}{2}}}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}} + \frac{a^+_{j+\frac{1}{2}} a^-_{j+\frac{1}{2}}}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}} \left( \bar{U}^+_{j+\frac{1}{2}} - \bar{U}^-_{j+\frac{1}{2}} \right),
\] (2.11)
are CU numerical fluxes with \( \mathbf{K} := (Q, K)^T \) and \( \mathbf{K}^+_{j+\frac{1}{2}} := \left( Q^+_{j+\frac{1}{2}}, K^+_{j+\frac{1}{2}} \right)^T \). In order to compute the one-sided point values used in (2.11), we proceed as follows.

First, we note that in order to develop a WB scheme, one should reconstruct the equilibrium variables (those that are constant at the steady states to be preserved) rather than \( U \). In order to develop a scheme capable of exactly preserving the moving-water equilibria (1.2), we will reconstruct the variables \( Q \) and \( E \), where \( E \) is given by (1.3). To this end, we need to evaluate the discrete values of \( E \) that contain an integral term \( I = \int_{x_1}^x g(S_f) \, dx \), which we evaluate in a recursive way as follows. We set \( \tilde{x} := x \), so that \( I_1 = 0 \), and then we apply the trapezoidal rule to obtain
\[
I_1 = I_1 + \frac{g \Delta x}{4} \left[ (S_f)_1 + (S_f)_{\frac{1}{2}} \right] = \frac{g \Delta x}{4} \left[ (S_f)_1 + (S_f)_{\frac{1}{2}} \right],
\]
where the value \( (S_f)_{\frac{1}{2}} \) is obtained using the prescribed boundary conditions, and then
\[
I_j = I_{j-1} + \frac{g \Delta x}{2} \left[ (S_f)_j + (S_f)_{j-1} \right], \quad j = 2, \ldots, N,
\]
where \( (S_f)_j = \left( n^2 \overline{Q}_j \overline{I}_j \right) / \left( \overline{A}_j R_j^{2/3} \right) \) and \( R_j = \left( \frac{1}{A_j \eta_j} + 2 \eta_j \right)^{-1} \). After computing the global terms \( I_j \), we will have \( E_j = \overline{Q}_j^2 / \left( 2 \overline{A}_j^2 \right) + g(\overline{A}_j \eta_j + Z_j) + I_j \).

Next, equipped with \( E_j \) we apply the piecewise linear generalized mimnod reconstruction described in Appendix A to the variables \( Q, E, \eta \) and \( Z \) and obtain the reconstructed one-sided point values \( Q^\pm_{j+\frac{1}{2}}, E^\pm_{j+\frac{1}{2}}, \eta^\pm_{j+\frac{1}{2}}, Z^\pm_{j+\frac{1}{2}} \).
\[ \eta^\pm_{\frac{1}{2}} \text{ and } Z^\pm_{\frac{1}{2}}. \] We then obtain the one-sided point values \( A^\pm_{j+\frac{1}{2}} \) by solving the equations

\[
E^\pm_{j+\frac{1}{2}} = \left( \frac{Q^\pm_{j+\frac{1}{2}}}{A^\pm_{j+\frac{1}{2}}} \right)^2 + g\left( A^+_{j+\frac{1}{2}} \eta^+_{j+\frac{1}{2}} + Z^+_{j+\frac{1}{2}} \right) + I_{j+\frac{1}{2}} \tag{2.12}
\]

for \( A^+_{j+\frac{1}{2}} \) and \( A^-_{j+\frac{1}{2}} \), respectively. When \( Q^\pm_{j+\frac{1}{2}} \neq 0 \), then the equations in (2.12) are cubic and we solve them using the exact solver described in Appendix B.

We then evaluate the one-sided values \( R^\pm_{j+\frac{1}{2}} \) of the global variable \( R \) defined in (1.6). We proceed along the lines of [29] and obtain the following recursive formulae:

\[
R^+_{j+\frac{1}{2}} = 0, \quad R^+_{j+\frac{1}{2}} = R^-_{j+\frac{1}{2}} + B^{(2)}_{\Psi,j+\frac{1}{2}}, \quad j = 0, \ldots, N, \\
R^-_{j+\frac{1}{2}} = R^+_{j-\frac{1}{2}} + B^{(2)}_j, \quad j = 1, \ldots, N,
\]

where \( B^{(2)}_{\Psi,j+\frac{1}{2}} \) and \( B^{(2)}_j \) are similar to the terms \( B^{(2)}_{\Psi,j+\frac{1}{2}} \) and \( B^{(2)}_j + S^{(2)}_j \) defined in (2.6) and (2.4), (2.7), respectively, but the integrals there are now approximated in a different, WB way as it has been done in [29]. To this end, we first note that it is easy to verify the following identity:

\[
\frac{g}{2} A^2 \eta_x + gA Z_x + gA S_f = AE_x + u Q_x - \frac{g}{2} (A^2 \eta)_x - \left( \frac{Q^2}{A} \right)_x,
\]

which is true inside each cell \( C_j \), where the solution is approximated using the linear pieces \( \bar{Q}_j, \bar{E}_j, \tilde{\eta}_j \) and \( \tilde{Z}_j \), and the corresponding smooth approximation of \( A \), which can be implicitly obtained using the definition of \( E \) in (1.3). We then use (2.13) to obtain

\[
B^{(2)}_j \approx \int_{C_j} \left[ \frac{g}{2} A^2 \eta_x + gA Z_x + gA S_f \right] dx = \int_{C_j} \left[ AE_x + u Q_x - \frac{g}{2} (A^2 \eta)_x - \left( \frac{Q^2}{A} \right)_x \right] dx
\]

\[
\approx \frac{A^-_{j+\frac{1}{2}} + A^+_{j-\frac{1}{2}}}{2} (E^-_{j+\frac{1}{2}} - E^+_{j-\frac{1}{2}}) + \frac{u^-_{j+\frac{1}{2}} + u^+_{j-\frac{1}{2}}}{2} (Q^-_{j+\frac{1}{2}} - Q^+_{j-\frac{1}{2}})
\]

\[
- \frac{g}{2} \left( \left( A^-_{j+\frac{1}{2}} \right)^2 \eta_{j+\frac{1}{2}} - \left( A^+_{j-\frac{1}{2}} \right)^2 \eta_{j-\frac{1}{2}} \right) - \frac{(Q^-_{j+\frac{1}{2}})^2}{A^-_{j+\frac{1}{2}}} + \frac{(Q^-_{j-\frac{1}{2}})^2}{A^+_{j-\frac{1}{2}}}.
\]

