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CONVERGENCE OF ENTROPY STABLE SCHEMES FOR DEGENERATE PARABOLIC EQUATIONS WITH A DISCONTINUOUS CONVECTION TERM

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Abstract. Three-point entropy stable schemes are extended for partial differential equations of the degenerate convection-diffusion type where a discontinuous space-dependent function is incorporated in the convective flux. Using the compensated compactness theory, convergence of the proposed entropy stable approximations to the entropy weak solution is proved. Assuming the so-called potential condition in the jump discontinuities, an estimate for entropy functions is demonstrated. Finally, using benchmark tests a validation of the efficiency of the entropy stable scheme is provided by comparison with an upwind-type solution.

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1. Introduction

In this work we focus on scalar degenerate parabolic-hyperbolic equations with discontinuous coefficients of the form

$$u_t + f(\gamma(x), u)_x = A(u)_{xx}, \quad (x,t) \in \Pi_T := \mathbb{R} \times [0,T], \tag{1.1}$$

$$u(x,0) = u_0(x), \qquad x \in \mathbb{R}, \tag{1.2}$$

where $u = u(x,t)$ represents a physical property of a system (mass, velocity, energy, etc.), f is the transport or convective flux of the physical property u, γ is a function modeling velocity variations dependent on space and A serves as the integrated diffusion coefficient of u. We consider a degenerate diffusion, that is, $A'(u)$ vanishes at one point or over an entire interval, which is commonly named strong degeneracy. Equations of the form (1.1) are used to model several phenomena such as flow in porous media [9, 10, 12], consolidation-sedimentation process in clarifier-thickener units [2, 4–6] and traffic flow in roads where the flux depends on the position or the driver's reaction [1].

Notice that if there are discontinuities in the equation coefficients or a degenerate diffusion, classical solutions of equation (1.1) might not exist. For such reason, it is necessary to seek solutions in a variational sense which are known as weak solutions.

Definition 1.1 ([19])**.** A weak solution of problem (1.1) and (1.2) is a measurable function $u(x,t)$ satisfying

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(i) $u(x,t) \in L^1(\Pi_T) \cap L^\infty(\Pi_T) \cap C(0,T,L^1(\mathbb{R}))$ and $A(u) \in L^2(0,T;H^1(\mathbb{R}))$.
(ii) For all test functions $\phi \in C_0^\infty(\Pi_T)$ such that $\phi|_{t=T} = 0$

$$\iint_{\Pi_T} (u\phi_t + f(\gamma(x),u)\phi_x + A(u)\phi_{xx})\,\mathrm{d}t\,\mathrm{d}x + \int_{\mathbb{R}} u_0(x)\phi(x,0)\,\mathrm{d}x = 0. \tag{1.3}$$

Weak solutions may not be unique; consequently, an extra condition should be imposed. For that, an entropy condition is derived based on physical properties of the system of interest and a convex function, $\eta(u)$, called entropy function is defined. For problem (1.1) and (1.2), Karlsen *et al.* in [19] proposed a Kružkov-type entropy condition by considering $\eta(u) = |u-c|$. Existence of Kružkov entropy weak solutions was proven in [18] and its uniqueness *via* L^1-stability in [21]. Moreover, the convergence of an upwind approximation to the proposed Kružkov entropy solution was also demonstrated in [19]. This last result is important, since equation (1.1) is not exactly solvable and the upwind solution may be used as a convergent reference solution [5, 11, 13, 22].

In this work, we are interested in proving the convergence of three-point entropy stable approximations to entropy solutions for the degenerate parabolic problem (1.1) and (1.2). An entropy stable framework for conservation laws was introduced by Tadmor in [25, 27], for nonconservative hyperbolic systems was derived in [7], extended in [17] for degenerate parabolic conservation laws and high-order entropy stable essentially nonoscillatory schemes were proposed in [15]. Entropy stable schemes proposed here is a family of space finite-difference semi-discretizations which are second-order accurate for smooth zones whereas the upwind discretization is first-order accurate. This feature differentiates how both methods deal with discontinuities and smooth regions of the solution, as we will show later. For the convergence proof we will use the compensated compactness theory allowing to deal with sign changes of the discontinuous function γ and the non convexity of the convective flux without further costs. This methodology has been used before to prove existence of weak solutions for degenerate parabolic equations [18] and convergence of numerical approximations for the scalar and multidimensional conservation laws with continuous and discontinuous coefficients [3, 8, 14, 20].

Let us start by defining entropy solutions based on all possible entropy 3-tuples:

Definition 1.2. If γ is piecewise continuous with a finite number of jump type discontinuities located at $\{\xi_m\}_{m=1}^M$, a weak solution u satisfies an entropy condition for problem (1.1) and (1.2) if for all C^2 convex functions $\eta : \mathbb{R} \to \mathbb{R}$

$$\begin{aligned}
&\iint_{\Pi_T} \Big(\eta(u)\phi_t + q(\gamma(x),u)\phi_x + r(u)_x\phi_x\Big)\,\mathrm{d}t\,\mathrm{d}x \\
&\quad + \iint_{\Pi_T\setminus\{\xi_m\}_{m=1}^M} \gamma'(x)\Big(\eta'(u)f_\gamma(\gamma(x),u) - q_\gamma(\gamma(x),u)\Big)\phi\ \,\mathrm{d}t\,\mathrm{d}x \\
&\quad + \int_0^T \sum_{m=1}^M \Big(\gamma'(x)(\eta'(u)f_\gamma(\gamma(x),u) - q_\gamma(\gamma(x),u))\Big)\Big|_{\xi_m^-}^{\xi_m^+} \phi(\xi_m,t)\,\mathrm{d}t \geqslant 0,
\end{aligned} \tag{1.4}$$

for any $0 \le \phi \in C_0^\infty(\Pi_T)$, where $q : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is the entropy flux and $r : \mathbb{R} \to \mathbb{R}$ is the diffusive entropy flux satisfying

$$\begin{aligned}
q_u(\gamma(x),u) &= \eta'(u)f_u(\gamma(x),u), \\
r'(u) &= \eta'(u)A'(u).
\end{aligned} \tag{1.5}$$

The equivalence between the previous definition and Kružkov-type entropy solutions will be discussed in Appendix A.

From now on the following conditions are assumed in order to guarantee the existence of the proposed entropy weak solutions:

(H1) $\gamma(x) \in BV(\mathbb{R})$.
(H2) $\gamma(x)$ is piecewise function $C^1(\mathbb{R})$ with a finite number of jump discontinuities and $\gamma(x) \neq 0$ for all $x \in \mathbb{R}$.
(H3) $f(\gamma(x), u)$ is Lipschitz continuous in each variable and $f(\gamma(x), 0) \in L^2(\mathbb{R} \times \mathbb{R})$.
(H4) $A(u)$ is Lipschitz continuous, piecewise smooth and satisfies $A(u) \leqslant A(w)$ if $u \leqslant w$.
(H5) Intervals $[\alpha, \beta]$ such that $A'(u) = 0, \forall u \in [\alpha, \beta]$ are allowed.
(H6) $\underline{u} \leqslant u_0(x) \leqslant \overline{u}$, $\forall x \in \mathbb{R}$ for certain values $\underline{u}$ and $\overline{u}$.

Moreover, we will prove *a priori* estimate for the entropy function in Theorem 4.2, which is an extension of Tadmor's result given in [27] for hyperbolic conservation laws. For that purpose, we consider a 3-tuple $(\eta(u; c), q(\gamma(x), u; c), r(u; c))$ parameterized by $c \in \mathbb{R}$ satisfying the following symmetry property:

$$\begin{aligned} \eta(u; c) &= \eta(c; u), \\ q(\gamma(x), u; c) &= q(\gamma(x), c; u), \\ r(u; c) &= r(c; u). \end{aligned} \tag{1.6}$$

In particular, Kružkov entropy function provides a 3-tuple with the previous desired symmetry property for the scalar case [21].

Let us now denote by $v = \eta'(u)$ the entropy variable. The mapping $v \to u$ is one to one and by inverting such function, $u(v)$, we define the potential function as follows:

$$\varphi(\gamma(x), u(v)) = v f(\gamma(x), u(v)) - q(\gamma(x), u(v)). \tag{1.7}$$

Let u and w be two entropy solutions of the Cauchy problem (1.1) and (1.2) satisfying the symmetry condition (1.6). The following two extra conditions are also imposed:

Assumption 1.3 (Potential condition)**.** *By considering $\gamma(x)$ as a function satisfying hypothesis (H1) and (H2), the entropy potential function φ satisfies*

$$\varphi_\gamma(\gamma(x), w; u) \cdot \gamma'(x) \leqslant 0.$$

for $x \neq \{\xi_m\}_{m=1}^M$.

Assumption 1.4 (Jump condition)**.** *For any discontinuity point ξ_m for $m = 1, \ldots, M$ the inequality*

$$(q(\gamma(x), w; u) - r(w; u)_x)\Big|_{\xi_m^-}^{\xi_m^+} \leqslant 0,$$

holds.

Let us exemplify the previous assumptions for the particular case of the Kružkov entropy 3-tuple. For Assumption 1.3, its condition is equivalent to

$$f_\gamma(\gamma(x), c)\gamma'(x) \leq 0, \quad \forall x.$$

Therefore, if there exists a value c such that $x \to f(\gamma(x), c)$ is constant, then Assumption 1.3 is always verified for the Kružkov 3-tuple. In particular,

$$f(\gamma(x), 0) = 0, \quad \forall x,$$

is a physically expected condition. Now, Assumption 1.4 is proven for Krûzkov entropy 3-tuples in [21] if for any states u and w the crossing condition

$$f(\gamma_+, u) - f(\gamma_-, u) < 0 < f(\gamma_+, w) - f(\gamma_-, w) \Rightarrow u < w$$

is assumed for any jump in γ, where (γ_-, γ_+) are the associated left and right limits.

The paper is organized as follows: In Section 2, we propose the entropy stable discretization for equation (1.1) which satisfies a discrete entropy inequality consistent with the entropy weak solution given in (1.2). In Section 3, we provide one of our main results: the L^∞ convergence of the entropy stable scheme. Which is proved based on the works of Karlsen *et al.* [18] and Fjordholm [14] using the compensated compactness methodology. Section 4 is devoted to prove the entropy estimate considering Assumptions 1.3 and 1.4. In Section 5 we illustrate the performance of the entropy stable approximation which captures correctly different dynamics like shock waves, rarefaction waves and oscillating solutions, improving in some cases the upwind approximation. Finally, conclusions are drawn in Section 6.

2. Entropy stable approximation

In this section, we adapt the entropy stable schemes proposed in [17] to the nonlinear parabolic equation (1.1). There are some previous works in the literature devoted to adapt numerical methods to treat the type of discontinuities in the advection term, for example upwind schemes [19, 22, 30], relaxation schemes [20] and Engquist-Osher schemes [5, 29]. In our case we consider some ideas from Karlsen *et al.*'s work [19] for the approximation of the function $\gamma(x)$.

Let us first discretize uniformly the spatial domain where the mesh points $\{x_i\}_{i\in\mathbb{N}}$ are obtained by the step-size $\Delta x = x_{i+1} - x_i$. To approximate the function γ into the convection flux we consider a piecewise function defined as follows:

$$\gamma_{i+\frac{1}{2}}(x) \approx \frac{1}{2}(\gamma(x_i) + \gamma(x_{i+1})), \quad \forall i \tag{2.1}$$

for all $x \in [x_i, x_{i+1})$. Here it is assumed that the possible discontinuities of γ are placed at the cell interfaces. Approximation (2.1) is simple and has given satisfactory results [19].

We now present the entropy conservative scheme for equation (1.1) proposed in [17]

$$\frac{\mathrm{d}}{\mathrm{d}t}u_i(t) = -\frac{1}{\Delta x}\Big(F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}\Big) + \frac{1}{(\Delta x)^2}\Big([A]_{i+\frac{1}{2}} - [A]_{i-\frac{1}{2}}\Big), \tag{2.2}$$

where $[A]_{i+\frac{1}{2}} = A(u_{i+1}) - A(u_i)$ and $F_{i+\frac{1}{2}}$ is any numerical flux satisfying

$$[v]_{i+\frac{1}{2}} F_{i+\frac{1}{2}} = [\varphi]_{i+\frac{1}{2}}, \tag{2.3}$$

being $[\varphi]_{i+\frac{1}{2}} = \varphi(\gamma_{i+\frac{1}{2}}, u_{i+1}) - \varphi(\gamma_{i+\frac{1}{2}}, u_i)$, $v = \eta'(u)$ is the entropy variable and φ is the potential function defined in (1.7). Moreover, we consider numerical fluxes of the form

$$F_{i+\frac{1}{2}} = H\Big(f\Big(\gamma_{i+\frac{1}{2}}, u_{i+\frac{1}{2}}\Big)\Big),$$

where H is an operator satisfying the Lipschitz condition:

$$\Big|H\Big(f\Big(\gamma_{i+\frac{1}{2}}, u_{i+\frac{1}{2}}\Big)\Big) - H\Big(f\Big(\gamma_{i-\frac{1}{2}}, u_{i-\frac{1}{2}}\Big)\Big)| \le k_H |f\Big(\gamma_{i+\frac{1}{2}}, u_{i+\frac{1}{2}}\Big) - f\Big(\gamma_{i-\frac{1}{2}}, u_{i-\frac{1}{2}}\Big)\Big|, \tag{2.4}$$

with k_H a positive constant depending on the operator H. For instance,

$$H\Big(f\Big(\gamma_{i+\frac{1}{2}}, u_{i+\frac{1}{2}}\Big)\Big) := \int_{\frac{-1}{2}}^{\frac{1}{2}} f\Big(\gamma_{i+\frac{1}{2}}, u\Big(v_{i+\frac{1}{2}}(\xi)\Big)\Big)\,\mathrm{d}\xi, \tag{2.5}$$

where $v_{i+\frac{1}{2}}(\xi) = \frac{1}{2}(v_{i+1} + v_i) + \xi\Delta v_{i+\frac{1}{2}}$.

Notice that the approximation of the integrated diffusion coefficient is valid for degenerate and nondegenerate diffusion terms. When the convection term dominates over the viscous term or the diffusive term vanishes (degenerate case), then some extra viscosity is required in order to get entropy stability [17, 25]. Thus, to

prevent spurious oscillations, some numerical viscosity is added to the scheme (2.2) and the entropy stable solution is then given by the formula

$$\frac{\mathrm{d}}{\mathrm{d}t}u_i(t) = -\frac{1}{\Delta x}\Big(F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}\Big) + \frac{1}{(\Delta x)^2}\Big([A]_{i+\frac{1}{2}} - [A]_{i-\frac{1}{2}}\Big) + \frac{\epsilon}{(\Delta x)^2}\Big([u]_{i+\frac{1}{2}} - [u]_{i-\frac{1}{2}}\Big). \tag{2.6}$$

In the next two sections, we will prove results to determine the convergence of the proposed scheme to the entropy solution in the sense of Definition 1.2.

