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HOMOGENIZATION OF SOUND-SOFT AND HIGH-CONTRAST ACOUSTIC METAMATERIALS IN SUBCRITICAL REGIMES

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Abstract. We propose a quantitative effective medium theory for two types of acoustic metamaterials constituted of a large number N of small heterogeneities of characteristic size s, randomly and independently distributed in a bounded domain. We first consider a "sound-soft" material, in which the total wave field satisfies a Dirichlet boundary condition on the acoustic obstacles. In the "sub-critical" regime $sN = O(1)$, we obtain that the effective medium is governed by a dissipative Lippmann–Schwinger equation which approximates the total field with a relative mean-square error of order $O(\max((sN)^2N^{-\frac{1}{3}}, N^{-\frac{1}{2}}))$. We retrieve the critical size $s \sim 1/N$ of the literature at which the effects of the obstacles can be modelled by a "strange term" added to the Helmholtz equation. Second, we consider high-contrast acoustic metamaterials, in which each of the N heterogeneities are packets of K inclusions filled with a material of density much lower than the one of the background medium. As the contrast parameter vanishes, $\delta \to 0$, the effective medium admits K resonant characteristic sizes $(s_i(\delta))_{1\leq i\leq K}$ and is governed by a Lippmann–Schwinger equation, which is diffusive or dispersive (with negative refractive index) for frequencies ω respectively slightly larger or slightly smaller than the corresponding K resonant frequencies $(\omega_i(\delta))_{1\leq i\leq K}$. These conclusions are obtained under the condition that (i) the resonance is of monopole type, and (ii) lies in the "subcritical regime" where the contrast parameter is small enough, *i.e.* $\delta = o(N^{-2})$, while the considered frequency is "not too close" to the resonance, *i.e.* $N\delta^{\frac{1}{2}} = O(|1 - s/s_i(\delta)|)$. Our mathematical analysis and the current literature allow us to conjecture that "solidification" phenomena are expected to occur in the "super-critical" regime $N\delta^{\frac{1}{2}}|1 - s/s_i(\delta)|^{-1} \to +\infty$.

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1. Introduction

Metamaterials [25, 34, 46, 53] offer promising perspectives in many applications of wave engineering, such as sensing [16, 28], imaging [5, 47, 63], focusing [10, 52], cloaking [8, 49] and guiding [17, 18, 61]. These are structures filled with heterogeneities much smaller than the wavelength, which behave, as the size of the heterogeneities become arbitrarily small, as apparently homogeneous effective media with special properties not found in usual

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materials. The mathematical and rigorous derivation of effective models for heterogeneous wave systems is of primary importance for the analysis, the understanding, and the numerical simulation of the physics of wave propagation in metamaterials. Various approaches have been proposed in the literature, based on the Foldy–Lax approximation [6, 15, 41], on two-scale expansions [23, 54, 59, 64, 69], or from physical models [45, 56].

In this paper, we propose a quantitative homogenization analysis of two kinds of acoustic structures filling a bounded domain Ω of the three-dimensional space $\mathbb{R}^3$, based on integral representations and layer potential techniques. Our motivation is to better understand the arising of effective medium properties for non-periodic heterogeneous media, which can achieve various dissipative or dispersive effects. A classical additional benefit of the quantitative homogenization process is to yield explicit homogenized equations whose solutions approximate the scattered field with clearly identified error bounds, leading to efficient numerical algorithms for estimating the total wave field.

1.1. General setting

We consider a system $D_{N,s}$ constituted of a total of $M = \sum_{i=1}^N K_i$ inclusions which are arranged in N packets $(y_i + sD_i)_{1\le i\le N}$ containing each K_i inclusions:

$$D_{N,s} = \bigcup_{1\le i\le N} (y_i + sD_i) \text{ with } D_i = \bigcup_{j=1}^{K_i} B_{ij},$$

where each resonator B_{ij} with $1 \le i \le N$, $1 \le j \le K_i$ has a single connected component. Each group of resonators D_i with $1 \le i \le N$ is rescaled by a factor $s > 0$ and is located close to its center $y_i \in \mathbb{R}^3$. The factor s is taken sufficiently small so that the packets D_i do not overlap. In order to derive effective medium theories, we consider two main uniformity assumptions on the heterogeneities.

(H1) The points $(y_i)_{1\le i\le N}$ are distributed randomly and independently according to a three-dimensional probability measure $\varrho\,\mathrm{d}x$ with density $\varrho \in L^\infty(\Omega)$ in a smooth bounded domain $\Omega \subset \mathbb{R}^3$. In particular, $\varrho \ge 0$ and $\int_\Omega \varrho\,\mathrm{d}x = 1$, and the law of large numbers implies the convergence

$$\sum_{i=1}^N \delta_{y_i} \to \varrho\,\mathrm{d}x \text{ as } N \to +\infty, \tag{1.1}$$

almost surely in $(y_i)_{1\le i\le N}$ and in the sense of distributions.

(H2) The packets of resonators are identical and constituted of K single components $(B_l)_{1\le l\le K}$:

$$D_i = D := \bigcup_{l=1}^K B_l, \qquad \forall 1 \le i \le N.$$

The background domain $\mathbb{R}^3\backslash D_{N,s}$ is a homogeneous medium characterized by its constant bulk modulus $\kappa > 0$, density $\rho > 0$, and speed of sound

$$v := \sqrt{\frac{\kappa}{\rho}}\cdot$$

An incident wave u_{in} is coming from the far field and generates a scattered wave by encountering the obstacles $D_{N,s}$. We assume that the incident field is time-harmonic regime with frequency ω, *i.e.* u_{in} is solution to the Helmholtz equation in the background medium:

$$\Delta u_{in} + k^2 u_{in} = 0 \text{ in } \mathbb{R}^3,$$

where $k := \omega/v$ is the wave number in the background medium. The resulting total wave field is denoted by $u_{N,s}$. The overall setting is illustrated in Figure 1.

We then consider two possible types of acoustic obstacles, leading to two kinds of acoustic metamaterials:

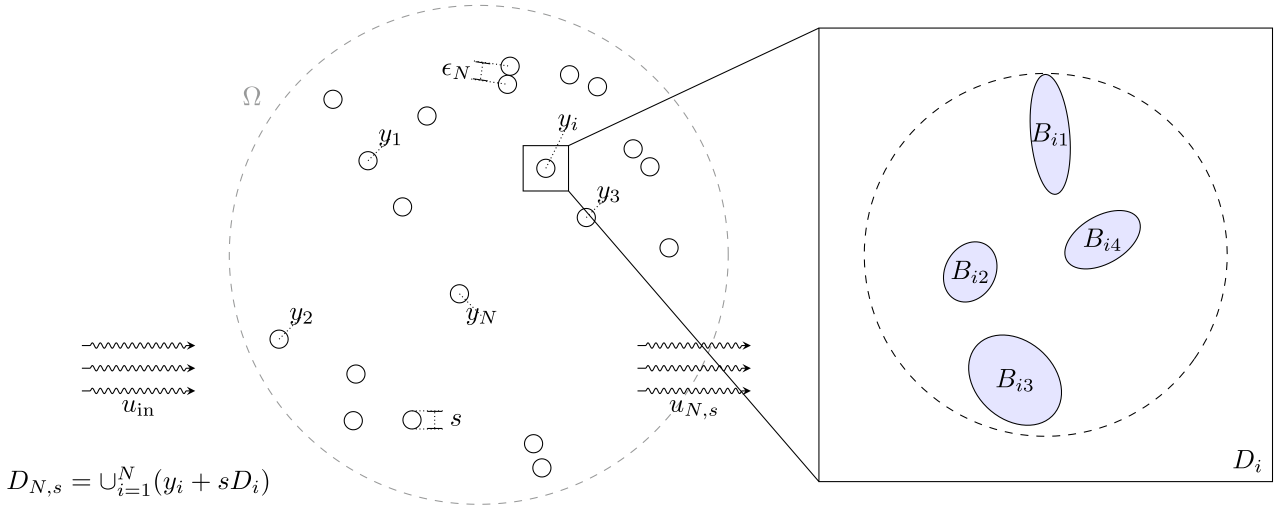

FIGURE 1. Setting of the homogenization problem. An incoming wave u_{in} generates a scattered wave $u_{N,s} - u_{in}$ by encountering a highly contrasted medium $D_{N,s}$ constituted of many small inclusions filling a bounded domain Ω. Each unit packet $D_i = \cup_{j=1}^{K_i} B_{ij}$ is rescaled by a small size factor s and translated in the vicinity of the point y_i to form the small acoustic obstacle $y_i + sD_i$. The smallest distance between the centers $(y_i)_{1\le i\le N}$ is denoted by ϵ_N (Eq. (2.1)).

(i) *Sound-soft obstacles*. In this case, the sound wave is "absorbed" by the obstacles, which corresponds to saying that $u_{N,s}$ is the solution to the Helmholtz equation in $\mathbb{R}^3\backslash D_{N,s}$, with a Dirichlet boundary condition on $\partial D_{N,s}$:

$$\begin{cases} \Delta u_{N,s} + k^2 u_{N,s} = 0 & \text{in } \mathbb{R}^3\backslash D_{N,s}, \\ u_{N,s} = 0 & \text{on } \partial D_{N,s}, \\ \left(\dfrac{\partial}{\partial|x|} - \mathrm{i}k\right)(u_{N,s}(x) - u_{in}(x)) = O\big(|x|^{-2}\big) & \text{as } |x| \to +\infty, \end{cases} \tag{1.2}$$

where the last equality is the outgoing Sommerfeld radiation condition for the scattered field;

(ii) *High-contrast obstacles*. In this configuration, the inclusions of $D_{N,s}$ are filled with a material of different bulk modulus κ_b and density ρ_b. The total field $u_{N,s}$ is then characterized as the solution to the following system of coupled Helmholtz equations:

$$\begin{cases} \operatorname{div}\left(\dfrac{1}{\rho_b}\nabla u_{N,s}\right) + \dfrac{\omega^2}{\kappa_b} u_{N,s} = 0 & \text{in } D_{N,s}, \\ \operatorname{div}\left(\dfrac{1}{\rho}\nabla u_{N,s}\right) + \dfrac{\omega^2}{\kappa} u_{N,s} = 0 & \text{in } \mathbb{R}^3\backslash D_{N,s}, \\ u_{N,s}|_+ - u_{N,s}|_- = 0 & \text{on } \partial D_{N,s}, \\ \dfrac{1}{\rho_b}\dfrac{\partial u_{N,s}}{\partial n}\Big|_- = \dfrac{1}{\rho}\dfrac{\partial u_{N,s}}{\partial n}\Big|_+ & \text{on } \partial D_{N,s}, \\ \left(\dfrac{\partial}{\partial|x|} - \mathrm{i}k\right)(u_{N,s} - u_{in}) = O\big(|x|^{-2}\big) & \text{as } |x| \to +\infty, \end{cases} \tag{1.3}$$

where $v|_+$ and $v|_-$ denote the outer and inner traces of a function v on $\partial D_{N,s}$ and $\boldsymbol{n}$ the outward normal. We consider the regime where the contrast parameter

$$\delta := \frac{\rho_b}{\rho} \tag{1.4}$$

is small: $\delta \to 0$. Such kind of system is naturally encountered when considering, for instance, air bubbles in water. The inclusions of $D_{N,s}$ behave as subwavelength "resonators" which significantly affect the acoustic properties of the background medium at subwavelength scales [11, 57].

The goal of this paper is to derive quantitative effective medium theories for both (1.2) and (1.3) when the size and the contrast of the heterogeneities converge to zero as their number becomes large:

$$s \to 0, \quad N \to +\infty, \quad \delta \to 0. \tag{1.5}$$

We emphasize that in this study, all the physical parameters other than δ, s and N (including the illuminating frequency ω) are kept constant (hence, of order $O(1)$). In both situations and as is classical in the homogenization of perforated problems, critical scalings for the parameters s, N and δ arise, at which the qualitative effective physical behavior of (1.2) and (1.3) change.

1.2. Effective medium theory for sound-soft obstacles

The analysis of the acoustic problem (1.2) in the context of evenly spaced centers $(y_i)_{1\leq i\leq N}$ and identical obstacles ($D_i = D$ for all $1 \leq i \leq N$) goes back at least to Rauch and Taylor [65, 66], who showed that the asymptotic behavior of the quantity sN determines the effective physical behaviour of the heterogeneous medium. Their study was restricted to the regimes $sN \to 0$ or $sN \to +\infty$ and the rates of convergence are not established. The intermediate regime where sN converges to a constant $\Lambda > 0$ was then studied by Chiado Piat and Codegone [31], still without providing convergence rates. Subsequently, Challa *et al.* [26, 27] proposed an analysis establishing the convergence of the far field of $u_{N,s}$ (away from the obstacles) to either (1.6), (1.7) or (1.8) for centers $(y_i)_{1\leq i\leq N}$ distributed according to a counting function with quantitative convergence rates.

In both settings, for evenly distributed obstacles as well as those distributed according to a counting function, these authors obtained three possible effective regimes which can be summarized as follows:

- if $sN \to 0$, the acoustic obstacles are too small and the scattered field converges to zero as $s \to 0$ and $N \to +\infty$, or in other words $u_{N,s}$ converges to the solution u_{in} of the Helmholtz equation (1.2) without the obstacles. The effective medium is transparent and is governed by a homogeneous Helmholtz equation:

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \mathbb{R}^3, \\ \left(\dfrac{\partial}{\partial |x|} - \mathrm{i}k\right)(u - u_{in}) = O(|x|^{-2}) & \text{as } |x| \to +\infty; \end{cases} \tag{1.6}$$

- if $sN \to \Lambda$ for a positive constant $\Lambda > 0$, then $u_{N,s} \to u$ where u is the solution to the dissipative Helmholtz equation

$$\begin{cases} \Delta u + k^2 u - \mu 1_\Omega u = 0 & \text{in } \mathbb{R}^3, \\ \left(\dfrac{\partial}{\partial |x|} - \mathrm{i}k\right)(u - u_{in}) = O(|x|^{-2}) & \text{as } |x| \to +\infty, \end{cases} \tag{1.7}$$

  where $\mu 1_\Omega$ is a positive Radon measure supported in Ω which describes the dissipative effects due to the obstacles. In [26, 27], the authors identify explicitly the measure to be given by $\mu = \Lambda \mathrm{cap}\,(D)\varrho$, where $\mathrm{cap}\,(D)$ is the capacity of the set of inclusions D (the definition is recalled in (2.18)) and ϱ is a counting function playing a role analogous to the random distribution of (H1).

– If $sN \to +\infty$, then $u_{N,s} \to u$ where u is the solution to the Helmholtz equation with a Dirichlet boundary condition on the whole set Ω:

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \mathbb{R}^3, \\ u = 0 & \text{on } \Omega, \\ \left(\dfrac{\partial}{\partial |x|} - \mathrm{i}k\right)(u - u_{in}) = O(|x|^{-2}) & \text{as } |x| \to +\infty. \end{cases} \tag{1.8}$$

Physically, the obstacles "solidify" and the effective medium Ω is opaque.

These results are comparable to those obtained in two-scale homogenization of the Poisson equation in a periodically perforated domain with periodicity ϵ: the well-known "strange term" (which is $-\mu 1_\Omega u$ in (1.7)) arises when the obstacle size satisfies the critical scaling $s \propto \epsilon^3$ [32, 35, 36, 39, 43, 44], *i.e.* $s \propto \frac{1}{N}$ since the cell-periodicity ϵ is of the order $N^{-\frac{1}{3}}$ for evenly distributed obstacles. The main difference in the study of (1.2) lies therefore in the treatment of the radiation condition.

In this paper, we improve these results to the case of randomly and independently distributed obstacles, and we provide quantitative convergence estimates. For the analysis of (1.2), we restrict ourselves, however, to the "subcritical regime" for which the size factor s is lower or equal to the critical size $1/N$:

(H3) There exists a constant $c > 0$ such that the parameters s and N satisfy

$$sN \leq c. \tag{1.9}$$

Note that the number K of connected component of D does not play any role in our analysis of (1.2) and (H2) may be replaced identically by "D is a Lipschitz subset of $\mathbb{R}^3$". Under this set of assumptions, we obtain in Proposition 3.4 the following main result regarding the convergence of (1.2):

Proposition 1.1. *Assume (H1)–(H3) and denote by u the solution to the Lippmann–Schwinger equation*

$$\begin{cases} \Delta u + (k^2 - sN\mathrm{cap}\,(D)\varrho 1_\Omega) u = 0 & \text{in } \mathbb{R}^3, \\ \left(\dfrac{\partial}{\partial |x|} - \mathrm{i}k\right)(u - u_{in}) = O\big(|x|^{-2}\big) & \text{as } |x| \to +\infty. \end{cases} \tag{1.10}$$

There exists an event $\mathcal{H}_{N_0}$ which holds with large probability $\mathbb{P}(\mathcal{H}_{N_0}) \to 1$ as $N_0 \to +\infty$, independent of s and δ, such that when $\mathcal{H}_{N_0}$ is realized, the function u is an approximation of the solution field $u_{N,s}$ to (1.2) with the following error estimates:

(i) *on any ball $B(0,r)$ containing the obstacles, $\Omega \subset B(0,r)$, there exists a constant $c > 0$ independent of s and N such that for any $N \geq N_0$:*

$$\mathbb{E}\Big[\|u_{N,s} - u\|^2_{L^2(B(0,r))} |\mathcal{H}_{N_0}\Big]^{\frac{1}{2}} \leq csN \max\Big((sN)^2 N^{-\frac{1}{3}}, N^{-\frac{1}{2}}\Big); \tag{1.11}$$

(ii) *on any bounded open subset $A \subset \mathbb{R}^3 \backslash \Omega$ away from the obstacles, there exists a constant $c > 0$ independent of s and N such that for any $N \geq N_0$:*

$$\mathbb{E}\Big[\|\nabla u_{N,s} - \nabla u\|^2_{L^2(A)} |\mathcal{H}_{N_0}\Big]^{\frac{1}{2}} \leq csN \max\Big((sN)^2 N^{-\frac{1}{3}}, N^{-\frac{1}{2}}\Big). \tag{1.12}$$

The relative error is of order $O(\max((sN)^2 N^{-\frac{1}{3}}, N^{-\frac{1}{2}}))$ because the scattered fields $u_{N,s} - u_{in}$ and $u - u_{in}$ are of order $O(sN)$.

These results show that the convergence of the scattered field towards the Helmholtz equations (1.6) and (1.7) still hold in the non-periodic case, where the mesure μ in the critical regime is given by $\mu = sN\mathrm{cap}\,(D)\varrho$. We conjecture that the convergence towards the "solidified equation" (1.8) remains true in the super-critical regime $sN \to +\infty$, however we were presently not able to obtain this result from our analysis resting on layer potential theory. The different regimes of convergence obtained are summarized in Table 1.

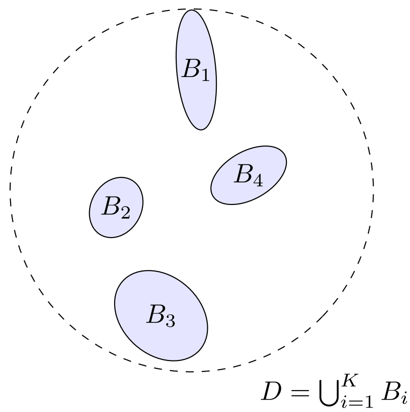

FIGURE 2. The elementary packet of resonators $D = \bigcup_{i=1}^K B_i$ illustrated with $K = 4$ connected elements. The set sD is obtained by rescaling D by the size-factor s.

1.3. Effective medium theory for high-contrast obstacles

The treatment of the high-contrast system (1.3) is more involved due to the arising of (subwavelength) resonances. A dedicated analysis shows that the elementary system constituted of K connected resonators $sD = \cup_{i=1}^K sB_i$ admits K "subwavelength" complex resonant frequencies $(\omega_i(\delta))_{1\le i\le K}$, with positive real parts and negative imaginary parts [4, 11, 20, 38]. Since the frequencies $(\omega_i(\delta))_{1\le i\le K}$ are close to the real axis, the scattered field is significantly enhanced for (real) incident frequencies ω close to the resonant frequencies. These resonances are called "subwavelength" because they correspond to incident wavelengths which are larger than the size of the resonators by several orders of magnitude.

One of the important outcomes of the analysis of [4, 11, 38] is that the values of the K resonant frequencies $(\omega_i(\delta))$ can be estimated by solving a $K\times K$ eigenvalue problem. Let $C \equiv (C_{ij})_{1\le i,j\le K}$ be the "capacitance matrix" associated with the unit set of obstacles $D = \cup_{i=1}^K B_i$ (illustrated on Fig. 2). The definition of this matrix is recalled in (4.35) below and its main properties can be found in Section 2 from [38]. Denote by $(\boldsymbol{a}_k)_{1\le k\le K}$ and $0 < \lambda_1 \le \lambda_2 \le \ldots \le \lambda_K$ the eigenvectors and eigenvalues of the (symmetric) generalized eigenvalue problem

$$C\boldsymbol{a}_j = \lambda_j V \boldsymbol{a}_j \text{ with } V := \operatorname{diag}(|B_i|)_{1\le i\le K}. \tag{1.13}$$

Then the leading order asymptotics of the resonant frequencies read

$$\omega_i(\delta) \sim \omega_{M,i} \text{ with } \omega_{M,i} := \frac{\delta^{\frac12}}{s}\lambda_i^{\frac12} v_b \text{ as } \delta \to 0, \tag{1.14}$$

where $v_b := \sqrt{\kappa_b/\rho_b}$ is the speed of sound inside the inclusion, and where the factor $1/s$ accounts for a rescaling of the packet of resonator by the arbitrary size factor s. Furthermore, throughough the paper, the notation $a \sim b$ means $a/b \to 1$ as $s, \delta \to 0$ and $N \to +\infty$.

The system (1.3) featuring N resonators ($K = 1$ and $D_i = B_1$ for $1 \le i \le N$) of size s has been studied by Ammari *et al.* [6, 15] by using a Foldy–Lax approximation method inspired from [39, 40]. The main difference between the two works lie in the fact that [6] assumes technical assumptions on the distribution of the centers $(y_i)_{1\le i\le N}$ which are difficult to realize in practice (see Rem. 2.2), while [15] assumes that these centers are distributed according to a counting function. Both authors consider frequencies ω lying close to but slightly away from the resonance: they more precisely assume the regime

$$1 - \frac{\omega_{M,1}^2}{\omega^2} \sim \frac{sN}{\Lambda} s^h \text{ and } sN \to 0, \tag{1.15}$$

for a real $h \in \mathbb{R}$, and a non-zero constant $\Lambda \in \mathbb{R}\backslash\{0\}$. The authors obtain then the following effective behaviors for the medium constituted of N resonators of size s with $N \to +\infty$ and $s \to 0$:

- if $h < 0$, then ω is too far from the resonant frequency and the effective medium is transparent [15]: $u_{N,s} \to u$ where u is the solution to (1.6);
- if $h = 0$, then $u_{N,s}$ converges to the solution u to the Lippmann–Schwinger equation (1.7) with $\mu = \Lambda\text{cap}\,(D)\varrho$, where ϱ is a distribution satisfying (1.1) in [6], and a counting function in [15]. An important qualitative difference with (1.2) lies in the fact that the effective medium is dissipative when ω is slightly above the resonant frequency $\omega_{M,1}$ ($\Lambda > 0$), and dispersive when ω is slightly below the resonant frequency $\omega_{M,1}$ ($\Lambda < 0$);
- if $h > 0$ and $\Lambda > 0$, then ω is very close to but slightly larger than $\Re(\omega_1(\delta))$, and the effective medium becomes opaque (this result was only obtained in [15]). The total wave field $u_{N,s}$ converges to the solution u of the problem (1.8).

The derivation of an effective medium theory for frequencies ω very close but slightly smaller to the resonant frequency $\omega_{M,i}$ ($h > 0$ and $\Lambda < 0$ in (1.15)) remains an open problem and is expected to be difficult since the medium becomes highly dispersive in this regime.

This paper improves the homogenization analysis of [6, 15] in several aspects. First, we consider no further hypothesis than the randomness and the independence assumptions for the distributions of the centers stated in (H1). Second, we generalize the above results to the case $K > 1$ where each packet of resonators has several connected components (when the resonance is of monopole type, see below). Third, our analysis gives both a generalization and a clear understanding about the origin of the hypothesis (1.15). We obtain indeed that the effective properties of the heterogeneous medium close to the resonant frequencies are determined by the asymptotic behavior of $sNQ(s, \delta)$, where $Q(s, \delta)$ is the quantity defined by

$$Q(s,\delta) := \sum_{i=1}^{K} \frac{\lambda_i}{\frac{s^2}{s_i(\delta)^2} - 1} (\boldsymbol{a}_i^T V \mathbf{1})^2 \text{ with } s_i(\delta) := \delta^{\frac{1}{2}} \frac{\lambda_i^{\frac{1}{2}}}{k_b} \text{ for } 1 \le i \le K, \tag{1.16}$$

where $\mathbf{1} = (1)_{1\le i\le K}$ is the vector of ones. Note that since we assume that ω is fixed by the size parameter s is variable, we consider resonant characteristic sizes $(s_i(\delta))_{1\le i\le K}$ rather than resonant frequencies $(\omega_{M,i})_{1\le i\le K}$ in the definition (1.16), which are related by the formula $\omega/\omega_{M,i} = s/s_i(\delta)$. The reader can then check that, in the present context with packets made of $K \ge 1$ resonators, the regime (1.15) can be rewritten $sNQ(s, \delta) \sim \Lambda s^{-h}$ with $s \sim s_1(\delta)$. The quantity $Q(s, \delta)$, which blows up as $s \sim s_i(\delta)$, appears somewhat naturally in our analysis; more precisely in the algebraic calculations of Lemma 4.4. Our methodology based on layer potentials constrains us, however, to the following "sub-critical" regime assumption:

(H4) s is close to a resonant characteristic size $s_i(\delta)$ with $1 \le i \le K$, whose associated eigenmode $\boldsymbol{a}_i$ is not V-orthogonal to the vector of ones:

$$\exists 1 \le i \le K, \quad s \sim s_i(\delta) \text{ with } \boldsymbol{a}_i^T V \mathbf{1} \ne 0, \tag{1.17}$$

and there exists a constant $c > 0$ independent of s, δ and N such that

$$sN|Q(s,\delta)| \le c. \tag{1.18}$$

We refer to the assumption (1.18) as a "sub-critical" regime because it covers the cases $h < 0$ and $h = 0$ in (1.15), but not the case $h > 0$ (which would be the super-critical regime). The condition $\boldsymbol{a}_i^T V \mathbf{1} \ne 0$ of equation (1.17) means that the resonance of a system of K resonators is of *monopole* type: the far field generated by a single resonator is proportional to $\Gamma^k(x)$ where Γ^k is the (outgoing) fundamental solution to the Helmholtz equation. This happens generically if the packets $D = \cup_{i=1}^K B_i$ does not have too many symmetries (see [38], Sect. 2). Note that (1.17) implies $|Q(s, \delta)| \to +\infty$, and (1.18) implies $sN \to 0$ as in (1.15).

In view of (1.16), assumption (H4) can also be more physically rephrased as

(H4) The contrast parameter is strictly smaller than $O(N^{-2})$:

$$\delta = o(N^{-2}),$$

and there exists a resonant characteristic size $s_i(\delta)$ with $1 \leq i \leq K$ such that $s \sim s_i(\delta)$ with $\boldsymbol{a}_i^T V \mathbf{1} \neq 0$ at a rate slower than $\delta^{\frac{1}{2}} N$: there exists a constant $c > 0$ such that

$$\exists 1 \leq i \leq K,\, c\delta^{\frac{1}{2}} N \leq \left| \frac{s}{s_i(\delta)} - 1 \right| \longrightarrow 0 \text{ as } \delta \to 0.$$

Then, equation (1.18) is a "subcritical regime" in the sense that the characteristic size s remains slightly away from $s_i(\delta)$; in other words, the incident frequency is not "too close" to the resonant frequency $\omega_{M,i}$ of (1.14).

With these assumptions, we obtain the following main result with quantitative error bounds (stated in Prop. 4.8).

Proposition 1.2. *Assume (H1), (H2) and (H4). Let u be the solution to the following Lippmann–Schwinger equation:*

$$\begin{cases} \left(\Delta + k^2 - sNQ(s,\delta)\varrho 1_\Omega\right) u = 0, \\ \left(\dfrac{\partial}{\partial |x|} - \mathrm{i}k\right)(u - u_{in}) = O\big(|x|^{-2}\big) \quad \text{as } |x| \to +\infty. \end{cases} \tag{1.19}$$

There exists an event $\mathcal{H}_{N_0}$ which holds with large probability $\mathbb{P}(\mathcal{H}_{N_0}) \to 1$ as $N_0 \to +\infty$, independent of s and δ, such that when $\mathcal{H}_{N_0}$ is realized, the function u is an approximation of the solution field $u_{N,s}$ to (1.3) with the following error estimates:

(i) *on any ball $B(0,r)$ such that $\Omega \subset B(0,r)$, there exists a constant $c > 0$ independent of s, N and δ such that for any $N \geq N_0$:*

$$\mathbb{E}\Big[\|u_{N,s} - u\|^2_{L^2(B(0,R))} | \mathcal{H}_{N_0}\Big]^{\frac{1}{2}} \leq csNQ(s,\delta) \max\Big(\delta^{\frac{1}{2}} N, N^{-\frac{1}{2}}\Big); \tag{1.20}$$

(ii) *on any bounded open subset $A \subset \mathbb{R}^3 \backslash \Omega$ away from the resonators, there exists a constant $c > 0$ independent of s, N and δ such that for any $N \geq N_0$:*

$$\mathbb{E}\Big[\|\nabla u_{N,s} - \nabla u\|_{L^2(A)} | \mathcal{H}^2_{N_0}\Big]^{\frac{1}{2}} \leq csNQ(s,\delta) \max\Big(\delta^{\frac{1}{2}} N, N^{-\frac{1}{2}}\Big). \tag{1.21}$$

The rate of convergence is of order $O(\max(\delta^{\frac{1}{2}} N, N^{-\frac{1}{2}}))$ because the scattered fields $u_{N,s-u_{in}}$ and $u - u_{in}$ are of order $O(sNQ(s,\delta))$.

