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CONVERGENCE ANALYSIS OF A LOCAL DISCONTINUOUS GALERKIN APPROXIMATION FOR NONLINEAR SYSTEMS WITH BALANCED ORLICZ-STRUCTURE

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Abstract. In this paper, we investigate a Local Discontinuous Galerkin (LDG) approximation for systems with balanced Orlicz-structure. We propose a new numerical flux, which yields optimal convergence rates for linear ansatz functions. In particular, our approach yields a unified treatment for problems with (p, δ)-structure for arbitrary $p \in (1, \infty)$ and $\delta \geq 0$.

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1. Introduction

We consider the numerical approximation of the nonlinear system

$$\begin{aligned} -\mathrm{div}\boldsymbol{\mathcal{A}}(\nabla \mathbf{u}) &= \mathbf{g} - \mathrm{div}\mathbf{G} && \text{in } \Omega, \\ \mathbf{u} &= \mathbf{u}_D && \text{on } \Gamma_D, \\ \boldsymbol{\mathcal{A}}(\nabla \mathbf{u})\mathbf{n} &= \mathbf{a}_N && \text{on } \Gamma_N, \end{aligned} \tag{1.1}$$

using a Local Discontinuous Galerkin (LDG) scheme. More precisely, for given data $\mathbf{g}$, $\mathbf{G}$, $\mathbf{u}_D$ and $\mathbf{a}_N$, we seek a vector field $\mathbf{u} = (u_1, \ldots, u_d)^\top : \Omega \to \mathbb{R}^d$ solving (1.1). Here, $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, is a polyhedral, bounded domain with Lipschitz continuous boundary $\partial\Omega$, which is disjointly divided into a Dirichlet part Γ_D, where we assume that $|\Gamma_D| > 0$, and a Neumann part Γ_N, *i.e.*, $\Gamma_D \cup \Gamma_N = \partial\Omega$ and $\Gamma_D \cap \Gamma_N = \emptyset$. By $\mathbf{n} : \partial\Omega \to \mathbb{S}^{n-1}$, we denote the unit normal vector field to $\partial\Omega$ pointing outward. The problem (1.1) is the prototype for many problems in applications, *e.g.*, if non-linear constitutive relations are used in elasticity or fluid mechanics. We refrain from using a concrete example for $\boldsymbol{\mathcal{A}}$ in (1.1), but consider the rather general case that $\boldsymbol{\mathcal{A}}\colon \mathbb{R}^{d\times n} \to \mathbb{R}^{d\times n}$ is a nonlinear operator having φ-structure for some balanced N-function φ (*cf.* Sect. 2.2). The relevant example falling into this class is

$$\boldsymbol{\mathcal{A}}(\nabla \mathbf{u}) = \frac{\varphi'(|\nabla \mathbf{u}|)}{|\nabla \mathbf{u}|}\nabla \mathbf{u}.$$

In the spacial case $\varphi(t) = \frac{1}{p}t^p$ problem (1.1) ist just the p-Laplace problem with $\boldsymbol{\mathcal{A}}(\nabla \mathbf{u}) = |\nabla \mathbf{u}|^{p-2}\nabla \mathbf{u}$, which for $p = 2$ reduces to the Laplace problem.

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Introducing the additional unknowns $\mathbf{L} : \Omega \to \mathbb{R}^{d\times n}$ and $\mathbf{A} : \Omega \to \mathbb{R}^{d\times n}$, the system (1.1) can be re-written as a "first order" system, *i.e.*,

$$\begin{aligned} \mathbf{L} = \nabla \mathbf{u}, \quad \mathbf{A} = \boldsymbol{\mathcal{A}}(\mathbf{L}), \quad -\mathrm{div}\mathbf{A} = \mathbf{g} - \mathrm{div}\mathbf{G} \qquad & \text{in } \Omega, \\ \mathbf{u} = \mathbf{u}_D \qquad & \text{on } \Gamma_D, \\ \mathbf{A}\mathbf{n} = \mathbf{a}_N \qquad & \text{on } \Gamma_N. \end{aligned} \tag{1.2}$$

Discontinuous Galerkin (DG) methods for elliptic problems have been introduced in the late 90's. They are by now well-understood and rigorously analysed in the context of linear elliptic problems (*cf.* [3] for the Poisson problem). In contrast to this, only few papers treat p-Laplace type problems with DG methods (*cf.* [12–14, 19, 27, 29]), or non-conforming methods (*cf.* [6]). There exists even fewer numerical investigations of problems with Orlicz-structure. Steady problems with Orlicz-structure are treated, using finite element methods (FEM) in [11, 17, 18][1] and non-conforming methods in [6]. Unsteady problems are investigated in [23, 31]. To the best of the author's knowledge, there are no studies of steady problems with Orlicz-structure using DG methods, except for Remark 2.3 of [19], where it is mentioned that the results for the p-Laplace can be extended to balanced N-functions. Note that the convergence rates in [19] are sub-optimal (*cf.* Rem. 5.6) since the continuous solution $\mathbf{u}$ of (1.1) satisfies the flux formulation of (1.1) with an additional term defined on interior and Dirichlet faces (*cf.* (5.1)). The main purpose of this paper is to overcome this drawback. Moreover, we state our results in the context of balanced N-functions to make clear that the usual distinction between the cases $p \geq 2$ and $p \leq 2$ for p-Laplace problems is not needed, and for general right-hand sides from $(W_0^{1,\varphi}(\Omega))^*$ represented *via* $\mathbf{g} - \mathrm{div}\mathbf{G}$ with $\mathbf{g} \in L^{\varphi^*}(\Omega)$, $\mathbf{G} \in L^{\varphi^*}(\Omega)$.

To this end, we introduce a new numerical flux (*cf.* (3.4)), which allows us to establish convergence of DG-solutions to a weak solution of the system (1.1) for general right-hand sides $\mathbf{g} - \mathrm{div}\mathbf{G}$ (*cf.* Thm. 4.8), and error estimates if $\mathbf{G} = \mathbf{0}$ (*cf.* Thm. 5.4, Cor. 5.5). The convergence rates are optimal for linear ansatz functions. Further, our approach yields a unified treatment of problems with (p, δ)-structure (*cf.* [19]), $p \in (1, \infty)$, $\delta \geq 0$ and recovers in the DG setting the results in [17, 22] (using FEM) and [6] (using special non-conforming methods). The presence of the shift in the new flux is analogous to the gradient shift in the natural distance. It takes into account the structure of the nonlinear problem (1.1) (*cf.* Prop. 2.8, Rems. 2.12, 5.6).

This paper is organized as follows: In Section 2, we introduce the employed notation, define the relevant function spaces, the basic assumptions on the nonlinear operator and its consequences, introduce discrete operators and discuss their properties. In Section 3, we define our numerical fluxes and derive the flux and the primal formulation of our problem. In Section 4, we prove the existence of DG solutions (*cf.* Prop. 4.5), the stability of the method, *i.e.*, *a priori* estimates (*cf.* Prop. 4.6, Cor. 4.7), and the convergence of DG solutions (*cf.* Thm. 4.8). In Section 5, we derive error estimates for our problem (*cf.* Thm. 5.4, Cor. 5.5). These are the first convergence rates for an LDG method for systems with balanced Orlicz-structure. In Section 6, we confirm our theoretical findings *via* numerical experiments. For the convenience of the reader, we collect in the Appendix A known results in the DG Orlicz setting, needed in the paper, and prove some new results.

2. Preliminaries

2.1. Function spaces

We use $c, C > 0$ to denote generic constants, that may change from line to line, but are not depending on the crucial quantities. Moreover, we write $f \sim g$ if and only if there exists constants $c, C > 0$ such that $c\, f \leq g \leq C\, f$.

For $k \in \mathbb{N}$ and $p \in [1, \infty]$, we will employ the customary Lebesgue spaces $L^p(\Omega)$ and Sobolev spaces $W^{k,p}(\Omega)$, where $\Omega \subseteq \mathbb{R}^n$, $n \geq 2$, is a bounded, polyhedral domain having a Lipschitz continuous boundary $\partial\Omega$, which is disjointly divided into a Dirichlet part $\Gamma_D \subseteq \partial\Omega$, where we assume that $|\Gamma_D| > 0$, and a Neumann part $\Gamma_N \subseteq \partial\Omega$, *i.e.*, $\Gamma_D \cup \Gamma_N = \partial\Omega$ and $\Gamma_D \cap \Gamma_N = \emptyset$. We denote by $\|\cdot\|_p$, the norm in $L^p(\Omega)$ and by $\|\cdot\|_{k,p}$, the

[1] Orlicz spaces are used to prove error estimate in the limiting case of the L^∞-norm for the Poisson problem in [21].

norm in $W^{k,p}(\Omega)$. The space $W^{1,p}_{\Gamma_D}(\Omega)$ is defined as those functions from $W^{1,p}(\Omega)$ whose trace vanishes on Γ_D. We equip $W^{1,p}_{\Gamma_D}(\Omega)$ with the gradient norm $\|\nabla \cdot\|_p$.

For a normed vector space X, we denote its (topological) dual space by X^*. We do not distinguish between function spaces for scalar, vector- or tensor-valued functions. However, we will denote vector-valued functions by boldface letters and tensor-valued functions by capital boldface letters. For $d \in \mathbb{N}$, the standard scalar product between two vectors $\mathbf{u} = (u_1, \ldots, u_d)^\top, \mathbf{v} = (v_1, \ldots, v_d)^\top \in \mathbb{R}^d$ is denoted by $\mathbf{u} \cdot \mathbf{v} = \sum_{i=1}^d u_i v_i$, and we use the notation $|\mathbf{u}| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$ for all $\mathbf{u} \in \mathbb{R}^d$. For $d, n \in \mathbb{N}$, the Frobenius scalar product between two tensors $\mathbf{P} = (P_{ij})_{i=1,\ldots,d,j=1,\ldots,n}, \mathbf{Q} = (Q_{ij})_{i=1,\ldots,d,j=1,\ldots,n} \in \mathbb{R}^{d\times n}$ is denoted by $\mathbf{P} : \mathbf{Q} = \sum_{i=1}^d \sum_{j=1}^n P_{ij} Q_{ij}$, and we write $|\mathbf{P}| = \sqrt{\mathbf{P} : \mathbf{P}}$ for all $\mathbf{P} \in \mathbb{R}^{d\times n}$. We denote by $|M|$, the n- or $(n-1)$-dimensional Lebesgue measure of a (Lebesgue) measurable set $M \subseteq \mathbb{R}^n$, $n \in \mathbb{N}$. The mean value of a locally integrable function f over a measurable set $M \subseteq \Omega$ is denoted by $\langle f \rangle_M := \fint_M f \,\mathrm{d}x := \frac{1}{|M|} \int_M f \,\mathrm{d}x$. Moreover, we use the notation $(f, g) := \int_\Omega f g \,\mathrm{d}x$, whenever the right-hand side is well-defined.

We will also use Orlicz and Sobolev–Orlicz spaces (*cf.* [30]). A real convex function $\psi : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ is said to be an ***N*-function**, if $\psi(0) = 0$, $\psi(t) > 0$ for all $t > 0$, $\lim_{t\to 0} \psi(t)/t = 0$, and $\lim_{t\to\infty} \psi(t)/t = \infty$. We call ψ a **regular *N*-function**, if it, in addition, belongs to $C^1(\mathbb{R}^{\geq 0}) \cap C^2(\mathbb{R}^{>0})$ and satisfies $\psi''(t) > 0$ for all $t > 0$. For a regular N-function, we have $\psi(0) = \psi'(0) = 0$, ψ' is increasing and $\lim_{t\to\infty} \psi'(t) = \infty$. We define the (convex) **conjugate *N*-function** ψ^* by $\psi^*(t) := \sup_{s\geq 0}(st - \psi(s))$ for all $t \geq 0$, which satisfies $(\psi^*)' = (\psi')^{-1}$. A given N-function ψ satisfies the Δ_2-condition (in short, $\varphi \in \Delta_2$), if there exists $K > 2$ such that for all $t \geq 0$, it holds $\psi(2\,t) \leq K\,\psi(t)$. We denote the smallest such constant by $\Delta_2(\psi) > 0$. If one assumes that both ψ and ψ^* satisfy the Δ_2-condition, there holds

$$\psi^* \circ \psi' \sim \psi. \tag{2.1}$$

We need the following refined version of the ε-Young inequality: for every $\varepsilon > 0$, there is a $c_\varepsilon > 0$, depending only on $\Delta_2(\psi), \Delta_2(\psi^*) < \infty$, such that for all $s, t \geq 0$, it holds

$$\begin{aligned} t\,s &\leq \varepsilon\,\psi(t) + c_\varepsilon\,\psi^*(s), \\ t\,\psi'(s) + \psi'(t)\,s &\leq \varepsilon\,\psi(t) + c_\varepsilon\,\psi(s). \end{aligned} \tag{2.2}$$

We denote by $L^\psi(\Omega)$ and $W^{1,\psi}(\Omega)$, the classical Orlicz and Sobolev–Orlicz spaces, *i.e.*, we have that $f \in L^\psi(\Omega)$ if the **modular** $\rho_\psi(f) = \rho_{\psi,\Omega}(f) := \int_\Omega \psi(|f|)\,\mathrm{d}x$ is finite and $f \in W^{1,\psi}(\Omega)$ if $f, \nabla f \in L^\psi(\Omega)$. Equipped with the induced Luxembourg norm, *i.e.*, $\|f\|_\psi := \inf\{\lambda > 0 \mid \int_\Omega \psi(|f|/\lambda)\,\mathrm{d}x \leq 1\}$, the space $L^\psi(\Omega)$ forms a Banach space. The same holds for the space $W^{1,\psi}(\Omega)$ if it is equipped with the norm $\|\cdot\|_\psi + \|\nabla \cdot\|_\psi$. Note that the dual space $(L^\psi(\Omega))^*$ of $L^\psi(\Omega)$ can be identified with the space $L^{\psi^*}(\Omega)$. The space $W^{1,\psi}_{\Gamma_D}(\Omega)$ is defined as those functions from $W^{1,\psi}(\Omega)$ whose trace vanishes on Γ_D. We equip $W^{1,\psi}_{\Gamma_D}(\Omega)$ with the gradient norm $\|\nabla \cdot\|_\psi$.

2.2. Basic properties of the nonlinear operator

In the whole paper, we always assume that the nonlinear operator $\boldsymbol{\mathcal{A}} : \mathbb{R}^{d\times n} \to \mathbb{R}^{d\times n}$ has φ-structure, which will be defined now. A detailed discussion and thorough proofs can be found in [9,16,33]. A regular N-function ψ is called **balanced**, if there exist constants $\gamma_1 \in (0, 1]$ and $\gamma_2 \geq 1$ such that for all $t > 0$, there holds

$$\gamma_1\,\psi'(t) \leq t\,\psi''(t) \leq \gamma_2\,\psi'(t). \tag{2.3}$$

The constants γ_1 and γ_2 are called **characteristics** of the balanced N-function ψ, and will be denoted as (γ_1, γ_2). The basic properties of balanced N-functions are collected in the following lemma (*cf.* [20] for related results of a slightly different approach).

Lemma 2.1. *Let ψ be a balanced N-function with characteristics (γ_1, γ_2). Then, the following statements apply:*

(i) *The conjugate N-function ψ^* is a balanced N-function with characteristics $(\frac{1}{\gamma_1}, \frac{1}{\gamma_2})$.*

(ii) *The N-functions ψ and ψ^* satisfy the Δ_2-condition, and the Δ_2-constants of ψ and ψ^* possess an upper bound depending only on the characteristics of ψ.*

(iii) *Uniformly with respect to $t > 0$, we have that $\psi(t) \sim \psi'(t)\, t \sim \psi''(t)\, t^2$, with constants of equivalence depending only on the characteristics of ψ.*

Proof. The assertion (i) is proved in Lemma 6.4 of [33]. The assertion (ii) is proved in [8] (*cf.* [9], Lem. 2.10). Assertion (iii) follows from (ii), since for N-functions satisfying the Δ_2-condition, uniformly with respect to $t > 0$, there holds

$$\psi'(t)\, t \sim \psi(t), \tag{2.4}$$

and the fact that ψ is balanced. □

Remark 2.2. It is well-known that the N-function ψ for which $\psi'(t) = (\delta + t)^{p-2} t$ for all $t \geq 0$, $p \in (1, \infty)$, $\delta \geq 0$, is balanced (*cf.* [17, 33]). Other examples are given *via* $\psi'(t) = (t^\alpha (\delta + t)^{1-\alpha})^{p-2} t$ for all $t \geq 0$, or $\psi'(t) = (t + \delta)^{p-2} t (\ln(1 + \delta + t))^\alpha$ for all $t \geq 0$, where $\alpha \geq 1$, $p \in (1, \infty)$, $\delta \geq 0$.