In order to evaluate \( B^{(2)}_{\Psi,j+\frac{1}{2}} \), we need to compute the integral across the cell interface as it was done in (2.6). However, the WB property of the resulting scheme will be guaranteed only if we modify the path \( \Psi_{j+\frac{1}{2}} \) by connecting the equilibrium variables, that is, by taking the linear path \( E_{j+\frac{1}{2}}(s) = E^-_{j+\frac{1}{2}} + s(E^+_{j+\frac{1}{2}} - E^-_{j+\frac{1}{2}}) \) instead of \( A_{j+\frac{1}{2}}(s) \) in (2.5), while implicitly obtaining a smooth function \( A_{j+\frac{1}{2}}(s) \) using the definition of \( E \) in (1.3), which gives

\[
E_{j+\frac{1}{2}}(s) = \left( \frac{Q_{j+\frac{1}{2}}(s)}{2(A_{j+\frac{1}{2}}(s))} \right)^2 + g\left( A_{j+\frac{1}{2}}(s) \eta_{j+\frac{1}{2}} + Z_{j+\frac{1}{2}}(s) \right) + I_{j+\frac{1}{2}}. \tag{2.14}
\]
Differentiating (2.14) with respect to \( s \) and multiplying \( E'_{j+\frac{1}{2}}(s) \) by \( A_{j+\frac{1}{2}}(s) \) lead to the following identity:

\[
\frac{g}{2} A_{j+\frac{1}{2}}^2(s) \eta_{j+\frac{1}{2}}'(s) + gA_{j+\frac{1}{2}}(s)Z_{j+\frac{1}{2}}'(s) = A_{j+\frac{1}{2}}(s)E'_{j+\frac{1}{2}}(s) + u_{j+\frac{1}{2}}(s)Q'_{j+\frac{1}{2}}(s)
\]

\[
- \frac{g}{2} \left( A_{j+\frac{1}{2}}^2(s) \eta_{j+\frac{1}{2}}(s) \right)' - \left( \frac{Q_{j+\frac{1}{2}}(s)}{A_{j+\frac{1}{2}}(s)} \right)',
\]

where \( u_{j+\frac{1}{2}}(s) := Q_{j+\frac{1}{2}}(s)/A_{j+\frac{1}{2}}(s) \). We then use (2.15) to obtain

\[
B_{\Psi,j+\frac{1}{2}}^{(2)} = \int_0^1 \left[ \frac{g}{2} A_{j+\frac{1}{2}}^2(s) \eta_{j+\frac{1}{2}}'(s) + gA_{j+\frac{1}{2}}(s)Z_{j+\frac{1}{2}}'(s) \right] \, ds \approx \frac{A_{j+\frac{1}{2}}^+ + A_{j+\frac{1}{2}}^-}{2} \left( E_{j+\frac{1}{2}}^+ - E_{j+\frac{1}{2}}^- \right)
\]

\[
+ \frac{u_{j+\frac{1}{2}}^+ + u_{j+\frac{1}{2}}^-}{2} \left( Q_{j+\frac{1}{2}}^+ - Q_{j+\frac{1}{2}}^- \right) - \frac{g}{2} \left[ \left( A_{j+\frac{1}{2}}^+ \right)^2 \eta_{j+\frac{1}{2}}^+ - \left( A_{j+\frac{1}{2}}^- \right)^2 \eta_{j+\frac{1}{2}}^- \right] - \frac{(Q_{j+\frac{1}{2}}^+ - Q_{j+\frac{1}{2}}^-)^2}{A_{j+\frac{1}{2}}^+} + \frac{(Q_{j+\frac{1}{2}}^+ - Q_{j+\frac{1}{2}}^-)^2}{A_{j+\frac{1}{2}}^-}.
\]

Next, equipped with \( R_{j+\frac{1}{2}}^\pm \), the values \( K_{j+\frac{1}{2}}^\pm \) required to compute the second component of the numerical fluxes in (2.11), are evaluated using the definition of \( K \) in (1.6), namely, we have

\[
K_{j+\frac{1}{2}}^\pm = \left( \frac{Q_{j+\frac{1}{2}}^\pm}{A_{j+\frac{1}{2}}^\pm} \right)^2 + \frac{g}{2} \left( A_{j+\frac{1}{2}}^\pm \right)^2 \eta_{j+\frac{1}{2}}^\pm + R_{j+\frac{1}{2}}^\pm.
\]

(2.16)

Remark 2.1. We note that following the proof of Theorem 5.1 from [29], one can show that whenever the data are at the discrete equilibrium (1.2), that is, if \( \bar{Q}_j \equiv \bar{Q} \) and \( E_j \equiv \bar{E} \) for all \( j \), then the one-sided point values (2.16) satisfy (1.7), that is, \( K_{j+\frac{1}{2}}^+ = K_{j+\frac{1}{2}}^- \equiv \bar{K} \) for all \( j \).