The bound (1.20) is also an improvement over [6, 15] because it shows the convergence of the scattered field $u_{N,s}$ to the homogenized field u in an L^2 sense even in regions very close to the resonators. In contrast, the error bounds of [6] apply only to a region slightly away from the resonators, while the analysis of [15] gives a convergence result only the far field pattern of $u_{N,s}$. The order of convergence obtained is natural, since $N^{-\frac{1}{2}}$ is the natural rate of convergence coming from the application of the law of large numbers for independent random variables, while $\delta^{\frac{1}{2}} N$ is the quantity occurring in (H4). From the result of Proposition 4.8, we obtain the existence of different regimes of convergence depending on the asymptotic behaviour of $sNQ(s,\delta)$ which are summarized in Table 1.

Let us finally mention the existence of further works related to the homogenization of resonant high-contrast wave media. The paper of Bouchitté *et al.* [22] studied the homogenization of a periodic array of high-contrast dielectric structures and proved the arising of resonances. However, an important difference with our work is that they study the high-contrast regime $\delta \to +\infty$ rather than $\delta \to 0$. Furthermore, they consider resonators

TABLE 1. Summary of the effective medium theories obtained for for respectively sound-soft metamaterials (system (1.2)) and high-contrast metamaterials (system (1.3)) with their different regimes of convergence under the assumptions (H1)–(H4). The supercritical regimes $sN \to +\infty$ or $sN|Q(s,\delta)| \to +\infty$ are not treated in our analysis, hence we are only able to formulate conjectures.

System	Scaling regime	Effective physics	Effective model
Sound-soft system (1.2) with $s \to 0$ and $N \to +\infty$	$sN \to 0$	Transparent medium	(1.6)
	$sN \to \Lambda$ with $\Lambda > 0$	Dissipative medium	(1.7)
	$sN \to +\infty$	Opaque medium (conjecture)	(1.8) (conjecture)
High-contrast system (1.3) with $\delta \to 0$, $s \sim s_i(\delta)$ and $N \to +\infty$	$sNQ(s,\delta) \to 0$	Transparent medium	(1.6)
	$sNQ(s,\delta) \to \Lambda$ with $\Lambda > 0$	Dissipative medium	(4.52)
	$sNQ(s,\delta) \to \Lambda$ with $\Lambda < 0$	Dispersive medium	
	$sNQ(s,\delta) \to +\infty$	Opaque medium (conjecture)	(1.8) (conjecture)
	$sNQ(s,\delta) \to -\infty$	Highly dispersive medium (conjecture)	Open problem

which have a size s comparable to that of the periodic cell ϵ_N; $s \propto \epsilon_N$. In contrast, the assumption (1.18) shows that subwavelength resonances are triggered for much smaller resonators: indeed, the minimum distance between two packets of resonators is of the order $\epsilon_N = O(N^{-\frac{2}{3}})$ for randomly distributed centers (see Prop. 2.1) while our homogenization result is obtained for size ratios smaller than $1/N$ due to (1.18). The resonances obtained in [22] and generalized to the full 3D Maxwell system in [24] are therefore not of the same type as ours. Recently [29, 30], developed a mathematical framework for analyzing the arising of resonances for wave propagating in infinitely periodic high-contrast media and the resulting dispersive effects in the situation $\delta \to 0$. However, they still assume that the periodic inclusions have a size comparable to the cell periodicity. Finally, Schweizer and Lamacz [51,67,68] have studied the homogenization of a metamaterial made of a periodic array of tiny Helmholtz resonators. The authors neglected the radiative effects by assuming that the material is enclosed by a Dirichlet boundary condition. This system is the closest in spirit to our high-contrast system (1.3), since it also features subwavelength resonances which lead to a homogenized equation similar to (3.18) for a particular set of scalings. By using two-scale convergence arguments, the authors obtained the qualitative convergence of the scattered field to a homogenized field solution to an effective equation analogous to (4.52).

1.4. Outline of the paper

As we shall see, the analysis we propose for the study of both these systems is similar and based on layer potentials; the system (1.2) may be seen as a simplified version of (1.3) which differs mainly due to the arising of resonances. Simultaneously, the system (1.2) is interesting in its own rights and we shall obtain original results.

The paper outlines as follows. Section 2 derives holomorphic expansions of the single layer potential and the Neumann–Poincaré operator associated to the domain $D_{N,s}$ with respect to the parameter s. We establish error bounds uniform with respect to the parameters s and N for the truncated holomorphic series, which bring into play the critical quantity sN of (H1), and which are at the basis of our homogenization method for the scattering problems (1.2) and (1.3).

The proofs of the homogenization results of Propositions 3.4 and 4.8 are the object of Section 3 for the sound-soft material, and Section 4 for the high-contrast metamaterial. We follow a common and systematic methodology for both cases: first, we write an integral reformulation of (1.2) and (1.3) which allows to find an explicit representation of the solution in terms of layer potentials. Then, we show that the integral equation can be reduced in the low frequency regime to a linear system with N unknown called "Foldy–Lax" approximation. Then, the mean-square convergence theory of [37] allows us to establish the convergence of this linear system to an integral equation. This allows us, in a last step, to read the resulting homogenized equation for $u_{N,s}$ and the corresponding error bounds.

Finally, a few useful technical results arising in the proofs are given in the appendix.

To conclude this introduction, let us mention that the methods developed in this work are quite general and could be used to study other types of metamaterials. Future investigations could concern metascreens as in [9, 15] in which the centers of the resonators are distributed on a surface rather than in a volume, and high-contrast metamaterials exhibiting a resonance which is not of monopole type. The condition (1.17) would not hold and a substantially different analysis is required. A formal study has been proposed in [14] in the case where the resonators are dimers $D = B_1 \cup B_2$ constituted of two identical spheres, which suggests that the effective medium is a double negative metamaterial.

In the whole paper, $c > 0$ denotes a universal independent constant which can change from line to line, and $(\boldsymbol{e}_i)_{1\le i\le N}$ is the canonical basis of $\mathbb{R}^N$.

2. Layer potentials in domains filled with a large number of small inclusions

This section introduces a number of useful preliminary results and notations which are at the basis of the homogenization procedure of both scattering problems (1.2) and (1.3). We start in Section 2.1 by providing probabilistic estimates of the minimum distance ϵ_N between two centers, and of a critical size ℓ_N arising in the expansions of the layer potentials for arbitrary distributions of points $((y_i)_{1\le i\le N})$. Section 2.2 introduces a rescaling operator $\mathcal{P}_{N,s}$ mapping $L^2(\mathcal{D}_{N,s})$ to the product space $L^2(D_1)\times\cdots\times L^2(D_N)$. This operator enables, in Section 2.3, to write the single layer potential and the Neumann–Poincaré operator on $\partial D_{N,s}$ in terms of an operator holomorphic in the characteristic size s, whose holomorphic series yields complete asymptotic expansions. Finally, Section 2.4 provides uniform estimates of single layer potentials viewed as a complex scalar field in $\mathbb{R}^3$, from the magnitude of the potential on $\partial D_{N,s}$.

2.1. Critical sizes

In this subsection, we introduce and estimate two parameters ϵ_N and ℓ_N homogeneous to a distance which play important roles in our analysis:

$$\epsilon_N := \min_{1\le i\ne j\le N} |y_i - y_j|, \qquad \ell_N := \left(\sum_{1\le i\ne j\le N} \frac{1}{|y_i-y_j|^2}\right)^{-\frac{1}{2}}. \tag{2.1}$$

The random variable ϵ_N is the minimum distance between the centers of the resonators; it would be the size of the unit cell if the resonators would be arranged in a periodic manner as in the classical setting of periodic homogenization [1,32]. The quantity ℓ_N arises naturally in the asymptotic expansion of the single layer potential $\mathcal{S}^k_{D_{N,s}}$ associated to the inclusions. As we shall see in Proposition 2.3 below, ℓ_N is a measure of the typical size under which it becomes possible to treat the interactions between different groups of resonators separately from those which takes place between the resonators of a same group. It is also the critical size above which "solidification" of the group of obstacle occurs (if $s \gg \ell_N$), or at which the "strange term" appears in (1.7) (if $s \sim \Lambda\ell_N$ for some constant $\Lambda > 0$).

The purpose of this part is to show that $\epsilon_N = O(N^{-\frac{2}{3}})$ and $\ell_N = O(N^{-1})$ in some probabilistic sense.

Proposition 2.1. *For any positive constants $t_0, t_1 > 0$, let $\mathcal{H}_{t_0}^{(0)}$ and $\mathcal{H}_{t_1}^{(1)}$ be the random events*

$$\mathcal{H}_{t_0}^{(0)} := \left\{ {t_0}^{-1} N^{-\frac{2}{3}} < \epsilon_N < t_0 N^{-\frac{2}{3}} \right\}, \tag{2.2}$$

$$\mathcal{H}_{t_1}^{(1)} := \left\{ t_1^{-1} N^{-1} < \ell_N < t_1 N^{-1} \right\}. \tag{2.3}$$

Assume (H1). The events $\mathcal{H}_{t_0}^{(0)}$ and $\mathcal{H}_{t_1}^{(1)}$ are satisfied with arbitrary large probability for sufficiently large constants $t_0, t_1 > 0$. More precisely, for any $c > 0$, there exist large enough $t_0, t_1 > 0$ such that for any $N \in \mathbb{N}$ large enough,

$$\mathbb{P}\left(\mathcal{H}_{t_0}^{(0)} \right) \geq 1 - c, \tag{2.4}$$

$$\mathbb{P}\left(\mathcal{H}_{t_1}^{(1)} | \mathcal{H}_{t_0}^{(0)} \right) \geq 1 - c, \tag{2.5}$$

where $\mathbb{P}(A|B)$ denotes the probability of the event A conditionally to B.

Proof. (1) Equation (2.4) follows from the result recalled in Proposition A.3 of the appendix.

(2) Denote by $f(y_i, y_j)$ the random variable

$$f(y_i, y_j) := \frac{1_{|y_i - y_j| > {t_0}^{-1} N^{-2/3}}}{|y_i - y_j|^2}.$$

We prove that there exists $\beta > 0$ large enough such that $\mathbb{E}[f(y_1, y_2)]$ satisfies

$$\beta^{-1} \leq \mathbb{E}[f(y_1, y_2)] \leq \beta. \tag{2.6}$$

Indeed, using an integration in polar coordinates, we find

$$\begin{aligned} \mathbb{E}[f(y_1, y_2)] &= \int_\Omega \int_\Omega \frac{1_{|y_1 - y_2| > t_0^{-1} N^{-\frac{2}{3}}}}{|y_1 - y_2|^2} \varrho(y_1) \varrho(y_2) \, \mathrm{d}y_1 \, \mathrm{d}y_2 \leq \|\varrho\|_{L^\infty(\Omega)}^2 \int_{y_1 \in \Omega} \int_{\mathbb{S}^2} \int_0^{\mathrm{diam}(\Omega)} \frac{1}{r^2} r^2 \, \mathrm{d}r \, \mathrm{d}\theta \\ &\leq 4\pi \|\varrho\|_{L^\infty(\Omega)}^2 |\Omega| \, \mathrm{diam}(\Omega). \end{aligned}$$

For the lower bound, we write

$$\mathbb{E}[f(y_1, y_2)] = \int_\Omega \int_\Omega \frac{\varrho(y_1) \varrho(y_2)}{|y_1 - y_2|^2} \, \mathrm{d}y_1 \, \mathrm{d}y_2 - \int_\Omega \int_\Omega \frac{1_{|y_1 - y_2| \leq t_0^{-1} N^{-\frac{2}{3}}}}{|y_1 - y_2|^2} \rho(y_1) \rho(y_2) \, \mathrm{d}y_1 \, \mathrm{d}y_2,$$

and we observe that the second integral vanishes as $N \to +\infty$ due to the bound

$$\int_\Omega \int_\Omega \frac{1_{|y_1 - y_2| \leq t_0^{-1} N^{-\frac{2}{3}}}}{|y_1 - y_2|^2} \rho(y_1) \rho(y_2) \, \mathrm{d}y_1 \, \mathrm{d}y_2 \leq \|\varrho\|_{L^\infty(\Omega)}^2 |\Omega| \int_{\mathbb{S}^2} \int_0^{t_0^{-1} N^{-\frac{2}{3}}} \frac{1}{r^2} r^2 \, \mathrm{d}r \, \mathrm{d}\theta \leq \|\varrho\|_{L^\infty(\Omega)}^2 4\pi t_0^{-1} N^{-\frac{2}{3}}.$$

Hence, equation (2.6) holds for N large enough with $\beta := \max \left(4\pi \|\varrho\|_{L^\infty(\Omega)}^2 |\Omega| \, \mathrm{diam}(\Omega), 2 \left(\int_{\Omega \times \Omega} \frac{\varrho(y_1)\varrho(y_2)}{|y_1 - y_2|^2} \, \mathrm{d}y_1 \, \mathrm{d}y_2 \right)^{-1} \right)$. Using similar integrations in polar coordinates, we also find, up to using a larger constant $\beta > 0$:

$$\mathbb{E}\left[f(y_1, y_2)^2 \right] \leq \beta N^{-2 + \frac{8}{3}} \text{ and } \mathbb{E}[f(y_1, y_2) f(y_1, y_3)] \leq \beta.$$

Then, the version of the law of large numbers of Proposition A.2 in the appendix implies, up to selecting a potentially larger constant $\beta > 0$:

$$\begin{aligned}\mathbb{E}\left[\left|\frac{2}{N(N-1)}\sum_{1\le i<j\le N} f(y_i,y_j) - \mathbb{E}[f(y_1,y_2)]\right|^2\right]\\ &\le \frac{4}{N^2(N-1)^2}\left(\frac{N(N-1)(N-2)}{6}\mathbb{E}[f(y_1,y_2)f(y_1,y_3)] + \frac{N(N-1)}{2}\mathbb{E}\Big[f(y_1,y_2)^2\Big]\right)\\ &\le \frac{\beta}{2}\Big(N^{-1}+N^{-2}N^{\frac{2}{3}}\Big) \le \beta N^{-1}.\end{aligned}$$

By using Markov inequality (Prop. A.1), we then find that for any $\gamma > 0$,

$$\mathbb{P}\left(\left|\frac{1}{N(N-1)}\ell_N^{-2} - \mathbb{E}[f(y_1,y_2)]\right| > \gamma \mathbb{E}[f(y_1,y_2)] \,|\, \mathcal{H}_t^{(0)}\right) \le (1-c)^{-1}\gamma^{-2}\beta^3 N^{-1}.$$

Consequently, setting $\gamma \equiv \gamma_N := hN^{-1/2}$ with h sufficiently large but independent of N, we find that there exists a constant c independent of N such that

$$\mathbb{P}\Big(\,\big|N^{-2}\ell_N^{-2} - \mathbb{E}[f(y_1,y_2)]\big| < hN^{-\frac{1}{2}} \,|\, \mathcal{H}_t^{(0)}\Big) \ge 1 - Ch^{-2}.$$

The result follows from (2.6) because the above inequality states that $\ell_N = \mathbb{E}[f(y_1,y_2)]^{-1}N^{-1} + O(N^{-\frac{3}{2}})$. ◻

Remark 2.1. The quantity $\ell_N \sim 1/N$ is known to be the critical scaling at which an ensemble of regularly spaced scattering obstacles "solidifies" (reflects entirely the sound wave), see [65, 66], or at which the "strange term" appears in two-scale homogenization of porous media [32, 35, 36, 39, 43, 44]. In this article, we show that $s \sim 1/N$ is also the critical size for obstacles randomly distributed in a volume. An analysis similar to Proposition 2.1 leads us to expect that for centers randomly distributed on a surface, the critical size is $s \sim 1/(N|\log N|^{\frac{1}{2}})$.

Remark 2.2. In the homogenization analysis of [6], the authors assume both $\min_{1\le i\ne j\le N}|y_i - y_j| \ge cN^{-\frac{1}{3}}$ and the ergodicity condition (1.1), which, in view of (2.3), cannot be achieved by randomly and independently distributed points $(y_i)_{1\le i\le N}$, and hence can be difficult to realize in practice. This difficulty was also pointed out by Gerard-Varet [42].

Remark 2.3. The consideration of the events $\mathcal{H}_{t_0}^{(0)}$ and $\mathcal{H}_{t_1}^{(1)}$ enables one to obtain error bounds without the need for extra hypotheses on the joint distribution of centers $(y_i)_{1\le i\le N}$. This allows to consider conveniently fully independent random distributions points $(y_i)_{1\le i\le N}$, in contrast with other possible settings considered in [2, 3, 42, 62].

It follows from Proposition 2.1 that one can choose sufficiently large constants $t_0, t_1 > 0$ such that for any fixed $N \ge 0$, the inequalities

$$t_0{}^{-1}N^{-2/3} < \epsilon_N < t_0 N^{-2/3} \text{ and } t_1^{-1}N^{-1} < \ell_N < t_1 N^{-1} \tag{2.7}$$

are satisfied with large probability (independent of N). Therefore, the reader may think of ϵ_N as $N^{-\frac{2}{3}}$ and of ℓ_N as N^{-1} when (H1) is realized. We keep the reference to ℓ_N and ϵ_N in the fully general setting where no assumption is made on the distribution of points $(y_i)_{1\le i\le N}$.

Finally, we denote by η_N the ratio between the distance between the centers $\epsilon_N \equiv O(N^{-2/3})$ and the size of the obstacles s. Throughout the paper, we assume the natural condition that η_N is smaller than a small fixed constant $c > 0$, which ensures that the acoustic obstacles do not overlap:

$$\eta_N := \frac{s}{\epsilon_N} < c. \tag{2.8}$$

In other words, this condition states that for arbitrary distributions of points, the natural range for the variations of the characteristic size is $s = O(N^{-\frac{2}{3}})$. The condition (2.8) is of course satisfied in the "subcritical" regimes of assumptions (H3) and (H4).

2.2. Rescaling operator and product spaces

We now introduce an operator $\mathcal{P}_{N,s}$ which performs a rescaling around each set of inclusions y_i+sD_i. The main motivation lies in the fact that the conjugations of the layer potentials by the operator $\mathcal{P}_{N,s}$ are holomorphic in the variable s (Prop. 2.3 below), which allows to determine conveniently their asymptotic expansions.

In all what follows, for any $y \in \mathbb{R}^3$ and $s > 0$, we denote by $\tau_{y,s}$ the affine transformation

$$\tau_{y,s}(t) := y + st, \quad t \in \mathbb{R}^3.$$

The transformations $(\tau_{y_i,s})_{1\le i\le N}$ enable one to replace the analysis on the whole set of tiny inclusions

$$D_{N,s} = \cup_{1\le i\le N}(y_i + sD_i)$$

with one on the product domain

$$\mathcal{D} := D_1 \times \cdots \times D_N.$$

Denoting by $L^2(\partial\mathcal{D})$ and $H^1(\partial\mathcal{D})$ the product spaces

$$L^2(\partial\mathcal{D}) := L^2(\partial D_1) \times \cdots \times L^2(\partial D_N),$$

$$H^1(\partial\mathcal{D}) := H^1(\partial D_1) \times \cdots \times H^1(\partial D_N),$$

we introduce the rescaling or pull-back operator

$$\begin{aligned} \mathcal{P}_{N,s} : L^2(\partial\mathcal{D}) \qquad &\longrightarrow L^2(\partial D_{N,s}) \\ (\phi_1,\dots,\phi_N) &\longmapsto \phi \text{ with } \phi_{|y_i+s\partial D_i} = \phi_i \circ \tau_{y_i,s}^{-1}. \end{aligned} \tag{2.9}$$

Clearly, the inverse of $\mathcal{P}_{N,s}$ is given by $\mathcal{P}_{N,s}^{-1}\phi = (\phi_{|y_i+s\partial D_i} \circ \tau_{y_i,s})_{1\le i\le N}$. With a small abuse of notation, we still denote by $\mathcal{P}_{N,s}$ the same operator acting from $H^1(\partial\mathcal{D})$ to $H^1(\partial D_{N,s})$.

By introducing appropriate norms on $L^2(\partial\mathcal{D})$ and $H^1(\partial\mathcal{D})$, we make $\mathcal{P}_{N,s}$ to be an isometry. The definition of these norms is motivated by the following lemma.

Lemma 2.1. *For any $\phi \in L^2(\partial D_{N,s})$ with $D_{N,s} = \cup_{i=1}^N(y_i + sD_i)$, it holds*

$$\|\phi\|_{L^2(\partial D_{N,s})} = s\left(\sum_{i=1}^N \|\phi\circ\tau_{y_i,s}\|^2_{L^2(\partial D_i)}\right)^{\frac{1}{2}}, \qquad \|\nabla_\Gamma\phi\|_{L^2(\partial D_{N,s})} = \left(\sum_{i=1}^N \|\nabla_\Gamma(\phi\circ\tau_{y_i,s})\|^2_{L^2(\partial D_i)}\right)^{\frac{1}{2}},$$

where ∇_Γ is the tangential gradient on $\partial D_{N,s}$.

Proof. By using a change of variables, we find

$$\|\phi\|^2_{L^2(\partial D_{N,s})} = \sum_{i=1}^N s^2 \int_{\partial D_i} (\phi \circ \tau_{y_i,s})^2 \, \mathrm{d}\sigma,$$

and therefore,

$$\|\nabla_\Gamma \phi\|^2_{L^2(\partial D_{N,s})} = \sum_{i=1}^N s^2 \int_{\partial D_i} \|(\nabla_\Gamma \phi) \circ \tau_{y_i,s}\|^2 \, \mathrm{d}\sigma = \sum_{i=1}^N \int_{\partial D_i} \|(\nabla_\Gamma \phi \circ \tau_{y_i,s})\|^2 \, \mathrm{d}\sigma.$$

□

In view of Lemma 2.1, we define the norm on $H^1(\partial D_{N,s})$ as follows:

$$\|\phi\|^2_{H^1(\partial D_{N,s})} := \|\phi\|^2_{L^2(\partial D_{N,s})} + s^2 \|\nabla_\Gamma \phi\|^2_{L^2(\partial D_{N,s})}. \tag{2.10}$$

Then it holds conveniently

$$\|\phi\|^2_{H^1(\partial D_{N,s})} = s^2 \sum_{i=1}^N \|\phi \circ \tau_{y_i,s}\|^2_{H^1(\partial D_i)}. \tag{2.11}$$

Endowing $L^2(\partial\mathcal{D})$ and $H^1(\partial\mathcal{D})$ with the norms

$$\|(\phi_1, \dots, \phi_N)\|_{L^2(\partial\mathcal{D})} := s\left(\sum_{i=1}^N \|\phi_i\|^2_{L^2(\partial D_i)}\right)^{\frac{1}{2}}, \qquad \|(\phi_1, \dots, \phi_N)\|_{H^1(\partial\mathcal{D})} := s\left(\sum_{i=1}^N \|\phi_i\|^2_{H^1(\partial D_i)}\right)^{\frac{1}{2}}, \tag{2.12}$$

we infer from (2.11) that $\mathcal{P}_{N,s}$ is an isometry:

$$\|\mathcal{P}_{N,s}(\phi_1, \dots, \phi_N)\|_{L^2(\partial D_{N,s})} = \|(\phi_1, \dots, \phi_N)\|_{L^2(\partial\mathcal{D})}, \quad \|\mathcal{P}_{N,s}(\phi_1, \dots, \phi_N)\|_{H^1(\partial D_{N,s})} = \|(\phi_1, \dots, \phi_N)\|_{H^1(\partial\mathcal{D})}.$$

We complete this part by stating a few elementary results which enable to estimate the norm of operators on the product domain $\mathcal{D}$. In all what follows, $|||\mathcal{A}|||_{V\to H}$ stands for the operator norm of a bounded operator $\mathcal{A} : V \to H$ on two given Banach spaces V and H:

$$|||\mathcal{A}|||_{V\to H} := \sup_{x\in V} \frac{\|\mathcal{A}x\|_H}{\|x\|_V}.$$

When the context is clear, we sometimes omit the subscript and we write $|||\mathcal{A}|||$ for $|||\mathcal{A}|||_{V\to H}$.

Proposition 2.2. *The norm of an operator $\mathcal{A} : L^2(\partial\mathcal{D}) \to L^2(\partial\mathcal{D})$ satisfies the following bound:*

$$|||\mathcal{A}|||_{L^2(\partial\mathcal{D})\to L^2(\partial\mathcal{D})} \le \max_{1\le i\le N} |||\mathcal{A}_{ii}|||_{L^2(\partial D_i)\to L^2(\partial D_i)} + \left(\sum_{1\le i\ne j\le N} |||\mathcal{A}_{ij}|||^2_{L^2(\partial D_j)\to L^2(\partial D_i)}\right)^{\frac{1}{2}}, \tag{2.13}$$

where $\mathcal{A}_{ij} : L^2(\partial D_j) \to L^2(\partial D_i)$ denotes the family of operators satisfying

$$\mathcal{A}\begin{bmatrix}\phi_1\\ \vdots \\ \phi_N\end{bmatrix} = \left(\sum_{j=1}^N \mathcal{A}_{ij}\phi_j\right)_{1\le i\le N}, \tag{2.14}$$

i.e. $\mathcal{A}_{ij}[\phi] := \boldsymbol{e}_i \cdot \mathcal{A}[\phi \boldsymbol{e}_j]$ for any $\phi \in L^2(\partial D_i)$.

Proof. Denoting by $\phi = (\phi_1, \dots, \phi_N)$, we have by using the triangle inequality:

$$\|\mathcal{A}[\phi]\|_{L^2(\partial\mathcal{D})} \le \|(\mathcal{A}_{ii}[\phi])_{1\le i\le N}\|_{L^2(\partial\mathcal{D})} + \left\| \left(\sum_{j\neq i} \mathcal{A}_{ij}[\phi_j] \right)_{1\le i\le N} \right\|_{L^2(\partial\mathcal{D})}. \tag{2.15}$$

Then both these terms can be bounded as follows:

$$\begin{aligned}
\left\|(\mathcal{A}_{ii}[\phi_i])_{1\le i\le N}\right\|^2_{L^2(\partial\mathcal{D})} &= s^2\sum_{i=1}^N \|\mathcal{A}_{ii}[\phi_i]\|^2_{L^2(\partial D_i)} \\
&\le s^2 \max_{1\le i\le N} |||\mathcal{A}_{ii}|||^2 \sum_{i=1}^N \|\phi_i\|^2_{L^2(\partial D_i)} = \max_{1\le i\le N} |||\mathcal{A}_{ii}|||^2\, \|\phi\|^2_{L^2(\partial\mathcal{D})}, \\
\left\| \left(\sum_{j\neq i} \mathcal{A}_{ij}[\phi_j] \right)_{1\le i\le N} \right\|^2_{L^2(\partial\mathcal{D})} &= s^2 \sum_{i=1}^N \left\| \sum_{j\neq i} \mathcal{A}_{ij}[\phi_j] \right\|^2_{L^2(\partial D_i)} \le s^2 \sum_{i=1}^N \left(\sum_{j\neq i}^N |||\mathcal{A}_{ij}|||\, \|\phi_j\|_{L^2(\partial D_j)} \right)^2 \\
&\le s^2 \sum_{1\le i\neq j\le N} |||\mathcal{A}_{ij}|||^2 \sum_{j=1}^N \|\phi_j\|^2_{L^2(\partial D_j)} \le \sum_{j\neq i}^N |||\mathcal{A}_{ij}|||^2\, \|\phi\|^2_{L^2(\partial\mathcal{D})}.
\end{aligned}$$

□

Remark 2.4. The same inequality holds by changing $L^2(\partial\mathcal{D})$ into $H^1(\partial\mathcal{D})$.

With a small abuse of notation, we identify in the next sections an operator $\mathcal{A}_{ij} : L^2(\partial D_j) \to L^2(\partial D_i)$ to its natural extension $\widetilde{\mathcal{A}}_{ij} : L^2(\partial\mathcal{D}) \to L^2(\partial\mathcal{D})$ by 0, *i.e.* satisfying $(\widetilde{\mathcal{A}}_{ij}\phi)_i = \mathcal{A}_{ij}\phi_j$ and $(\widetilde{\mathcal{A}}_{ij}\phi)_l = 0$ for $l \neq i$.

2.3. Holomorphic expansions of layer potentials with respect to the size parameter

This part introduces the main results of this section, which are the holomorphic expansions of the single layer potential and the Neumann–Poincaré operator given in Corollary 2.1.