Assumption 2.3 (Nonlinear operator)**.** *We assume that the nonlinear operator $\boldsymbol{\mathcal{A}}\colon \mathbb{R}^{d\times n} \to \mathbb{R}^{d\times n}$ belongs to $C^0(\mathbb{R}^{d\times n}, \mathbb{R}^{d\times n}) \cap C^1(\mathbb{R}^{d\times n} \setminus \{\mathbf{0}\}, \mathbb{R}^{d\times n})$ and satisfies $\boldsymbol{\mathcal{A}}(\mathbf{0}) = \mathbf{0}$. Moreover, we assume that the operator $\boldsymbol{\mathcal{A}}$ has φ-**structure**, i.e., there exist a regular N-function φ and constants $\gamma_3 \in (0, 1]$, $\gamma_4 > 1$ such that*

$$\sum_{j,l=1}^{n} \sum_{i,k=1}^{d} \partial_{kl} \mathcal{A}_{ij}(\mathbf{P}) Q_{ij} Q_{kl} \geq \gamma_3 \frac{\varphi'(|\mathbf{P}|)}{|\mathbf{P}|} |\mathbf{Q}|^2, \tag{2.5a}$$

$$\big|\partial_{kl} \mathcal{A}_{ij}(\mathbf{P})\big| \leq \gamma_4 \frac{\varphi'(|\mathbf{P}|)}{|\mathbf{P}|}, \tag{2.5b}$$

*are satisfied for all $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{d\times n}$ with $\mathbf{P} \neq \mathbf{0}$ and all $i, k = 1, \ldots, d$, $j, l = 1, \ldots, n$. The constants γ_3, γ_4, and $\Delta_2(\varphi)$ are called the **characteristics** of $\boldsymbol{\mathcal{A}}$ and will be denoted by $(\gamma_3, \gamma_4, \Delta_2(\varphi))$.*

Closely related to the non-linear operator $\boldsymbol{\mathcal{A}}\colon \mathbb{R}^{d\times n} \to \mathbb{R}^{d\times n}$ with φ-structure are the non-linear functions $\mathbf{F}, \mathbf{F}^*\colon \mathbb{R}^{d\times n} \to \mathbb{R}^{d\times n}$, for every $\mathbf{P} \in \mathbb{R}^{d\times n}$ defined *via*

$$\mathbf{F}(\mathbf{P}) := \sqrt{\frac{\varphi'(|\mathbf{P}|)}{|\mathbf{P}|}}\, \mathbf{P}, \qquad \mathbf{F}^*(\mathbf{P}) := \sqrt{\frac{(\varphi^*)'(|\mathbf{P}|)}{|\mathbf{P}|}}\, \mathbf{P}. \tag{2.6}$$

Remark 2.4. Note that $\mathbf{F}$ and $\mathbf{F}^*$ are derived from the potentials

$$\kappa(t) := \int_0^t \sqrt{\varphi'(s)\, s}\, \mathrm{d}s \quad \text{and} \quad \kappa^*(t) := \int_0^t \sqrt{(\varphi^*)'(s)\, s}\, \mathrm{d}s \quad \text{for all } t \geq 0,$$

respectively. One easily sees that these potentials are balanced N-functions.

Another important tool are the **shifted N-functions** (*cf.* [16, 32, 33]). For a given N-function ψ, we define the shifted N-functions $\psi_a : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$, $a \geq 0$, *via*

$$\psi_a(t) := \int_0^t \psi_a'(s)\, \mathrm{d}s \quad \text{with} \quad \psi_a'(t) := \psi'(a + t) \frac{t}{a + t} \quad \text{for all } t \geq 0. \tag{2.7}$$

The following properties of shifted N-functions are proved in [16, 33].

Lemma 2.5. *Let the N-functions ψ, ψ^* satisfy the Δ_2-condition. Then, it holds:*

(i) *The family of shifted N-functions $\psi_a : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$, $a \geq 0$, satisfy the Δ_2-condition uniformly with respect to $a \geq 0$ with $\Delta_2(\psi_a)$ depending only on $\Delta_2(\psi)$.*

(ii) *The conjugate function satisfies $(\psi_a)^*(t) \sim (\psi^*)_{\psi'(a)}(t)$ uniformly with respect to $t, a \geq 0$ with constants depending only on $\Delta_2(\psi), \Delta_2(\psi^*)$.*

Lemma 2.6. *Let ψ be an N-function satisfying the Δ_2-condition. Then, there exists a constant $c > 0$ such that for every $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{d\times n}$ and $t \geq 0$, we have that*

$$\psi'_{|\mathbf{P}|}(t) \leq c\,\psi'_{|\mathbf{Q}|}(t) + c\,\psi'_{|\mathbf{P}|}(|\mathbf{P}-\mathbf{Q}|). \tag{2.8}$$

Moreover, $\psi'_{|\mathbf{Q}|}(|\mathbf{P}-\mathbf{Q}|) \sim \psi'_{|\mathbf{P}|}(|\mathbf{P}-\mathbf{Q}|)$ holds uniformly with respect to $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{d\times n}$.

Lemma 2.7 (Change of shift)**.** *Let ψ be an N-function such that ψ and ψ^* satisfy the Δ_2-condition. Then, for every $\delta \in (0,1)$, there exists $c_\delta > 0$ such that for every $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{d\times n}$ and $t \geq 0$, we have that*

$$\begin{aligned} \psi_{|\mathbf{P}|}(t) &\leq c_\delta\,\psi_{|\mathbf{Q}|}(t) + \delta\,\psi_{|\mathbf{P}|}(|\mathbf{P}-\mathbf{Q}|), \\ \left(\psi_{|\mathbf{P}|}\right)^*(t) &\leq c_\delta\,\left(\psi_{|\mathbf{Q}|}\right)^*(t) + \delta\,\psi_{|\mathbf{P}|}(|\mathbf{P}-\mathbf{Q}|). \end{aligned} \tag{2.9}$$

Moreover, $\psi_{|\mathbf{Q}|}(|\mathbf{P}-\mathbf{Q}|) \sim \psi_{|\mathbf{P}|}(|\mathbf{P}-\mathbf{Q}|)$ holds uniformly with respect to $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{d\times n}$.

The connection between $\boldsymbol{\mathcal{A}}, \mathbf{F}, \mathbf{F}^* \colon \mathbb{R}^{d\times n} \to \mathbb{R}^{d\times n}$ and $\varphi_a, (\varphi^*)_a : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$, $a \geq 0$, is best explained by the following proposition (*cf.* [16,33]).

Proposition 2.8. *Let $\boldsymbol{\mathcal{A}}$ satisfy Assumption 2.3 for a balanced N-function φ. Then, uniformly with respect to $\mathbf{P}, \mathbf{Q} \in \mathbb{R}^{d\times n}$, we have that*

$$(\boldsymbol{\mathcal{A}}(\mathbf{P}) - \boldsymbol{\mathcal{A}}(\mathbf{Q})) : (\mathbf{P}-\mathbf{Q}) \sim |\mathbf{F}(\mathbf{P}) - \mathbf{F}(\mathbf{Q})|^2 \sim \varphi_{|\mathbf{P}|}(|\mathbf{P}-\mathbf{Q}|), \tag{2.10}$$

$$|\mathbf{F}^*(\mathbf{P}) - \mathbf{F}^*(\mathbf{Q})|^2 \sim (\varphi^*)_{|\mathbf{P}|}(|\mathbf{P}-\mathbf{Q}|), \tag{2.11}$$

$$|\boldsymbol{\mathcal{A}}(\mathbf{P}) - \boldsymbol{\mathcal{A}}(\mathbf{Q})| \sim \varphi'_{|\mathbf{P}|}(|\mathbf{P}-\mathbf{Q}|). \tag{2.12}$$

Moreover, uniformly with respect to $\mathbf{Q} \in \mathbb{R}^{d\times n}$, we have that

$$\boldsymbol{\mathcal{A}}(\mathbf{Q}) \cdot \mathbf{Q} \sim |\mathbf{F}(\mathbf{Q})|^2 \sim \varphi(|\mathbf{Q}|), \tag{2.13}$$

$$|\boldsymbol{\mathcal{A}}(\mathbf{P})| \sim \varphi'(|\mathbf{P}|). \tag{2.14}$$

The constants in (2.10)–(2.14) depend only on the characteristics of $\boldsymbol{\mathcal{A}}$ and φ.

Combining (2.1) and (2.11)–(2.13), we obtain, uniformly with respect to $\mathbf{Q} \in \mathbb{R}^{d\times n}$,

$$|\mathbf{F}^*(\boldsymbol{\mathcal{A}}(\mathbf{Q}))|^2 \sim \varphi^*(|\boldsymbol{\mathcal{A}}(\mathbf{Q})|) \sim |\mathbf{F}(\mathbf{Q})|^2 \sim \varphi(|\mathbf{Q}|). \tag{2.15}$$

In the same way as in Lemmas 2.8, 2.10, Corollary 2.9 of [19], one can show that

Lemma 2.9. *Let $\boldsymbol{\mathcal{A}}$ satisfy Assumption 2.3 for a balanced N-function φ. Then, uniformly with respect to $t \geq 0$ and $\mathbf{Q}, \mathbf{P} \in \mathbb{R}^{d\times n}$, we have that*

$$(\varphi^*)_{|\boldsymbol{\mathcal{A}}(\mathbf{P})|}(t) \sim (\varphi_{|\mathbf{P}|})^*(t), \tag{2.16}$$

$$(\varphi^*)_{|\boldsymbol{\mathcal{A}}(\mathbf{P})|}(|\boldsymbol{\mathcal{A}}(\mathbf{Q}) - \boldsymbol{\mathcal{A}}(\mathbf{P})|) \sim \varphi_{|\mathbf{P}|}(|\mathbf{Q}-\mathbf{P}|), \tag{2.17}$$

$$|\mathbf{F}^*(\boldsymbol{\mathcal{A}}(\mathbf{Q})) - \mathbf{F}^*(\boldsymbol{\mathcal{A}}(\mathbf{P}))|^2 \sim |\mathbf{F}(\mathbf{Q}) - \mathbf{F}(\mathbf{P})|^2 \tag{2.18}$$

with constants depending only on the characteristics of $\boldsymbol{\mathcal{A}}$ and φ.

Lemma 2.10. *Let $\boldsymbol{\mathcal{A}}$ satisfy Assumption 2.3 for a balanced N-function φ. Then, uniformly with respect to $t \geq 0$ and $\mathbf{Q}, \mathbf{P} \in \mathbb{R}^{d\times n}$, we have that*

$$\begin{aligned}
\big(\varphi_{|\mathbf{P}|}\big)(t) &\leq c\,\big(\varphi_{|\mathbf{Q}|}\big)(t) + c\,|\mathbf{F}(\mathbf{P}) - \mathbf{F}(\mathbf{Q})|^2,\\
\big(\varphi_{|\mathbf{P}|}\big)^*(t) &\leq c\,\big(\varphi_{|\mathbf{Q}|}\big)^*(t) + c\,|\mathbf{F}(\mathbf{P}) - \mathbf{F}(\mathbf{Q})|^2,\\
(\varphi^*)_{|\boldsymbol{\mathcal{A}}(\mathbf{P})|}(t) &\leq c\,(\varphi^*)_{|\boldsymbol{\mathcal{A}}(\mathbf{Q})|}(t) + c\,|\mathbf{F}(\mathbf{P}) - \mathbf{F}(\mathbf{Q})|^2,\\
(\varphi^*)_{|\mathbf{P}|}(t) &\leq c\,(\varphi^*)_{|\mathbf{Q}|}(t) + c\,|\mathbf{F}^*(\mathbf{P}) - \mathbf{F}^*(\mathbf{Q})|^2
\end{aligned}$$

with constants depending only on the characteristics of $\boldsymbol{\mathcal{A}}$ and φ.

Remark 2.11 (Natural energy spaces)**.** If $\boldsymbol{\mathcal{A}}$ satisfies Assumption 2.3 for a balanced N-function φ, we see from (2.13) and (2.15) that $\mathbf{u} \in W^{1,\varphi}(\Omega)$, $\mathbf{L} = \nabla\mathbf{u} \in L^{\varphi}(\Omega)$ and $\mathbf{A} = \boldsymbol{\mathcal{A}}(\mathbf{L}) \in L^{\varphi^*}(\Omega)$.

Remark 2.12 (Natural distance)**.** If $\boldsymbol{\mathcal{A}}$ satisfies Assumption 2.3 for a balanced N-function φ, we see from the previous results that for all $\mathbf{u}, \mathbf{w} \in W^{1,\varphi}(\Omega)$

$$(\boldsymbol{\mathcal{A}}(\nabla\mathbf{u}) - \boldsymbol{\mathcal{A}}(\nabla\mathbf{w}), \nabla\mathbf{u} - \nabla\mathbf{w}) \sim \|\mathbf{F}(\nabla\mathbf{u}) - \mathbf{F}(\nabla\mathbf{w})\|_2^2 \sim \int_\Omega \varphi_{|\nabla\mathbf{u}|}(|\nabla\mathbf{u} - \nabla\mathbf{w}|)\,\mathrm{d}x,$$

where the constants depend only on the characteristics of $\boldsymbol{\mathcal{A}}$ and φ. In the context of p-Laplace problems, the quantity $\mathbf{F}$ was introduced in [1], while the last expression equals the quasi-norm introduced in [5] raised to the power $\rho = \max\{p, 2\}$. We refer to all three equivalent quantities as the **natural distance**. This name expresses the fact, that the natural distance provides the appropriate error measure for problem (1.1) rather than the $W^{1,p}(\Omega)$-semi-norm.

2.3. DG spaces, jumps and averages

Let $(\mathcal{T}_h)_{h>0}$ be a family of triangulations of our domain Ω consisting of n-dimensional simplices K. Here, the parameter $h > 0$, refers to the maximal mesh-size, *i.e.*, if we set $h_K := \mathrm{diam}(K)$ for all $K \in \mathcal{T}_h$, then $h := \max_{K\in\mathcal{T}_h} h_K$. For simplicity, we always assume, in the paper, that $h \leq 1$. For a simplex $K \in \mathcal{T}_h$, we denote by $\rho_K > 0$, the supremum of diameters of inscribed balls. We assume that there exists a constant $\omega_0 > 0$, independent of $h > 0$, such that $h_K\rho_K^{-1} \leq \omega_0$ for all $K \in \mathcal{T}_h$. The smallest such constant is called the chunkiness of $(\mathcal{T}_h)_{h>0}$. Note that, in the following, all constants may depend on the chunkiness ω_0, but are independent of $h > 0$. For $K \in \mathcal{T}_h$, let S_K denote the neighborhood of K, *i.e.*, the patch S_K is the union of all simplices of $\mathcal{T}_h$ touching K. We assume further for our triangulation that the interior of each S_K is connected. Under these assumptions, $|K| \sim |S_K|$ uniformly in $K \in \mathcal{T}_h$ and $h > 0$, and the number of simplices in S_K and patches to which a simplex belongs to are uniformly bounded with respect to $K \in \mathcal{T}_h$ and $h > 0$. We define the faces of $\mathcal{T}_h$ as follows: an interior face of $\mathcal{T}_h$ is the non-empty interior of $\partial K \cap \partial K'$, where K, K' are two adjacent elements of $\mathcal{T}_h$. For the face $\gamma := \partial K \cap \partial K'$, we use the notation $S_\gamma := K \cup K'$. A boundary face of $\mathcal{T}_h$ is the non-empty interior of $\partial K \cap \partial\Omega$, where K is a boundary element of $\mathcal{T}_h$. For the face $\gamma := \partial K \cap \partial\Omega$, we use the notation $S_\gamma := K$. By Γ_h^i, Γ_D, and Γ_N, we denote the interior, the Dirichlet and the Neumann faces, resp., and put $\Gamma_h := \Gamma_h^i \cup \Gamma_D \cup \Gamma_N$. We assume that each $K \in \mathcal{T}_h$ has at most one face from $\Gamma_D \cup \Gamma_N$. We introduce the following scalar products on Γ_h

$$\langle f, g\rangle_{\Gamma_h} := \sum_{\gamma\in\Gamma_h} \langle f, g\rangle_\gamma, \quad \text{where} \quad \langle f, g\rangle_\gamma := \int_\gamma fg\,\mathrm{d}s \quad \text{for all } \gamma \in \Gamma_h,$$

if all the integrals are well-defined. Similarly, we define $\langle\cdot,\cdot\rangle_{\Gamma_D}$, $\langle\cdot,\cdot\rangle_{\Gamma_N}$ and $\langle\cdot,\cdot\rangle_{\Gamma_h^i}$. We extend the notation of modulars to the sets Γ_h^i, Γ_D, and $\Gamma_h^i \cup \Gamma_D$, *i.e.*, we define $\rho_{\psi,B}(f) := \int_B \psi(|f|)\,\mathrm{d}s$ for every $f \in L^\psi(B)$, where $B = \Gamma_h^i$ or $B = \Gamma_D$ or $B = \Gamma_h^i \cup \Gamma_D$.

For $m \in \mathbb{N}_0$ and $K \in \mathcal{T}_h$, we denote by $\mathcal{P}_m(K)$, the space of scalar, vector-valued or tensor-valued polynomials of degree at most m on K. Given a triangulation of Ω with the above properties, given an N-function ψ, and given $k \in \mathbb{N}_0$, we define

$$\begin{aligned} U_h^k &:= \big\{\mathbf{u}_h \in L^1(\Omega)^d \,\big|\, \mathbf{u}_h|_K \in \mathcal{P}_k(K)^d \text{ for all } K \in \mathcal{T}_h\big\}, \\ X_h^k &:= \big\{\mathbf{X}_h \in L^1(\Omega)^{d\times n} \,\big|\, \mathbf{X}_h|_K \in \mathcal{P}_k(K)^{d\times n} \text{ for all } K \in \mathcal{T}_h\big\}, \\ W^{1,\psi}(\mathcal{T}_h) &:= \big\{\mathbf{w}_h \in L^1(\Omega) \,\big|\, \mathbf{w}_h|_K \in W^{1,\psi}(K) \text{ for all } K \in \mathcal{T}_h\big\}. \end{aligned} \tag{2.19}$$

Note that $W^{1,\psi}(\Omega) \subseteq W^{1,\psi}(\mathcal{T}_h)$ and $U_h^k \subseteq W^{1,\psi}(\mathcal{T}_h)$. We denote by $\Pi_h^k : L^1(\Omega)^d \to U_h^k$, the (local) L^2-projection into U_h^k, which for every $\mathbf{u} \in L^1(\Omega)$ and $\mathbf{z}_h \in U_h^k$ is defined *via*

$$\big(\Pi_h^k \mathbf{u}, \mathbf{z}_h\big) = (\mathbf{u}, \mathbf{z}_h). \tag{2.20}$$

Analogously, we define the (local) L^2-projection into X_h^k, *i.e.*, $\Pi_h^k : L^1(\Omega)^{d\times n} \to X_h^k$.

