A HYBRIDIZABLE DISCONTINUOUS GALERKIN METHOD FOR THE FULLY COUPLED TIME-DEPENDENT STOKES/DARCY-TRANSPORT PROBLEM

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Abstract. We present a high-order hybridized discontinuous Galerkin (HDG) method for the fully coupled time-dependent Stokes–Darcy-transport problem where the fluid viscosity and source/sink terms depend on the concentration and the dispersion/diffusion tensor depends on the fluid velocity. This HDG method is such that the discrete flow equations are compatible with the discrete transport equation. Furthermore, the HDG method guarantees strong mass conservation in the \( H^{1} \) sense and naturally treats the interface conditions between the Stokes and Darcy regions via facet variables. We employ a linearizing decoupling strategy where the Stokes/Darcy and the transport equations are solved sequentially by time-lagging the concentration. We prove well-posedness and optimal \( a \) priori error estimates for the velocity and the concentration in the energy norm. We present numerical examples that respect compatibility of the flow and transport discretizations and demonstrate that the discrete solution is robust with respect to the problem parameters.

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1. Introduction

Coupled free fluid and porous media flow is encountered in many engineering applications [24, 33] and can be modeled by the Stokes/Darcy equations. Adding a transport equation to this coupled system brings forth a model that can be used to simulate the spread of contaminants towards groundwater resources [4] or biochemical transport in hemodynamics [21].

The accuracy and stability of numerical discretizations of the stationary Stokes/Darcy equations [3, 11–13, 25, 29, 36, 39, 41] and advection-diffusion type transport equations [10, 19, 40, 51] are well studied. However, accuracy and stability are not automatically guaranteed when these discretizations are coupled. In particular, compatible discretizations, as defined by Dawson et al. [22], are desired to avoid loss of accuracy and loss of conservation properties of the numerical methods used for the transport equation.

The first numerical study on the coupling of the stationary Stokes/Darcy equations with a transport equation was given in [50] where a mixed finite element method (MFEM) is used for the flow problem and the...
local discontinuous Galerkin method is used for the transport problem. They considered one-way coupling; the concentration is affected by the flow velocity, but the velocity is not affected by a change in concentration.

The same problem was studied in [43] by using discontinuous Galerkin (DG) methods for both flow and transport equations. In [28], Ervin et al. considered a fully time-dependent version of the one-way coupled problem where they developed partitioned time-stepping methods by imposing the interface conditions weakly using penalties. One-way coupling was considered also in [16] in which the flow problem was discretized by a strongly mass conservative Embedded-Hybridized DG (EDG-HDG) method while the transport equation was discretized by an EDG method.

Less studied is the fully-coupled problem in which, apart from the transport equation depending on the flow velocity, the flow solution is time-dependent and the fluid viscosity and source/sink terms depend on the concentration.

To the best of our knowledge, there are only three papers that focus on this fully coupled problem. First, [14] presented an analysis of a weak solution for the case where free flow is governed by the Navier–Stokes equations. The analysis in [14], however, also holds when free flow is governed by the Stokes equations. There are two numerical papers on this topic. The first one [45], introduced a stabilized mixed finite element method using nonconforming piecewise linear Crouzeix–Raviart finite elements for the velocity, a piecewise constant approximation for the pressure, and a conforming, piecewise linear, finite element method for the transport equation based on a skew-symmetric formulation. The second one [52], used a finite element approximation with different time steps on different physical variables under an appropriate time step assumption.

In this paper, we extend the work in [16] to the fully coupled case. To deal with the non-linearity, we consider a linearizing decoupling strategy, where the Stokes/Darcy and the transport equations are solved sequentially by time-lagging the concentration. We use HDG methods [20] for both the Stokes/Darcy and transport sub-problems at each time step and prove well-posedness and \textit{a priori} error estimates. These results can easily be extended to the EDG-HDG discretization used for the Stokes/Darcy problem in [16]. While [45] proved that their method is first order accurate in space, we show that the HDG method presented here is higher order accurate. Furthermore, unlike [45, 52], our HDG method is strongly mass conserving, \textit{i.e.}, the velocity field is $H(\text{div})$-conforming and exactly divergence free on the elements in the absence of source/sink terms. This renders our scheme robust with respect to the problem parameters. By choosing the polynomial degree in a specific way our flow/transport scheme is also compatible.

Here is an outline for the remainder of this article. In Section 2, we present the fully coupled Stokes/Darcy-transport model and specify the assumptions on the problem parameters. Section 3 sets notation, describes in detail the semi-discrete HDG scheme, and lists the attractive properties of the numerical discretization. Next, Section 4 summarizes standard inequalities and shows continuity, coercivity, and the inf-sup condition for the discretization of the Stokes/Darcy sub-problem. A full discretization of the problem based on a sequential decoupling strategy is introduced in Section 5 while the main results, \textit{i.e.}, \textit{a priori} error estimates for the velocity, pressure, and concentration, are presented in Section 6. Finally, we present some numerical experiments in Section 7 followed by conclusions in Section 8.

2. The Stokes/Darcy-transport system

Let $\Omega \subset \mathbb{R}^{\text{dim}}$, $\text{dim} = 2, 3$, be a bounded polygonal domain. We denote its boundary by $\partial \Omega$ and the outward unit normal to $\partial \Omega$ by $n$. Domain $\Omega$ consists of two non-overlapping polygonal regions, a free flow region $\Omega^s$ and a Darcy flow region $\Omega^d$, such that $\Omega = \Omega^s \cup \Omega^d$. The polygonal interface between $\Omega^s$ and $\Omega^d$ is denoted by $\Gamma^I$ and the external boundary of $\Omega^j$ is denoted by $\Gamma^j := \partial \Omega \cap \partial \Omega^j$, $j = s, d$. See Figure 1 for a depiction of a domain when $\text{dim} = 2$.

We denote the time interval of interest by $J = [0, T]$. The fully coupled Stokes/Darcy-transport system for the velocity field $u : \Omega \times J \rightarrow \mathbb{R}^{\text{dim}}$, fluid pressure $p : \Omega \times J \rightarrow \mathbb{R}$ and concentration $c : \Omega \times J \rightarrow \mathbb{R}$ is given by

\[
\partial_t u - \nabla \cdot (2\mu(c)\varepsilon(u)) + \nabla p = f^s(c) \quad \text{in } \Omega^s \times J,
\]  

(1a)
Figure 1. A depiction of a domain $\Omega \subset \mathbb{R}^2$ with its two sub-domains $\Omega^s$ and $\Omega^d$. 

\[ \mathbb{K}^{-1}(c)u + \nabla p = \mathbb{K}^{-1}(c)f^d(c) \quad \text{in } \Omega^d \times J, \]  
\[ -\nabla \cdot u = \chi_d(g_p - g_i) \quad \text{in } \Omega \times J, \]  
\[ \phi \partial_c(\mathbb{K}^{-1}(c)u) \cdot \nabla c = \chi_d(c_1g_i - c_0g_p) \quad \text{in } \Omega \times J, \]  
\[ u = 0 \quad \text{on } \Gamma^s \times J, \]  
\[ u \cdot n = 0 \quad \text{on } \Gamma^d \times J, \]  
\[ \tilde{D}(u)\nabla c \cdot n = 0 \quad \text{on } \partial \Omega \times J, \]

where $\varepsilon(u) := \left(\nabla u + (\nabla u)^T\right)/2$ is the strain rate tensor and $\chi_d$ is the characteristic function that takes the value 1 in $\Omega^d$ and 0 in $\Omega^s$. Here the fluid viscosity $\mu$, the matrix $\mathbb{K} = \kappa/\mu$, where $\kappa$ is the permeability matrix of the porous medium, and the body force terms $f^s$ and $f^d$ are concentration dependent functions. The porosity $\phi$ of the medium in $\Omega^d$ is a spatially varying function. In $\Omega^s$ we set $\phi = 1$. The functions $g_i$ and $g_p$ denote the source and sink terms related to injection and production wells and $c_I$ is the injected concentration. Furthermore, in the Stokes region $\tilde{D}(u) = dI$, where $I$ is the $\dim \times \dim$ identity matrix and $d$ is the diffusion coefficient. In the Darcy region $\tilde{D}(u) = D(u)$, where $D(u)$ denotes the diffusion dispersion tensor in $\Omega^d$.

We will denote the restriction of the velocity $u$, pressure $p$, and concentration $c$ to sub-domain $\Omega^j$, $j = s, d$ by, respectively, $u^j$, $p^j$, and $c^j$. Then, on the interface $\Gamma^j$, choosing the unit normal vector $n$ to be pointing from $\Omega^s$ to $\Omega^d$, we prescribe the following interface conditions that hold for $t \in J$:

\[ u^s \cdot n = u^d \cdot n, \]  
\[ p^s - (2\mu(c^s)\varepsilon(u^s)n) \cdot n = p^d, \]  
\[ -2(\varepsilon(u^s)n) \cdot \tau^\ell = \gamma^\ell u^s \cdot \tau^\ell, \quad \ell = 1, \ldots, \dim - 1, \]  
\[ c^s = c^d, \]  
\[ d\nabla c^s \cdot n = D(u^d)\nabla c^d \cdot n, \]

where $\tau^\ell$, $\ell = 1, \ldots, \dim - 1$ denote the unit tangent vectors on $\Gamma^j$.

These conditions enforce the normal continuity of the velocity (2a), the normal continuity of the normal component of the stress (2b), continuity of the concentration (2d), and normal continuity of the concentration flux (2e). Equation (2c), where $\gamma^\ell = \alpha/\sqrt{\tau^\ell \cdot \kappa \tau^\ell}$ with $\alpha > 0$ a constant, is the Beavers–Joseph–Saffman law which enforces a condition on the tangential component of the normal stress [5,46].

To close the model, we assume the following initial conditions:

\[ u^s(x, 0) = u_0^s(x) \quad \text{in } \Omega^s, \]  
\[ c(x, 0) = c_0(x) \quad \text{in } \Omega. \]
We end this section by discussing some assumptions we make on the various functions used in the Stokes/Darcy-transport model. The dispersion-diffusion tensor $D(u)$ in $\Omega^d$ satisfies for $u, v \in \mathbb{R}^{\dim}$ [37, 49]:

\[
D_{\min}(1 + |u|)|\xi|^2 \leq \xi^T D(u) \xi \leq D_{\max}(1 + |u|)|\xi|^2 \quad \forall \xi \in \mathbb{R}^{\dim},
\]

where $D_{\min}, D_{\max}$, and $D_L$ are positive constants and $| \cdot |$ denotes the Euclidean norm. We assume that $\mu$ is Lipschitz continuous in $c$ with Lipschitz constant $\mu_L$ and that there exist constants $\phi_*, \phi^*, \mu_*, \mu^* > 0$ such that

\[
\phi_* \leq \phi(x) \leq \phi^* \quad \forall x \in \Omega^d, \\
\mu_* \leq \mu(c) \leq \mu^* \quad \forall c \in \mathbb{R}.
\]

The permeability matrix $\kappa$ is symmetric, uniformly bounded, and elliptic, that is, there exist positive constants $\kappa_* < \kappa^*$ such that

\[
\kappa_* |\xi|^2 \leq \xi^T \kappa(x) \xi \leq \kappa^* |\xi|^2 \quad \forall \xi \in \mathbb{R}^{\dim}, \quad \forall x \in \Omega^d.
\]

From (6) and (5b), we deduce that

\[
K_* |\xi|^2 \leq \xi^T \kappa(x) \xi \leq K^* |\xi|^2 \quad \forall \xi \in \mathbb{R}^{\dim}, \quad \forall (c, x) \in \mathbb{R} \times \Omega^d,
\]

where $K_*=\kappa_*/\mu^*$ and $K^*=\kappa^*/\mu_*$.

The body force functions $f^s$ and $f^d$ are assumed to be Lipschitz continuous in $c$ with Lipschitz constants $L_f^s$ and $L_f^d$. Note that $f^s$ and $f^d$ depend on $x$ and $c$, but they do not depend explicitly on $t$. We will further assume that $0 \leq c_I \leq 1$ a.e. in $\Omega^d$ and that $g_i, g_p \geq 0$, $g_i, g_p \in L^\infty(J; L^2(\Omega^d))$ are such that

\[
\int_{\Omega^d} (g_i(x, t) - g_p(x, t)) \, dx = 0 \quad \forall t \in J.
\]

A weak formulation of the problem defined by equations (1)–(3) was presented in [45]. The analysis for a weak solution of a more general version of this problem, in which the free fluid flow is governed by the Navier–Stokes equations, can be found in [14].

3. The hybridized discontinuous Galerkin method

3.1. Preliminaries

We use the same notation that we used previously in [16, 17]. Let $T^j := \{K\}$ be a shape-regular triangulation of $\Omega^j$, $j = s, d$, into non-overlapping elements (we only consider simplices) such that $T^s$ and $T^d$ match at the interface $\Gamma^j$. We define the triangulation of the entire domain $\Omega$ as $T := T^s \cup T^d$. The maximum diameter over all elements is $h = \max_{K \in T} h_K$, where $h_K$ stands for the diameter of an element $K$. The boundary of an element $K$ and its outward unit normal are denoted by $\partial K$ and $n$, respectively. A facet of an element boundary is an interior facet if it is shared by two neighboring elements and it is a boundary facet if it is a part of $\partial \Omega$. The set of all interior facets and all boundary facets in $\Omega^j$ are denoted by $\mathcal{F}_i^j$ and $\mathcal{F}_b^j$, $j = s, d$, respectively. We also collect the facets that lie on the interface $\Gamma^j$ in the set $\mathcal{F}^j$. The set of all facets that lie in $\bar{\Omega}$ and in $\bar{\Omega}^j$ are denoted by $\mathcal{F}$ and $\mathcal{F}^j$, respectively. We point out that $\mathcal{F}^j = \mathcal{F}_i^j \cup \mathcal{F}_b^j \cup \mathcal{F}^j$, $j = s, d$. Furthermore, we define $\Gamma_0 := \cup_{F \in \mathcal{F}} F$ and $\Gamma_0^j := \cup_{F \in \mathcal{F}^j} F$, $j = s, d$.

The finite element function spaces on $\Omega$ for the velocity and pressure are given by

\[
V_h := \left\{ v_h \in \left[ L^2(\Omega) \right]^{\dim} : v_h \in \left[ P_{k_i}(K) \right]^{\dim}, \forall K \in T \right\}, \\
Q_h := \left\{ q_h \in L^2(\Omega) : q_h \in P_{k_f-1}(K), \forall K \in T \right\} \cap L_0^2(\Omega),
\]
where \( P_h(K) \) denotes the space of polynomials of degree at most \( k \) defined on the element \( K \). The finite element spaces for the velocity and pressure traces are given by

\[
\tilde{V}_h := \left\{ \tilde{v}_h \in \left[ L^2(\Gamma_0^s) \right]^{\dim} : \tilde{v}_h \in [ P_{k_f}(F) ]^{\dim}, \forall F \in \mathcal{F}, \tilde{v}_h = 0 \text{ on } \Gamma^s \right\},
\]

\[
\bar{Q}_h := \left\{ \bar{q}_h \in L^2(\Gamma_0^s) : \bar{q}_h \in P_{k_f}(F) \forall F \in \mathcal{F} \right\}, \quad j = s, d.
\]

(9)

Here \( P_k(F) \) denotes the space of polynomials of degree at most \( k \) defined on the facet \( F \). Note that functions in \( \bar{V}_h \) are not defined on \( \Gamma_0^s \setminus \Gamma_f \). The finite element function spaces for the concentration and its trace are defined as

\[
C_h = \{ c_h \in L^2(\Omega) : c_h \in P_{k_c}(K), \forall K \in \mathcal{T} \}, \quad \bar{C}_h = \{ \bar{c}_h \in L^2(\Gamma_0^s) : \bar{c}_h \in P_{k_c}(F) \forall F \in \mathcal{F} \}.
\]

(10)

**Remark 3.1.** As pointed out in [22], when fluid flow is coupled to a transport equation, how the numerical scheme handles the mass conservation equation can impact the accuracy and global conservation properties of the transport discretization. Therefore, Dawson et al. [22] introduced the concept of “compatibility” as the minimal requirement on the flow discretization to preserve accuracy and global conservation. Compatibility requires that: (1) the numerical scheme is zeroth-order accurate, that is, the method has the ability to reproduce the constant concentration solution; and (2) the numerical scheme is globally conservative. In [16], we considered the one-way coupled Stokes-Darcy-transport problem and showed that compatibility can be achieved by choosing \( k_c = k_f - 1 \) and demonstrated in Section 7.2 of [16] that an incompatible discretization cannot preserve the constant solution.

To reduce the notational burden, we define \( V_h := \tilde{V}_h \times \bar{V}_h \), \( Q_h := Q_h \times \bar{Q}_h \times \bar{Q}_h \), and \( Q_h^j := Q_h^j \times \bar{Q}_h^j \), \( j = s, d \). We denote elements in these product spaces by \( v_h := (v_h, \tilde{v}_h) \in V_h \), \( q_h := (q_h, \bar{q}_h, \bar{q}_h) \in Q_h \), and \( q_h^j := (q_h^j, \bar{q}_h^j, \bar{q}_h^j) \in Q_h^j \), \( j = s, d \). In addition, we set \( X_h := V_h \times Q_h \). Similarly, we introduce \( C_h = C_h \times \bar{C}_h \) and denote the corresponding elements by \( c_h := (c_h, \bar{c}_h) \in C_h \).