2.3.1. Definitions and notation conventions

In all what follows, we denote by $\Gamma^k(x) := -\frac{e^{ik|x|}}{4\pi|x|}$ the (outgoing) fundamental solution to the Helmholtz equation with wave number $k > 0$, *i.e.*

$$(\Delta + k^2)\Gamma^k = \delta_0,$$

where δ_0 is the Dirac distribution. For a smooth bounded open set $D \subset \mathbb{R}^d$, we denote by $\mathcal{S}_D^k$ and $\mathcal{K}_D^{k*}$ respectively the single layer potential and the adjoint of the Neumann–Poincaré operator on ∂D: for any $\phi \in L^2(\partial D)$,

$$\mathcal{S}_D^k[\phi](x) := \int_{\partial D} \Gamma^k(x-y)\phi(y)\,\mathrm{d}\sigma(y), \qquad x \in \mathbb{R}^3, \tag{2.16}$$

$$\mathcal{K}_D^{k*}[\phi](x) := \int_{\partial D} \nabla_x\Gamma^k(x-y)\cdot \boldsymbol{n}(x)\phi(y)\,\mathrm{d}\sigma(y), \qquad x \in \partial D, \tag{2.17}$$

where $\mathrm{d}\sigma$ is the surface measure of ∂D and $\boldsymbol{n}$ is its outward normal. We use the notations $\mathcal{S}_D^{k_b}$ and $\mathcal{K}_D^{k_b*}$ for the same operators with k replaced with k_b, and we use the short-hand notation $\Gamma := \Gamma^0$, $\mathcal{S}_D := \mathcal{S}_D^0$ and $\mathcal{K}_D^* := \mathcal{K}_D^{0*}$ for the fundamental solution of the Laplace operator, its associated single layer potential and the adjoint of its Neumann–Poincaré operator. We recall that $\mathcal{K}_D^{k*}$ is a compact operator on $L^2(\partial D)$ and that $\mathcal{S}_D^k$ is an invertible operator from $L^2(\partial D)$ to $H^1(\partial D)$ when k^2 is not a Dirichlet eigenvalue of $-\Delta$ on D, whose inverse is denoted by $(\mathcal{S}_D^k)^{-1}: H^1(\partial D) \to L^2(\partial D)$, see *e.g.* [7]. Finally, the harmonic capacity of a set D is denoted by cap (D):

$$\operatorname{cap}(D) = -\int_{\partial D} \mathcal{S}_D^{-1}[1_{\partial D}]\, \mathrm{d}\sigma. \tag{2.18}$$

For the analysis in the most general setting where (H1) and (H2) are not necessarily satisfied, we consider the following uniform boundedness assumptions on the resonator packets D_i:

(i) the points $(y_i)_{i\in\mathbb{N}}$ belong to a bounded domain $\Omega \subset \mathbb{R}^3$:

$$\sup_{i\neq j} |y_i - y_j| < +\infty; \tag{2.19}$$

(ii) the number of resonators K_i per packet D_i is bounded: $\sup_{i\in\mathbb{N}} K_i < +\infty$;
(iii) the sets D_i have uniformly bounded perimeters:

$$\sup_{i\in\mathbb{N}} |\partial D_i| < +\infty. \tag{2.20}$$

This implies that they also have bounded diameters: $\sup_{i\in\mathbb{N}} \operatorname{diam}(D_i) < +\infty$;
(iv) the sets D_i have a "uniformly bounded capacity" in the following sense:

$$\sup_{i\in\mathbb{N}} |||\mathcal{S}_{D_i}|||_{L^2(\partial D_i)\to H^1(\partial D_i)} < +\infty, \qquad \sup_{i\in\mathbb{N}} \left|\left|\left|(\mathcal{S}_{D_i})^{-1}\right|\right|\right|_{H^1(\partial D_i)\to L^2(\partial D_i)} < +\infty. \tag{2.21}$$

These assumptions are naturally fulfilled when considering the assumption (H2) in which all the packets D_i are identical.

Throughout the paper, we denote for any $p \in \mathbb{N}$ and $y = (y_1, y_2, y_3) \in \mathbb{R}^3$ by $\nabla^p \Gamma^k(x)$ and by y^p the p-th order tensors:

$$\nabla^p\Gamma^k(x) = \left(\partial^p_{i_1\dots i_p}\Gamma^k(x)\right)_{1\leq i_1\dots i_p\leq 3}, \qquad y^p := (y_{i_1}y_{i_2}\dots y_{i_p})_{1\leq i_1\dots i_p\leq 3},$$

and we denote by $\nabla^p\Gamma^k(x)\cdot y^p$ the contraction

$$\nabla^p\Gamma^k(x)\cdot y^p := \sum_{1\leq i_1\dots i_p\leq d} \partial^p_{i_1\dots i_p}\Gamma^k(x) y_{i_1}\dots y_{i_p}.$$

2.3.2. Holomorphic expansions of the single layer potential and the Neumann–Poincaré operator in the heterogeneous domain

The next proposition provides a full asymptotic expansion of the single layer potential as $s \to 0$ with truncation estimates independent of s and N.

Proposition 2.3. *The following factorization holds for the single layer potential on $D_{N,s}$ for any $N \in \mathbb{N}$ and any distribution of points $(y_i)_{1\leq i\leq N}$ satisfying (i)–(iv):*

$$\mathcal{S}^k_{D_{N,s}} = \mathcal{P}_{N,s}\left(\sum_{p=0}^{+\infty} s^{p+1}\sum_{i=1}^N k^p\mathcal{S}_{D_i,p} + \sum_{p=0}^{+\infty} s^{p+2}\sum_{1\leq i\neq j\leq N}\nabla^p\Gamma^k(y_i-y_j)\cdot\mathcal{T}^p_{D_i,D_j}\right)\mathcal{P}^{-1}_{N,s}, \tag{2.22}$$

where $\mathcal{S}_{D_i,p}: L^2(\partial D_i) \to H^1(\partial D_i)$ and $\mathcal{T}^p_{D_i,D_j}: L^2(\partial D_j) \to H^1(\partial D_i)$ are respectively the operators and the p-th order operator-valued tensors defined by

$$\mathcal{S}_{D_i,p}[\phi](t) := -\frac{i^p}{4\pi p!}\int_{\partial D_i} |t-t'|^{p-1}\phi(t')\,\mathrm{d}\sigma(t'), \qquad \phi\in L^2(\partial D_i),\ t\in\partial D_i,$$

$$\mathcal{T}^p_{D_i,D_j}[\phi](t) := \frac{1}{p!}\int_{\partial D_j}(t-t')^p\phi(t')\,\mathrm{d}\sigma(t'), \qquad \phi\in L^2(\partial D_j),\, t\in\partial D_i.$$

The series (2.22) *converges in operator norm for any s satisfying* (2.8)*. Furthermore, there exist constants $c>0$ independent of s and N such that for any $p\in\mathbb{N}$:*

$$\left|\left|\left|\mathcal{P}_{N,s}\left(s^{p+1}\sum_{i=1}^N k^p\mathcal{S}_{D_i,p}\right)\mathcal{P}^{-1}_{N,s}\right|\right|\right|_{L^2(\partial D_{N,s})\to H^1(\partial D_{N,s})} \leq cs^{p+1}, \tag{2.23}$$

$$\left|\left|\left|\mathcal{P}_{N,s}\left(s^{p+2}\sum_{1\leq i\neq j\leq N}\nabla^p\Gamma^k(y_i-y_j)\cdot\mathcal{T}^p_{D_i,D_j}\right)\mathcal{P}^{-1}_{N,s}\right|\right|\right|^2_{L^2(\partial D_{N,s})\to H^1(\partial D_{N,s})} \leq cs\big(s\ell_N^{-1}\big)\eta_N^p. \tag{2.24}$$

Proof. Let us denote for $1\leq i,j\leq N$ by $\mathcal{A}_{ij}: L^2(\partial D_j)\to H^1(\partial D_i)$ the operators associated to $\mathcal{P}^{-1}_{N,s}\mathcal{S}^k_{D_{N,s}}\mathcal{P}_{N,s}$ as in (2.14). We have for $\phi\in L^2(\partial D_i)$ and $t\in\partial D_i$:

$$\begin{aligned}\mathcal{A}_{ii}[\phi](t) &= \mathcal{S}^k_{y_i+sD_i}[\phi\circ\tau^{-1}_{y_i,s}]\circ\tau_{y_i,s}(t)\\ &= \int_{y_i+s\partial D_i}\Gamma^k(y_i+st-y')\phi\circ\tau^{-1}_{y_i,s}(y')\,\mathrm{d}\sigma(y') = s^2\int_{\partial D_i}\Gamma^k(s(t-t'))\phi(t')\,\mathrm{d}\sigma(t')\\ &= s\mathcal{S}^{sk}_{D_i}[\phi](t).\end{aligned}$$

The first part of the expansion follows by using the identity $\mathcal{S}^{sk}_{D_i}[\phi]=\sum_{p=0}^{+\infty}s^pk^p\mathcal{S}_{D_i,p}[\phi]$. For $i\neq j$, we have instead for $t\in\partial D_i$ and $\phi\in L^2(\partial D_j)$:

$$\begin{aligned}\mathcal{A}_{ij}[\phi](t) &= \mathcal{S}^k_{y_j+sD_j}[\phi\circ\tau^{-1}_{y_j,s}]\circ\tau_{y_i,s}(t) = s^2\int_{\partial D_j}\Gamma^k(y_i-y_j+s(t-t'))\phi(t')\,\mathrm{d}\sigma(t')\\ &= \sum_{p=0}^{+\infty}\frac{s^{2+p}}{p!}\nabla^p\Gamma^k(y_i-y_j)\cdot\int_{\partial D_j}(t-t')^p\phi(t')\,\mathrm{d}\sigma(t'),\end{aligned}$$

from where the second term of the expansion follows. The bound on the operator norm of the diagonal part of $\mathcal{P}^{-1}_{N,s}\mathcal{S}^k_{D_{N,s}}\mathcal{P}_{N,s}$ is obtained by recalling that $\mathcal{P}_{N,s}$ is an isometry and by making use of (2.13):

$$\begin{aligned}\left|\left|\left|\mathcal{P}_{N,s}\left(\sum_{i=1}^N k^p\mathcal{S}_{D_i,p}\right)\mathcal{P}^{-1}_{N,s}\right|\right|\right|_{L^2(\partial D_{N,s})\to H^1(\partial D_{N,s})} &\leq \left|\left|\left|\sum_{i=1}^N k^p\mathcal{S}_{D_i,p}\right|\right|\right|_{L^2(\partial\mathcal{D})\to H^1(\partial\mathcal{D})}\\ &\leq k^p\max_{1\leq i\leq N}|||\mathcal{S}_{D_i,p}|||_{L^2(\partial D_i)\to H^1(\partial D_i)},\end{aligned}$$

where the triple norm is bounded by assumption for $p=0$ (Eq. (2.21)) and by using (2.20) for $p\geq1$:

$$k^p|||\mathcal{S}_{D_i,p}|||_{L^2(\partial D_i)\to H^1(\partial D_i)} \leq \frac{k^p}{4\pi p!}\Big(|\operatorname{diam}(\partial D_i)|^{\frac{p-1}{2}}+(p-1)|\operatorname{diam}(\partial D_i)|^{\frac{p-2}{2}}\Big)|\partial D_i|^{\frac12} \text{ for } p\geq1.$$

Similarly using (2.13) and the result of Proposition B.1, we obtain the existence of constants $c,\beta>0$ such that

$$\begin{aligned}&\left|\left|\left|\mathcal{P}_{N,s}\left(s^{p+2}\sum_{1\leq i\neq j\leq N}\nabla^p\Gamma^k(y_i-y_j)\cdot\mathcal{T}^p_{D_i,D_j}\right)\mathcal{P}^{-1}_{N,s}\right|\right|\right|^2_{L^2(\partial D_{N,s})\to H^1(\partial D_{N,s})}\\ &\leq s^{2p+4}\sum_{1\leq i\neq j\leq N}|\nabla^p\Gamma^k(y_i-y_j)|^2\,|||\mathcal{T}^p_{D_i,D_j}|||^2 \leq \beta^p s^{2p+4}\sum_{1\leq i\neq j\leq N}\frac{1}{|y_i-y_j|^{2p+2}}\\ &\leq \beta^p s^{2p+4}\epsilon_N^{-2p}\ell_N^{-2} \leq cs^2\big(s^2\ell_N^{-2}\big)\eta_N^{2p},\end{aligned} \tag{2.25}$$

where we used the assumption (2.8) on η_N. Finally, let us note that the series (2.22) must converge as soon as (2.8) is satisfied because $\mathcal{S}^k_{D_{N,s}}$ as a function of s has no poles on $\mathcal{H}$, and coincides with this series on a non-trivial neighborhood of 0. □

Remark 2.5. Formula (2.22) has two terms: the first one features the operators $\mathcal{S}_{D_i,p}$ which are "diagonal" terms describing the interactions occuring between each components $B_{i,1},\dots,B_{i,k}$ of the resonator packet D_i. The second term involves the extra-diagonal operators $\mathcal{T}^p_{D_i,D_j}$ which account for the interactions occuring in between the groups D_i and D_j for $i\neq j$. The estimation (2.24) shows that the diagonal term is of order $O(s)$ while the extra-diagonal interactions are of order $O(s(s\ell_N^{-1}))$. The "critical" size ℓ_N corresponds therefore to the characteristic size s under which (when $s\ll\ell_N$) the diagonal interaction is dominant.

In order to study the asymptotic of the resonant problem (1.3), we need a similar holomorphic expansion of the adjoint of the Neumann–Poincaré operator.

Proposition 2.4. *The following factorization holds for the adjoint of the Neumann–Poincaré operator on $D_{N,s}$, for any $N\in\mathbb{N}$ and for any distribution of points $(y_i)_{1\le i\le N}$ satisfying (i)–(iv):*

$$\mathcal{K}^{k*}_{D_{N,s}}=\mathcal{P}_{N,s}\left(\sum_{p=0}^{+\infty}s^p\sum_{i=1}^N k^p\mathcal{K}^*_{D_i,p}+\sum_{p=0}^{+\infty}s^{p+2}\sum_{1\le i\neq j\le N}\nabla^{p+1}\Gamma^k(y_i-y_j)\cdot\mathcal{M}^{p+1}_{D_i,D_j}\right)\mathcal{P}^{-1}_{N,s},\tag{2.26}$$

where $\mathcal{K}^*_{D_i,p}:L^2(\partial D_i)\to L^2(\partial D_i)$ and $\mathcal{M}^{p+1}_{D_i,D_j}:L^2(\partial D_j)\to L^2(\partial D_i)$ are respectively the operators and the operator-valued tensors defined by

$$\mathcal{K}^*_{D_i,p}[\phi](t):=-\frac{i^p}{4\pi p!}\int_{\partial D_i}\boldsymbol{n}(t)\cdot\nabla|t-t'|^{p-1}\phi(t')\,\mathrm{d}\sigma(t'),\qquad \phi\in L^2(\partial D_i),\ t\in\partial D_i,$$
$$\mathcal{M}^{p+1}_{D_i,D_j}[\phi](t):=\frac{1}{p!}\int_{\partial D_j}\boldsymbol{n}(t)\otimes(t-t')^p\phi(t')\,\mathrm{d}\sigma(t'),\qquad \phi\in L^2(\partial D_j),\ t\in\partial D_i.$$

The series (2.26) converges for any s satisfying the condition (2.8). Furthermore, there exists a constant $c>0$ independent of s and N such that for any $p\in\mathbb{N}$,

$$\left|\left|\left|\mathcal{P}_{N,s}\left(s^p\sum_{i=1}^N k^p\mathcal{K}^*_{D_i,p}\right)\mathcal{P}^{-1}_{N,s}\right|\right|\right|_{L^2(\partial D_{N,s})\to L^2(\partial D_{N,s})}\le cs^p,\tag{2.27}$$

$$\left|\left|\left|\mathcal{P}_{N,s}\left(s^{p+2}\sum_{1\le i\neq j\le N}\nabla^{p+1}\Gamma^k(y_i-y_j)\cdot\mathcal{M}^{p+1}_{D_i,D_j}\right)\mathcal{P}^{-1}_{N,s}\right|\right|\right|^2_{L^2(\partial D_{N,s})\to H^1(\partial D_{N,s})}\le c(s\ell_N^{-1})\eta_N^{p+1}.\tag{2.28}$$

Proof. The proof is analogous to that of Proposition 2.3. □

Equations (2.30) and (2.26) are rewritten in a more usable manner in the following corollary, where we introduce the operators $\mathcal{S}^k_{\mathcal{D}}(s)$ and $\mathcal{K}^{k*}_{\mathcal{D}}(s)$ holomorphic in the variable s.

Corollary 2.1. *The single layer and the Neumann–Poincaré operators can be expressed in terms of holomorphic operators on the cartesian product $\mathcal{D}=D_1\times\dots\times D_N$ through the following factorizations:*

$$\mathcal{S}^k_{D_{N,s}}=\mathcal{P}_{N,s}\mathcal{S}^k_{\mathcal{D}}(s)\mathcal{P}^{-1}_{N,s},\quad \mathcal{K}^{k*}_{D_{N,s}}=\mathcal{P}_{N,s}\mathcal{K}^{k*}_{\mathcal{D}}(s)\mathcal{P}^{-1}_{N,s}.\tag{2.29}$$

(i) *The operator $\mathcal{S}^k_{\mathcal{D}}(s)$ is given by*

$$\mathcal{S}^k_{\mathcal{D}}(s):=s\mathcal{S}_{\mathcal{D},0}+s^2\mathcal{S}^k_{\mathcal{D},1}+\sum_{p=2}^{+\infty}s^{p+1}\mathcal{S}^k_{\mathcal{D},p},\tag{2.30}$$

where the operators $\mathcal{S}^k_{\mathcal{D},p}$ *are defined by*

$$\mathcal{S}_{\mathcal{D},0} \equiv \mathcal{S}^k_{\mathcal{D},0} := \sum_{i=1}^N \mathcal{S}_{D_i,0} \text{ and } \mathcal{S}^k_{\mathcal{D},p} = \sum_{i=1}^N k^p \mathcal{S}_{D_i,p} + \sum_{1\le i\neq j\le N} \nabla^{p-1}\Gamma^k(y_i-y_j)\cdot \mathcal{T}^{p-1}_{D_i,D_j} \text{ for any } p\ge 1.$$

Moreover, the terms of the series of (2.30) *decay geometrically in the operator norm:*

$$|||s^p \mathcal{S}^k_{\mathcal{D},p}|||_{L^2(\partial\mathcal{D})\to H^1(\partial\mathcal{D})} \le c \times \begin{cases} 1 & \text{if } p=0,\\ s\ell_N^{-1} & \text{if } p=1,\\ s\ell_N^{-1}\eta_N^{p-1} & \text{if } p\ge 2.\end{cases} \tag{2.31}$$

(ii) *The operator* $\mathcal{K}^{k*}_{\mathcal{D}}(s)$ *is given by*

$$\mathcal{K}^{k*}_{\mathcal{D}}(s) = \mathcal{K}^*_{\mathcal{D}} + \sum_{p=2}^{+\infty} s^p \mathcal{K}^{k*}_{\mathcal{D},p}, \tag{2.32}$$

where $\mathcal{K}^*_{\mathcal{D}}$ *and* $\mathcal{K}^{k*}_{\mathcal{D},p}$ *are the operators defined by*

$$\mathcal{K}^*_{\mathcal{D}} = \sum_{i=1}^N \mathcal{K}^*_{D_i}, \qquad \mathcal{K}^{k*}_{\mathcal{D},p} = \sum_{i=1}^N k^p \mathcal{K}^*_{D_i,p} + \sum_{1\le i\neq j\le N} \nabla^{p-1}\Gamma^k(y_i-y_j)\cdot \mathcal{M}^{p-1}_{D_i,D_j} \text{ for any } p\ge 2. \tag{2.33}$$

Moreover, the terms of the series of (2.32) *decay geometrically in the operator norm:*

$$\left|\left|\left| s^p\mathcal{K}^{k*}_{\mathcal{D},p}\right|\right|\right| \le c\big(s\ell_N^{-1}\big)\eta_N^{p-1} \text{ for } p\ge 2. \tag{2.34}$$

Remark 2.6. We shall use below the following identity

$$\mathcal{S}^k_{D_{N,s}}[\mathcal{P}_{N,s}[\phi]](x) = \sum_{i=1}^N s^2\int_{\partial D_i} \Gamma^k(x-y_i-st')\phi_i(t')\,\mathrm{d}\sigma(t'), \qquad x\in\mathbb{R}^3,\ \phi\equiv(\phi_i)_{1\le i\le N}\in L^2(\partial\mathcal{D}), \tag{2.35}$$

which implies in particular that the operator $\mathcal{S}^k_{\mathcal{D}}(s)$ reads explicitly

$$\big(\mathcal{S}^k_{\mathcal{D}}[\phi]\big)_i(t) = s^2\sum_{j=1}^N \int_{\partial D_j} \Gamma^k(y_i-y_j+s(t-t'))\phi_j(t')\,\mathrm{d}\sigma(t'), \qquad t\in\partial D_i, (\phi)_{1\le i\le N}\in L^2(\partial\mathcal{D}). \tag{2.36}$$

We obtain a norm estimate of the inverse of the single layer potential in the subcritical regime $s=O(\ell_N^{-1})$ (corresponding to (H3) if (H1) holds).

Corollary 2.2. *The following norm estimate holds for the inverse of the single layer potential in the subcritical regime* $s=O(\ell_N^{-1})$*:*

$$\left|\left|\left|\left(\mathcal{S}^k_{D_{N,s}}\right)^{-1}\right|\right|\right|_{H^1(\partial\mathcal{D})\to L^2(\partial\mathcal{D})} \le cs^{-1},$$

for a constant $c>0$ *independent of* s *and* N.

Proof. From (2.30) and (2.31), the Neumann series

$$s^{-1}\left[\big(\mathcal{S}^k_{\mathcal{D},0}\big)^{-1} + \sum_{p=2}^{+\infty} s^p\sum_{j=1}^p(-1)^j \sum_{\substack{1\le i_1\dots i_j\le p\\ i_1+\dots+i_j=p}} \big(\mathcal{S}^k_{\mathcal{D},0}\big)^{-1}\mathcal{S}^k_{\mathcal{D},i_1}\big(\mathcal{S}^k_{\mathcal{D},0}\big)^{-1}\cdots\big(\mathcal{S}^k_{\mathcal{D},0}\big)^{-1}\mathcal{S}^k_{\mathcal{D},i_j}\big(\mathcal{S}^k_{\mathcal{D},0}\big)^{-1}\right] \tag{2.37}$$

is convergent (and equal to $(\mathcal{S}_{\mathcal{D}}^k(s))^{-1}$) as soon as $s\ell_N^{-1}=O(1)$ and the condition (2.8) is satisfied with a sufficiently small constant c. This implies $\||(\mathcal{S}_{\mathcal{D}}^k)^{-1}\||_{H^1(\partial\mathcal{D})\to L^2(\partial\mathcal{D})}=O(s^{-1})$ and the result, recalling that $\mathcal{P}_{N,s}$ is an isometry. ☐

We complete this part by stating a few useful properties which relate the holomorphic expansions (2.22) and (2.26).

Lemma 2.2. *The following identities hold for any $1\le i\le N$, $1\le j\le K_i$ and $\phi\in L^2(\partial D_i)$:*

(i) $\mathcal{K}_{D_i,p}^*[\phi]=\boldsymbol{n}\cdot\nabla_x\mathcal{S}_{D_i,p}[\phi]$ *for any* $p\ge 1$;
(ii) $\int_{\partial B_{ij}}\mathcal{K}_{D_i}^*[\phi]\,\mathrm{d}\sigma=\frac{1}{2}\int_{\partial B_{ij}}\phi\,\mathrm{d}\sigma$;
(iii) $\int_{\partial B_{ij}}\mathcal{K}_{D_i,p}^*[\phi]\,\mathrm{d}\sigma=-\int_{B_{ij}}\mathcal{S}_{D_i,p-2}[\phi]\,\mathrm{d}x$ *for any* $p\ge 2$.

Moreover, the following identities hold for any $1\le i_1\ne i_2\le N$, $1\le j_1\le K_{i_1}$ $\phi\in L^2(\partial D_{i_2})$ and $p\ge 0$:

(iv) $\mathcal{M}_{D_{i_1},D_{i_2}}^{p+1}[\phi](t)=\boldsymbol{n}(t)\otimes\mathcal{T}_{D_{i_1},D_{i_2}}^{p}[\phi](t)$ *and*

$$\int_{\partial B_{i_1j_1}}\mathcal{M}_{D_{i_1},D_{i_2}}^{p+1}[\phi]\,\mathrm{d}\sigma=\int_{B_{i_1j_1}}\partial_l\mathcal{T}_{D_{i_1},D_{i_2}}^{p}[\phi]\otimes e_l\,\mathrm{d}x=\int_{B_{i_1j_1}}\mathrm{I}\otimes\mathcal{T}_{D_{i_1},i_2}^{p-1}[\phi]\,\mathrm{d}x.$$

In particular,

$$\int_{\partial B_{i_1j_1}}\mathcal{M}_{D_{i_1},D_{i_2}}^{1}[\phi]\,\mathrm{d}\sigma=0\text{ and }\int_{\partial B_{i_1j_1}}\mathcal{M}_{D_{i_1},D_{i_2}}^{2}[\phi]\,\mathrm{d}\sigma=\left(\int_{\partial D_{i_2}}\phi\,\mathrm{d}\sigma\right)|B_{i_1j_1}|\mathrm{I}\text{ for any }1\le j_1\le K_{i_1}.$$

Proof. The points (i)–(iii) are classical, see *e.g.* [12,14]. For the point (iv), we use the identity $\partial_l y^p\otimes e_l=py^{p-1}\otimes\mathrm{I}$, which holds in the space of symmetric tensors. ☐

2.4. Uniform norm estimates in a heterogeneous medium

Throughout the paper, we denote by $r>0$ a sufficiently large, fixed positive number such that $D_{N,s}\subset B(0,r)$ for any $N\in\mathbb{N}$ and s satisfying the condition (2.8). The following proposition establishes uniform norm estimates for the single layer potential and its gradient on $\partial D_{N,s}$ or on bounded subdomains of $\mathbb{R}^3$.

Proposition 2.5. *There exists a constant c independent of N,s and $\phi\in L^2(\partial D_{N,s})$ such that*

(i) $\|\mathcal{S}_{D_{N,s}}^k[\phi]\|_{H^1(\partial D_{N,s})}\le cs(1+s\ell_N^{-1})\|\phi\|_{L^2(\partial D_{N,s})}$,
(ii) $\|\mathcal{S}_{D_{N,s}}^k[\phi]\|_{L^2(B(0,r))}\le cN^{\frac{1}{2}}s\|\phi\|_{L^2(\partial D_{N,s})}$,
(iii) $\|\nabla\mathcal{S}_{D_{N,s}}^k[\phi]\|_{L^2(B(0,r))}\le cs^{\frac{1}{2}}(1+s^{\frac{1}{2}}\ell_N^{-\frac{1}{2}})\|\phi\|_{L^2(\partial D_{N,s})}$,
(iv) $\|\mathcal{S}_{D_{N,s}}^{k_b}[\phi]\|_{L^2(D_{N,s})}\le cs^{\frac{3}{2}}(1+s\ell_N^{-1})\|\phi\|_{L^2(\partial D_{N,s})}$,
(v) $\|\nabla\mathcal{S}_{D_{N,s}}^{k_b}[\phi]\|_{L^2(D_{N,s})}\le cs^{\frac{1}{2}}(1+s^{\frac{1}{2}}\ell_N^{-\frac{1}{2}})\|\phi\|_{L^2(\partial D_{N,s})}$.

Furthermore, on any bounded open set A such that $A\cap D_{N,s}=\emptyset$,

(i) $\|\mathcal{S}_{D_{N,s}}^k[\phi]\|_{L^2(A)}\le cN^{\frac{1}{2}}s\|\phi\|_{L^2(\partial D_{N,s})}$,
(ii) $\|\nabla\mathcal{S}_{D_{N,s}}^k[\phi]\|_{L^2(A)}\le cN^{\frac{1}{2}}s\|\phi\|_{L^2(D_{N,s})}$.

Proof. (i) This result is a consequence of the inequalities (2.23) and (2.24) of Proposition 2.3.
(ii) We write for $x\in B(0,r)\backslash\partial D_{N,s}$,

$$\left|\mathcal{S}_{D_{N,s}}^k[\phi](x)\right|=\left|\int_{\partial D_{N,s}}\Gamma^k(x-y)\phi(y)\,\mathrm{d}\sigma(y)\right|\le\left(\int_{\partial D_{N,s}}\left|\Gamma^k(x-y)\right|^2\mathrm{d}\sigma(y)\right)^{\frac{1}{2}}\left(\int_{\partial D_{N,s}}|\phi(y)|^2\,\mathrm{d}\sigma(y)\right)^{\frac{1}{2}}.$$

Computing the square of this expression and integrating over $B(0,r)$, we find

$$\int_{B(0,r)} \left|\mathcal{S}^k_{D_{N,s}}[\phi]\right|^2 \mathrm{d}x \leq \|\phi\|^2_{L^2(\partial D_{N,s})} \int_{B(0,r)} \int_{\partial D_{N,s}} \left|\Gamma^k(x-y)\right|^2 \mathrm{d}x \, \mathrm{d}\sigma(y)$$
$$\leq \|\phi\|^2_{L^2(\partial D_{N,s})} \int_{\partial D_{N,s}} \left(\sup_{y' \in \mathbb{R}^3} \int_{B(0,r)} \left|\Gamma^k(x-y')\right|^2 \mathrm{d}x \right) \mathrm{d}\sigma(y) \leq cNs^2 \|\phi\|^2_{L^2(\partial D_{N,s})},$$

by using (2.20).