For $\mathbf{w}_h \in W^{1,\psi}(\mathcal{T}_h)$, we denote by $\nabla_h \mathbf{w}_h \in L^\psi(\Omega)$ the **local gradient**, defined *via* $(\nabla_h \mathbf{w}_h)|_K := \nabla(\mathbf{w}_h|_K)$ for all $K \in \mathcal{T}_h$. For every $K \in \mathcal{T}_h$, $\mathbf{w}_h \in W^{1,\psi}(\mathcal{T}_h)$ admits an interior trace $\mathrm{tr}^K(\mathbf{w}_h) \in L^\psi(\partial K)$. For each face $\gamma \in \Gamma_h$ of a given simplex $T \in \mathcal{T}_h$, we define this interior trace by $\mathrm{tr}_\gamma^K(\mathbf{w}_h) \in L^\psi(\gamma)$. Then, for $\mathbf{w}_h \in W^{1,\psi}(\mathcal{T}_h)$ and interior faces $\gamma \in \Gamma_I$ shared by adjacent elements $K_\gamma^-, K_\gamma^+ \in \mathcal{T}_h$, we denote by

$$\{\mathbf{w}_h\}_\gamma := \frac{1}{2}\Big(\mathrm{tr}_\gamma^{K^+}(\mathbf{w}_h) + \mathrm{tr}_\gamma^{K^-}(\mathbf{w}_h)\Big) \in L^\psi(\gamma), \tag{2.21}$$

$$[\![\mathbf{w}_h \otimes \mathbf{n}]\!]_\gamma := \mathrm{tr}_\gamma^{K^+}(\mathbf{w}_h) \otimes \mathbf{n}_\gamma^+ + \mathrm{tr}_\gamma^{K^-}(\mathbf{w}_h) \otimes \mathbf{n}_\gamma^- \in L^\psi(\gamma), \tag{2.22}$$

the **average** and the **normal jump**, resp., of $\mathbf{w}_h$ on γ. Moreover, for every $\mathbf{w}_h \in W^{1,\psi}(\mathcal{T}_h)$ and boundary faces $\gamma \in \Gamma_D$, we define boundary averages and boundary jumps, resp., *via*

$$\{\mathbf{w}_h\}_\gamma := \mathrm{tr}_\gamma^\Omega(\mathbf{w}_h) \in L^\psi(\gamma), \tag{2.23}$$

$$[\![\mathbf{w}_h \otimes \mathbf{n}]\!]_\gamma := \mathrm{tr}_\gamma^\Omega(\mathbf{w}_h) \otimes \mathbf{n} \in L^\psi(\gamma), \tag{2.24}$$

where $\mathbf{n} : \partial\Omega \to \mathbb{S}^{d-1}$ denotes the unit normal vector field to Ω pointing outward. Analogously, we define $\{\mathbf{X}_h\}_\gamma$ and $[\![\mathbf{X}_h \mathbf{n}]\!]_\gamma$ for every $\mathbf{X}_h \in X_h^k$ and $\gamma \in \Gamma_h^i \cup \Gamma_D$. We omit the index γ if there is no danger of confusion. For every $k \in \mathbb{N}_0$, we define for a given face $\gamma \in \Gamma_h^i \cup \Gamma_D$, the **(local) jump operator** $\boldsymbol{\mathcal{R}}_{h,\gamma}^k : W^{1,\psi}(\mathcal{T}_h) \to X_h^k$ for every $\mathbf{w}_h \in W^{1,\psi}(\mathcal{T}_h)$ (using Riesz representation) *via*

$$\Big(\boldsymbol{\mathcal{R}}_{h,\gamma}^k(\mathbf{w}_h), \mathbf{X}_h\Big) := \langle [\![\mathbf{w}_h \otimes \mathbf{n}]\!]_\gamma, \{\mathbf{X}_h\}_\gamma \rangle_\gamma \quad \text{for all } \mathbf{X}_h \in X_h^k, \tag{2.25}$$

and the **(global) jump operator** *via*

$$\boldsymbol{\mathcal{R}}_h^k = \boldsymbol{\mathcal{R}}_{h,\Gamma_D}^k := \sum_{\gamma \in \Gamma_h^i \cup \Gamma_D} \boldsymbol{\mathcal{R}}_{\gamma,h}^k : W^{1,\psi}(\mathcal{T}_h) \to X_h^k, \tag{2.26}$$

which, by definition, for every $\mathbf{w}_h \in W^{1,\psi}(\mathcal{T}_h)$ and $\mathbf{X}_h \in X_h^k$ satisfies

$$\Big(\boldsymbol{\mathcal{R}}_h^k \mathbf{w}_h, \mathbf{X}_h\Big) = \langle [\![\mathbf{w}_h \otimes \mathbf{n}]\!], \{\mathbf{X}_h\} \rangle_{\Gamma_h^i \cup \Gamma_D}. \tag{2.27}$$

Furthermore, for every $k \in \mathbb{N}_0$, we define the **discrete gradient operator** or **DG gradient operator** $\boldsymbol{\mathcal{G}}_h^k = \boldsymbol{\mathcal{G}}_{h,\Gamma_D}^k : W^{1,\psi}(\mathcal{T}_h) \to L^\psi(\Omega)$, for every $\mathbf{w}_h \in W^{1,\psi}(\mathcal{T}_h)$ *via*[2]

$$\boldsymbol{\mathcal{G}}_h^k \mathbf{w}_h = \boldsymbol{\mathcal{G}}_{h,\Gamma_D}^k \mathbf{w}_h := \nabla_h \mathbf{w}_h - \boldsymbol{\mathcal{R}}_{h,\Gamma_D}^k \mathbf{w}_h \quad \text{in } L^\psi(\Omega). \tag{2.28}$$

[2]Note that we use for discrete gradients the notation from [15], which is different from the one used in [19].

In particular, for every $\mathbf{w}_h \in W^{1,\psi}(\mathcal{T}_h)$ and $\mathbf{X}_h \in X_h^k$, we have that

$$\Big(\boldsymbol{\mathcal{G}}_h^k \mathbf{w}_h, \mathbf{X}_h\Big) = (\nabla_h \mathbf{w}_h, \mathbf{X}_h) - \big\langle [\![\mathbf{w}_h \otimes \mathbf{n}_h]\!], \{\mathbf{X}_h\}\big\rangle_{\Gamma_h^i \cup \Gamma_D}. \tag{2.29}$$

Note that for every $\mathbf{u} \in W^{1,\psi}_{\Gamma_D}(\Omega)$, it holds $\boldsymbol{\mathcal{G}}_h^k \mathbf{u} = \nabla \mathbf{u}$ in $L^\psi(\Omega)$.

We define the pseudo-modular $m_{\psi,h} = m_{\psi,h,\Gamma_D}$ and the modular $M_{\psi,h} = M_{\psi,h,\Gamma_D}$ on $W^{1,\psi}(\mathcal{T}_h)$ and $W^{1,\psi}_{\Gamma_D}(\mathcal{T}_h)$ *via*

$$\begin{aligned} m_{\psi,h}(\mathbf{w}_h) &= m_{\psi,h,\Gamma_D}(\mathbf{w}_h) := h\, \rho_{\psi,\Gamma_h^i \cup \Gamma_D}\big(h^{-1}[\![\mathbf{w}_h \otimes \mathbf{n}]\!]\big), \\ M_{\psi,h}(\mathbf{w}_h) &= M_{\psi,h,\Gamma_D}(\mathbf{w}_h) := \rho_{\psi,\Omega}(\nabla_h \mathbf{w}_h) + m_{\psi,h,\Gamma_D}(\mathbf{w}_h). \end{aligned} \tag{2.30}$$

The induced Luxembourg norm of the modular $M_{\psi,h}$ is denoted by $\|\cdot\|_{M_{\psi,h}}$. Note that for every $\mathbf{u} \in W^{1,\psi}_{\Gamma_D}(\Omega)$, it holds $m_{\psi,h}(\mathbf{u}) = 0$ and $M_{\psi,h}(\mathbf{u}) = \rho_{\psi,\Omega}(\nabla \mathbf{u})$, so $M_{\psi,h}$ forms an extension of the modular $\rho_{\psi,\Omega}(\nabla\,\cdot)$ on $W^{1,\psi}_{\Gamma_D}(\Omega)$ to the DGsetting. In most cases, we will omit the index Γ_D in $\boldsymbol{\mathcal{R}}_{h,\Gamma_D}^k$, $\boldsymbol{\mathcal{G}}_{h,\Gamma_D}^k$, m_{ψ,h,Γ_D}^k, and M_{ψ,h,Γ_D}^k, and simply write $\boldsymbol{\mathcal{R}}_h^k$, $\boldsymbol{\mathcal{G}}_h^k$, $m_{\psi,h}^k$, and $M_{\psi,h}^k$, respectively.

Remark 2.13. In the case $\psi = \varphi$, due to (2.13), for every $\mathbf{w}_h \in W^{1,\varphi}(\mathcal{T}_h)$, it holds

$$\begin{aligned} m_{\varphi,h}(\mathbf{w}_h) &\sim h\, \big\|\mathbf{F}\big(h^{-1}[\![\mathbf{w}_h \otimes \mathbf{n}]\!]\big)\big\|^2_{2,\Gamma_h^i \cup \Gamma_D}, \\ M_{\varphi,h}(\mathbf{w}_h) &\sim \|\mathbf{F}(\nabla_h \mathbf{w}_h)\|^2_{2,\Omega} + h\, \big\|\mathbf{F}\big(h^{-1}[\![\mathbf{w}_h \otimes \mathbf{n}]\!]\big)\big\|^2_{2,\Gamma_h^i \cup \Gamma_D}. \end{aligned}$$

3. Fluxes and LDG formulations

In order to obtain the LDG formulation of (1.1) for given $k \in \mathbb{N}$, we multiply the equations in $(1.2)_1$ by $\mathbf{X}_h \in X_h^k$, $\mathbf{Y}_h \in X_h^k$ and $\mathbf{z}_h \in U_h^k$, resp., use partial integration, replace in the volume integrals the fields $\mathbf{u}$, $\mathbf{L}$, $\mathbf{A}$ and $\mathbf{G}$ by the discrete objects $\mathbf{u}_h$, $\mathbf{L}_h$, $\mathbf{A}_h$ and $\Pi_h^k \mathbf{G}$, resp., and in the surface integrals $\mathbf{u}$, $\mathbf{A}$ and $\mathbf{G}$ by the numerical fluxes $\widehat{\mathbf{u}}_h := \widehat{\mathbf{u}}(\mathbf{u}_h)$, $\widehat{\mathbf{A}}_h := \widehat{\mathbf{A}}(\mathbf{u}_h, \mathbf{A}_h, \mathbf{L}_h)$ and $\widehat{\mathbf{G}}_h := \widehat{\mathbf{G}}(\Pi_h^k \mathbf{G})$, and obtain

$$\begin{aligned} \int_K \mathbf{L}_h : \mathbf{X}_h \,\mathrm{d}x &= -\int_K \mathbf{u}_h \cdot \mathrm{div}\mathbf{X}_h \,\mathrm{d}x + \int_{\partial K} \widehat{\mathbf{u}}_h \cdot (\mathbf{X}_h \mathbf{n}) \,\mathrm{d}s, \\ \int_K \mathbf{A}_h : \mathbf{Y}_h \,\mathrm{d}x &= \int_K \boldsymbol{\mathcal{A}}(\mathbf{L}_h) : \mathbf{Y}_h \,\mathrm{d}x, \\ \int_K \mathbf{A}_h : \nabla_h \mathbf{z}_h \,\mathrm{d}x &= \int_K \mathbf{g} \cdot \mathbf{z}_h + \Pi_h^k \mathbf{G} : \nabla_h \mathbf{z}_h \,\mathrm{d}x + \int_{\partial K} \mathbf{z}_h \cdot \Big(\widehat{\mathbf{A}}_h \mathbf{n} - \widehat{\mathbf{G}}_h \mathbf{n}\Big) \,\mathrm{d}s. \end{aligned} \tag{3.1}$$

For given boundary data $\mathbf{u}_D : \Gamma_D \to \mathbb{R}^d$ and $\mathbf{a}_N : \Gamma_N \to \mathbb{R}^d$, we denote by $\mathbf{u}_D^* \in W^{1,\varphi}(\Omega)$ an extension of $\mathbf{u}_D$, which exists if $\mathbf{u}_D$ belongs to the trace space of $W^{1,\varphi}(\Omega)$, that is characterized in [28]. For the nonlinear operator $\boldsymbol{\mathcal{A}} : \mathbb{R}^{d\times n} \to \mathbb{R}^{d\times n}$ having φ-structure, we define for every $a \geq 0$ and $\mathbf{P} \in \mathbb{R}^{d\times n}$

$$\boldsymbol{\mathcal{A}}_a(\mathbf{P}) := \frac{\varphi_a'(|\mathbf{P}|)}{|\mathbf{P}|}\,\mathbf{P} \quad \text{in } \mathbb{R}^{d\times n}, \tag{3.2}$$

where φ_a, $a \geq 0$, is the shifted N-function of the balanced N-function φ. Using these notions, the numerical fluxes are defined *via*[3]

$$\widehat{\mathbf{u}}(\mathbf{u}_h) := \begin{cases} \{\mathbf{u}_h\} & \text{on } \Gamma_h^i, \\ \mathbf{u}_D^* & \text{on } \Gamma_D, \\ \mathbf{u}_h & \text{on } \Gamma_N, \end{cases} \tag{3.3}$$

[3]Due to $[\![\mathbf{u}_D^* \otimes \mathbf{n}]\!] = \mathbf{0}$ on Γ_h^i, the flux $\widehat{\mathbf{A}}$ depends only on $\mathbf{u}_D : \Gamma_D \to \mathbb{R}^d$. We have chosen to formulate the flux in this form for a more compact notation.

$$\widehat{\mathbf{A}}(\mathbf{u}_h, \mathbf{A}_h, \mathbf{L}_h) := \begin{cases} \{\mathbf{A}_h\} - \alpha\, \boldsymbol{\mathcal{A}}_{\{|\Pi_h^0 \mathbf{L}_h|\}}\big(h^{-1}[\![(\mathbf{u}_h - \mathbf{u}_D^*) \otimes \mathbf{n}]\!]\big) & \text{on } \Gamma_h^i \cup \Gamma_D, \\ \mathbf{a}_N \otimes \mathbf{n} & \text{on } \Gamma_N, \end{cases} \tag{3.4}$$

and

$$\widehat{\mathbf{G}}(\Pi_h^k \mathbf{G}) := \begin{cases} \{\Pi_h^k \mathbf{G}\} & \text{on } \Gamma_h^i \cup \Gamma_D, \\ \mathbf{0} & \text{on } \Gamma_N, \end{cases} \tag{3.5}$$

resp., where $\alpha > 0$ is some constant. Note that we actually would like to use the shift $|\nabla \mathbf{u}|$ in the flux (3.4), which apparently is not possible since $\nabla \mathbf{u}$ is not known *a priori*. Since the distance of the discrete shift $\{|\Pi_h^0 \mathbf{L}_h|\}$ to $|\nabla \mathbf{u}|$ is controlled (*cf.* Lem. 5.3), we resort to the discrete shift $\{|\Pi_h^0 \mathbf{L}_h|\}$.

The fluxes are conservative since they are single-valued. The fluxes $\widehat{\mathbf{u}}$ and $\widehat{\mathbf{A}}$ are consistent, since $\widehat{\mathbf{u}}(\mathbf{u}) = \mathbf{u}$, $\widehat{\mathbf{A}}(\mathbf{u}, \mathbf{A}, \mathbf{L}) = \mathbf{A}$ for regular functions $\mathbf{u}$ and $\mathbf{A}$ satisfying $\mathbf{u} = \mathbf{u}_D^*$ on Γ_D and $\boldsymbol{\mathcal{A}}(\nabla \mathbf{u})\mathbf{n} = \mathbf{a}_N$ on Γ_N.

Note that for the flux $\widehat{\mathbf{u}}(\mathbf{u}_h)$ in (3.3), we have that $\mathbf{L}_h = \boldsymbol{\mathcal{G}}_h^k \mathbf{u}_h$ if $\mathbf{u}_D = \mathbf{0}$ (*cf.* (3.7)). Thus, this choice of the flux is the natural DG equivalent of $\mathbf{L} = \nabla \mathbf{u}$. Moreover, equation (3.7) also implies that the flux $\widehat{\mathbf{A}}(\mathbf{u}_h, \mathbf{A}_h, \mathbf{L}_h)$ is actually only depending on $\mathbf{A}_h$ and $\mathbf{u}_h$. It is a natural generalization of the corresponding fluxes for the Laplace problem (*cf.* [3]) and the p-Laplace problem (*cf.* [13]) taking into account the φ-structure of (1.1). Note that for the Laplace problem, *i.e.*, $\varphi(t) = \frac{1}{2}t^2$, the flux in (3.4) coincides with a standard flux, while in the case of the p-Laplace problem, *i.e.*, $\varphi(t) = \frac{1}{p}t^p$, the flux in (3.4) for $\mathbf{u}_D = \mathbf{0}$, $\Gamma_D = \partial\Omega$ reads

$$\widehat{\mathbf{A}} = \{\mathbf{A}_h\} - \alpha \left(\left\{ \left| \Pi_h^0 \boldsymbol{\mathcal{G}}_h^k \mathbf{u}_h \right| \right\} + h^{-1} |[\![\mathbf{u}_h \otimes \mathbf{n}]\!]| \right)^{p-2} h^{-1} [\![\mathbf{u}_h \otimes \mathbf{n}]\!] \quad \text{on } \Gamma_h^i \cup \partial\Omega,$$

where $(\Pi_h^0 \boldsymbol{\mathcal{G}}_h^k \mathbf{u}_h)|_T = \langle \boldsymbol{\mathcal{G}}_h^k \mathbf{u}_h \rangle_T$, $T \in \mathcal{T}_h$. The usage of the shift $\{|\Pi_h^0 \mathbf{L}_h|\}$ in the operator $\boldsymbol{\mathcal{A}}_{\{|\Pi_h^0 \mathbf{L}_h|\}}$ in (3.4) instead of a zero shift in $\boldsymbol{\mathcal{A}}_0$ in equation (2.33) of [19] and [13] is the key to better approximation properties (*cf.* Thm. 5.4, Cor. 5.5, Rem. 5.6). The flux $\widehat{\mathbf{G}}$ is designed such that it yields the weak form in (3.6).