Next, let us define the function spaces

\[
V := \left\{ v \in \left[ L^2(\Omega) \right]^{\dim} : v^s \in \left[ H^2(\Omega^s) \right]^{\dim}, v^d \in \left[ H^1(\Omega^d) \right]^{\dim}, \quad v = 0 \text{ on } \Gamma^s, \quad v \cdot n = 0 \text{ on } \Gamma^d, \quad v^s \cdot n = v^d \cdot n \text{ on } \Gamma^f \right\},
\]

\[
Q := \left\{ q \in L^2(\Omega) : q^s \in H^1(\Omega^s), \quad q^d \in H^2(\Omega^d) \right\},
\]

\[
C := H^2(\Omega),
\]

and set \( X := V \times Q \). As before, we use a superscript \( j \) to specify the restriction of these spaces to \( \Omega^j \), \( j = s, d \). The trace spaces of \( V \) restricted to \( \Gamma_0^s \), \( Q^j \) restricted to \( \Gamma_0^j \), and \( C \) restricted to \( \Gamma_0^s \) are denoted by, respectively, \( \tilde{V} \), \( \bar{Q}^j \), and \( \bar{C} \). The trace operator \( \gamma_c : V^s \to \tilde{V} \) restricts functions in \( V^s \) to \( \Gamma_0^s \), and similarly the trace operators \( \gamma_Q : Q^j \to \bar{Q}^j \) restrict functions in \( Q^j \) to \( \Gamma_0^j \), \( j = s, d \). However, when it is clear from the context, we omit the subscript in the trace operator. Analogous to the discrete case, we introduce \( \bar{V} := V \times \tilde{V}, \quad \bar{Q} := Q \times \bar{Q}^s \times \bar{Q}^d, \) and \( \bar{C} := C \times \bar{C} \). We then define extended function spaces as

\[
V(h) := V_h + \bar{V}, \quad Q(h) := Q_h + \bar{Q}, \quad C(h) := C_h + \bar{C},
\]

and set \( X(h) := V(h) \times Q(h) \).

We close this section by listing various norms on the spaces described above. We refer the reader to [1] for the definitions of the standard Sobolev spaces \( W^{m,p}(D) \) and their corresponding norms \( \| \cdot \|_{W^{m,p}(D)} \). For ease
of notation, we write \( \| \cdot \|_{m,p,D} \) instead of \( \| \cdot \|_{W^{m,p}(D)} \) with the following simplifications. When \( m = 0 \), \( W^{0,p}(D) \) coincides with \( L^p(D) \) and when \( p = 2 \), \( H^m(D) = W^{m,2}(D) \). For \( p = 2 \), we write \( \| \cdot \|_{m,D} \) to denote \( \| \cdot \|_{W^{m,2}(D)} \) and for \( m = 0, p = 2 \), we write \( \| \cdot \|_D \) instead of \( \| \cdot \|_{0,D} \).

On \( V^s(h) \) we define the standard HDG-norm and its strengthened version as follows:

\[
\| v \|_{v,s}^2 := \sum_{K \in T} \left( \| \nabla v \|_K^2 + h_K^{-1} \| v - \bar{v} \|_{\partial K}^2 \right), \quad \| v \|_{v,s}^2 := \| v \|_{v,s}^2 + \sum_{K \in T} h_K^2 |v|^2_{H^2(K)}.
\]

On \( V(h) \) we then introduce the norms

\[
\| v \|_{v}^2 := \| v \|_{v,s}^2 + \| v \|_{\Omega_d}^2 + \sum_{j=1}^{\dim -1} \gamma_j^j \| \bar{v} \|_{\Gamma_j}^2, \\
\| v \|_{v}^2 := \| v \|_{v}^2 + \sum_{K \in T} h_K^2 |v|^2_{H^2(K)} = \| v \|_{v,s}^2 + \| v \|_{\Omega_d}^2 + \sum_{j=1}^{\dim -1} \gamma_j^j \| \bar{v} \|_{\Gamma_j}^2,
\]

and note that \( \| v \|_{v} \) and \( \| v \|_{v,s} \) are equivalent on \( V_h \) due to the fact that \( \| \cdot \|_{v,s} \) and \( \| \cdot \|_{v,s}^2 \) are equivalent on \( V^s_h \) (see, e.g., [51], Eq. (5.5)).

On the pressure spaces \( Q^j(h) \), \( j = s, d \) and \( Q(h) \), we define, respectively,

\[
\| q \|_{p,j}^2 := \| q \|_{\Omega}^2 + \sum_{K \in T^j} h_K \| q \|_{\partial K}^2, \\
\| q \|_{p}^2 := \sum_{j=s,d} \| q \|_{p,j}^2.
\]

Finally, for \( w_h \in C(h) \), we define the following semi-norm:

\[
\| w_h \|_c^2 := \sum_{K \in T} \left( \| \nabla w_h \|_K^2 + h_K^{-1} \| w_h - \bar{w}_h \|_{\partial K}^2 \right).
\] (11)

### 3.2. Semi-discrete HDG scheme

The semi-discrete method we propose for the Stokes/Darcy-transport system in equations (1) and (2) is as follows: for \( t > 0 \), find \( (u_h(t), p_h(t)) \in X_h \) and \( c_h(t) \in C_h \) such that

\[
\sum_{K \in T} \int_K \partial_t u_h \cdot v_h \, dx + B^{dd}_h(u_h; (u_h, p_h), (v_h, q_h)) = \sum_{K \in T} \int_K f^s(c_h) \cdot v_h \, dx + \sum_{K \in T} \int_K (g_p - g_i) q_h \, dx \tag{12a}
\]

and

\[
\sum_{K \in T} \int_K \phi \partial_t c_h w_h \, dx + B^{dr}_h(u_h; c_h, w_h) + \sum_{K \in T} \int_K c_h g_p \, w_h \, dx = \sum_{K \in T} \int_K c_I g_i \, w_h \, dx, \tag{12b}
\]

for all \( (v_h, q_h) \in X_h \) and \( w_h \in C_h \). The form \( B^{dd}_h \) in equation (12a) collects the discretization terms for the Stokes/Darcy momentum and mass conservation equations as follows:

\[
B^{dd}_h(c; (u, p), (v, q)) := a_h(c; u, v) + \sum_{j=s,d} \left( b^j_h(p^j, v) + b^{j, j}_h(\bar{p}^j, \bar{v}) \right) + \sum_{j=s,d} \left( b^j_h(q^j, u) + b^{j, j}_h(\bar{q}^j, \bar{u}) \right). \tag{13}
\]

Here \( a_h(\cdot, \cdot) \) is defined as

\[
a_h(c; u, v) := a^s_h(c; u, v) + a^d_h(c; u, v) + a^l_h(\bar{c}; \bar{u}, \bar{v}), \tag{14}
\]
where

\[
\begin{align*}
a^d_h(c; u, v) & := \sum_{K \in T} \int_K 2\mu(c)\varepsilon(u) : \varepsilon(v) \, dx + \sum_{K \in T} \int_{\partial K} 2\beta_s \mu(c) \frac{h_K}{h_K^2} (u - \bar{u}) \cdot (v - \bar{v}) \, ds \\
& \quad - \sum_{K \in T} \int_{\partial K} 2\nu(c)\varepsilon(u) \cdot (v - \bar{v}) \, ds - \sum_{K \in T} \int_{\partial K} 2\mu(c)\varepsilon(v) \cdot (u - \bar{u}) \, ds, \\
a^g_h(c; u, v) & := \int_{\Omega^d} K^{-1}(c) u \cdot v \, dx, \\
a^f_h(\bar{c}; \bar{u}, \bar{v}) & := \sum_{\ell = 1}^{\dim - 1} \int_{\Gamma^f} \gamma^f \mu(c)(\bar{u} \cdot \tau^f) (\bar{v} \cdot \tau^f) \, ds,
\end{align*}
\]

and \(\beta_s > 0\) is a penalty parameter. The bilinear forms \(b^j_h(\cdot, \cdot)\) and \(b^{f,j}_h (\cdot, \cdot)\) in equation (13), \(j = s, d\) are defined as

\[
\begin{align*}
\begin{align*}
b^j_h(p^j, v) & := - \sum_{K \in T} \int_K p \nabla \cdot v \, dx + \sum_{K \in T} \int_{\partial K} \bar{p}^j v \cdot n^j \, ds, \quad (15a) \\
b^{f,j}_h(\bar{p}^j, \bar{v}) & := - \int_{\Gamma^f} \bar{p}^j \bar{v} \cdot n^j \, ds. \quad (15b)
\end{align*}
\end{align*}
\]

Before defining the terms related to the transport equation, we point out that equations (13)–(15) are the same as in [17] when the viscosity and \(\kappa\) are both constants.

The form \(B^{tr}_h(u; c(t), w)\) in equation (12b) discretizes the advective and diffusive parts of the transport equation:

\[
B^{tr}_h(u; c, w) = B^g_h(u; c, w) + B^d_h(u; c, w). \quad (16)
\]

The advective part is defined as

\[
\begin{align*}
B^g_h(u; c, w) & := - \sum_{K \in T} \int_K c u \cdot \nabla w \, dx + \sum_{K \in T} \int_{\partial K} c u \cdot n (w - \bar{w}) \, ds - \sum_{K \in T} \int_{\partial K^{in}} u \cdot n (c - \bar{c}) (w - \bar{w}) \, ds, \\
B^d_h(u; c, w) & := \sum_{K \in T} \int_{\partial K} \tilde{D}(u) \nabla c \cdot \nabla w \, dx + \sum_{K \in T} \int_{\partial K} \frac{\partial c}{\partial h_K} \tilde{D}(u) n (c - \bar{c}) \cdot (w - \bar{w}) n \, ds \\
& \quad - \sum_{\ell = 1}^{\dim - 1} \int_{\partial \Gamma^f} \tilde{D}(u) \nabla c \cdot n (w - \bar{w}) \, ds - \sum_{K \in T} \int_{\partial K} \left( \tilde{D}(u) \nabla w \right) \cdot n (c - \bar{c}) \, ds, \quad (18)
\end{align*}
\]

where \(\beta_{tr} > 0\) is a penalty parameter. Here we pause again to mention that equations (16)–(18) are the same as in [16], and a standard extension of the discretization analyzed in [51].

To complete the discretization, we project the initial conditions \(u_0\) and \(c_0\) equation (3) into \(V_h\) and \(C_h\), respectively.

### 3.3. Properties of the numerical scheme

The semi-discrete HDG scheme presented in Section 3.2 has various attractive features. Besides local momentum conservation, a property of all HDG methods, this particular HDG method also conserves mass strongly, according to the definition defined in [34].

To be specific, the discrete velocity enjoys the following properties:

\[
- \nabla \cdot u_h = \chi^d \Pi_Q (g_p - g_i) \quad \forall x \in K, \forall K \in T, \quad (19a)
\]
\[ [u_h \cdot n] = 0 \quad \forall x \in F, \forall F \in F \setminus F^i, \tag{19b} \]
\[ u_h^j \cdot n = \bar{u}_h \cdot n \quad \forall x \in F, \forall F \in F^i, \quad j = s, d, \tag{19c} \]

where \([ \cdot ]\) is the usual jump operator and \(n\) is the unit normal vector on \(F\). Note that equations (19b) and (19c) imply that \(u_h\) is \(H(\text{div})\)-conforming on the whole domain. More details on equation (19) can be found in Section 3.3 from [17]. Additionally, the scheme is consistent, that is, the solution to equations (1) and (3) satisfies equation (12), as we discuss next.

**Lemma 3.2 (Consistency).** Suppose that the solution \((u, p, c)\) to the Stokes/Darcy-transport system equations (1)–(3) satisfies \((u, p) \in L^2(0, T; X), \partial_t u \in L^2(0, T; L^2(\Omega^s)), c \in L^2(0, T; C), \) and \(\partial_t c \in L^2(0, T; L^2(\Omega))\). Then \((u(t), p(t), c(t))\), where \(u := (u, \gamma(u)), p := (p, \gamma(p^s), \gamma(p^d)), c := (c, \gamma(c))\), satisfy the semi-discrete HDG scheme equation (12) for all \(t \in (0, T)\).

**Proof.** The proof is as in Lemma 1 of [17] and Lemma 6 of [16] with minor modifications. We do not repeat the proof here, but mention that it is based on integration by parts in \(a_h^u(c; u, v), \sum_{j=s,d} b^j_h(p^j, v), B_h^c(u, c, w)\), and \(B_h^d(u, c, w)\), using \(\gamma(u) = u \) on \(\Gamma^s_0\), \(\gamma(p^j) = p^j\) on \(\Gamma^d_0\), \(j = s, d\), \(\gamma(c) = c\) on \(\Gamma_0\), the smoothness of the solution, the continuity of \(\mu\) and \(D\), and equations (1) and (2). \(\square\)

### 4. Continuity, Coercivity, and an inf-sup condition

Let us recollect various known inequalities. Throughout this article we denote by \(C > 0\) a generic constant that is independent of the mesh size and the time step. From Lemma 1.46, Remark 1.47 of [23], for any \(K \in T\), we have

\[ \|v\|_{\partial K} \leq C h_K^{-1/2} \|v\|_K \quad \forall v \in P_k(K), \tag{20} \]

We will also use the following versions of the continuous trace inequality ([8], Theorem 1.6.6, (10.3.8)):

\[ \|v\|_{\partial K} \leq C \left( h_K^{-1} \|v\|_K^2 + h_K \|v\|_{1,K}^2 \right) \quad \forall v \in H^1(K), \tag{21} \]
\[ \|v\|_{0, \infty, \partial K} \leq C \quad \forall v \in W^{1, \infty}(K), \tag{22} \]

where \(C\) in equation (22) depends on \(\|v\|_{1, \infty, K}\). Regarding the trace on the interface, we have equation (1.24) of [30] and Theorem 1.6.6 of [8]:

\[ \|v\|_{\Gamma^s} \leq C \|\nabla v\|_{\Omega^s} \quad \forall v \in \left\{ v \in H^1(\Omega^s) : v = 0 \text{ on } \Gamma^s \right\}, \tag{23} \]
\[ \|v\|_{\Gamma^d} \leq C \|v\|_{1, \Omega^d} \quad \forall v \in H^1(\Omega^d). \tag{24} \]

Furthermore, by Theorem 4.4 of [30], for any \(v_h \in V_h\), for \(k_f \geq 1\),

\[ \|v_h^s\|_{\Gamma^s} \leq C \|v_h\|_{v,s} \leq C \|v_h\|_v. \tag{25} \]

Similarly, for any \(w_h \in C_h\), for \(k_c \geq 1\),

\[ \|w_h^c\|_{\Gamma^c} \leq C \|w_h\|_c. \tag{26} \]

Next, we recall some inverse inequalities from Lemmas 1.44 and 1.50 of [23]:

\[ \|\nabla v\|_K \leq C h_K^{-1} \|v\|_K \quad \forall v \in P_k(K), \tag{27} \]
\[ \|v\|_{0, \infty, K} \leq C h_K^{-\text{dim}/2} \|v\|_K \quad \forall v \in P_k(K). \tag{28} \]

The following Poincaré-type inequality follows from Proposition 4.5 of [30] and Remark 1.1 of [6]:

\[ \|v\|_{\Omega^s} \leq C \|v\|_{v,s} \quad \forall v := (v, \mu) \in H^1(T_h^s) \times \tilde{V}_h. \tag{29} \]
The following version of Korn’s first inequality is a consequence of equation (1.19) of [7], Proposition 4.7 of [30] and [42], p.110:

\[
\|\mathbf{v}_h\|_{\bar{v},s} \leq C \sum_{K \in \mathcal{T}} \left( \|\mathbf{e}(\mathbf{v}_h)\|^2_K + h^{-1}_K \|\mathbf{v}_h - \bar{\mathbf{v}}_h\|^2_{\partial K} \right).
\] (30)

Continuity and coercivity \(a_h(\cdot, \cdot)\), follow from Lemmas 2 and 3 [17] keeping in mind that \(\mu\) satisfies equation (5b) and \(K\) satisfies equation (7). They can be stated as follows:

**Lemma 4.1** (Continuity and coercivity of \(a_h\)). There exists a constant \(C > 0\), independent of \(h\), such that for all \(\mathbf{u}, \mathbf{v} \in \mathbf{V}(h)\) and \(c \in \mathcal{C}(h)\),

\[
a_h(c; \mathbf{u}, \mathbf{v}) \leq C\|\mathbf{u}\|_{\bar{v}r}\|\mathbf{v}\|_{\bar{v}r}.
\] (31)

In addition, there exists a constant \(C_a > 0\), independent of \(h\) but dependent on \(\kappa^*, \mu^*, \) and \(\mu^*\), and a constant \(\beta^* > 0\) such that if \(\beta_h > \beta^*_0\), then

\[
a_h(c_h; \mathbf{v}_h, \mathbf{v}_h) \geq C_a\|\mathbf{v}_h\|^2_{\bar{v}r} \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \quad \forall c_h \in \mathcal{C}_h.
\] (32)

The inf-sup condition on the discrete spaces \(\mathbf{V}_h\) and \(\mathcal{Q}_h\) was proved in [17] in the case of a continuous discrete velocity trace space \(\bar{\mathbf{V}}_h \cap [C^0(\Gamma^*_0)]^\text{dim}\). It is straightforward to show that the inf-sup condition also holds when the discrete velocity trace space is the larger discontinuous \(\bar{\mathbf{V}}_h\) space.