2.1. Discrete entropy inequality

Here we will demonstrate that scheme (2.2) satisfies a discrete entropy inequality consistent with entropy weak solutions given by (1.4). We start giving the following definition:

Definition 2.1. Given an entropy 3-tuple (η, q, r), the numerical method (2.6) is said to be entropy stable for equation (1.1) if it satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\eta(u_i(t)) + \frac{1}{\Delta x}\Big(G_{i+\frac{1}{2}} - G_{i-\frac{1}{2}}\Big) - \frac{1}{\Delta x}\Big(R_{i+\frac{i}{2}} - R_{i-\frac{i}{2}}\Big) + Z_i \leqslant 0, \tag{2.7}$$

for some consistent entropy numerical fluxes, $G_{i+\frac{1}{2}}$ and $R_{i+\frac{i}{2}}$, and with a function Z that approximates the product $\varphi_\gamma \cdot \gamma'$.

The numerical flux $F_{i+\frac{1}{2}}$ given in (2.3) is consistent with the physical flux $f(\gamma, u)$. Moreover, as a consequence of the facts that function A is increasing and η is convex (η' is increasing as well) then

$$(\eta'(u_r) - \eta'(u_l))(A(u_r) - A(u_l)) \geqslant 0 \qquad \forall u_l, u_r \in \mathbb{R}, \tag{2.8}$$

holds. Where u_l and u_r denote the left and right limits of u respectively. Following Theorem 3.2 given in [17], we will obtain that scheme (2.2) is an entropy stable approximation since it satisfies inequality (2.7). Let us first multiply scheme (2.6) to the left by the value v_i, obtaining

$$\frac{\mathrm{d}}{\mathrm{d}t}\eta(u_i(t)) = -\frac{1}{\Delta x}v_i\Big(F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}\Big) + \frac{v_i}{(\Delta x)^2}\Big(\Big([A]_{i+\frac{1}{2}} - [A]_{i-\frac{1}{2}}\Big) + \epsilon\Big([u]_{i+\frac{1}{2}} - [u]_{i-\frac{1}{2}}\Big)\Big). \tag{2.9}$$

Denoting $\tilde{v}_{i+\frac{1}{2}} = \frac{v_{i+1}+v_i}{2}$, we get

$$v_i\Big(F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}\Big) = \tilde{v}_{i+\frac{1}{2}}F_{i+\frac{1}{2}} - \tilde{v}_{i-\frac{1}{2}}F_{i-\frac{1}{2}} - \frac{1}{2}\Big([v]_{i+\frac{1}{2}}F_{i+\frac{1}{2}} + [v]_{i-\frac{1}{2}}F_{i-\frac{1}{2}}\Big)$$

and

$$v_i\Big([A]_{i+\frac{1}{2}} - [A]_{i-\frac{1}{2}}\Big) = \tilde{v}_{i+\frac{1}{2}}[A]_{i+\frac{1}{2}} - \tilde{v}_{i-\frac{1}{2}}[A]_{i-\frac{1}{2}} - \frac{1}{2}\Big([v]_{i+\frac{1}{2}}[A]_{i+\frac{1}{2}} + [v]_{i-\frac{1}{2}}[A]_{i-\frac{1}{2}}\Big).$$

Considering that $[v]_{i+\frac{1}{2}}F = [\varphi]_{i+\frac{1}{2}}$ and $\varphi = vf - q$ and after some algebraic manipulations we obtain

$$\begin{aligned}[vf]_{i+\frac{1}{2}} + [vf]_{i-\frac{1}{2}} = &\Big(v_{i+1}f\Big(\gamma_{i+\frac{1}{2}}, u_{i+1}\Big) + v_i f\Big(\gamma_{i+\frac{1}{2}}, u_i\Big)\Big) - \Big(v_{i-1}f\Big(\gamma_{i-\frac{1}{2}}, u_{i-1}\Big) + v_i f\Big(\gamma_{i-\frac{1}{2}}, u_i\Big)\Big)\\ &- 2\left(\frac{v_i f\Big(\gamma_{i+\frac{1}{2}}, u_i\Big) - v_i f\Big(\gamma_{i-\frac{1}{2}}, u_i\Big)}{\gamma_{i+\frac{1}{2}} - \gamma_{i-\frac{1}{2}}}\right)\Big(\gamma_{i+\frac{1}{2}} - \gamma_{i-\frac{1}{2}}\Big),\end{aligned}$$

and

$$\begin{aligned}[q]_{i+\frac{1}{2}} + [q]_{i-\frac{1}{2}} = &\Big(q\Big(\gamma_{i+\frac{1}{2}}, u_{i+1}\Big) + q\Big(\gamma_{i+\frac{1}{2}}, u_i\Big)\Big) - \Big(q\Big(\gamma_{i-\frac{1}{2}}, u_{i-1}\Big) + q\Big(\gamma_{i-\frac{1}{2}}, u_i\Big)\Big)\\ &- 2\left(\frac{q\Big(\gamma_{i+\frac{1}{2}}, u_i\Big) - q\Big(\gamma_{i-\frac{1}{2}}, u_i\Big)}{\gamma_{i+\frac{1}{2}} - \gamma_{i-\frac{1}{2}}}\right)\Big(\gamma_{i+\frac{1}{2}} - \gamma_{i-\frac{1}{2}}\Big).\end{aligned}$$

Using the previous identities, scheme (2.9) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\eta(u_i(t)) + \frac{1}{\Delta x}\Big(G_{i+\frac{1}{2}} - G_{i-\frac{1}{2}}\Big) - \frac{1}{\Delta x}\Big(R_{i+\frac{i}{2}} - R_{i-\frac{i}{2}}\Big) + Z_i = -\frac{1}{2(\Delta x)^2}\Big(E_{i+\frac{i}{2}} + E_{i-\frac{i}{2}}\Big), \tag{2.10}$$

where

$$\begin{aligned}
G_{i+\frac{1}{2}} &= \tilde{q}_{i+\frac{1}{2}}|_{\gamma=\gamma_{i+1}} + \tilde{v}_{i+\frac{1}{2}}F_{i+\frac{1}{2}} - (\tilde{vf})_{i+\frac{1}{2}}|_{\gamma=\gamma_{i+1}} - \frac{\epsilon}{\Delta x}\tilde{v}_{i+\frac{1}{2}}[u]_{i+\frac{1}{2}},\\
R_{i+\frac{i}{2}} &= \frac{1}{\Delta x}\tilde{v}_{i+\frac{1}{2}}[A]_{i+\frac{1}{2}},\\
Z_i &= \frac{1}{2\Delta x}\left(\frac{v_i[f]^{\gamma}_{i+\frac{1}{2}}}{\gamma_{i+\frac{1}{2}} - \gamma_{i-\frac{1}{2}}}\Big(\gamma_{i+\frac{1}{2}} - \gamma_{i-\frac{1}{2}}\Big) + \frac{v_i[f]^{\gamma}_{i-\frac{1}{2}}}{\gamma_{i+\frac{1}{2}} - \gamma_{i-\frac{1}{2}}}\Big(\gamma_{i+\frac{1}{2}} - \gamma_{i-\frac{1}{2}}\Big) - 2\frac{[q]^{\gamma}_{i-\frac{1}{2}}}{\gamma_{i+\frac{1}{2}} - \gamma_{i-\frac{1}{2}}}\Big(\gamma_{i+\frac{1}{2}} - \gamma_{i-\frac{1}{2}}\Big)\right),\\
E_{i+\frac{i}{2}} &= \Big([v]_{i+\frac{1}{2}}[A]_{i+\frac{1}{2}} + \epsilon[v]_{i+\frac{1}{2}}[u]_{i+\frac{1}{2}}\Big),
\end{aligned}$$

where $[f]^{\gamma}_{i+\frac{1}{2}} = f\Big(\gamma_{i+\frac{1}{2}}, u_i\Big) - f\Big(\gamma_{i-\frac{1}{2}}, u_i\Big)$ (analogous for q^{γ}) and $\tilde{q}_{i+\frac{1}{2}}|_{\gamma=\gamma_{i+1}} = \frac{1}{2}(q(\gamma_{i+1}, u_{i+1}) + q(\gamma_{i+1}, u_i))$.

Thus, the numerical flux $G_{i+\frac{1}{2}}$ is consistent with $f(\gamma(x), u)$ and $R_{i+\frac{1}{2}}$ is consistent with r_x. Also, it is straightforward to verify that the term Z_i approximates $\varphi_\gamma \cdot \gamma'$. Finally, using (2.8) and $[v]_{i+\frac{1}{2}}[u]_{i+\frac{1}{2}} \geq 0$ (since η is strictly convex its gradient v is monotone and we may consider $[u]_{i+\frac{1}{2}} \approx u_v[v]_{i+\frac{1}{2}}$ where $u_v = \frac{du}{dv}$ is a positive real number) the right side of scheme (2.10) is negative and scheme (2.6) satisfies an inequality of the form (2.7).

3. L^∞ CONVERGENCE

Following the ideas of Karlsen *et al.* in [18], we will prove the strong convergence of the approximations sequence $\{u^\Delta\}_{\Delta>0}$ defined in (2.6) to entropy solutions for problem (1.1) and (1.2). We will establish strong compactness of the diffusion sequence $\{A\big(u^\Delta\big)\}_{\Delta x>0}$ in $L^2(\Pi_T)$ and precompactness of $\{\eta\big(u^\Delta\big) + q\big(\gamma(x), u^\Delta\big)_x\}$ in $H^{-1}(\Pi_T)$. The theory of compensated compactness from Murat and Tartar [23,28] will allow us to complete the proof. Furthermore, a total flux estimation will be also necessary. Let us start with this last issue.

Lemma 3.1. *Assume hypothesis (H1)–(H6) and consider $u_0(x) \in L^\infty(\mathbb{R})$ with compact support. If the entropy stable solution u^Δ defined in (2.6) satisfies*

$$\frac{\Delta t}{\Delta x}k_H \max\{|f_u|, |f_\gamma|\} + 2\frac{\Delta t}{(\Delta x)^2}\Big(\max_u |A'| + \epsilon\Big) < 1, \tag{3.1}$$

where k_H is the Lipschitz constant in (2.4), then for suitable functions γ the inequality

$$\|u^\Delta\|_\infty \leqslant \|u_0\|_\infty \tag{3.2}$$

holds.

Proof. From hypothesis (H6), we have $\underline{u} \leqslant u_0(x) \leqslant \overline{u}$, $\forall x \in \mathbb{R}$. Assume for any $x_j \in \Pi$ that $\underline{u} \leqslant u_i^j \leqslant \overline{u}$. By induction we will prove that $\underline{u} \leqslant u_i^{j+1} \leqslant \overline{u}$, $\forall i$. Let us consider a time forward approximation in scheme (2.6)

$$u_i^{j+1} = u_i^j - \frac{\Delta t}{\Delta x}\Big(F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}\Big) + \frac{\Delta t}{(\Delta x)^2}\Big([A]_{i+\frac{1}{2}} - [A]_{i-\frac{1}{2}}\Big) + \frac{\epsilon\Delta t}{(\Delta x)^2}\Big([u]_{i+\frac{1}{2}} - [u]_{i-\frac{1}{2}}\Big).$$

After some algebraic manipulations, we can rewrite the previous scheme as follows:

$$u_i^{j+1} = u_i^j - \frac{\alpha}{2}\Big(u_{i+1}^j - u_{i-1}^j\Big) + \beta\Big(u_{i+1}^j - u_i^j\Big) - \theta\Big(u_i^j - u_{i-1}^j\Big) + \sigma\Big(u_{i+1}^j - 2u_i^j + u_{i-1}^j\Big),$$

where

$$\alpha = \frac{\Delta t}{\Delta x}\frac{\left(F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}\right)}{\left(\tilde{u}_{i+\frac{1}{2}} - \tilde{u}_{i-\frac{1}{2}}\right)}; \quad \beta = \frac{\Delta t}{(\Delta x)^2}\frac{[A]_{i+\frac{1}{2}}}{[u]_{i+\frac{1}{2}}}, \quad \theta = \frac{\Delta t}{(\Delta x)^2}\frac{[A]_{i-\frac{1}{2}}}{[u]_{i-\frac{1}{2}}}, \quad \sigma = \epsilon\frac{\Delta t}{(\Delta x)^2},$$

recall that $\tilde{u}_{i+\frac{1}{2}} = \frac{u_{i+1}+u_i}{2}$. Regrouping terms, we get

$$u_i^{j+1} = (1-\beta-\theta-2\sigma)u_i^j + \left(-\frac{\alpha}{2}+\beta+\sigma\right)u_{i+1}^j + \left(\frac{\alpha}{2}+\theta+\sigma\right)u_{i-1}^j.$$

From condition (2.4) and Lipschitz continuity of f, we obtain

$$\left|F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}\right| \leq k_H \max\{|f_u|, |f_\gamma|\}\left(\left|\tilde{u}_{i+\frac{1}{2}} - \tilde{u}_{i-\frac{1}{2}}\right| + \left|\gamma_{i+\frac{1}{2}} - \gamma_{i-\frac{1}{2}}\right|\right).$$

Consider only functions γ such that $|\gamma_{i+\frac{1}{2}} - \gamma_{i-\frac{1}{2}}| \leq c_\gamma|\tilde{u}_{i+\frac{1}{2}} - \tilde{u}_{i-\frac{1}{2}}|$ for some positive constant c_γ (particularly for $c_\gamma = 1$). Using inequality (3.1), it follows that u_i^{j+1} is a convex combination of $u_{i-1}^j, u_i^j, u_{i+1}^j$ and $u_i^{j+1} \in [\underline{u}, \overline{u}]$. Thus, condition (3.2) is satisfied. □

Remark 3.2. Notice that for $\epsilon = 0$ and $k_H = 1$ (particular case of operator (2.5)), inequality (3.1) establishes a standard CFL condition for viscous conservation laws. Lemma 3.1 provides a generalization of such condition when numerical viscosity is introduced.

In the following result, we will obtain a bound for the diffusion increment.

Lemma 3.3. *Assume hypothesis (H1)–(H6) and consider $u_0(x) \in L^\infty(\mathbb{R})$ with compact support. Let u^Δ be the entropy stable solution defined in (2.6) taking $\epsilon = \alpha\Delta x$ for $\alpha > 0$, then there exists a constant C independent of Δt and Δx such that*

$$\left\|A\left(u_{i+m}^{j+k}\right) - A\left(u_i^j\right)\right\|_{L^2(\Pi_{T-k\Delta t})} \leqslant C\left(|m|\,\Delta x + \sqrt{k\,\Delta t}\right). \tag{3.3}$$

In particular, this implies that $\{A(u^\Delta)\}_{\Delta>0}$ is strongly compact in $L^2(\Pi_T)$.