Finally, in order to ensure the WB property of the resulting flux globalization based PCCU scheme (2.10)–(2.11), we need to guarantee that the term \( \tilde{U}_{j+\frac{1}{2}}^+ - \tilde{U}_{j+\frac{1}{2}}^- \) in (2.11) vanishes whenever the data are at the equilibrium (1.2). We first note that \( \tilde{U}_{j+\frac{1}{2}}^+ - \tilde{U}_{j+\frac{1}{2}}^- \) used in the original CU numerical flux (2.3) is not necessarily zero as \( A_{j+\frac{1}{2}}^\pm \) may not be equal to \( A_{j+\frac{1}{2}}^\mp \). We therefore replace the first component of the difference \( \tilde{U}_{j+\frac{1}{2}}^+ - \tilde{U}_{j+\frac{1}{2}}^- \) with \( \tilde{A}_{j+\frac{1}{2}}^+ - \tilde{A}_{j+\frac{1}{2}}^- \), where the values \( \tilde{A}_{j+\frac{1}{2}}^\pm \) are obtained by solving the equations

\[
E_{j+\frac{1}{2}}^\pm = \left( \frac{Q_{j+\frac{1}{2}}^\pm}{A_{j+\frac{1}{2}}^\pm} \right)^2 + g \left( A_{j+\frac{1}{2}}^\pm \eta_{j+\frac{1}{2}}^\pm + Z_{j+\frac{1}{2}}^\pm \right) + I_{j+\frac{1}{2}},
\]

(2.17)

which are almost the same as the cubic equations (2.12), but with \( \eta_{j+\frac{1}{2}}^\pm \) and \( Z_{j+\frac{1}{2}}^\pm \) replaced with \( \eta_{j+\frac{1}{2}} := \left( \eta_{j+\frac{1}{2}}^+ + \eta_{j+\frac{1}{2}}^- \right)/2 \) and \( Z_{j+\frac{1}{2}} := \left( Z_{j+\frac{1}{2}}^+ + Z_{j+\frac{1}{2}}^- \right)/2 \), respectively. When \( Q_{j+\frac{1}{2}}^\pm \neq 0 \), then the equations in (2.17) are cubic and we solve them using the exact solver described in Appendix B. It is obvious now that as long as \( Q_{j+\frac{1}{2}}^+ = Q_{j+\frac{1}{2}}^- \) and \( E_{j+\frac{1}{2}}^+ = E_{j+\frac{1}{2}}^- \), \( \tilde{A}_{j+\frac{1}{2}}^+ \) and \( \tilde{A}_{j+\frac{1}{2}}^- \) will be equal to \( \tilde{A}_{j+\frac{1}{2}} \), and thus \( \tilde{U}_{j+\frac{1}{2}}^+ - \tilde{U}_{j+\frac{1}{2}}^- = \left( \tilde{A}_{j+\frac{1}{2}}^+ - \tilde{A}_{j+\frac{1}{2}}^- \right) (Q_{j+\frac{1}{2}}^+ - Q_{j+\frac{1}{2}}^-) \right)^T = 0 \).

Clearly if the data correspond to the discrete version of the equilibrium (1.2), then the flux globalization based numerical flux in (2.11) will satisfy \( \mathbf{K}_{j+\frac{1}{2}} \equiv (\bar{Q}, \bar{K})^T \) for all \( j \), and hence the RHS of (2.10) will vanish.
3. Numerical examples

In this section, we test the developed flux globalization based WB PCCU scheme on a number of numerical examples. We compare the results obtained by the proposed scheme with the one computed by the PCCU scheme constructed along the lines of [7], and the flux globalization based WB CU scheme constructed following [9] (this scheme is briefly described in Appendix C). Below the studied schemes will be referred to as the WB PCCU, PCCU, and WB CU schemes.

In all of the examples, we take the constant gravitational acceleration \( g = 9.812 \).

Example 1 – “lake-at-rest” steady state

In the first test, which is a modification of the example from [40], we demonstrate the ability of the WB PCCU, WB CU, and PCCU schemes to preserve the “lake-at-rest” steady state.

The initial data, \( w(x,0) \equiv 12 \) and \( u(x,0) \equiv 0 \), are considered on the computational domain \([0,1500] \) with the free boundary conditions imposed at its both ends. The channel width is

\[
b(x) = 2 \left\{ 1 + \exp \left[ - \left( \frac{x - 1000}{250} \right)^2 \right] \right\},
\]

and the bottom topography is

\[
Z(x) = \begin{cases} 
0.02x, & 0 \leq x < 500, \\
10, & 500 \leq x < 1000, \\
30 - 0.02x, & 1000 \leq x \leq 1500. 
\end{cases}
\]

We take the Manning friction coefficient \( n = 0.04 \) and compute the numerical solution until the final time \( t = 10 \) by the WB PCCU, WB CU, and PCCU schemes using \( N = 600 \) uniform cells. We calculate the \( L^\infty\)-error for the water depth \( h \) computed by the studied schemes and these errors are in the range \( 8 \times 10^{-15} - 2 \times 10^{-14} \) even though \( \Delta x = 2.5 \) is quite large. This confirms the ability of all of the three studied schemes to preserve the “lake-at-rest” steady state within the machine accuracy.

Example 2 – moving-water steady states (continuous \( b \) and \( Z, n = 0 \))

In this example, which is a modification of Example 2 from [9], we consider the frictionless case and first study the convergence of the solutions computed by the WB PCCU, WB CU, and PCCU schemes towards the steady flow at large times.

We take the continuous irregular channel width on the computational domain \([0,25] \),

\[
b(x) = 1 + \frac{1}{2} e^{-\left( \frac{x-15}{3} \right)^2}, \quad (3.1)
\]

and the continuous bottom topography

\[
Z(x) = \begin{cases} 
0.2 - 0.05(x-10)^2, & \text{if } 8 \leq x \leq 12, \\
0, & \text{otherwise.}
\end{cases}
\]

The initial data are

\[
h(x,0) = 2 - Z(x), \quad q(x,0) \equiv 0.
\]

We consider two different sets of boundary conditions, which correspond to either the supercritical,

\[
h(0,t) = 2, \quad q(0,t) = 24, \quad (3.2)
\]

or the subcritical flow:

\[
q(0,t) = 4.42, \quad h(25,t) = 2. \quad (3.3)
\]
Figure 1. Example 2 (moving-water steady state, supercritical case, continuous $b$ and $Z$): $w$, $q$, $Q$, and $E$ computed by the WB PCCU, WB CU, and PCCU schemes.