**Theorem 4.2.** There exists a constant \(c_{\text{inf}}^* > 0\), independent of \(h\), such that for any \(\mathbf{q}_h \in \mathcal{Q}_h\),

\[
c_{\text{inf}}^*\|\mathbf{q}_h\|_{\ell, K} \leq \sup_{\mathbf{v}_h \in \mathbf{V}_h \neq 0} \frac{\sum_{j=s,d} \left( b^j_h (\mathbf{q}_h^j; \mathbf{v}_h) + b^j_h (\bar{q}_h^j; \bar{\mathbf{v}}_h) \right)}{\|\mathbf{v}_h\|_{\bar{v}r}}.
\] (33)

Note that the proof of this theorem as well as the error analysis requires appropriate interpolation operators onto \(\mathbf{V}_h\) and \(\bar{\mathbf{V}}_h\). For \(\mathbf{V}_h\) we consider the BDM interpolation operator \(\Pi_V: [H^1(\Omega)]^\text{dim} \to \mathbf{V}_h\) which is such that if \(u \in [H^{k_f+1}(K)]^\text{dim}, K \in \mathcal{T}\), then (see, e.g., [32], Lem. 7 and [9], Sect. III.3):

\[
\begin{align*}
\int_K q(\nabla \cdot u - \nabla \cdot \Pi_V u) \, dx &= 0, \quad \forall q \in P_{k_f-1}(K), \\
\int_F \bar{q}(u - \Pi_V u) \cdot n \, ds &= 0, \quad \forall \bar{q} \in P_{k_f}(F), \quad F \subset \partial K, \\
\Pi_V u &\in H(\text{div}; \Omega), \\
\|u - \Pi_V u\|_{m, K} &\leq C h^{\ell-m}_K \|u\|_{\ell, K}, \quad m = 0, 1, 2, \quad \max(1, m) \leq \ell \leq k_f + 1,
\end{align*}
\] (34a,b,c,d)

where we remark that \(F\) is an edge if \(\text{dim} = 2\) and a face if \(\text{dim} = 3\). Furthermore, for \(u \in W^{1,\infty}(K)\) ([31], (2.33)),

\[
\|u - \Pi_V u\|_{\infty, K} + h_K \|\nabla(u - \Pi_V u)\|_{\infty, K} \leq Ch_K \|u\|_{1, \infty, K}.
\] (35)

The interpolant onto the trace space \(\bar{\mathbf{V}}_h\) is defined by the \(L^2\)-projection onto \(\bar{\mathbf{V}}_h\). It is straightforward to deduce the following estimates using equation (21) and the fact that \(h = \max_{K \in \mathcal{T}} h_K\): For \(v \in [H^\ell(\Omega^*)]^\text{dim}, 1 \leq \ell \leq k_f + 1\),

\[
\|v - \Pi_V v\|_{\partial K} \leq C h^{\ell-1/2}_K \|v\|_{\ell, K},
\] (36)

\[
\|\Pi_V v - \bar{\Pi}_V v\|_{\partial K} \leq C h^{\ell-1/2}_K \|v\|_{\ell, K}.
\] (37)
We finish this section by noting that by equations (34a)–(34c) the solution \( u \) of equations (1) and (2) under the assumption that \( u \in [H^{k+1}(K)]^{dim} \), for all \( K \in T \), satisfies

\[
\sum_{j=s,d} \left( b^j_h (q^j_h, u - \Pi_V u) + b^{t,j}_h (q^{t,j}_h, \gamma (u^s) - \bar{\Pi}_V u) \right) = 0 \quad \forall q_h \in Q_h. \tag{38}
\]

5. FULLY DISCRETE NUMERICAL SCHEME

Let us now describe the fully discrete HDG method and decoupling strategy used to solve the Stokes/Darcy and transport problems sequentially.

For the time discretization, we partition the time interval \( J \) as: \( 0 = t^0 < t^1 < \ldots < t^N = T \). For simplicity, we assume a uniform partition with \( t^{n+1} - t^n = \Delta t \) for \( 0 \leq n \leq N - 1 \). We denote a function \( h(t) \) at time level \( t^n \) by \( h_n := h(t^n) \) and for a sequence \( \{ u^n \}_{n \geq 1} \) we denote by \( d_t u^n = (u^n - u^{n-1})/\Delta t \) a first order difference operator.

In the first step of our sequential algorithm, given an initial velocity \( u^0_h \) in the Stokes domain and an initial concentration \( c^0_h \), we solve the Stokes/Darcy problem and obtain a velocity in the entire region. This velocity, with properties given by equation (19), is then substituted into the concentration problem. This approach is repeated for all time steps with the initial velocity and concentration being replaced by the last computed velocity and concentration solutions. We summarize the fully discrete problem in Algorithm 1.

\textbf{Algorithm 1.} Sequential algorithm.

\begin{algorithmic}
\FOR{n = 1, \ldots, N}
\STATE 1. Find \( (u_h^n, p_h^n) \in X_h \) such that for all \( (v_h, q_h) \in X_h \)
\STATE \hspace{1cm} \sum_{K \in T} \int_K d_t u^n_h \cdot v_h \, dx + B^d_h \left( c^{n-1}_h, (u^n_h, p^n_h), (v_h, q_h) \right) = \sum_{K \in T} \int_K f^s \left( c^{n-1}_h \right) \cdot v_h \, dx + \sum_{K \in T} \int_K \int_K \left( c^{n-1}_h \right) \cdot v_h \, dx + \sum_{K \in T} \int_K \left( g^n_p - g^n_i \right) q_h \, dx. \tag{39}
\STATE \hspace{1cm} 2. Find \( c^n_h \in C_h \) such that for all \( w_h \in C_h \)
\STATE \hspace{1cm} \sum_{K \in T} \int_K \phi d_t c^n_h \cdot w_h \, dx + B^{tr}_h (c^n_h, w_h) + \sum_{K \in T} \int_K g^n_p c^n_h \cdot w_h \, dx = \sum_{K \in T} \int_K c_I g^n_i \cdot w_h \, dx. \tag{40}
\ENDFOR
\end{algorithmic}

\textbf{Remark 5.1.} In Algorithm 1, \( \Pi_C \) denotes the \( L^2 \)-projection onto \( C_h \) and \( \Pi_V u_0 \) is understood as \( \Pi_V \) applied to the extension of \( u_0 \) to \( \Omega \) by zero assuming \( u_0 \in H^1_0 (\Omega^s) \). We note that this choice of \( u_0^n \) satisfies normal continuity across the interfaces in \( \Gamma^{s}_h \) and has zero divergence in \( \Omega^s \) under the additional assumption that \( \nabla \cdot u_0 = 0 \) in \( \Omega^s \). We further remark that the properties in equation (19) hold for \( u^n_h \) for each time step \( n = 0, \ldots, N \).

We conclude this section by stating some preliminary results obtained by Taylor’s theorem ([15], Lem. 3.2).

For a function \( z \) defined on \( D \times [0, T] \), assuming enough regularity, we have the following results:

\[
\sum_{m=1}^n \| \partial_t z^m - d_t z^m \|^2_D \leq C \Delta t \| \partial_t z \|_{L^2(0, T; L^2(D))}. \tag{41a}
\]
\[
\Delta t \sum_{m=1}^{n} \|d_t z^m\|_{\ell,D}^2 \leq \|\partial_t z\|_{L^2(0,T;H^\ell(D))}^2, \quad \ell = 0, 1,
\]  
\tag{41b}
\]

where \(\|f\|_{L^2(a,b;X)} := (\int_a^b \|f(t)\|^2_X \, dt)^{1/2}\). Note that the inequalities in equation (41) for \(\ell = 0\) were presented in Lemma 3.2 of [15] and that it is straightforward to extend equation (41b) to \(\ell = 1\).

### 6. Main results

In this section we present our main results. For the error estimates we will make use of the following definition of the discrete in time norm:

\[
\|f\|_{\ell^2(0,T;X)} = \left( \Delta t \sum_{m=1}^{n} \|f^m\|^2_X \right)^{1/2}.
\]

The following theorem states well-posedness of the discrete Stokes/Darcy problem equation (39).

**Theorem 6.1.** Let \(\beta_s > \beta_0^s\) be as in Lemma 4.1 and \(n \geq 1\). Then given \(u_{h}^{n-1} \in V_h\) and \(c_{h}^{n-1} \in C_h\), there exists a unique solution \((u_{h}^n, p_{h}^n) \in X_h\) to equation (39) that satisfies

\[
\|u_{h}^n\|_v + \|p_{h}^n\|_p \leq C \left( \frac{1}{\Delta t} \|u_{h}^{n-1}\|_{\Omega^s} + \|f^s(\gamma_{h}^{n-1})\|_{\Omega^d} + \frac{1}{K_s} \|f^d(\gamma_{h}^{n-1})\|_{\Omega^d} \right) + \|g_{p}^n - g_{l}^n\|_{\Omega^d}. \quad \tag{42}
\]

**Proof.** The result follows by applying the abstract theory for saddle point problems ([27], Thm. 2.34) together with Theorem 4.2 and equation (32).

In Sections 6.1 and 6.2, we prove *a priori* error estimates for the discrete velocity and pressure, respectively. We show well-posedness of the discrete transport problem equation (40) in Section 6.3 and an *a priori* error estimate for the concentration in Section 6.4.

### 6.1. Error estimates for the discrete velocity

In this section we derive estimates for the error \(u_{h}^n - u^n\), for each \(n \geq 0\), given error estimates for the discrete concentration in previous time steps. To do so, we define the following:

\[
\begin{align*}
\xi_{u}^n & := u^n - \Pi_{V} u^n, \\
\zeta_{u}^n & := u_{h}^{n-1} - \Pi_{V} u^n, \\
\xi_{p}^n & := p^n - \Pi_{Q} p^n, \\
\zeta_{p}^n & := p_{h}^{n-1} - \Pi_{Q} p^n, \\
\bar{\xi}_{u}^n & := \gamma(u^n) - \bar{\Pi}_{V} u^n, \\
\bar{\zeta}_{u}^n & := \gamma_{h}^{n-1} - \bar{\Pi}_{V} u^n, \\
\bar{\xi}_{p}^n & := \gamma(p^n) - \bar{\Pi}_{Q} p^n, \\
\bar{\zeta}_{p}^n & := \gamma_{h}^{n-1} - \bar{\Pi}_{Q} p^n,
\end{align*}
\]

where \(\Pi_{Q}\) is the \(L^2\)-projection onto \(Q_h\) and \(\bar{\Pi}_{j}^j\) is the \(L^2\)-projection onto \(\bar{Q}_{j}^j\), \(j = s, d\). For the case \(n = 0\), \(\Pi_{V} u_0\) is understood as \(\Pi_{V}\) applied to the extension of \(u_0\) to \(\Omega\) by zero assuming \(u_0 \in H^1_0(\Omega^s)\). Therefore, \(\zeta_0 = 0\).

Furthermore, note that the following identities hold:

\[
\begin{align*}
\xi_{u}^n - u_{h}^n & = \zeta_{u}^n - \zeta_{u}^n, \\
p^n - p_{h}^{n} & = \xi_{p}^n - \zeta_{p}^n,
\end{align*}
\]

\(\gamma(u^n) - \bar{u}_{h}^{n} = \bar{\xi}_{u}^{n} - \bar{\zeta}_{u}^{n}\), \(\gamma(p^n) - \bar{p}_{h}^{n} = \bar{\xi}_{p}^{n} - \bar{\zeta}_{p}^{n}\), \(j = s, d\). \(\tag{43}\)

\(\tag{44}\)

To be consistent with the notation used in previous sections, we set \(\ell^n_{u} = (\ell_{u}^n, \bar{\ell}_{u}^{n})\), \(\ell^n_{p} := (\ell_{p}^n, \bar{\ell}_{p}^{n})\), for \(\ell = \xi, \zeta, \text{and } j = s, d\).

Here we recall the following results on the interpolation errors ([17], Lemmas 7 and 8). Suppose that \(u\) is such that \(u^s \in [H^\ell(\Omega^s)]^{\dim}\) and \(u^d \in [H^{\ell-1}(\Omega^d)]^{\dim}\) for \(2 \leq \ell \leq k_f + 1\), and that \(p^j \in H^r(\Omega^j)\) for \(0 \leq r \leq k_f\) and \(j = s, d\). Then

\[
\|\ell_{u}^{n}\|_{\ell^2(0,T;X)} \leq C h^{\ell-1} \|u\|_{H^\ell(\Omega^s)}. \quad \tag{45a}
\]
Lemma 6.2. Let \((u, p)\) be the velocity solution of equations (1)–(3), \(\bar{u} = (u^s)\). Then for any \(n \geq 1\),

\[
\sum_{j=s,d} (b^j_n (\xi^j_n, v_h) + b^j_n (\tilde{\zeta}^j_n, \bar{v}_h)) = 0, \quad \forall v_h \in V_h,
\]

\[
\sum_{j=s,d} (b^j_n (q^j_n, \xi^j_n) + b^j_n (\tilde{q}^j_n, \tilde{\zeta}^j_n)) = 0, \quad \forall q_h \in Q_h.
\]

Proof. The proof is based on the properties of the numerical scheme equation (19), the properties of the BDM projection \(\Pi_V\) in equation (34), and the properties of the \(L^2\)-projections \(\Pi_Q\) and \(\Pi^j_Q, j = s, d\). Indeed, equation (46) follows after noting that \(\nabla \cdot v_h \in P_{k-1}(K), v_h \cdot n^j, \bar{v}_h \cdot n^j \in P_k(F)\), and using the definitions of the \(L^2\)-projections \(\Pi_Q\) and \(\Pi^j_Q, j = s, d\),

\[
\sum_{j=s,d} \left( b^j_n (\xi^j_n, v_h) + b^j_n (\tilde{\zeta}^j_n, \bar{v}_h) \right) = \sum_{j=s,d} \left( - \sum_{K \in T^j} \int_K (p^n - \Pi_Q p^n) \nabla \cdot v_h \, dx + \int_{\partial K} \left( \gamma (p^n) - \Pi_Q^j p^n \right) v_h \cdot n^j \, ds - \int_{\Gamma} \left( \gamma (p^n) - \Pi_Q^j p^n \right) \bar{v}_h \cdot n^j \, ds \right) = 0,
\]

while equation (47) is exactly the same as equation (38), evaluated at \(t = t_n\), and rewritten by using the definitions of \(\xi^j_n\) and \(\tilde{\zeta}^j_n\). \(\square\)

Theorem 6.3 (Error equation for Eq. (39)). There holds

\[
\sum_{K \in T^d} \int_K \left( d_t u^n_h - \partial_t u^n \right) \cdot v_h \, dx + a_h (c_h^{n-1}; \xi^j_n, v_h) + \sum_{j=s,d} \left( b^j_n (\xi^j_n, v_h) + b^j_n (\tilde{\zeta}^j_n, \bar{v}_h) \right) + \sum_{j=s,d} \left( b^j_n (q^j_n, \xi^j_n) + b^j_n (\tilde{q}^j_n, \tilde{\zeta}^j_n) \right)
\]

\[
= a_h (c_h^{n-1}; \xi^j_n, v_h) + a_h (c_h^{n-1}; \xi^j_n, v_h) - a_h (c_h^{n-1}; u^n, v_h) + \sum_{K \in T^d} \int_K \left[ \mathbb{K}^{-1} (c_h^{n-1}) f^d (c_h^{n-1}) - \mathbb{K}^{-1} (c^n) f^d (c^n) \right] \cdot v_h \, dx + \sum_{K \in T^d} \int_K \mathbb{K}^{-1} (c^n) f^d (c^n) \cdot v_h \, dx.
\]

Proof. By Lemma 3.2 at time \(t = t^n\), we have that for all \((v_h, q_h) \in X_h\),

\[
\sum_{K \in T} \int_K \partial_t u^n \cdot v_h \, dx + B_h^{ed} (c^n; (u^n, p^n), (v_h, q_h)) = \sum_{K \in T} \int_K f^s (c^n) \cdot v_h \, dx + \sum_{K \in T^d} \int_K \mathbb{K}^{-1} (c^n) f^d (c^n) \cdot v_h \, dx + \sum_{K \in T^d} \int_K (p^n - q^n) q_h \, dx.
\]

Subtracting this equation from equation (39), we obtain

\[
\sum_{K \in T} \int_K \left( d_t u^n_h - \partial_t u^n \right) \cdot v_h \, dx + B_h^{ed} (c_h^{n-1}; (u^n, p^n), (v_h, q_h)) - B_h^{ed} (c^n; (u^n, p^n), (v_h, q_h))
\]
for all \((v_h, q_h) \in X_h\). Then, by equation (43), the \(B^sd_h\) terms can be rewritten as

\[
B^sd_h \left( c_h^{n-1}; (u^n_h, p^n_h), (v_h, q_h) \right) - B^sd_h \left( c^n; (u^n, p^n), (v_h, q_h) \right)
\]

\[
= B^sd_h \left( c_h^{n-1}; (u^n_h - u^n, p^n_h - p^n), (v_h, q_h) \right) + B^sd_h \left( c_h^{n-1}; (u^n - u^n_h, p^n_h), (v_h, q_h) \right) - B^sd_h \left( c^n; (u^n, p^n), (v_h, q_h) \right)
\]

\[
= a_h \left( c_h^{n-1}; c_u^n, v_h \right) - a_h \left( c_h^{n-1}; c_u^n, v_h \right) + a_h \left( c_h^{n-1}; u^n - u^n_h, v_h \right) - a_h \left( c^n; u^n_h, v_h \right)
\]

\[
+ \sum_{j=s,d} \left( b^j_h \left( \xi^{n-1}_d, v_h \right) + b^j \left( \xi^{n-1}_d, \bar{v}_h \right) \right) + \sum_{j=s,d} \left( b^j_h \left( \xi^n, c_u^n \right) + b^j \left( \xi^n, \bar{c}_u^n \right) \right),
\]