Proof. Let us start finding a bound for the spatial variation of the diffusion coefficient. Multiplying by $\Delta t\Delta x\, u_i^{j+\frac{1}{2}}$ the entropy stable scheme (2.6) and discretizing the time derivative, we get

$$\begin{aligned}\Delta t\Delta x\, u_i^{j+\frac{1}{2}}\,\frac{u_i^{j+1}-u_i^j}{\Delta t} &+ \frac{\Delta t\Delta x}{\Delta x}u_i^{j+\frac{1}{2}}\left(F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}\right) - \frac{\epsilon\Delta t\Delta x}{(\Delta x)^2}u_i^{j+\frac{1}{2}}\left([u]_{i+\frac{1}{2}} - [u]_{i-\frac{1}{2}}\right)\\ &= \frac{\Delta t\Delta x}{(\Delta x)^2}\,u_i^{j+\frac{1}{2}}\left([A]_{i+\frac{1}{2}} - [A]_{i-\frac{1}{2}}\right).\end{aligned}$$

We may consider $u_i^{j+\frac{1}{2}} \approx \left(u_i^{j+1} + u_i^j\right)/2$ when Δx is sufficiently small. Adding now over the discretized domain Π_T and using in the first term the formula $\sum_{j=0}^{N-1}(g^{j+1}+g^j)(g^{j+1}-g^j) = (g^N)^2 - (g^0)^2$, then

$$\begin{aligned}\frac{\Delta x}{2}\sum_{i=-\infty}^{\infty}\left(u_i^N\right)^2 - \left(u_i^0\right)^2 &+ \Delta t\sum_{j=0}^{N-1}\sum_{i=-\infty}^{\infty}u_i^{j+\frac{1}{2}}\left(F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}\right)\\ &- \frac{\epsilon\Delta t}{\Delta x}\sum_{j=0}^{N-1}\sum_{i=-\infty}^{\infty}u_i^{j+\frac{1}{2}}\left([u]_{i+\frac{1}{2}} - [u]_{i-\frac{1}{2}}\right) = \frac{\Delta t}{\Delta x}\sum_{j=0}^{N-1}\sum_{i=-\infty}^{\infty}u_i^{j+\frac{1}{2}}\left([A]_{i+\frac{1}{2}} - [A]_{i-\frac{1}{2}}\right).\end{aligned}$$

We now use summation by parts and assume that u^Δ vanishes sufficiently fast when $i \to \infty$ and $u^{j+\frac{1}{2}} \approx u^j$

$$\frac{\Delta x}{2}\sum_{i=-\infty}^{\infty}\left(u_i^N\right)^2-\left(u_i^0\right)^2-\Delta t\sum_{j=0}^{N-1}\sum_{i=-\infty}^{\infty}F_{i+1}[u^j]_{i+\frac{1}{2}}$$
$$+\frac{\epsilon\Delta t}{\Delta x}\sum_{j=0}^{N-1}\sum_{i=-\infty}^{\infty}[u^j]^2_{i+\frac{1}{2}}=-\frac{\Delta t}{\Delta x}\sum_{j=0}^{N-1}\sum_{i=-\infty}^{\infty}\frac{[A^j]_{i+\frac{1}{2}}}{[u^j]_{i+\frac{1}{2}}}[u^j]^2_{i+\frac{1}{2}}.$$

Thus, the term of the right hand side of the above equation can be bounded as follows:

$$\begin{aligned}
&\frac{\Delta t}{\Delta x}\sum_{j=0}^{N-1}\sum_{i=-\infty}^{\infty}\frac{[A^j]_{i+\frac{1}{2}}}{[u^j]_{i+\frac{1}{2}}}[u^j]^2_{i+\frac{1}{2}}\\
&\quad=\frac{1}{2}\Big(\|u^0\|^2_{L^2(\mathbb{R})}-\|u^N\|^2_{L^2(\mathbb{R})}\Big)+\Delta t\sum_{j=0}^{N-1}\sum_{i=-\infty}^{\infty}F_{i+1}[u^j]_{i+\frac{1}{2}}-\frac{\epsilon\Delta t}{\Delta x}\sum_{j=0}^{N-1}\sum_{i=-\infty}^{\infty}[u^j]^2_{i+\frac{1}{2}}\\
&\quad\leqslant\|u^0\|^2_{L^2(\mathbb{R})}+\Delta t\sum_{j,i}F_{i+1}[u^j]_{i+\frac{1}{2}}.
\end{aligned}\tag{3.4}$$

The last term of the right hand side in the previous inequality is bounded since f is a Lipschitz function from definition (2.3) and u^Δ is bounded. Discretizing the identity $(\partial_x(A(u)))^2=(A'(u))^2(\partial_x u)^2$, we get

$$\Delta t\Delta x\sum_{j,i}\frac{\left(A^j_{i+1}-A^j_i\right)^2}{\left(u^j_{i+1}-u^j_i\right)^2}\frac{\left(u^j_{i+1}-u^j_i\right)^2}{(\Delta x)^2}\leqslant\max_u|A'(u)|\Delta t\Delta x\sum_{j,i}\frac{\left(A^j_{i+1}-A^j_i\right)}{\left(u^j_{i+1}-u^j_i\right)}\frac{\left(u^j_{i+1}-u^j_i\right)^2}{(\Delta x)^2}\leq C.$$

Which implies

$$\|A\Big(u^j_{i+m}\Big)-A\Big(u^j_i\Big)\|_{L^2(\Pi_{T-k\Delta t})}\leqslant C|m|\Delta x.\tag{3.5}$$

Let us now bound the L^2-norm of the time variation of the diffusion coefficient over the domain $\Pi_{T-k\Delta t}$

$$\begin{aligned}
&\sum_{j=0}^{N-k}\sum_{i=-\infty}^{\infty}\left(A_i^{j+k}-A_i^j\right)^2\Delta t\Delta x\\
&\quad\leqslant\|A\|_{Lip}\sum_{j,i}\Big(A_i^{j+k}-A_i^j\Big)|u_i^{j+k}-u_i^j|\Delta t\Delta x\\
&\quad\leqslant\|A\|_{Lip}\sum_{j,i}\Bigg\{\Big(A_i^{j+k}-A_i^j\Big)\sum_{\alpha=j}^{j+k}\Bigg(-\frac{\Delta t}{\Delta x}\Big(F^\alpha_{i+\frac{1}{2}}-F^\alpha_{i-\frac{1}{2}}\Big)\\
&\qquad+\frac{\Delta t}{(\Delta x)^2}\Big([A^\alpha]_{i+\frac{1}{2}}-[A^\alpha]_{i-\frac{1}{2}}\Big)+\frac{\epsilon\Delta t}{(\Delta x)^2}\Big([u^\alpha]_{i+\frac{1}{2}}-[u^\alpha]_{i-\frac{1}{2}}\Big)\Bigg)\Bigg\}\\
&\quad\leqslant\Delta t\|A\|_{Lip}\sum_j\sum_{\alpha=j}^{j+k}\Bigg\{C_1+\sum_i\Bigg(\frac{1}{\Delta x}F^\alpha_{i+1}\Big([A^{j+k}]_{i+\frac{1}{2}}-[A^j]_{i+\frac{1}{2}}\Big)\\
&\qquad-\frac{1}{(\Delta x)^2}[A^\alpha]_{i+\frac{1}{2}}\Big([A^{j+k}]_{i+\frac{1}{2}}-[A^j]_{i+\frac{1}{2}}\Big)-\frac{\epsilon}{(\Delta x)^2}[u^\alpha]_{i+\frac{1}{2}}\Big([A^{j+k}]_{i+\frac{1}{2}}-[A^j]_{i+\frac{1}{2}}\Big)\Bigg)\Bigg\}\\
&\quad\leqslant\tilde{C}_1k\Delta t\|A\|_{Lip}\Big(\|f\|_{L^2(\Pi_T)}\|A_x\|_{L^2(\Pi_T)}+2\|A_x\|^2_{L^2(\Pi_T)}+\frac{2\epsilon}{\Delta x}\|A_x\|_{L^2(\Pi_T)}\|u^\Delta\|_\infty\Big)\\
&\quad\leqslant Ck\Delta t,
\end{aligned}$$

where we have used the summation by parts formula, Hölder's inequality, the Lipschitz continuity of the functions f and A, inequality (3.5) and $\epsilon = \alpha\Delta x$. Thus, condition (3.3) is proved with the spatial and time variation bound obtained. Furthermore, by the Kolmogorov compactness criterion [16], it is concluded that $\{A(u^\Delta)\}_{\Delta>0}$ is strongly compact in $L^2(\Pi_T)$. □

To complete the study of the compactness of the diffusion function, we state the fundamental theorem of compensated compactness theory.

Theorem 3.4. *Let $K \subset \mathbb{R}$ be a bounded open set, and $u^\Delta : \Pi_T \to K$. Then there exists a family of probability measures $\{\nu_{(x,t)}(\zeta) \in \mathrm{Prob}(\mathbb{R}^n)\}_{(x,t)\in\Pi_T}$ (depending weak–$\star$ measurably on (x,t)) such that*

$$\operatorname{supp} \nu_{(x,t)} \subset \bar{K} \text{ for a.e. } (x,t) \in \Pi_T.$$

Furthermore, for any continuous function $\Phi : K \to \mathbb{R}$, we have along a subsequence

$$\Phi(u^\Delta) \rightharpoonup^* \bar{\Phi} \text{ in } L^\infty(\Pi_T) \text{ as } \Delta \downarrow 0,$$

where

$$\bar{\Phi}(x,t) := \int_{\mathbb{R}} \Phi(\zeta)\, \mathrm{d}\nu_{(x,t)}(\zeta) \text{ for a.e. } (x,t) \in \Pi_T.$$

The function $\nu(x,t)$ is known as a Young measure and $\mathrm{Prob}(\mathbb{R}^n)$ is the set of all probability measures in the Borel sets. As a consequence of this theorem, a uniformly bounded sequence $\{u^\Delta\}_{\Delta>0}$ converges to u a.e. on Π_T if and only if $\nu_{(x,t)} = \delta_{u(x,t)}$ where δ_u is a Dirac measure located at $u(x,t)$.

Assuming $u_0(x) \in L^\infty(\mathbb{R})$ with compact support, from Lemma 3.1 we have $\|u^\Delta\|_\infty \leqslant \|u_0\|_\infty$. Then there exists $L := \max |A(u^\Delta)|$. So for any function $g \in C([0,L])$ there exists $\tilde{M} > 0$ such that

$$\|g(A(u^\Delta))\|_{L^\infty(\Pi_t)} \leq \tilde{M}.$$

and from Theorem 3.4 satisfying

$$g(\overline{A}(x,t)) = \lim_{\Delta x \longrightarrow 0} g(A(u^\Delta(x,t))) = \int_{\mathbb{R}} g(A(\zeta))\, \mathrm{d}\nu_{(x,t)}(\zeta), \qquad \text{for a.e. } (x,t) \in \Pi_T. \tag{3.6}$$

Since $A(u^\Delta) \in [0,L]$, then taking g as the identity function, from (3.6) we have $\overline{A}(x,t) \in [0,L]$ for a.e. $(x,t) \in \Pi_T$.

Finally, we are ready to prove the compactness property of the diffusive part.

Theorem 3.5. *Assume hypothesis (H1)–(H6) and consider $u_0(x) \in L^\infty(\mathbb{R})$ with compact support. Let u^Δ be a sequence of entropy stable approximations for problem (1.1) and (1.2). Then, a subsequence of $\{A(u^\Delta)\}_{\Delta>0}$ converges strongly to $A(u)$ in $L^2_{\mathrm{loc}}(\Pi_T)$, where u is the weak–$\star$ limit of entropy stable solution u^Δ in $L^\infty(\Pi_T)$. Moreover, $A(u) \in L^\infty(\Pi_T) \cap L^2(0,T;H^1(\mathbb{R}))$.*

Proof. It is straightforward from Theorems 3.1, 3.3 and 3.4. □

Now we study the compactness of $\{\eta_t + q_x\}$. For that, we extend Lemma 3.2 in Fjordholm [14] for parabolic conservation laws. Let us define $\{\eta_t + q_x\}$ and $\{\eta_t + G_x - R_x + Z\}$ as measures and let B be a borelian in $\mathbb{R}$, then

$$(\eta_t + q_x)(B \times [t_1,t_2]) = \int_{t_1}^{t_2} \left[\left(\int_B \eta(u^\Delta(x,t))_t \, \mathrm{d}x \right) + \sum_{i=0}^{M-1} (q(\gamma_i,u_i) - q(\gamma_{i-1},u_{i-1}))\delta_{x_{i-\frac{1}{2}}}(B) \right] \mathrm{d}t,$$

$$(\eta_t + G_x + Z - R_x)(B \times [t_1,t_2]) = \int_{t_1}^{t_2} \left[\left(\int_B \eta(u^\Delta(x,t))_t \, \mathrm{d}x \right) + \sum_{i=0}^{M-1} \left(G_{i+\frac{1}{2}} - G_{i-\frac{1}{2}} \right) \delta_{x_{i-\frac{1}{2}}}(B) \right.$$

$$+\sum_{i=0}^{M-1} Z_i \delta_{x_{i-\frac{1}{2}}}(B) - \sum_{i=0}^{M-1}\Big(R_{i+\frac{1}{2}} - R_{i-\frac{1}{2}}\Big)\delta_{x_{i-\frac{1}{2}}}(B)\Bigg]\,\mathrm{d}t,$$

where u^Δ is the numerical solution considered as a piecewise constant function. Let $\Omega \subset \Pi_T$ be an open and bounded set with $\operatorname{supp}\big(u^\Delta\big) \subset \Omega$ for all $\Delta x > 0$. For a function $\psi \in W^{-1,\infty}(\Omega)$ with $\psi_{t=0,T} = 0$ we can define $\{\eta_t + q_x\}$ and $\{\eta_t + G_x - R_x + Z\}$ as functionals of the form

$$(\eta_t + q_x)(\psi) := \int_\Omega \psi\, d(\eta_t + q_x),$$
$$(\eta_t + G_x + Z - R_x)(\psi) := \int_\Omega \psi\, d(\eta_t + G_x + Z - R_x).$$

The following result is known as the Murat's lemma and it will be useful to prove the compactness in $H^{-1}_{\mathrm{loc}}(\Omega)$ of the sequence $\{\eta_t + q_x\}$.