For both (3.2) and (3.3), we compute the numerical solutions using a uniform mesh with $N = 100$ cells by the WB PCCU, WB CU, and PCCU schemes until the very large final time $t = 1000$. The obtained results suggest that all three schemes converge to roughly the same steady states; see $w$ and $q$ computed in the supercritical (Fig. 1) and subcritical (Fig. 2) cases. However, the equilibrium quantities $Q$ and $E$ are not necessarily constant. In the supercritical case (Fig. 1, lower row) only $E$ computed by the PCCU scheme contains substantial oscillations, while in the subcritical case (Fig. 2, lower row) both $Q$ and $E$ computed by the PCCU scheme are quite oscillatory and $E$ computed by the WB CU scheme is clearly nonconstant. The latter is rather expected as the WB CU scheme can only preserve the steady state (1.7), unlike the WB PCCU scheme which is capable of preserving both (1.7) and (1.2) at the discrete level.

We then follow Example 5 of [9] and test the ability of the studied schemes to capture the propagation of a small perturbation of the obtained moving-water equilibria. To this end, we denote the obtained steady states by $h_S(x)$, $q_S(x)$ and then consider the following initial data:

$$h(x, 0) = h_S(x) + \begin{cases} 
10^{-4} & \text{if } 0.45 \leq x \leq 0.55, \\
0 & \text{otherwise,}
\end{cases} 
q(x, 0) \equiv q_S(x).$$

We compute the solutions by the three studied schemes until the final time $t = 1$ using both coarse and fine meshes with $N = 100$ and 1000, respectively. The obtained differences $h(x, 1) - h_S(x)$ are plotted in Figure 3 for both the supercritical (upper row) and subcritical (lower row) cases. As one can see, in the subcritical case, all of the three schemes capture the perturbation quite accurately though the PCCU scheme develops small oscillations when the coarse mesh is used. However, in the supercritical case, the coarse mesh PCCU solution contains much bigger oscillations, whose magnitude grows (rather than decaying) when the mesh is refined. In order to better understand this phenomenon, we further investigate the performance of the PCCU scheme and in Figure 4, we plot the difference $h(x, 1) - h_S(x)$ computed using several uniform meshes. The obtained results are rather counter-intuitive: the magnitude of oscillations substantially grows when the mesh is refined.
from $N = 400$ to $1200$ (notice the difference in the vertical scales between the left and right panels of Fig. 4). When the mesh is further refined (from $N = 1200$ to $10,000$), the magnitude of oscillations starts decaying but they spread and affect the computed solution in about a half of the computational domain. We stress that compared with the magnitude of $h(x, 1)$ the oscillations are not very large, so the developed instability is rather weak. We believe that this instability is attributed to the choice of a linear path in (2.5). At the same time, we emphasize that both the WB PCCU and WB CU schemes produce accurate and non-oscillatory results in both the supercritical and subcritical cases.

Example 3 – moving water steady states (discontinuous $b$, continuous $Z$, $n = 0$)

In this example, we use precisely the same setting as in Example 2, but we now take a discontinuous channel width

$$
\begin{align*}
b(x) &= \begin{cases} 
2, & x < 10, \\
2.4, & x > 10.
\end{cases}
\end{align*}
$$

(3.4)

We first compute the discrete steady states for the three studied schemes by evolving the solution until a very large time $t = 1000$ using $N = 100$ uniform cells. The results, reported in Figures 5 and 6 for the supercritical and subcritical cases, respectively, clearly demonstrate that in the supercritical case both WB PCCU, WB CU, and PCCU schemes capture $w$, $q$, and $Q$ components of the solutions quite accurately, while the PCCU scheme fails to converge to the uniform state in $E$. In the subcritical case, the PCCU scheme seems to converge to a wrong solution and the $w$ profile computed by the WB PCCU scheme seems to be sharper than the one computed by the WB CU scheme.

We then denote the computed discrete steady states by $h_S(x)$, $q_S(x)$ (note that for the three studied schemes these states are different) and add a small perturbation to the water depth. We consider the following initial
Figure 3. Example 2, small perturbation of the moving-water steady states (discontinuous $b$, continuous $Z$): The difference $h(x, 1) - h_s(x)$ computed by the WB PCCU, WB CU, and PCCU schemes for the supercritical (upper row) and subcritical (lower row) flows using $N = 100$ (left column) and 1000 (right column) uniform cells.

Figure 4. Example 2, small perturbation of the moving-water steady state (discontinuous $b$, continuous $Z$): The difference $h(x, 1) - h_s(x)$ computed by the PCCU scheme for the supercritical flow using $N = 200, 400, 600, 1200, 4000$, and $10000$ uniform cells.

data:

$$h(x, 0) = h_s(x) + \begin{cases} 
10^{-4} & \text{if } 0.45 \leq x \leq 0.55, \\
0 & \text{otherwise},
\end{cases} \quad q(x, 0) \equiv q_s(x),$$

and compute the WB PCCU, WB CU, and PCCU numerical solutions until the final time $t = 1$ using either $N = 100$ or 1000 uniform cells. The obtained differences $h(x, 1) - h_s(x)$ are plotted in Figure 7. One can see that as in Example 2, both the WB PCCU and WB CU schemes produce very accurate non-oscillatory results.
Figure 5. Example 3 (moving-water steady state, supercritical case, discontinuous $b$, continuous $Z$): $w$, $q$, $Q$, and $E$ computed by the WB PCCU, WB CU, and PCCU schemes.

Figure 6. Example 3 (moving-water steady state, subcritical case, discontinuous $b$, continuous $Z$): Same as in Figure 5.
Figure 7. Example 3, small perturbation of the moving-water steady states (discontinuous \( b \), continuous \( Z \)): The difference \( h(x,1) - h_S(x) \) computed by the WB PCCU, WB CU, and PCCU schemes for the supercritical (upper row) and subcritical (lower row) flow using 100 (left column) and 1000 (right column) uniform cells.

even when the coarse mesh is used. The PCCU scheme, however, produces oscillations, whose magnitude does not grow as severely as in Example 2, but the frequency of oscillations increases when the mesh is refined.

**Example 4 – transcritical steady state with a shock (discontinuous \( b \) and \( Z \), \( n = 0 \))**

In this example, we consider the frictionless case and study the convergence of the solutions computed by the WB PCCU, WB CU, and PCCU schemes towards the transcritical steady flow at large times.