(50)

where we applied equations (46) and (47). Combining this with equation (49) yields the result. \(\square\)

The velocity error at each time step depends on the error in concentration from the previous time step. Therefore, for the velocity error estimates, we will need some auxiliary results related to the concentration error. To estimate the error of the concentration, we use the continuous interpolant \(\mathcal{I} c \in C_h \cap C^0(\Omega)\) of \(c\) [8], and we set \(\mathcal{I} c(t) = \mathcal{I} c|_\Gamma(t) \in C_h\). Denoting the restriction of \(c\) to \(\Gamma_0\) by \(\bar{c}\), we define

\[
\xi^n = c^n - \mathcal{I} c^n, \quad \xi^n = c^n - \mathcal{I} c^n, \quad \xi^n = (\xi^n, \bar{\xi}^n),
\]

(51)

\[
\bar{\xi}_c = \bar{c} - \bar{\mathcal{I}} c^n, \quad \bar{\xi}_c = \bar{c} - \bar{\mathcal{I}} c^n, \quad \bar{\xi}_c = (\bar{\xi}_c, \bar{\bar{\xi}}_c).
\]

Note that:

\[
c^n - c^n = \xi^n - \xi^n, \quad \bar{c} - \bar{\mathcal{I}} c^n = \bar{\xi}_c - \bar{\xi}_c.
\]

Furthermore, we have the following interpolation estimate ([8], Sect. 4.4) for \(2 \leq \ell \leq k_c + 1, 0 \leq r \leq \ell\):

\[
\sum_{K \in T} \|\xi_c\|_{r,K}^2 \leq C h^{2(\ell - r)} \|c\|_{\ell,\Omega}^2.
\]

(53)

**Theorem 6.4.** Let \(c^0 \in H^{k_c+1}(\Omega^s), c^n \in L^2(0, T; H^{k_c+1}(\Omega^s)), c^d \in H^1(0, T; L^2(\Omega^d))\) such that \(\partial_t c^c \in L^2(0, T; H^1(\Omega^s))\), and \(\bar{c} = \gamma(c)\) on \(\Gamma_0\). Then we have the following estimates:

\[
\Delta t \sum_{m=1}^n \sum_{K \in T^s} h_K^2 \|c^m - c_h^{m-1}\|_{1,K}^2 \leq C \left( h^2(\Delta t)^2 \|\partial_c c\|_{L^2(0, T; H^1(\Omega^s))}^2 + \Delta t \sum_{m=1}^n \sum_{K \in T^s} h_K^2 \|c^{m-1} - c_h^{m-1}\|_{1,K}^2 \right),
\]

(54a)

\[
\Delta t \sum_{m=1}^n \|c^m - c_h^{m-1}\|_{\Omega^s}^2 \leq C \left( \Delta t \sum_{m=1}^n \|\xi_c^{m-1}\|_c^2 + (\Delta t)^2 \|\partial_c c\|_{L^2(0, T; H^1(\Omega^s))}^2 \right) + h^{2k_c+1} \left( \Delta t \|c_0\|_{k_c+1,\Omega^s}^2 + \|c\|_{1,\Omega^d}^2 \right),
\]

(54b)
and
\[
\Delta t \sum_{m=1}^{n} \left( \sum_{K \in T} \left\| c^m - c_{h,m-1}^m \right\|_K^2 \right) \leq C \left( (\Delta t)^2 \| \partial_t c \|_{L^2(0,T;L^2(\Omega))}^2 + \Delta t \sum_{m=1}^{n} \left( \sum_{K \in T} \left\| c_{h,m-1}^m - c_{h,m-1}^m \right\|_K^2 \right) \right). \quad (54c)
\]

**Proof.** We first prove equation (54a). By the triangle inequality and the definition of \( \partial_t c \),
\[
\sum_{K \in T} h_K^2 \left\| c^m - c_{h,m-1}^m \right\|_{1,K}^2 \leq \sum_{K \in T} 2h_K^2 \left( (\Delta t)^2 \| \partial_t c \|_{1,K}^2 + \| c_{h,m-1}^m - c_{h,m-1}^m \|_{1,K}^2 \right). \quad (55)
\]

Multiplying this by \( \Delta t \), summing from \( m = 1 \) to \( n \), and using equation (41b), we obtain equation (54a). We now prove equation (54b). We have
\[
\left\| c^m - c_{h,m-1}^m \right\|_{\Gamma, I}^2 \leq 2 \left( \left\| c^m - c_{h,m-1}^m \right\|_{\Gamma, I}^2 + \left\| c_{h,m-1}^m - c_{h,m-1}^m \right\|_{\Gamma, I}^2 \right). \quad (56)
\]

Splitting the second term on the right side of equation (55) by using \( \mathcal{I}c = \bar{\mathcal{I}}c \) for any \( F \in \mathcal{F} \) gives
\[
\left\| c^m - c_{h,m-1}^m \right\|_{\Gamma, I}^2 \leq 2 \left( \left\| c^m - \mathcal{I}c_{m-1} \right\|_{\Gamma, I}^2 + \left\| c_{h,m-1}^m - \bar{\mathcal{I}}c_{m-1} \right\|_{\Gamma, I}^2 \right) = 2 \left( \left\| c^m - c_{m-1} \right\|_{\Gamma, I}^2 + \left\| c_{c,m-1} \right\|_{\Gamma, I}^2 \right). \quad (57)
\]

The first term on the right hand side of equation (57) is bounded by equations (21) and (53) as follows:
\[
\left\| c_{c,m-1} \right\|_{\Gamma, I}^2 \leq \sum_{K \in T} C \left( \left\| \frac{1}{h_K} \right\|_{K}^2 \left\| c_{c,m-1} \right\|_{K}^2 + h_K \left\| c_{c,m-1} \right\|_{1,K}^2 \right) \leq C h^{2k_{c+1}} \left\| c_{m-1} \right\|_{k_{c+1}, \Omega}^2. \quad (58)
\]

Using equation (26) and the definition of \( \| \cdot \|_{c} \),
\[
\left\| c_{c,m-1} \right\|_{\Gamma, I}^2 \leq 2 \left( \left\| c_{c,m-1} - (c_{c,m-1}) \right\|_{\Gamma, I}^2 + \left\| (c_{c,m-1}) \right\|_{\Gamma, I}^2 \right) \leq 2 \left( h \sum_{K \in T} \left\| c_{c,m-1} \right\|_{\Gamma, I}^2 + \left\| (c_{c,m-1}) \right\|_{\Gamma, I}^2 \right) \leq C \left\| c_{c,m-1} \right\|_{c}^2. \quad (59)
\]

Collecting equations (55)–(59), we obtain
\[
\left\| c^m - c_{h,m-1}^m \right\|_{\Gamma, I}^2 \leq C \left( (\Delta t)^2 \left\| d_{t} c_{m} \right\|_{1, \Omega}^2 + h^{2k_{c+1}} \left\| c_{m-1} \right\|_{k_{c+1}, \Omega}^2 + \left\| c_{c,m-1} \right\|_{c}^2 \right). \quad (54b)
\]

Equation (54b) now follows after multiplying the above inequality by \( \Delta t \), summing from 1 to \( n \), and using equation (41b). We next prove equation (54c). By the triangle inequality,
\[
\sum_{K \in T} \left\| c^m - c_{h,m-1}^m \right\|_{K}^2 \leq \sum_{K \in T} 2 \left( (\Delta t)^2 \left\| d_{t} c_{m} \right\|_{K}^2 + \left\| c_{m-1} - c_{h,m-1}^m \right\|_{K}^2 \right). \quad (60)
\]

The results follows by multiplying equation (60) by \( \Delta t \), summing from \( m = 1 \) to \( n \), and using equation (41b) as before. \( \square \)
Now that the auxiliary result is established, we proceed with proving error estimates for the velocity.

**Theorem 6.5.** Let \( c \) and \( u \) be the solutions of equations (1)–(3) such that

\[
\begin{align*}
  u^s & \in L^2\left(0, T; \left[H^{k_s+1}(\Omega^s)\right]^\text{dim}\right) \cap L^\infty\left(0, T; \left[W^{1,\infty}(\Omega^s)\right]^\text{dim} \cap \left[H^{k_f}(\Omega^f)\right]^\text{dim}\right), \\
  \partial_t u^s & \in L^2\left(0, T; \left[H^{k_s}(\Omega^s)\right]^\text{dim}\right), \partial_t^2 u^s \in L^2\left(0, T; \left[L^2(\Omega^s)\right]^\text{dim}\right), \\
  u^d & \in L^2\left(0, T; \left[H^{k_d}(\Omega^d)\right]^\text{dim}\right) \cap L^\infty\left(0, T; \left[W^{1,\infty}(\Omega^d)\right]^\text{dim}\right), \\
  p & \in L^2\left(0, T; L^2(\Omega)\right), \nabla p^d \in L^\infty\left(0, T; \left[L^\infty(\Omega^d)\right]^\text{dim}\right), \\
  c & \in L^2\left(0, T; H^{k_c+1}(\Omega)\right), \partial_t c^e \in L^2\left(0, T; H^1(\Omega^s)\right), \partial_t c^d \in L^\infty\left(0, T; L^2(\Omega^d)\right), \\
  u_0 & \in \left[H^1_0(\Omega^s)\right]^\text{dim}, \nabla \cdot u^0 = 0, c_0^e \in H^{k_c+1}(\Omega^s).
\end{align*}
\]

Suppose that \( u_0^s, \ldots, u_{n-1}^s \in V_h \) and \( c_0^e, \ldots, c_{n-1}^e \in C_h \), the solutions of equations (39) and (40), respectively, are known and satisfy for \( 1 \leq i \leq n \),

\[
\left\| c_{i-1}^e - c_{i-1}^e \right\|^2_\Omega + \Delta t \sum_{m=1}^{i-1} \left\| c_{m-1}^e - c_{m-1}^e \right\|^2_c \leq C\left( (\Delta t)^2 + h^{2k_f} + h^{2k_c} \right).
\]  \( \tag{61} \)

Then \( \zeta_n^e \) satisfies:

\[
\left\| \zeta_n^e \right\|^2_{\Omega^s} + (\Delta t)^2 \sum_{m=1}^{n} \left\| d_m \zeta_m^e \right\|^2_{\Omega^s} + \Delta t \sum_{m=1}^{n} \left\| \zeta_m^e \right\|^2_v \leq C\left( (\Delta t)^2 + h^{2k_f} + h^{2k_c} \right),
\]  \( \tag{62} \)

where the constants depend on \( \mu^s, \kappa^s, \mu_L, L_f, L_f^4, \sum_{j=1}^{\text{dim}-1} \gamma_j \), and the regularity of \( u_0, u, c_0 \), and \( c \) but are independent of the mesh size.

**Remark 6.6.** Assumption equation (61) is proved in Corollary 6.14.

**Proof.** Setting \((v_h, q_h) = (\zeta_n^e, -\zeta_p^e)\) in Theorem 6.3, using \( a(a - b) = \frac{1}{2}(a^2 - b^2 + (a - b)^2) \), and the coercivity of \( a_h \) equation (32) yields

\[
\begin{align*}
  \frac{1}{2\Delta t} \left( \left\| \zeta_n^e \right\|_{\Omega^s}^2 - \left\| \zeta_{n-1}^e \right\|_{\Omega^s}^2 \right) & + \frac{\Delta t}{2} \| d_i \zeta_n^e \|_{\Omega^s}^2 + C_a \| \zeta_n^e \|_v^2 \\
  &= \sum_{K \in T_K} \int_K (\partial_t u^n - d_t \Pi V u^n) \cdot \zeta_n^e \, dx + a_h\left( c_{n-1}^e; \zeta_n^e, \zeta_n^e \right) + \left[ a_h\left( c^n; u^n, \zeta_n^e \right) - a_h\left( c_{n-1}^e; u^n, \zeta_n^e \right) \right] \\
  &+ \int_{\Omega^d} \left[ f^s\left( c_{n-1}^e \right) - f^s\left( c^n \right) \right] \cdot \zeta_n^e \, dx + \int_{\Omega^d} \left[ \mathbb{K}^{-1}\left( c_{n-1}^e \right) f^d\left( c_{n-1}^e \right) - \mathbb{K}^{-1}\left( c^n \right) f^d\left( c^n \right) \right] \cdot \zeta_n^e \, dx \\
  &=: I_1 + \ldots + I_5.
\end{align*}
\]  \( \tag{63} \)

Using equation (29) and employing Young’s inequality for some \( \epsilon > 0 \),

\[
I_1 \leq \left( \left\| \partial_t u^n - d_t u^n \right\|_{\Omega^s} + \| d_i \zeta_n^e \|_{\Omega^s} \right) \| \zeta_n^e \|_{\Omega^s} \leq C\left( \left\| \partial_t u^n - d_t u^n \right\|_{\Omega^s}^2 + \| d_i \zeta_n^e \|_{\Omega^s}^2 \right) + \epsilon \| \zeta_n^e \|_v^2.
\]  \( \tag{64} \)
It follows from equation (31) and Young’s inequality that
\[ I_2 \leq C \| \xi^n_u \|_{\Omega^d} \| \zeta^n_u \|_{\nu} \leq C \| \xi^n_u \|_{\Omega^d}^2 + \epsilon \| \zeta^n_u \|_{\nu}^2. \]  
(65)

Consider now \( I_3 \):
\[
I_3 = \sum_{K \in T^*} \int K 2 \left[ \mu(c^n) - \mu(c_h^{n-1}) \right] \varepsilon(u^n) : \varepsilon(\zeta^n_u) \, dx - \int \Omega^d \left[ \mu(c^n) - \mu(c_h^{n-1}) \right] \varepsilon(u^n) n^s \cdot (\zeta^n_u - \bar{\zeta}_n) \, ds \\
+ \int \Omega^d \left[ \mathbb{K}^{-1}(c^n) - \mathbb{K}^{-1}(c_h^{n-1}) \right] u^n \cdot \varepsilon(\zeta^n_u) \, dx + \sum_{j=1}^{\dim-1} \int_{\Gamma^j} \gamma^j \left[ \mu(c^n) - \mu(c_h^{n-1}) \right] (u^{sn} \cdot \tau^j) (\bar{\zeta}_u \cdot \tau^j) \, ds \\
=: I_{31} + \ldots + I_{34}.
\]  
(66)

The first term on the right side of equation (66) can be bounded as follows using Lipschitz continuity of \( \mu \), the generalized Hölder’s inequality for integrals and sums, and Young’s inequality:
\[
I_{31} \leq 2 \mu_L \sum_{K \in T^*} \left\| \varepsilon(\zeta^n_u) \right\|_{0, \infty, K} \left\| \nabla \zeta^n_u \right\|_K \leq 2 \mu_L \left\| \nabla u^n \right\|_{0, \infty, \Omega^*} \left( \sum_{K \in T^*} \left\| \varepsilon(\zeta^n_u) \right\|_{0, \infty, K} \left\| \nabla \zeta^n_u \right\|_K \right)^{1/2} \leq C \mu_L^2 \left\| \nabla u^n \right\|_{0, \infty, \Omega^*}^2 \sum_{K \in T^*} \left\| \varepsilon(\zeta^n_u) \right\|_{0, \infty, K} \left\| \nabla \zeta^n_u \right\|_K. 
\]
(67)

Next we bound \( I_{32} \). By Lipschitz continuity of \( \mu \), equation (21), generalized Hölder’s inequality, and Young’s inequality,
\[
|I_{32}| \leq 2 \mu_L \sum_{K \in T^*} \left\| \varepsilon(\zeta^n_u) \right\|_{0, \infty, K} \left\| \nabla u^n \right\|_{0, \infty, K} \left\| \zeta^n_u - \bar{\zeta}_n \right\|_{0, \infty, K} \leq 2 \mu_L \left\| \nabla u^n \right\|_{0, \infty, \Omega^*} \left( \sum_{K \in T^*} \left\| \varepsilon(\zeta^n_u) \right\|_{0, \infty, K} \left\| \nabla u^n \right\|_{0, \infty, K} \right)^{1/2} \leq C \mu_L^2 \left\| \nabla u^n \right\|_{0, \infty, \Omega^*}^2 \sum_{K \in T^*} \left\| \varepsilon(\zeta^n_u) \right\|_{0, \infty, K} \left\| \nabla u^n \right\|_{0, \infty, K}. 
\]
(68)

We bound \( I_{33} \) by using the assumption on \( \mathbb{K} \) given in equation (6):
\[
I_{33} \leq \frac{\mu_L}{\kappa_s} \left\| \varepsilon(\zeta^n_u) \right\|_{\Omega^d} \left\| u^n \right\|_{0, \infty, \Omega^*} \left\| \zeta^n_u \right\|_{\nu} \leq \frac{\mu_L^2}{\kappa_s} \left\| u^n \right\|_{0, \infty, \Omega^*} \left( \sum_{j=1}^{\dim-1} \gamma^j (\gamma^j)^2 \right)^{1/2} \left\| \Omega^d \right\}_{\Omega^d}^2 + \epsilon \left\| \zeta^n_u \right\|_{\nu}^2. 
\]
(69)