Lemma 3.6. *Let $\Lambda \subset \mathbb{R}^m$ be a bounded open set and $\{\mu_j\}_{j\in\mathbb{N}}$ a bounded sequence in $W^{-1,\infty}(\Lambda)$. Assume that $\mu_j = \chi_j + \pi_j$ where $\{\chi_j\}$ is contained in a compact subset of $H^{-1}_{\mathrm{loc}}(\Lambda)$ and $\{\pi_j\}$ is bounded in $\mathcal{M}_{\mathrm{loc}}(\Lambda)$. Then, $\{\mu_j\}$ is contained in a compact subset of $H^{-1}_{\mathrm{loc}}(\Lambda)$.*

Lemma 3.7. *Let $\Omega \subset \Pi_T$ be a bounded open set, (η, q) an entropy pair and G an entropy numerical flux that satisfies the Lipschitz condition*

$$|q(\gamma, u_i) - G(\gamma, u_{i-m+1}, \ldots, u_{i+m})| \leqslant C(|u_{i-m+2} - u_{i-m+1}| + \ldots + |u_{i+m} - u_{i+m-1}|), \quad \forall u_{i-m}, \ldots, u_{i+m}. \tag{3.7}$$

Assume the sequence of functions u^Δ with $\Delta x > 0$ satisfies:

$$\text{There exists} \quad \tilde{M} > 0 \quad \text{such that} \quad \|u^\Delta\|_{L^\infty(\Omega)} \leqslant \tilde{M}, \ \ \forall \Delta x > 0 \tag{3.8}$$

$$\operatorname{supp}\big(\mathrm{u}^\Delta\big) \subset \Omega, \quad \text{for all} \quad \Delta \mathrm{x} > 0 \tag{3.9}$$

$$\int_0^T \sum_{i=0}^{M-1} |u^\Delta_{i+1} - u^\Delta_i|^p \Delta x\, \mathrm{d}t \to 0, \quad \text{when} \quad \Delta x \to 0 \quad \text{for some} \quad p \in [2, \infty) \tag{3.10}$$

$$|\eta_t + G_x + Z - R_x|(\Omega) \leqslant C, \quad \text{for all} \quad \Delta x > 0. \tag{3.11}$$

Then, the sequence $\eta\big(u^\Delta\big)_t + q\big(\gamma(x), u^\Delta\big)_x$ is contained in a compact subset of $H^{-1}_{\mathrm{loc}}(\Omega)$.

Proof. Decomposing the functional as

$$(\eta_t + q_x)(\psi) = \underbrace{\int_\Omega \psi d(\eta_t + G_x + Z - R_x)}_{E_1(\psi)} + \underbrace{\int_\Omega \psi d((\eta_t + q_x) - (\eta_t + G_x + Z - R_x))}_{E_2(\psi)},$$

and using inequality (3.11), we get that $E_1(\psi)$ is bounded in $\mathcal{M}(\Omega)$ since

$$|E_1(\psi)| = \left|\int_\Omega \psi d(\eta_t + G_x + Z - R_x)\right| \leqslant \|\psi\|_{L^\infty(\Omega)} |\eta_t + G_x + Z - R_x|(\Omega) \leqslant C\|\psi\|_{L^\infty(\Omega)}.$$

On the other hand

$$\begin{aligned}
|E_2(\psi)| &= \left|\int_\Omega \psi d((\eta_t + q_x) - (\eta_t + G_x + Z - R_x))\right| \\
&= \left|\int_\Omega \psi d(q_x + G_x + Z - R_x)\right| \\
&\leqslant \underbrace{\left|\int_\Omega \psi d(q_x - G_x)\right|}_{E_{2,1}(\psi)} + \underbrace{\left|\int_\Omega \psi d(R_x)\right|}_{E_{2,2}(\psi)} + \underbrace{\left|\int_\Omega \psi d(Z)\right|}_{E_{2,3}(\psi)}.
\end{aligned}$$

We will prove that the term $E_2(\psi) \in H^{-1}(\Omega)$ by verifying that each term $E_{2,i}(\psi)$, for $i = 1, 2, 3$, is precompact in $H^{-1}(\Omega)$. For $E_{2,1}(\psi)$, we have

$$\begin{aligned}
|E_{2,1}(\psi)| &= \left| \int_\Omega \psi \, d(q_x - G_x) \right| \\
&= \left| \int_0^T \sum_{i=0}^{M-1} \Big(\psi\Big(x_{i+\frac{1}{2}}, t\Big) - \psi\Big(x_{i-\frac{1}{2}}, t\Big) \Big) \Big(q(\gamma_i, u_i) - G_{i+\frac{1}{2}} \Big) \, \mathrm{d}t \right| \\
&\leqslant C \|\psi_x\|_{L^2(\Omega)} \left(\int_0^T \sum_{i=0}^{M-1} |u_{i+1}^\Delta - u_i^\Delta|^2 \Delta x \, \mathrm{d}t \right)^{\frac{1}{2}} \qquad \text{(by Holder and (3.7))} \\
&\leqslant C \|\psi_x\|_{L^2(\Omega)} |\Omega|^{\frac{1}{k}} \left(\int_0^T \sum_{i=0}^{M-1} |u_{i+1}^\Delta - u_i^\Delta|^p \Delta x \, \mathrm{d}t \right)^{\frac{1}{p}} \longrightarrow 0 \qquad \text{as} \qquad \Delta x \to 0.
\end{aligned}$$

This last inequality is obtained using Holder's inequality with p from hypothesis (3.10) and k satisfying $\frac{1}{p} + \frac{1}{k} = \frac{1}{2}$ with $(p \geqslant 2)$. Denoting by $\eta_{u_{i+\frac{1}{2}}} = v_{i+\frac{1}{2}}$, then for $E_{2,2}(\psi)$

$$\begin{aligned}
|E_{2,2}(\psi)| = \left| \int_\Omega \psi d(R_x) \right| &= \left| \int_0^T \sum_{i=0}^{M-1} \Big(\psi\Big(u_{i+\frac{1}{2}}\Big) - \psi\Big(u_{i-\frac{1}{2}}\Big) \Big) \frac{\tilde{\eta}_{u_{i+\frac{1}{2}}} [A]_{i+\frac{1}{2}}}{\Delta x} \, \mathrm{d}t \right| \\
&\leqslant \|\psi_x\|_{L^2(\Omega)} \int_0^T \sum_{i=0}^{M-1} \frac{\tilde{\eta}_{u_{i+\frac{1}{2}}} [A]_{i+\frac{1}{2}}}{\Delta x} \, \Delta x \, \mathrm{d}t \\
&\leqslant \|\psi_x\|_{L^2(\Omega)} \|\eta_u\|_{L^\infty(\Omega)} |\Omega|^{\frac{1}{2}} \|A_x\|_{L^2(\Omega)} \\
&\leqslant C \Delta x \, \|\psi_x\|_{L^2(\Omega)} \|\eta_u\|_{L^\infty(\Omega)} |\Omega|^{\frac{1}{2}} \longrightarrow 0 \qquad \text{as} \qquad \Delta x \to 0.
\end{aligned}$$

Holder's inequality, Theorem 3.3 and the continuity of functions $\eta'(u)$ and ψ_x were used in order to obtain the previous inequalities. Therefore, $E_2(\varphi)$ is precompact in $H^{-1}(\Omega)$. Consider now $E_{2,3}(\psi)$

$$|E_{2,3}(\psi)| = \left| \int_\Omega \psi \, d(Z) \right| = \int_0^T \sum_{i=0}^{M-1} \psi_i Z_i \, \mathrm{d}t \leqslant \|\psi\|_{L^\infty(\Omega)} |Z|(\Omega) \leqslant C \|\psi\|_{L^\infty(\Omega)}.$$

Recall that $Z_i = \frac{1}{2}\Big(v_i [f]^\gamma_{i+\frac{1}{2}} - [q]^\gamma_{i+\frac{1}{2}} + v_i [f]^\gamma_{i-\frac{1}{2}} - [q]^\gamma_{i-\frac{1}{2}} \Big)$. Since η' is bounded in Ω, f is Lipschitz continuous in $\gamma \in BV(\Omega)$ and by the definition of q, Z_i is bounded in Ω. Thus, by Lemma 3.6 we can conclude that $\eta_t\big(u^\Delta\big) + q\big(\gamma(x), u^\Delta\big)_x$ is a sequence of functions parameterized by $\Delta x > 0$ which is precompact in $H^{-1}(\Omega)$. □

We prove now that the entropy stable scheme defined in (2.6) satisfies conditions (3.7)–(3.11) from Lemma 3.7:

(i) Substituting $G_{i+\frac{1}{2}}$ and considering $\gamma = \gamma_{i+\frac{1}{2}}$ and $\epsilon = \alpha \Delta x$ for some fixed $\alpha > 0$, we get

$$\begin{aligned}
&\left| q\Big(\gamma_{i+\frac{1}{2}}, u_i\Big) - \tilde{q}_{i+\frac{1}{2}} + \tilde{v}_{i+\frac{1}{2}} F_{i+\frac{1}{2}} - (\tilde{vf})_{i+\frac{1}{2}} - \frac{\epsilon}{\Delta x} \tilde{v}_{i+\frac{1}{2}} [u]_{i+\frac{1}{2}} \right| \\
&\quad = \left| \frac{1}{2} [q]_{i+\frac{1}{2}} + \tilde{v}_{i+\frac{1}{2}} \Big(\tilde{f}_{i+\frac{1}{2}} - F_{i+\frac{1}{2}} \Big) + \frac{1}{4} [v]_{i+\frac{1}{2}} [f]_{i+\frac{1}{2}} + \alpha \tilde{v}_{i+\frac{1}{2}} [u]_{i+\frac{1}{2}} \right| \\
&\quad \leqslant C |u_{i+1} - u_i|,
\end{aligned}$$

where $C = \Big(\frac{\|q_u\|}{2} + \|v\| \|f_u\| + \|v'\| \|f_u\| + \alpha \|v\| \Big)$. The previous inequalities were obtained since η' and η'' are bounded in a compact set and q and f are Lipschitz continuous. Thus, condition (3.7) is satisfied for scheme (2.6).

(ii) By Lemma 3.1 and taking into account that the initial condition u_0 is a compact support function, the approximation sequence u^Δ satisfies conditions (3.8), (3.9) and

$$\int_0^T \sum_{i=0}^{M-1} |u_{i+1}^\Delta - u_i^\Delta|^2 \,\mathrm{d}t \leqslant \int_0^T \sum_{i=0}^{M-1} \left(2\|u_0\|_{L^\infty(\Omega)}\right)^2 \mathrm{d}t \leqslant 4 \cdot \mathrm{diam}(\Omega)\|u_0\|_{L^\infty}^2.$$

Thus, we also have the condition (3.10).

(iii) Using the definition of the entropy stable scheme given in (2.10), we get

$$\begin{aligned}|\eta_t + G_x + Z - R_x|(\Omega) &= \int_0^T \sum_{i=0}^{M-1} \left|E_{i+\frac{1}{2}}\right| \,\mathrm{d}t \\ &= \int_0^T \sum_{i=0}^{M-1} \left([v]_{i+\frac{1}{2}}[A]_{i+\frac{1}{2}} + \epsilon[v]_{i+\frac{1}{2}}[u]_{i+\frac{1}{2}}\right) \mathrm{d}t \\ &\leqslant C \int \sum_{i=0}^{M-1} \left|u_{i+1}^\Delta - u_i^\Delta\right|^2 \mathrm{d}t,\end{aligned}$$

by Lipschitz continuity of $A(u)$ and considering $[u]_{i+\frac{1}{2}} \approx u_v[v]_{i+\frac{1}{2}}$ such that $u_v \in \mathbb{R}_+$. Thus, by (ii) condition (3.11) is proved.

We now demonstrate that the succession of entropy stable approximations has as limit an entropy weak solution which satisfies Definition 1.2.

Theorem 3.8. *Assume $u_0(x) \in L^\infty(\mathbb{R})$ with compact support. If the entropy stable scheme u^Δ proposed in (2.6) taking $\epsilon = \alpha\Delta x$ ($\alpha > 0$) has a limit, then it is a weak solution to the problem (1.1) and (1.2) in the sense of Definition 1.1.*

Proof. Let us multiply the entropy stable scheme (2.6) by $\Delta x\Delta t\phi_i^j$ as follows

$$\begin{aligned}&\Delta t\Delta x\phi_i^j \frac{\mathrm{d}}{\mathrm{d}t}u_i(t) + \frac{\Delta t\Delta x}{\Delta x}\phi_i^j\left(F_{i+\frac{1}{2}} - F_{i-\frac{1}{2}}\right) \\ &\quad - \frac{\Delta t\Delta x}{(\Delta x)^2}\phi_i^j(A_{i+1} - 2A_i + A_{i-1}) = \frac{\epsilon\Delta t\Delta x}{(\Delta x)^2}\phi_i^j(u_{i+1} - 2u_i + u_{i-1}).\end{aligned} \tag{3.12}$$

where ϕ_i^j is a compact support function verifying $\phi_0^j = 0$ with j the time index and $\phi_i^j = 0$ for all $|x_i| \geqslant M\Delta x$. Approximating the partial derivative of time of (3.12) and using the summation by parts formula, we get

$$\begin{aligned}&\Delta t\Delta x \sum_{j=1}^{N}\sum_{i=-\infty}^{\infty} u_i^j\left(\frac{\phi_i^j - \phi_i^{j-1}}{\Delta t}\right) + \Delta x \sum_{i=-\infty}^{\infty} u_i^0\phi_i^0 + \Delta x\Delta t\sum_{i,j} F_{i+\frac{1}{2}}^j\left(\frac{\phi_{i+\frac{1}{2}}^j - \phi_{i-\frac{1}{2}}^j}{\Delta x}\right) \\ &\quad + \frac{\Delta t\Delta x}{(\Delta x)^2}\sum_{j=1}^{N}\sum_{i=-\infty}^{\infty} A_i\left(\phi_{i+1}^j - 2\phi_i^j + \phi_{i-1}^j\right) = \frac{\epsilon\Delta t\Delta x}{(\Delta x)^2}\sum_{j=1}^{N}\sum_{i=-\infty}^{\infty}\left(u_{i+1}^j - u_i^j\right)\left(\phi_{i+1}^j - \phi_i^j\right).\end{aligned}$$

Since $u_0(x)$ is a compact support function, consequently, the approximation u^Δ is also a compact support function for any j. Thus, the previous formula approaches the weak solution (1.3) as $\Delta t, \Delta x \to 0$ since $\epsilon = \alpha\Delta x$. □

Theorem 3.9. *Assume $u_0(x) \in L^\infty(\mathbb{R})$ with compact support. If the entropy stable scheme u^Δ proposed in (2.6) has a limit, then it is an entropy solution of problem (1.1) and (1.2) in the sense of Definition 1.2.*