Proceeding as in [19] and, in addition, using that Π_h^k is self-adjoint, we arrive at the **flux formulation** of (1.1): For given data $\mathbf{u}_D \in \operatorname{tr} W^{1,\varphi}(\Omega)$, $\mathbf{g} \in L^{\varphi^*}(\Omega)$, $\mathbf{G} \in L^{\varphi^*}(\Omega)$ and $\mathbf{a}_N \in L^{\varphi^*}(\Gamma_N)$ find $(\mathbf{u}_h, \mathbf{L}_h, \mathbf{A}_h)^\top \in U_h^k \times X_h^k \times X_h^k$ such that for all $(\mathbf{X}_h, \mathbf{Y}_h, \mathbf{z}_h)^\top \in X_h^k \times X_h^k \times U_h^k$, it holds

$$\begin{aligned} (\mathbf{L}_h, \mathbf{X}_h) &= \Big(\boldsymbol{\mathcal{G}}_h^k \mathbf{u}_h + \boldsymbol{\mathcal{R}}_h^k \mathbf{u}_D^*, \mathbf{X}_h \Big), \\ (\mathbf{A}_h, \mathbf{Y}_h) &= (\boldsymbol{\mathcal{A}}(\mathbf{L}_h), \mathbf{Y}_h), \\ \Big(\mathbf{A}_h, \boldsymbol{\mathcal{G}}_h^k \mathbf{z}_h \Big) &= (\mathbf{g}, \mathbf{z}_h) + \Big(\mathbf{G}, \boldsymbol{\mathcal{G}}_h^k \mathbf{z}_h \Big) + \langle \mathbf{a}_N, \mathbf{z}_h \rangle_{\Gamma_N} \\ &\quad - \alpha \Big\langle \boldsymbol{\mathcal{A}}_{\{|\Pi_h^0 \mathbf{L}_h|\}}\big(h^{-1}[\![(\mathbf{u}_h - \mathbf{u}_D^*) \otimes \mathbf{n}]\!]\big), [\![\mathbf{z}_h \otimes \mathbf{n}]\!] \Big\rangle_{\Gamma_h^i \cup \Gamma_D}. \end{aligned} \tag{3.6}$$

Next, we want to eliminate in the system (3.6) the variables $\mathbf{L}_h \in X_h^k$ and $\mathbf{A}_h \in X_h^k$ to derive a system only expressed in terms of the single variable $\mathbf{u}_h \in U_h^k$. To this end, first note that from (3.6), it follows that

$$\mathbf{L}_h = \boldsymbol{\mathcal{G}}_h^k \mathbf{u}_h + \boldsymbol{\mathcal{R}}_h^k \mathbf{u}_D^* \quad \text{in } X_h^k, \tag{3.7}$$

$$\mathbf{A}_h = \Pi_h^k \boldsymbol{\mathcal{A}}(\mathbf{L}_h) \quad \text{in } X_h^k. \tag{3.8}$$

If we insert (3.7) and (3.8) into $(3.6)_3$, we find that for every $\mathbf{z}_h \in U_h^k$, there holds

$$\begin{aligned} \Big(\boldsymbol{\mathcal{A}}(\boldsymbol{\mathcal{G}}_h^k \mathbf{u}_h + \boldsymbol{\mathcal{R}}_h^k \mathbf{u}_D^*), \boldsymbol{\mathcal{G}}_h^k \mathbf{z}_h \Big) &= (\mathbf{g}, \mathbf{z}_h) + \Big(\mathbf{G}, \boldsymbol{\mathcal{G}}_h^k \mathbf{z}_h \Big) + \langle \mathbf{a}_N, \mathbf{z}_h \rangle_{\Gamma_N} \\ &\quad - \alpha \Big\langle \boldsymbol{\mathcal{A}}_{\{|\Pi_h^0 (\boldsymbol{\mathcal{G}}_h^k \mathbf{u}_h + \boldsymbol{\mathcal{R}}_h^k \mathbf{u}_D^*)|\}}\big(h^{-1}[\![(\mathbf{u}_h - \mathbf{u}_D^*) \otimes \mathbf{n}]\!]\big), [\![\mathbf{z}_h \otimes \mathbf{n}]\!] \Big\rangle_{\Gamma_h^i \cup \Gamma_D}. \end{aligned} \tag{3.9}$$

This is the **primal formulation** of our system. It can be equivalently formulated as

$$(\mathbf{B}_h \mathbf{u}_h, \mathbf{z}_h) = (\mathbf{b}_h, \mathbf{z}_h) \quad \text{for all } \mathbf{z}_h \in U_h^k, \tag{3.10}$$

where the nonlinear operator $\mathbf{B}_h\colon U_h^k \to (U_h^k)^*$ and the linear functional $\mathbf{b}_h\colon U_h^k \to \mathbb{R}$ for every $\mathbf{u}_h, \mathbf{z}_h \in U_h^k$ are defined *via*

$$
\begin{aligned}
(\mathbf{B}_h\mathbf{u}_h, \mathbf{z}_h) &:= \Big(\boldsymbol{\mathcal{A}}(\boldsymbol{\mathcal{G}}_h^k\mathbf{u}_h + \boldsymbol{\mathcal{R}}_h^k\mathbf{u}_D^*), \boldsymbol{\mathcal{G}}_h^k\mathbf{z}_h\Big) \\
&\quad + \alpha\Big\langle \boldsymbol{\mathcal{A}}_{\{|\Pi_h^0(\boldsymbol{\mathcal{G}}_h^k\mathbf{u}_h+\boldsymbol{\mathcal{R}}_h^k\mathbf{u}_D^*)|\}}(h^{-1}[\![(\mathbf{u}_h - \mathbf{u}_D^*)\otimes\mathbf{n}]\!]), [\![\mathbf{z}_h\otimes\mathbf{n}]\!]\Big\rangle_{\Gamma_h^i\cup\Gamma_D}, \qquad (3.11)\\
(\mathbf{b}_h, \mathbf{z}_h) &:= (\mathbf{g}, \mathbf{z}_h) + \Big(\mathbf{G}, \boldsymbol{\mathcal{G}}_h^k\mathbf{z}_h\Big) + \langle \mathbf{a}_N, \mathbf{z}_h\rangle_{\Gamma_N}. \qquad (3.12)
\end{aligned}
$$

4. Well-posedness, stability and weak convergence

In this section, for $k \in \mathbb{N}$ and $\alpha > 0$, we establish the existence of a solution of (3.6), (3.9) and (3.10) (*i.e.*, well-posedness), resp., their stability (*i.e.*, an *a priori* estimate), and the weak convergence of the discrete solutions to a solution of problem (1.1). Our approach extends the treatment in [19]. Although the existence of discrete solution resorts to standard tools, the rigorous argument is involved due to the presence of the shift $\{|\Pi_h^0(\boldsymbol{\mathcal{G}}_h^k\mathbf{u}_h + \boldsymbol{\mathcal{R}}_h^k\mathbf{u}_D^*)|\}$ in the stabilization term. Hence, we present a detailed treatment. We start showing several estimates needed for the existence of discrete solutions and their stability.

Lemma 4.1. *Let $\boldsymbol{\mathcal{A}}$ satisfy Assumption 2.3 for a balanced N-function φ. Then, for every $\mathbf{u}_h \in U_h^k$, we have that*

$$
\begin{aligned}
\big(\mathbf{B}_h\mathbf{u}_h, \mathbf{u}_h - \Pi_h^k\mathbf{u}_D^*\big) &\geq c\,\min\{1,\alpha\}\,M_{\varphi,h}(\mathbf{u}_h - \mathbf{u}_D^*) + c\,\alpha\, m_{\varphi\{|\Pi_h^0(\boldsymbol{\mathcal{G}}_h^k\mathbf{u}_h+\boldsymbol{\mathcal{R}}_h^k\mathbf{u}_D^*)|\}}(\mathbf{u}_h - \mathbf{u}_D^*) - c_\alpha\,\rho_{\varphi,\Omega}(\nabla\mathbf{u}_D^*)\\
&\geq c\,\min\{1,\alpha\}\,M_{\varphi,h}(\mathbf{u}_h) - c_\alpha\,\rho_{\varphi,\Omega}(\nabla\mathbf{u}_D^*) - c\,\min\{1,\alpha\}\,\rho_{\varphi,\Omega}\big(h^{-1}\mathbf{u}_D^*\big) \qquad (4.1)
\end{aligned}
$$

with a constant $c > 0$ depending only on the characteristics of $\boldsymbol{\mathcal{A}}$ and φ, and the chunkiness $\omega_0 > 0$, and a constant $c_\alpha > 0$ additionally depending on $\max\{1,\alpha\}$.

Proof. In view of (3.7), we write $\boldsymbol{\mathcal{A}}_{\{|\Pi_h^0\mathbf{L}_h|\}}$ instead of $\boldsymbol{\mathcal{A}}_{\{|\Pi_h^0(\boldsymbol{\mathcal{G}}_h^k\mathbf{u}_h+\boldsymbol{\mathcal{R}}_h^k\mathbf{u}_D^*)|\}}$ to shorten the notation. Resorting to (2.28), for every $\mathbf{u}_h \in U_h^k$, we find that

$$
\boldsymbol{\mathcal{G}}_h^k\big(\mathbf{u}_h - \Pi_h^k\mathbf{u}_D^*\big) = \Big(\boldsymbol{\mathcal{G}}_h^k\mathbf{u}_h + \boldsymbol{\mathcal{R}}_h^k\mathbf{u}_D^*\Big) - \Big(\boldsymbol{\mathcal{R}}_h^k\mathbf{u}_D^* - \boldsymbol{\mathcal{R}}_h^k\Pi_h^k\mathbf{u}_D^*\Big) - \nabla_h\Pi_h^k\mathbf{u}_D^* \quad \text{in } X_h^k. \qquad (4.2)
$$

Using (4.2), we immediately deduce that for every $\mathbf{u}_h \in U_h^k$, we have that

$$
\begin{aligned}
\big(\mathbf{B}_h\mathbf{u}_h, \mathbf{u}_h - \Pi_h^k\mathbf{u}_D^*\big) &= \Big(\boldsymbol{\mathcal{A}}\Big(\boldsymbol{\mathcal{G}}_h^k\mathbf{u}_h + \boldsymbol{\mathcal{R}}_h^k\mathbf{u}_D^*\Big), \boldsymbol{\mathcal{G}}_h^k\mathbf{u}_h + \boldsymbol{\mathcal{R}}_h^k\mathbf{u}_D^*\Big)\\
&\quad + \alpha\big\langle \boldsymbol{\mathcal{A}}_{\{|\Pi_h^0\mathbf{L}_h|\}}\big(h^{-1}[\![(\mathbf{u}_h - \mathbf{u}_D^*)\otimes\mathbf{n}]\!]\big), [\![(\mathbf{u}_h - \mathbf{u}_D^*)\otimes\mathbf{n}]\!]\big\rangle_{\Gamma_h^i\cup\Gamma_D}\\
&\quad - \Big(\boldsymbol{\mathcal{A}}\Big(\boldsymbol{\mathcal{G}}_h^k\mathbf{u}_h + \boldsymbol{\mathcal{R}}_h^k\mathbf{u}_D^*\Big), \nabla_h\Pi_h^k\mathbf{u}_D^*\Big) - \Big(\boldsymbol{\mathcal{A}}\Big(\boldsymbol{\mathcal{G}}_h^k\mathbf{u}_h + \boldsymbol{\mathcal{R}}_h^k\mathbf{u}_D^*\Big), \boldsymbol{\mathcal{R}}_h^k\big(\mathbf{u}_D^* - \Pi_h^k\mathbf{u}_D^*\big)\Big)\\
&\quad - \alpha\Big\langle \boldsymbol{\mathcal{A}}_{\{|\Pi_h^0\mathbf{L}_h|\}}\big(h^{-1}[\![(\mathbf{u}_h - \mathbf{u}_D^*)\otimes\mathbf{n}]\!]\big), [\![\big(\Pi_h^k\mathbf{u}_D^* - \mathbf{u}_D^*\big)\otimes\mathbf{n}]\!]\Big\rangle_{\Gamma_h^i\cup\Gamma_D}\\
&=: I_1 + \alpha\, I_2 + I_3 + I_4 + I_5. \qquad (4.3)
\end{aligned}
$$

Before we estimate the terms I_i, $i = 3,\dots,5$, we collect the information we obtain from the terms I_1 and I_2. From Proposition 2.8, for every $\mathbf{u}_h \in U_h^k$, it follows that

$$
I_1 \sim \rho_{\varphi,\Omega}\Big(\boldsymbol{\mathcal{G}}_h^k\mathbf{u}_h + \boldsymbol{\mathcal{R}}_h^k\mathbf{u}_D^*\Big), \quad I_2 \sim m_{\varphi_{\{|\Pi_h^0\mathbf{L}_h|\}},h}(\mathbf{u}_h - \mathbf{u}_D^*). \qquad (4.4)
$$

Resorting, again, to (2.28), for every $\mathbf{u}_h \in U_h^k$, we observe that

$$
\boldsymbol{\mathcal{G}}_h^k\mathbf{u}_h + \boldsymbol{\mathcal{R}}_h^k\mathbf{u}_D^* = \boldsymbol{\mathcal{G}}_h^k(\mathbf{u}_h - \mathbf{u}_D^*) + \nabla\mathbf{u}_D^* \quad \text{in } L^\varphi(\Omega), \qquad (4.5)
$$

which together with the properties of φ implies for every $\mathbf{u}_h \in U_h^k$ that

$$\rho_{\varphi,\Omega}\Big(\boldsymbol{\mathcal{G}}_h^k(\mathbf{u}_h - \mathbf{u}_D^*)\Big) \le c\, I_1 + c\, \rho_{\varphi,\Omega}(\nabla \mathbf{u}_D^*). \tag{4.6}$$

The shift change from Lemma 2.7, (3.7), (4.4), the trace inequality (A.16), (4.5), the L^φ-stability of Π_h^k in (A.8) with $k = 0$, and (4.6), for every $\mathbf{u}_h \in U_h^k$, yield

$$\begin{aligned}
m_{\varphi,h}(\mathbf{u}_h - \mathbf{u}_D^*) &\le c\, m_{\varphi_{\{|\Pi_h^0 \mathbf{L}_h|\}},h}(\mathbf{u}_h - \mathbf{u}_D^*) + c\, h\, \rho_{\varphi,\Gamma_h^i \cup \Gamma_D}\Big(\Big\{\Big|\Pi_h^0\Big(\boldsymbol{\mathcal{G}}_h^k \mathbf{u}_h + \boldsymbol{\mathcal{R}}_h^k \mathbf{u}_D^*\Big)\Big|\Big\}\Big)\\
&\le c\, I_2 + c\, \rho_{\varphi,\Omega}\Big(\Pi_h^0\Big(\boldsymbol{\mathcal{G}}_h^k(\mathbf{u}_h - \mathbf{u}_D^*) + \nabla \mathbf{u}_D^*\Big)\Big)\\
&\le c\, I_2 + c\, \rho_{\varphi,\Omega}\Big(\boldsymbol{\mathcal{G}}_h^k(\mathbf{u}_h - \mathbf{u}_D^*) + \nabla \mathbf{u}_D^*\Big)\\
&\le c\, I_2 + c\, \rho_{\varphi,\Omega}\Big(\boldsymbol{\mathcal{G}}_h^k(\mathbf{u}_h - \mathbf{u}_D^*)\Big) + c\, \rho_{\varphi,\Omega}(\nabla \mathbf{u}_D^*)\\
&\le c\, I_2 + c\, I_1 + c\, \rho_{\varphi,\Omega}(\nabla \mathbf{u}_D^*).
\end{aligned} \tag{4.7}$$

From (4.6), (4.7) and the equivalent expression for $M_{\varphi,h}$ in Lemma A.2, it follows that

$$M_{\varphi,h}(\mathbf{u}_h - \mathbf{u}_D^*) \le c\, I_1 + c\, I_2 + c\, \rho_{\varphi,\Omega}(\nabla \mathbf{u}_D^*),$$

which implies that

$$\alpha\, m_{\varphi_{\{|\Pi_h^0 \mathbf{L}_h|\}},h}(\mathbf{u}_h - \mathbf{u}_D^*) + \min\{1,\alpha\}\, M_{\varphi,h}(\mathbf{u}_h - \mathbf{u}_D^*) \le c\, I_1 + c\, \alpha\, I_2 + c\, \rho_{\varphi,\Omega}(\nabla \mathbf{u}_D^*). \tag{4.8}$$

Now we can estimate the remaining terms. Using (2.14), the ε-Young inequality (2.2), (2.1), the L^φ-gradient stability of Π_h^k in (A.9), and (4.6), we have that

$$\begin{aligned}
|I_3| &\le \varepsilon\, \rho_{\varphi^*,\Omega}\Big(\varphi'\Big(\Big|\boldsymbol{\mathcal{G}}_h^k \mathbf{u}_h + \boldsymbol{\mathcal{R}}_h^k \mathbf{u}_D^*\Big|\Big)\Big) + c_\varepsilon\, \rho_{\varphi,\Omega}\big(\nabla_h \Pi_h^k \mathbf{u}_D^*\big)\\
&\le \varepsilon\, \rho_{\varphi,\Omega}\Big(\boldsymbol{\mathcal{G}}_h^k \mathbf{u}_h + \boldsymbol{\mathcal{R}}_h^k \mathbf{u}_D^*\Big) + c_\varepsilon\, \rho_{\varphi,\Omega}(\nabla \mathbf{u}_D^*)\\
&\le \varepsilon\, c\, I_1 + c_\varepsilon\, \rho_{\varphi,\Omega}(\nabla \mathbf{u}_D^*),
\end{aligned} \tag{4.9}$$

and with the stability of $\boldsymbol{\mathcal{R}}_h^k$ in (A.1) and the approximation property of Π_h^k in (A.14)

$$\begin{aligned}
|I_4| &\le \varepsilon\, \rho_{\varphi^*,\Omega}\Big(\varphi'\Big(\Big|\boldsymbol{\mathcal{G}}_h^k \mathbf{u}_h + \boldsymbol{\mathcal{R}}_h^k \mathbf{u}_D^*\Big|\Big)\Big) + c_\varepsilon\, \rho_{\varphi,\Omega}\Big(\boldsymbol{\mathcal{R}}_h^k\big(\mathbf{u}_D^* - \Pi_h^k \mathbf{u}_D^*\big)\Big)\\
&\le \varepsilon\, \rho_{\varphi,\Omega}\Big(\boldsymbol{\mathcal{G}}_h^k \mathbf{u}_h + \boldsymbol{\mathcal{R}}_h^k \mathbf{u}_D^*\Big) + c_\varepsilon\, m_{\varphi,h}\big(\mathbf{u}_D^* - \Pi_h^k \mathbf{u}_D^*\big)\\
&\le \varepsilon\, c\, I_1 + c_\varepsilon\, \rho_{\varphi,\Omega}(\nabla \mathbf{u}_D^*).
\end{aligned} \tag{4.10}$$