Again by the Lipschitz property of \( \mu \) and Hölder’s inequality,
\[
I_{34} \leq \mu_L \left\| \varepsilon(\zeta^n_u) \right\|_{\Gamma^j} \left( \sum_{j=1}^{\dim-1} \gamma^j \right)^{1/2} \left\| u^{sn} \right\|_{0, \infty, \Gamma^j} \left( \sum_{j=1}^{\dim-1} \gamma^j (\gamma^j)^2 \right)^{1/2} \left\| \Omega^d \right\}_{\Omega^d}^2 \leq \mu_L \left\| \zeta^n_u \right\|_{\nu} \left( \sum_{j=1}^{\dim-1} \gamma^j \right)^{1/2} \left\| u^n \right\|_{0, \infty, \Omega^*} \left\| \zeta^n_u \right\|_{\nu}. 
\]
Combining the above bounds for $I_1$ to $I_5$, with equation (63), letting $\epsilon = \frac{C_u}{16}$ ($C_u$ is the coercivity constant), multiplying by $2\Delta t$, summing from 1 to $n$, noting that $\xi_u^0 = 0$, and applying equation (41), we obtain:

$$
\|\xi_u^n\|^2_{\Omega^*} + (\Delta t)^2 \sum_{m=1}^n \|d_t \xi_u^m\|^2_{\Omega^*} + C_a \Delta t \sum_{m=1}^n \|\xi_u^m\|^2_v \\
\leq C \left( (\Delta t)^2 \|\partial_t u\|^2_{L^2(0,T;L^2(\Omega^*))} + \|\partial_t \xi_u\|^2_{L^2(0,T;L^2(\Omega^*)))} + 2\Delta t \sum_{m=1}^n \|\xi_u^m\|^2_v \right)
$$

Next, using equations (34d) and (45b),

$$
\|\xi_u^n\|^2_{\Omega^*} + (\Delta t)^2 \sum_{m=1}^n \|d_t \xi_u^m\|^2_{\Omega^*} + C_a \Delta t \sum_{m=1}^n \|\xi_u^m\|^2_v
$$
\[ \leq C \left[ (\Delta t)^2 \|\partial_t u\|_{L^2(0,T;L^2(\Omega^r))}^2 + h^{2kj} \left( \|\partial_t u\|_{L^2(0,T;[H^{k,j+1}(\Omega^r)]^2)}^2 + \|u\|_{L^2(0,T;[H^{k,j+1}(\Omega^r)]^2)}^2 \right) 
+ \|u\|_{L^2(0,T;[H^{k,j}(\Omega^r)]^2)}^2 \right] 
+ \|\nabla u\|_{L^\infty(0,T;[L^\infty(\Omega^r)]^2)}^2 \Delta t \sum_{m=1}^n \sum_{K \in T^d} h_K \left\| c^m - c_h^{m-1} \right\|_{1,K}^2 
+ \left( \|u\|_{L^\infty(0,T;[L^\infty(\Omega^r)]^2)}^2 + \|\nabla u\|_{L^\infty(0,T;[L^\infty(\Omega^r)]^2)}^2 \right) \Delta t \sum_{m=1}^n \left\| c^m - c_h^{m-1} \right\|_{\Omega^r}^2 
+ \left( 1 + \|\nabla u\|_{L^\infty(0,T;[L^\infty(\Omega^r)]^2)}^2 \right) \Delta t \sum_{m=1}^n \left\| c^m - c_h^{m-1} \right\|_{\Omega^r}^2, \]

where the constant $C > 0$ depends on $\mu_L, \kappa_*, K^*, \gamma^j, L^r_*$ but is independent of $h$ and $\Delta t$. Equation (62) follows by equations (54a)–(54c), and assumption equation (61). \qed

The following is a straightforward consequence of Theorem 6.5.

**Corollary 6.7.** Assume that the regularity assumptions, equation (61), and $u^n$ and $u_h^n$ are as given in Theorem 6.5. Then for all $n \geq 1$,

\[ \|u^n - u_h^n\|_{L^\infty(\Omega^r)}^2 + \Delta t \sum_{m=1}^n \|u^m - u_h^m\|_{L^2(\Omega^r)}^2 \leq C \left( (\Delta t)^2 + h^{2kj} + h^{2kc} \right), \]

\[ \sum_{K \in T^d} h_K \|u^n - u_h^n\|_{L^2(\Omega^r)}^2 + \Delta t \sum_{m=1}^n \sum_{K \in T^d} h_K \|u^m - u_h^m\|_{L^2(\Omega^r)}^2 \leq C \left( (\Delta t)^2 + h^{2kj} + h^{2kc} \right). \]

**Proof.** By the triangle inequality, equations (34d), (36), (37), (43) and (62) (see [17], (48b)),

\[ \|u^n - u_h^n\|_{L^\infty(\Omega^r)}^2 + \Delta t \sum_{m=1}^n \|u^n - u_h^n\|_{L^2(\Omega^r)}^2 \leq \left( \|\zeta_u^n\|_{L^\infty(\Omega^r)}^2 + \Delta t \sum_{m=1}^n \|\zeta_u^m\|_{L^2(\Omega^r)}^2 \right) + \left( \|\zeta_u^n\|_{L^\infty(\Omega^r)}^2 + \Delta t \sum_{m=1}^n \|\zeta_u^m\|_{L^2(\Omega^r)}^2 \right) \]

\[ \leq C \left( (\Delta t)^2 + h^{2kj} + h^{2kc} \right) \]

\[ + h^{2kj} \left( \|u^n\|_{L^\infty(0,T;[H^{k,j+1}(\Omega^r)]^2)} + \|u^n\|_{L^2(0,T;[H^{k,j+1}(\Omega^r)]^2)} \right) \]

Now we prove equation (74b). Using equations (20), (21), (43) and (62) together with equation (34d), we obtain

\[ \sum_{K \in T^d} h_K \|u^n - u_h^n\|_{L^2(\Omega^r)}^2 + \Delta t \sum_{m=1}^n \sum_{K \in T^d} h_K \|u^m - u_h^m\|_{L^2(\Omega^r)}^2 \]

\[ \leq C \sum_{K \in T^d} h_K \left( \|\zeta_u^n\|_{L^2(\Omega^r)}^2 + \|\zeta_u^m\|_{L^2(\Omega^r)}^2 \right) + \Delta t \sum_{m=1}^n \sum_{K \in T^d} h_K \left( \|\zeta_u^n\|_{L^2(\Omega^r)}^2 + \|\zeta_u^m\|_{L^2(\Omega^r)}^2 \right) \]

\[ \leq \left( C \sum_{K \in T^d} \|\zeta_u^n\|_{L^2(\Omega^r)}^2 + \Delta t \sum_{m=1}^n \sum_{K \in T^d} \|\zeta_u^m\|_{L^2(\Omega^r)}^2 \right) \]
+ C \left( \sum_{K \in T^s} \left( \| \xi^n_p \|^2_K + h_K^2 \| \xi^n_p \|^2_{1,K} \right) + \Delta t \sum_{m=1}^n \sum_{K \in T^d} \left( \| \xi^n_c \|^2_K + h_K^2 \| \xi^n_c \|^2_{1,K} \right) \right)
\leq C \left( (\Delta t)^2 + h^{2k_f} + h^{2k_c} \right) + h^{2k_f} \left( \| u \|_{L^\infty(0,T;H^k(f(\Omega^s)))} + \| u^d \|_{L^2(0,T;H^{k_f}(\Omega^d))} \right).

This completes the proof. \qed

6.2. Error estimate for the pressure

In this section, we briefly discuss the a priori error estimate for the pressure approximation.

Lemma 6.8. Assume that the regularity assumptions, equation (61), and \( u^n \) and \( u^n_h \) are as given in Theorem 6.5. If \( p \) and \( p_h \) are the pressure solutions to equations (1)–(3) and (39), respectively, then

\[
\Delta t \sum_{m=1}^n \| \xi^n_m \|^2_p \leq C \left( \Delta t \sum_{m=1}^n \| \xi^n_m \|^2_v + \Delta t \sum_{m=1}^n \| d_i \xi^n_m \|^2_{\Omega^s} + \Delta t \sum_{m=1}^n \| \xi^n_{m-1} \|^2_c + \Delta t \sum_{m=1}^n \sum_{K \in T} \| c^{m-1} - c^{m-1}_h \|^2_K 
+ \Delta t \sum_{m=1}^n \sum_{K \in T^s} h_K^2 \| c^{m-1} - c^{m-1}_h \|^2_{1,K} \right) + (\Delta t)^2 + h^{2k_f} + h^{2k_c+1}.
\quad (75)
\]

Proof. Setting \( q_h = 0 \) in the error equation in Theorem 6.3, we obtain:

\[
\sum_{j=s,d} \left( h_j^b \left( \xi^n_j, v_h \right) + h_j^{1,j} \left( \xi^n_j, v_h \right) \right) 
= \left( a_h \left( c_{(n)}_h; \xi^n_u, v_h \right) - a_h \left( c_{(n)}_h; \xi^n_v, v_h \right) \right) + \left( a_h \left( c^n; u^n, v_h \right) - a_h \left( c^n; u^n, v_h \right) \right) 
+ \int_{\Omega^s} \left( f_s \left( c_{(n)}_h \right) - f_s \left( c^n \right) \right) \cdot v_h \, dx + \int_{\Omega^d} \left[ K^{-1} \left( c_{(n)}_h \right) f^d \left( c_{(n)}_h \right) - K^{-1} \left( c^n \right) f^d \left( c^n \right) \right] \cdot v_h \, dx 
- \sum_{K \in T^s} \int_{K} (d_i u^n_h - \partial_i u^n) \cdot v_h \, dx 
= : H_1 + \ldots + H_5.
\quad (76)
\]

By equation (31) and Young’s inequality, and using that \( \| \cdot \|_v \) and \( \| \cdot \|_{v'} \) are equivalent on \( V_h \), we have

\[
H_1 \leq C \left( \| \xi^n_u \|_v + \| \xi^n_v \|_{v'} \right) \| v_h \|_v.
\quad (77)
\]

Following the proof of equation (71), we can show that

\[
H_2 \leq C \left( \mu_L \| \nabla u^n \|_{0,\infty,\Omega^s} \left( \sum_{K \in T^s} \left( \| c^n - c_{(n)}_h \|^2_K + h^2_K \| c^n - c_{(n)}_h \|^2_{1,K} \right) \right) \right)^{1/2}
+ \kappa^{-1}_s \mu_L \| u^n \|_{0,\infty,\Omega^s} \| c^n - c_{(n)}_h \|_{2,\Gamma^f} + \mu_L \left( \sum_{j=1}^{\text{dim}_1} \gamma_j \right) \right)^{1/2} \| u^n \|_{0,\infty,\Delta^s} \| c^n - c_{(n)}_h \|_{1,\Gamma^f} \| v_h \|_{v'}.
\quad (78)
\]

As in equations (72) and (73), we find

\[
H_3 \leq L_{\| f \|_{1,\Gamma^f}} \| c_{(n)}_h - c^n \|_{\Omega^s} \| v_h \|_v,
\quad (79)
\]
Remark 6.9. Therefore, the result follows by equations (54a)–(54c) under the assumptions on the exact solution given in

Using Cauchy–Schwarz and triangle inequalities, and equation (34d),

\[
H_3 = -\sum_{K \in T^s} \int_K (d_\ell \zeta u^n + (d_\ell \Pi \nu u^n - \partial_t \Pi \nu u^n) - \partial_t \zeta u^n) \cdot v_h \, dx
\leq \left(\|d_\ell \zeta u^n\|_\Omega + \|d_\ell \Pi \nu u^n - \partial_t \Pi \nu u^n\|_\Omega + Ch^{k_f} \|\partial_t u^n\|_{k_f, \Omega}^r\right)\|\nu_h\|_\nu.
\]

Therefore, combining (76)–(81), dividing both sides by \(\|\nu_h\|_\nu\), taking the supremum over \(\nu_h \in V_h\), and using Theorem 4.2, we obtain

\[
ce^*_m \|\xi^m_p\|_p \leq C \left(\|\xi^m_v\|_v + \|\xi^m_v\|_{v'} + \mu_L \|\nabla u^n\|_{0, \infty, \Omega^s} \left(\sum_{K \in T^s} \left(\|c^n - c_h^{n-1}\|_K + h^2 K \|c^n - c_h^{n-1}\|_{1, K}\right)\right)^{1/2}
+ \mu_L \left(\sum_{j=1}^{\dim - 1} \gamma^j\right)^{1/2} \|u^n\|_{0, \infty, \Omega^s} \|c^n - c_h^{n-1}\|_{\Gamma} + L^2 \|c_h^{n-1} - c^n\|_{\Omega^s}
+ \left(\mu_L \kappa^s_\star^{-1} \left(\|f^d(c^n)\|_{0, \infty, \Omega^d} + \|u^n\|_{0, \infty, \Omega^d}\right) + K^s_\star^{-1} L^d_f\right) \|c_h^{n-1} - c^n\|_{\Omega^d}
+ \|d_\ell \zeta u^n\|_\Omega + \|d_\ell \Pi \nu u^n - \partial_t \Pi \nu u^n\|_\Omega + Ch^{k_f} \|\partial_t u^n\|_{k_f, \Omega}^r\right).
\]

Squaring both sides, multiplying by \((c^*_m)^{-2}\Delta t\), summing from 1 to \(n\), using equations (41a) and (45b), stability of \(\Pi \nu\), and the regularity assumptions on \(u^s\), \(u^d\), and \(\nabla p^d\) yields

\[
\Delta t \sum_{m=1}^n \|\xi^m_p\|_p^2 \leq C \left(\Delta t \sum_{m=1}^n \|\xi^m_v\|_v^2 + \Delta t \sum_{m=1}^n \|d_\ell \zeta u^n\|_{\Omega^s}^2
+ \Delta t \sum_{m=1}^n \|c^m - c_h^{m-1}\|_K^2 + \Delta t \sum_{m=1}^n \sum_{K \in T^s} h^2 K \|c^m - c_h^{m-1}\|_{1, K}^2
+ \Delta t \sum_{m=1}^n \|c^m - c_h^{m-1}\|_{\Gamma}^2 + (\Delta t)^2 \|\partial_t u\|_{L^2(0, T; [H^k_f(\Omega^s)]^d)}^2
+ h^{2k_f} \left(\|u\|_{L^2(0, T; [H^k_f(\Omega^s)]^d)}^2 + \|u\|_{L^2(0, T; [H^k_f(\Omega^s)]^d)}^2 + \|\partial_t u\|_{L^2(0, T; [H^k_f(\Omega^s)]^d)}^2\right)\right).
\]

Therefore, the result follows by equations (54a)–(54c) under the assumptions on the exact solution given in Theorem 6.5. 

**Remark 6.9.** An immediate consequence of Lemma 6.8 and Theorem 6.5 is

\[
\Delta t \sum_{m=1}^n \|\xi^m_p\|_p^2 \leq C \left(\Delta t + (\Delta t)^{-1} \left(h^{2k_f} + h^{2k_\ell}\right) + \Delta t \sum_{m=1}^n \|\xi^m_v\|_v^2 \right).
\]
\( \Delta t \sum_{m=1}^{n} \sum_{K \in \mathcal{T}} \left\| c^{m-1} - c^{m-1}_n \right\|_K^2 + \Delta t \sum_{m=1}^{n} \sum_{K \in \mathcal{T}^{-}} \left| n_k^2 \left\| c^{m-1} - c^{m-1}_n \right\|_1^2 \right. \)

This loss of \( \Delta t \) is due to \( \Delta t \sum_{m=1}^{n} \| d_i \xi^m \|_{\Omega^n}^2 \) in equation (75). However, an improved estimate can be obtained by bounding this term as follows: testing equation (48) with \( v_h = d_i \xi^m \), multiplying by \( \Delta t \), summing from \( m = 1 \) to \( n \), employing a summation-by-parts formula on the terms on the right hand side that are contained in \( a_h(\cdots, d_i \xi^m) \) to transfer the discrete time derivative on \( d_i \xi^m \) to the other terms, and assuming that the exact solution is sufficiently smooth in time, leads to

\[ \Delta t \sum_{m=1}^{n} \| d_i \xi^m \|_{\Omega^n}^2 \leq C \left( \frac{1}{h^{2k_f}} + \frac{1}{h^{2k_c}} + (\Delta t)^2 \right). \]

We do not provide the details of this proof here, but instead refer to [18], p.42.

### 6.3. Existence and uniqueness of the concentration solution

In this section, we will prove existence and uniqueness of the discrete concentration solution \( c^n_h \in C_h \) to equation (40). To achieve this, given the velocity solution \( u^n_h \) at time step \( t = t^n \), let us first introduce

\[ \| w_h \|_{c, u^n_h}^2 = \sum_{K \in \mathcal{T}} \left[ \| \tilde{D} (u^n_h)^{1/2} \nabla w_h \|_K^2 + h_K^{-1} \| (1 + \chi_d |u^n_h|)^{1/2} (w_h - \bar{w}_h) \|_{\partial K}^2 \right] \quad \forall w_h \in C_h. \quad (82) \]

Due to assumption equation (4a), note that

\[ \| w_h \|_c \leq C \| w_h \|_{c, u^n_h} \quad (83) \]

at any time step \( t = t^n \). Also observe that from equation (4a),

\[ \tilde{D}_{\min} (1 + \chi_d |u|) |\xi|^2 \leq \xi^T \tilde{D} (u) \xi \leq \tilde{D}_{\max} (1 + \chi_d |u|) |\xi|^2 \quad \forall \xi \in \mathbb{R}^{\dim}, \quad (84) \]

where \( \tilde{D}_{\max} := \max(d, D_{\max}) \) and \( \tilde{D}_{\min} := \min(d, D_{\min}) \).