Proof. The scheme (2.6) satisfies the discrete entropy condition 2.7. Multiplying inequality (2.7) by $\Delta x \Delta t \phi_i^j$ and adding over the entire domain, we obtain

$$\Delta x \Delta t \sum_{j=1}^{n} \sum_{i=-\infty}^{\infty} \phi_i^j \left(\frac{\mathrm{d}}{\mathrm{d}t} \eta(u_i) + \frac{1}{\Delta x} \left(G_{i+\frac{1}{2}} - G_{i-\frac{1}{2}} \right) - \frac{1}{\Delta x} \left(R_{i+\frac{i}{2}} - R_{i-\frac{i}{2}} \right) + Z_i \right) \leqslant 0,$$

where ϕ_i^j is a compact support function. Considering the addition by parts in the previous inequality, we get the following:

$$-\Delta x \Delta t \sum_{j=1}^{N} \sum_{i=-\infty}^{\infty} \left(\eta\left(u_i^j\right) \frac{\phi_i^j - \phi_i^{j-1}}{\Delta t} + G_i \frac{\phi_{i+\frac{1}{2}}^j - \phi_{i-\frac{1}{2}}^j}{\Delta x} + R_i \frac{\phi_{i+\frac{1}{2}}^j - \phi_{i-\frac{1}{2}}^j}{\Delta x} + Z_i \phi_i^j \right) \leqslant 0. \tag{3.13}$$

Taking the limits in (3.13) as $\Delta x, \Delta t \to 0$, we have

$$\iint_{\Pi_T} (\eta(u_i(t))\phi_t + q(\gamma(x), u)\phi_x + r_x(u)\phi_x)\, \mathrm{d}x\, \mathrm{d}t + \iint_{\Pi_T} \gamma'(x)(q_\gamma(\gamma(x), u) - \eta' f_\gamma(\gamma(x), u))\phi \geqslant 0. \tag{3.14}$$

The second integral in this previous inequality can be written as the sum of two integrals. One that considers the continuous part of γ and the other that includes its discontinuities. Thus, obtaining the entropy inequality (1.4). □

Finally, we are already to prove the L^∞-convergence of the entropy stable scheme. For that, we state the following result which is a direct extension of Lemma 2.2 given in [18], where it is proven for the case where γ is multiplicative, so we omit its proof here. The proof uses the main theorem of compensated compactness (Thm. 3.4).

Theorem 3.10 ([18])**.** *Assume $\{u^\Delta\} \subset L^\infty(\Pi_T)$ uniformly. Assume also that for any C^2 function $\eta : \mathbb{R} \longrightarrow \mathbb{R}$, the sequence of distributions $\{\eta_t(u^\Delta) + \left(q(\gamma(x), u^\Delta)\right)_x\}$ is contained in a compact subset of $H^{-1}(\Omega)$, where $q : \mathbb{R} \longrightarrow \mathbb{R}$ is defined by $q_u(u) = \eta'(u) f_u(\gamma(x), u)$. Then, along a subsequence $u^\Delta \longrightarrow_\star u$, $f\left(\gamma(x), u^\Delta\right) \longrightarrow_\star f(\gamma(x), u)$. Moreover, if $\gamma(x) \neq 0$ for almost all $x \in \mathbb{R}$ and there is no interval where $f(\gamma, \cdot)$ is linear. Then, a subsequence of $\{u^\Delta\}$ converges to u almost everywhere in $L^\infty(\Pi_T)$.*

We can now state our main convergence theorem.

Theorem 3.11. *Let (η, q, r) be an entropy 3-tuple associated to equation (1.1). Assume that $u_0(x) \in L^\infty(\mathbb{R})$ with compact support. Assume also that hypothesis (H1)–(H6) and conditions (3.1) and (3.7)–(3.11) hold. Considering that the discrete equation (2.10) conforms an entropy stable scheme such that $\epsilon = \alpha \Delta x$ for some $\alpha > 0$. Then, there exists a weak solution u of the parabolic problem (1.1) that can be constructed as the limit of the approximations u^Δ given by the entropy stable scheme (2.6) and this structure guarantees that u is also the entropy solution.*

Proof. Given equation (1.1), assuming conditions (H1)–(H6) and considering the entropy stable scheme (2.6), we have that this scheme satisfies conditions (3.7)–(3.11) from Lemma 3.7. Moreover, Theorems 3.10 and 3.4 guarantee that $u^\Delta \to u$ along a subsequence almost everywhere in Π_T when $\Delta x \to 0$. Lemma 3.1 also guarantees that the limit belongs to $L^1(\Pi_T) \cap L^\infty(\Pi_T)$. Therefore, convergence holds for any $L^p(\Pi_T)$ with $p \in [1, \infty)$.

Additionally, Lemma 3.5 implies that $A(u) \in L^2\left(0, T, H^1(\mathbb{R})\right)$. The Definition 3.8 states that if u^Δ converges, then the limit is a weak solution of problem (1.1). Finally, by construction, the approximation u^Δ given in (2.6) is an entropy solution, which was shown in Theorem 3.9. □

Remark 3.12. The previous convergence proof has been made considering the fully discrete numerical method obtained by applying forward Euler for the time stepping and entropy stable discretization for the spatial approximation. Thus, the resulting method is the first-order of accuracy in time and second-order in space.

4. *A priori* entropy estimate

This section is devoted to prove that the class of entropy solutions satisfying Definition 1.2 satisfies an *a priori* estimate that is a generalization of a result from Tadmor (see Thm. 2.1 from [26]). First we start proving an entropy inequality for a symmetric 3-tuple of the type (1.6). Let us consider test functions that vanish around the discontinuities of γ, that is, $0 \leqslant \phi \in D(\Pi_T)$ be a test function satisfying

$$\phi(x,t) = 0 \quad \forall (x,t) \in [\xi_m - h_m, \xi_m + h_m] \times [0,T],$$

for some $h_m > 0,\ m = 1, \dots, M$, where ξ_m are the discontinuity jumps of γ. Thus, $\phi \in D\big(\Pi_T \setminus \{\xi_m\}_{m=1}^M\big)$.

Theorem 4.1. *Let us consider a symmetric 3-tuple of the form* (1.6) *and assume Assumption 1.3. For any pair of entropy solutions $w = w(x,t)$ and $u = u(x,t)$ of the convection-diffusion equation* (1.1) *satisfying condition* (1.4)

$$-\iint_{\Pi_T} (\eta(w(x,t);u(x,t))\phi_t(x,t) + q(\gamma(x), w(x,t); u(x,t))\phi_x(x,t) + r(w(x,t);u(x,t))\phi_{xx}(x,t))\,\mathrm{d}t\,\mathrm{d}x \leqslant 0, \quad (4.1)$$

is satisfied for any $0 \leqslant \phi \in D\big(\Pi_T \setminus \{\xi_m\}_{m=1}^M\big)$.

Proof. For equation (1.1), an entropy solution with a symmetric entropy 3-tuple (1.6) verifies condition (1.4) as follows:

$$\begin{aligned}
&\iint_{\Pi_T} (\eta(u;c)\phi_t + (q(\gamma(x),u;c))\phi_x + r(u;c)_x\phi_x)\,\mathrm{d}t\,\mathrm{d}x \\
&\quad + \iint_{\Pi_T\setminus\{\xi_m\}_{m=1}^M} \gamma'(x)(\eta'(u;c)f_\gamma(\gamma,u) - q_\gamma(\gamma,u;c))\phi\ \mathrm{d}t\,\mathrm{d}x \\
&\quad + \int_0^T \sum_{m=1}^M (\gamma'(x)(\eta_u(u;c)f_\gamma(\gamma,u)(u) - q_\gamma(\gamma,u;c)))\Big|_{\xi_m^-}^{\xi_m^+} \phi(\xi_m,t)\,\mathrm{d}t \geqslant 0.
\end{aligned}$$

Since $0 \leqslant \phi \in D\big(\Pi_T \setminus \{\xi_m\}_{m=1}^M\big)$, then the last term on the left hand side of the previous inequality vanishes. Therefore, considering only the intervals where γ is continuous, we get

$$\begin{aligned}
&\iint_{\Pi_T} (\eta(u;c)\phi_t + (q(\gamma(x),u;c))\phi_x + r(u;c)_x\phi_x)\,\mathrm{d}t\,\mathrm{d}x \\
&\quad + \iint_{\Pi_T\setminus\{\xi_m\}_{m=1}^M} \gamma'(x)(\eta_u(u;c)f_\gamma(\gamma,u) - q_\gamma(\gamma,u;c))\phi\ \mathrm{d}t\,\mathrm{d}x \geqslant 0.
\end{aligned}$$

Considering now $0 \leqslant \phi \in D\Big(\big(\Pi_T \setminus \{\xi_m\}_{m=1}^M\big)^2\Big)$ and denoting $\phi := \phi(x,t,s,y)$, $w := w(x,t)$ and $c := u(y,s)$, we rewrite the previous inequality as

$$\begin{aligned}
&\iint_{\Pi_T} (\eta(w;u)\phi_t + (q(\gamma,w;u))\phi_x + r(w;u)_x\phi_x)\,\mathrm{d}t\,\mathrm{d}x \\
&\quad + \iint_{\Pi_T\setminus\{\xi_m\}_{m=1}^M} \gamma'(x)(\eta_w(w;u)f_\gamma(\gamma,v) - q_\gamma(\gamma,w;u))\phi\ \mathrm{d}t\,\mathrm{d}x \geqslant 0.
\end{aligned}$$

Integrating over $(y,s) \in \Pi_T$

$$\begin{aligned}\iiiint_{(\Pi_T)^2} &(\eta(w;u)\phi_t + (q(\gamma,w;u))\phi_x + r(w;u)_x\phi_x)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s \\ &+ \iiiint_{(\Pi_T\setminus\{\xi_m\}_{m=1}^M)^2} \gamma'(x)(\eta_w(w;u)f_\gamma(\gamma,v) - q_\gamma(\gamma,w;u))\phi\ \mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s \geqslant 0.\end{aligned} \tag{4.2}$$

Analogously, taking an entropy solution $u = u(y,s)$ and the constant $c = w(x,t)$, we have

$$\begin{aligned}\iiiint_{(\Pi_T)^2} &\Big(\eta(u;w)\phi_s + (q(\gamma,u;w))\phi_y + r(u;w)_y\phi_y\Big)\,\mathrm{d}y\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}x \\ &+ \iiiint_{(\Pi_T\setminus\{\xi_m\}_{m=1}^M)^2} \gamma'(y)(\eta_u(u;w)f_\gamma(\gamma,u) - q_\gamma(\gamma,u;w))\phi\ \mathrm{d}y\,\mathrm{d}s\,\mathrm{d}t\,\mathrm{d}x \geqslant 0.\end{aligned} \tag{4.3}$$

Let us denote

$$\partial_{t+s} := \partial_t + \partial_s, \quad \partial_{x+y} := \partial_x + \partial_y \ \text{ and } \partial^2_{x+y} := \partial^2_x + 2\partial_x\partial_y + \partial^2_y.$$

Adding inequalities (4.2) and (4.3) and using the symmetry relationships (1.6), we get

$$\begin{aligned}\iiiint_{(\Pi_T)^2} &\Big(\eta(w;u)\phi_{t+s} + \frac{1}{2}(q(\gamma(x),w;u) + q(\gamma(y),u;w))\phi_{x+y} \\ &+ \frac{1}{2}(q(\gamma(x),w;u) - q(\gamma(y),u;w))\phi_x + \frac{1}{2}(q(\gamma(y),u;w) - q(\gamma(x),w;u))\phi_y \\ &+ r(w;u)\partial^2_{x+y}\phi - 2r(w;u)\partial_x\phi_y\Big)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s \\ &+ \iiiint_{(\Pi_T\setminus\{\xi_m\}_{m=1}^M)^2} (\gamma'(x)(\eta_w(w;u)f_\gamma(\gamma(x),w) - q_\gamma(\gamma(x),w;u))\phi \\ &+ \gamma'(y)(\eta_u(u;w)f_\gamma(\gamma(y),u) - q_\gamma(\gamma(y),u;w))\phi)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s \geqslant 0.\end{aligned}$$

Grouping terms in the previous inequality and multiplying by (-1), we have

$$\begin{aligned}-\iiiint_{(\Pi_T)^2} &\Big(\eta(w;u)\phi_{t+s} + \frac{1}{2}(q(\gamma(x),w;u) + q(\gamma(y),u;w))\phi_{x+y} + r(w;u)\partial^2_{x+y}\phi\Big)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s \\ &- \iiiint_{(\Pi_T\setminus\{\xi_m\}_{m=1}^M)^2} \Big[(\gamma'(x)\eta_w(v;u)f_\gamma(\gamma(x),w) + \gamma'(y)\eta_u(u;w)f_\gamma(\gamma(y),u)) \\ &- (\gamma'(x)q_\gamma(\gamma(x),w;u) + \gamma'(y)q_\gamma(\gamma(y),u;w))\Big]\phi\ \mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s \\ &- \iiiint_{(\Pi_T)^2} \Big(\frac{1}{2}(q(\gamma(x),w;u) - q(\gamma(y),u;w))\phi_x \\ &+ \frac{1}{2}(q(\gamma(y),u;w) - q(\gamma(x),w;u))\phi_y\Big)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\end{aligned}$$

$$\leqslant -\iiiint\limits_{(\Pi_T)^2} 2r(w;u)\partial_x\phi_y \,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s. \tag{4.4}$$

By integration by parts the right-hand side of the last inequality can be shown to be less than or equal to zero. Take $\delta \in C_0^\infty$ a non-negative function that satisfies $\delta(z) = \delta(-z)$, $\delta(z) = 0$ for $|z| \geqslant 1$ and $\int_{\mathbb{R}} \delta(z)dz = 1$. For $\rho > 0$ and $z \in \mathbb{R}$, we define $\delta_\rho(z) = \frac{1}{\rho}\delta\left(\frac{z}{\rho}\right)$. Consider the test function $\phi(x,t,y,s) = \phi\left(\frac{x+y}{2}, \frac{t+s}{2}\right)\delta_\rho\left(\frac{x-y}{2}\right)\delta_\rho\left(\frac{t-s}{2}\right)$ with $0 \leqslant \phi \in D\Big(\big(\Pi_T \setminus \{\xi_m\}_{m=1}^M\big)^2\Big)$ where $\phi(x,t) = 0 \;\; \forall (x,t) \in [\xi_m - h_m, \xi_m + h_m] \times [0,T]$ for small $h_m > 0$, $m = 1,\dots,M$, with ξ_m are the discontinuity points of γ. Taking $\rho = \min\limits_{m=1,\dots,M} h_m$, we can guarantee that $\phi \in \big(\Pi_T \setminus \{\xi_m\}_{m=1}^M\big)^2$. Hence,

$$\begin{aligned}
\partial_{t+s}\phi(x,t,y,s) &= \partial_{t+s}\phi\left(\frac{x+y}{2}, \frac{t+s}{2}\right)\delta_\rho\left(\frac{x-y}{2}\right)\delta_\rho\left(\frac{t-s}{2}\right)\\
\partial_{x+y}\phi(x,t,y,s) &= \partial_{x+y}\phi\left(\frac{x+y}{2}, \frac{t+s}{2}\right)\delta_\rho\left(\frac{x-y}{2}\right)\delta_\rho\left(\frac{t-s}{2}\right)\\
\partial^2_{x+y}\phi(x,t,y,s) &= \partial^2_{x+y}\phi\left(\frac{x+y}{2}, \frac{t+s}{2}\right)\delta_\rho\left(\frac{x-y}{2}\right)\delta_\rho\left(\frac{t-s}{2}\right).
\end{aligned}$$