Using the definition of $\boldsymbol{\mathcal{A}}_{\{|\Pi_h^0 \mathbf{L}_h|\}}$ in (3.2), the ε-Young inequality (2.2) with $\varphi_{\{|\Pi_h^0 \mathbf{L}_h|\}}$, (2.1), the shift change in Lemma 2.7, (3.7), the approximation property of Π_h^k in (A.14), and proceeding as in (4.7) to handle $h\rho_{\varphi,\Gamma_h^i \cup \Gamma_D}(\{|\Pi_h^0(\boldsymbol{\mathcal{G}}_h^k \mathbf{u}_h + \boldsymbol{\mathcal{R}}_h^k \mathbf{u}_D^*)|\})$, we get

$$\begin{aligned}
|I_5| &\le \alpha\, h \Big|\Big\langle \varphi'_{\{|\Pi_h^0 \mathbf{L}_h|\}}\big(\big|h^{-1}[\![(\mathbf{u}_h - \mathbf{u}_D^*) \otimes \mathbf{n}]\!]\big|\big), h^{-1}[\![(\mathbf{u}_D^* - \Pi_h^k \mathbf{u}_D^*) \otimes \mathbf{n}]\!]\Big\rangle_{\Gamma_h^i \cup \Gamma_D}\Big|\\
&\le \varepsilon\, h\, \alpha\, \rho_{\varphi^*_{\{|\Pi_h^0 \mathbf{L}_h|\}},\Gamma_h^i \cup \Gamma_D}\Big(\varphi'_{\{|\Pi_h^0 \mathbf{L}_h|\}}\big(\big|h^{-1}[\![(\mathbf{u}_h - \mathbf{u}_D^*) \otimes \mathbf{n}]\!]\big|\big)\Big) + c_\varepsilon\, \alpha\, m_{\varphi_{\{|\Pi_h^0 \mathbf{L}_h|\}},h}\big(\mathbf{u}_D^* - \Pi_h^k \mathbf{u}_D^*\big)\\
&\le \varepsilon\, c\, \alpha\, m_{\varphi_{\{|\Pi_h^0 \mathbf{L}_h|\}},h}(\mathbf{u}_h - \mathbf{u}_D^*) + c_\varepsilon\, c_\kappa\, \alpha\, m_{\varphi,h}\big(\mathbf{u}_D^* - \Pi_h^k \mathbf{u}_D^*\big) + \kappa\, c_\varepsilon\, \alpha\, h\, \rho_{\varphi,\Gamma_h^i \cup \Gamma_D}\Big(\Big\{\Big|\Pi_h^0\Big(\boldsymbol{\mathcal{G}}_h^k \mathbf{u}_h + \boldsymbol{\mathcal{R}}_h^k \mathbf{u}_D^*\Big)\Big|\Big\}\Big)\\
&\le \varepsilon\, c\, \alpha\, I_2 + \kappa\, c_\varepsilon\, c\, \max\{1,\alpha\}\, I_1 + c_\kappa\, c_\varepsilon\, \max\{1,\alpha\}\, \rho_{\varphi,\Omega}(\nabla \mathbf{u}_D^*).
\end{aligned} \tag{4.11}$$

Choosing first $\varepsilon > 0$ and, then, $\kappa > 0$ sufficiently small, we conclude from (4.3), (4.8)–(4.11) that the first inequality in (4.1) applies. Then, the second one follows from the first one and

$$\begin{aligned} M_{\varphi,h}(\mathbf{u}_h - \mathbf{u}_D^*) &\geq c\, M_{\varphi,h}(\mathbf{u}_h) - c\, M_{\varphi,h}(\mathbf{u}_D^*) \\ &\geq c\, M_{\varphi,h}(\mathbf{u}_h) - c\, \rho_{\varphi,\Omega}(\nabla \mathbf{u}_D^*) - c\, \rho_{\varphi,\Omega}\big(h^{-1}\mathbf{u}_D^*\big), \end{aligned} \tag{4.12}$$

where we used the definition of $M_{\varphi,h}$ in (2.30) and the trace inequality (A.15). □

Lemma 4.2. *Let $\boldsymbol{\mathcal{A}}$ satisfy Assumption 2.3 for a balanced N-function φ. Then, for every $\mathbf{u}_h \in U_h^k$ there holds*

$$|(\mathbf{B}_h \mathbf{u}_h, \mathbf{u}_h)| \geq c\, M_{\varphi,h}(\mathbf{u}_h) - c\,\big(\rho_{\varphi,\Omega}(\nabla \mathbf{u}_D^*) + \rho_{\varphi,\Omega}(h^{-1}\mathbf{u}_D^*)\big)$$

with a constant $c > 0$ depending only on $\alpha > 0$, the characteristics of $\boldsymbol{\mathcal{A}}$ and φ, and the chunkiness $\omega_0 > 0$.

Proof. In view of (3.7), we write $\boldsymbol{\mathcal{A}}_{\{|\Pi_h^0 \mathbf{L}_h|\}}$ instead of $\boldsymbol{\mathcal{A}}_{\{|\Pi_h^0(\boldsymbol{\mathcal{G}}_h^k \mathbf{u}_h + \boldsymbol{\mathcal{R}}_h^k \mathbf{u}_D^*)|\}}$. Then, for every $\mathbf{u}_h \in U_h^k$, we have that

$$\begin{aligned} (\mathbf{B}_h \mathbf{u}_h, \mathbf{u}_h) &= \Big(\boldsymbol{\mathcal{A}}(\boldsymbol{\mathcal{G}}_h^k \mathbf{u}_h + \boldsymbol{\mathcal{R}}_h^k \mathbf{u}_D^*), \boldsymbol{\mathcal{G}}_h^k \mathbf{u}_h + \boldsymbol{\mathcal{R}}_h^k \mathbf{u}_D^*\Big) \\ &\quad + \alpha \Big\langle \boldsymbol{\mathcal{A}}_{\{|\Pi_h^0 \mathbf{L}_h|\}}\big(h^{-1}[\![(\mathbf{u}_h - \mathbf{u}_D^*) \otimes \mathbf{n}]\!]\big), [\![(\mathbf{u}_h - \mathbf{u}_D^*) \otimes \mathbf{n}]\!]\Big\rangle_{\Gamma_h^i \cup \Gamma_D} \\ &\quad - \Big(\boldsymbol{\mathcal{A}}\Big(\boldsymbol{\mathcal{G}}_h^k \mathbf{u}_h + \boldsymbol{\mathcal{R}}_h^k \mathbf{u}_D^*\Big), \boldsymbol{\mathcal{R}}_h^k \mathbf{u}_D^*\Big) \\ &\quad + \alpha \Big\langle \boldsymbol{\mathcal{A}}_{\{|\Pi_h^0 \mathbf{L}_h|\}}\big(h^{-1}[\![(\mathbf{u}_h - \mathbf{u}_D^*) \otimes \mathbf{n}]\!]\big), [\![\mathbf{u}_D^* \otimes \mathbf{n}]\!]\Big\rangle_{\Gamma_h^i \cup \Gamma_D} \\ &=: I_1 + \alpha\, I_2 + I_3 + I_4. \end{aligned} \tag{4.13}$$

Proceeding as in the proof of Lemma 4.1, for every $\mathbf{u}_h \in U_h^k$, we find that (*cf.* (4.8))

$$I_1 + \alpha\, I_2 \geq c\, m_{\varphi_{\{|\Pi_h^0 \mathbf{L}_h|\}},h}(\mathbf{u}_h - \mathbf{u}_D^*) + c\, M_{\varphi,h}(\mathbf{u}_h - \mathbf{u}_D^*) - c\, \rho_{\varphi,\Omega}(\nabla \mathbf{u}_D^*)$$

with $c > 0$ depending on $\alpha > 0$. Using the ε-Young inequality (2.2), (2.1), the stability of $\boldsymbol{\mathcal{R}}_h^k$ in (A.1), (4.5), the trace inequality (A.15), (A.2), for all $\mathbf{u}_h \in U_h^k$, we get (*cf.* (4.10))

$$\begin{aligned} |I_3| &\leq \varepsilon\, \rho_{\varphi,\Omega}\Big(\boldsymbol{\mathcal{G}}_h^k \mathbf{u}_h + \boldsymbol{\mathcal{R}}_h^k \mathbf{u}_D^*\Big) + c_\varepsilon\, m_{\varphi,h}(\mathbf{u}_D^*) \\ &\leq \varepsilon\, c\, \rho_{\varphi,\Omega}\Big(\boldsymbol{\mathcal{G}}_h^k (\mathbf{u}_h - \mathbf{u}_D^*)\Big) + c_\varepsilon\, \rho_{\varphi,\Omega}(\nabla \mathbf{u}_D^*) + c_\varepsilon\, \rho_{\varphi,\Omega}\big(h^{-1}\mathbf{u}_D^*\big) \\ &\leq \varepsilon\, c\, M_{\varphi,h}(\mathbf{u}_h - \mathbf{u}_D^*) + c_\varepsilon\, \rho_{\varphi,\Omega}(\nabla \mathbf{u}_D^*) + c_\varepsilon\, \rho_{\varphi,\Omega}\big(h^{-1}\, \mathbf{u}_D^*\big). \end{aligned}$$

Proceeding as in the estimate (4.11) and, in addition, using the equivalent expression for $M_{\varphi,h}$ in (A.2), for every $\mathbf{u}_h \in U_h^k$, we obtain

$$\begin{aligned} |I_4| &\leq \alpha\, h \Big| \Big\langle \varphi'_{\{|\Pi_h^0 \mathbf{L}_h|\}}\big(|h^{-1}[\![(\mathbf{u}_h - \mathbf{u}_D^*) \otimes \mathbf{n}]\!]|\big), h^{-1}[\![\mathbf{u}_D^* \otimes \mathbf{n}]\!]\Big\rangle_{\Gamma_h^i \cup \Gamma_D} \Big| \\ &\leq \varepsilon\, m_{\varphi_{\{|\Pi_h^0 \mathbf{L}_h|\}},h}(\mathbf{u}_h - \mathbf{u}_D^*) + c_\kappa\, c_\varepsilon\, m_{\varphi,h}(\mathbf{u}_D^*) + \kappa\, c_\varepsilon\, h\, \rho_{\varphi,\Gamma_h^i \cup \Gamma_D}\Big(\Big\{\Big|\Pi_h^0\Big(\boldsymbol{\mathcal{G}}_h^k \mathbf{u}_h + \boldsymbol{\mathcal{R}}_h^k \mathbf{u}_D^*\Big)\Big|\Big\}\Big) \\ &\leq \varepsilon\, m_{\varphi_{\{|\Pi_h^0 \mathbf{L}_h|\}},h}(\mathbf{u}_h - \mathbf{u}_D^*) + c_\kappa\, c_\varepsilon\, \rho_{\varphi,\Omega}(\nabla \mathbf{u}_D^*) + c_\kappa\, c_\varepsilon\, \rho_{\varphi,\Omega}\big(h^{-1}\, \mathbf{u}_D^*\big) + \kappa\, c_\varepsilon\, \rho_{\varphi,\Omega}\Big(\boldsymbol{\mathcal{G}}_h^k (\mathbf{u}_h - \mathbf{u}_D^*)\Big) \\ &\leq \varepsilon\, m_{\varphi_{\{|\Pi_h^0 \mathbf{L}_h|\}},h}(\mathbf{u}_h - \mathbf{u}_D^*) + c_\kappa\, c_\varepsilon\, \rho_{\varphi,\Omega}(\nabla \mathbf{u}_D^*) + c_\kappa\, c_\varepsilon\, \rho_{\varphi,\Omega}\big(h^{-1}\, \mathbf{u}_D^*\big) + \kappa\, c_\varepsilon\, M_{\varphi,h}(\mathbf{u}_h - \mathbf{u}_D^*) \end{aligned}$$

with constants $c_\kappa, c_\varepsilon > 0$ depending on $\alpha > 0$. Choosing first $\varepsilon > 0$ and than $\kappa > 0$ sufficiently small, the last three estimates and (4.12) prove the assertion. □

Lemma 4.3. *Let $\boldsymbol{\mathcal{A}}$ satisfy Assumption 2.3 for a balanced N-function φ. Then, for every $\varepsilon > 0$, there exists a constant $c_\varepsilon > 0$, depending on the characteristics of $\boldsymbol{\mathcal{A}}$ and φ and the chunkiness ω_0, such that for every $\mathbf{u}_h \in U_h^k$, we have that*

$$\left|\left(\mathbf{b}_h, \mathbf{u}_h - \Pi_h^k \mathbf{u}_D^*\right)\right| \leq \varepsilon\, M_{\varphi,h}(\mathbf{u}_h - \mathbf{u}_D^*) + c_\varepsilon\, \rho_{\varphi^*,\Omega}(\mathbf{g}) + c_\varepsilon\, \rho_{\varphi^*,\Omega}(\mathbf{G}) + c_\varepsilon\, \rho_{\varphi^*,\Gamma_N}(\mathbf{a}_N) + c_\varepsilon\, \rho_{\varphi,\Omega}(\nabla \mathbf{u}_D^*).$$

Proof. Using the identities (4.2) and (4.5), for every $\mathbf{u}_h \in U_h^k$, we observe that

$$\begin{aligned}\left(\mathbf{b}_h, \mathbf{u}_h - \Pi_h^k \mathbf{u}_D^*\right) &= (\mathbf{g}, \mathbf{u}_h - \mathbf{u}_D^*) + \left(\mathbf{g}, \mathbf{u}_D^* - \Pi_h^k \mathbf{u}_D^*\right) + \left(\mathbf{G}, \boldsymbol{\mathcal{G}}_h^k(\mathbf{u}_h - \mathbf{u}_D^*)\right) + \left(\mathbf{G}, \boldsymbol{\mathcal{R}}_h^k\left(\mathbf{u}_D^* - \Pi_h^k \mathbf{u}_D^*\right)\right) \\ &\quad + \langle \mathbf{a}_N, \mathbf{u}_h - \mathbf{u}_D^*\rangle_{\Gamma_N} + \left\langle \mathbf{a}_N, \mathbf{u}_D^* - \Pi_h^k \mathbf{u}_D^*\right\rangle_{\Gamma_N} \\ &=: J_1 + \ldots + J_6.\end{aligned} \tag{4.14}$$

Resorting to the ε-Young inequality (2.2), Poincaré's inequality (A.25) and the approximation property of Π_h^k in (A.6) in doing so, for every $\mathbf{u}_h \in U_h^k$, we find that

$$\begin{aligned}|J_1 + J_2| &\leq \varepsilon\, \rho_{\varphi,\Omega}(\mathbf{u}_h - \mathbf{u}_D^*) + c_\varepsilon\, \rho_{\varphi^*,\Omega}(\mathbf{g}) + c\, \rho_{\varphi,\Omega}\left(\mathbf{u}_D^* - \Pi_h^k \mathbf{u}_D^*\right) \\ &\leq \varepsilon\, c\, M_{\varphi,h}(\mathbf{u}_h - \mathbf{u}_D^*) + c_\varepsilon\, \rho_{\varphi^*,\Omega}(\mathbf{g}) + c\, \rho_{\varphi,\Omega}(h\, \nabla \mathbf{u}_D^*) \\ &\leq \varepsilon\, c\, M_{\varphi,h}(\mathbf{u}_h - \mathbf{u}_D^*) + c_\varepsilon\, \rho_{\varphi^*,\Omega}(\mathbf{g}) + c\, \rho_{\varphi,\Omega}(\nabla \mathbf{u}_D^*).\end{aligned}$$

Resorting, again, to the ε-Young inequality (2.2), using the equivalent expression for $M_{\varphi,h}$ in (A.2), the stability of $\boldsymbol{\mathcal{R}}_h^k$ in (A.1) and the approximation property of Π_h^k in (A.14), for every $\mathbf{u}_h \in U_h^k$, we deduce that

$$\begin{aligned}|J_3 + J_4| &\leq \varepsilon\, \rho_{\varphi,\Omega}\left(\boldsymbol{\mathcal{G}}_h^k(\mathbf{u}_h - \mathbf{u}_D^*)\right) + c_\varepsilon\, \rho_{\varphi^*,\Omega}(\mathbf{G}) + c\, \rho_{\varphi,\Omega}\left(\boldsymbol{\mathcal{R}}_h^k\left(\mathbf{u}_D^* - \Pi_h^k \mathbf{u}_D^*\right)\right) \\ &\leq \varepsilon\, c\, M_{\varphi,h}(\mathbf{u}_h - \mathbf{u}_D^*) + c_\varepsilon\, \rho_{\varphi^*,\Omega}(\mathbf{G}) + c\, m_{\varphi,h}\left(\mathbf{u}_D^* - \Pi_h^k \mathbf{u}_D^*\right) \\ &\leq \varepsilon\, c\, M_{\varphi,h}(\mathbf{u}_h - \mathbf{u}_D^*) + c_\varepsilon\, \rho_{\varphi^*,\Omega}(\mathbf{G}) + c\, \rho_{\varphi,\Omega}(\nabla \mathbf{u}_D^*).\end{aligned}$$

Finally, we estimate, using the ε-Young inequality (2.2), the trace inequality (A.26), the L^φ-gradient stability of Π_h^k in (A.7) and the approximation property of Π_h^k in (A.14) to conclude for every $\mathbf{u}_h \in U_h^k$ that

$$\begin{aligned}|J_5 + J_6| &\leq c_\varepsilon\, \rho_{\varphi^*,\Gamma_N}(\mathbf{a}_N) + \varepsilon\, \rho_{\varphi,\Gamma_N}(\mathbf{u}_h - \mathbf{u}_D^*) + c\, \rho_{\varphi,\Gamma_N}\left(\mathbf{u}_D^* - \Pi_h^k \mathbf{u}_D^*\right) \\ &\leq c_\varepsilon\, \rho_{\varphi^*,\Gamma_N}(\mathbf{a}_N) + \varepsilon\, c\, M_{\varphi,h}(\mathbf{u}_h - \mathbf{u}_D^*) + c\, M_{\varphi,h}\left(\mathbf{u}_D^* - \Pi_h^k \mathbf{u}_D^*\right) \\ &\leq c_\varepsilon\, \rho_{\varphi^*,\Gamma_N}(\mathbf{a}_N) + \varepsilon\, c\, M_{\varphi,h}(\mathbf{u}_h - \mathbf{u}_D^*) + c\, \rho_{\varphi,\Omega}(\nabla \mathbf{u}_D^*).\end{aligned}$$

□

Lemma 4.4. *Let $\boldsymbol{\mathcal{A}}$ satisfy Assumption 2.3 for a balanced N-function φ. Then, for every $\varepsilon > 0$, there exists a constant $c_\varepsilon > 0$ such that for every $\mathbf{u}_h \in U_h^k$, we have that*

$$|(\mathbf{b}_h, \mathbf{u}_h)| \leq \varepsilon\, M_{\varphi,h}(\mathbf{u}_h) + c_\varepsilon\, \rho_{\varphi^*,\Omega}(\mathbf{g}) + c_\varepsilon\, \rho_{\varphi^*,\Omega}(\mathbf{G}) + c_\varepsilon\, \rho_{\varphi^*,\Gamma_N}(\mathbf{a}_N).$$

Proof. The assertion follows, using the same tools as in the proof of Lemma 4.3. □

Now we have everything at our disposal to prove the existence of discrete solutions and their stability.