We take the computational domain \([0, 25]\), where we prescribe the discontinuous channel width

\[
b(x) = \begin{cases} 
2.4 & \text{if } 8 \leq x \leq 12, \\
2 & \text{otherwise}, 
\end{cases}
\]

the discontinuous bottom topography

\[
Z(x) = \begin{cases} 
0.2 & \text{if } 8 \leq x \leq 12, \\
0 & \text{otherwise}, 
\end{cases}
\]

the initial data

\[
h(x,0) = 0.33 - Z(x), \quad q(x,0) \equiv 0,
\]

and the boundary conditions

\[
q(0,t) = 0.18, \quad h(25,t) = 0.33.
\]

As \( t \to \infty \), the solution of this initial-boundary value problem is expected to converge to the steady state containing a shock wave. At the steady state, the equilibrium variable \( E \) is expected to be piecewise constant.
as it would jump across the shock. At the same time, the other equilibrium variable $Q$ should be constant as $t \to \infty$.

We compute the numerical solutions on a uniform mesh with $N = 100$ cells by the WB PCCU, WB CU, and PCCU schemes until the very large final time $t = 1000$. The obtained results suggest that the three schemes converge to different discrete steady states; see $w$ and $E$ plotted in Figure 8 together with the $q$ and $Q$ components. As one can see, none of the three studied schemes is capable of accurately capturing the steady state containing a shock wave as significant over- and undershoots appear at both $Q$ and $E$ to the right of the shock located at $x = 12$. At the same time, the proposed WB PCCU scheme still outperforms the WB CU and PCCU ones. Indeed one can see that the PCCU scheme fails to capture the left component part of $E$, which contains a small jump at about $x = 8$. One can also observe that the values of $w$ at $x = 0$ computed by the three schemes are quite different. In order to better understand what the correct value should be, we perform a mesh refinement study; see Figure 9, where we plot $w$ and $E$ computed on finer meshes with $N = 200$ and 400. As one can observe, the numerical results computed by the WB CU scheme approach those computed by the WB PCCU scheme; see the $w$ component in Figure 9. This suggests that the solution computed by the WB PCCU scheme is more accurate than the WB CU solution despite both of them containing a $\mathcal{O}(1)$-error near the shock.

**Example 5 – moving water steady states ($n \neq 0$)**

This example is an extension of Examples 2 and 3: we now include the Manning friction term. We take the Manning coefficient $n = 0.005$ for supercritical flows and $n = 0.05$ for subcritical flows. The result for the continuous channel width (3.1) calculated by the WB PCCU, WB CU, and PCCU schemes at $t = 1000$ using $N = 100$ uniform cells are shown in Figures 10 and 11. As one can see, the solutions computed by the three studied schemes converge to the same steady states, but the PCCU and WB CU solutions fail to converge to...
the constant state $E$ in the subcritical case (Fig. 11) and the PCCU solution fails to converge to the constant state $Q$ in the supercritical case (Fig. 10).

The results for the discontinuous channel width (3.4) are shown in Figures 12 and 13. As one can see, the steady state captured by the WB CU scheme is slightly different from the WB PCCU and PCCU steady states. This is due to the fact that in the WB CU scheme, the channel width $b$ is approximated using a continuous function. At the same time, the advantage of the proposed WB PCCU scheme over the PCCU one is quite obvious as the PCCU scheme cannot accurately capture $E$ (in both the supercritical and subcritical cases) and $Q$ in the subcritical case.

**Example 6 – Riemann problem with unique solution**

In the final example, we test the performance of the WB PCCU, WB CU, and PCCU schemes on the Riemann problem, which is a modification of Test7 from [32].

We take the following initial data:

$$h(x,0) = \begin{cases} 1, & x < 0, \\ 0.8, & x > 0 \end{cases}, \quad u(x,0) = \begin{cases} 2, & x < 0, \\ 4, & x > 0 \end{cases},$$

with the bottom topography and the channel width containing jumps at $x = 0$:

$$Z(x) = \begin{cases} 1.1, & x < 0, \\ 1, & x > 0 \end{cases}, \quad b(x) = \begin{cases} 2.4, & x < 0, \\ 2, & x > 0. \end{cases}$$

The solution of this Riemann problem consists of three waves: the left- and right-moving rarefaction waves and a composite wave containing a steady contact wave at $x = 0$ followed by a rarefaction.
Figure 10. Example 5 (moving-water steady state, supercritical case, continuous $b$ and $Z$): $w$, $q$, $Q$, and $E$ computed by the WB PCCU, WB CU, and PCCU schemes.

Figure 11. Example 5 (moving-water steady state, subcritical case, continuous $b$ and $Z$): Same as in Figure 10.
Figure 12. Example 5 (moving-water steady state, supercritical case, discontinuous $b$, continuous $Z$): Same as in Figures 10–11.

Figure 13. Example 5 (moving-water steady state, subcritical case, discontinuous $b$, continuous $Z$): Same as in Figures 10–12.
We compute the numerical solutions by the three studied schemes until the final time $t = 0.03$ using the uniform meshes with $\Delta x = 1/500$. The obtained results are shown in Figure 14 (top row). One can observe that the solutions computed by both the WB CU and PCCU schemes are oscillatory to the left of $x = 0$, whereas the WB PCCU solution is oscillation-free. While the oscillations in the PCCU solution are clearly nonphysical near $x = 0$, the advantage of the WB PCCU scheme over the WB CU scheme is less obvious. In order to verify that the WB PCCU solution is the correct one, we refine the mesh to $\Delta x = 1/5000$ and compare the high-resolution WB PCCU and WB CU solutions, shown in Figure 14 (bottom row). This mesh-refinement study clearly indicates that at low resolution the WB PCCU scheme outperforms its WB CU counterpart.