**Theorem 6.10.** There exists a constant \( \beta_{tr}^{d} > 0 \) such that if \( \beta_{tr} > \beta_{tr}^{d} \), then for all \( w_h \in C_h \),

\[ B_{tr}^{d}(u^n_h; w_h, w_h) \geq C_{tr} \| w_h \|_{c, u^n_h}^2 + \frac{1}{2} \sum_{K \in \mathcal{T}^{-}} \int_K \nabla \cdot u^n_h w_h^2 \, dx, \quad (85) \]

where \( C_{tr} > 0 \) is a constant that depends on \( d, D_{\min}, \) and \( D_{\max} \).

**Proof.** From Lemma 2 of [16],

\[ B_{tr}^{u}(u^n_h, w_h, w_h) = \frac{1}{2} \sum_{K \in \mathcal{T}^{-}} \int_K \nabla \cdot u^n_h w_h^2 \, dx + \frac{1}{2} \sum_{K \in \mathcal{T}} \| u^n_h \cdot n |^{1/2} (w_h - \bar{w}_h) \|_{\partial K}^2 \quad (86) \]

Therefore, to complete the proof, we will focus on \( B_{tr}^{d} \). By equation (18),

\[ B_{tr}^{d}(u^n_h; w_h, w_h) = \sum_{K \in \mathcal{T}} \| \tilde{D} (u^n_h)^{1/2} \nabla w_h \|_K^2 + \sum_{K \in \mathcal{T}^{-}} \frac{\partial}{\partial K} \left[ \tilde{D} (u^n_h) n \cdot n \right] (w_h - \bar{w}_h)^2 \, ds \]

\[ - 2 \sum_{K \in \mathcal{T}^{-}} \int_{\partial K} \left[ \tilde{D} (u^n_h) \nabla w_h \right] \cdot n (w_h - \bar{w}_h) \, ds := J_1 + J_2 + J_3. \quad (88) \]
For $\delta > 0$, we have coercivity, and next to bound $J_3$ we recall from Lemma A.1 of [37] that for any $K \in T^d$, if $F$ is a face of $K$, then

\[
\left\| \tilde{D}(u_h^n)^{1/2}\nabla w_h \right\|_F = \left\| \tilde{D}(u_h^n)^{1/2}\nabla w_h \right\|_F \lesssim h_K^{-1/2} \left( \left\| \nabla w_h \right\|_K^2 + \left\| \tilde{D}(u_h^n)^{1/2}\nabla w_h \right\|_K^2 \right)^{1/2}.
\]

Next to bound $J_3$ we recall from Lemma A.1 of [37] that for any $K \in T^d$, if $F$ is a face of $K$, then

\[
\left\| \tilde{D}(u_h^n)^{1/2}\nabla w_h \right\|_F = \left\| \tilde{D}(u_h^n)^{1/2}\nabla w_h \right\|_F \lesssim d^{1/2} h_K^{-1/2} \left\| \nabla w_h \right\|_K.
\]

For $K \in T^d$, by the discrete trace inequality equation (20), we have

\[
\left\| \tilde{D}(u_h^n)^{1/2}\nabla w_h \right\|_F = d^{1/2} \left\| \nabla w_h \right\|_F \lesssim d^{1/2} h_K^{-1/2} \left\| \nabla w_h \right\|_K.
\]

Therefore, for any $K \in T$,

\[
\left\| \tilde{D}(u_h^n)^{1/2}\nabla w_h \right\|_F \leq C h_K^{-1/2} \left( \left\| (1 + \chi_d |u_h^n|)^{1/2}\nabla w_h \right\|_K \right)^2.
\]

Using this and applying equation (84) twice,

\[
\int_{\partial K} \left[ \tilde{D}(u_h^n) \nabla w_h \right] \cdot n (w_h - \bar{w}_h) \, ds = \int_{\partial K} \left( \tilde{D}(u_h^n)^{1/2}\nabla w_h \right) \cdot \left( \tilde{D}(u_h^n)^{1/2} n \right) (w_h - \bar{w}_h) \, ds
\]

where we used Young’s inequality with constant $\delta > 0$. Therefore,

\[
J_3 \geq -\frac{C^2 \tilde{D}_{\max}}{\delta \tilde{D}_{\min}} \sum_{K \in T} h_K^{-1} \left( \left\| (1 + \chi_d |u_h^n|)^{1/2}(w_h - \bar{w}_h) \right\|_K^2 - \delta \sum_{K \in T} \left\| \tilde{D}(u_h^n)^{1/2}\nabla w_h \right\|_K^2. \]

Choosing $\delta = \frac{1}{2}$ and gathering equations (87), (89) and (93), we obtain

\[
B_h^C(u_h^n; w_h, w_h) \geq \frac{1}{2} C \left( \tilde{D}_{\min} \beta_{tr} - 2C^2 \tilde{D}_{\min}^{-1} \tilde{D}_{\max} \right) h_K^{-1} \left\| (1 + \chi_d |u_h^n|)^{1/2}(w_h - \bar{w}_h) \right\|_K^2.
\]

The result follows by choosing $\beta_{tr}^C \geq 2C^2 \tilde{D}_{\max} \tilde{D}_{\min}^{-1}$.

Note that recalling equation (83), we also have

\[
B_h^C(u_h^n; w_h, w_h) \geq C \left\| w_h \right\|_K^2 + \frac{1}{2} \sum_{K \in T} \int_K \left\| \nabla \cdot u_h^n w_h \right\|_K^2.
\]

Now that we have coercivity, we proceed with the existence and stability proof for the discrete concentration.
Theorem 6.11. Let $c_0 \in L^2(\Omega)$ and $g_1, g_p \in L^2(0, T; L^\infty(\Omega^d))$. Let $n \geq 1$ and let $u^n_h$ be the solution to equation (39). If $d_n \Delta t < 1$, where $d_n = \frac{1}{h} \left(1 + C \| g^n_h - g^n_0 \|_{0, \infty, \Omega^d} \right)$, then there exists a unique solution $c^n_h \in C_h$ to equation (40). Furthermore, if $K := \sum_{m=1}^n d_m / (1 - \Delta t d_m)$, then
\[
\phi_\star \| c^n_h \|^2_{L^2(\Omega)} + C \Delta t \sum_{m=1}^n \| c^m_h \|^2_{C} \leq e^{K \Delta t} \left( \phi_\star \| c_0 \|^2_{L^2(\Omega)} + \| g_i \|^2_{L^2(0, T; L^2(\Omega^d))} \right).
\]

Proof. Let $w_h = c^n_h$ in equation (40). From the algebraic inequality $(a - b)a \geq \frac{1}{2} (a^2 - b^2)$, equation (95), and the assumption that $0 < c_I \leq 1$ a.e., we have
\[
\frac{\phi_\star}{2 \Delta t} \left( \| c^n_h \|^2_{L^2(\Omega)} - \| c^{n-1}_h \|^2_{L^2(\Omega)} \right) + C \| c^n_h \|^2_{C} + \int_{\Omega^d} g^n_p(c^n_h)^2 \, dx \leq \sum_{K \in T^d} \int_K c_I g^n_h \, dx - \frac{1}{2} \sum_{K \in T^d} \int_K \nabla \cdot u^n_h(c^n_h)^2 \, dx \leq \| g^n_I \|^2_{L^2(\Omega^d)} + \frac{1}{2} \| \nabla \cdot u^n_h \|_{0, \infty, \Omega^d}^2 \| c^n_h \|^2_{L^2(\Omega^d)}.
\]

Multiplying this inequality by $2 \Delta t$, summing from $m = 1$ to $n$, noting that $g_p \geq 0$, and recalling equation (19a) with stability of the $L^2$-projections $\Pi_C$ and $\Pi_Q$, we obtain
\[
\phi_\star \| c^n_h \|^2_{L^2(\Omega)} + 2C \Delta t \sum_{m=1}^n \| c^m_h \|^2_{C} \leq \phi_\star \| c^n_0 \|^2_{L^2(\Omega)} + \frac{\phi_\star}{2 \Delta t} \sum_{m=1}^n \left( \| g^n_m \|^2_{L^2(\Omega^d)} + \| c^n_m \|^2_{C} \right) + \Delta t \sum_{m=1}^n \| \Pi_Q (g^n_m - g^n_p) \|_{0, \infty, \Omega^d}^2 \| c^n_m \|^2_{C} \leq \phi_\star \| c_0 \|^2_{L^2(\Omega)} + \Delta t \sum_{m=1}^n \left( 1 + C \| g^n_m - g^n_p \|_{0, \infty, \Omega^d} \right) \| c^n_m \|^2_{L^2(\Omega^d)}.
\]

Equation (96) follows after applying Gronwall’s inequality ([35], Lem. 27). This stability bound then implies the existence of a unique solution since the system is finite dimensional and linear.

\[\]

6.4. Error estimate for the discrete concentration

This section is devoted to proving an error estimate for the discrete concentration.

Lemma 6.12 (Error equation for Eq. (40)).
\[
\sum_{K \in T} \int_K \phi \, d_t c^n_h \, w_h \, dx + B^n_t(u^n_h; c^n_h, w_h) + \int_{\Omega^d} g^n_p c^n_h \, w_h \, dx = \sum_{K \in T} \int_K \phi \, d_t c^n_h \, w_h \, dx - \sum_{K \in T} \int_K \phi \, (d_t c^n - \partial_t c^n) \, w_h \, dx + B^n_t(u^n_h; c^n_h, w_h) + B^n_t(u^n_h; \xi^n_h, w_h) + B^n_t(u^n; \xi^n_h, w_h) + \int_{\Omega^d} g^n_p c^n_h \, w_h \, dx \label{eq:6.12}
\]

Proof. By Lemma 3.2, for $t = t^n$, we have
\[
\sum_{K \in T} \int_K \phi \, \partial_t c^n_h \, w_h \, dx + B^n_t(u^n_h; c^n_h, w_h) + \int_{\Omega^d} g^n_p c^n_h \, w_h \, dx = \sum_{K \in T} \int_K c_I g^n_h \, w_h \, dx \quad \forall w_h \in C_h, \label{eq:6.13}
\]
where $c^n = (c^n, \gamma(c^n))$. Subtracting equation (98) from equation (40) yields that for all $w_h \in C_h$,
\[
\sum_{K \in T} \int_K \phi \, (d_t c^n_h - \partial_t c^n) \, w_h \, dx + B^n_t(u^n_h; c^n_h, w_h) - B^n_t(u^n; c^n_h, w_h) + \int_{\Omega^d} g^n_p(c^n_h - c^n) \, w_h \, dx = 0. \label{eq:6.14}
\]
Next, we rewrite the $B_1^n$ terms in equation (99) by observing that $B_1^n$ is linear in the second slot and that $B_1^n$ is linear in the first slot:

$$B_1^n(u_h^n; c_h^n, w_h) - B_1^n(u^n; c^n, w_h)$$

$$= B_1^n(u_h^n; c_h^n - c^n, w_h) + B_1^n(u^n - u_h^n; c^n, w_h) + B_1^n(u_h^n; c^n, w_h) - B_1^n(u^n; c^n, w_h)$$

$$= B_1^n(u_h^n; c_h^n, w_h) - B_1^n(u^n - u_h^n; c^n, w_h) + B_1^n(u_h^n; c^n, w_h) - B_1^n(u^n; c^n, w_h)$$

$$+ B_1^n(u_h^n; Ic^n, w_h) - B_1^n(u^n; Ic^n, w_h) - B_1^n(u^n; c^n, w_h).$$

Equation (100) uses equation (5a), the Cauchy–Schwarz inequality, Young’s inequality with constant $\gamma$. In addition to the assumptions in Theorem 6.5, suppose that

$$\Box$$

Using equation (52), again the linearity of $B_1^n$ in the second slot, and equations (99) and (100) completes the proof.

**Theorem 6.13.** In addition to the assumptions in Theorem 6.5, suppose that

$$c_0 \in H^{k_r}(\Omega), \quad c \in L^2\left(0, T; H^{k_r+1}(\Omega)\right) \cap L^\infty\left(0, T; W^{1,\infty}(\Omega)\right), \quad \partial_t c \in L^2\left(0, T; H^{k_r}(\Omega)\right),$$

$$\partial_{tt} c \in L^2\left(0, T; L^2(\Omega)\right), \quad g_r, g_p \in L^\infty\left(0, T; L^\infty(\Omega)\right)\right).$$

Then for sufficiently small $\Delta t$,

$$\left\|\xi^n_c\right\|_{L^\Omega}^2 + \Delta t \sum_{m=1}^n \left\|\xi^n_c\right\|_{L^\Omega}^2 \leq C \left(h^{2k_r} + h^{2k_r} + (\Delta t)^2\right),$$

(101)

where $C$ depends on $\phi_*, \phi^*$, $d$, $D$ and the regularity of the solution but is independent of $h$ and $\Delta t$.

**Proof.** Setting $w_h = \xi^n_c$ in Lemma 6.12, using the inequality $a(a - b) \geq \frac{a^2 - b^2}{2}$, and Theorem 6.10,

$$\frac{\phi_8}{2\Delta t} \left(\left\|\xi^n_c\right\|_{L^\Omega}^2 - \left\|\xi^n_c\right\|_{L^\Omega}^2\right) + C_{tr}\left\|\xi^n_c\right\|_{L^\Omega}^2 + \int_{t=0}^T \int_{\Omega} \sum_{m=1}^n \left\|\xi^n_c\right\|_{L^\Omega}^2 \leq C \left(h^{2k_r} + h^{2k_r} + (\Delta t)^2\right).$$

Using equation (5a), the Cauchy–Schwarz inequality, Young’s inequality with constant $\gamma$, and equation (53),

$$I_1 \leq \phi^* \left(1 - \frac{\phi^*}{\sqrt{\Delta t}} \int_{t=0}^T \int_{\Omega} \left\|\xi^n_c\right\|_{L^\Omega}^2 \leq \phi^* \left(\frac{\phi^*}{\sqrt{\Delta t}} \int_{t=0}^T \left\|\xi^n_c\right\|_{L^\Omega}^2 \leq C \left(h^{2k_r} + h^{2k_r} + (\Delta t)^2\right).$$

Again by equation (5a), this time using Taylor’s theorem in integral form, and applying Young’s inequality,

$$I_2 = \sum_{K \in T} \int_K \phi \left(\partial_t c^n - \partial_t c^n\right) c^n dx \leq \phi^* \left(1 - \frac{\phi^*}{\sqrt{\Delta t}} \int_{t=0}^T \left\|\xi^n_c\right\|_{L^\Omega}^2 \leq C \left(h^{2k_r} + h^{2k_r} + (\Delta t)^2\right).$$
Hölder’s inequality yields
\[
I_{3} = B_{h}^{n}(u_{h}^{n}; \xi_{c}^{n}, \zeta_{c}^{n}) = - \sum_{K \in T} \xi_{c}^{n} (u_{h}^{n} - u^{n}) \cdot \nabla \zeta_{c}^{n} \, dx - \sum_{K \in T} \zeta_{c}^{n} u^{n} \cdot \nabla \zeta_{c}^{n} \, dx \\
+ \sum_{K \in T} \xi_{c}^{n} (u_{h}^{n} - u^{n}) \cdot n(\zeta_{c}^{n} - \tilde{\zeta}_{c}^{n}) \, ds + \sum_{K \in T} \zeta_{c}^{n} u^{n} \cdot n(\zeta_{c}^{n} - \tilde{\zeta}_{c}^{n}) \, ds =: I_{31} + \ldots + I_{34}. \tag{102}
\]

Hölder’s inequality yields
\[
I_{31} \leq C \| \xi_{c}^{n} \|_{0, \infty; \Omega} \| u_{h}^{n} - u^{n} \|_{\Omega} \| \nabla \zeta_{c}^{n} \|_{\Omega} \leq C \| c^{n} \|_{0, \infty; \Omega} \| u_{h}^{n} - u^{n} \|_{\Omega} \| \zeta_{c}^{n} \|_{c}. \tag{103}
\]

The term $I_{32}$ can be bounded by Hölder’s inequality, and equation (53):
\[
I_{32} \leq C \| u^{n} \|_{0, \infty; \Omega} \| \zeta_{c}^{n} \|_{\Omega} \| \nabla \zeta_{c}^{n} \|_{\Omega} \leq C h^{k_{c}} \| c^{n} \|_{k_{c}; \Omega} \| \zeta_{c}^{n} \|_{c}. \tag{104}
\]

By Hölder’s inequality and using equation (22),
\[
I_{33} \leq \sum_{K \in T} \| \xi_{c}^{n} \|_{0, \infty; \partial K} \| u_{h}^{n} - u^{n} \|_{\partial K} \| \zeta_{c}^{n} - \tilde{\zeta}_{c}^{n} \|_{\partial K} \leq C \left( \sum_{K \in T} h_{K} \| u_{h}^{n} - u^{n} \|_{\partial K}^{2} \right)^{1/2} \| \zeta_{c}^{n} \|_{c}. \tag{105}
\]