(iii) The function $u := \mathcal{S}^k_{D_{N,s}}[\phi]$ is the solution to the following Helmholtz equation:

$$\begin{cases} \Delta u + k^2 u = 0 & \text{in } \mathbb{R}^3 \backslash \partial D_{N,s}, \\ \left[\!\left[\dfrac{\partial u}{\partial n} \right]\!\right] = \phi & \text{on } \partial D_{N,s}, \\ \dfrac{\partial u}{\partial |x|} - \mathrm{i}ku = O\big(|x|^{-1}\big) & \text{as } |x| \to +\infty, \end{cases} \tag{2.38}$$

where $[\![\partial u/\partial n]\!]$ is the jump of the normal derivative accross $\partial D_{N,s}$. Hence, we can evaluate its gradient on $B(0,r)$ thanks to the following integration by parts:

$$0 = \int_{B(0,r)} \big(-|\nabla u|^2 + k^2|u|^2\big) \, \mathrm{d}x + \int_{\partial B(0,r)} \frac{\partial u}{\partial n} \overline{u} \, \mathrm{d}\sigma - \int_{\partial D_{N,s}} \left[\!\left[\frac{\partial u}{\partial n} \right]\!\right] \overline{u} \, \mathrm{d}\sigma.$$

Therefore, we find

$$\|\nabla u\|^2_{L^2(B(0,r))} = \int_{B(0,r)} k^2 |u|^2 \, \mathrm{d}x - \int_{\partial D_{N,s}} \phi \overline{u} \, \mathrm{d}\sigma + \int_{\partial B(0,r)} \frac{\partial u}{\partial n} \overline{u} \, \mathrm{d}\sigma. \tag{2.39}$$

We now evaluate the three terms. The first term can be bounded by cNs^2 according to the point (ii). The third term satisfies the same bound, since the following majorations hold for $y \in \partial B(0,r)$:

$$|u(y)| \leq \int_{\partial D_{N,s}} \left|\Gamma^k(y-y')\phi(y')\right| \mathrm{d}\sigma(y') \leq \int_{\partial D_{N,s}} \sup_{y' \in B(0,r')} \left|\Gamma^k(y-y')\right|^2 \mathrm{d}\sigma(y') \|\phi\|_{L^2(\partial D_{N,s})}$$
$$\leq cN^{\frac{1}{2}} s \|\phi\|_{L^2(\partial D_{N,s})},$$

and

$$|\nabla u(y)| \leq \int_{\partial D_{N,s}} \left|\nabla\Gamma^k(y-y')\phi(y')\right| \mathrm{d}\sigma(y') \leq \int_{\partial D_{N,s}} \sup_{y' \in B(0,r')} \left|\nabla\Gamma^k(y-y')\right|^2 \mathrm{d}\sigma(y') \|\phi\|_{L^2(\partial D_{N,s})}$$
$$\leq cN^{\frac{1}{2}} s \|\phi\|_{L^2(\partial D_{N,s})},$$

where $r' > 0$ is a characteristic size such that we have the strict inclusions $D_{N,s} \Subset B(0,r') \Subset B(0,r)$. Finally, we use point (i) to bound the second term of (2.39):

$$\left| \int_{\partial D_{N,s}} \phi \overline{u} \, \mathrm{d}\sigma \right| \leq \|\phi\|_{L^2(\partial D_{N,s})} \|\mathcal{S}^k_{D_{N,s}}[\phi]\|_{L^2(\partial D_{N,s})} \leq cs\big(1 + s\ell_N^{-1}\big) \|\phi\|^2_{L^2(\partial D_{N,s})}. \tag{2.40}$$

Hence, we have obtained $\|\nabla \mathcal{S}^k[\phi]\|^2_{L^2(B(0,r))} \leq cs(1+sN+s\ell_N^{-1})\|\phi\|^2_{L^2(\partial D_{N,s})}$. The estimate follows because $N = O(\ell_N^{-1})$ in view of (2.1) and assumption (2.19).

(iv) If $x \in y_i + sD_i$, we find the following inequalities:

$$\begin{aligned}
\mathcal{S}_{D_{N,s}}^{k_b}[\phi](x) &= \int_{\partial D_{N,s}} \Gamma^{k_b}(x-y)\phi(y)\,\mathrm{d}\sigma(y) \\
&\leq \sum_{j\neq i}\left(\int_{y_j+s\partial D_j} \left|\Gamma^{k_b}(x-y)\right|^2 \mathrm{d}\sigma(y)\right)^{\frac{1}{2}}\left(\int_{y_j+s\partial D_j} |\phi(y)|^2\,\mathrm{d}\sigma(y)\right)^{\frac{1}{2}} \\
&\quad + \left(\int_{y_i+s\partial D_i} \left|\Gamma^{k_b}(x-y)\right|\mathrm{d}\sigma(y)\right)^{\frac{1}{2}}\left(\int_{y_i+s\partial D_i} \left|\Gamma^{k_b}(x-y)\right| |\phi(y)|^2\,\mathrm{d}\sigma(y)\right)^{\frac{1}{2}} \\
&\leq c\sum_{j\neq i}\frac{s}{|y_i-y_j|}\|\phi\|_{L^2(y_j+s\partial D_j)} + cs^{\frac{1}{2}}\left(\int_{y_i+s\partial D_i}\left|\Gamma^{k_b}(x-y)\right||\phi(y)|^2\,\mathrm{d}\sigma(y)\right)^{\frac{1}{2}} \\
&\leq cs\left(\sum_{j\neq i}\frac{1}{|y_i-y_j|^2}\right)^{\frac{1}{2}}\|\phi\|_{L^2(\partial D_{N,s})} + cs^{\frac{1}{2}}\left(\int_{y_i+s\partial D_i}\left|\Gamma^{k_b}(x-y)\right||\phi(y)|^2\,\mathrm{d}\sigma(y)\right)^{\frac{1}{2}}.
\end{aligned}$$

Computing the square and integrating over $D_{N,s}$ yields then

$$\begin{aligned}
\left\|\mathcal{S}_{D_{N,s}}^{k_b}[\phi]\right\|_{L^2(D_{N,s})}^2 &\leq 2cs^3s^2\ell_N^{-2}\|\phi\|_{L^2(\partial D_{N,s})}^2 + 2cs\sup_{1\leq i\leq N}\sup_{y\in y_i+sD_i}\int_{y_i+sD_i}\left|\Gamma^{k_b}(x-y)\right|\mathrm{d}x\|\phi\|_{L^2(\partial D_{N,s})}^2 \\
&\leq 2cs^3s^2\ell_N^{-2}\|\phi\|_{L^2(\partial D_{N,s})}^2 + 2css^2\sup_{1\leq i\leq N}\sup_{u\in D_i}\int_{D_i}\frac{1}{4\pi|u-t|}\,\mathrm{d}\sigma(t)\|\phi\|_{L^2(D_{N,s})}^2.
\end{aligned}$$

The estimate follows by using the assumption (2.20) to bound the second term of the above inequality.

(v) From the jump identities on the normal derivative of the single layer potential, the function $u = \mathcal{S}_{\mathcal{D}}^{k_b}[\phi]$ solves the following Helmholtz equation in $D_{N,s}$:

$$\begin{cases}
\Delta u + k_b^2 u = 0 & \text{in } D_{N,s}, \\
\left.\dfrac{\partial u}{\partial n}\right|_- = -\dfrac{1}{2}\phi + \mathcal{K}_{D_{N,s}}^{k_b *}[\phi] & \text{on } \partial D_{N,s}.
\end{cases}$$

From the expansion of $\mathcal{K}_{D_{N,s}}^{k_b *}$ of Proposition 2.4, we have the inequality $\|\mathcal{K}_{D_{N,s}}^{k_b *}[\phi]\|_{L^2(\partial D_{N,s})} \leq c\|\phi\|_{L^2(\partial D_{N,s})}$. Therefore, using again an integration by parts with the results of the items (i) and (iv), we find

$$\|\nabla u\|_{L^2(D_{N,s})}^2 = k_b^2\|u\|_{L^2(D_{N,s})}^2 + \int_{\partial D_{N,s}}\frac{\partial u}{\partial n}\overline{u}\,\mathrm{d}\sigma \leq cs^3\left(1+s^2\ell_N^{-2}\right)\|\phi\|_{L^2(\partial D_{N,s})}^2 + cs\left(1+s\ell_N^{-1}\right)\|\phi\|_{L^2(\partial D_{N,s})}^2.$$

(vi) The proof of (ii) is unchanged if $B(0,r)$ is replaced by A.

(vii) In the proof of (iii), the integral on $\partial D_{N,s}$ is not present in (2.39) if $B(0,r)$ is replaced with A, yielding an upper bound proportional to $sN^{\frac{1}{2}}$.

□

3. Homogenization of a sound-soft metamaterial

In this section, we use the holomorphic expansion (2.30) in order to establish a quantitative effective medium theory for the sound-soft material (1.2) in the subcritical regime $sN = O(1)$. Our analysis relies on the following single layer potential representation of the solution:

$$\begin{aligned} u_{N,s} &= u_{in} - \mathcal{S}^k_{D_{N,s}}\left[\left(\mathcal{S}^k_{D_{N,s}}\right)^{-1}[u_{in}]\right] \\ &= u_{in} - \mathcal{S}^k_{D_{N,s}}\left[\mathcal{P}_{N,s}\left(\mathcal{S}^k_{\mathcal{D}}\right)^{-1}\mathcal{P}^{-1}_{N,s}[u_{in}]\right]. \end{aligned} \tag{3.1}$$

We proceed in two parts. In Section 3.1, we reduce, for arbitrary distributions of centers $(y_i)_{1\le i\le N}$, the inversion of the single layer potential $\mathcal{S}^k_{\mathcal{D}}$ in (3.1) to the invertibility of an algebraic linear system. Following [26], we call this algebraic system the Foldy–Lax approximation of (3.1) since the coefficients of the solution determine the far field expansion of $u_{N,s}$. Using the theory of [37], we show the convergence of the algebraic system to an effective integral equation in Section 3.2. Then, we derive in Section 3.3 an effective medium theory in the setting where the packets of resonators $(D_i)_{1\le i\le N}$ are identical and in the subcritical regime $sN = O(1)$ (assumptions (H1)–(H3)). Our main result is given in Proposition 3.4, which states the quantitative convergence of the total field to the solution of a dissipative Lippmann–Schwinger equation.

3.1. The Foldy–Lax approximation for an arbitrary distribution of small obstacles

Formula (3.1) allows to compute asymptotic expansions of $u_{N,s}$ from an asymptotic expansion of $(\mathcal{S}^k_{\mathcal{D}})^{-1}$. From (2.31), it is clear that $s\mathcal{S}_{\mathcal{D},0} + s^2\mathcal{S}^k_{\mathcal{D},1}$ is the leading order term in the series expansion (2.30) of $\mathcal{S}^k_{\mathcal{D}}$. It turns out that this operator has an almost explicit inverse as outlined in the next lemma.

Lemma 3.1. *The operator $\mathcal{S}^k_{\mathcal{D},0} + s\mathcal{S}^k_{\mathcal{D},1} : L^2(\partial\mathcal{D}) \to H^1(\partial\mathcal{D})$ is invertible if and only for any right-hand side $f \equiv (f_i)_{1\le i\le N} \in H^1(\partial\mathcal{D})$, there exists a unique solution $z^N \equiv (z_i^N)_{1\le i\le N}$ to the following N-dimensional linear system:*

$$\left(1 + \frac{\mathrm{i}ks}{4\pi}\mathrm{cap}\,(D_i)\right)z_i^N - \mathrm{cap}\,(D_i)s\sum_{j\ne i}\Gamma^k(y_i - y_j)z_j^N = \int_{\partial D_i}\mathcal{S}^{-1}_{D_i} f_i \,\mathrm{d}\sigma, \qquad 1\le i\le N. \tag{3.2}$$

When it is the case, the solution $\phi \equiv (\phi_i)_{1\le i\le N} \in L^2(\partial\mathcal{D})$ to the problem

$$\left(\mathcal{S}^k_{\mathcal{D},0} + s\mathcal{S}^k_{\mathcal{D},1}\right)[\phi] = f \tag{3.3}$$

is explicitly given by the formula

$$\begin{aligned} \phi_i &= \mathcal{S}^{-1}_{D_i} f_i + \left(\frac{\mathrm{i}ks}{4\pi}z_i^N - s\sum_{j\ne i}\Gamma^k(y_i-y_j)z_j^N\right)\mathcal{S}^{-1}_{D_i}[1_{\partial D_i}] \\ &= \left(\mathcal{S}^{-1}_{D_i} f_i + \frac{1}{\mathrm{cap}\,(D_i)}\left(\int_{\partial D_i}\mathcal{S}^{-1}_{D_i} f_i\,\mathrm{d}\sigma\right)\mathcal{S}^{-1}_{D_i}[1_{\partial D_i}]\right) - \frac{z_i^N}{\mathrm{cap}\,(D_i)}\mathcal{S}^{-1}_{D_i}[1_{\partial D_i}], \quad 1\le i\le N. \end{aligned} \tag{3.4}$$

Proof. Equation (3.3) can be written as

$$\mathcal{S}_{D_i}\phi_i - \frac{\mathrm{i}ks}{4\pi}\int_{\partial D_i}\phi_i\,\mathrm{d}\sigma 1_{\partial D_i} + s\sum_{j\ne i}\Gamma^k(y_i-y_j)\int_{\partial D_j}\phi_j\,\mathrm{d}\sigma 1_{\partial D_i} = f_i, \qquad 1\le i\le N,$$

which is equivalent to

$$\phi_i - \frac{\mathrm{i}ks}{4\pi}\int_{\partial D_i}\phi_i\,\mathrm{d}\sigma\mathcal{S}^{-1}_{D_i}[1_{\partial D_i}] + s\sum_{j\ne i}\Gamma^k(y_i-y_j)\int_{\partial D_j}\phi_j\,\mathrm{d}\sigma\mathcal{S}^{-1}_{D_i}[1_{\partial D_i}] = \mathcal{S}^{-1}_{D_i} f_i. \tag{3.5}$$

Integrating on ∂D_i and denoting $z_i^N = \int_{\partial D_i}\phi_i\,\mathrm{d}\sigma$, we find that (3.3) admits a solution given by (3.4) if and only if (3.2) is invertible. Then, substituting $\int_{\partial D_i}\phi_i\,\mathrm{d}\sigma$ by z_i^N back in (3.5) yields the formula (3.4). □

The invertibility of the linear system (3.4) is not clear for arbitrary distribution of points $(y_i)_{1\leq i\leq N}$ and packets of obstacles $(D_i)_{1\leq i\leq N}$. However, we shall obtain it under the randomness assumption on the centers and the uniformity assumption that the packets of resonators are identical and constituted of K single components $(B_l)_{1\leq l\leq K}$. In the remainder of this section, we therefore assume (H1) and (H2).

In the context of the asymptotic expansion based on the representation (3.1), the right-hand side of (3.3) is given by $f = \mathcal{P}_{N,s}^{-1}[u_{in}]$, whose leading order expansion is given by

$$\begin{aligned}\mathcal{P}_{N,s}^{-1}[u_{in}] &= (u_{in}\circ\tau_{y_i,s})_{1\leq i\leq N}\\ &= (u_{in}(y_i)1_{\partial D_i} + O(s)) = (u_{in}(y_i)1_{\partial D_i})_{1\leq i\leq N} + O_{\|\cdot\|_{H^1(\partial\mathcal{D})}}\left(s^2N^{\frac{1}{2}}\right).\end{aligned}\tag{3.6}$$

Substituting the leading order term into (3.2) and using (H2) yields the following linear system for the coefficients $(z_i^N)_{1\leq i\leq N}$:

$$\left(1+\frac{\mathrm{i}ks}{4\pi}\mathrm{cap}\,(D)\right)z_i^N - \mathrm{cap}\,(D)s\sum_{1\leq j\neq i\leq N}\Gamma^k(y_i-y_j)z_j^N = -\mathrm{cap}\,(D)u_{in}(y_i), \qquad 1\leq i\leq N. \tag{3.7}$$

The finite dimensional system (3.7) turns to be exactly the so-called the Foldy–Lax system associated to the scattering problem (1.2); the obstacles $D_{N,s}$ behave as N point sources with intensity $-sz_i^N$ [26,27], as retrieved in the next proposition.

Proposition 3.1. *The following expansion holds for a fixed $x\in\mathbb{R}^3\backslash\Omega$ away from the resonators:*

$$u_{N,s}(x) - u_{in}(x) = -\sum_{i=1}^N sz_i^N\Gamma^k(x-y_i) + O(s(sN)), \tag{3.8}$$

where (z_i^N) is the solution to the algebraic system (3.7).

Proof. First, for $x\in\mathbb{R}^3\backslash D_{N,s}$ and $\phi\equiv(\phi_i)_{1\leq i\leq N}\in L^2(\partial\mathcal{D})$, a Taylor expansion in (2.35) yields

$$\mathcal{S}_{D_{N,s}}^k[\mathcal{P}_{N,s}[\phi]](x) = \sum_{i=1}^N s^2\left(\Gamma^k(x-y_i) + O\left(\frac{s}{d(x,\partial D_{N,s})^2}\right)\right)\int_{\partial D_i}\phi_i\,\mathrm{d}\sigma. \tag{3.9}$$

Let us then consider the function $\phi\in L^2(\partial\mathcal{D})$ given by

$$\begin{aligned}\phi = \left(\mathcal{S}_{\mathcal{D}}^k\right)^{-1}\mathcal{P}_{N,s}^{-1}[u_{in}] &= \left(\mathcal{S}_{\mathcal{D}}^k\right)^{-1}\left[(u_{in}(y_i)1_{\partial D_i})_{1\leq i\leq N}\right] + O\left(sN^{\frac{1}{2}}\right)_{L^2(\partial\mathcal{D})}\\ &= \phi_0 - \left(\frac{s^{-1}z_i^N}{\mathrm{cap}\,(D_i)}\mathcal{S}_{D_i}^{-1}[1_{\partial D_i}]\right)_{1\leq i\leq N} + O\left(sN^{\frac{1}{2}}\right)_{L^2(\partial\mathcal{D})},\end{aligned}$$

where $(z_i)^N$ is the solution to (3.7) and ϕ_0 a function whose coordinates $\phi_0\equiv(\phi_{0,i})_{1\leq i\leq N})$ satisfy $\int_{\partial D_i}\phi_{0,i}\,\mathrm{d}\sigma = 0$ for $1\leq i\leq N$. Using (3.9), we find that the scattered field is given by

$$\begin{aligned}u_{N,s}(x) - u_{in}(x) &= -\mathcal{S}_{D_{N,s}}^k\left[\mathcal{P}_{N,s}\left(\mathcal{S}_{\mathcal{D}}^k\right)^{-1}\mathcal{P}_{N,s}^{-1}[u_{in}]\right](x) = -\mathcal{S}_{D_{N,s}}^k[\mathcal{P}_{N,s}[\phi]]\\ &= -\sum_{i=1}^N s\left(\Gamma^k(x-y_i) + O(s)\right)z_i^N + O(s(sN)).\end{aligned}$$

The result follows by using the result of Proposition 3.3 belows which states that $\left(\sum_{i=1}^N|z_i^N|^2\right)^{\frac{1}{2}} = O(N^{\frac{1}{2}})$. ∎

Remark 3.1. The Foldy–Lax approximation (3.8) holds even for arbitrary obstacles as soon as the system (3.2) is invertible and well-conditioned. This has been partially obtained in Lemma 2.22 of [26] under the natural condition (2.8) and $\min_{1\leq i\neq j\leq N}\cos(k|y_i-y_j|)>t$ with $t>0$, however the proof features a mistake at the end of page 103 of this reference.

3.2. Convergence of the Foldy–Lax system to an integral equation

As $N\to+\infty$ and under the randomness assumption (H1) of the centers $(y_i)_{1\leq i\leq N}$, it can be expected that the Foldy–Lax system (3.7) can be approximated by the following Lippmann–Schwinger equation:

$$\left(1+\frac{\mathrm{i}ks}{4\pi}\mathrm{cap}\,(D)\right)z(y)-\mathrm{cap}\,(D)sN\int_\Omega\Gamma^k(y-y')z(y')\varrho(y')\,\mathrm{d}y'=-\mathrm{cap}\,(D)u_{in}(y),\qquad y\in\Omega,\tag{3.10}$$

where the discrete Monte-Carlo sum in (3.7) has been replaced with the expectation with respect to the measure $\varrho\,\mathrm{d}x$. The object of this part is to give a precise statement justifying the invertibility of the Foldy–Lax system and its convergence towards (3.10). We rely on the theory of our recent work [37] for this purpose.

To start with, the well-posedness of the Lippmann–Schwinger equation (3.10) is a classical result, see *e.g.* [33, 50]. It follows from the following statement on the spectrum of the volume potential.

Proposition 3.2. *Let us denote by* $\mathcal{V}^{k,\varrho}$ *the volume potential*

$$\begin{aligned}\mathcal{V}^{k,\varrho}:L^2(\Omega)&\to H^2(\Omega)\\ z&\mapsto\textstyle\int_\Omega\Gamma^k(\cdot-y')z(y')\varrho(y')\,\mathrm{d}y'.\end{aligned}\tag{3.11}$$

(i) $\mathcal{V}^{k,\varrho}:L^2(\Omega)\to L^2(\Omega)$ *is a compact operator, hence its essential spectrum is the set* $\{0\}$*;*
(ii) *the point spectrum of* $\mathcal{V}^{k,\varrho}$ *belongs to the negative complex plane* $\mathbb{C}_-=\{\lambda\in\mathbb{C}\,|\,\Im(\lambda)<0\}$:

$$\mathrm{sp}\big(\mathcal{V}^{k,\varrho}\big)\subset\mathbb{C}_-\cup\{0\}.$$

Proof. Points (i) is classical, see [55, 58]. For the point (ii), we prove that $\lambda\mathrm{I}-\mathcal{V}^{k,\varrho}:L^2(\Omega)\to L^2(\Omega)$ is invertible if $\lambda\in\mathbb{C}\backslash\{0\}$ with $\Im(\lambda)\geq0$. From the Fredholm alternative, it is sufficient to show that $\lambda\mathrm{I}-\mathcal{V}^{k,\varrho}$ has a trivial kernel.

Let $\phi\in L^2(\Omega)$ be an element of this kernel; $\mathcal{V}^{k,\varrho}[\phi]=\lambda\phi$. The function $u:=\mathcal{V}^{k,\varrho}[\phi]$ satisfies

$$\begin{cases}\Delta u+k^2u=\varrho\phi 1_\Omega=\dfrac{1}{\lambda}\varrho u1_\Omega & \text{in }\mathbb{R}^3,\\ \partial_{|x|}u-\mathrm{i}ku=O\big(|x|^{-2}\big) & \text{as }|x|\to+\infty.\end{cases}$$

Multiplying by $\overline{u}$ and integrating by parts in the ball $B(0,R)$ for a sufficiently large R yields

$$\int_{B(0,R)}\big(-|\nabla u|^2+k^2|u|^2\big)\,\mathrm{d}x+\int_{\partial B(0,R)}\frac{\partial u}{\partial n}\overline{u}\,\mathrm{d}\sigma=\frac{1}{\lambda}\int_\Omega\varrho|u|^2\,\mathrm{d}x.\tag{3.12}$$

Taking the imaginary part, we find that the radiated flux at infinity is given by

$$\Im\left(\int_{\partial B(0,R)}\frac{\partial u}{\partial n}\overline{u}\,\mathrm{d}\sigma\right)=\Im\left(\frac{1}{\lambda}\right)\int_\Omega\varrho|u|^2\,\mathrm{d}x.$$

Since $\Im(\lambda)\geq0$, it follows that $\Im(\frac{1}{\lambda})\leq0$ and the above flux is non-positive, which entails $u=0$ in $\mathbb{R}^3\backslash\Omega$, and then $u=\lambda\phi=0$ in $\mathbb{R}^3$ by the unique continuation principle [33, 50]. $\square$

In the remainder of this section, we assume (H3): we consider the setting where the quantity sN multiplying the compact operator in (3.10) is bounded. We denote by A_N^k the matrix

$$A_N^k := N^{-1}\big(\Gamma^k(y_i - y_j)\big)_{1\le i\ne j\le N} \tag{3.13}$$

occuring in the system (3.7), and by $|||\cdot|||_2$ is the triple norm defined for a matrix A by

$$|||A|||_2 := \max_{z=(z_i)_{1\le i\le N}} \frac{\|Az\|_2}{\|z\|_2} \text{ where } \|z\|_2 := \left(\sum_{1\le i\le N} |z_i|^2\right)^{\frac{1}{2}}.$$

By using the upper-boundedness with respect to the Frobenius norm and Proposition 2.1, we infer the existence of a constant $c > 0$ independent of N such that

$$|||A_N^k|||_2 \le c, \tag{3.14}$$

almost surely with respect to the distribution of the points $(y_i)_{i\in\mathbb{N}}$.

The theory of [37] yields the convergence of the linear system (3.7) towards the integral equation (3.10).

Proposition 3.3. *Assume (H1)–(H3). There exists an event $\mathcal{H}_{N_0}$, independent of s and δ, which holds with probability $\mathbb{P}(\mathcal{H}_{N_0}) \to 1$ as $N_0 \to +\infty$ such that, conditionally to $\mathcal{H}_{N_0}$ and with a constant $c > 0$ independent of s and N:*

(i) *the linear system* (3.7) *is invertible for $N \ge N_0$, and well-conditioned:*

$$\left|\left|\left|\left(\left(1 + \frac{\mathrm{i}ks}{4\pi}\mathrm{cap}\,(D)\right)\mathrm{I} - sN\mathrm{cap}\,(D)A_N^k\right)^{-1}\right|\right|\right|_2 \le c; \tag{3.15}$$

(ii) *the operator $\mathcal{S}_{\mathcal{D},0} + s\mathcal{S}_{D,1}^k$ is invertible for $N \ge N_0$, and its inverse satisfies*

$$\left|\left|\left|\big(\mathcal{S}_{\mathcal{D},0}^k + s\mathcal{S}_{\mathcal{D},1}^k\big)^{-1}\right|\right|\right|_{H^1(\partial\mathcal{D})\to L^2(\partial\mathcal{D})} \le c. \tag{3.16}$$

Consequently, the inverse of $\mathcal{S}_{\mathcal{D}}^k$ admits the following asymptotic expansion:

$$\big(\mathcal{S}_{\mathcal{D}}^k\big)^{-1} = s^{-1}\big(\mathcal{S}_{\mathcal{D},0} + s\mathcal{S}_{\mathcal{D},1}^k\big)^{-1} + O\big(s^{-1}(s\ell_N^{-1})\eta_N\big);$$

(iii) *the solution (z_i^N) to the linear system* (3.7) *can be approximated by the solution z to the Lippmann–Schwinger equation* (3.10) *in the following mean-square senses:*

(a) $\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^N \big|z_i^N - z(y_i)\big|^2 |\mathcal{H}_{N_0}\right]^{\frac{1}{2}} \le cN^{-\frac{1}{2}} sN\|u_{in}\|_{L^2(\Omega)};$

(b) $\mathbb{E}\Big[\|z^N - z\|_{L^2(\Omega)}^2 |\mathcal{H}_{N_0}\Big]^{\frac{1}{2}} \le cN^{-\frac{1}{2}} sN\|u_{in}\|_{L^2(\Omega)}, \qquad \forall N \ge N_0,$
where z^N the Nystrom interpolating function of the system (3.7) *defined by*

$$z^N(y) := -\frac{\mathrm{cap}\,(D)}{1 + \frac{iks}{4\pi}\mathrm{cap}\,(D)}\left(u_{in}(y) - s\sum_{i=1}^N \Gamma^k(y - y_i) z_i^N\right). \tag{3.17}$$

Proof. (i) Let ω be the open subset

$$\omega := \left\{\lambda \in \mathbb{C} \,|\, \Re(\lambda) > \frac{1}{c\,\mathrm{cap}\,(D)}\right\},$$

where c is the constant of (H3) such that $sN \leq c$. Since $0 \notin \omega$, ω contains only a finite number of eigenvalues of $\mathcal{V}^{k,\varrho}$ with negative imaginary part. Therefore, up to selecting a larger constant c, we can assume that ω is a subset of the resolvent set of $\mathcal{V}^{k,\varrho}$: $\omega \subset \mathbb{C}\backslash \mathrm{sp}(\mathcal{V}^{k,\varrho})$. According to Proposition 2.8 of [37], there exists an event $\mathcal{H}_{N_0}$ satisfying $\mathbb{P}(\mathcal{H}_{N_0}) \to 1$ as $N_0 \to +\infty$ such that ω is also contained in the resolvent set of the matrix A_N^k for $N \geq N_0$. Denote by $\lambda_{N,s}$ the quantity

$$\lambda_{N,s} := \frac{1 + \frac{\mathrm{i}ks}{4\pi}\mathrm{cap}\,(D)}{sN\mathrm{cap}\,(D)}.$$

Since $\lambda_{N,s} \in \omega$, the matrix $\lambda_{N,s}\mathrm{I} - A_N^k$ is invertible for $N \geq N_0$, which is equivalent to the invertibility of (3.7). Furthermore, since ω is included in the resolvent set of A_N^k, the distance of $1/(sN\mathrm{cap}\,(D))$ to the spectrum of A_N^k is bounded from below:

$$d\left(\frac{1}{sN\mathrm{cap}\,(D)}, \mathrm{sp}(A_N^k)\right) \geq c',$$

for a constant $c' > 0$ independent of s and N. By using the inequality of Proposition C.1 in the appendix, this implies the existence of a constant $c'' > 0$ such that

$$|||\left(\lambda_{s,N}\mathrm{I} - A_N^k\right)^{-1}|||_2 \leq c'',$$

from where (3.15) follows easily.

(ii) The invertibility of $\mathcal{S}_{\mathcal{D},0} + s\mathcal{S}_{\mathcal{D},1}^k$ and the conditioning is obtained from Lemma 3.1 and the formula (3.4). The expansion for the inverse follows from computing the Neumann series of (2.30) with the estimates of (2.31).

(iii) These bounds follow from the previous points and by applying respectively the Corollary 3.2 and the Proposition 3.6 of [37].

□

3.3. Effective medium theory for sound-soft metametarials up to the critical regime

In this final subsection, we use the result of Proposition 3.3 to obtain that the solution $u_{N,s}$ to (1.2), represented by (3.1), can be approximated by $u_{N,s}(x) \simeq -\mathrm{cap}\,(D)z(x)$ where z is the solution to the integral equation (3.10). From there, an few additional steps yield the following homogenization theorem.