**Appendix A. Generalized minmod reconstruction**

Assume that the discrete values (either its cell averages or point values) of a certain function $\omega$ are given at the cell centers $x = x_j$. We denote these values by $\omega_j$ and then perform a piecewise linear reconstruction to obtain $\tilde{\omega}(x) = \sum_j \tilde{\omega}_j(x) \chi_j(x)$, where

$$\tilde{\omega}_j(x) = \omega_j + (\omega_x)_j (x - x_j) \quad (A.1)$$

and $\chi_j$ is the characteristic function of the cell $C_j$. We then use (A.1) to compute the one-sided point values $\omega^{+\pm}_{j+\frac{1}{2}}$ at the cell interfaces $x = x_{j+\frac{1}{2}}$:

$$\omega^{-}_{j+\frac{1}{2}} = \lim_{x \to x_{j+\frac{1}{2}}^-} \tilde{\omega}(x) = \omega_j + \frac{\Delta x}{2} (\omega_x)_j, \quad \omega^{+}_{j+\frac{1}{2}} = \lim_{x \to x_{j+\frac{1}{2}}^+} \tilde{\omega}(x) = \omega_{j+1} - \frac{\Delta x}{2} (\omega_x)_{j+1}.$$
In order to ensure a non-oscillatory nature of this reconstruction, we compute the slopes \((\omega_x)_j\) in (A.1) using a generalized minmod limiter [33, 37, 41]:

\[
(\omega_x)_j = \minmod \left( \theta \frac{\omega_j - \omega_{j-1}}{\Delta x}, \frac{\omega_{j+1} - \omega_{j-1}}{2\Delta x}, \theta \frac{\omega_{j+1} - \omega_j}{\Delta x} \right), \quad \theta \in [1, 2].
\]

Here, the minmod function is defined as

\[
\minmod(z_1, z_2, \ldots) := \begin{cases} 
\min_j \{z_j\}, & \text{if } z_j > 0 \text{ \forall } j, \\
\max_j \{z_j\}, & \text{if } z_j < 0 \text{ \forall } j, \\
0, & \text{otherwise,}
\end{cases}
\]

and the parameter \(\theta\) can be used to control the amount of numerical viscosity present in the resulting scheme: larger values of \(\theta\) typically lead to smaller numerical dissipation, but (slightly) more oscillatory results. In all of the numerical experiments reported in Section 3, we have taken \(\theta = 1.3\).

**APPENDIX B. Solution of the cubic equations (2.12), (2.17) and (C.4)**

We consider equations (2.12), (2.17) and (C.4) for \(A^+_{j+\frac{1}{2}}\) and \(\hat{A}^+_{j+\frac{1}{2}}\) only, as the corresponding equation for \(A^-_{j+\frac{1}{2}}\) and \(\hat{A}^-_{j+\frac{1}{2}}\) can be solved similarly. Assume that \(Q^+_{j+\frac{1}{2}} \neq 0\) and then introduce the variable \(\tau\), which is

\[
\tau = A^+_{j+\frac{1}{2}} + \frac{3\eta_{j+\frac{1}{2}}^2 - E^+_{j+\frac{1}{2}}}{3g\eta_{j+\frac{1}{2}}} \quad \text{for equation (2.12)},
\]

\[
\tau = \hat{A}^+_{j+\frac{1}{2}} + \frac{3\eta_{j+\frac{1}{2}}^2 - E^+_{j+\frac{1}{2}}}{3g\eta_{j+\frac{1}{2}}} \quad \text{for equation (2.17)},
\]

and

\[
\tau = A^+_{j+\frac{1}{2}} \quad \text{for equation (C.4)}.
\]

In this case, all of the studied cubic equations can be written as

\[
\tau^3 + \alpha \tau + \beta = 0,
\]

where the coefficients \(\alpha\) and \(\beta\) can be easily obtained for each of the equations. One then can verify that \(\alpha < 0\) and thus the cubic polynomial in (B.1) has a local minimum at \(\tau = \sqrt{-\frac{\beta}{\alpha}}\). There are three possible cases that need to be considered.

First, if \(f(\sqrt{-\frac{\beta}{\alpha}}) < 0\), then (B.1) has three real roots and they are

\[
\tau = 2\sqrt{-\frac{\alpha}{3}} \cos \left( \frac{1}{3} \left( \Theta + 2\pi k \right) \right), \quad \Theta := \arccos \left( -\frac{\beta}{P^2} \right), \quad k = 0, 1, 2.
\]

These three roots correspond to the three root of \(A^+_{j+\frac{1}{2}}\) (for equations (2.12) and (C.4)) and \(\hat{A}^+_{j+\frac{1}{2}}\) (for equation (2.17)). In order to select the physically relevant one, we perform a generalized minmod reconstruction described in Appendix A to the water surface. This results in \(w^+_{j+\frac{1}{2}}\) and out of the three aforementioned roots we choose the one which is closest to \(\left( w^+_{j+\frac{1}{2}} - Z^+_{j+\frac{1}{2}} \right)/\eta^+_{j+\frac{1}{2}}\).

Second, if \(f(\sqrt{-\frac{\beta}{\alpha}}) = 0\), then the physically relevant root of the cubic polynomial in (B.1) is \(\tau = \sqrt{-\frac{\beta}{\alpha}}\), which gives the corresponding value of \(A^+_{j+\frac{1}{2}}\) or \(\hat{A}^+_{j+\frac{1}{2}}\).

Third, if \(f(\sqrt{-\frac{\beta}{\alpha}}) > 0\), then the only zero of of the cubic polynomial in (B.1) corresponds to a nonphysical negative value of \(A^+_{j+\frac{1}{2}}\) or \(\hat{A}^+_{j+\frac{1}{2}}\) (one can easily check this using equation (2.12), (C.4), or (2.17)). In this case, we reconstruct (using the generalized minmod limiter described in Appendix A) the water surface \(w\) and obtain the point values \(w^\pm_{j+\frac{1}{2}}\), and then compute \(A^+_{j+\frac{1}{2}}\) or \(\hat{A}^+_{j+\frac{1}{2}}\) by \(\frac{w^+_{j+\frac{1}{2}} - Z^+_{j+\frac{1}{2}}}{\eta^+_{j+\frac{1}{2}}} \). Notice that the obtained point values of \(A\) do not satisfy equation (2.12), (C.4), or (2.17) and we therefore modify \(E^+_{j+\frac{1}{2}}\) or \(K^+_{j+\frac{1}{2}}\) by substituting \(A^+_{j+\frac{1}{2}}\) into (2.12) or (C.4), or by substituting \(\hat{A}^+_{j+\frac{1}{2}}\) into (2.17) to keep (2.12), (C.4), and (2.17) valid throughout the computational domain.
APPENDIX C. Flux globalization based WB CU scheme