Using Hölder’s inequality and this time employing equations (21), (22) and (53),
\[
I_{34} = \sum_{K \in T} \| \xi_{c}^{n} \|_{\partial K} \| u^{n} \|_{0, \infty; \partial K} \| \zeta_{c}^{n} - \tilde{\zeta}_{c}^{n} \|_{\partial K} \leq C \left( \sum_{K \in T} \left( \| \xi_{c}^{n} \|_{K}^{2} + h_{K}^{2} \| \zeta_{c}^{n} \|_{1,K}^{2} \right) \right)^{1/2} \left( \sum_{K \in T} h_{K}^{-1} \| \zeta_{c}^{n} - \tilde{\zeta}_{c}^{n} \|_{\partial K}^{2} \right)^{1/2} \leq C h^{k_{c}} \| c^{n} \|_{k_{c}; \Omega} \| \zeta_{c}^{n} \|_{c}. \tag{106}
\]

Putting equations (102)–(106) together, using Young’s inequality and equation (83), we find
\[
I_{3} \leq C \left( \| c \|_{0, \infty; \Omega} \| u_{h}^{n} - u^{n} \|_{\Omega}^{2} + \sum_{K \in T} h_{K} \| u_{h}^{n} - u^{n} \|_{\partial K}^{2} + h^{2k_{c}} \| c^{n} \|_{k_{c}; \Omega}^{2} \right) + \epsilon \| \zeta_{c}^{n} \|_{c, u_{h}^{n}}. \tag{107}
\]

Since $c = \gamma(c)$ on $\partial K$,
\[
I_{4} = - \sum_{K \in T} \int_{K} c^{n} (u^{n} - u_{h}^{n}) \cdot \nabla \zeta_{c}^{n} \, dx + \sum_{K \in T} \int_{\partial K} c^{n} (u^{n} - u_{h}^{n}) \cdot n(\zeta_{c}^{n} - \tilde{\zeta}_{c}^{n}) \, ds =: I_{41} + I_{42}. \tag{108}
\]

Hölder’s and Young’s inequalities give
\[
I_{41} \leq \| u^{n} - u_{h}^{n} \|_{\Omega} \| c^{n} \|_{0, \infty; \Omega} \| \zeta_{c}^{n} \|_{c} \leq C \| c^{n} \|_{0, \infty; \Omega} \| u^{n} - u_{h}^{n} \|_{\Omega}^{2} + \epsilon \| \zeta_{c}^{n} \|_{c}^{2}, \tag{109}
\]

and
\[
I_{42} \leq C \| c^{n} \|_{0, \infty; \Omega} \left( \sum_{K \in T_{h}} h_{K} \| u^{n} - u_{h}^{n} \|_{\partial K}^{2} \right)^{1/2} \| \zeta_{c}^{n} \|_{c} \leq C \| c^{n} \|_{0, \infty; \Omega} \sum_{K \in T_{h}} h_{K} \| u^{n} - u_{h}^{n} \|_{\partial K}^{2} + \epsilon \| \zeta_{c}^{n} \|_{c}^{2}. \tag{110}
\]
Collecting equations (108)–(110) and using equation (83) leads to

\[I_4 \leq 2c||\xi^n||_{c,u^n}^2 + C||e^n||_{0,\infty,\Omega}^2 \left(\|u^n - u_h^n\|^2 + \sum_{K \in T_h} h_K \|u^n - u_h^n\|^2_{\partial K}\right).\]

Since \(I_c = \bar{I}_c\) on \(\Gamma_0\), and \(\tilde{D}(u^n) - \tilde{D}(u_h^n) = 0\) in \(\Omega^s\),

\[I_5 = \sum_{K \in T^d} \int_K [D(u^n) - D(u_h^n)] \nabla Ic^n \cdot \nabla \zeta^n dx - \sum_{K \in T^d} \int_{\partial K} \left((D(u^n) - D(u_h^n)) \nabla Ic^n\right) \cdot n (\zeta^n - \bar{\zeta^n}) ds =: I_{51} + I_{52}.\]

Using the Lipschitz property of \(D\) equation (4b), Hölder’s and Young’s inequalities,

\[I_{51} \leq D_L \|u^n - u_h^n\|_{\Omega^d}\|\nabla Ic^n\|_{0,\infty,\Omega^d}\|\nabla \zeta^n\|_{\Omega^d} \leq C||e^n||_{1,\infty,\Omega^d} \|u^n - u_h^n\|^2_{\Omega^d} + \epsilon\|\zeta^n\|^2_\Omega.\]

Similarly, by equation (22),

\[I_{52} \leq \sum_{K \in T^d} D_L \|u^n - u_h^n\|_{\partial K}\|\nabla Ic^n\|_{0,\infty,\partial K}\|\zeta^n - \bar{\zeta^n}\|_{\partial K} \leq C||e^n||_{1,\infty,\Omega^d} \left(\sum_{K \in T^d} h_K \|u^n - u_h^n\|^2_{\partial K}\right)^{1/2} \|\zeta^n\|_c.\]

Therefore, substituting equations (112) and (113) in equation (111) and using equation (83) we obtain

\[I_5 \leq 2c||\xi^n||_{c,u^n}^2 + C||e^n||_{1,\infty,\Omega^d} \left(\sum_{K \in T^d} h_K \|u^n - u_h^n\|^2_{\partial K} + \|u^n - u_h^n\|^2_{\Omega^d}\right).\]

Since \(\xi^n = \bar{\zeta^n}\) on \(\partial K\),

\[I_6 = B_h^n(u^n; \xi^n, \zeta^n) = \sum_{K \in T} \int_K \tilde{D}(u^n) \nabla \xi^n \cdot \nabla \zeta^n dx - \sum_{K \in T} \int_{\partial K} \left(\tilde{D}(u^n) \nabla \zeta^n\right) \cdot n (\zeta^n - \bar{\zeta^n}) ds =: I_{61} + I_{62}.\]

By Hölder’s inequality and equation (84),

\[I_{61} \leq \sum_{K \in T} \left\|\tilde{D}(u^n)^{1/2} \nabla \xi^n\right\|_K \left\|\tilde{D}(u_h^n)^{1/2} \nabla \xi^n\right\|_K \leq \tilde{D}_{\max} \sum_{K \in T} \left\|(1 + \chi_d |u^n|)^{1/2} \nabla \xi^n\right\|_K \left\|(1 + \chi_d |u_h^n|)^{1/2} \nabla \xi^n\right\|_K \leq \tilde{D}_{\max} \tilde{D}_{\min}^{1/2} \sum_{K \in T} \left(1 + \chi_d u^n + L_{\infty}(K)\right)^{1/2} \left\|\nabla \xi^n\right\|_K \left\|\tilde{D}(u_h^n)^{1/2} \nabla \zeta^n\right\|_K \leq C \left(1 + \|u^n\|_{L_{\infty}(\Omega)}\right) \sum_{K \in T} \left\|\nabla \xi^n\right\|_K^2 + \epsilon\|\zeta^n\|^2_\Omega.\]
Finally, using H"older's inequality and equation (53), and Young's inequality,
\[
I_{62} = \sum_{K \in T} \int_{\partial K} \left[ \tilde{D}(u^n)^{1/2} \nabla \xi^n_c \cdot \tilde{D}(u^n)^{1/2} n \left( \zeta^n - \tilde{\zeta}^n_c \right) \right] ds
\leq \sum_{K \in T} \left\| \tilde{D}(u^n)^{1/2} \nabla \xi^n_c \right\|_{\partial K} \left\| \tilde{D}(u^n)^{1/2} n \left( \zeta^n - \tilde{\zeta}^n_c \right) \right\|_{\partial K}
\leq \tilde{D}_{\max} \sum_{K \in T} \left( 1 + \|\chi_d u^n\|_{L^\infty(\partial K)} \right) \left\| \nabla \xi^n_c \right\|_{\partial K} \left( 1 + \|\chi_d u^n\|_{L^\infty(\partial K)} \right) \left( 1 + \|\chi_d u^n\|_{L^\infty(\partial K)} \right) \left\| \nabla \xi^n_c \right\|_{\partial K}
\leq \tilde{D}_{\max} \sum_{K \in T} \left( 1 + \|\chi_d u^n\|_{L^\infty(\partial K)} \right) \left\| \nabla \xi^n_c \right\|_{\partial K} \left( 1 + \|\chi_d u^n\|_{L^\infty(\partial K)} \right) \left( 1 + \|\chi_d u^n\|_{L^\infty(\partial K)} \right) \left\| \nabla \xi^n_c \right\|_{\partial K}
\leq C \sum_{K \in T} \left( 1 + \|\chi_d u^n\|_{L^\infty(\partial K)} \right) h_K \left\| \nabla \xi^n_c \right\|_{\partial K} + \epsilon \left\| \xi^n_c \right\|^2_{c,u^n_H},
\tag{117}
\]

Hence, the combination of equations (21), (53) and (115)–(117) results in:
\[
I_6 \leq C \sum_{K \in T} \left( \left\| \nabla \xi^n_c \right\|^2_{K} + h_K \left\| \nabla \xi^n_c \right\|^2_{\partial K} \right) + 2\epsilon \left\| \xi^n_c \right\|^2_{c,u^n_H}
\leq C \sum_{K \in T} \left( \left\| \nabla \xi^n_c \right\|^2_{K} + \left\| \nabla \xi^n_c \right\|^2_{K} + h_K \left\| \xi^n_c \right\|^2_{2,K} \right) + 2\epsilon \left\| \xi^n_c \right\|^2_{c,u^n_H}
\leq C h^{2k_e} \left\| c^n \right\|^2_{k_e+1,\Omega} + 2\epsilon \left\| \xi^n_c \right\|^2_{c,u^n_H},
\]

where $C$ depends on $\|u^n\|_{W^{1,\infty}(\Omega)}$, $d$, $D_{\max}$, and $D_{\min}$. By equation (19a), the stability of the $L^2$-projection $\Pi_Q$ and H"older’s inequality,
\[
I_7 = -\frac{1}{2} \sum_{K \in T_d} \int_{K} \Pi_Q \left( g^n_p - \tilde{g}^n_p \right) \left( \zeta^n_c \right)^2 dx \leq \frac{1}{2} \phi_s \left\| g^n_p - \tilde{g}^n_p \right\|_{0,\infty,\Omega} \left( \phi_s \left\| \zeta^n_c \right\|^2_{\Omega} \right).
\]

Finally, using H"older’s inequality, equation (53), and Young’s inequality,
\[
I_8 \leq \left\| g^n_p \right\|_{0,\infty,\Omega} \left\| \xi^n_c \right\|_{\Omega} \left\| \zeta^n_c \right\|_{\Omega} \leq C h^{k_e} \left\| g^n_p \right\|_{0,\infty,\Omega} \left\| c^n \right\|_{k_e,\Omega} \left\| \zeta^n_c \right\|_{\Omega}
\leq C h^{2k_e} \phi_s^{-1} \left\| g^n_p \right\|^2_{0,\infty,\Omega} \left\| c^n \right\|^2_{k_e,\Omega} + \gamma \left( \phi_s \left\| \zeta^n_c \right\|^2_{\Omega} \right).
\]

Collecting all bounds, choosing $\epsilon = C_{tr}/14$ ($C_{tr}$ is the coercivity constant), $\gamma = 1/6$, and recalling that $g^n_p \geq 0$, we find:
\[
\begin{align*}
\frac{\phi_s}{2\Delta t} \left( \left\| \zeta^n_c \right\|^2_{\Omega} - \left\| \zeta^{n-1}_c \right\|^2_{\Omega} \right) + \frac{C_{tr}}{2} \left\| \zeta^n_c \right\|^2_{c,u^n_H} & \leq C \left( h^{2k_e} \left\| \partial_t c \right\|^2_{L^2(I_{n-1},I^n; H^k(\Omega))} + \Delta t \left\| \partial_{tt} c \right\|^2_{L^2(I_{n-1},I^n; L^2(\Omega))} \right. \\
& \left. + \left( 1 + \left\| c^n \right\|^2_{0,\infty,\Omega} \right) \left( \left\| u^n - u^n_H \right\|^2_{\Omega} + \sum_{K \in T} h_K \left\| u^n - u^n_H \right\|^2_{\delta K} \right) \right).
\end{align*}
\]
Corollary 6.14. Suppose that the regularity assumptions of Theorem 6.13 hold. Then

\[ ||c^n_h - c^n||^2_{\Omega} + \Delta t \sum_{m=1}^{n} ||c_m^n - c_m||^2_{c} \leq C((\Delta t)^2 + h^{2k_j} + h^{2k_c}), \quad \forall n \geq 1. \]  

(119)

We end this section with the following corollary, proving assumption equation (61) in Theorem 6.5 as well as the a priori error estimate for the concentration.

Corollary 6.14. Suppose that the regularity assumptions of Theorem 6.13 hold. Then

\[ ||c^n_h - c^n||^2_{\Omega} + \Delta t \sum_{m=1}^{n} ||c_m^n - c_m||^2_{c} \leq C((\Delta t)^2 + h^{2k_j} + h^{2k_c}), \quad \forall n \geq 1. \]  

(119)
\[ \|c^n_k - c^n\|_\Omega^2 \leq 2 \left( \|\zeta^n_c\|_\Omega^2 + \sum_{K \in T} \|\xi^n_c\|_K^2 \right) \leq C \left( h^{2k_f} + h^{2k_c} + (\Delta t)^2 \right) + Ch^{2k_c} \|c^n\|_{k_c, \Omega}^2. \] (120)

Similarly, since \( \xi^m_c - \tilde{\xi}^m_c = 0 \) on \( \partial K \) for any \( K \in T \),

\[ \|c^m_k - c^m\|_c^2 \leq 2 \left( \|\zeta^m_c\|_c^2 + \|\xi^m_c\|_c^2 \right) = 2 \left( \|\zeta^m_c\|_c^2 + \sum_{K \in T} \|\nabla \xi^m_c\|_K^2 \right) \leq 2\|\zeta^m_c\|_c^2 + C \sum_{K \in T} h^{2k_c} \|c^m\|_{k_c+1, \Omega}^2. \]

Summing this from \( m = 1 \) to \( n \) and multiplying by \( \Delta t \), we get

\[ \Delta t \sum_{m=1}^n \|c^m_k - c^m\|_c^2 \leq 2\Delta t \sum_{m=1}^n \|\zeta^m_c\|_c^2 + C\Delta t \sum_{m=1}^n h^{2k_c} \|c^m\|_{k_c+1, \Omega}^2. \] (121)

Equation (119) now follows from equations (120), (121) and (101).

\[ \Box \]

7. Numerical Examples

Algorithm 1 is implemented in the higher-order finite element library Netgen/NGSolve [47,48]. In all numerical examples we choose \( \Omega = [0, 1]^2 \) with subregions \( \Omega^d = [0, 1] \times [0, 0.5] \) and \( \Omega^s = [0, 1] \times [0.5, 1] \). We furthermore choose the penalty parameters as \( \beta_s = 6k_f^2 \) and \( \beta_{tr} = 6k_c^2 \) ([2], Lem. 1, Sect. 5), where \( k_c = k_f - 1 \) (see Rem. 3.1).

7.1. Example 1

We first consider the constant coefficient case, i.e., the time-dependent one-way coupled problem in which the numerical solution to the Stokes/Darcy problem is unaffected by the concentration. Let \( \alpha = \frac{1}{2} \left( 1 + 4\pi^2 \right) \sqrt{\kappa} \), \( \tilde{D} = D = \begin{bmatrix} 0.01 & 0.005 \\ 0.005 & 0.02 \end{bmatrix} \) on \( \Omega \), and \( T = 0.1 \). The source terms and boundary conditions for the Stokes/Darcy-transport problem are chosen such that the exact solution is given by

\[ u^s = \left( \frac{-1}{2\pi^2} \sin(\pi x_1 + t)e^{(x_2 + t)^2/2}, \frac{1}{\pi} \cos(\pi x_1 + t)e^{(x_2 + t)^2/2} \right)^T \] (122a)

\[ u^d = \left( -2 \sin(\pi x_1 + t)e^{(x_2 + t)^2/2}, \frac{1}{\pi} \cos(\pi x_1 + t)e^{(x_2 + t)^2/2} \right)^T, \] (122b)

\[ p^s = \kappa \frac{\mu}{\kappa} \cos(\pi x_1 + t)e^{(x_2 + t)^2/2}; \] (122c)

\[ p^d = -\frac{2}{\kappa} \cos(\pi x_1 + t)e^{(x_2 + t)^2/2}; \] (122d)

\[ c = \sin(2\pi (x_1 - t)) \cos(2\pi (x_2 - t)). \] (122e)

Note that this solution satisfies all the interface conditions and that \( \nabla \cdot u^s = 0 \) in \( \Omega^s \).