Proposition 3.4. *Assume (H1)–(H3) and denote by u the solution to the Lippmann–Schwinger equation*

$$\begin{cases} \Delta u + \left(k^2 - sN\mathrm{cap}\,(D)\varrho 1_\Omega\right)u = 0 & \text{in } \mathbb{R}^3, \\ \left(\dfrac{\partial}{\partial |x|} - \mathrm{i}k\right)(u - u_{in}) = O\left(|x|^{-2}\right) & \text{as } |x| \to +\infty. \end{cases} \tag{3.18}$$

There exists an event $\mathcal{H}_{N_0}$, independent of s and δ, which holds with large probability $\mathbb{P}(\mathcal{H}_{N_0}) \to 1$ as $N_0 \to +\infty$ such that when $\mathcal{H}_{N_0}$ is realized, the function u is an approximation of the solution field $u_{N,s}$ to (1.2) with the following error estimates:

(i) *on any ball $B(0,r)$ containing the obstacles, $\Omega \subset B(0,r)$, there exists a constant $c > 0$ independent of s and N such that for any $N \geq N_0$:*

$$\mathbb{E}\left[\|u_{N,s} - u\|^2_{L^2(B(0,r))}|\mathcal{H}_{N_0}\right]^{\frac{1}{2}} \leq csN\max\left((sN)^2N^{-\frac{1}{3}}, N^{-\frac{1}{2}}\right); \tag{3.19}$$

(ii) *on any bounded open subset $A \subset \mathbb{R}^3 \backslash \Omega$ away from the obstacles, there exists a constant $c > 0$ independent of s and N such that for any $N \geq N_0$:*

$$\mathbb{E}\Big[\|\nabla u_{N,s} - \nabla u\|^2_{L^2(A)} | \mathcal{H}_{N_0}\Big]^{\frac{1}{2}} \leq csN \max\Big((sN)^2 N^{-\frac{1}{3}}, N^{-\frac{1}{2}}\Big). \tag{3.20}$$

The relative error is of order $O(\max((sN)^2N^{-\frac{1}{3}}, N^{-\frac{1}{2}}))$ because the scattered fields $u_{N,s} - u_{in}$ and $u - u_{in}$ are of order $O(sN)$.

Remark 3.2. It can be shown by adapting the arguments below that the effective medium is not changed at first order if the resonators are rotated according to a rotation field $y \to R(y)$, but we keep this setting for simplicity.

Remark 3.3. The convergence rate $\max((sN)^2N^{-\frac{1}{3}}, N^{-\frac{1}{2}})$ is a competition of two terms: $N^{-\frac{1}{2}}$ is a natural rate associated to the convergence of Monte-Carlo estimators, while $(sN)^2N^{-1/3} = (sN)s\epsilon_N^{-1}$ brings into play the ratio $\eta_N \equiv s/\epsilon_N$ between the size of a resonator and the minimum distance between the centers $(y_i)_{1\leq i\leq N}$.

The remaining part of the section is dedicated to the proof of this proposition, which is based on the representation formula (3.1) and is divided in several steps. First, we establish an asymptotic expansion based on the asymptotic expansion of layer potentials obtained in Section 2.3 which involves the solution $(z_i^N)_{1\leq i\leq N}$ to the Foldy–Lax system (3.7).

Lemma 3.2. *The following approximation holds for the scattered field:*

$$\begin{aligned} u_{N,s} &= u_{in} - \mathcal{S}^k_{D_{N,s}}\Big[\mathcal{P}_{N,s}\big(\mathcal{S}^k_{\mathcal{D}}\big)^{-1}\mathcal{P}^{-1}_{N,s}[u_{in}]\Big] \\ &= u_{in} + \frac{1}{\operatorname{cap}(D)}\int_{\partial D}\left(\frac{1+\frac{iks}{4\pi}\operatorname{cap}(D)}{\operatorname{cap}(D)} z^N(\cdot - st) + u_{in}(\cdot - st)\right)\mathcal{S}_D^{-1}[1_{\partial D}](t)\,\mathrm{d}\sigma(t) \\ &\quad + O\big(\max(s\ell_N^{-1}\eta_N, s)sN\big)_{L^2(B(0,r))}, \end{aligned} \tag{3.21}$$

recalling the definition (3.17) for the Nystrom interpolating function z^N. This approximation is also valid by replacing the norm of $L^2(B(0,r))$ with the one of $H^1(A)$ for A any bounded open set outside the obstacles ($A \subset \mathbb{R}^3\backslash\Omega$).

Proof. We assume that the highly probable events $\mathcal{H}^{(1)}_{t_1}$ and $\mathcal{H}_{N_0}$ are realized. First, the conditioning bound (3.16) enables to compute the asymptotic expansion of the inverse of the single layer potential: the expansion (2.30) can be rewritten as

$$\mathcal{S}^k_{\mathcal{D}} = s\big(\mathcal{S}_{\mathcal{D},0} + s\mathcal{S}^k_{\mathcal{D},1}\big)\left(\mathrm{I} + \big(\mathcal{S}_{\mathcal{D},0} + s\mathcal{S}^k_{\mathcal{D},1}\big)^{-1}\sum_{p=2}^{+\infty} s^p \mathcal{S}^k_{\mathcal{D},p}\right) = s\big(\mathcal{S}_{\mathcal{D},0} + s\mathcal{S}^k_{\mathcal{D},1}\big)\big(\mathrm{I} + O\big(s\ell_N^{-1}\eta_N\big)\big),$$

where $s\ell_N^{-1}$ is bounded under the event $\mathcal{H}^{(1)}_{t_1}$. Using a Neumann-Series, we infer

$$\big(\mathcal{S}^k_{\mathcal{D}}\big)^{-1} = s^{-1}\big(\mathcal{S}_{\mathcal{D},0} + s\mathcal{S}^k_{\mathcal{D},1}\big)^{-1}\big(\mathrm{I} + O\big(s\ell_N^{-1}\eta_N\big)\big) = s^{-1}\big(\mathcal{S}_{\mathcal{D},0} + s\mathcal{S}^k_{\mathcal{D},1}\big)^{-1} + O\big(s^{-1}s\ell_N^{-1}\eta_N\big).$$

Inserting now (3.6) in this expansion, we obtain

$$\begin{aligned} \big(\mathcal{S}^k_{\mathcal{D}}\big)^{-1}\mathcal{P}^{-1}_{N,s}[u_{in}] &= \big(\mathcal{S}^k_{\mathcal{D}}\big)^{-1}\Big[(u_{in}(y_i)1_{\partial D_i})_{1\leq i\leq N}\Big] + O\Big(sN^{\frac{1}{2}}\Big)_{L^2(\partial\mathcal{D})} \\ &= s^{-1}\big(\mathcal{S}_{\mathcal{D},0} + s\mathcal{S}^k_{\mathcal{D},1}\big)^{-1}\Big[(u_{in}(y_i)1_{\partial D_i})_{1\leq i\leq N}\Big] + O\Big(s^{-1}s\ell_N^{-1}\eta_N sN^{\frac{1}{2}}\Big)_{L^2(\partial\mathcal{D})} + O\Big(sN^{\frac{1}{2}}\Big)_{L^2(\partial\mathcal{D})}. \end{aligned}$$

Using the expression (3.4) of the inverse $(\mathcal{S}^k_{\mathcal{D},0}+\mathcal{S}^k_{\mathcal{D},1})^{-1}$, we arrive at

$$\begin{aligned}\left(\mathcal{S}^k_{\mathcal{D}}\right)^{-1}\mathcal{P}^{-1}_{N,s}[u_{in}] &= s^{-1}\left(u_{in}(y_i)\mathcal{S}_D^{-1}[1_{\partial D}] - \frac{\mathrm{cap}\,(D)}{\mathrm{cap}\,(D)}u_{in}(y_i)\mathcal{S}_D^{-1}[1_{\partial D}] - \frac{z_i^N}{\mathrm{cap}\,(D)}\mathcal{S}_D^{-1}[1_{\partial D}]\right)_{1\le i\le N}\\ &\quad + O\Big(\max\big(s\ell_N^{-1}\eta_N, s\big)s^{-1}sN^{\frac12}\Big)_{1\le i\le N}\\ &= -s^{-1}\left(\frac{z_i^N}{\mathrm{cap}\,(D)}\mathcal{S}_D^{-1}[1_{\partial D}]\right)_{1\le i\le N} + O\Big(\max\big(s\ell_N^{-1}\eta_N, s\big)s^{-1}sN^{\frac12}\Big)_{1\le i\le N}.\end{aligned}$$

Using the bound (ii) of Proposition 2.5, the fact that $\mathcal{P}_{N,s}$ is an isometry, and formula (2.35), we obtain

$$\begin{aligned}u_{N,s} &= u_{in} - \mathcal{S}^k_{D_{N,s}}\left[\mathcal{P}_{N,s}\left(\mathcal{S}^k_{\mathcal{D}}\right)^{-1}\mathcal{P}^{-1}_{N,s}[u_{in}]\right]\\ &= u_{in} + \frac{1}{\mathrm{cap}\,(D)}\sum_{i=1}^N s z_i^N\int_{\partial D}\Gamma^k(\cdot - y_i - st)\mathcal{S}_D^{-1}[1_{\partial D}](t)\,\mathrm{d}\sigma(t) + O\big(\max\big(s\ell_N^{-1}\eta_N, s\big)sN\big)_{L^2(B(0,r))},\end{aligned}$$

which leads to the the approximation formula (3.21) after using (3.17). Note that in virtue of the points (vi) and (vii) of Proposition 2.5, the same estimate is also valid by replacing the norm of $L^2(B(0,r))$ with the one of $H^1(A)$ for A any bounded open set outside the obstacles ($A\subset\mathbb{R}^3\backslash\Omega$). $\square$

Taking the limit $s\to 0$ and $N\to+\infty$ in the above expression, we expect the convergence

$$u_{N,s}\simeq u_{in} + \frac{1}{\mathrm{cap}\,(D)}\left(\int_{\partial D}\mathcal{S}_D^{-1}[1_{\partial D}]\,\mathrm{d}\sigma\right)\left(\frac{1}{\mathrm{cap}\,(D)}z + u_{in}\right)\simeq -\frac{1}{\mathrm{cap}\,(D)}z \text{ as } s\to 0 \text{ and } N\to+\infty, \tag{3.22}$$

which would imply the result of Proposition 3.4 because $-z/\mathrm{cap}\,(D)$ is (up to an error of order $O(s)$) the solution to (3.18). This asymptotic behavior is justified by the next three technical lemmas. The first one is an improvement of the point (iii)b of Proposition 3.3.

Lemma 3.3. *For any $r>0$, there exists a constant $c>0$ independent of N and s such that*

$$\mathbb{E}\left[\left\|z^N - z\right\|^2_{L^2(B(0,r))}\right]^{\frac12}\le csNN^{-\frac12}. \tag{3.23}$$

For any bounded open set $A\subset\mathbb{R}^3\backslash\Omega$ which does not contain the obstacles, it also holds

$$\mathbb{E}\left[\left\|\nabla z^N - \nabla z\right\|^2_{L^2(A)}\right]^{\frac12}\le csNN^{-\frac12}. \tag{3.24}$$

Proof. Let r_N be the discrepancy

$$r_N(y) := \frac{\mathrm{cap}\,(D)sN}{1+\frac{iks}{4\pi}\mathrm{cap}\,(D)}\left(\frac1N\sum_{i=1}^N\Gamma^k(y-y_i)z(y_i) - \int_\Omega\Gamma^k(y-y')z(y')\varrho(y')\,\mathrm{d}y'\right).$$

By the law of large number and using the fact that $\Gamma^k\in L^2(B(0,r))$ and $\nabla\Gamma^k\in L^2(A)$, we have the estimates

$$\mathbb{E}\left[\|r_N\|^2_{L^2(B(0,r))}\right]^{\frac12}\le csNN^{-\frac12},\qquad \mathbb{E}\left[\|\nabla r_N\|^2_{L^2(A)}\right]^{\frac12}\le csNN^{-\frac12}. \tag{3.25}$$

Then, subtracting (3.7) from (3.10) yields

$$z^N(y)-z(y) = \frac{\mathrm{cap}\,(D)sN}{1+\frac{iks}{4\pi}\mathrm{cap}\,(D)}\left(\frac1N\sum_{i=1}^N\Gamma^k(y-y_i)(z_{i,N}-z(y_i))\right) + r_N(y).$$

Hence, equation (3.25) and the point (iii)a of Proposition 3.3 imply the bound

$$\mathbb{E}\Big[\big\|z^N - z\big\|^2_{L^2(B(0,r))}\Big]^{\frac{1}{2}} \leq csN\left(\frac{1}{N}\sum_{i=1}^N \mathbb{E}\Big[\big\|\Gamma^k(\cdot - y_i)\big\|^2_{L^2(B(0,r))}\Big]\right)^{\frac{1}{2}}\left(\frac{1}{N}\sum_{i=1}^N \mathbb{E}\Big[\big|z_{i,N} - z(y_i)\big|^2\Big]\right)^{\frac{1}{2}}$$
$$+\,\mathbb{E}\Big[\|r_N\|^2_{L^2(B(0,r))}\Big]^{\frac{1}{2}} \leq 2csNN^{-\frac{1}{2}},$$

from where (3.23) follows. The result of (3.24) is obtained similarly. ☐

We then need uniform estimates of convolution integrals of the form $\int_{\partial D} v(\cdot - st)\phi(t)\,\mathrm{d}\sigma(t)$, which occur in (3.21).

Lemma 3.4. *Let $A \subset A' \subset \mathbb{R}^3$ be bounded open subsets such that*

$$A - st := \{x - st \,|\, x \in A\} \subset A' \tag{3.26}$$

for any $t \in \partial D$ and s sufficiently small. The following uniform bound holds for any $\phi \in L^2(\partial D)$ and $v \in L^2(A')$:

$$\left\|\int_{\partial D} v(\cdot - st)\phi(t)\,\mathrm{d}\sigma(t)\right\|_{L^2(A)} \leq |\partial D|\,\|v\|_{L^2(A')}\|\phi\|_{L^2(\partial D)}. \tag{3.27}$$

Proof. This is a direct consequence of the Cauchy–Schwartz inequality:

$$\int_A\left|\int_{\partial D} v(x - st)\phi(t)\,\mathrm{d}\sigma(t)\right|^2\mathrm{d}x \leq \int_{\partial D}\|v\|^2_{L^2(A-st)}\,\mathrm{d}\sigma(t)\|\phi\|_{L^2(\partial D)} \leq |\partial D|\,\|v\|_{L^2(A')}\|\phi\|_{L^2(\partial D)}.$$

☐

Our third lemma establishes the convergence of $\int_{\partial D} v(\cdot - st)\phi(t)\,\mathrm{d}\sigma(t)$ to the function $\big(\int_{\partial D}\phi\,\mathrm{d}\sigma\big)v$ as $s \to 0$.

Lemma 3.5. *Let $A \subset A' \subset \mathbb{R}^3$ be bounded open subsets satisfying (3.27). The following convergence holds for any $v \in H^1(A')$:*

$$\left\|\int_{\partial D} v(\cdot - st)\phi(t)\,\mathrm{d}\sigma(t) - \left(\int_{\partial D}\phi\,\mathrm{d}\sigma\right)v\right\|_{L^2(A)} \leq cs\|\nabla v\|_{L^2(A')}. \tag{3.28}$$

Proof. The following inequality holds for any $x \in A$,

$$\left|\int_{\partial D}(v(x - st) - v(x))\phi(t)\,\mathrm{d}\sigma(t)\right|^2 = \left|-\int_{\partial D}\int_0^s \nabla v(x - ut)\cdot t\phi(t)\,\mathrm{d}u\,\mathrm{d}\sigma(t)\right|^2$$
$$\leq \int_{\partial D}\left(\int_0^s \nabla v(x - ut)\cdot t\,\mathrm{d}u\right)^2\mathrm{d}\sigma(t)\|\phi\|^2_{L^2(\partial D)} \leq \int_{\partial D}\int_0^s|\nabla v(x - ut)|^2\,\mathrm{d}u\big(s|t|^2\big)\,\mathrm{d}\sigma(t)\|\phi\|^2_{L^2(\partial D)}.$$

Integrating on A, we therefore obtain

$$\left\|\int_{\partial D} v(\cdot - st)\phi(t)\,\mathrm{d}\sigma(t) - \left(\int_{\partial D}\phi\,\mathrm{d}\sigma\right)v\right\|^2_{L^2(A)} \leq \sup_{t\in\partial D, 0\leq u\leq s}\|\nabla v\|^2_{L^2(A-ut)}s^2\int_{\partial D}|t|^2\,\mathrm{d}\sigma(t)\|\phi\|_{L^2(\partial D)}.$$

The result follows because the supremum can be bounded by $\|\nabla v\|^2_{L^2(A')}$. ☐

Proof of Proposition 3.4. We start by proving the convergence (3.22). Consider $0 < r < r'$ such that $\Omega \subset B(0,r) \subset B(0,r')$. Using (3.23) and (3.27) with $A \equiv B(0,r)$, $A' \equiv B(0,r')$, $v \equiv z - z^N$, we can write the asymptotic expansion

$$\int_{\partial D} z^N(\cdot - st)\mathcal{S}_D^{-1}[1_{\partial D}](t)\,\mathrm{d}\sigma(t) = \int_{\partial D} z(\cdot - st)\mathcal{S}_D^{-1}[1_{\partial D}](t)\,\mathrm{d}\sigma(t) + O\Big(sNN^{-\frac{1}{2}}\Big)_{\mathbb{E}\left[\|\cdot\|^2_{L^2(B(0,r))}\right]^{\frac{1}{2}}}.$$

Then, the result of Lemma 3.5 yields

$$\begin{aligned}\int_{\partial D}&\left(\frac{1+\frac{iks}{4\pi}\mathrm{cap}\,(D)}{\mathrm{cap}\,(D)}z(\cdot - st) + u_{in}(\cdot - st)\right)\mathcal{S}_D^{-1}[1_{\partial D}](t)\,\mathrm{d}\sigma(t)\\ &= -\mathrm{cap}\,(D)\frac{1+\frac{iks}{4\pi}\mathrm{cap}\,(D)}{\mathrm{cap}\,(D)}z - \mathrm{cap}\,(D)u_{in} + O_{L^2(B(0,r))}(s).\end{aligned} \tag{3.29}$$

Coming back to (3.21) and remembering that $\eta_N = s\epsilon_N^{-1} = O(sN^{\frac{2}{3}})$, we obtain the following asymptotic expansion for $u_{N,s}$:

$$\begin{aligned}u_{N,s} &= -\frac{1}{\mathrm{cap}\,(D)}z + O\Big(\max\Big(sNsN^{\frac{2}{3}}, s, N^{-\frac{1}{2}}, N^{-1}\Big)sN\Big)_{\mathbb{E}\left[\|\cdot\|^2_{L^2(B(0,r))}\right]^{\frac{1}{2}}}\\ &= -\frac{1}{\mathrm{cap}\,(D)}z + O\Big(\max\Big((sN)^2N^{-\frac{1}{3}}, N^{-\frac{1}{2}}\Big)sN\Big)_{\mathbb{E}\left[\|\cdot\|^2_{L^2(B(0,r))}\right]^{\frac{1}{2}}}.\end{aligned} \tag{3.30}$$

Finally, it remains to observe that

$$-\frac{z}{\mathrm{cap}\,(D)} = u + O_{L^2(B(0,r))}(s),$$

where u is the solution to the exterior problem (3.18). The convergence estimate (3.19) follows. The same proof applies by replacing $B(0,r)$ with a subset $A \subset \mathbb{R}^3\backslash\Omega$ and $\nabla u_{N,s}$ instead of $u_{N,s}$, yielding the convergence estimate (3.24). ☐

Remark 3.4. The above arguments cannot be easily adapted to treat the "supercritical" regime $sN \to +\infty$ due to several deep mathematical reasons. First, one cannot expect a better estimate than (3.16) for the inverse of $\mathcal{S}^k_{\mathcal{D},0} + s\mathcal{S}^k_{\mathcal{D},1}$. Then the geometric series associated to the inverse of (2.30) would feature remainder terms of order $O(\eta_N^p)$ which are of order $O(1)$ when $s \sim \kappa\epsilon_N$ even for a small constant κ.

4. Homogenization of a high-contrast acoustic metamaterial

In this section, we consider the scattering problem (1.3) in the high-contrast medium. We apply the same method as in the previous sections to show the convergence to an effective medium and to obtain quantitative convergence estimates. The main difference with the previous section lies in the need to account for resonances.

Our analysis is again divided into three parts. In Section 4.1, we reduce the scattering problem (1.3) to an integral equation of the form

$$\mathcal{A}(s,\delta)[\Phi] = F, \tag{4.1}$$

where $\mathcal{A}$ is an integral operator holomorphic in s and δ over $L^2(\partial\mathcal{D}) \times L^2(\partial\mathcal{D})$. Following the analysis of our previous work [38], we reduce (4.1) to a M dimensional linear system

$$A(s,\delta)\boldsymbol{x} = \boldsymbol{F}, \tag{4.2}$$

which is singular at exactly $2M$ complex resonant values of the characteristic size s. We call the system (2.31) the Foldy–Lax approximation of (1.3) because the solution $\boldsymbol{x}$ determines the far field expansion of $u_{N,s}$ similarly as in the Proposition 4.3 for the dissipative medium of (1.2).

In Section 4.2, we assume (H1)–(H4) and we show that the solution $\boldsymbol{x}^N$ of the algebraic system (4.2) can be approximated in terms of the solution of a limit integral equation. Finally, these results are used to establish in Section 4.3 a quantitative effective medium theory and the homogenization result of Proposition 4.8.

4.1. The Foldy–Lax system for an arbitrary system of many tiny resonators

Following [10,11], the solution $u_{N,s}$ of (1.3) can be represented as

$$u_{N,s}(x) = \begin{cases} \mathcal{S}^{k_b}_{D_{N,s}}[\varphi](x) & \text{if } x \in D_{N,s}, \\ u_{in}(x) + \mathcal{S}^{k}_{D_{N,s}}[\psi](x) & \text{if } x \in \mathbb{R}^3 \backslash D_{N,s}, \end{cases} \tag{4.3}$$

where the functions $(\varphi,\psi) \in L^2(\partial D_{N,s}) \times L^2(\partial D_{N,s})$ solve the system of integral equations

$$\begin{cases} \mathcal{S}^{k_b}_{D_{N,s}}[\varphi] - \mathcal{S}^{k}_{D_{N,s}}[\psi] = u_{in}, \\ \left(-\dfrac{1}{2}\mathrm{I} + \mathcal{K}^{k_b*}_{D_{N,s}}\right)[\varphi] - \delta\left(\dfrac{1}{2}\mathrm{I} + \mathcal{K}^{k*}_{D_{N,s}}\right)[\psi] = \delta\dfrac{\partial u_{in}}{\partial n}. \end{cases} \tag{4.4}$$

Using the factorization (2.29), we can rewrite (4.4) as an equation for $(\mathcal{P}^{-1}_{N,s}[\varphi], \mathcal{P}^{-1}_{N,s}[\psi]) \in L^2(\partial\mathcal{D}) \times L^2(\partial\mathcal{D})$ in terms of the holomorphic operators $\mathcal{S}^{k}_{\mathcal{D}}(s)$, $\mathcal{S}^{k_b}_{\mathcal{D}}(s)$, $\mathcal{K}^{k*}_{\mathcal{D}}(s)$ and $\mathcal{K}^{k_b*}_{\mathcal{D}}(s)$:

$$\mathcal{A}(s,\delta)\begin{bmatrix} \mathcal{P}^{-1}_{N,s}[\varphi] \\ \mathcal{P}^{-1}_{N,s}[\psi] \end{bmatrix} = \begin{bmatrix} \mathcal{P}^{-1}_{N,s}[u_{in}] \\ \mathcal{P}^{-1}_{N,s}[\delta\frac{\partial u_{in}}{\partial n}], \end{bmatrix} \tag{4.5}$$

where the operator $\mathcal{A}(s,\delta)$ is given by

$$\mathcal{A}(s,\delta) = \begin{bmatrix} \mathcal{S}^{k_b}_{\mathcal{D}}(s) & -\mathcal{S}^{k}_{\mathcal{D}}(s) \\ -\frac{1}{2}\mathrm{I} + \mathcal{K}^{k_b*}_{\mathcal{D}}(s) & -\delta\left(\frac{1}{2}\mathrm{I} + \mathcal{K}^{k*}_{\mathcal{D}}(s)\right) \end{bmatrix}. \tag{4.6}$$

Equation (4.6) has the exact same structure as the resonance problem studied for resonators with fixed size in [38], where the parameter s plays here the role of the subwavelength frequency ω considered in [38], and where the operators act on the cartesian product space $L^2(\partial\mathcal{D})$ (defined in Sect. 2.2) with $\mathcal{D} = \partial D_1 \times \partial D_2 \times \ldots \partial D_N$. The same analysis would yield $M = \sum_{i=1}^{N} K_i$ pairs of complex resonant sizes $s^{\pm}_{k}(\delta)$ of order $O(\delta^{\frac{1}{2}})$, defined as the poles of $\mathcal{A}(s,\delta)^{-1}$ or equivalently the values of s for which $\mathcal{A}(s,\delta)$ has a non-trivial kernel. In what follows, we follow the steps of [38] to analyze the invertibility of (4.6).

By using a Schur complement, the integral equation (4.5) can be rewritten as the following system of two equations for $(\mathcal{P}^{-1}_{N,s}[\varphi], \mathcal{P}^{-1}_{N,s}[\psi])$:

$$\begin{cases} \mathcal{P}^{-1}_{N,s}[\psi] = \left(\mathcal{S}^{k}_{\mathcal{D}}\right)^{-1}\mathcal{S}^{k_b}_{\mathcal{D}}\mathcal{P}^{-1}_{N,s}[\varphi] - \left(\mathcal{S}^{k}_{\mathcal{D}}\right)^{-1}\mathcal{P}^{-1}_{N,s}[u_{in}], \\ \left[-\dfrac{1}{2}\mathrm{I} + \mathcal{K}^{k_b*}_{\mathcal{D}} - \delta\left(\dfrac{1}{2}\mathrm{I} + \mathcal{K}^{k*}_{\mathcal{D}}\right)\left(\mathcal{S}^{k}_{\mathcal{D}}\right)^{-1}\mathcal{S}^{k_b}_{\mathcal{D}}\right]\mathcal{P}^{-1}_{N,s}[\varphi] = \delta\mathcal{P}^{-1}_{N,s}\left[\dfrac{\partial u_{in}}{\partial n}\right] - \delta\left(\dfrac{1}{2}\mathrm{I} + \mathcal{K}^{k*}_{\mathcal{D}}\right)\left(\mathcal{S}^{k}_{\mathcal{D}}\right)^{-1}\mathcal{P}^{-1}_{N,s}[u_{in}]. \end{cases} \tag{4.7}$$

Therefore, the invertibility of $\mathcal{A}(s,\delta)$ is equivalent to that of the operator

$$\mathcal{L}(s,\delta) := -\frac{1}{2}\mathrm{I} + \mathcal{K}^{k_b*}_{\mathcal{D}}(s) - \delta\left(\frac{1}{2}\mathrm{I} + \mathcal{K}^{k*}_{\mathcal{D}}(s)\right)\left(\mathcal{S}^{k}_{\mathcal{D}}(s)\right)^{-1}\mathcal{S}^{k_b}_{\mathcal{D}}(s).$$

Using the asymptotic expansions (2.30) and (2.32), $\mathcal{L}(s,\delta)$ can be rewritten as

$$\mathcal{L}(s,\delta) = -\frac{1}{2}\mathrm{I} + \mathcal{K}^{*}_{\mathcal{D}} + s^2\mathcal{B}_1(s) + \delta\mathcal{B}_2(s) \tag{4.8}$$

where $\mathcal{B}_1(s)$ and $\mathcal{B}_2(s)$ are the holomorphic and compact operators defined by

$$s^2\mathcal{B}_1(s) := \sum_{p=2}^{+\infty} s^p\mathcal{K}^{k_b*}_{\mathcal{D},p}, \quad \mathcal{B}_2(s) := -\left(\frac{1}{2}\mathrm{I} + \mathcal{K}^{k*}_{D}(s)\right)\left(\mathcal{S}^{k}_{\mathcal{D}}(s)\right)^{-1}\mathcal{S}^{k_b}_{\mathcal{D}}(s). \tag{4.9}$$

For the derivation of quantitative error bounds for the homogenization of (1.3) in the regime $s \to 0$ and $N \to +\infty$, it is important to derive estimates uniform in s and N. Anticipating the analysis of Section 4.3 which considers the subregime of (H4) whereby sN is smaller than the deviation to the resonance, we assume in the remainder of this section that

$$sN \to 0 \text{ as } s \to 0 \text{ and } N \to +\infty. \tag{4.10}$$

Remark 4.1. We note that, if $sN \sim \Lambda$ for some constant $\Lambda > 0$, there can be resonance phenomena due to the interactions between the resonators independently of the value of the contrast parameter δ. For instance when $K = 1$ and $\delta = 0$, replacing formally (4.38) below with its continuous limit leads us to expect a resonance when there is a non-zero solution ϕ to

$$\operatorname{cap}(D)^{-1}\phi(y) - sN\int_\Omega \varrho(y')\Gamma^{k_b}(y-y')\phi(y')\,\mathrm{d}y' = 0,$$

i.e. when sN is close to $1/(\operatorname{cap}(D)\mu)$ for μ an eigenvalue of the volume potential $\mathcal{V}^{k_b,\varrho}$.

Under the assumption (4.10), we have the following operator norm estimates for $\mathcal{B}_1(s)$ and $\mathcal{B}_2(s)$.