Taking this ϕ as a test function, then inequality (4.4) becomes

$$\begin{aligned}
&-\iiiint\limits_{(\Pi_T)^2}\Big(\eta(w;u)\phi_{t+s} + \frac{1}{2}(q(\gamma(x),w;u) + q(\gamma(y),u;w))\phi_{x+y}\\
&\qquad + r(w;u)\partial^2_{x+y}\phi\Big)\delta_\rho\left(\frac{x-y}{2}\right)\delta_\rho\left(\frac{t-s}{2}\right)\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\\
&\qquad \leqslant \iiiint\limits_{\left(\Pi_T\setminus\{\xi_m\}_{m=1}^M\right)^2} (\gamma'(x)\eta_w(w;u)f_\gamma(\gamma(x),w)\\
&\qquad\quad + \gamma'(y)(\eta_u(u;w)f_\gamma(\gamma(y),u) - \gamma'(x)q_\gamma(\gamma(x),w;u)\\
&\qquad\quad - \gamma'(y)q_\gamma(\gamma(y),u;w))\phi\,\delta_\rho\left(\frac{x-y}{2}\right)\delta_\rho\left(\frac{t-s}{2}\right)\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\\
&\qquad\quad + \iiiint\limits_{(\Pi_T)^2}\frac{1}{2}(q(\gamma(x),w;u) - q(\gamma(y),u;w))\\
&\qquad\quad \left(\partial_x\phi\,\delta_\rho\left(\frac{x-y}{2}\right) + \phi\,\partial_x\,\delta_\rho\left(\frac{x-y}{2}\right)\right)\delta_\rho\left(\frac{t-s}{2}\right)\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\\
&\qquad\quad + \iiiint\limits_{(\Pi_T)^2}\frac{1}{2}(q(\gamma(y),u;w) - q(\gamma(x),w;u))\\
&\qquad\qquad \left(\partial_y\phi\,\delta_\rho\left(\frac{x-y}{2}\right) + \phi\,\partial_y\delta_\rho\left(\frac{x-y}{2}\right)\right)\delta_\rho\left(\frac{t-s}{2}\right)\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s.
\end{aligned} \tag{4.5}$$

Notice that $\partial_x\phi\left(\frac{x+y}{2}, \frac{t+s}{2}\right) = \partial_y\phi\left(\frac{x+y}{2}, \frac{t+s}{2}\right)$. Therefore, the first terms of the last two integrals canceled each other out.

Let us consider the change of variables

$$\widetilde{x} = \frac{x+y}{2}, \quad \widetilde{t} = \frac{t+s}{2}, \quad z = \frac{x-y}{2}, \quad \tau = \frac{t-s}{2},$$

that maps $(\Pi_T)^2$ into $\Upsilon = \{(\widetilde{x},\widetilde{t},z,\tau) \in \mathbb{R} : 0 < \widetilde{t}+\tau < T\} \subset \mathbb{R}^4$ and $\left(\Pi_T \setminus \{\xi_m\}_{m=1}^M\right)^2$ into $\Upsilon_\xi = \{(\widetilde{x},\widetilde{t},z,\tau) \in \Upsilon : \widetilde{x} \pm z \neq \xi_m,\ m = 1,\dots,M\}$. Then, we can rewrite

$$\begin{aligned}
\partial_{t+s}\phi\left(\frac{x+y}{2},\frac{t+s}{2}\right) &= \partial_{\widetilde{t}}\phi(\widetilde{x},\widetilde{t})\\
\partial_{x+y}\phi\left(\frac{x+y}{2},\frac{t+s}{2}\right) &= \partial_{\widetilde{x}}\phi(\widetilde{x},\widetilde{t})\\
\partial^2_{x+y}\phi\left(\frac{x+y}{2},\frac{t+s}{2}\right) &= \partial^2_{\widetilde{x}}\phi(\widetilde{x},\widetilde{t}),
\end{aligned}$$

and inequality (4.5) can be written as

$$I_0(\widetilde{x},\widetilde{t},z,\tau) \leqslant I_1(\widetilde{x},\widetilde{t},z,\tau) + I_2(\widetilde{x},\widetilde{t},z,\tau), \tag{4.6}$$

where

$$\begin{aligned}
I_0(\widetilde{x},\widetilde{t},z,\tau) &= -\iiiint_{\Upsilon}\Big[\eta(w;u)\partial_{\widetilde{t}}\phi(\widetilde{x},\widetilde{t}) + \frac{1}{2}(q(\gamma(\widetilde{x}+z),w;u) + q(\gamma(\widetilde{x}-z),u;w))\partial_{\widetilde{x}}\phi(\widetilde{x},\widetilde{t})\\
&\quad + r(w;u)\partial^2_{\widetilde{x}}\phi(\widetilde{x},\widetilde{t})\Big]\delta_\rho(z)\delta_\rho(\tau)\,\mathrm{d}\Upsilon,\\
I_1(\widetilde{x},\widetilde{t},z,\tau) &= \iiiint_{\Upsilon\setminus\{\xi\}_m}(\gamma'(\widetilde{x}+z)\eta_w(w;u)f_\gamma(\gamma(\widetilde{x}+z),w)\\
&\quad + \gamma'(\widetilde{x}-z)\eta_u(u;w)f_\gamma(\gamma(\widetilde{x}-z),u) - \gamma'(\widetilde{x}+z)q_\gamma(\gamma(\widetilde{x}+z),w;u)\\
&\quad - \gamma'(\widetilde{x}-z)q_\gamma(\gamma(\widetilde{x}-z),u;w))\phi(\widetilde{x},\widetilde{t})\delta_\rho(z)\delta_\rho(\tau)\,d\widetilde{x}\,d\widetilde{t}\,dz\,d\tau,\\
I_2(\widetilde{x},\widetilde{t},z,\tau) &= \frac{1}{4}\iiiint_{\Upsilon}(q(\gamma(\widetilde{x}+z),w;u) - q(\gamma(\widetilde{x}-z),u;w)\phi(\widetilde{x},\widetilde{t})\partial_z\delta_\rho(z)\delta_\rho(\tau)\,\mathrm{d}\Upsilon\\
&\quad - \frac{1}{4}\iiiint_{\Upsilon}(q(\gamma(\widetilde{x}-z),u;w) - q(\gamma(\widetilde{x}+z),w;u)\phi(\widetilde{x},\widetilde{t})\partial_z\delta_\rho(z)\delta_\rho(\tau)\,\mathrm{d}\Upsilon,
\end{aligned}$$

with $u = u(\widetilde{x}-z,\widetilde{t}-\tau)$ and $w = w(\widetilde{x}+z,\widetilde{t}+\tau)$. Taking the limit when $\rho \longrightarrow 0$, we get

$$\begin{aligned}
\lim_{\rho\longrightarrow 0} I_0(\widetilde{x},\widetilde{t},z,\tau) = -\iint_{\Pi_T}&(\eta(w(x,t);u(x,t))\phi_t(x,t) + q(\gamma(x),w(x,t);u(x,t))\phi_x(x,t)\\
&+ r(w(x,t);u(x,t))\phi_{xx}(x,t))\,\mathrm{d}t\,\mathrm{d}x.
\end{aligned}$$

Taking Assumption 1.3, we obtain

$$\begin{aligned}
\lim_{\rho\longrightarrow 0} I_1(\widetilde{x},\widetilde{t},z,\tau) = \iint_{\Pi_T\setminus\{\xi_m\}_{m=1}^M}&(\gamma'(x)\eta_w(w(x,t);u(x,t))f_\gamma(\gamma(x),w(x,t)) + \gamma'(x)\eta_u(u(x,t);w(x,t))f_\gamma(\gamma(x),u(x,t))\\
&- 2\gamma'(x)q_\gamma(\gamma(x),w(x,t);u(x,t)))\phi(x,t)\,\mathrm{d}t\,\mathrm{d}x \leqslant 0,
\end{aligned}$$

where we also use the fact that ϕ is positive and bounded. Finally, since we are in a domain where γ is continuous and q is continuous with respect to γ, we have $|\gamma(\widetilde{x}+z) - \gamma(\widetilde{x}-z)| \leqslant L\,|z|$ and

$$|(q(\gamma(\widetilde{x}+z),w;u) - q(\gamma(\widetilde{x}-z),u;w)| \leqslant \|q_\gamma\|\,|\gamma(\widetilde{x}+z) - \gamma(\widetilde{x}-z)| \leqslant L\,|z|.$$

Thus

$$I_2(\widetilde{x},\widetilde{t},z,\tau) = \frac{1}{4}\iiiint_{\Upsilon} (q(\gamma(\widetilde{x}+z),w;u) - q(\gamma(\widetilde{x}-z),u;w))\phi(\widetilde{x},\widetilde{t})\partial_z\delta_\rho(z)\delta_\rho(\tau)\,\mathrm{d}\Upsilon$$
$$\leqslant \iiiint_{\Upsilon} L|z|\,|\phi(\widetilde{x},\widetilde{t})|\,|\partial_z\delta_\rho(z)|\,\mathrm{d}\Upsilon.$$

Therefore, $\lim_{\rho\longrightarrow 0} I_2(\widetilde{x},\widetilde{t},z,\tau) = 0$. Using all the previous results, we can obtain

$$-\iint_{\Pi_T} (\eta(w(x,t);u(x,t))\phi_t(x,t) + q(\gamma(x),w(x,t);u(x,t))\phi_x(x,t) + r(w(x,t);u(x,t))\phi_{xx}(x,t))\,\mathrm{d}t\,\mathrm{d}x \leqslant 0,$$

which concludes the proof. □

Using the previous theorem, we will prove the *a priori* estimate mentioned before.

Theorem 4.2. *Let $w(x,t)$ and $u(x,t)$ be two entropy solutions of equation (1.1) satisfying (1.4) with initial conditions $v_0, u_0 \in L^1 \cap L^\infty(\mathbb{R})$, respectively. If Assumptions 1.3 and 1.4 hold, then for a.e. $t \in [0,T]$,*

$$\int_{\mathbb{R}} \eta(w(x,t);u(x,t))\,\mathrm{d}x \leqslant \int_{\mathbb{R}} \eta(w(x,0);u(x,0))\,\mathrm{d}x. \tag{4.7}$$

Proof. In the previous theorem, we proved that any two entropy solutions that satisfy Definition 1.2 and

$$-\iint_{\Pi_T} (\eta(w(x,t);u(x,t))\phi_t(x,t) + q(\gamma(x),w(x,t);u(x,t))\phi_x(x,t) + r(w(x,t);u(x,t))\phi_{xx}(x,t))\,\mathrm{d}t\,\mathrm{d}x \leqslant 0,$$

for $0 \leqslant \phi \in D(\Pi_T \setminus \{\xi_m\}_{m=1}^M)$. Now our goal is to find a stability result that can be guaranteed when the test function does not vanish around the discontinuities of γ.

For $h > 0$, let us introduce the following Lipschitz function:

$$\mu_h = \begin{cases} \frac{1}{h}(x+2h), & \text{if } x \in [-2h,-h] \\ 1, & \text{if } x \in [-h,h] \\ \dfrac{1}{h}(2h-x), & \text{if } x \in [h,2h] \\ 0, & \text{if } |x| \geqslant 2h. \end{cases} \tag{4.8}$$

Let us define $\Psi_h(x) = 1 - \sum_{m=1}^M \mu_h(x-\xi_m)$ and notice that $\Psi \to 1$ in $L^1(\mathbb{R})$ and Ψ_h becomes 0 around ξ_m. Taking a test function $0 \leqslant \phi \in D(\Pi_T)$, it is straightforward to verify that $0 \leqslant \phi = \phi\Psi_h \in D(\Pi_T \setminus \{\xi_m\}_{m=1}^M)$. Then we can introduce this test function in (4.1) to obtain

$$-\iint_{\Pi_T} (\eta(w(x,t);u(x,t))(\phi\Psi_h)_t + q(\gamma(x),w(x,t);u(x,t))(\phi\Psi_h)_x + r(w(x,t);u(x,t))(\phi\Psi_h)_{xx})\,\mathrm{d}t\,\mathrm{d}x \leqslant 0.$$

Applying integration by parts, we have

$$-\iint_{\Pi_T} (\eta(w;u)(\phi_t\Psi_h) + q(\gamma(x),w;u)(\phi_x\Psi_h) - r(w;u)_x(\phi_x\Psi_h))\,\mathrm{d}t\,\mathrm{d}x$$
$$\underbrace{-\iint_{\Pi_T} (q(\gamma(x),w;u) - r(w;u)_x)\phi\Psi_h'\,\mathrm{d}t\,\mathrm{d}x}_{J(h)} \leqslant 0.$$

Since $\Psi_h \to 1$ in $L^1(\mathbb{R})$, we get

$$-\iint\limits_{\Pi_T} (\eta(w;u)\phi_t + q(\gamma(x),w;u)\phi_x - r(w;u)_x\phi_x)\,\mathrm{d}t\,\mathrm{d}x \leqslant \lim_{h\to 0} J(h).$$

By computing the limit, we obtain

$$\begin{aligned}\lim_{h\to 0} J(h) &= \sum_{m=1}^{M} \lim_{h\to 0} \frac{1}{h} \int\limits_0^T \int\limits_{\xi_m+h}^{\xi_m+2h} (q(\gamma(x),w;u) - r(w;u)_x)\phi(x,t)\,\mathrm{d}t\,\mathrm{d}x \\ &\quad - \sum_{m=1}^{M} \lim_{h\to 0} \frac{1}{h} \int\limits_0^T \int\limits_{\xi_m-2h}^{\xi_m-h} (q(\gamma(x),w;u) - r(w;u)_x)\phi(x,t)\,\mathrm{d}t\,\mathrm{d}x \\ &= \sum_{m=1}^{M} \int\limits_0^T (q(\gamma(x),w;u) - r(w;u)_x)\Big|_{\xi_m^-}^{\xi_m^+} \phi(\xi_m,t)\,\mathrm{d}t.\end{aligned}$$