Proposition 4.5 (Well-posedness)**.** *Let $\boldsymbol{\mathcal{A}}$ satisfy Assumption 2.3 for a balanced N-function φ. Then, for given data $\mathbf{u}_D \in \operatorname{tr} W^{1,\varphi}(\Omega)$, $\mathbf{g} \in L^{\varphi^*}(\Omega)$, $\mathbf{G} \in L^{\varphi^*}(\Omega)$ and $\mathbf{a}_N \in L^{\varphi^*}(\Gamma_N)$ as well as given $k \in \mathbb{N}$ and $\alpha > 0$, $h > 0$, there exist $\mathbf{u}_h \in U_h^k$ solving (3.9), and $(\mathbf{L}_h, \mathbf{A}_h)^\top \in X_h^k \times X_h^k$ such that $(\mathbf{u}_h, \mathbf{L}_h, \mathbf{A}_h)^\top$ solves (3.6).*

Proof. Lemmas 4.2 and 4.4 with $\varepsilon > 0$ small enough yield for every $\mathbf{u}_h \in U_h^k$

$$\begin{aligned}(\mathbf{B}_h\mathbf{u}_h, \mathbf{u}_h) - (\mathbf{b}_h, \mathbf{u}_h) \geq c\, M_{\varphi,h}(\mathbf{u}_h) - c\,\rho_{\varphi,\Omega}(\nabla \mathbf{u}_D^*) - c\,\rho_{\varphi,\Omega}\big(h^{-1}\mathbf{u}_D^*\big) \\ - c\,\rho_{\varphi^*,\Omega}(\mathbf{g}) - c\,\rho_{\varphi^*,\Omega}(\mathbf{G}) - c\,\rho_{\varphi^*,\Gamma_N}(\mathbf{a}_N).\end{aligned} \tag{4.15}$$

We equip U_h^k with the norm $\|\cdot\|_{M_{\varphi,h}}$. Using that $\|\mathbf{u}_h\|_{M_{\varphi,h}} \leq M_{\varphi,h}(\mathbf{u}_h)$ if $\|\mathbf{u}_h\|_{M_{\varphi,h}} \geq 1$, we find that the right-hand side in (4.15) converges to infinity for $\|\mathbf{u}_h\|_{M_{\varphi,h}} \to \infty$. Thus, Brouwer's fixed point theorem yields the existence of $\mathbf{u}_h \in U_h^k$ solving (3.9). In turn, we also conclude the existence of $\mathbf{L}_h \in X_h^k$ and $\mathbf{A}_h \in X_h^k$ from (3.7) and (3.8), respectively, which shows the solvability of (3.6). □

Proposition 4.6 (Stability)**.** *Let $\boldsymbol{\mathcal{A}}$ satisfy Assumption 2.3 for a balanced N-function φ. Moreover, let $\mathbf{u}_h \in U_h^k$ be a solution of (3.9) for $\alpha > 0$, $h > 0$, and $k \in \mathbb{N}$. Then, it holds*

$$\begin{aligned}M_{\varphi,h}(\mathbf{u}_h - \mathbf{u}_D^*) + m_{\varphi_{\{|\Pi_h^0(\boldsymbol{\mathcal{G}}_h^k\mathbf{u}_h + \boldsymbol{\mathcal{R}}_h^k\mathbf{u}_D^*)|\}},h}(\mathbf{u}_h - \mathbf{u}_D^*) \\ \leq c\,\rho_{\varphi,\Omega}(\nabla \mathbf{u}_D^*) + c\,\rho_{\varphi^*,\Omega}(\mathbf{g}) + c\,\rho_{\varphi^*,\Omega}(\mathbf{G}) + c\,\rho_{\varphi^*,\Gamma_N}(\mathbf{a}_N)\end{aligned} \tag{4.16}$$

with a constant $c > 0$ depending only on $\alpha > 0$, the characteristics of $\boldsymbol{\mathcal{A}}$ and φ, $k \in \mathbb{N}$, and the chunkiness $\omega_0 > 0$.

Proof. The assertion follows by combining Lemmas 4.1 and 4.3 for $\varepsilon > 0$ sufficiently small, if we choose $\mathbf{z}_h = \mathbf{u}_h - \Pi_h^k\mathbf{u}_D^* \in U_h^k$ in (3.10). □

Corollary 4.7. *Let $\boldsymbol{\mathcal{A}}$ satisfy Assumption 2.3 for a balanced N-function φ, and let $(\mathbf{u}_h, \mathbf{L}_h, \mathbf{A}_h)^\top \in U_h^k \times X_h^k \times X_h^k$ be a solution of (3.6) for $\alpha > 0$, $h > 0$ and $k \in \mathbb{N}$. Then, it holds*

$$\begin{aligned}\rho_{\varphi,\Omega}(\mathbf{L}_h) + \rho_{\varphi^*,\Omega}(\mathbf{A}_h) + h\rho_{\varphi^*,\Gamma_h^i\cup\Gamma_D}\Big(\boldsymbol{\mathcal{A}}_{\{|\Pi_h^0\mathbf{L}_h|\}}\big(h^{-1}[\![(\mathbf{u}_h - \mathbf{u}_D^*)\otimes\mathbf{n}]\!]\big)\Big) \\ \leq c\,\rho_{\varphi,\Omega}(\nabla \mathbf{u}_D^*) + c\,\rho_{\varphi^*,\Omega}(\mathbf{g}) + c\,\rho_{\varphi^*,\Omega}(\mathbf{G}) + c\,\rho_{\varphi^*,\Gamma_N}(\mathbf{a}_N)\end{aligned}$$

with a constant $c > 0$ depending only on $\alpha > 0$, the characteristics of $\boldsymbol{\mathcal{A}}$ and φ, and the chunkiness $\omega_0 > 0$.

Proof. From (3.7) and (4.5), it follows that $\mathbf{L}_h = \boldsymbol{\mathcal{G}}_h^k(\mathbf{u}_h - \mathbf{u}_D^*) + \nabla\mathbf{u}_D^*$ in $L^\varphi(\Omega)$, which together with the properties of φ and the equivalent expression for the modular $M_{\varphi,h}$ in Lemma A.2, implies that

$$\begin{aligned}\rho_{\varphi,\Omega}(\mathbf{L}_h) \leq c\,\rho_{\varphi,\Omega}\Big(\boldsymbol{\mathcal{G}}_h^k(\mathbf{u}_h - \mathbf{u}_D^*)\Big) + c\,\rho_{\varphi,\Omega}(\nabla\mathbf{u}_D^*) \\ \leq c\,M_{\varphi,h}(\mathbf{u}_h - \mathbf{u}_D^*) + c\,\rho_{\varphi,\Omega}(\nabla\mathbf{u}_D^*).\end{aligned} \tag{4.17}$$

From (3.8) we know that $\mathbf{A}_h = \Pi_h^k\boldsymbol{\mathcal{A}}(\mathbf{L}_h)$ in X_h^k, so the stability of Π_h^k in (A.8), (2.14) and (2.1) imply that

$$\rho_{\varphi^*,\Omega}(\mathbf{A}_h) \leq c\,\rho_{\varphi^*,\Omega}(\varphi'(|\mathbf{L}_h|)) \leq c\,\rho_{\varphi,\Omega}(\mathbf{L}_h). \tag{4.18}$$

In addition, using the shift change in (2.8) and (2.12), we find that

$$\begin{aligned}\Big|\boldsymbol{\mathcal{A}}_{\{|\Pi_h^0\mathbf{L}_h|\}}\big(h^{-1}[\![(\mathbf{u}_h - \mathbf{u}_D^*)\otimes\mathbf{n}]\!]\big)\Big| &= \varphi'_{\{|\Pi_h^0\mathbf{L}_h|\}}\big(h^{-1}|[\![(\mathbf{u}_h - \mathbf{u}_D^*)\otimes\mathbf{n}]\!]|\big) \\ &\leq c\,\varphi'\big(h^{-1}|[\![(\mathbf{u}_h - \mathbf{u}_D^*)\otimes\mathbf{n}]\!]|\big) + c\,\varphi'\big(\{|\Pi_h^0\mathbf{L}_h|\}\big),\end{aligned}$$

which, using (2.1), the discrete trace inequality (A.12), and the L^φ-stability of Π_h^0 in (A.8), implies that

$$\begin{aligned}h\rho_{\varphi^*,\Gamma_h^i\cup\Gamma_D}\Big(\boldsymbol{\mathcal{A}}_{\{|\Pi_h^0\mathbf{L}_h|\}}\big(h^{-1}[\![(\mathbf{u}_h - \mathbf{u}_D^*)\otimes\mathbf{n}]\!]\big)\Big) &\leq c\,h\,\rho_{\varphi^*,\Gamma_h^i\cup\Gamma_D}\big(\varphi'(h^{-1}|[\![(\mathbf{u}_h - \mathbf{u}_D^*)\otimes\mathbf{n}]\!]|)\big) \\ &\quad + c\,h\,\rho_{\varphi^*,\Gamma_h^i\cup\Gamma_D}\big(\varphi'\big(\{|\Pi_h^0\mathbf{L}_h|\}\big)\big) \\ &\leq c\,m_{\varphi,h}(\mathbf{u}_h - \mathbf{u}_D^*) + c\,h\,\rho_{\varphi,\Gamma_h^i\cup\Gamma_D}\big(\{\Pi_h^0\mathbf{L}_h\}\big) \\ &\leq c\,m_{\varphi,h}(\mathbf{u}_h - \mathbf{u}_D^*) + c\,\rho_{\varphi,\Omega}\big(\Pi_h^0\mathbf{L}_h\big) \\ &\leq c\,M_{\varphi,h}(\mathbf{u}_h - \mathbf{u}_D^*) + c\,\rho_{\varphi,\Omega}(\mathbf{L}_h).\end{aligned} \tag{4.19}$$

Resorting to Proposition 4.6, we conclude the assertion from (4.17) to (4.19). □

Theorem 4.8 (Convergence)**.** *Let $\boldsymbol{\mathcal{A}}$ satisfy Assumption 2.3 for a balanced N-function φ. For every $h > 0$, let $(\mathbf{u}_h, \mathbf{L}_h, \mathbf{A}_h)^\top \in U_h^k \times X_h^k \times X_h^k$ be a solution of (3.6) for $\alpha > 0$ and $k \in \mathbb{N}$. Moreover, let $(h_n)_{n\in\mathbb{N}} \subseteq \mathbb{R}^{>0}$ be any sequence such that $h_n \to 0$ $(n \to \infty)$ and define $(\mathbf{u}_n, \mathbf{L}_n, \mathbf{A}_n)^\top := (\mathbf{u}_{h_n}, \mathbf{L}_{h_n}, \mathbf{A}_{h_n})^\top \in U_{h_n}^k \times X_{h_n}^k \times X_{h_n}^k$, $n \in \mathbb{N}$. Then, it holds*

$$\begin{aligned} \mathbf{u}_n &\rightharpoonup \mathbf{u} && \text{in } L^\varphi(\Omega) && (n \to \infty), \\ \mathbf{L}_n &\rightharpoonup \nabla\mathbf{u} && \text{in } L^\varphi(\Omega) && (n \to \infty), \\ \mathbf{u}_n &\rightharpoonup \mathbf{u} && \text{in } L^\varphi(\Gamma_N) && (n \to \infty), \end{aligned} \tag{4.20}$$

where $\mathbf{u} \in W^{1,\varphi}(\Omega)$ is the unique weak solution of (1.1), i.e., $\mathbf{u} = \mathbf{u}_D$ in $L^\varphi(\Gamma_D)$ and for every $\mathbf{z} \in W^{1,\varphi}_{\Gamma_D}(\Omega)$, it holds

$$(\boldsymbol{\mathcal{A}}(\nabla\mathbf{u}), \nabla\mathbf{z}) = (\mathbf{g}, \mathbf{z}) + (\mathbf{G}, \nabla\mathbf{z}) + \langle \mathbf{a}_N, \mathbf{z}\rangle_{\Gamma_N}. \tag{4.21}$$

Proof. Resorting to the theory of monotone operators (*cf.* [35]), we observe that (1.1) admits a unique weak solution, since $\boldsymbol{\mathcal{A}}$ is strictly monotone. Thus, to prove (4.20) for the entire sequence to the unique weak solution $\mathbf{u} \in W^{1,\varphi}(\Omega)$ of (1.1), it is sufficient to show that each sequence has a subsequence that satisfies (4.20). Then, the assertion for the entire sequence follows from the standard convergence principle. To this end, we adapt Minty's trick to our situation. Appealing to Proposition 4.6, it holds $\sup_{n\in\mathbb{N}} M_{\varphi,h_n}(\mathbf{u}_n - \mathbf{u}_D^*) < \infty$, which, resorting to Lemma A.11, yields a not relabeled subsequence and a function $\tilde{\mathbf{u}} \in W^{1,\varphi}_{\Gamma_D}(\Omega)$ such that

$$\begin{aligned} \mathbf{u}_n - \mathbf{u}_D^* &\rightharpoonup \tilde{\mathbf{u}} && \text{in } L^\varphi(\Omega) && (n \to \infty), \\ \boldsymbol{\mathcal{G}}_{h_n}^k(\mathbf{u}_n - \mathbf{u}_D^*) &\rightharpoonup \nabla\tilde{\mathbf{u}} && \text{in } L^\varphi(\Omega) && (n \to \infty), \\ \mathbf{u}_n - \mathbf{u}_D^* &\rightharpoonup \tilde{\mathbf{u}} && \text{in } L^\varphi(\Gamma_N) && (n \to \infty). \end{aligned} \tag{4.22}$$

Therefore, introducing the notation $\mathbf{u} := \tilde{\mathbf{u}} + \mathbf{u}_D^* \in W^{1,\varphi}(\Omega)$, in particular, exploiting that $\boldsymbol{\mathcal{G}}_{h_n}^k(\mathbf{u}_n - \mathbf{u}_D^*) = \mathbf{L}_n - \nabla\mathbf{u}_D^*$ (*cf.* (4.5)), equation (4.22) is exactly (4.20) and yields directly that $\mathbf{u} = \mathbf{u}_D$ in $L^\varphi(\Gamma_D)$. Resorting to Corollary 4.7 and the reflexivity of $L^{\varphi^*}(\Omega)$, we obtain a further not relabeled subsequence and a function $\mathbf{A} \in L^{\varphi^*}(\Omega)$ such that

$$\mathbf{A}_n \rightharpoonup \mathbf{A} \quad \text{in } L^{\varphi^*}(\Omega) \quad (n \to \infty). \tag{4.23}$$

Next, let $\mathbf{z} \in W^{1,\varphi}_{\Gamma_D}(\Omega)$ be arbitrary and define $\mathbf{z}_n := \Pi_{h_n}^k \mathbf{z} \in V_{h_n}^k$ for every $n \in \mathbb{N}$. Then, appealing to the convergence properties for $\Pi_{h_n}^k$ in Lemma A.9, Corollary A.4, and the trace inequality in (A.26), we obtain for $n \to \infty$ that

$$\rho_{\varphi,\Omega}\Big(\boldsymbol{\mathcal{G}}_{h_n}^k(\mathbf{z}_n - \mathbf{z})\Big) + \rho_{\varphi,\Omega}(\mathbf{z}_n - \mathbf{z}) + M_{\varphi,h_n}(\mathbf{z}_n - \mathbf{z}) + \rho_{\varphi,\Gamma_N}(\mathbf{z}_n - \mathbf{z}) \to 0. \tag{4.24}$$

Apart from that, appealing to Corollary 4.7, we have that

$$\sup_{n\in\mathbb{N}} h_n\, \rho_{\varphi^*,\Gamma_h^i\cup\Gamma_D}\big(\boldsymbol{\mathcal{A}}_{\{|\Pi_{h_n}^0 \mathbf{L}_{h_n}|\}}\big(h_n^{-1}[\![(\mathbf{u}_n - \mathbf{u}_D^*)\otimes\mathbf{n}]\!]\big)\big) \le c. \tag{4.25}$$