Lemma 4.1. *The following norm estimates hold for the operators $s^2\mathcal{B}_1(s)$ and $\delta\mathcal{B}_2(s)$ independently of s and N:*

$$|||s^2\mathcal{B}_1(s)|||_{L^2(\partial\mathcal{D})\to L^2(\partial\mathcal{D})} = O\big(s\ell_N^{-1}\eta_N\big),\ |||\delta\mathcal{B}_2(s)|||_{L^2(\partial\mathcal{D})\to L^2(\partial\mathcal{D})} = O(\delta). \tag{4.11}$$

Furthermore, there exists a constant $c > 0$ independent of s and N such that the following inequality holds for any $\phi \in L^2(\partial\mathcal{D})$:

$$s\left(\sum_{\substack{1\le i_1\le N\\ 1\le j_1\le K_{i_1}}}\left|\int_{\partial B_{i_1j_1}}\big((s^2\mathcal{B}_1(s)+\delta\mathcal{B}_2(s))[\phi]\big)_{i_1}\,\mathrm{d}\sigma\right|^2\right)^{\frac{1}{2}} \le c\big(s^2+\delta\big)\|\phi\|_{L^2(\partial\mathcal{D})}. \tag{4.12}$$

Proof. The estimate (4.11) results from (2.31) and (2.34). Let us prove (4.12). Since $|||\mathcal{B}_2(s)||| = O(1)$, it is clear from the Cauchy–Schwarz inequality and the definition (2.12) of the $L^2(\partial\mathcal{D})$ norm that it is sufficient to prove only the bound on the term involving $\mathcal{B}_1(s)$. We have, due to Lemma 2.2 for a given $1 \le i_1 \le N$ and $1 \le j_1 \le K_{i_1}$:

$$\begin{aligned}
&\left|\int_{\partial B_{i_1j_1}} s^2(\mathcal{B}_1(s)[\phi])_{i_1}\,\mathrm{d}\sigma\right|\\
&\quad= \left|\sum_{p=2}^{+\infty} s^p\int_{\partial B_{i_1j_1}} k_b^p\mathcal{K}^*_{D_{i_1},p}[\phi_{i_1}]\,\mathrm{d}\sigma + \sum_{1\le i_2\ne i_1\le N}\sum_{p=2}^{+\infty} s^p\nabla^{p-1}\Gamma^{k_b}(y_{i_1}-y_{i_2})\cdot\int_{B_{i_1j_1}} \mathrm{I}\otimes\mathcal{T}^{p-3}_{D_{i_1},D_{i_2}}[\phi_{i_2}]\,\mathrm{d}x\right|\\
&\quad\le s^2c\|\phi_{i_1}\|_{L^2(\partial B_{i_1j_1})} + \left|\sum_{1\le i_2\ne i_1\le N}\sum_{p=3}^{+\infty} s^p\nabla^{p-3}(\Delta\Gamma^{k_b})(y_{i_1}-y_{i_2})\cdot\int_{B_{i_1j_1}}\mathcal{T}^{p-3}_{D_{i_1},D_{i_2}}[\phi^*_{i_2}]\,\mathrm{d}x\right|\\
&\quad\le s^2c\|\phi_{i_1}\|_{L^2(\partial B_{i_1j_1})} + \left|\sum_{1\le i_2\ne i_1\le N}\sum_{p=3}^{+\infty} s^p k_b^2\nabla^{p-3}\Gamma^{k_b}(y_{i_1}-y_{i_2})\cdot\int_{B_{i_1j_1}}\mathcal{T}^{p-1}_{D_{i_1},D_{i_2}}[\phi^*_{i_2}]\,\mathrm{d}x\right|\\
&\quad\le cs^2\|\phi_{i_1}\|_{L^2(\partial B_{i_1j_1})} + c\sum_{1\le i_2\ne i_1\le N}\sum_{p=1}^{+\infty}\frac{s^{p+2}}{|y_{i_1}-y_{i_2}|^p}\|\phi_{i_2}\|_{L^2(\partial D_{i_2})},
\end{aligned} \tag{4.13}$$

where we used Proposition B.1 at the last line. Then observe that

$$\begin{aligned}\sum_{1\le i_2\neq i_1\le N}\sum_{p=1}^{+\infty}\frac{s^{p+2}}{|y_{i_1}-y_{i_2}|^p}\|\phi_{i_2}\|_{L^2(\partial D_{i_2})} &\le \sum_{1\le i_2\neq i_1\le N}\sum_{p=1}^{+\infty}\frac{s^{p+2}}{|y_{i_1}-y_{i_2}|\epsilon_N^{p-1}}\|\phi_{i_2}\|_{L^2(\partial D_{i_2})}\\ &\le c\sum_{p=1}^{+\infty}\eta_N^{p-1}\sum_{1\le i_2\neq i_1\le N}\frac{s^3}{|y_{i_1}-y_{i_2}|}\|\phi_{i_2}\|_{L^2(\partial D_{i_2})}\le cs^3\left(\sum_{1\le i_2\neq i_1\le N}\frac{1}{|y_{i_1}-y_{i_2}|^2}\right)^{\frac12}\|\phi\|_{L^2(\partial\mathcal{D})}.\end{aligned}\tag{4.14}$$

This implies by using the Cauchy–Schwarz inequality and the definition (2.12) of the norm:

$$\begin{aligned}s^2\sum_{\substack{1\le i_1\le N\\ 1\le j_1\le K_{i_1}}}\left|\int_{\partial B_{i_1j_1}}\left(s^2\mathcal{B}_1(s)[\phi]\right)_{i_1}\mathrm{d}\sigma\right|^2 &\le cs^4\|\phi\|^2_{L^2(\partial\mathcal{D})}+cs^2s^4\ell_N^{-2}\|\phi\|^2_{L^2(\partial\mathcal{D})}\\ &\le cs^4\left(1+\left(s\ell_N^{-1}\right)^2\right).\end{aligned}$$

The result follows because $s\ell_N^{-1}$ is bounded by the assumption (4.10). □

The next step is to characterize the kernel of $-(1/2)\mathrm{I}+\mathcal{K}^*_{\mathcal{D}}$, which is the zero-th order part of $\mathcal{L}(s,\delta)$ in (4.8).

Lemma 4.2. *The kernel of the operator $-\frac12\mathrm{I}+\mathcal{K}^*_{\mathcal{D}}$ is of dimension $M=\sum_{i=1}^N K_i$ and is given by*

$$\mathrm{Ker}\left(-\frac12\mathrm{I}+\mathcal{K}^*_{\mathcal{D}}\right)=\bigtimes_{\substack{1\le i\le N\\ 1\le j\le K_i}}\mathrm{span}(\psi^*_{ij}),$$

where for any $1\le i\le N$, the functions $(\psi^*_{ij})_{1\le j\le K_i}$ form a basis of $\mathrm{Ker}(\mathcal{K}^*_{D_i})$ and are defined by:

$$\psi^*_{ij}:=(\mathcal{S}_{D_i})^{-1}[1_{\partial B_{ij}}],\qquad 1\le j\le K_i.$$

The range of the operator $-\frac12\mathrm{I}+\mathcal{K}^*_{\mathcal{D}}$ is the space of functions $\phi=(\phi_i)_{1\le i\le N}$ with zero averages on the group of resonators D_i:

$$\mathrm{Ran}\left(-\frac12\mathrm{I}+\mathcal{K}^*_{\mathcal{D}}\right)=L^2_0(\partial D):=L^2_0(\partial D_1)\times\cdots\times L^2_0(\partial D_N),$$

where $L^2_0(\partial D_i)=\{\phi\in\partial D_i\,|\,\int_{\partial D_i}\phi\,\mathrm{d}\sigma=0\}$. It is of codimension M. Furthermore, we have the direct sum decomposition

$$L^2(\partial D)=\mathrm{Ker}\left(-\frac12\mathrm{I}+\mathcal{K}^*_{\mathcal{D}}\right)\oplus L^2_0(\partial D),$$

and $-\frac12\mathrm{I}+\mathcal{K}^*_{\mathcal{D}}$ is invertible as an operator $L^2_0(\partial D)\to L^2_0(\partial D)$.

In order to compute the inverse of the operator $\mathcal{L}(s,\delta)$ of (4.8) we introduce a constant finite range operator $\mathcal{H}$ such that $-\frac12\mathrm{I}+\mathcal{K}^*_{\mathcal{D}}+\mathcal{H}$ is invertible. In what follows, we denote by $(C_i)_{1\le i\le N}$ the capacitance matrices associated to each group of resonators $D_i=\cup_{1\le j\le K_i}B_{ij}$, which are defined by the following identity:

$$C_{i,jl}=-\int_{\partial B_{ij}}\psi^*_{il}\,\mathrm{d}\sigma,\qquad 1\le i\le N,1\le j,l\le K_i.\tag{4.15}$$

We recall that C_i is a symmetric positive definite matrix for $1\le i\le N$ (see *e.g.* [38], Sect. 2 for a list of properties of the capacitance matrix). In order to define the operator $\mathcal{H}$, we introduce the basis of functions $(\phi^*_{ij})_{1\le j\le K_i}$ of $\mathrm{Ker}\left(-\frac12\mathrm{I}+\mathcal{K}^*_{D_i}\right)$ which satisfy the property

$$\int_{\partial B_{il}}\phi^*_{ij}\,\mathrm{d}\sigma=\delta_{jl}\text{ for any }1\le l\le K_i.\tag{4.16}$$

These functions are explicitly given by

$$\phi_{ij}^* := -\sum_{l=1}^{K_i} (C_i^{-1})_{jl} \psi_{il}^*, \text{ for any } 1 \le i \le N, 1 \le j \le K_i. \tag{4.17}$$

Each function ϕ_{ij}^* of the above definition belongs to $L^2(\partial D_i)$ for any $1 \le j \le K_i$. Then, in what follows and with a slight abuse of notation, we still denote by $\phi_{ij}^* \equiv (0, \dots, 0, \phi_{ij}^*, 0, \dots, 0) \in L^2(\mathcal{D})$ the function with N coordinates whose coordinate i is given by (4.17) and which is zero on the other coordinates.

Definition 4.1. We denote by $\mathcal{H} : L^2(\partial\mathcal{D}) \to L^2(\partial\mathcal{D})$ the finite range projection operator satisfying $\mathrm{Ran}(\mathcal{H}) = \mathrm{Ker}(-\frac{1}{2}\mathrm{I} + \mathcal{K}_{\mathcal{D}}^*)$ and $\mathrm{Ker}(\mathcal{H}) = \mathcal{L}_0^2(\partial D)$. The operator $\mathcal{H}$ reads explicitly

$$\mathcal{H}[\phi] = \left(\sum_{j=1}^{K_i} \left(\int_{\partial B_{ij}} \phi_i \, \mathrm{d}\sigma \right) \phi_{ij}^* \right)_{1 \le i \le N} \quad \text{with } \phi \equiv (\phi_i)_{1 \le i \le N} \in L^2(\partial\mathcal{D}).$$

Proposition 4.1. *The operator $\mathcal{L}(s, \delta)$ defined in (4.8) can be decomposed as*

$$\mathcal{L}(s, \delta) = \mathcal{L}_0 - \mathcal{H} + s^2 \mathcal{B}_1(s) + \delta \mathcal{B}_2(s), \tag{4.18}$$

where $\mathcal{L}_0 := -\frac{1}{2}\mathrm{I} + \mathcal{K}_{\mathcal{D}}^* + \mathcal{H}$ is an invertible Fredholm operator. The inverse of $\mathcal{L}_0$ reads explicitly

$$\mathcal{L}_0^{-1}[\phi] = \left(-\frac{1}{2}\mathrm{I} + \mathcal{K}_{\mathcal{D}}^* \right)^{-1} [\phi - \mathcal{H}[\phi]] + \mathcal{H}[\phi],$$

where $(-\frac{1}{2}\mathrm{I} + \mathcal{K}_{\mathcal{D}}^*)^{-1}$ is the inverse of the operator $(-\frac{1}{2}\mathrm{I} + \mathcal{K}_{\mathcal{D}}^*) : L_0^2(\partial D) \to L_0^2(\partial D)$. Furthermore, the following properties hold true:

- $\mathcal{H}[\phi_{ij}^*] = \mathcal{L}_0[\phi_{ij}^*] = \phi_{ij}^*$ *for any* $1 \le i \le N$ *and* $1 \le j \le K_i$.
- $\int_{\partial B_{ij}} (\mathcal{L}_0^{-1}[\phi])_i \, \mathrm{d}\sigma = \int_{\partial B_{ij}} \phi_i \, \mathrm{d}\sigma$ *for any* $\phi \equiv (\phi_i)_{1 \le i \le N} \in L^2(\partial\mathcal{D})$, $1 \le i \le N$ *and* $1 \le j \le K_i$
- $\phi = (\phi - \mathcal{H}[\phi]) + \mathcal{H}[\phi]$ *is the direct sum decomposition of* ϕ *on* $\mathcal{L}_0^2(\partial\mathcal{D}) \oplus \mathrm{Ker}\big(-\frac{1}{2}\mathrm{I} + \mathcal{K}_{\mathcal{D}}^*\big)$.

The decomposition (4.18) reads

$$\mathcal{L}(s, \delta) = \mathcal{G}(s, \delta) - \mathcal{H}, \tag{4.19}$$

where $\mathcal{G}(s, \delta)$ is the operator

$$\mathcal{G}(s, \delta) = \mathcal{L}_0 + s^2 \mathcal{B}_1(s) + \delta \mathcal{B}_2(s).$$

Since $\mathcal{L}_0$ is invertible, $\mathcal{G}(s, \delta)$ is a holomorphic Fredholm operator whose inverse can easily be computed thanks to a Neumann series.

Lemma 4.3. *The operator $\mathcal{G}(s, \delta)$ is invertible for sufficiently small s and δ and it holds*

$$\mathcal{G}(s, \delta)^{-1} = \mathcal{L}_0^{-1} - \mathcal{C}(s, \delta),$$

where $\mathcal{C}(s, \delta)$ is the compact operator of order $O(s\ell_N^{-1}\eta_N + \delta)$ defined by the following Neumann series:

$$\mathcal{C}(s, \delta) = \sum_{p=1}^{+\infty} (-1)^{p+1} \mathcal{L}_0^{-1} \big((s^2 \mathcal{B}_1(s) + \delta \mathcal{B}_2(s)) \mathcal{L}_0^{-1} \big)^p.$$

The decomposition (4.19) "invertible+finite range" enables to reduce the problem (4.5) to the inversion of a finite dimensional holomorphic $M \times M$ matrix $A(s, \delta)$.

Proposition 4.2. *The operator $\mathcal{A}(s,\delta)$ is invertible if and only if the $M\times M$ matrix*

$$A(s,\delta)\equiv (A(s,\delta)_{i_1j_1,i_2j_2})_{\substack{1\le i_1\le N,\,1\le j_1\le K_{i_1}\\ 1\le i_2\le N, 1\le j_2\le K_{i_2}}},$$

defined by

$$A(s,\delta)_{i_1j_1,i_2j_2}:=\int_{\partial B_{i_1j_1}}\big(\mathcal{C}(s,\delta)\big[\phi^*_{i_2j_2}\big]\big)_{i_1}\,\mathrm{d}\sigma,\qquad 1\le i_1,i_2\le N, 1\le j_1\le K_{i_1}, 1\le j_2\le K_{j_2}$$

is invertible. When it is the case, the solution $(\mathcal{P}_{N,s}^{-1}[\varphi],\mathcal{P}_{N,s}^{-1}[\psi])$ to (4.4) reads

$$\begin{cases}\mathcal{P}_{N,s}^{-1}[\varphi]=\displaystyle\sum_{\substack{1\le i\le N\\1\le j\le K_i}} x^N_{ij}\mathcal{G}^{-1}(s,\delta)\big[\phi^*_{ij}\big]+\mathcal{G}^{-1}(s,\delta)[f],\\ \mathcal{P}_{N,s}^{-1}[\psi]=\displaystyle\sum_{\substack{1\le i\le N\\1\le j\le K_i}} x^N_{ij}(\mathcal{S}^k_{\mathcal{D}})^{-1}\mathcal{S}^{k_b}_{\mathcal{D}}\mathcal{G}^{-1}(s,\delta)[\phi^*_{ij}]+\big(\mathcal{S}^k_{\mathcal{D}}\big)^{-1}\mathcal{S}^{k_b}_{\mathcal{D}}\mathcal{G}^{-1}(s,\delta)[f]-\big(\mathcal{S}^k_{\mathcal{D}}\big)^{-1}\mathcal{P}^{-1}_{N,s}[u_{in}],\end{cases}\tag{4.20}$$

where $f\in\mathcal{L}^2(\partial\mathcal{D})$ is the function

$$f:=\delta\mathcal{P}^{-1}_{N,s}\left[\frac{\partial u_{in}}{\partial n}\right]-\delta\left(\frac{1}{2}\mathrm{I}+\mathcal{K}^{k*}_{\mathcal{D}}\right)\big(\mathcal{S}^k_{\mathcal{D}}\big)^{-1}\mathcal{P}^{-1}_{N,s}[u_{in}],\tag{4.21}$$

and where the coefficients $\boldsymbol{x}^N:=(x^N_{ij})_{1\le i\le N,1\le j\le K_i}$ are solving the finite dimensional problem

$$A(s,\delta)\boldsymbol{x}^N=\boldsymbol{F}\ \text{with}\ \boldsymbol{F}:=\left(\int_{\partial B_{ij}}\big(\mathcal{G}^{-1}(s,\delta)[f]\big)_i\,\mathrm{d}\sigma\right)_{1\le i\le N,1\le j\le K_i}.\tag{4.22}$$

Proof. The proof is identical to the one of Proposition 2.4 in [38]. □

The next result justifies that (4.22) can be called the "Foldy–Lax" approximation of the high-contrast system (1.3).

Proposition 4.3. *The following point-wise expansion holds for any $x\in\mathbb{R}^3\backslash\Omega$:*

$$u_{N,s}(x)-u_{in}(x)=-s^2(1+O(sN))\sum_{i=1}^N\left(\sum_{1\le j\le K_i}x^N_{ij}\right)\Gamma^k(x-y_i)+O(s),$$

which shows that $D_{N,s}$ behaves outside the medium as a system of point-source scatterers located at the points $(y_i)_{1\le i\le N}$ with intensities $\left(-s^2\sum_{1\le j\le K_i}x^N_{ij}\right)_{1\le i\le N}$.

Proof. We proceed as in the proof of Proposition 3.1. A Taylor expansion yields:

$$\begin{aligned}u_{N,s}(x)-u_{in}(x)&=\mathcal{S}^k_{D_{N,s}}\Big[\mathcal{P}_{N,s}\Big[\mathcal{P}^{-1}_{N,s}[\psi]\Big]\Big]=\sum_{1\le i\le N}s^2\int_{\partial D_i}\Gamma^k(x-y_i-st)\Big(\mathcal{P}^{-1}_{N,s}[\psi]\Big)_i(t)\,\mathrm{d}\sigma(t)\\&=s^2\sum_{1\le i\le N}\big(\Gamma^k(x-y_i)+O(s)\big)\int_{\partial D_i}\Big(\mathcal{P}^{-1}_{N,s}\Big)_i[\psi]\,\mathrm{d}\sigma.\end{aligned}\tag{4.23}$$

Then (4.20) and the analysis below reveals that

$$\int_{\partial D_i}\Big(\mathcal{P}^{-1}_{N,s}[\psi]\Big)_i\,\mathrm{d}\sigma=\sum_{1\le j\le K_i}x^N_{ij}(1+O(sN))+O\big(s^{-1}\big).$$

The result is obtained by inserting this expansion into (4.23). □

In the next proposition, we compute the asymptotic of $A(s,\delta)$ at the order $O(\eta_N(s^2+\delta))$. For a $M\times M$ matrix $A\equiv (A_{i_1j_1,i_2j_2})_{\substack{1\le i_1,i_2\le N,\\ 1\le j_1\le K_{i_1}\\ 1\le j_2\le K_{j_2}}}$, we denote by $|||A|||_2$ the triple norm defined by

$$|||A|||_2 = \max_{z\equiv (z_{i_2j_2})_{\substack{1\le i_2\le N\\ 1\le j_2\le K_{i_2}}}} \frac{\|Az\|_2}{\|z\|_2}, \text{ where } \|z\|_2 := \sum_{1\le i_1\le N}\sum_{1\le j_1\le K_{i_1}} |z_{i_1j_1}|^2. \tag{4.24}$$

We write in a block-wise sense $A=\sum_{i=1}^N A_{ii}+\sum_{1\le i_1\ne i_2\le N} A_{i_1i_2}$ with $A_{i_1i_2}\in\mathbb{C}^{K_{i_1}\times K_{i_2}}$ to mean that $A_{i_1j_1,i_2j_2} := (A_{i_1i_1})_{j_1j_2}\delta_{i_1i_2}+(A_{i_1i_2})_{j_1j_2}1_{i_1\ne i_2}$ for any $1\le i_1,i_2\le N$ and $1\le j_1\le K_{i_1}$ and $1\le j_2\le K_{i_2}$.

Proposition 4.4. *The following asymptotic holds true for small s and δ:*

$$\begin{aligned}A(s,\delta) = &\sum_{1\le i\le N}\left(s^2k_b^2V_iC_i^{-1}-\delta I_{K_i\times K_i}\right)\\ &-\sum_{i_1\ne i_2}\Big(s^3k_b^2\Gamma^{k_b}(y_{i_1}-y_{i_2})V_{i_1}\mathbf{1}_{K_{i_1}}\mathbf{1}_{K_{i_2}}^T+\delta s\big(\Gamma^k(y_{i_1}-y_{i_2})-\Gamma^{k_b}(y_{i_1}-y_{i_2})\big)C_{i_1}\mathbf{1}_{K_{i_1}}\mathbf{1}_{K_{i_2}}^T\Big)\\ &+O\big((s^2+\delta)\big(s\ell_N^{-1}\eta_N+\delta\big)\big),\end{aligned}\tag{4.25}$$

where C_i is the $K_i\times K_i$ capacitance matrix defined by (4.15) *and*

$$V_i=\operatorname{diag}(|B_{il}|)_{1\le l\le K_i},\quad I_{K_i\times K_i}=(\delta_{lm})_{1\le l,m\le K_i},\quad \mathbf{1}_{K_i}=(1)_{1\le l\le K_i} \text{ for any } 1\le i\le N, \tag{4.26}$$

and where the $O((s^2+\delta)(s\ell_N^{-1}\eta_N+\delta))$ remainder is estimated with the norm (4.24).

Proof. Let us consider an arbitrary complex vector $z=(z_{i_2j_2})_{1\le i_2\le N,1\le j_2\le N}$ and $\varphi_z := \sum_{\substack{1\le i_2\le N\\ 1\le j_2\le N}} z_{i_2j_2}\phi^*_{i_2j_2}$. The norm of φ_z satisfies

$$\begin{aligned}\|\varphi_z\|_{L^2(\partial\mathcal{D})} = \Bigg\|\sum_{\substack{1\le i_2\le N\\ 1\le j_2\le N}} z_{i_2j_2}\phi^*_{i_2j_2}\Bigg\|_{L^2(\partial\mathcal{D})} &= s\left(\sum_{1\le i_2\le N}\Bigg\|\sum_{1\le j_2\le K_{i_2}} z_{i_2j_2}\phi^*_{i_2j_2}\Bigg\|_{L^2(\partial D_{i_2})}\right)^{\frac12}\\ &\le s\sup_{1\le i_2\le N}\left(\sum_{1\le j_2\le K_{i_2}}\|\phi^*_{i_2j_2}\|^2_{L^2(\partial D_{i_2})}\right)^{\frac12}\|z\|_2\\ &\le cs\|z\|_2.\end{aligned}\tag{4.27}$$

Then, by using (4.12), we have

$$\begin{aligned}A(s,\delta)_{i_1j_1,i_2j_2} &= \int_{\partial B_{i_1j_1}}\left(\mathcal{C}(s,\delta)\left[\phi^*_{i_2j_2}\right]\right)_{i_1}\mathrm{d}\sigma\\ &= \int_{\partial B_{i_1j_1}}\left(\mathcal{L}_0^{-1}\left(s^2\mathcal{B}_1(s)+\delta\mathcal{B}_2(s)\right)\left[\phi^*_{i_2j_2}\right]\right)_{i_1}\mathrm{d}\sigma + R(s,\delta)_{i_1j_1,i_2j_2}\\ &= \int_{\partial B_{i_1j_1}}\left(\left(s^2\mathcal{B}_1(s)+\delta\mathcal{B}_2(s)\right)\left[\phi^*_{i_2j_2}\right]\right)_{i_1}\mathrm{d}\sigma + R(s,\delta)_{i_1j_1,i_2j_2},\end{aligned}\tag{4.28}$$

where the remainder $R(s,\delta)$ satisfies

$$|||R(s,\delta)|||_2 = O\big((s^2+\delta)\big(s\ell_N^{-1}+\delta\big)\big). \tag{4.29}$$

Indeed, we have by using (4.12):

$$\|R(s,\delta)z\|_2=\left(\sum_{\substack{1\le i_1\le N\\1\le j_1\le N}}\left|\int_{\partial B_{i_1j_1}}\left(\left(s^2\mathcal{B}_1(s)+\delta\mathcal{B}_2(s)\right)\mathcal{R}(s,\delta)[\phi_z]\right)_{i_1}\mathrm{d}\sigma\right|^2\right)^{\frac{1}{2}}\le cs^{-1}\left(s^2+\delta\right)\|\mathcal{R}(s,\delta)[\phi_z]\|_{L^2(\partial\mathcal{D})}$$

where $\mathcal{R}(s,\delta)$ is an operator satisfying $|||\mathcal{R}(s,\delta)|||_{L^2(\partial\mathcal{D})\to L^2(\partial\mathcal{D})}=O(s\ell_N^{-1}\eta_N+\delta)$. The estimate (4.29) follows from (4.27). We then compute each term $\mathcal{B}_2(s)$ in (4.28). Repeating the arguments of Lemma 4.1 and using Lemma 2.2, we find that:

$$\begin{aligned}
&\int_{\partial B_{i_1j_1}}s^2\left(\mathcal{B}_1(s)\left[\phi^*_{i_2j_2}\right]\right)_{i_1}\mathrm{d}\sigma\\
&=s^2k_b^2\int_{\partial B_{i_1j_1}}\mathcal{K}^*_{D_{i_1},2}\left[\phi^*_{i_1j_2}\right]\mathrm{d}\sigma\delta_{i_1i_2}+s^3\nabla\Gamma^{k_b}(y_{i_1}-y_{i_2})\cdot\int_{\partial_{B_{i_1j_1}}}\mathcal{M}^1_{D_{i_1},D_{i_2}}\left[\phi^*_{i_2j_2}\right]\mathrm{d}\sigma 1_{i_1\neq i_2}+O\left(s^2\left(s\ell_N^{-1}\right)\eta_N\right)\\
&=-s^2k_b^2\int_{B_{i_1j_1}}\mathcal{S}_{D_{i_1}}\left[\phi^*_{i_1j_2}\right]\mathrm{d}y\delta_{i_1i_2}-s^3k_b^2\Gamma^{k_b}(y_{i_1}-y_{i_2})|B_{i_1j_1}|1_{i_1\neq i_2}+O\left(s^2\left(s\ell_N^{-1}\eta_N\right)\right)\\
&=s^2k_b^2|B_{i_1j_1}|(C_{i_1})^{-1}_{j_1j_2}\delta_{i_1i_2}-s^3k_b^2\Gamma^{k_b}(y_{i_1}-y_{i_2})|B_{i_1j_1}|+O\left(s^2\left(s\ell_N^{-1}\right)\eta_N\right),
\end{aligned}\tag{4.30}$$

where the quantity $O(s^2(s\ell_N)^{-1}\eta_N)$ is estimated here and in the next lines with the $|||\cdot|||_2$ norm as in (4.29). Using the hypothesis (4.10), the term $\delta\mathcal{B}_2(s)$ is then developed as follows:

$$\begin{aligned}
\delta\mathcal{B}_2(s)&=-\delta\left(\frac{1}{2}\mathrm{I}+\mathcal{K}^{k*}_{\mathcal{D}}\right)\left(\mathcal{S}^k_{\mathcal{D}}\right)^{-1}\mathcal{S}^{k_b}_{\mathcal{D}}\\
&=-\delta\left(\frac{1}{2}\mathrm{I}+\mathcal{K}^*_{\mathcal{D}}+O\left(s\ell_N^{-1}\eta_N\right)\right)\left(\mathcal{S}_{\mathcal{D},0}+s\mathcal{S}^k_{\mathcal{D},1}+O\left(s\ell_N^{-1}\eta_N\right)\right)^{-1}\left(\mathcal{S}_{\mathcal{D},0}+s\mathcal{S}^{k_b}_{D,1}+O\left(s\ell_N^{-1}\eta_N\right)\right)\\
&=-\delta\left(\frac{1}{2}\mathrm{I}+\mathcal{K}^*_{\mathcal{D}}\right)\left(\mathcal{S}^{-1}_{\mathcal{D},0}-s\mathcal{S}^{-1}_{\mathcal{D},0}\mathcal{S}^k_{\mathcal{D},1}\mathcal{S}^{-1}_{\mathcal{D},0}\right)\left(\mathcal{S}_{\mathcal{D},0}+s\mathcal{S}^{k_b}_{\mathcal{D},1}\right)+O\left(s\ell_N^{-1}\eta_N\delta\right)\\
&=-\delta\left(\frac{1}{2}\mathrm{I}+\mathcal{K}^*_{\mathcal{D}}\right)+\delta s\left(\frac{1}{2}\mathrm{I}+\mathcal{K}^*_{\mathcal{D}}\right)\mathcal{S}^{-1}_{\mathcal{D},0}\left(\mathcal{S}^k_{\mathcal{D},1}-\mathcal{S}^{k_b}_{\mathcal{D},1}\right)+O\left(s\ell_N^{-1}\eta_N\delta\right).
\end{aligned}\tag{4.31}$$

We then use the identity $\mathcal{S}^k_{D_{i_1}}[\phi^*_{i_1j_2}]=-\mathrm{i}/4\pi 1_{\partial D_{i_1}}$, which implies

$$\int_{\partial B_{i_1j_1}}\mathcal{S}^{-1}_{D_{i_1}}\mathcal{S}^k_{D_{i_1},1}[\phi^*_{i_1j_2}]\,\mathrm{d}\sigma=\frac{\mathrm{i}}{4\pi}\sum_{l=1}^{K_{i_1}}(C_{i_1})_{j_1l}\text{ and }\int_{\partial B_{i_1j_1}}\mathcal{S}^{-1}_{D_{i_1}}\mathcal{T}^0_{D_{i_1},D_{i_2}}[\phi^*_{i_2j_2}]\,\mathrm{d}\sigma=-\sum_{l=1}^{K_{i_1}}(C_{i_1})_{j_1l}.$$

Hence if $i_1=i_2$:

$$\begin{aligned}
\int_{\partial B_{i_1j_1}}\delta\left(\mathcal{B}_2(s)\phi^*_{i_2j_2}\right)_{i_1}\mathrm{d}\sigma&=-\delta\int_{\partial B_{i_1j_1}}\left(\frac{1}{2}\mathrm{I}+\mathcal{K}^*_{D_{i_1}}\right)\left(\phi^*_{i_1j_2}-s(k-k_b)\mathcal{S}^{-1}_{D_{i_1}}\mathcal{S}_{D_{i_1},1}\left[\phi^*_{i_1j_2}\right]\right)\mathrm{d}\sigma+O\left(s\ell_N^{-1}\eta_N\delta\right)\\
&=-\delta\delta_{j_1j_2}+\frac{i\delta s}{4\pi}(k-k_b)\sum_{l=1}^{K_{i_1}}(C_{i_1})_{j_1l}+O\left(s\ell_N^{-1}\eta_N\delta\right)=-\delta\delta_{j_1j_2}+O\left(s\ell_N^{-1}\eta_N\delta\right),
\end{aligned}\tag{4.32}$$

and if $i_1 \neq i_2$:

$$\begin{aligned}
&\int_{\partial B_{i_1 j_1}} \delta\big(\mathcal{B}_2(s)\phi^*_{i_2 j_2}\big)_{i_1}\,\mathrm{d}\sigma \\
&\qquad = \delta s\big(\Gamma^k(y_{i_1}-y_{i_2})-\Gamma^{k_b}(y_{i_1}-y_{i_2})\big)\int_{\partial B_{i_1 j_1}}\left(\frac{1}{2}\mathrm{I}+\mathcal{K}^*_{D_{i_1}}\right)\mathcal{S}^{-1}_{D_{i_1}}\mathcal{T}^0_{D_{i_1},D_{i_2}}\big[\phi^*_{i_2 j_2}\big]\,\mathrm{d}\sigma + O\big(s\ell_N^{-1}\eta_N\delta\big). \\
&\qquad = -\delta s\big(\Gamma^k(y_{i_1}-y_{i_2})-\Gamma^{k_b}(y_{i_1}-y_{i_2})\big)\sum_{l=1}^{K_{i_1}}(C_{i_1})_{j_1 l} + O\big(s\ell_N^{-1}\eta_N\delta\big).
\end{aligned}$$

Therefore we find finally

$$\begin{aligned}
C(s,\delta)_{i_1 j_1, i_2 j_2} &= \big(s^2 k_b^2 |B_{i_1 j_1}|(C_{i_1})^{-1}_{j_1 j_2} - \delta\delta_{j_1 j_2}\big)\delta_{i_1 i_2} \\
&\quad - \left(s^3 k_b^2 |B_{i_1 j_1}|\Gamma^{k_b}(y_{i_1}-y_{i_2}) + \delta s\big(\Gamma^k(y_{i_1}-y_{i_2})-\Gamma^{k_b}(y_{i_1}-y_{i_2})\big)\sum_{l=1}^{K_{i_1}}(C_{i_1})_{j_1 l}\right)1_{i_1\neq i_2} \\
&\quad + O\big(\big(s^2+\delta\big)\big(s\ell_N^{-1}\eta_N+\delta\big)\big).
\end{aligned} \tag{4.33}$$

which is the result to obtain. □

Remark 4.2. The nonlinear eigenvalue problem $A(s,\delta)\boldsymbol{x}=0$ is substantially different from the one characterizing scattering resonances in the non-dilute regime of [19, 38] featuring a single packet of resonators, because the matrix (4.25) takes into account the interactions $\Gamma^k(y_i - y_j)$ which are of order $O((s^2+\delta)s\ell_N^{-1})$. As we see below in Remark 4.6, this predicts that a shift in the scattering resonance may be observed due to the interactions.