We present our numerical results for a wide range of values for \( \kappa \) and \( \mu \): \( \kappa = \mu = 1; \ \kappa = 10^3, \ \mu = 10^{-6}; \ \kappa = 1, \ \mu = 10^{-6}; \) and \( \kappa = 10^{-3}, \ \mu = 10^{-6}. \) Since we are primarily interested in the spatial error, to minimize the temporal error as much as possible, we use the third order backward differentiation formulae (BDF3) as time stepping method even though the sequential Algorithm 1 is only first order accurate in time. We choose
The diffusion dispersion tensor in $\Omega$ and the viscosity according to equation (122). We define the diffusion dispersion tensor and the dependence of the viscosity on the concentration solution. The source terms for the Stokes/Darcy-transport problem (1) are chosen such that the exact solution is the diffusion dispersion tensor and the viscosity is defined as the quarter-power mixing rule [38] with where

$$\nu = 0.9, \mu_1 = 1.3.$$
From the rightmost columns in Tables 7 and 8 we observe that the discretization is exactly mass conserving. Indeed, from Table 9 we observe that the rate of convergence reduces from the concentration approximation. Therefore, for the velocity we expect an order of at least 1 to achieve compatibility and is consistent with our a priori estimates in the energy norm in Corollary 6.7 which imply that the rate of convergence of the velocity approximation is polluted by the concentration approximation. We use the rate of convergence estimates in the energy norm with order 1/4, 1/8, 1/16, 1/32, for the test case in Section 7.1 using $k_f = 2$, $k_c = k_f - 1$, and BDF3 time stepping with $\Delta t = 0.1h^2/3$.

<table>
<thead>
<tr>
<th>$h$</th>
<th>Dofs</th>
<th>$|u_h - u|_{\Omega^d}$ Rate</th>
<th>$|p_h - p|_{\Omega^d}$ Rate</th>
<th>$|\Pi_Q(\nabla \cdot (u_h - u))|_{\Omega^d}$ Rate</th>
</tr>
</thead>
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<td>8.4e-05</td>
</tr>
<tr>
<td>$\kappa = 10^3, \mu = 10^{-6}$</td>
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<td>1/4</td>
<td>745</td>
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We use $k_f = 3$, $k_c = 2$, and BDF3 time stepping with $\Delta t = 0.1h^3/4$. We present numerical results for $\kappa = 10^3, 1, 10^{-3}$. We observe from Tables 7 and 8 that when $\mu$ is changed from a constant to a concentration dependent function the rate of convergence reduces from $k_f + 1$ to a value between $k_f$ and $k_f + 1$. This is due to our choice $k_c = k_f - 1$ to achieve compatibility and is consistent with our a priori estimates in the energy norm in Corollary 6.7 which imply that the rate of convergence of the velocity approximation is polluted by the concentration approximation. Indeed, from Table 9 we observe that $c_h$ converges in the $L^2$-norm with order $k_f$. Therefore, for the velocity we expect an order of at least $k_f - 1$ in the energy norm and $k_f$ in the $L^2$-norm.

From the rightmost columns in Tables 7 and 8 we observe that the discretization is exactly mass conserving.
Table 4. Errors and rates of convergence at final time $T = 0.1$ for $u_h$ and $p_h$ in the Stokes region $\Omega^s$ for the test case in Section 7.1 using $k_f = 3$, $k_c = k_f - 1$, and BDF3 time stepping with $\Delta t = 0.1h^3/4$.

<table>
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<tr>
<th>$h$</th>
<th>Dofs</th>
<th>$|u_h - u|_{\Omega^s}$ Rate</th>
<th>$|p_h - p|_{\Omega^s}$ Rate</th>
<th>$|\nabla \cdot u_h|_{\Omega^s}$ Rate</th>
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</tr>
<tr>
<td></td>
<td></td>
<td>$\kappa = 10^3, \mu = 10^{-6}$</td>
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</tbody>
</table>

Figure 2. The permeability field in $\Omega^d = [0, 1] \times [0, 0.5]$ defined by equation (124).

7.3. Example 3

In this final example, we simulate a more realistic problem similar to Section 6.2 of [17] in which the permeability field in the Darcy region is highly heterogeneous. For this, let the boundary of the Stokes region be partitioned as $\Gamma^s = \Gamma^s_1 \cup \Gamma^s_2 \cup \Gamma^s_3$ where

$$\Gamma^s_1 := \{ x \in \Gamma^s : x_1 = 0 \}, \quad \Gamma^s_2 := \{ x \in \Gamma^s : x_1 = 1 \}, \quad \Gamma^s_3 := \{ x \in \Gamma^s : x_2 = 1 \}.$$  

Similarly, let $\Gamma^d = \Gamma^d_1 \cup \Gamma^d_2$ where

$$\Gamma^d_1 := \{ x \in \Gamma^d : x_1 = 0 \text{ or } x_1 = 1 \}, \quad \Gamma^d_2 := \{ x \in \Gamma^d : x_2 = 0 \}.$$  

We impose the following boundary conditions:

$$u = \left( x_2(3/2 - x_2)/5, 0 \right) \quad \text{on } \Gamma^s_1.$$
\[ \tau \] denotes the unit tangent vector on \( \Gamma \).

Table 5. Errors and rates of convergence at final time \( T = 0.1 \) for \( u_h \) and \( p_h \) in the Darcy region \( \Omega^d \) for the test case in Section 7.1 using \( k_f = 3 \), \( k_c = k_f - 1 \), and BDF3 time stepping with \( \Delta t = 0.1 h^3/4 \).

<table>
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<tr>
<th>( h )</th>
<th>Dofs</th>
<th>( | u_h - u |_{\Omega^d} )</th>
<th>Rate</th>
<th>( | p_h - p |_{\Omega^d} )</th>
<th>Rate</th>
<th>( \left| \Pi_Q \left( \nabla \cdot (u_h - u) \right) \right|_{\Omega^d} )</th>
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</table>

\( \kappa = 1, \mu = 10^{10} \)

<table>
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<tr>
<th>( h )</th>
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<th>( | u_h - u |_{\Omega^d} )</th>
<th>Rate</th>
<th>( | p_h - p |_{\Omega^d} )</th>
<th>Rate</th>
<th>( \left| \Pi_Q \left( \nabla \cdot (u_h - u) \right) \right|_{\Omega^d} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/4</td>
<td>1161</td>
<td>1.2e-04</td>
<td>-</td>
<td>5.4e-07</td>
<td>-</td>
<td>4.1e-12</td>
</tr>
<tr>
<td>1/8</td>
<td>5993</td>
<td>3.6e-06</td>
<td>5.1</td>
<td>4.0e-08</td>
<td>3.8</td>
<td>8.2e-13</td>
</tr>
<tr>
<td>1/16</td>
<td>22081</td>
<td>2.2e-07</td>
<td>4.0</td>
<td>5.1e-09</td>
<td>3.0</td>
<td>3.1e-12</td>
</tr>
<tr>
<td>1/32</td>
<td>90241</td>
<td>1.3e-08</td>
<td>4.1</td>
<td>6.1e-10</td>
<td>3.1</td>
<td>1.2e-11</td>
</tr>
</tbody>
</table>

\( \kappa = 1, \mu = 10^{10} \)

<table>
<thead>
<tr>
<th>( h )</th>
<th>Dofs</th>
<th>( | u_h - u |_{\Omega^d} )</th>
<th>Rate</th>
<th>( | p_h - p |_{\Omega^d} )</th>
<th>Rate</th>
<th>( \left| \Pi_Q \left( \nabla \cdot (u_h - u) \right) \right|_{\Omega^d} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/4</td>
<td>1161</td>
<td>1.2e-04</td>
<td>-</td>
<td>5.4e-01</td>
<td>-</td>
<td>4.1e-12</td>
</tr>
<tr>
<td>1/8</td>
<td>5993</td>
<td>3.6e-06</td>
<td>5.1</td>
<td>4.0e-02</td>
<td>3.8</td>
<td>8.9e-13</td>
</tr>
<tr>
<td>1/16</td>
<td>22081</td>
<td>2.2e-07</td>
<td>4.0</td>
<td>5.1e-03</td>
<td>3.0</td>
<td>3.3e-12</td>
</tr>
<tr>
<td>1/32</td>
<td>90241</td>
<td>1.3e-08</td>
<td>4.1</td>
<td>6.1e-04</td>
<td>3.1</td>
<td>1.3e-11</td>
</tr>
</tbody>
</table>

\( \kappa = 10^3, \mu = 10^{-6} \)

Table 6. Errors and rates of convergence at final time \( T = 0.1 \) for \( c_h \) in \( \Omega \), on a mesh with \( h = 1/4, 1/8, 1/16, 1/32 \), for the test case in Section 7.1 using \( k_f = 3 \), \( k_c = k_f - 1 \), and BDF3 time stepping with \( \Delta t = 0.1 h^3/4 \).

\( -2\mu \varepsilon(u) + p\mathbf{1} \) \( n = 0 \) on \( \Gamma_2^s \),
\( u \cdot n = 0 \) and \( -2\mu \varepsilon(u) \cdot \tau = 0 \) on \( \Gamma_3^s \),
\( u \cdot n = 0 \) on \( \Gamma_1^d \),
\( p = -0.05 \) on \( \Gamma_2^d \),

where \( \tau \) denotes the unit tangent vector on \( \Gamma_3^s \). The first boundary condition on the left boundary \( \Gamma_1^s \) of \( \Omega^s \) imposes a parabolic velocity profile. We set the permeability to

\[
\kappa = 700 \left( 1 + 0.5 \left( \sin(10\pi x_1) \cos(20\pi x_2^2) + \cos^2(6.4\pi x_1) \sin(9.2\pi x_2) \right) \right) + 100,
\]

(124)
The tensor is defined as source/sink terms are set to zero. In the Darcy region \( \Omega^d \). The other parameters are set as Table 7.

<table>
<thead>
<tr>
<th>( h )</th>
<th>Dofs</th>
<th>( |u_h - u|_{\Omega^s} ) Rate</th>
<th>( |p_h - p|_{\Omega^s} ) Rate</th>
<th>( |\nabla \cdot u_h|_{\Omega^s} ) Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \kappa = 10^2 )</td>
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</tr>
<tr>
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<td>5993</td>
<td>3.7e−06</td>
<td>4.6</td>
<td>3.9e−04</td>
</tr>
<tr>
<td>1/16</td>
<td>22 081</td>
<td>2.4e−07</td>
<td>4.0</td>
<td>4.1e−05</td>
</tr>
<tr>
<td>1/32</td>
<td>90 241</td>
<td>2.1e−08</td>
<td>3.5</td>
<td>4.7e−06</td>
</tr>
<tr>
<td>( \kappa = 1 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/4</td>
<td>1161</td>
<td>9.6e−05</td>
<td>8.1e−03</td>
<td>–</td>
</tr>
<tr>
<td>1/8</td>
<td>5993</td>
<td>3.8e−06</td>
<td>4.7</td>
<td>3.9e−04</td>
</tr>
<tr>
<td>1/16</td>
<td>22 081</td>
<td>2.3e−07</td>
<td>4.1</td>
<td>4.1e−05</td>
</tr>
<tr>
<td>1/32</td>
<td>90 241</td>
<td>1.8e−08</td>
<td>3.6</td>
<td>4.7e−06</td>
</tr>
<tr>
<td>( \kappa = 10^{-3} )</td>
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<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/4</td>
<td>1161</td>
<td>1.8e−03</td>
<td>–</td>
<td>6.9e−01</td>
</tr>
<tr>
<td>1/8</td>
<td>5993</td>
<td>5.0e−05</td>
<td>5.1</td>
<td>5.4e−02</td>
</tr>
<tr>
<td>1/16</td>
<td>22 081</td>
<td>3.6e−06</td>
<td>3.8</td>
<td>6.3e−03</td>
</tr>
<tr>
<td>1/32</td>
<td>90 241</td>
<td>4.0e−07</td>
<td>3.2</td>
<td>7.7e−04</td>
</tr>
</tbody>
</table>

Table 8. Errors and rates of convergence at final time \( T = 0.1 \) for \( u_h \) and \( p_h \) in the Darcy region \( \Omega^d \) for the test case in Section 7.2 using \( k_f = 3, k_c = k_f - 1 \), and BDF3 time stepping with \( \Delta t = 0.1 h^3/4 \).

<table>
<thead>
<tr>
<th>( h )</th>
<th>Dofs</th>
<th>( |u_h - u|_{\Omega^d} ) Rate</th>
<th>( |p_h - p|_{\Omega^d} ) Rate</th>
<th>( |\Pi_Q(V \cdot (u_h - u))|_{\Omega^d} ) Rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \kappa = 10^2 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1/4</td>
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<td>3.5e−03</td>
<td>–</td>
<td>6.6e−07</td>
</tr>
<tr>
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<td>3.4</td>
<td>4.2e−08</td>
</tr>
<tr>
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<td>22 081</td>
<td>3.8e−05</td>
<td>3.1</td>
<td>5.3e−09</td>
</tr>
<tr>
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<td>90 241</td>
<td>4.4e−06</td>
<td>3.1</td>
<td>6.3e−10</td>
</tr>
<tr>
<td>( \kappa = 1 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>1/4</td>
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<td>3.5e−03</td>
<td>–</td>
<td>6.6e−04</td>
</tr>
<tr>
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<td>22 081</td>
<td>3.8e−05</td>
<td>3.1</td>
<td>5.3e−06</td>
</tr>
<tr>
<td>1/32</td>
<td>90 241</td>
<td>4.4e−06</td>
<td>3.1</td>
<td>6.3e−07</td>
</tr>
<tr>
<td>( \kappa = 10^{-3} )</td>
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<td>1/4</td>
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<td>3.9e−03</td>
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<td>6.0e−01</td>
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<tr>
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<td>5993</td>
<td>3.4e−04</td>
<td>3.5</td>
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<td>90 241</td>
<td>4.4e−06</td>
<td>3.1</td>
<td>6.2e−04</td>
</tr>
</tbody>
</table>

A plot of which is given in Figure 2. The viscosity is defined by the quarter-power mixing rule as in equation (123). The other parameters are set as \( \mu = 0.1, \alpha = 0.5, k_f = 3, h = 1/80, \Delta t = 10^{-3}, T = 15 \), and the source/sink terms are set to zero. In the Darcy region \( \Omega^d \) the porosity is set to \( \phi = 0.4 \). The dispersion/diffusion tensor is defined as

\[
\tilde{D}(u) = \begin{cases} 
\delta I, & \text{in } \Omega^s; \\
\phi d_m I + d_t |u| T + d_t |u| (I - uu^T)/|u|^2, & \text{in } \Omega^d,
\end{cases}
\]
Table 9. Errors and rates of convergence at final time $T = 0.1$ for $c_h$ in $\Omega$, on a mesh with $h = 1/4, 1/8, 1/16, 1/32$, for the test case in Section 7.2 using $k_f = 3$, $k_c = k_f - 1$, and BDF3 time stepping with $\Delta t = 0.1h^3/4$.

<table>
<thead>
<tr>
<th>Dofs</th>
<th>$\kappa = 10^3$</th>
<th>$\kappa = 1$</th>
<th>$\kappa = 10^{-3}$</th>
</tr>
</thead>
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<tr>
<td>$</td>
<td></td>
<td>c - c_h</td>
<td></td>
</tr>
<tr>
<td>$</td>
<td>\text{Rate}</td>
<td>$</td>
<td>$</td>
</tr>
<tr>
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<td>1.4e-02</td>
<td>1.4e-02</td>
</tr>
<tr>
<td>1644</td>
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<td>1.2e-03</td>
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<tr>
<td>6144</td>
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<td>3.2</td>
<td>1.3e-04</td>
</tr>
<tr>
<td>24846</td>
<td>1.3e-05</td>
<td>3.3</td>
<td>1.3e-05</td>
</tr>
</tbody>
</table>

Figure 3. The velocity field after one time step (left) and at the final time (right) for the example in Section 7.3. The color represents the magnitude of the velocity.

We compute the solution using BDF3 time stepping. Figure 3 shows the computed velocity field after one time step and at the final time. In the Darcy region $\Omega^d$, the flow field avoids areas with low permeability as expected. Figure 4 shows the pressure contours at various times which demonstrates the effect of the concentration on the pressure around the plume of contaminants, especially in the Stokes region. Figure 5 presents the plume of contaminant spreading through the surface water region and infiltrating the porous medium. We plot the solution 6 different instances in time. The contaminant plume stays compact while in the surface water region. Once it reaches the subsurface region it spreads out following a path dictated by the heterogeneous permeability structure of the porous medium.
Figure 4. Pressure contours at times $t = \Delta t, 3, 6, 9, 12, 15$ for the example in Section 7.3. The color represents the concentration values. (A) $t = \Delta t$. (B) $t = 3$. (C) $t = 6$. (D) $t = 9$. (E) $t = 12$. (F) $t = 15$. 
Figure 5. The plume of contaminant at times $t = \Delta t, 3, 6, 9, 12, 15$ for the example in Section 7.3. The color represents the concentration values. (A) $t = \Delta t$. (B) $t = 3$. (C) $t = 6$. (D) $t = 9$. (E) $t = 12$. (F) $t = 15$. 
8. Conclusions

In this paper, we introduced and analyzed a fully discrete sequential method for the fully coupled Stokes/Darcy-transport problem. The spatial discretization uses the HDG method which is higher-order accurate, strongly mass conservative, and compatible. We remark that the analysis also easily extends to the EDG-HDG method considered in [16, 17]. The sequential method discussed in the article linearizes the problem by time-lagging the concentration and decoupling the Stokes/Darcy and transport subproblems. We proved well-posedness and obtained a priori estimates in the energy norm. Finally, we presented numerical results demonstrating mass conservation and robustness with respect to varying permeability and optimal convergence in the $L^2$-norm for one-way coupling.

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References


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