Since $\mathbf{z}_n \in U_{h_n}^k$ is an admissible test function in (3.6), for every $n \in \mathbb{N}$, we have that

$$\begin{aligned} \Big(\mathbf{A}_n, \boldsymbol{\mathcal{G}}_{h_n}^k \mathbf{z}_n\Big) &= (\mathbf{g}, \mathbf{z}_n) + \Big(\mathbf{G}, \boldsymbol{\mathcal{G}}_{h_n}^k \mathbf{z}_n\Big) + \langle \mathbf{a}_N, \mathbf{z}_n\rangle_{\Gamma_N} \\ &\quad - \alpha\Big\langle \boldsymbol{\mathcal{A}}_{\{|\Pi_{h_n}^0 \mathbf{L}_{h_n}|\}}\big(h_n^{-1}[\![(\mathbf{u}_n - \mathbf{u}_D^*)\otimes\mathbf{n}]\!]\big), [\![(\mathbf{z}_n - \mathbf{z})\otimes\mathbf{n}]\!]\Big\rangle_{\Gamma_h^i\cup\Gamma_D}, \end{aligned} \tag{4.26}$$

where we used that $[\![\mathbf{z}\otimes\mathbf{n}]\!] = \mathbf{0}$ on $\Gamma_h^i \cup \Gamma_D$. This together with (4.24) and (4.25) yields for $n \to \infty$ that for every $\mathbf{z} \in W^{1,\varphi}_{\Gamma_D}(\Omega)$

$$(\mathbf{A}, \nabla\mathbf{z}) = (\mathbf{g}, \mathbf{z}) + (\mathbf{G}, \nabla\mathbf{z}) + \langle \mathbf{a}_N, \mathbf{z}\rangle_{\Gamma_N}. \tag{4.27}$$

For arbitrary $\mathbf{z} \in W^{1,\varphi}(\Omega)$, we set $\mathbf{z}_n := \Pi^k_{h_n}\mathbf{z} \in V^k_{h_n}$, $n \in \mathbb{N}$. Then, recalling that $\mathbf{A}_n = \Pi^k_{h_n}\boldsymbol{\mathcal{A}}(\mathbf{L}_n)$ in $X^k_{h_n}$, $n \in \mathbb{N}$, the monotonicity of $\boldsymbol{\mathcal{A}}$, the self-adjointness of $\Pi^k_{h_n}$, and (4.26), for every $n \in \mathbb{N}$ and $\mathbf{z} \in W^{1,\varphi}(\Omega)$, further yield that

$$\begin{aligned}
0 &\le (\boldsymbol{\mathcal{A}}(\mathbf{L}_n) - \boldsymbol{\mathcal{A}}(\nabla_{h_n}\mathbf{z}_n), \mathbf{L}_n - \nabla_{h_n}\mathbf{z}_n)\\
&= (\mathbf{A}_n - \boldsymbol{\mathcal{A}}(\nabla_{h_n}\mathbf{z}_n), \mathbf{L}_n - \nabla_{h_n}\mathbf{z}_n)\\
&= \Big(\mathbf{A}_n, \boldsymbol{\mathcal{G}}^k_{h_n}\big(\mathbf{u}_n - \Pi^k_{h_n}\mathbf{u}^*_D\big)\Big) + \Big(\mathbf{A}_n, \boldsymbol{\mathcal{G}}^k_{h_n}\big(\Pi^k_{h_n}\mathbf{u}^*_D - \mathbf{u}^*_D\big) + \nabla\mathbf{u}^*_D - \nabla_{h_n}\mathbf{z}_n\Big)\\
&\quad + (\boldsymbol{\mathcal{A}}(\nabla_{h_n}\mathbf{z}_n), \nabla_{h_n}\mathbf{z}_n - \mathbf{L}_n)\\
&= \big(\mathbf{g}, \mathbf{u}_n - \Pi^k_{h_n}\mathbf{u}^*_D\big) + \Big(\mathbf{G}, \boldsymbol{\mathcal{G}}^k_{h_n}\big(\mathbf{u}_n - \Pi^k_{h_n}\mathbf{u}^*_D\big)\Big) + \big\langle \mathbf{a}_N, \mathbf{u}_n - \Pi^k_{h_n}\mathbf{u}^*_D\big\rangle_{\Gamma_N}\\
&\quad - \alpha\Big\langle \boldsymbol{\mathcal{A}}_{\{|\Pi^0_{h_n}\mathbf{L}_{h_n}|\}}\big(h_n^{-1}[\![(\mathbf{u}_n - \mathbf{u}^*_D)\otimes\mathbf{n}]\!]\big), [\![(\mathbf{u}_n - \Pi^k_{h_n}\mathbf{u}^*_D \pm \mathbf{u}^*_D)\otimes\mathbf{n}]\!]\Big\rangle_{\Gamma^i_h\cup\Gamma_D}\\
&\quad + \Big(\mathbf{A}_n, \boldsymbol{\mathcal{G}}^k_{h_n}\big(\Pi^k_{h_n}\mathbf{u}^*_D - \mathbf{u}^*_D\big) + \nabla\mathbf{u}^*_D - \nabla_{h_n}\mathbf{z}\Big) + (\boldsymbol{\mathcal{A}}(\nabla_{h_n}\mathbf{z}_n), \nabla_{h_n}\mathbf{z}_n - \mathbf{L}_n).
\end{aligned} \tag{4.28}$$

Moreover, using $\boldsymbol{\mathcal{A}}_{\{|\Pi^0_h\mathbf{L}_h|\}}(\mathbf{P}) : \mathbf{P} \ge 0$ for every $\mathbf{P} \in \mathbb{R}^{d\times n}$, we conclude from (4.28) that for every $n \in \mathbb{N}$, there holds

$$\begin{aligned}
0 &\le (\mathbf{A}_n - \boldsymbol{\mathcal{A}}(\nabla_{h_n}\mathbf{z}_n), \mathbf{L}_n - \nabla_{h_n}\mathbf{z}_n)\\
&\le \big(\mathbf{g}, \mathbf{u}_n - \Pi^k_{h_n}\mathbf{u}^*_D\big) + \Big(\mathbf{G}, \boldsymbol{\mathcal{G}}^k_{h_n}\big(\mathbf{u}_n - \Pi^k_{h_n}\mathbf{u}^*_D\big)\Big) + \big\langle \mathbf{a}_N, \mathbf{u}_n - \Pi^k_{h_n}\mathbf{u}^*_D\big\rangle_{\Gamma_N}\\
&\quad + \alpha\big\langle \boldsymbol{\mathcal{A}}_{\{|\Pi^0_{h_n}\mathbf{L}_{h_n}|\}}\big(h_n^{-1}[\![(\mathbf{u}_n - \mathbf{u}^*_D)\otimes\mathbf{n}]\!]\big), [\![(\Pi^k_{h_n}\mathbf{u}^*_D - \mathbf{u}^*_D)\otimes\mathbf{n}]\!]\big\rangle_{\Gamma^i_h\cup\Gamma_D}\\
&\quad + \Big(\mathbf{A}_n, \boldsymbol{\mathcal{G}}^k_{h_n}\big(\Pi^k_{h_n}\mathbf{u}^*_D - \mathbf{u}^*_D\big) + \nabla\mathbf{u}^*_D - \nabla_{h_n}\mathbf{z}_n\Big) + (\boldsymbol{\mathcal{A}}(\nabla_{h_n}\mathbf{z}_n), \nabla_{h_n}\mathbf{z}_n - \mathbf{L}_n).
\end{aligned} \tag{4.29}$$

Hence, by passing for $n \to \infty$ in (4.29), taking into account (4.22)–(4.24), the convergence properties of $\Pi^k_{h_n}$ in Lemma A.9, Corollary A.4, the trace inequality in (A.26), the estimate (4.25), and the fact that $\boldsymbol{\mathcal{A}}$ generates a Nemyckii operator, we conclude for every $\mathbf{z} \in W^{1,\varphi}(\Omega)$ that

$$\begin{aligned}
0 &\le (\mathbf{g}, \mathbf{u} - \mathbf{u}^*_D) + (\mathbf{G}, \nabla(\mathbf{u} - \mathbf{u}^*_D)) + \langle \mathbf{a}_N, \mathbf{u} - \mathbf{u}^*_D\rangle_{\Gamma_N} + (\mathbf{A}, \nabla(\mathbf{u}^*_D - \mathbf{z})) + (\boldsymbol{\mathcal{A}}(\nabla\mathbf{z}), \nabla\mathbf{z} - \nabla\mathbf{u})\\
&= (\mathbf{A}, \nabla(\mathbf{u} - \mathbf{u}^*_D)) + (\mathbf{A}, \nabla(\mathbf{u}^*_D - \mathbf{z})) + (\boldsymbol{\mathcal{A}}(\nabla\mathbf{z}), \nabla\mathbf{z} - \nabla\mathbf{u})\\
&= (\mathbf{A} - \boldsymbol{\mathcal{A}}(\nabla\mathbf{z}), \nabla\mathbf{u} - \nabla\mathbf{z}),
\end{aligned} \tag{4.30}$$

where we used for the first equality sign that $\mathbf{u} - \mathbf{u}^*_D = \tilde{\mathbf{u}} \in W^{1,\varphi}_{\Gamma_D}(\Omega)$ and, thus, equation (4.27) applies with $\mathbf{z} = \mathbf{u} - \mathbf{u}^*_D \in W^{1,\varphi}_{\Gamma_D}(\Omega)$. Eventually, choosing $\mathbf{z} := \mathbf{u} \pm \tau\tilde{\mathbf{z}} \in W^{1,\varphi}(\Omega)$ in (4.30) for arbitrary $\tau \in (0,1)$ and $\tilde{\mathbf{z}} \in W^{1,\varphi}_{\Gamma_D}(\Omega)$, diving by $\tau > 0$ and passing for $\tau \to 0$, for every $\tilde{\mathbf{z}} \in W^{1,\varphi}_{\Gamma_D}(\Omega)$, we conclude that $(\mathbf{A} - \boldsymbol{\mathcal{A}}(\nabla\mathbf{u}), \nabla\tilde{\mathbf{z}}) = 0$, which, due to (4.27), implies that $\mathbf{u} \in W^{1,\varphi}(\Omega)$ satisfies (4.21). □

Remark 4.9. There are only few numerical investigations for nonlinear problems with Orlicz-structure showing the convergence of discrete solutions to a weak solution. We are only aware of the studies [11,17,18,23,31]. None of these contributions uses DG methods. For the subclass of nonlinear problems of p-Laplace type DG methods have been used in [12–14, 19, 27, 29]. In all these investigations, only the case $\mathbf{G} = \mathbf{0}$ is treated. Thus, to the best of the author's knowledge Theorem 4.8 is the first convergence result for a DG scheme in an Orlicz-setting for general right-hand sides from $(W^{1,\varphi}_0(\Omega))^*$ represented *via* $\mathbf{g} - \mathrm{div}\mathbf{G}$ with $\mathbf{g} \in L^{\varphi^*}(\Omega)$, $\mathbf{G} \in L^{\varphi^*}(\Omega)$.

5. Error estimates

In order to establish error estimates, we need to find a system similar to (3.6), which is satisfied by a solution of our original problem (1.1) in the case $\mathbf{G} = \mathbf{0}$. Using the notation $\mathbf{L} = \nabla\mathbf{u}$, $\mathbf{A} = \boldsymbol{\mathcal{A}}(\mathbf{L})$, we find

that $(\mathbf{u},\mathbf{L},\mathbf{A})^\top \in W^{1,\varphi}(\Omega)\times L^{\varphi}(\Omega)\times L^{\varphi^*}(\Omega)$. If, in addition, $\mathbf{A}\in W^{1,1}(\Omega)$, we observe as in [19], *i.e.*, using integration-by-parts, the boundary conditions, the properties of Π_h^k, the definition of the discrete gradient, and of the jump functional, that

$$\begin{aligned}(\mathbf{L},\mathbf{X}_h) &= (\nabla\mathbf{u},\mathbf{X}_h),\\ (\mathbf{A},\mathbf{Y}_h) &= (\boldsymbol{\mathcal{A}}(\mathbf{L}),\mathbf{Y}_h),\\ \Big(\mathbf{A},\boldsymbol{\mathcal{G}}_h^k\mathbf{z}_h\Big) &= (\mathbf{g},\mathbf{z}_h)+\langle\mathbf{a}_N,\mathbf{z}_h\rangle_{\Gamma_N}+\big\langle\{\mathbf{A}\}-\{\Pi_h^k\mathbf{A}\},[\![\mathbf{z}_h\otimes\mathbf{n}]\!]\big\rangle_{\Gamma_h^i\cup\Gamma_D}\end{aligned}\tag{5.1}$$

is satisfied for all $(\mathbf{X}_h,\mathbf{Y}_h,\mathbf{z}_h)^\top\in X_h^k\times X_h^k\times U_h^k$. Using this and (3.9), we arrive at

$$\begin{aligned}&\Big(\boldsymbol{\mathcal{A}}\Big(\boldsymbol{\mathcal{G}}_h^k\mathbf{u}_h+\boldsymbol{\mathcal{R}}_h^k\mathbf{u}_D^*\Big)-\boldsymbol{\mathcal{A}}(\nabla\mathbf{u}),\boldsymbol{\mathcal{G}}_h^k\mathbf{z}_h\Big)+\alpha\Big\langle\boldsymbol{\mathcal{A}}_{\{|\Pi_h^0(\boldsymbol{\mathcal{G}}_h^k\mathbf{u}_h+\boldsymbol{\mathcal{R}}_h^k\mathbf{u}_D^*)|\}}\big(h^{-1}[\![(\mathbf{u}_h-\mathbf{u}_D^*)\otimes\mathbf{n}]\!]\big),[\![\mathbf{z}_h\otimes\mathbf{n}]\!]\Big\rangle_{\Gamma_h^i\cup\Gamma_D}\\ &=\big\langle\{\Pi_h^k\mathbf{A}\}-\{\mathbf{A}\},[\![\mathbf{z}_h\otimes\mathbf{n}]\!]\big\rangle_{\Gamma_h^i\cup\Gamma_D},\end{aligned}\tag{5.2}$$

which is satisfied for all $\mathbf{z}_h\in U_h^k$.

Before we prove the convergence rate in Theorem 5.4, we derive some estimates.

Lemma 5.1. *Let $\boldsymbol{\mathcal{A}}$ satisfy Assumption 2.3 for a balanced N-function φ and $k\in\mathbb{N}$. Moreover, let $\mathbf{u}\in W^{1,\varphi}(\Omega)$ satisfy $\mathbf{F}(\nabla\mathbf{u})\in W^{1,2}(\Omega)$. Then, we have that*

$$\rho_{\varphi_{|\nabla\mathbf{u}|},\Omega}\Big(\boldsymbol{\mathcal{R}}_h^k\big(\mathbf{u}-\Pi_h^k\mathbf{u}\big)\Big)\le c\,h^2\,\|\nabla\mathbf{F}(\nabla\mathbf{u})\|_2^2\tag{5.3}$$

with a constant $c>0$ depending only on the characteristics of $\boldsymbol{\mathcal{A}}$ and φ, and the chunkiness $\omega_0>0$. The same assertion is valid for the Scott–Zhang interpolation operator Π_{SZ}^k, defined in [34], instead of Π_h^k.

Proof. Since for every $\mathbf{v}\in W^{1,p}(\mathcal{T}_h)$ we have $\boldsymbol{\mathcal{R}}_h^k\mathbf{v}=\sum_{\gamma\in\Gamma_h^i\cup\Gamma_D}\boldsymbol{\mathcal{R}}_{h,\gamma}^k\mathbf{v}$ with $\operatorname{supp}\boldsymbol{\mathcal{R}}_{h,\gamma}^k\mathbf{v}\subseteq S_\gamma$, where for each face $\gamma\in\Gamma_h^i\cup\Gamma_D$ the set S_γ consist of at most two elements $K\in\mathcal{T}_h$, and the fact that $\Omega=\bigcup_{\gamma\in\Gamma_h^i\cup\Gamma_D}S_\gamma$, it is sufficient to treat $\boldsymbol{\mathcal{R}}_{h,\gamma}^k(\mathbf{u}-\Pi_h^k\mathbf{u})$ on S_γ to obtain the global result (5.3) by summation. The shift change in Lemma 2.7, the local stability properties of $\boldsymbol{\mathcal{R}}_{h,\gamma}^k$ in Lemma A.1, Proposition 2.8, $\mathbf{u}-\Pi_h^k\mathbf{u}=\mathbf{u}-\Pi_h^k\mathbf{u}+\Pi_h^k(\mathbf{u}-\Pi_h^k\mathbf{u})$, the approximation property of Π_h^k in (A.13) for $\mathbf{u}-\Pi_h^k\mathbf{u}$, Poincaré's inequality on S_γ together with Lemma A.12 of [7], and again a shift change in Lemma 2.7 together with Poincaré's inequality on S_γ, yield

$$\begin{aligned}&\int_{S_\gamma}\varphi_{|\nabla\mathbf{u}|}\Big(\Big|\boldsymbol{\mathcal{R}}_{h,\gamma}^k\big(\mathbf{u}-\Pi_h^k\mathbf{u}\big)\Big|\Big)\,\mathrm{d}x\\ &\le c\int_{S_\gamma}\varphi_{\left|\langle\nabla\mathbf{u}\rangle_{S_\gamma}\right|}\Big(\Big|\boldsymbol{\mathcal{R}}_{h,\gamma}^k\big(\mathbf{u}-\Pi_h^k\mathbf{u}\big)\Big|\Big)+\varphi_{|\nabla\mathbf{u}|}\Big(\Big|\nabla\mathbf{u}-\langle\nabla\mathbf{u}\rangle_{S_\gamma}\Big|\Big)\,\mathrm{d}x\\ &\le c\,h\int_\gamma\varphi_{\left|\langle\nabla\mathbf{u}\rangle_{S_\gamma}\right|}\big(h^{-1}\big|[\![(\mathbf{u}-\Pi_h^k\mathbf{u})\otimes\mathbf{n}]\!]\big|\big)\,\mathrm{d}s+\int_{S_\gamma}\Big|\mathbf{F}(\nabla\mathbf{u})-\mathbf{F}\Big(\langle\nabla\mathbf{u}\rangle_{S_\gamma}\Big)\Big|^2\,\mathrm{d}x\\ &\le c\int_{S_\gamma}\varphi_{\left|\langle\nabla\mathbf{u}\rangle_{S_\gamma}\right|}\big(\big|\nabla_h\big(\mathbf{u}-\Pi_h^k\mathbf{u}\big)\big|\big)\,\mathrm{d}x+c\int_{S_\gamma}h^2\,|\nabla\mathbf{F}(\nabla\mathbf{u})|^2\,\mathrm{d}x\\ &\le c\int_{S_\gamma}\varphi_{|\nabla\mathbf{u}|}\big(\big|\nabla_h\big(\mathbf{u}-\Pi_h^k\mathbf{u}\big)\big|\big)\,\mathrm{d}x+c\int_{S_\gamma}h^2\,|\nabla\mathbf{F}(\nabla\mathbf{u})|^2\,\mathrm{d}x.\end{aligned}$$