In this appendix, we briefly describe an extension of the flux globalization based WB CU scheme from [9] to SW flows in channels. The scheme reads as (2.10) with

\[ \mathbf{K}_{j+\frac{1}{2}} = \frac{a^+_{j+\frac{1}{2}} K^-_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}} K^+_{j+\frac{1}{2}}}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}} + \frac{a^+_{j+\frac{1}{2}} a^-_{j+\frac{1}{2}}}{a^+_{j+\frac{1}{2}} - a^-_{j+\frac{1}{2}}} \left( U^+_{j+\frac{1}{2}} - U^-_{j+\frac{1}{2}} \right). \]

The main idea here is to follow [9, 11–13, 28] and perform the generalized minmod reconstruction of the equilibrium variables \( Q \) and \( R \) rather than \( Q \) and \( E \). In order to do this, we first need to compute the values of \( K \) at the cell centers \( x = x_j \). We begin with the global quantity \( R \) defined in (1.6), select the lower boundary of integration to be \( \hat{x} = x_{j+\frac{1}{2}} \), so that \( R_{\frac{1}{2}} = 0 \) and evaluate the corresponding discrete values of \( R \) using the following recursive formulae:

\[
R_j = R_{j-\frac{1}{2}} + \frac{g}{2} \left( \mathbf{A}_j A_{j-\frac{1}{2}} \right) \left( \eta_j - \eta_{j-\frac{1}{2}} \right) + \frac{g}{2} \left( \mathbf{A}_j A_{j-\frac{1}{2}} \right) \left( Z_j - Z_{j-\frac{1}{2}} \right)
\]

\[ + \frac{g}{2} \left( \mathbf{A}_j (S_f)_j + A_{j-\frac{1}{2}} (S_f)_{j-\frac{1}{2}} \right), \quad j = 1, \ldots, N, \]

\[
R_{j+\frac{1}{2}} = R_j + \frac{g}{2} \left( A_j \overline{A}_j \right) \left( \eta_{j+\frac{1}{2}} - \eta_j \right) + \frac{g}{2} \left( A_{j+\frac{1}{2}} + \overline{A}_j \right) \left( Z_{j+\frac{1}{2}} - Z_j \right)
\]

\[ + \frac{g}{2} \left[ A_{j+\frac{1}{2}} (S_f)_{j+\frac{1}{2}} + \overline{A}_j (S_f)_j \right], \quad j = 1, \ldots, N. \]

(C.1)

Here, the point values of \( Z \) and \( \eta \) are obtained with the help of a continuous piecewise linear interpolants

\[
\hat{Z}(x) = Z_{j-\frac{1}{2}} + \frac{Z_{j+\frac{1}{2}} - Z_{j-\frac{1}{2}}}{\Delta x} \left( x - x_{j-\frac{1}{2}} \right), \quad \hat{\eta}(x) = \eta_{j-\frac{1}{2}} + \frac{\eta_{j+\frac{1}{2}} - \eta_{j-\frac{1}{2}}}{\Delta x} \left( x - x_{j-\frac{1}{2}} \right), \quad x \in C_j,
\]

where \( Z_{j+\frac{1}{2}} := Z(x_{j+\frac{1}{2}}) \) when \( Z \) is continuous at \( x = x_{j+\frac{1}{2}} \) and \( Z_{j+\frac{1}{2}} := \left( Z(x_{j+\frac{1}{2}}^-) + Z(x_{j+\frac{1}{2}}^+) \right)/2 \) if \( Z \) is discontinuous there, and \( \eta_{j+\frac{1}{2}} \) are computed in the same way. The point values of \( Z \) and \( \eta \) at the cell centers are then equal to \( Z_j = \hat{Z}(x_j) = \left( Z_{j-\frac{1}{2}} + Z_{j+\frac{1}{2}} \right)/2 \) and \( \eta_j = \hat{\eta}(x_j) = \left( \eta_{j-\frac{1}{2}} + \eta_{j+\frac{1}{2}} \right)/2 \). Finally, \( A_{j+\frac{1}{2}} \) is evaluated using the average value of the water surface \( w \), namely, \( w_{j+\frac{1}{2}} := (\overline{w}_j + w_{j+1})/2 \), where \( \overline{w}_j = \overline{A}_j \eta_j + Z_j \), and hence

\[
A_{j+\frac{1}{2}} = \frac{w_{j+\frac{1}{2}} - Z_{j+\frac{1}{2}}}{\eta_{j+\frac{1}{2}}} = \overline{A}_j \eta_j + Z_j - 2Z_{j+\frac{1}{2}} + Z_{j+1}.
\]

(C.2)

The discrete values of the friction on the RHS of (C.1) are given by

\[
(S_f)_j = \frac{n^2 |Q_j| \overline{Q}_j}{A_j^2} \left. \left[ \frac{1}{\overline{A}_j} + 2\eta_j \right] \right)^{\frac{1}{2}}, \quad (S_f)_{j+\frac{1}{2}} = \frac{n^2 |Q_{j+\frac{1}{2}}| \overline{Q}_{j+\frac{1}{2}}}{A_{j+\frac{1}{2}}^2} \left. \left[ \frac{1}{A_{j+\frac{1}{2}}} + 2\eta_{j+\frac{1}{2}} \right] \right)^{\frac{1}{2}},
\]

where \( Q_{j+\frac{1}{2}} := \left( Q^+_{j+\frac{1}{2}} + Q^-_{j+\frac{1}{2}} \right)/2 \), and the one-side values \( Q^\pm_{j+\frac{1}{2}} \) are obtained using the generalized minmod reconstruction described in Appendix A.

**Remark C.1.** We note that unlike the flux globalization based WB PCCU scheme presented in Section 2.2, the global variable \( R \) is continuous at the cell interface \( x = x_{j+\frac{1}{2}} \). It should be also pointed out that the quadrature used to obtain (C.2) is WB as we demonstrate in Appendix C.1.