Remark 4.3. It seems complicated to analyze the KN characteristic values of the full operator $A(s,\delta)$ of (4.22), *i.e.* the exact values of s such that $A(s,\delta)\boldsymbol{x}$ has a non-trivial solution, from where an explicit analysis could be inferred as in our previous work [38]. Indeed, $A(s,\delta)$ is a non-Hermitian perturbation in δ of the operator $A(s,0)$, which has $s=0$ as a characteristic value with geometric multiplicity $2KN$. In the present approach, we find an approximation of $A(s,\delta)^{-1}$ at radii s slightly away from the resonance $s_i(\delta)$, which does not require to characterize explicitly the splitting of the KN characteristic values.

In the next proposition, we compute the asymptotic expansion of the right-hand sides f and $\boldsymbol{F}$ of (4.21) and (4.22).

Proposition 4.5. *The right-hand side $f \equiv (f_i)_{1\le i\le N}$ of (4.21) admits the following asymptotic expansion:*

$$\begin{aligned}
f = &-s^{-1}\delta\left(\left(u_{in}(y_i) + s\sum_{j\neq i}\Gamma^k(y_i-y_j)\mathrm{cap}\,(D_j)u_{in}(y_j)\right)\sum_{l=1}^{K_i}\psi^*_{il}\right)_{1\le i\le N} \\
&+ O\Big(\delta s^{-1}\max\big(s\ell_N^{-1}\eta_N, s\big)sN^{\frac{1}{2}}\Big)_{L^2(\partial\mathcal{D})},
\end{aligned}$$

where the relative error is of order $O(\max(s\ell_N^{-1}\eta_N, s))$. Then the right-hand side $\boldsymbol{F} \equiv (F_{ij})_{\substack{1\le i\le N\\ 1\le j\le K_i}}$ of (4.22) reads:

$$\begin{aligned}
\boldsymbol{F} = &\left(s^{-1}\delta\left(u_{in}(y_i) + s\sum_{j\neq i}\Gamma^k(y_i-y_j)\mathrm{cap}\,(D_j)u_{in}(y_j)\right)(C_i\mathbf{1}_{K_i})_j\right)_{\substack{1\le i\le N\\ 1\le j\le K}} \\
&+ O(\delta s^{-1}\Big(\max\big(s\ell_N^{-1}\eta_N, s\big)N^{\frac{1}{2}}\Big)_{\|\cdot\|_2}.
\end{aligned} \tag{4.34}$$

Proof. We expand (4.21), recalling $\mathcal{P}_{N,s}^{-1}[u_{in}] = (u_{in}(y_i)1_{\partial D_i})_{1\leq i\leq N} + O(s^2N^{\frac{1}{2}})_{L^2(\mathcal{D})}$:

$$\begin{aligned}
f &= O\Big(\delta s N^{\frac{1}{2}}\Big)_{L^2(\partial\mathcal{D})} - \delta s^{-1}\left(\frac{1}{2}\mathrm{I} + \mathcal{K}_{\mathcal{D}}^* + O\big(s\ell_N^{-1}\eta_N\big)\right)\\
&\quad\times\left(\mathcal{S}_{\mathcal{D}}^{-1} - s\mathcal{S}_{\mathcal{D}}^{-1}\mathcal{S}_{\mathcal{D},1}^{k}\mathcal{S}_{\mathcal{D}}^{-1} + O\big(s\ell_N^{-1}\eta_N\big)\right)\left[(u_{in}(y_i)1_{\partial D_i})_{1\leq i\leq N} + O\Big(s^2N^{\frac{1}{2}}\Big)_{L^2(\partial\mathcal{D})}\right]\\
&= -\delta s^{-1}\left(\frac{1}{2} + \mathcal{K}_{\mathcal{D}}^*\right)\left(\mathcal{S}_{\mathcal{D}}^{-1} - s\mathcal{S}_{\mathcal{D}}^{-1}\mathcal{S}_{\mathcal{D},1}^{k}\mathcal{S}_{\mathcal{D}}^{-1}\right)(u_{in}(y_i)1_{\partial D_i})_{1\leq i\leq N} + O\Big(\delta s^{-1}s\ell_N^{-1}\eta_N sN^{\frac{1}{2}}\Big) + O\Big(\delta sN^{\frac{1}{2}}\Big)_{L^2(\partial\mathcal{D})}\\
&= -\delta s^{-1}\left(\frac{1}{2} + \mathcal{K}_{\mathcal{D}}^*\right)\left(\left(\left(1 - \frac{\mathrm{i}sk}{4\pi}\mathrm{cap}\,(D_i)\right)u_{in}(y_i) + s\sum_{j\neq i}\Gamma^k(y_i-y_j)\mathrm{cap}\,(D_j)u_{in}(y_j)\right)\sum_{l=1}^{K}\psi_{il}^*\right)_{1\leq i\leq N}\\
&\quad + O\Big(\delta s^{-1}\big(\max\big(s\ell_N^{-1}\eta_N, s\big)\big)_{L^2(\partial\mathcal{D})}sN^{\frac{1}{2}}\Big),
\end{aligned}$$

where the first result follows by using the fact that $\mathcal{K}_{D_i}^*[\psi_{il}^*] = \psi_{il}^*/2$. Then we use the identity

$$\begin{aligned}
F_{ij} &= \int_{\partial B_{ij}}\big(\mathcal{G}^{-1}(s,\delta)[f]\big)_i\,\mathrm{d}\sigma = \int_{\partial B_{ij}} f_i\,\mathrm{d}\sigma + O\Big(\big(s^2+\delta\big)\delta s^{-1}N^{\frac{1}{2}}\Big)_{\|\cdot\|_2}\\
&= -s^{-1}\delta\sum_{l=1}^{K_i}\int_{\partial B_{ij}}\psi_{il}^*\,\mathrm{d}\sigma\left(u_{in}(y_i) + s\sum_{j\neq i}\Gamma^k(y_i-y_j)\mathrm{cap}\,(D_j)u_{in}(y_j)\right) + O\Big(\delta s^{-1}\max\big(s\ell_N^{-1}\eta_N, s\big)N^{\frac{1}{2}}\Big)_{\|\cdot\|_2}\\
&= 0 + s^{-1}\delta\left(u_{in}(y_i) + s\sum_{j\neq i}\Gamma^k(y_i-y_j)\mathrm{cap}\,(D_j)u_{in}(y_j)\right)\sum_{l=1}^{K_i}(C_i)_{jl} + O\Big(\delta s^{-1}\max\big(s\ell_N^{-1}\eta_N, s\big)N^{\frac{1}{2}}\Big)_{\|\cdot\|_2}.
\end{aligned}$$

□

4.2. Convergence of the Foldy–Lax system to an integral equation

In what follows, we consider the setting of (H1) and (H2) whereby the packets $(D_i)_{1\leq i\leq N}$ are identical ($D_i = D$ for $1\leq i\leq N$) and constituted of K resonators $(B_j)_{1\leq j\leq K}$, and whose centers $(y_i)_{1\leq i\leq N}$ are distributed randomly and independently in Ω. We denote by $C := C_1 = \cdots = C_N \in \mathbb{R}^{K\times K}$ the common capacitance matrix:

$$C := \left(-\int_{\partial B_j}\mathcal{S}_D^{-1}[1_{\partial B_i}]\,\mathrm{d}\sigma\right)_{1\leq i,j\leq K}, \tag{4.35}$$

by $V := V_1 = \cdots = V_N = \mathrm{diag}(|B_j|)_{1\leq j\leq K}$ the common volume matrix (Eq. (4.26)), and by $\psi_j^* := \psi_{1j}^* = \cdots = \psi_{Nj}^*$ and $\phi_j^* = \phi_{1j}^* = \cdots = \phi_{Kj}^*$. the functions

$$\psi_j^* := \mathcal{S}_D^{-1}[1_{\partial B_j}], \quad \phi_j^* := -\sum_{l=1}^{K}C_{jl}^{-1}\psi_l^*.$$

We also consider ψ^* the function of $L^2(\partial D)$ given by:

$$\psi^* = \mathcal{S}_D^{-1}[1_{\partial D}] = \sum_{l=1}^{K}\psi_l^* = -\sum_{1\leq l,k\leq K}C_{lk}\phi_l^*.$$

We denote by $(\boldsymbol{a}_k)_{1\leq k\leq K}$ and $0<\lambda_1\leq\lambda_2\leq\ldots\leq\lambda_K$ the eigenvectors and eigenvalues of the symmetric eigenvalue problem

$$C\boldsymbol{a}_j = \lambda_j V\boldsymbol{a}_j, \tag{4.36}$$

and by $(s_i(\delta))_{1\le i\le K}$ the K "resonant" values

$$s_i(\delta) := \delta^{\frac{1}{2}}\frac{\lambda_i^{\frac{1}{2}}}{k_b}, \quad 1\le i\le K, \tag{4.37}$$

such that the matrix $s^2k_b^2VC^{-1} - \delta I_{K\times K}$ is not invertible when $s = s_i(\delta)$. Finally, as anticipated in the introduction, we introduce the quantities $\boldsymbol{Q}(s,\delta)$ and $Q(s,\delta)$ defined by

$$\boldsymbol{Q}(s,\delta) := \sum_{i=1}^{K}\frac{\lambda_i}{\frac{s^2}{s_i(\delta)^2}-1}(\boldsymbol{a}_i^TV\mathbf{1})V\boldsymbol{a}_i, \qquad Q(s,\delta) := \mathbf{1}^T\boldsymbol{Q}(s,\delta) = \sum_{i=1}^{K}\frac{\lambda_i}{\frac{s^2}{s_i(\delta)^2}-1}(\boldsymbol{a}_i^TV\mathbf{1})^2.$$

The values $s_i(\delta)$ correspond to the leading order of the resonant characteristic sizes s at which the scattering operator $\mathcal{A}(s,\delta)$ is not invertible. The quantity $Q(s,\delta)$ appears naturally in the analysis and is a measure of the inverse of the deviation from s to a resonant characteristic size $s_i(\delta)$ when $\boldsymbol{a}_i^TV\mathbf{1}\neq 0$.

For convenience, we denote by

$$\boldsymbol{x}_i^N = \left(x_{ij}^N\right)_{1\le j\le K}\in\mathbb{C}^K, \qquad 1\le i\le N,$$

the $N\times K$ components of the solution $\boldsymbol{x} = (x_{ij}^N)_{\substack{1\le i\le N\\ 1\le j\le K}}$ to (4.22). The purpose of this part is to show that the algebraic solution $\boldsymbol{x}^N$ can be approximated by

$$\boldsymbol{x}_i^N \simeq w(y_i)\frac{V\boldsymbol{a}_i}{\mathbf{1}^TV\boldsymbol{a}_i}, \qquad 1\le i\le N,$$

where w is the solution to an integral equation (given in (4.46) hereafter). This convergence is established in Proposition 4.7 below.

Neglecting error terms, the finite dimensional problem (4.22) can be rewritten as the following approximate N-dimensional linear systems for the K-dimensional vectors $(\boldsymbol{x}_i^N)_{1\le i\le N}$:

$$\begin{aligned}&\left(s^2k_b^2VC^{-1}-\delta\mathrm{I}_{K\times K}\right)\boldsymbol{x}_i^N - \sum_{1\le j\neq i\le N}\left(s^3k_b^2\Gamma^{k_b}(y_i-y_j)V\mathbf{1}\mathbf{1}^T + \delta s\left(\Gamma^k(y_i-y_j)-\Gamma^{k_b}(y_i-y_j)\right)C\mathbf{1}\mathbf{1}^T\right)\boldsymbol{x}_j^N\\ &\quad = \delta s^{-1}\left(u_{in}(y_i) + s\sum_{1\le j\neq i\le N}\Gamma^k(y_i-y_j)\mathrm{cap}\,(D)u_{in}(y_j)\right)_{1\le i\le N} C\mathbf{1}, \qquad 1\le i\le N,\end{aligned} \tag{4.38}$$

where we have denoted $\mathbf{1}=\mathbf{1}_K$ since the context is clear. Equation (4.38) can be reduced to a linear system of N equations after some algebraic manipulations.

Lemma 4.4. *Assume that s is not equal to any of the resonant scalings $s_i(\delta)$ for $1\le i\le N$. Let us denote by $b^N(y_i)$ the quantity*

$$b^N(y_i) := u_{in}(y_i) + s\mathrm{cap}\,(D)\sum_{1\le j\neq i\le N}\Gamma^k(y_i-y_j)u_{in}(y_j)\,, \qquad 1\le i\le N. \tag{4.39}$$

The linear system (4.38) admits a solution if and only if the linear system

$$z_i^N - sQ(s,\delta)\sum_{1\le j\neq i\le N}\Gamma^k(y_i-y_j)z_j^N - s\mathrm{cap}\,(D)\sum_{1\le j\neq i\le N}\Gamma^{k_b}(y_i-y_j)z_j^N = s^{-1}Q(s,\delta)b^N(y_i), \quad 1\le i\le N \tag{4.40}$$

has a solution. When it is the case, the solution $(\boldsymbol{x}_i^N)$ to (4.38) is given by

$$\boldsymbol{x}_i^N = s\left(\sum_{1\le j\neq i\le N}\Gamma^k(y_i-y_j)z_j^N\right)\boldsymbol{Q}(s,\delta) + s\left(\sum_{1\le j\neq i\le N}\Gamma^{k_b}(y_i-y_j)z_j^N\right)C\mathbf{1} + s^{-1}b^N(y_i)\boldsymbol{Q}(s,\delta), \quad 1\le i\le N, \tag{4.41}$$

and we have further $z_i^N = \mathbf{1}^T\boldsymbol{x}_i^N$ for $1\le i\le N$.

Proof. Equation (4.38) can be rewritten as

$$\begin{aligned}\left(s^2k_b^2VC^{-1}-\delta\mathrm{I}_{K\times K}\right)\boldsymbol{x}_i^N \quad &-\delta s\sum_{1\le j\neq i\le N}\Gamma^k(y_i-y_j)z_j^N C\mathbf{1}\\ &-s\sum_{1\le j\le\neq i\le N}\Gamma^{k_b}(y_i-y_j)z_j^N\left(s^2k_b^2VC^{-1}-\delta I_{K\times K}\right)C\mathbf{1}=\delta s^{-1}b^N(y_i)C\mathbf{1}.\end{aligned}\tag{4.42}$$

For $s\neq s_i(\delta)$ with $1\le i\le K$, this equation is equivalent to

$$\begin{aligned}&\boldsymbol{x}_i^N-\delta s\sum_{1\le j\neq i\le N}\Gamma^k(y_i-y_j)z_j^N\left(s^2k_b^2VC^{-1}-\delta\mathrm{I}_{K\times K}\right)^{-1}C\mathbf{1}-s\sum_{1\le j\neq i\le N}\Gamma^{k_b}(y_i-y_j)z_j^N C\mathbf{1}\\ &=\delta s^{-1}b^N(y_i)(s^2k_b^2VC^{-1}-\delta\mathrm{I}_{K\times K})^{-1}C\mathbf{1}.\end{aligned}\tag{4.43}$$

Since

$$(s^2k_b^2VC^{-1}-\delta I_{K\times K})^{-1}C\mathbf{1}=\sum_{i=1}^K\frac{\lambda_i}{s^2k_b^2\lambda_i^{-1}-\delta}(\mathbf{1}^TV\boldsymbol{a}_i)V\boldsymbol{a}_i=\sum_{i=1}^K\frac{\lambda_i\delta^{-1}}{\frac{s^2}{s_i(\delta)^2}-1}(\mathbf{1}^TV\boldsymbol{a}_i)V\boldsymbol{a}_i=\boldsymbol{Q}(s,\delta),$$

we obtain the linear system (4.40) for the coefficients $(z_i^N)_{1\le i\le N}$ after taking the inner product of (4.43) with the vector $\mathbf{1}$. Then (4.43) implies that $\boldsymbol{x}_i^N$ is given by the formula (4.41). ☐

In the non-resonant setting of Section 3, the analogous algebraic system (3.7) was obtained with $sNQ(s,\delta)$ instead of sN. In what follows, we therefore assume (H4) which is the natural subcritical regime associated to (1.3): the characteristic size s converges to one of the resonant values $s_i(\delta)$, $\mathbf{1}^TV\boldsymbol{a}_i\neq 0$ and $sNQ(s,\delta)$ remains bounded.

Remark 4.4. If $s\to s_i(\delta)$ while $\mathbf{1}^TV\boldsymbol{a}_i=0$, the analysis becomes substantially different because $Q(s,\delta)$ remains bounded, and the natural subcritical regime becomes $sN=O(1)$. Due to a Perron–Frobenius type theorem for the capacitance matrix [38], this cannot happen for the first resonant value $s_1(\delta)$. However, $\mathbf{1}^TV\boldsymbol{a}_i=0$ can happen in presence of strong symmetries of the packet of resonators D, for instance if D is a dimer constituted of two symmetrical spheres. Then a different analysis must be performed to capture the right effective medium in the regime $sN=O(1)$, set aside for a future work. The reader is referred to [14] for a formal approach based on a Foldy–Lax approximation.

Since (H4) implies $sQ(s,\delta)N=O(1)$ and $sN=o(1)$, equation (4.40) is a perturbed version of the following linear system

$$w_i^N-sQ(s,\delta)\sum_{1\le j\neq i\le N}\Gamma^k(y_i-y_j)w_j^N=s^{-1}Q(s,\delta)u_{in}(y_i),\qquad 1\le i\le N.\tag{4.44}$$

Reasoning as in Section 3.3, we obtain the following approximation theorem.

Proposition 4.6. *Assume (H1)–(H4). There exists an event $\mathcal{H}_{N_0}$, independent of s and δ, which holds with probability $\mathbb{P}(\mathcal{H}_{N_0})\to 1$ as $N_0\to+\infty$ such that, when $\mathcal{H}_{N_0}$ is realized:*

(i) *the linear system (4.44) is invertible for $N\ge N_0$ and*

$$\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^N\left|w_i^N-w(y_i)\right|^2|\mathcal{H}_{N_0}\right]^{\frac12}\le c\Big(N^{-\frac12}sQ(s,\delta)N\Big)s^{-1}Q(s,\delta),\tag{4.45}$$

where w is the solution to the Lippmann–Schwinger equation

$$w-sNQ(s,\delta)\int_\Omega\Gamma^k(\cdot-y)w(y)\varrho(y)\,\mathrm{d}y=s^{-1}Q(s,\delta)u_{in}.\tag{4.46}$$

We also have the convergence of the Nystrom interpolant

$$w^N(y) := s^{-1}Q(s,\delta)u_{in}(y) + sQ(s,\delta)\sum_{i=1}^{N}\Gamma^k(y-y_j)w_j^N, \qquad y\in\mathbb{R}^3, \tag{4.47}$$

in the following mean-square sense:

$$\mathbb{E}\Big[\big\|w^N - w\big\|^2_{L^2(\Omega)}|\mathcal{H}_{N_0}\Big]^{\frac{1}{2}} \le c\Big(N^{-\frac{1}{2}}sNQ(s,\delta)\Big)s^{-1}Q(s,\delta);$$

(ii) *the matrix associated to the linear system* (4.44) *is well-conditioned: there exists a constant $c>0$ independent of s, δ and N such that*

$$\Big|\Big|\Big|\big(\mathrm{I} - sNQ(s,\delta)A_N^k\big)^{-1}\Big|\Big|\Big|_2 \le c, \tag{4.48}$$

where A_N^k is the matrix defined in (3.13).

Using a standard perturbation argument, we obtain the following result for the solution (z_i^N) to (4.40).

Corollary 4.1. *Assume (H1)–(H4). The solution $z^N = (z_i^N)_{1\le i\le N}$ to the linear system* (4.40) *can be approximated by w^N up to a relative error of order $O(sN)$:*

$$\left(\frac{1}{N}\sum_{1\le i\le N}|z_i^N - w_i^N|^2\right)^{\frac{1}{2}} \le csNs^{-1}Q(s,\delta). \tag{4.49}$$

Consequently, z^N can also be approximated by the solution w to the Lippmann–Schwinger equation (4.46)*:*

$$\mathbb{E}\left[\frac{1}{N}\sum_{i=1}^{N}|z_i^N - w(y_i)|^2|\mathcal{H}_{N_0}\right]^{\frac{1}{2}} \le c\max\Big(N^{-\frac{1}{2}}sNQ(s,\delta), sN\Big)s^{-1}Q(s,\delta). \tag{4.50}$$

Remark 4.5. We could technically write explicitly the asymptotic satisfied by $(z_i^N)_{1\le i\le N}$ by using a Neumann series, or to state an approximation theorem similar to Proposition 4.6 by considering the continuous limit equation to (4.40):

$$z - sNQ(s,\delta)\int_\Omega \Gamma^k(\cdot - y)z(y)\varrho(y)\,\mathrm{d}y - sN\mathrm{cap}\,(D)\int_\Omega \Gamma^{k_b}(\cdot - y)z(y)\varrho(y)\,\mathrm{d}y = s^{-1}Q(s,\delta)b^N(y).$$

However, we restrict our analysis to the approximation provided by the simplest model (4.46) for the sake of simplicity.

We infer an approximation formula for the solution $\boldsymbol{x}^N$ to the discrete problem (4.22).

Proposition 4.7. *Assume (H1)–(H4) and the event $\mathcal{H}_{N_0}$ to be realized. The linear system* (4.22) *is invertible and the solution $\boldsymbol{x}_i^N$ admits the following asymptotic expansion:*

$$\begin{aligned}\boldsymbol{x}_i^N &= w_i^N\frac{\boldsymbol{Q}(s,\delta)}{Q(s,\delta)} + O\Big(sN\Big(s^{-1}Q(s,\delta)N^{\frac{1}{2}}\Big)\Big)_{\|\cdot\|_2}\\ &= w_i^N\frac{V\boldsymbol{a}_i}{\boldsymbol{1}^TV\boldsymbol{a}_i} + O\Big(Q(s,\delta)^{-1}\Big(s^{-1}Q(s,\delta)N^{\frac{1}{2}}\Big)\Big)_{\|\cdot\|_2}.\end{aligned}$$

Proof. Denote by $\hat{A}(s,\delta)$ the matrix of the linear system (4.38). Recall that due to (2.7) and the inequality (2.13), the matrices $A_N^k = N^{-1}(\Gamma^k(y_i-y_j))_{1\leq i\neq j\leq N}$, $A_N^{k_b} = N^{-1}(\Gamma^{k_b}(y_i-y_j))_{1\leq i\neq j\leq N}$ have bounded norm with large probability:

$$\left|\left|\left|A_N^{k_b}\right|\right|\right|_2 \leq c, \qquad \left|\left|\left|A_N^{k}\right|\right|\right|_2 \leq c.$$

From the formula (4.41) and the norm estimate result (4.48), we infer that the conditioning of the matrix $\hat{A}(s,\delta)$ satisfies

$$\left|\left|\left|\hat{A}(s,\delta)^{-1}\right|\right|\right|_2 \leq c\delta^{-1}Q(s,\delta).$$

By left-multiplying (4.22) with $\hat{A}(s,\delta)^{-1}$ and using (4.25) (recall from (2.7) that $\ell_N^{-1} = O(N)$ with large probability), we obtain

$$(\mathrm{I} + O(Q(s,\delta)(sN\eta_N+\delta)))\boldsymbol{x}^N = \hat{A}(s,\delta)^{-1}\boldsymbol{F},$$

which implies, since $\delta = o(sN\eta_N)$ and by using a Neumann series:

$$\boldsymbol{x}^N = \hat{A}(s,\delta)^{-1}\boldsymbol{F} + O\Big(s^{-1}Q(s,\delta)N^{\frac{1}{2}}(sNQ(s,\delta)\eta_N)\Big)_{\|\cdot\|_2}.$$

Finally, recall from (4.34) that

$$\boldsymbol{F} = \delta s^{-1}b^N(y_i)C\mathbf{1} + O\Big(\delta s^{-1}\max(sN\eta_N, s)N^{\frac{1}{2}}\Big)_{\|\cdot\|_2},$$

and that the quantity $\hat{A}(s,\delta)^{-1}(\delta s^{-1}b^N(y_i)C\mathbf{1})$ is exactly given by the formula (4.41). Furthermore, the vector $(b^N(y_i))_{1\leq i\leq N}$ of (4.39) can be approximated by

$$\big(b^N(y_i)\big)_{1\leq i\leq N} = (u_{in}(y_i))_{1\leq i\leq N} + O\Big(sNN^{\frac{1}{2}}\Big)_{\|\cdot\|_2}.$$

Then, equation (4.49) enables to substitute z^N with the vector $w^N = (w_i^N)_{1\leq i\leq N}$ up to an error of order sN. All in all, we obtain

$$\begin{aligned}\boldsymbol{x}_i^N &= sNA_N^k w^N\boldsymbol{Q}(s,\delta) + s^{-1}u_{in}(y_i)\boldsymbol{Q}(s,\delta) + O\Big(\max(sN, sNQ(s,\delta)\eta_N)s^{-1}Q(s,\delta)N^{\frac{1}{2}}\Big)_{\|\cdot\|_2}\\ &= w_i^N\frac{\boldsymbol{Q}(s,\delta)}{Q(s,\delta)} + O\Big(\max(sN, sNQ(s,\delta)\eta_N)s^{-1}Q(s,\delta)N^{\frac{1}{2}}\Big)_{\|\cdot\|_2},\end{aligned}$$

where we have used the characterization (4.44) at the last line. The result follows from

$$\frac{\boldsymbol{Q}(s,\delta)}{Q(s,\delta)} = \frac{V\boldsymbol{a}_i}{\mathbf{1}^TV\boldsymbol{a}_i} + O(Q(s,\delta)^{-1}), \tag{4.51}$$

and by noticing that $sN = O(Q(s,\delta)^{-1})$ and $sNQ(s,\delta)\eta_N = sNQ(s,\delta)sNN^{-\frac{1}{3}} = o(sN) = o(Q(s,\delta)^{-1})$. □

Remark 4.6. We can infer an important physical consequence from the limit equation (4.46) which models the leading order effects of the interactions between the acoustic obstacles. It allows to formally derive the leading order effective damping and effective shifts of the resonant characteristic size $s_i(\delta)$ from the effective equation (4.46). Let us consider a complex characteristic size $s_{\mathrm{eff},i}(\delta) \in \mathbb{C}$ such that

$$\phi - sNQ(s,\delta)\mathcal{V}^{k,\varrho}[\phi] = 0$$

has a non-trivial solution $\phi \in L^2(\Omega)$ for $s = s_{\mathrm{eff},i}(\delta)$. This is possible if $s_{\mathrm{eff},i}(\delta)NQ(s_{\mathrm{eff},i}(\delta),\delta) = \frac{1}{\mu}$, for μ being a (complex) eigenvalue of the volume potential $\mathcal{V}^{k,\varrho}$. To the leading order, we obtain

$$\lambda_i(\mathbf{1}^TV\boldsymbol{a}_i)^2 s_{\mathrm{eff},i}(\delta)N \simeq \frac{1}{\mu}\left(\frac{s_{\mathrm{eff},i}^2}{s_i(\delta)^2} - 1\right),$$

which yields an effective resonant characteristic size

$$s_{\mathrm{eff},i}(\delta) \simeq s_i(\delta)\Big(1 + \frac{\mu}{2}\lambda_i(\mathbf{1}^T V \boldsymbol{a}_i)^2 s_i(\delta) N\Big).$$

If one considers a material with *fixed* scaling characteristic size s (as in [6]), and one is interested in effective resonant frequencies $\omega_{\mathrm{eff},i}(\delta)$, inverting the relation

$$s \simeq \delta^{\frac{1}{2}}\lambda_i^{\frac{1}{2}}\frac{v_b}{\omega_{\mathrm{eff},i}(\delta)}\left(1 + \frac{\mu}{2}\lambda_i(\mathbf{1}^T V \boldsymbol{a}_i)^2 \delta^{\frac{1}{2}}\lambda_i^{\frac{1}{2}}\frac{v_b}{\omega_{\mathrm{eff},i}(\delta)} N\right)$$

yields the effective resonant frequency

$$\omega_{\mathrm{eff},i}(\delta) \simeq \frac{\delta^{\frac{1}{2}}\lambda_i^{\frac{1}{2}} v_b}{s}\Big(1 + \frac{\mu}{2}\lambda_i(\mathbf{1}^T V \boldsymbol{a}_i)^2 s N\Big) \quad \simeq \frac{\delta^{\frac{1}{2}}\lambda_i^{\frac{1}{2}} v_b}{s} + \frac{\mu}{2}\lambda_i^{\frac{3}{2}} v_b(\mathbf{1}^T V \boldsymbol{a}_i)^2 N \delta^{\frac{1}{2}}.$$

This analysis predicts that the multiple scattering interactions induce a damping and an effective shift of the Minnaert resonance by a factor of order $sN = O(N\delta^{\frac{1}{2}})$ (on respectively the real and imaginary parts). This is significantly different from the situation with a single system of K resonators of size s, where the leading order correction for the Minnaert resonance is of order $O(\delta^{\frac{3}{2}}/s)$ (corresponding to a factor δ), and the damping is of order $O(\delta/s)$ (corresponding to a factor $\delta^{\frac{1}{2}}$) (see [6, 38]). This particularly emphasizes that interactions between the N groups of K resonators generate attenuation effects which cannot be predicted by the leading order corrections of a single system of K resonators.