Taking into account Assumption 1.4, for any $0 \leqslant \phi \in D(\Pi_T)$ it is satisfied that

$$-\iint\limits_{\Pi_T} (\eta(w;u)\phi_t + q(\gamma(x),w;u)\phi_x + r(w;u)\phi_{xx})\,\mathrm{d}t\,\mathrm{d}x \leqslant 0. \tag{4.9}$$

Let us now define, for $r > 1$ a function $\alpha_r : \mathbb{R} \longrightarrow [0,1]$ satisfying

$$\alpha_r(x) = \begin{cases} 1, & \text{if } |x| \leqslant r \\ 0, & \text{if } |x| \geqslant r+1. \end{cases}$$

Let s, s_0 be fixed such that $0 < s_0 < s < T$. For any $\tau > 0$ and $k > 0$ with $0 < s_0 + \tau < s + k < T$, take $\beta_{\tau,k} : [0,T] \longrightarrow \mathbb{R}$ a Lipschitz function that is linear in $[s_0, s_0+\tau] \cup [s, s+k]$ and satisfies

$$\beta_{\tau,k}(t) = \begin{cases} 0, & \text{si } t \in [0,s_0] \ \vee \ t \in [s+k,T], \\ 1, & \text{si } t \in [s_0, s+k]. \end{cases}$$

Function $\phi = \alpha_r\beta_{\tau,k}$ is an admissible test function for inequality (4.9). Thus

$$\begin{aligned}&\frac{1}{k}\int\limits_s^{s+k}\int\limits_{\mathbb{R}} \eta(w(x,t);u(x,t))\alpha_r(x)\,\mathrm{d}t\,\mathrm{d}x - \frac{1}{\tau}\int\limits_{s_0}^{s_0+\tau}\int\limits_{\mathbb{R}} \eta(w(x,t);u(x,t))\alpha_r(x)\,\mathrm{d}t\,\mathrm{d}x \\ &\quad \leqslant \|\alpha_r'\|_{L^\infty(\mathbb{R})} \int\limits_{s_0}^{s+k} \int\limits_{r\leqslant|x|\leqslant r+1} q(\gamma(x),w(x,t);u(x,t))\,\mathrm{d}t\,\mathrm{d}x \\ &\quad + \|\alpha_r''\|_{L^\infty(\mathbb{R})} \int\limits_{s_0}^{s+k} \int\limits_{r\leqslant|x|\leqslant r+1} r(w(x,t);u(x,t))\,\mathrm{d}t\,\mathrm{d}x.\end{aligned}$$

As $s_0 \to 0$, we have

$$\frac{1}{k}\int\limits_s^{s+k}\int\limits_{-r}^{r} \eta(w(x,t);u(x,t))\,\mathrm{d}t\,\mathrm{d}x - \frac{1}{\tau}\int\limits_{s_0}^{s_0+\tau}\int\limits_{-r}^{r} \eta(w(x,t);u(x,t))\,\mathrm{d}t\,\mathrm{d}x \leqslant o\left(\frac{1}{r}\right).$$

The term $o\left(\frac{1}{r}\right)$ comes from the fact that the two integrals on the right hand side approaches to 0 when $r \to \infty$. As $\tau \to 0$ then $r \longrightarrow \infty$ and we obtain

$$\frac{1}{k}\int_s^{s+k}\int_{\mathbb{R}} \eta(w(x,t);u(x,t))\,\mathrm{d}t\,\mathrm{d}x \leqslant \int_{\mathbb{R}} \eta(w(x,0);u(x,0))\,\mathrm{d}x.$$

Finally, taking limits as $k \to 0$, we have the desired inequality

$$\int_{\mathbb{R}} \eta(w(x,s);u(x,s))\,\mathrm{d}x \leqslant \int_{\mathbb{R}} \eta(w(x,0);u(x,0))\,\mathrm{d}x,$$

for a.e $t \in [0,T]$. □

Notice that if the entropy function from Kružkov, $\eta(u) = |u-c|$ $(c \in \mathbb{R})$ is taken, then inequality (4.7) implies the uniqueness of the solution for problem (1.1) and (1.2) as a consequence of a L^1 contraction result [21].

5. Numerical examples

Here the entropy stable approximation (2.6) is applied to several degenerate parabolic tests with discontinuous coefficients in order to illustrate its performance. Taking into account that the upwind scheme has been proved convergent for such problems, it is also interesting to compare both numerical solutions. To obtain the numerical solutions, we consider a semi-discretization in space of equation (1.1) as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}u(x_i,t) = L_\Delta(u(t)),$$

which is solved using the TVD-RK2

$$\begin{aligned} U^{(1)} &= U^n + \Delta t L_\Delta(U^n),\\ U^{n+1} &= \frac{1}{2}U^n + \frac{1}{2}U^{(1)} + \frac{1}{2}\Delta t L_\Delta\left(U^{(1)}\right). \end{aligned}$$

The operator L_Δ in each (x_{i-1}, x_{i-1}) is defined as

$$L_\Delta(U^n) = -\frac{1}{\Delta x}\left(F^E_{i+\frac{1}{2}} - F^E_{i-\frac{1}{2}}\right) + \frac{1}{(\Delta x)^2}\left([A]_{i+\frac{1}{2}} - [A]_{i-\frac{1}{2}}\right) + \frac{\epsilon}{(\Delta x)^2}\left([U^n]_{i+\frac{1}{2}} - [U^n]_{i-\frac{1}{2}}\right),$$

in the case of the entropy stable method, being $F^E_{i+1/2}$ the numerical entropy flux defined in (2.3) and

$$L_\Delta(U^n) = -\frac{1}{\Delta x}\left(F^U_{i+\frac{1}{2}} - F^U_{i-\frac{1}{2}}\right) + \frac{1}{(\Delta x)^2}\left([A]_{i+\frac{1}{2}} - [A]_{i-\frac{1}{2}}\right),$$

in the case of the upwind scheme, where $F^U_{i+\frac{1}{2}} = f^+(U_i) + f^-(U_{i+1})$.

It is important to point out that the proposed TVD-RK2 scheme is stable under the same CFL condition that the explicit Euler time-discretization [24]. Therefore, the stability condition for our numerical scheme is (3.1).

TABLE 1. Error and convergence rate for the entropy stable scheme for problem (5.1) and (5.2) at time $T = 0.15$.

N	10	20	40	80	160	320
L^∞-error	0.646277	0.030289	0.011608	0.002502	0.000590	0.000148
Order L^∞	–	3.83	1.38	2.21	2.08	1.99

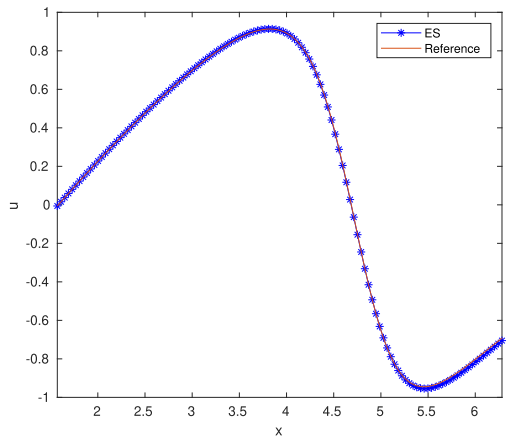

FIGURE 1. Entropy stable solution for parabolic problem (5.1) and (5.2) at time $T = 0.15$ with $\Delta x = 0.03$, $\epsilon = 0.05$ and CFL $= 0.9$.

5.1. Viscous Burgers' equation with continuous coefficients

We start analyzing the error and convergence rate of the proposed entropy stable (ES) scheme. For that, we consider a viscous Burgers' equation with a continuous coefficient $\gamma(x)$ and an integrated diffusion coefficient without degeneracy

$$u_t + \left((0.6x+1)\frac{u^2}{2}\right)_x = u_{xx}, \quad (x,t) \in \left[\frac{\pi}{2}, 2\pi\right] \times [0,T], \tag{5.1}$$

along with the initial condition

$$u_0(x) = (-2\cos(x))/(2+\sin(x)). \tag{5.2}$$

In Table 1 it is shown the results of the numerical convergence by considering the L$^\infty$-norm. The error is obtained using an upwind reference solution with a spatial mesh of 1280 nodes. Notice that the convergence order of the entropy stable (ES) method is initially greater than three and at the end gets closer to two. In Figure 1, a comparison between the entropy stable approximation and the reference solution for problem (5.1) and (5.2) is presented.

5.2. Linear equation with oscillating solution

Let us consider the linear advection equation

$$u_t + u_x = 0, \qquad (x,t) \in [0,7] \times [0,T], \tag{5.3}$$

with initial condition

$$u(0,x) = \begin{cases} -\left(x^2\sin(3\pi x)\right)/15 & \text{if } 0 \le x \le 4, \\ 0 & \text{otherwise.} \end{cases} \tag{5.4}$$

TABLE 2. Error and convergence rate for the entropy stable and upwind schemes for problem (5.3) and (5.4) at time $T = 2$.

	N	25	75	125	175	225	275
L^2-error	UP	0.308471	0.166431	0.118452	0.090467	0.072168	0.059297
	ES	0.387318	0.174129	0.060500	0.029652	0.017770	0.011911
Order L^2	UP	–	0.55	0.66	0.80	0.89	0.97
	ES	–	0.71	2.05	2.11	2.03	1.99

For such Cauchy problem, the exact solution is known and it exhibits a fast oscillatory behavior, see Figure 2. We show the convergence rates and error of the upwind (UP) and entropy stable methods for problem (5.3) and (5.4) in Table 2. We observe that the ES scheme is of second order accuracy while the UP method has difficulties in reaching the first order, which is related with the oscillating dynamics of the solution.

Based on the linear equation (5.3), we now consider the following degenerate parabolic equation:

$$u_t + (\gamma(x)u)_x = A(u)_{xx}, \qquad (x,t) \in [0,7] \times [0,T], \tag{5.5}$$

with the same initial condition, where

$$A(u) = \begin{cases} \lambda(u+1) & \text{if } u < 0.1, \\ 0.00275 & \text{if } 0.1 \le u \le 0.4, \\ \lambda(u+0.7) & \text{if } 0.4 < u, \end{cases}$$

with $\lambda = 0.0025$ being a parameter. For the function $\gamma(x)$, we test two different discontinuous fluxes

$$\gamma_1(x) = \begin{cases} 0.5 & \text{if } x < 3.8, \\ 1 & \text{if } x \ge 3.8, \end{cases}$$

and

$$\gamma_2(x) = \begin{cases} 0.5 & \text{if } x < 3.5, \\ 1 & \text{if } 3.5 \le x \ge 4.5, \\ 0.25 & \text{if } x > 4.5. \end{cases}$$

To formulate the entropy stable approximation, we take the entropy function $\eta(u) = u^2/2$ and from definition (2.3), we obtain the numerical flux

$$F^E_{i+\frac{1}{2}}(u_i, u_{i+1}) = \frac{1}{2}\gamma_{i+\frac{1}{2}}(u_i + u_{i+1}).$$

In Figure 2, we can observe the numerical solutions for the hyperbolic and degenerate parabolic problems. It is interesting to notice that the dispersion of the upwind method for the hyperbolic case is maintained in the degenerate parabolic case. In Figure 3, we show the performance of both methods for two different advective fluxes considering two piecewise functions, $\gamma_1(x)$ and $\gamma_2(x)$. It is important to note that the UP method requires further refinement of the mesh to leave behind the inherent scatter error that still appears in Figure 3. On the other hand, the stability of the ES approximation is regulated by the viscosity parameter ϵ, which must be higher as the degenerate diffusion and discontinuous coefficients are incorporated.

5.3. Burgers' equation with a rarefaction solution

Let us consider a well-known nonlinear hyperbolic equation

$$u_t + \left(\frac{u^2}{2}\right)_x = 0, \quad (x,t) \in [-1,1] \times [0,T], \tag{5.6}$$

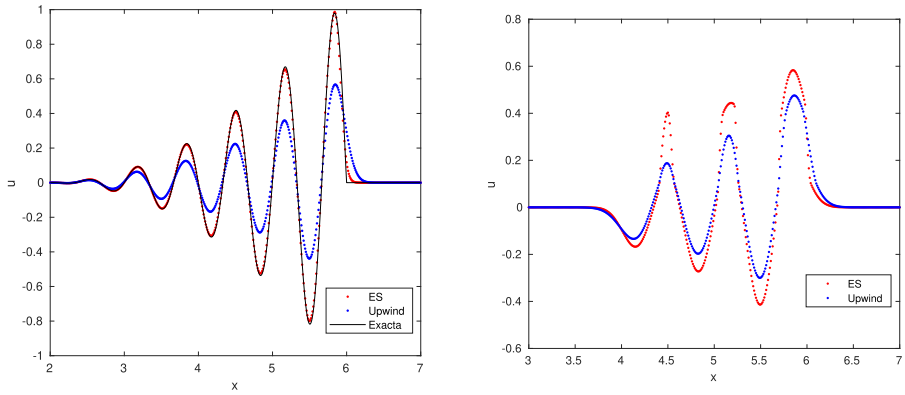

FIGURE 2. Numerical solution for the hyperbolic problem (5.3) and (5.4) (*left*) and for the parabolic problem (5.5) and (5.4) with $\gamma(x) = 1$ (*right*) at time $T = 2$, taking $\Delta x = 0.01$ and CFL $= 0.3$. For the ES scheme the parameter ϵ is 0.015 (*left*) and 0.05 (*right*).

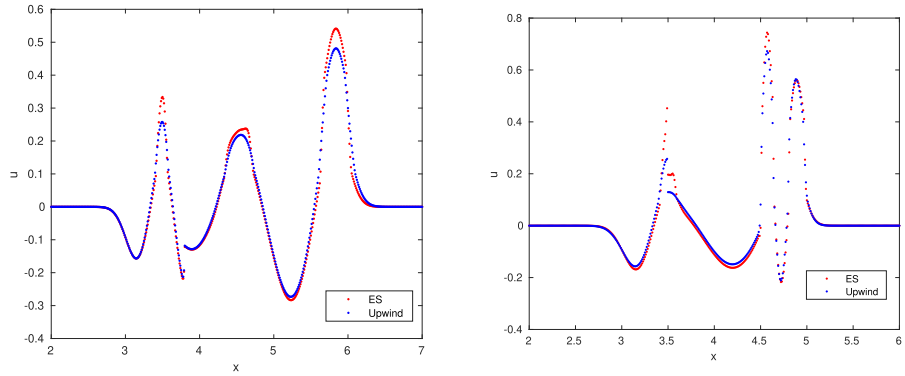

FIGURE 3. Numerical solution for the parabolic problem (5.5) and (5.4) at time $T = 2$ taking $\Delta x = 0.01$ and CFL $= 0.45$. Graph of the left hand side was obtained considering $\gamma(x) = \gamma_1(x)$ and $\epsilon = 0.1$ (ES scheme). Graph of the right hand side with $\gamma(x) = \gamma_2(x)$ and $\epsilon = 0.08$.

with initial condition

$$u_0(x) := \begin{cases} -1 & \text{if } x \leq -1/3, \\ 1 & \text{if } x > -1/3. \end{cases} \tag{5.7}$$

The entropy solution of this problem is a continuous solution named as a rarefaction wave. It is well known that a standard upwind method develops a non-existent discontinuous jump, see Figure 4. Notice that the entropy stable approximation does not develop such discontinuity.