Equipped with \( R_j \), we compute

\[
K_j = \frac{n^2 |Q_j| \overline{Q}_j}{A_j^2} + \frac{g}{2} \overline{A}_j \eta_j + R_j,
\]

(C.3)
and then obtain the one-sided point values $K_{j+\frac{1}{2}}^\pm$ using the generalized minmod reconstruction described in Appendix A.

Finally, we recover $A_{j+\frac{1}{2}}^\pm$ by solving the equations

$$
K_{j+\frac{1}{2}}^\pm \equiv \frac{(Q_{j+\frac{1}{2}}^\pm)^2}{A_{j+\frac{1}{2}}^\pm} + \frac{g}{2} (A_{j+\frac{1}{2}}^\pm)^2 \eta_j + R_{j+\frac{1}{2}},
$$

(C.4)

for $A_{j+\frac{1}{2}}^\pm$. When $Q_{j+\frac{1}{2}}^\pm \neq 0$, these equations are cubic and we solve them using the exact solver described in Appendix B.

### C.1. Well-balanced (WB) property

In order to show the WB property of the flux globalization based WB CU scheme, we note that as long as the discrete data at a certain time level $t$ satisfies (1.7), namely, if

$$
\overline{Q}_j \equiv \hat{Q} = \text{Const}, \quad K_j \equiv \hat{K} = \text{Const}, \quad \forall j,
$$

then $Q_{j+\frac{1}{2}}^+ = Q_{j+\frac{1}{2}}^- \equiv \hat{Q}$ and $K_{j+\frac{1}{2}}^+ = K_{j+\frac{1}{2}}^- \equiv \hat{K}$, which together with (C.4) gives $A_{j+\frac{1}{2}}^+ = A_{j+\frac{1}{2}}^-$ so that $U_{j+\frac{1}{2}}^+ - U_{j+\frac{1}{2}}^- = 0$, and it immediately implies that the RHS of (2.10) vanishes and the discrete version of the steady states (1.7) is preserved.

In addition, the flux globalization based WB CU scheme is capable of preserving the “lake-at-rest” steady states (1.4). Notice that these steady states correspond to a steady state (1.7) with $\overline{Q} = 0$. This, however, is not automatically true for discrete versions of (1.4) and (1.7). Therefore, it is not obvious that the scheme can preserve the discrete “lake-at-rest” states satisfying

$$
\overline{Q}_j = 0, \quad \overline{A}_j \eta_j + Z_j \equiv \hat{\omega} = \text{Const}, \quad \forall j.
$$

(C.5)

In order to verify this, we prove that $K_j = K_{j+1}$ for all $j$ as long as (C.5) is satisfied (and thus the reconstructed point values $K_{j+\frac{1}{2}}^\pm$ will take the same value). In fact, one has from (C.1), (C.3), and $\overline{Q}_j \equiv 0$ that

$$
K_{j+1} - K_j \equiv \frac{g}{2} \left[ (\overline{A}_{j+1}^{-1} \eta_{j+1} - \overline{A}_j \eta_j) + R_{j+1} - R_j \right] = \frac{g}{2} \left[ (\overline{A}_{j+1} \eta_{j+1} - \overline{A}_j \eta_j) + R_{j+1} - R_{j+\frac{1}{2}} + R_{j+\frac{1}{2}} - R_j \right]
$$

(C.6)

$$
= \frac{g}{2} \left[ (\overline{A}_{j+1} \eta_{j+1} - \overline{A}_j \eta_j) + \overline{A}_{j+1} A_{j+\frac{1}{2}} (\eta_{j+1} - \eta_{j+\frac{1}{2}}) + A_{j+\frac{1}{2}} \overline{A}_j (\eta_j - \eta_{j+\frac{1}{2}}) \right] + \left[ (\overline{A}_{j+1} + A_{j+\frac{1}{2}}) (Z_{j+1} - Z_{j+\frac{1}{2}}) + (A_{j+\frac{1}{2}} + \overline{A}_j) (Z_{j+\frac{1}{2}} - Z_{j+1}) \right].
$$

We then note that (C.5) implies $Z_{j+1} - Z_j = \overline{A}_j \eta_j - \overline{A}_{j+1} \eta_{j+1}$, which together with (C.2) yields

$$
\overline{A}_{j+1} A_{j+\frac{1}{2}} (\eta_{j+1} - \eta_{j+\frac{1}{2}}) + A_{j+\frac{1}{2}} \overline{A}_j (\eta_{j+\frac{1}{2}} - \eta_j) = A_{j+\frac{1}{2}} (Z_j - Z_{j+1}) - (\hat{\omega} - Z_{j+\frac{1}{2}}) (\overline{A}_{j+1} - \overline{A}_j).
$$

(C.7)

Next,

$$
\left( \overline{A}_{j+1} + A_{j+\frac{1}{2}} \right) (Z_{j+1} - Z_{j+\frac{1}{2}}) + \left( A_{j+\frac{1}{2}} + \overline{A}_j \right) (Z_{j+\frac{1}{2}} - Z_j)
$$

$$
= \left( \overline{A}_{j+1} + A_{j+\frac{1}{2}} \right) (Z_{j+1} - Z_j + Z_j - Z_{j+\frac{1}{2}}) + \left( A_{j+\frac{1}{2}} + \overline{A}_j \right) (Z_{j+\frac{1}{2}} - Z_j)
$$

(C.8)

$$
= \left( \overline{A}_{j+1} + A_{j+\frac{1}{2}} \right) (Z_{j+1} - Z_j) - (\overline{A}_{j+1} - \overline{A}_j) (Z_{j+\frac{1}{2}} - Z_j).$$
Finally, it follows from (C.5) to (C.8) that
\[ K_{j+1} - K_j = \frac{q}{2} [\bar{A}_{j+1}(\hat{w} - Z_{j+1}) - \bar{A}_j(\hat{w} - Z_j) + \bar{A}_{j+1}(Z_{j+1} - Z_j) - (\bar{A}_{j+1} - \bar{A}_j)(\hat{w} - Z_j)] = 0, \]
which indicates that the “lake-at-rest” state (C.5) is preserved.

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