We note that this discussion could be correlated to the study of [64], where the authors also come to the conclusion, by resorting to a different modelling, that an acoustic bubble within an array has a much larger radiative damping than a single bubble.

4.3. Effective medium theory for a monopole-type resonant system up to a critical scale

This third and final subsection is dedicated to the proof of the following homogenization theorem.

Proposition 4.8. *Assume (H1)–(H4). Let u be the solution to the following Lippmann–Schwinger equation:*

$$\begin{cases} (\Delta + k^2 - sNQ(s,\delta)\varrho 1_\Omega)u = 0, \\ \left(\dfrac{\partial}{\partial |x|} - \mathrm{i}k\right)(u - u_{in}) = O\big(|x|^{-2}\big) \text{ as } |x| \to +\infty. \end{cases} \tag{4.52}$$

There exists an event $\mathcal{H}_{N_0}$, independent of s and δ, which holds with large probability $\mathbb{P}(\mathcal{H}_{N_0}) \to 1$ as $N_0 \to +\infty$ such that when $\mathcal{H}_{N_0}$ is realized, the function u is an approximation of the solution field $u_{N,s}$ to (1.3) with the following error estimates:

(i) *on any ball $B(0,r)$ such that $\Omega \subset B(0,r)$, there exists a constant $c > 0$ independent of s, N and δ such that for any $N \geq N_0$:*

$$\mathbb{E}\Big[\|u_{N,s} - u\|^2_{L^2(B(0,R))} | \mathcal{H}_{N_0}\Big]^{\frac{1}{2}} \leq csNQ(s,\delta)\max\Big(\delta^{\frac{1}{2}}N, N^{-\frac{1}{2}}\Big);$$

(ii) *on any bounded open subset $A \subset \mathbb{R}^3 \backslash \Omega$ away from the resonators, there exists a constant $c > 0$ independent of s, N and δ such that for any $N \geq N_0$:*

$$\mathbb{E}\Big[\|\nabla u_{N,s} - \nabla u\|_{L^2(A)} | \mathcal{H}^2_{N_0}\Big]^{\frac{1}{2}} \leq csNQ(s,\delta)\max\Big(\delta^{\frac{1}{2}}N, N^{-\frac{1}{2}}\Big).$$

We can conclude several important results from (4.52):

(i) in the (strictly) *subcritical* regime $sNQ(s,\delta)\to 0$, *i.e.*

$$\frac{\delta^{\frac{1}{2}}N}{\left|\frac{s}{s_i(\delta)}-1\right|}\longrightarrow 0 \text{ as } \delta\to 0, N\to 0 \text{ and } \frac{s}{s_i(\delta)}\to 1,$$

the effective medium is transparent at first order, and we have the convergence $u\to u_{in}$ in $L^2(B(0,r))$;

(ii) in the *critical* regime, *i.e.* $sNQ(s,\delta)\sim\Lambda$ for some constant $\Lambda\in\mathbb{R}$ or

$$\frac{\delta^{\frac{1}{2}}N}{\frac{s}{s_i(\delta)}-1}\to\Lambda' \text{ as } \delta\to 0, N\to+\infty \text{ and } \frac{s}{s_i(\delta)}\to 1$$

for a related constant Λ' which can be inferred from $Q(s,\delta)$, then $u_{N,s}\to u$ where u is the solution to the effective equation

$$\begin{cases} f(\Delta+k^2-\Lambda\varrho 1_\Omega)u=0\\ \left(\dfrac{\partial}{\partial|x|}-\mathrm{i}k\right)(u-u_{in})=O\left(|x|^{-2}\right) \text{ as } |x|\to+\infty. \end{cases} \tag{4.53}$$

We note that the constant Λ can be negative and (4.53) may exhibit dispersive effects;

(iii) in the *subcritical* regime $sNQ(s,\delta)\to+\infty$, we expect some solidification effects to arise as in the sound-soft problem (1.2). A different analysis of this regime is required and is left for a future work, referring to [13,15] for a different but related analysis.

Remark 4.7. Due to resonances, the setting of (1.3) is fundamentally different than the one of (1.2), although the homogenized equation for the latter is obtained by simply replacing cap (D) by $Q(s,\delta)$. The coefficient $Q(s,\delta)$ is indeed variable and has different asymptotic behaviours depending on how s compares to the resonant size $s_i(\delta)$, while still satisfying $s\to 0$. This shows that the limits $\delta\to 0$ and $s\to 0$ cannot be inverted, and one must consider suitable scaling regimes to obtain nontrivial effective behaviours. More physically said, the regime $\delta\to 0$ requires the scattered field $u_{N,s}-u_{in}$ to be constant in every inclusion, but these constants are different from zero. The capacitance matrix analysis (and in particular equation (4.58) below together with (4.51)) shows that the constants are proportional to the eigenvectors of the capacitance eigenvalue problem (1.13).

The proof relies on the expression (4.20) for the derivation of the homogenized resonant scattered field.

Lemma 4.5. *We have the following asymptotic expansions with the norm of $L^2(\partial\mathcal{D})$ defined in (2.12):*

(i) $\mathcal{G}^{-1}(s,\delta)=\mathcal{L}_0^{-1}+O(sN\eta_N)_{L^2(\partial\mathcal{D})\to L^2(\partial\mathcal{D})}$,

(ii) $\mathcal{G}^{-1}(s,\delta)[f]=O(\delta N^{\frac{1}{2}})_{L^2(\partial\mathcal{D})}$,

(iii) $(\mathcal{S}_\mathcal{D}^k)^{-1}\mathcal{S}_\mathcal{D}^{k_b}\mathcal{G}^{-1}(s,\delta)=\mathcal{L}_0^{-1}+O(sN)_{L^2(\partial\mathcal{D})\to L^2(\partial\mathcal{D})}$,

(iv) $(\mathcal{S}_\mathcal{D}^k)^{-1}\mathcal{S}_\mathcal{D}^{k_b}\mathcal{G}^{-1}(s,\delta)[f]=O(\delta N^{\frac{1}{2}})_{L^2(\partial\mathcal{D})}$,

(v) $\left(\mathcal{S}_\mathcal{D}^k\right)^{-1}\mathcal{P}_{N,s}^{-1}[u_{in}]=\left(s^{-1}u_{in}(y_i)\psi^*\right)_{1\le i\le N}+O(s^{-1}sNsN^{\frac{1}{2}})_{L^2(\partial\mathcal{D})}=O\left(N^{\frac{1}{2}}\right)_{L^2(\partial\mathcal{D})}$.

Proof. (i) is a consequence of Lemma 4.3.

(ii) This identity comes from $\|f\|_{L^2(\partial\mathcal{D})}=O(\delta N^{\frac{1}{2}})$, see Proposition 4.5.

(iii) $\left(\mathcal{S}_\mathcal{D}^k\right)^{-1}\mathcal{S}_\mathcal{D}^{k_b}\mathcal{G}^{-1}(s,\delta)=(\mathrm{I}+O(sN))(\mathcal{L}_0^{-1}+O(sN\eta_N))=\mathcal{L}_0^{-1}+O(sN)$.

(iv) Is obtained identically as (4.5).

(v)
$$\begin{aligned}(\mathcal{S}_\mathcal{D}^k)^{-1}\mathcal{P}_{N,s}^{-1}[u_{in}]&=s^{-1}(\mathcal{S}_{\mathcal{D},0}^{-1}+O(sN))[(u_{in}(y_i)1_{\partial D})+O(ssN^{\frac{1}{2}})_{L^2(\partial\mathcal{D})}]\\&=s^{-1}(u_{in}(y_i)\psi^*)_{1\le i\le N}+O(s^{-1}sNsN^{\frac{1}{2}})_{L^2(\partial\mathcal{D})}+O(s^{-1}ssN^{\frac{1}{2}})_{L^2(\partial\mathcal{D})}.\end{aligned}$$

□

Consequently, we obtain the following approximations for the solution $(\mathcal{P}_{N,s}^{-1}[\varphi],\mathcal{P}_{N,s}^{-1}[\psi])$ of (4.20).

Lemma 4.6. *The following asymptotic formulas hold for the solution* $(\mathcal{P}_{N,s}^{-1}[\varphi], \mathcal{P}_{N,s}^{-1}[\psi])$ *of* (4.5):

$$\begin{cases} \mathcal{P}_{N,s}^{-1}[\varphi] = \sum\limits_{\substack{1\leq i\leq N\\ 1\leq j\leq K}} w_i^N \dfrac{\boldsymbol{e}_j^T \boldsymbol{Q}(s,\delta)}{Q(s,\delta)} \phi_{ij}^* + O\Big((sN)Q(s,\delta)N^{\frac{1}{2}}\Big)_{L^2(\partial\mathcal{D})}, \\ \mathcal{P}_{N,s}^{-1}[\psi] = \sum\limits_{\substack{1\leq i\leq N\\ 1\leq j\leq K}} w_i^N \dfrac{\boldsymbol{e}_j^T \boldsymbol{Q}(s,\delta)}{Q(s,\delta)} \phi_{ij}^* + O\Big((sN)Q(s,\delta)N^{\frac{1}{2}}\Big)_{L^2(\partial\mathcal{D})}, \end{cases} \tag{4.54}$$

where $(\boldsymbol{e}_j)_{1<j\leq K}$ *stands for the canonical basis of* $\mathbb{R}^K$ *and* $O(sN)$ *is the relative error.*

Proof. First, the result of Proposition 4.7 yields

$$\begin{aligned} \sum_{\substack{1\leq i\leq N\\ 1\leq j\leq K}} x_{ij}^N \phi_{ij}^* &= \sum_{\substack{1\leq i\leq N\\ 1\leq j\leq K}} \left(w_i^N \frac{\boldsymbol{e}_j^T \boldsymbol{Q}(s,\delta)}{Q(s,\delta)} + O\Big(sNs^{-1}Q(s,\delta)N^{\frac{1}{2}}\Big)_{\|\cdot\|_2} \right) \phi_{ij}^* \\ &= \sum_{\substack{1\leq i\leq N\\ 1\leq j\leq K}} w_i^N \frac{\boldsymbol{e}_j^T \boldsymbol{Q}(s,\delta)}{Q(s,\delta)} \phi_{ij}^* + O\Big((sN)Q(s,\delta)N^{\frac{1}{2}}\Big)_{L^2(\partial\mathcal{D})}. \end{aligned}$$

Consequently, by using (i) and (ii) of Lemma 4.5, we obtain

$$\begin{aligned} \mathcal{P}_{N,s}^{-1}[\varphi] &= \sum_{\substack{1\leq i\leq N\\ 1\leq j\leq K}} w_i^N \frac{\boldsymbol{e}_j^T \boldsymbol{Q}(s,\delta)}{Q(s,\delta)} \phi_{ij}^* + O\Big(sN\eta_N Q(s,\delta)N^{\frac{1}{2}}\Big)_{L^2(\partial\mathcal{D})}) + O\Big(sNQ(s,\delta)N^{\frac{1}{2}}\Big)_{L^2(\partial\mathcal{D})} + O(\delta N^{\frac{1}{2}}) \\ &= \sum_{\substack{1\leq i\leq N\\ 1\leq j\leq K}} w_i^N \frac{\boldsymbol{e}_j^T \boldsymbol{Q}(s,\delta)}{Q(s,\delta)} \phi_{ij}^* + O\Big(Q(s,\delta)^{-1}Q(s,\delta)N^{\frac{1}{2}}\Big)_{L^2(\partial\mathcal{D})}. \end{aligned}$$

Similarly, we find

$$\begin{aligned} \mathcal{P}_{N,s}^{-1}[\psi] &= \big(\mathcal{S}_{\mathcal{D}}^{k}\big)^{-1} \mathcal{S}_{\mathcal{D}}^{k_b} \mathcal{G}^{-1}(s,\delta) \left[\sum_{\substack{1\leq i\leq N\\ 1\leq j\leq K}} x_{ij}^N \phi_{ij}^* \right] + \big(s^{-1}u_{in}(y_i)\psi^*\big)_{1\leq i\leq N} + O\Big(\delta N^{\frac{1}{2}}\Big) + O\Big(s^{-1}sNsN^{\frac{1}{2}}\Big)_{L^2(\partial\mathcal{D})} \\ &= \sum_{\substack{1\leq i\leq N\\ 1\leq j\leq K}} w_i^N \frac{\boldsymbol{e}_j^T \boldsymbol{Q}(s,\delta)}{Q(s,\delta)} \phi_{ij}^* + O\Big(sNss^{-1}Q(s,\delta)N^{\frac{1}{2}}\Big)_{L^2(\partial\mathcal{D})} + O\Big(sNQ(s,\delta)N^{\frac{1}{2}}\Big)_{L^2(\partial\mathcal{D})} + O\Big(N^{\frac{1}{2}}\Big)_{L^2(\partial\mathcal{D})}, \end{aligned}$$

from where the result follows. ∎

We infer the following approximations for the potentials $\mathcal{S}_{D_{N,s}}^{k_b}[\varphi]$ and $\mathcal{S}_{D_{N,s}}^{k}[\psi]$.

Proposition 4.9. *The following asymptotic expansions hold on* $D_{N,s}$*, on the ball* $B(0,r)$ *containing the inclusions, and on any open set* $A \subset \mathbb{R}^3 \backslash B(0,r)$ *outside the inclusions:*

$$\begin{aligned} \mathcal{S}_{D_{N,s}}^{k_b}[\varphi] &= \sum_{\substack{1\leq i\leq N\\ 1\leq l\leq K}} s w_i^N \frac{\boldsymbol{e}_j^T \boldsymbol{Q}(s,\delta)}{Q(s,\delta)} \mathcal{S}_D^{sk_b}[\phi_l^*] \circ \tau_{y_i,s}^{-1} + O\Big(sNQ(s,\delta)s^{\frac{3}{2}}N^{\frac{1}{2}}\Big)_{L^2(D_{N,s})} \\ &= O\Big(Q(s,\delta)s^{\frac{3}{2}}N^{\frac{1}{2}}\Big)_{L^2(D_{N,s})}, \end{aligned} \tag{4.55}$$

$$\mathcal{S}^k_{D_{N,s}}[\psi] = \sum_{\substack{1\leq i\leq N\\1\leq l\leq K}} s w_i^N \frac{\boldsymbol{e}_j^T \boldsymbol{Q}(s,\delta)}{Q(s,\delta)} \mathcal{S}^{sk}_D[\phi_l^*]\circ\tau_{y_i,s}^{-1} + O(sNQ(s,\delta)sN)_{L^2(B(0,r))}, \tag{4.56}$$

$$\nabla\mathcal{S}^k_{D_{N,s}}[\psi] = \sum_{\substack{1\leq i\leq N\\1\leq l\leq K}} s w_i^N \frac{\boldsymbol{e}_j^T \boldsymbol{Q}(s,\delta)}{Q(s,\delta)} \nabla\mathcal{S}^{sk}_D[\phi_l^*]\circ\tau_{y_i,s}^{-1} + O(sNQ(s,\delta)sN)_{L^2(A)}, \tag{4.57}$$

where sN is the relative error.

Proof. These estimates are obtained by applying the bounds of Proposition 2.5 and by using the identity

$$\mathcal{S}^k_{y_i+sD}\left[\phi_{il}^*\circ\tau_{y_i,s}^{-1}\right] = s\mathcal{S}^k_D[\phi_l^*]\circ\tau_{y_i,s}^{-1}.$$

For (4.55), we recall that $sN\to 0$ due to (1.18) in (H4) and $|Q(s,\delta)|\to+\infty$. □

Proof of Proposition 4.8. From the previous estimates (4.55) and (4.56), we obtain the following asymptotic expansion for $u_{N,s}$ in the ball $B(0,r)$:

$$\begin{aligned}
u_{N,s} &= u_{in} + \mathcal{S}^k_{D_{N,s}}\left[\mathcal{P}_{N,s}\left[\mathcal{P}_{N,s}^{-1}[\psi]\right]\right] + O\Big(Q(s,\delta)s^{\frac{3}{2}}N^{\frac{1}{2}}\Big)\\
&= u_{in} + \sum_{\substack{1\leq i\leq N\\1\leq j\leq K}} s^2 w_i^N \frac{\boldsymbol{e}_j^T\boldsymbol{Q}(s,\delta)}{Q(s,\delta)} \int_{\partial D}\Gamma^k(\cdot - y_i - st)\phi_j^*(t)\,\mathrm{d}\sigma(t) + O((sN)Q(s,\delta)sN)_{L^2(B(0,r))}\\
&= u_{in} + \sum_{1\leq j\leq K}\boldsymbol{e}_j^T\frac{\boldsymbol{Q}(s,\delta)}{Q(s,\delta)}\int_{\partial D}\left(\frac{s}{Q(s,\delta)}w^N(\cdot - st) - u_{in}(\cdot - st)\right)\phi_j^*(t)\,\mathrm{d}\sigma(t)\\
&\quad + O((sN)Q(s,\delta)sN)_{L^2(B(0,r))},
\end{aligned}\tag{4.58}$$

where w^N is the Nystrom interpolant (4.47). Using the same methodology as in Section 3.3, we read the following convergences:

$$\begin{aligned}
u_{N,s} &= u_{in} + \sum_{1\leq j\leq K}\boldsymbol{e}_j^T\frac{\boldsymbol{Q}(s,\delta)}{Q(s,\delta)}\int_{\partial D}\left(\frac{s}{Q(s,\delta)}w(\cdot - st) - u_{in}(\cdot - st)\right)\phi_j^*(t)\,\mathrm{d}\sigma(t)\\
&\quad + O\Big((sN)Q(s,\delta)\max\Big(sN, N^{-\frac{1}{2}}\Big)\Big)_{\mathbb{E}\left[\|\cdot\|^2_{L^2(B(0,r))}\right]^{\frac{1}{2}}}\\
&= u_{in} + \sum_{1\leq j\leq K}\boldsymbol{e}_j^T\frac{\boldsymbol{Q}(s,\delta)}{Q(s,\delta)}\left(\frac{s}{Q(s,\delta)}w - u_{in}\right)\\
&\quad + O\Big(\max\Big(sN, N^{-\frac{1}{2}}\Big)sNQ(s,\delta)\Big)_{\mathbb{E}\left[\|\cdot\|^2_{L^2(B(0,r))}\right]^{\frac{1}{2}}} + O(s)_{\mathbb{E}\left[\|\cdot\|^2_{L^2(B(0,r))}\right]^{\frac{1}{2}}}\\
&= \frac{s}{Q(s,\delta)}w + O\Big(\max\Big(sN, N^{-\frac{1}{2}}, N^{-1}Q(s,\delta)^{-1}\Big)sNQ(s,\delta)\Big)_{L^2(B(0,r)}.
\end{aligned}$$

The result follows because $s/Q(s,\delta)w$ is the solution to (4.52). □

Remark 4.8. When the packet of obstacles D is constituted of a single resonator $D\equiv B$ ($K=1$), we have (see [11,38]):

$$\lambda_1 = \frac{\operatorname{cap}(D)}{|D|},\quad \boldsymbol{a}_1 = |D|^{-\frac{1}{2}},\quad V = |D|.$$

Therefore,

$$Q(s,\delta) = \frac{\lambda_1}{\frac{s^2}{s_1(\delta)^2} - 1}(\boldsymbol{a}_1^T V \mathbf{1})^2 = \frac{\operatorname{cap}(D)}{\frac{\omega^2}{\omega_M^2} - 1}, \text{ with } \omega_M := v_b \sqrt{\frac{\operatorname{cap}(D)}{|D|}} \frac{\delta^{\frac{1}{2}}}{s},$$

and the effective equation reads, as $\omega \to \omega_M$ in the subcritical regime (H4):

$$\begin{cases} \left(\Delta + k^2 - \dfrac{sN}{\frac{\omega^2}{\omega_M^2} - 1} \operatorname{cap}(D) \varrho 1_\Omega\right) u = 0 \\ \qquad\qquad\qquad\qquad \big(\partial_{|x|} - \mathrm{i}k\big) u = O\big(|x|^{-2}\big) \text{ as } |x| \to +\infty. \end{cases}$$

We retrieve therefore the result of the seminal paper [6].

Appendix A. Markov inequality and law of large numbers

We used the following result in Proposition 2.1.

Proposition A.1. *Let X be a square integrable real random variable ($\mathbb{E}[X^2] < +\infty$) and $a \in \mathbb{R}$. The following Markov inequality holds:*

$$\mathbb{P}(|X|^2 \geq |a|^2) \leq \frac{\mathbb{E}[|X|^2]}{|a|^2}.$$

Proposition A.2. *Let $(y_i)_{i\in\mathbb{N}}$ be a sequence of independent real random variables and $f : \mathbb{R}^p \to \mathbb{R}$ be a square integrable function:*

$$\mathbb{E}[f(y_1,\dots,y_p)^2] < +\infty.$$

Let S denote the set of pair of increasing indices $((i_1,\dots,i_p),(j_1,\dots,j_p))$ with at least one element in common:

$$S := \left\{ ((i_1,\dots,i_p),(j_1,\dots,j_p)) \in \mathbb{N}^p \times \mathbb{N}^p \;\middle|\; \begin{array}{r} 1 \leq i_1 < \dots < i_p \leq N, \\ 1 \leq j_1 < \dots < j_p \leq N \\ \exists 1 \leq k, l \leq p \text{ such that } i_k = j_l \end{array} \right\}.$$

The following law of large numbers holds: for N sufficiently large,

$$\begin{aligned} &\mathbb{E}\left[\left|\frac{p!(N-p)!}{N!} \sum_{i_1<i_2<\dots i_p} f\big(y_{i_1},\dots,y_{i_p}\big) - \mathbb{E}[f(y_1,\dots,y_p)]\right|^2\right]^{\frac{1}{2}} \\ &\quad \leq \frac{p!(N-p)!}{N!} \left(\sum_{((i_1,\dots,i_p),(j_1,\dots,j_p))\in S} \Big(\mathbb{E}\big[f\big(y_{i_1},\dots,y_{i_p}\big) f\big(y_{j_1},\dots,y_{j_p}\big)\big] - \mathbb{E}[f(y_1,\dots,y_p)]^2 \Big) \right)^{\frac{1}{2}} \\ &\quad \leq 2pN^{-\frac{1}{2}} \mathbb{E}\Big[\big|f(y_1,\dots,y_p) - \mathbb{E}[f(y_1,\dots,y_p)]\big|^2\Big]^{\frac{1}{2}}. \end{aligned} \tag{A.1}$$

Proof. The number of elements of S is

$$\begin{aligned} |S| &= \left(\frac{N!}{(N-p)!p!}\right)^2 - \frac{N!}{p!(N-p)!} \frac{(N-p)!}{p!(N-2p)!} \\ &= \left(\frac{N!}{(N-p)!p!}\right)^2 \left(1 - \frac{(N-p)!^2}{N!(N-2p)!}\right) = \left(\frac{N!}{(N-p)!p!}\right)^2 \left(\frac{p^2}{N} + O\left(\frac{1}{N^2}\right)\right). \end{aligned} \tag{A.2}$$

Then by independence, we have

$$\begin{aligned}
&\mathbb{E}\left[\left|\frac{p!(N-p)!}{N!}\sum_{i_1<i_2<\dots i_p} f(y_{i_1},\dots,y_{i_p})-\mathbb{E}[f(y_1,\dots,y_p)]\right|^2\right]\\
&=\left(\frac{p!(N-p)!}{N!}\right)^2\sum_{\substack{1\leq i_1<\dots<i_p\leq N\\ 1\leq j_1<\dots<j_p\leq N}}\mathbb{E}\big[\big(f(y_{i_1},\dots,y_{i_p})-\mathbb{E}\big[f(y_{i_1},\dots,y_{i_p})\big]\big)\big(f(y_{j_1},\dots,y_{j_p})-\mathbb{E}\big[f(y_{j_1},\dots,y_{j_p})\big]\big)\big]\\
&=\left(\frac{p!(N-p)!}{N!}\right)^2\sum_{((i_1,\dots,i_p),(j_1,\dots,j_p))\in S}\left(\mathbb{E}\big[f(y_{i_1},\dots,y_{i_p})f(y_{j_1},\dots,y_{j_p})\big]-\mathbb{E}[f(y_1,\dots,y_p)]^2\right)\\
&\leq\left(\frac{p!(N-p)!}{N!}\right)^2|S|\left(\mathbb{E}\Big[|f(y_1,\dots,y_p)|^2\Big]-\mathbb{E}[f(y_1,\dots,y_p)]^2\right)\\
&\leq\frac{4p^2}{N}\mathbb{E}\Big[\big(f(y_1,\dots,y_p)-\mathbb{E}[f(y_1,\dots,y_p)]\big)^2\Big],
\end{aligned}$$

where we used the Cauchy–Schwartz inequality at the fourth line and (A.2) for N sufficiently large at the last line. □

Remark A.1. The second inequality in (A.1) yields more precise estimates than the second one.

Finally we recall the limit of the probability distribution of the minimum distance between independently distributed random points.

Proposition A.3 ([48, 60])**.** *Let $(y_i)_{i\in\mathbb{N}}$ be independent random points distributed according to a probability measure $\varrho\,\mathrm{d}\mu$ with support of dimension d (e.g. $\mathrm{d}\mu=\mathrm{d}x$ if $d=3$, $\mathrm{d}\mu=\mathrm{d}\sigma$ if $d=2$, $\mathrm{d}\mu=\mathrm{d}\ell$ if $d=1$). Assume that ϱ is square integrable. Then*

$$\mathbb{P}\left(\min_{1\leq i\neq j\leq N}|y_i-y_j|<tN^{-\frac{2}{3}}\right)\xrightarrow{N\to+\infty}1-\exp\left(-\frac{1}{2}t^dc_d\int\varrho^2\,\mathrm{d}\mu\right),$$

where $c_d=\pi^{d/2}\Gamma(1+d/2)$ is the volume of the d-dimensional unit ball. As a consequence, for any $c>0$, there exists $t>0$ such that for any $N\in\mathbb{N}$,

$$\mathbb{P}\left(t^{-1}N^{-2/d}\leq\min_{1\leq i\neq j\leq N}|y_i-y_j|\leq tN^{-2/d}\right)>1-c.$$

Appendix B. Higher order derivatives of the Helmholtz fundamental solution

We recall the following result which was used in Proposition 2.3.

Proposition B.1 ([38], Lem. 5.1)**.** *There exists a constant $c>0$ independent of k such that for any $p\in\mathbb{N}$,*

$$\forall x\in\mathbb{R}^3,\quad\nabla^{p+1}\Gamma^k(x)\leq c6^pp!|x|^{-1}\big(|x|^{-p}+k^p\big).\tag{B.1}$$

Appendix C. Resolvent estimates of Schatten operators

The proof of Proposition 3.3 is based on the following result from Bandtlow [21] which bounds the norm of the resolvent of a possibly nonnormal Hilbert–Schmidt operator in terms of the distance to the spectrum $\sigma(A)$. We denote by $\rho(A)$ the resolvent set of an operator A.

Proposition C.1. *Let A be a Hilbert–Schmidt operator. For any $\lambda \in \rho(A)$, the following inequality holds:*

$$|||(\lambda \mathrm{I} - A)^{-1}|||_2 \leq \frac{1}{d(\lambda, \sigma(A))} \exp\left(\frac{1}{2}\left(\frac{\mathrm{Tr}(\overline{A^T}A)}{d(\lambda, \sigma(A))^2} + 1\right)\right), \tag{C.1}$$

where $d(\lambda, \sigma(A))$ is the distance of λ to the spectrum of A:

$$d(\lambda, \sigma(A)) := \inf_{\mu \in \sigma(A)} |\lambda - \mu|.$$

Proof. See Theorem 4.1 in [21]. □

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