We also propose a degenerate parabolic case associated to the previous equation

$$u_t + \left(\gamma(x)\frac{u^2}{2}\right)_x = A(u)_{xx}, \quad (x,t) \in [-1,1] \times [0,T], \tag{5.8}$$

where the integrated diffusion coefficient is

$$A(u) = \begin{cases} 10\lambda(u+1) & \text{if } \ u < -0.3, \\ 7\lambda & \text{if } \ u \geq -0.3. \end{cases}$$

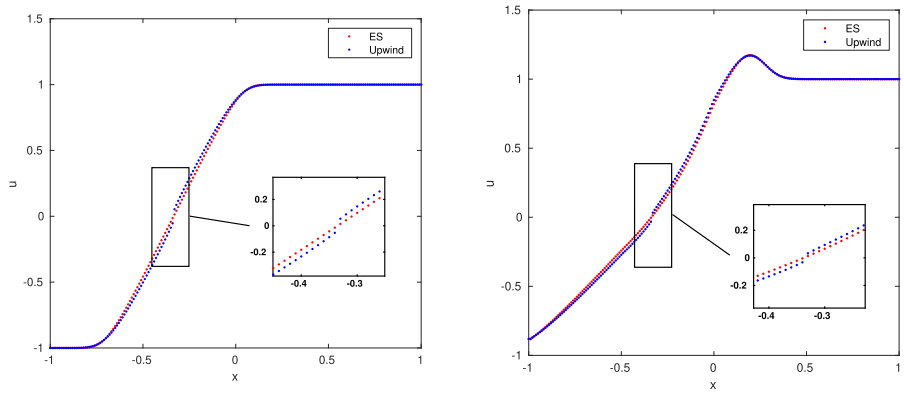

FIGURE 4. Numerical solution for the hyperbolic problem (5.6) and (5.7) at time $T = 0.35$, taking $\Delta x = 0.01$ and CFL $= 0.1$. Consider $\gamma(x) = 1$ (*left*) and $\gamma(x)$ given in (5.9) (*right*). For the ES scheme the parameter ϵ is 0.3 in both cases.

$$\gamma_3(x) = \begin{cases} 2 & \text{if } \ x < -0.5, \\ -2x+1 & \text{if } \ x \geq -0.5 \quad \text{and} \quad x < 0, \\ 1 & \text{if } \ x \geq 0. \end{cases} \tag{5.9}$$

and

$$\gamma_4(x) = \begin{cases} 2 & \text{if } \ x < 0, \\ 1 & \text{if } \ x \geq 0. \end{cases} \tag{5.10}$$

The entropy stable approximation is obtained by considering $\eta(u) = u^2/2$ and

$$F^E_{i+\frac{1}{2}}(u_i, u_{i+1}) = \gamma_{i+\frac{1}{2}} \frac{1}{6}\big(u_i^2 + u_i u_{i+1} + u_{i+1}^2\big). \tag{5.11}$$

In Figure 4, it is shown how the wrong shock produced by the UP method is maintained despite the inclusion of a diffusion term in the equation. Moreover, it is important to point out that the ES approximation converges with CFL≤ 0.6, but the upwind approximation introduces several shocks for CFL values higher than 0.1. In Figure 5, we can observe the numerical solutions when a discontinuous flux, $\gamma(x)$, is considered. Both methods (ES and UP) reproduce well the jump of the solution as a consequence of the discontinuous flux. However, there is a small gap in the upwind solution caused by the wrong shock which is not eliminated even with a greater refinement in the mesh.

5.4. Traffic flow equation with a shock solution

Finally, we consider again a nonlinear Riemann problem with a discontinuous flux of the form [19]

$$u_t + \big(\gamma(x)u(1-u)\big)_x = 0, \quad (x,t) \in [-1,1] \times [0,T], \tag{5.12}$$

being the initial condition $u_0(x) = 0.6$ for $x < 0$ and $x > 0$ and where

$$\gamma(x) = \begin{cases} 0.05 & \text{if } \ x < 0, \\ 0.1 & \text{if } \ x \geq 0. \end{cases} \tag{5.13}$$

For this problem, a rarefaction wave and a jump discontinuity connect two states of the solution, $u = 0.6$ and $u \approx 0.14$, see Figure 6. The entropy stable method is constructed considering $\eta(u) = u(\log(u) - 1)$, obtaining

$$F^E_{i+\frac{1}{2}} = \gamma_{i+\frac{1}{2}} \frac{1}{2}\big(u_{i+1} - \big(u_i^2 + u_{i+1}^2\big) + u_i\big).$$

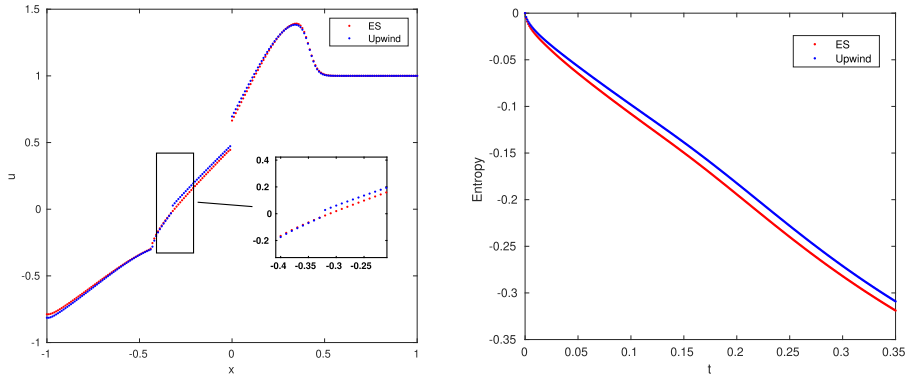

FIGURE 5. Numerical solution for the parabolic problem (5.5) and (5.7) with $\gamma(x)$ given in (5.10) (*left*) at time $T = 0.35$ taking $\Delta x = 0.01$ and CFL $= 0.9$. On the right hand side, the total entropy decay rate is shown. We used $\epsilon = 0.52$ for both graphs.

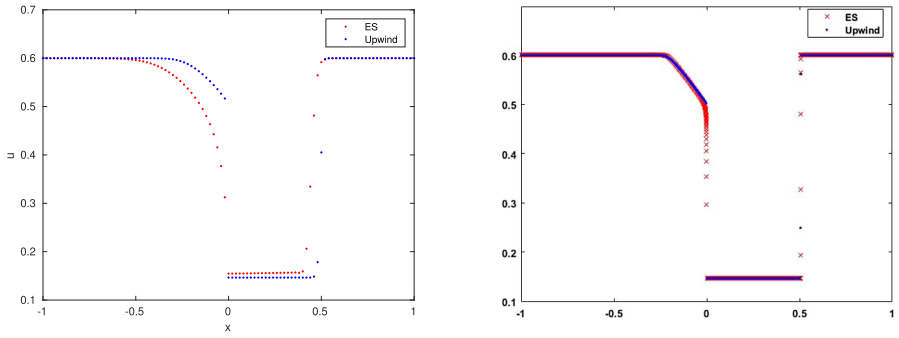

FIGURE 6. Numerical solutions for the hyperbolic problem (5.12) with function $\gamma(x)$ given in (5.13) at time $T = 20$. Taking $\Delta x = 0.02$, CFL $= 0.9$ and $\epsilon = 0.028$ (*left*); $\Delta x = 0.0002$ and CFL $= 0.9$ and $\epsilon = 0.01$ (*right*).

To solve a degenerate parabolic problem associated with equation (5.12), we consider the integrated diffusion coefficient

$$A(u) = \begin{cases} \lambda u & \text{if } \ 0 \le u \le 0.45, \\ 0.45\lambda & \text{if } \ 0.45 \le u \le 0.55, \\ \lambda(u - 0.1) & \text{if } \ 0.55 < u \le 1. \end{cases} \tag{5.14}$$

In this example, we focus on studying the behavior of the UP and ES approximations when a mesh refinement is applied. In Figure 6, numerical simulations for the hyperbolic problem are shown. It is observed that the ES method needs further refinement to achieve the correct spatial position of the jump discontinuity.

For the degenerate parabolic problem, the refinement of the mesh does not offer a significant difference in the numerical solutions, see Figure 7. Another interesting feature is that the ES method does not require extra viscosity for the degenerate parabolic problem and the ϵ parameter is zero, see Remark 3.2.

6. CONCLUSIONS

In this work we proposed three-point entropy stable finite-difference schemes for degenerate parabolic equations with a discontinuous convection term. The proposed entropy stable discretization is based on the entropy

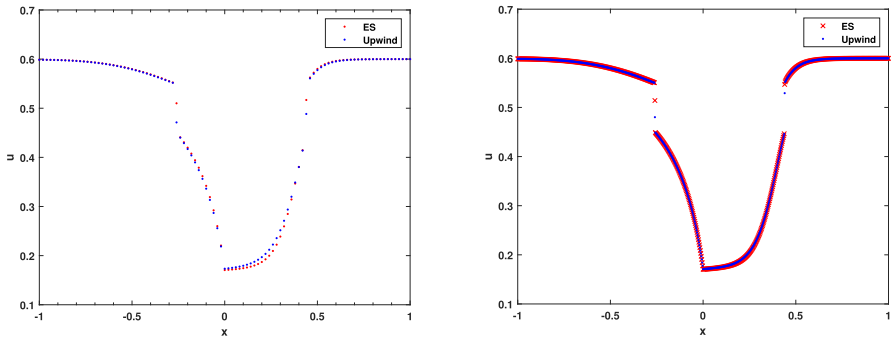

FIGURE 7. Numerical solutions for the degenerate parabolic problem given by (5.14) at time $T = 20$. Taking $\Delta x = 0.02$, CFL $= 0.9$ and $\epsilon = 0$ (*left*); $\Delta x = 0.0025$, CFL $= 0.9$ and $\epsilon = 0$ (*right*).

stable formulation given initially by Tadmor for conservation laws. Using the arithmetical mean as an estimation in each interval we included in a simple way the approximation of the discontinuous coefficient. The entropy stable solution was proven to converge to entropy weak solutions *via* the compensated compactness methodology. Such methodology allowed us to consider discontinuities and sign changes of the discontinuous coefficient. The entropy stable sequence has a free parameter, for which we found a theoretical bound; in the past, this parameter was found by trial and error.

To validate and analyze the efficiency of the proposed three-point entropy stable methods, numerical simulations were provided for benchmark problems. We conclude that the spatial entropy stable discretization has the advantages of a second-order accuracy method with the added bonus of capturing shocks correctly. Thus, the entropy stable solution can be an alternative to the upwind solution as a reference solution to solve degenerate convection-diffusion problems and to validate similar numerical methods.

In the near future, we would like to construct high-order entropy stable solutions for degenerate parabolic equations and extend them to solve multidimensional problems. For applications, it is important to search for entropy functions that perform well in several situations.

APPENDIX A. EQUIVALENCE OF ENTROPY SOLUTIONS

Let us first consider the Kružkov entropy condition for the degenerate problem (1.1) and (1.2) [18]

$$\begin{aligned}
&\iint_{\Pi_T} |u-c|\phi_t + [\operatorname{sign}(u-c)(f(\gamma(x),u) - f(\gamma(x),c))]\phi_x + |A(u)-A(c)|\phi_{xx}\,\mathrm{d}t\,\mathrm{d}x \\
&\quad + \iint_{\Pi_T\setminus\{\xi_m\}_{m=1}^M} \operatorname{sign}(u-c) f(\gamma(x),c)_x \phi\,\mathrm{d}t\,\mathrm{d}x \\
&\quad + \int_0^T \sum_{m=1}^M |f(\gamma(\xi_m+),c) - f(\gamma(\xi_m-),c)|\phi(\xi_m,t)\,\mathrm{d}t \geqslant 0,
\end{aligned} \tag{A.1}$$

for any $0 \leq \phi \in C_0^\infty(\Pi_T)$.

Entropy solutions satisfying the above condition are known as Kružkov entropy solutions. Moreover, it is known that a Kružkov-type entropy solution is a weak solution of the problem of interest. Let us now demonstrate the equivalence between inequalities (A.1) and (1.4):

– Assume that a weak solution u satisfies the entropy 3-tuples inequality (1.4). To prove that u also satisfies the Kružkov inequality (A.1), we suppose conditions (H1)–(H5) and use similar arguments as given in Burger *et al.* [4].
– Conversely, let u be a weak solution of problem (1.1) and (1.2) that satisfies inequality (1.4). Assume u takes values in $[a, b]$ and consider a convex continuous entropy function $\eta(u)$ with corresponding entropy fluxes q and r. By using an adaptive linear interpolation on a sufficiently fine grid, there exists for all $\alpha > 0$, an entropy η_α which is a convex, piecewise affine function given by

$$\eta_\alpha(s) = b_0 + b_1 s + \sum_j a_j |s - c_j|, \quad \text{with} \quad a_j > 0.$$

satisfying

$$\eta(s) \leq \eta_\alpha(s) \leq \eta(s) + \alpha \quad \text{for} \quad s \in [a, b].$$

That is, there is a continuous path between the graphs of $\eta(s)$ and $\eta(s) + \alpha$. Also there are associated fluxes q_α and r_α:

$$q_\alpha(s) = b_1 f(\gamma, s) + \sum_j a_j (f(\gamma, s) - f(\gamma, c_j)) \operatorname{sign}(s - c_j),$$
$$r_\alpha(s) = b_1 A(s) + \sum_j a_j (A(s) - A(c_j)) \operatorname{sign}(s - c_j).$$

By the Kružkov type entropy inequality (A.1) and (1.3), (1.4) is valid for η_α, q_α and r_α. Then, since η_α, q_α and r_α converge uniformly to η, q and r respectively on [a, b] we can interchange the limit with the integral. Therefore, the entropy inequality (1.4) is valid for the given η, which was consider an arbitrary entropy function.

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