By summation over $\gamma\in\Gamma_h^i\cup\Gamma_D$, using Proposition 2.8 and Lemma A.5 in doing so, we conclude that

$$\begin{aligned}\rho_{\varphi_{|\nabla\mathbf{u}|},\Omega}\Big(\boldsymbol{\mathcal{R}}_h^k\big(\mathbf{u}-\Pi_h^k\mathbf{u}\big)\Big)&\le c\,\rho_{\varphi_{|\nabla\mathbf{u}|},\Omega}\big(\nabla\mathbf{u}-\nabla_h\Pi_h^k\mathbf{u}\big)+c\,h^2\,\|\nabla\mathbf{F}(\nabla\mathbf{u})\|_2^2\\ &\le c\,\big\|\mathbf{F}(\nabla\mathbf{u})-\mathbf{F}\big(\nabla_h\Pi_h^k\mathbf{u}\big)\big\|_2^2+c\,h^2\,\|\nabla\mathbf{F}(\nabla\mathbf{u})\|_2^2\\ &\le c\,h^2\,\|\nabla\mathbf{F}(\nabla\mathbf{u})\|_2^2.\end{aligned}$$

This proves the assertion for Π_h^k. Since the Scott–Zhang interpolation operator has the same properties (*cf.* [17, 19]), the assertion for Π_{SZ}^k follows analogously. □

Lemma 5.2. *Let $\boldsymbol{\mathcal{A}}$ satisfy Assumption 2.3 for a balanced N-function φ. Moreover, let $\mathbf{X} \in L^{\varphi}(\Omega)$, $\mathbf{Y} \in L^{\varphi^*}(\Omega)$, and let $\mathbf{u} \in W^{1,\varphi}(\Omega)$ satisfy $\mathbf{F}(\nabla\mathbf{u}) \in W^{1,2}(\Omega)$. Then, for any $j \in \mathbb{N}_0$, it holds*

$$\Big\|\mathbf{F}(\nabla\mathbf{u}) - \mathbf{F}\Big(\Pi_h^j\mathbf{X}\Big)\Big\|_2^2 \le c\,h^2\|\nabla\mathbf{F}(\nabla\mathbf{u})\|_2^2 + c\,\|\mathbf{F}(\nabla\mathbf{u}) - \mathbf{F}(\mathbf{X})\|_2^2,$$
$$\Big\|\mathbf{F}^*(\boldsymbol{\mathcal{A}}(\nabla\mathbf{u})) - \mathbf{F}^*\Big(\Pi_h^j\boldsymbol{\mathcal{A}}(\mathbf{Y})\Big)\Big\|_2^2 \le c\,h^2\|\nabla\mathbf{F}(\nabla\mathbf{u})\|_2^2 + c\,\|\mathbf{F}^*(\boldsymbol{\mathcal{A}}(\nabla\mathbf{u})) - \mathbf{F}^*(\boldsymbol{\mathcal{A}}(\mathbf{Y}))\|_2^2$$

with a constant $c > 0$ depending only on the characteristics of $\boldsymbol{\mathcal{A}}$ and φ, and the chunkiness $\omega_0 > 0$.

Proof. Resorting to Poincaré's inequality on each $K \in \mathcal{T}_h$ and Proposition 2.8, we find that

$$\begin{aligned}\Big\|\mathbf{F}(\nabla\mathbf{u}) - \mathbf{F}\Big(\Pi_h^j\mathbf{X}\Big)\Big\|_2^2 &\le 2\,\big\|\mathbf{F}(\nabla\mathbf{u}) - \mathbf{F}\big(\Pi_h^0\nabla\mathbf{u}\big)\big\|_2^2 + 2\,\Big\|\mathbf{F}\big(\Pi_h^0\nabla\mathbf{u}\big) - \mathbf{F}\Big(\Pi_h^j\mathbf{X}\Big)\Big\|_2^2\\ &\le c\,h^2\|\nabla\mathbf{F}(\nabla\mathbf{u})\|_2^2 + c\,\rho_{\varphi_{|\Pi_h^0\nabla\mathbf{u}|},\Omega}\Big(\Pi_h^0\nabla\mathbf{u} - \Pi_h^j\mathbf{X}\Big).\end{aligned} \tag{5.4}$$

The L^{φ}-stability of Π_h^k in (A.8), Proposition 2.8, and again Poincaré's inequality on each $K \in \mathcal{T}_h$ yield

$$\begin{aligned}\rho_{\varphi_{|\Pi_h^0\nabla\mathbf{u}|},\Omega}\Big(\Pi_h^0\nabla\mathbf{u} - \Pi_h^j\mathbf{X}\Big) &= \rho_{\varphi_{|\Pi_h^0\nabla\mathbf{u}|},\Omega}\Big(\Pi_h^j\big(\Pi_h^0\nabla\mathbf{u} - \mathbf{X}\big)\Big)\\ &\le c\,\rho_{\varphi_{|\Pi_h^0\nabla\mathbf{u}|},\Omega}\big(\Pi_h^0\nabla\mathbf{u} - \mathbf{X}\big)\\ &\le c\,\big\|\mathbf{F}\big(\Pi_h^0\nabla\mathbf{u}\big) - \mathbf{F}(\nabla\mathbf{u})\big\|_2^2 + c\,\|\mathbf{F}(\nabla\mathbf{u}) - \mathbf{F}(\mathbf{X})\|_2^2\\ &\le c\,h^2\|\nabla\mathbf{F}(\nabla\mathbf{u})\|_2^2 + c\,\|\mathbf{F}(\nabla\mathbf{u}) - \mathbf{F}(\mathbf{X})\|_2^2.\end{aligned} \tag{5.5}$$

Combining (5.4) and (5.5), we conclude the first assertion. The second assertion follows analogously, if we additionally use Lemma 2.9 and (*cf.* [19], Lem. 4.4)

$$\big\|\mathbf{F}(\nabla\mathbf{u}) - \mathbf{F}\big(\Pi_h^0\nabla\mathbf{u}\big)\big\|_2^2 \sim \big\|\mathbf{F}^*(\boldsymbol{\mathcal{A}}(\nabla\mathbf{u})) - \mathbf{F}^*\big(\Pi_h^0\boldsymbol{\mathcal{A}}(\nabla\mathbf{u})\big)\big\|_2^2.$$

□

Lemma 5.3. *Let $\boldsymbol{\mathcal{A}}$ satisfy Assumption 2.3 for a balanced N-function φ and $k \in \mathbb{N}_0$. Moreover, let $\mathbf{X} \in L^{\varphi}(\Omega)$ and let $\mathbf{u} \in W^{1,\varphi}(\Omega)$ satisfy $\mathbf{F}(\nabla\mathbf{u}) \in W^{1,2}(\Omega)$. Then, we have that*

$$h\,\rho_{\varphi_{|\nabla\mathbf{u}|},\Gamma_h^i\cup\Gamma_D}\big(\big||\nabla\mathbf{u}| - \big\{\big|\Pi_h^0\mathbf{X}\big|\big\}\big|\big) \le c\,h^2\,\|\nabla\mathbf{F}(\nabla\mathbf{u})\|_2^2 + c\,\|\mathbf{F}(\nabla\mathbf{u}) - \mathbf{F}(\mathbf{X})\|_2^2 \tag{5.6}$$

with a constant $c > 0$ depending only on the characteristics of $\boldsymbol{\mathcal{A}}$ and φ, and the chunkiness $\omega_0 > 0$.

Proof. Using $\big||\nabla\mathbf{u}| - \{|\Pi_h^0\mathbf{X}|\}\big| \le \{|\nabla\mathbf{u} - \Pi_h^0\mathbf{X}|\}$ on $\Gamma_h^i \cup \Gamma_D$, the convexity of $\varphi_{|\nabla\mathbf{u}|}$, (2.10) and the trace inequality (A.11), for every $\gamma \in \Gamma_h^i \cup \Gamma_D$, we find that

$$\begin{aligned}h\int_\gamma \varphi_{|\nabla\mathbf{u}|}\big(\big||\nabla\mathbf{u}| - \big\{\big|\Pi_h^0\mathbf{X}\big|\big\}\big|\big)\,\mathrm{d}s &\le c\,h\int_\gamma \varphi_{|\nabla\mathbf{u}|}\big(\big\{|\nabla\mathbf{u} - \Pi_h^0\mathbf{X}|\big\}\big)\,\mathrm{d}s\\ &\le c\,h\sum_{K\in\mathcal{T}_h;K\subseteq S_\gamma}\int_\gamma \varphi_{|\nabla\mathbf{u}|}\big(|\nabla\mathbf{u} - (\Pi_h^0\mathbf{X})|_K|\big)\,\mathrm{d}s\\ &\le c\,h\sum_{K\in\mathcal{T}_h;K\subseteq S_\gamma}\int_\gamma \big|\mathbf{F}(\nabla\mathbf{u}) - \mathbf{F}((\Pi_h^0\mathbf{X})|_K)\big|^2\,\mathrm{d}s\\ &\le c\,h^2\int_{S_\gamma}|\nabla\mathbf{F}(\nabla\mathbf{u})|^2\,\mathrm{d}x + c\int_{S_\gamma}\big|\mathbf{F}(\nabla\mathbf{u}) - \mathbf{F}(\Pi_h^0\mathbf{X})\big|^2\,\mathrm{d}x.\end{aligned}$$

Then, the assertion follows by summing with respect to $\gamma \in \Gamma_h^i \cup \Gamma_D$ and resorting to Lemma 5.2 with $j = 0$. □

Theorem 5.4. *Let $\boldsymbol{\mathcal{A}}$ satisfy Assumption 2.3 for a balanced N-function φ. Moreover, let $\mathbf{u} \in W^{1,\varphi}(\Omega)$ be a solution of (1.1) with $\mathbf{G} = \mathbf{0}$ which satisfies $\mathbf{F}(\nabla\mathbf{u}) \in W^{1,2}(\Omega)$ and let $\mathbf{u}_h \in U_h^k$ be a solution of (3.9) for $\alpha > 0$ and $k \in \mathbb{N}$. Then, we have that*

$$\left\|\mathbf{F}\Big(\boldsymbol{\mathcal{G}}_h^k\mathbf{u}_h + \boldsymbol{\mathcal{R}}_h^k\mathbf{u}_D^*\Big) - \mathbf{F}(\nabla\mathbf{u})\right\|_2^2 + \alpha\, m_{\varphi_{\left\{\left|\Pi_h^0\left(\boldsymbol{\mathcal{G}}_h^k\mathbf{u}_h+\boldsymbol{\mathcal{R}}_h^k\mathbf{u}_D^*\right)\right|\right\}},h}(\mathbf{u}_h - \mathbf{u}) \le c\, h^2 \|\nabla\mathbf{F}(\nabla\mathbf{u})\|_2^2$$

with a constant $c > 0$ depending only on the characteristics of $\boldsymbol{\mathcal{A}}$ and φ, $k \in \mathbb{N}$, the chunkiness $\omega_0 > 0$, and $\alpha^{-1} > 0$.

Proof. To shorten the notation, we use again $\mathbf{L}_h$ instead of $\boldsymbol{\mathcal{G}}_h^k\mathbf{u}_h + \boldsymbol{\mathcal{R}}_h^k\mathbf{u}_D^*$ in the shifts. Let $\mathbf{z}_h = \mathbf{u}_h - \Pi_h^k\mathbf{u} \in U_h^k$, then we have that

$$\boldsymbol{\mathcal{G}}_h^k\mathbf{z}_h = \Big(\boldsymbol{\mathcal{G}}_h^k\mathbf{u}_h + \boldsymbol{\mathcal{R}}_h^k\mathbf{u}_D^* - \nabla\mathbf{u}\Big) - \Big(\boldsymbol{\mathcal{G}}_h^k\Pi_h^k\mathbf{u} + \boldsymbol{\mathcal{R}}_h^k\mathbf{u}_D^* - \nabla\mathbf{u}\Big),$$

so the error equation (5.2) gives us

$$\begin{aligned}
&\Big(\boldsymbol{\mathcal{A}}\Big(\boldsymbol{\mathcal{G}}_h^k\mathbf{u}_h + \boldsymbol{\mathcal{R}}_h^k\mathbf{u}_D^*\Big) - \boldsymbol{\mathcal{A}}(\nabla\mathbf{u}), \Big(\boldsymbol{\mathcal{G}}_h^k\mathbf{u}_h + \boldsymbol{\mathcal{R}}_h^k\mathbf{u}_D^*\Big) - \nabla\mathbf{u}\Big)\\
&\qquad + \alpha\Big\langle \boldsymbol{\mathcal{A}}_{\{|\Pi_h^0\mathbf{L}_h|\}}\big(h^{-1}[\![(\mathbf{u}_h - \mathbf{u}_D^*)\otimes\mathbf{n}]\!]\big), [\![(\mathbf{u}_h - \mathbf{u}_D^*)\otimes\mathbf{n}]\!]\Big\rangle_{\Gamma_h^i\cup\Gamma_D}\\
&\quad = \Big(\boldsymbol{\mathcal{A}}\Big(\boldsymbol{\mathcal{G}}_h^k\mathbf{u}_h + \boldsymbol{\mathcal{R}}_h^k\mathbf{u}_D^*\Big) - \boldsymbol{\mathcal{A}}(\nabla\mathbf{u}), \Big(\boldsymbol{\mathcal{G}}_h^k\Pi_h^k\mathbf{u} + \boldsymbol{\mathcal{R}}_h^k\mathbf{u}_D^*\Big) - \nabla\mathbf{u}\Big)\\
&\qquad + \alpha\Big\langle \boldsymbol{\mathcal{A}}_{\{|\Pi_h^0\mathbf{L}_h|\}}\big(h^{-1}[\![(\mathbf{u}_h - \mathbf{u}_D^*)\otimes\mathbf{n}]\!]\big), [\![(\Pi_h^k\mathbf{u} - \mathbf{u}_D^*)\otimes\mathbf{n}]\!]\Big\rangle_{\Gamma_h^i\cup\Gamma_D}\\
&\qquad + \big\langle\{\Pi_h^k\mathbf{A}\} - \{\mathbf{A}\}, [\![(\mathbf{u}_h - \Pi_h^k\mathbf{u})\otimes\mathbf{n}]\!]\big\rangle_{\Gamma_h^i\cup\Gamma_D}\\
&\quad =: K_1 + \alpha\, K_2 + K_3.
\end{aligned} \tag{5.7}$$

Due to Proposition 2.8, the first term on the left-hand side of (5.7) is equivalent to

$$\rho_{\varphi,\Omega}\Big(\boldsymbol{\mathcal{G}}_h^k\mathbf{u}_h + \boldsymbol{\mathcal{R}}_h^k\mathbf{u}_D^* - \nabla\mathbf{u}\Big) \sim \left\|\mathbf{F}\Big(\boldsymbol{\mathcal{G}}_h^k\mathbf{u}_h + \boldsymbol{\mathcal{R}}_h^k\mathbf{u}_D^*\Big) - \mathbf{F}(\nabla\mathbf{u})\right\|_2^2, \tag{5.8}$$

while (2.4) yields that the second term on the left-hand side of (5.7) is equivalent to

$$\alpha\, m_{\varphi_{\{|\Pi_h^0\mathbf{L}_h|\}},h}(\mathbf{u}_h - \mathbf{u}_D^*). \tag{5.9}$$

It is important to use that $[\![\mathbf{u}_D^*\otimes\mathbf{n}]\!] = [\![\mathbf{u}\otimes\mathbf{n}]\!]$ on $\Gamma_h^i\cup\Gamma_D$, which allows to replace in (5.9) $\mathbf{u}_h - \mathbf{u}_D^*$ by $\mathbf{u}_h - \mathbf{u}$. Thus, equations (5.8) and (5.9) yield that the left-hand side of (5.7) is bounded from below by

$$c\left\|\mathbf{F}\Big(\boldsymbol{\mathcal{G}}_h^k\mathbf{u}_h + \boldsymbol{\mathcal{R}}_h^k\mathbf{u}_D^*\Big) - \mathbf{F}(\nabla\mathbf{u})\right\|_2^2 + \alpha\, c\, m_{\varphi_{\{|\Pi_h^0\mathbf{L}_h|\}},h}(\mathbf{u}_h - \mathbf{u}). \tag{5.10}$$

To treat the term K_1, we exploit that, owing to $\boldsymbol{\mathcal{R}}_h^k\mathbf{u} = \boldsymbol{\mathcal{R}}_h^k\mathbf{u}_D$, it holds

$$\boldsymbol{\mathcal{G}}_h^k\Pi_h^k\mathbf{u} + \boldsymbol{\mathcal{R}}_h^k\mathbf{u}_D^* - \nabla\mathbf{u} = \big(\nabla_h\Pi_h^k\mathbf{u} - \nabla\mathbf{u}\big) + \boldsymbol{\mathcal{R}}_h^k\big(\mathbf{u} - \Pi_h^k\mathbf{u}\big). \tag{5.11}$$

Thus, Proposition 2.8, the ε-Young inequality (2.2) with $\psi = \varphi_{|\nabla\mathbf{u}|}$, Lemmas A.5 and 5.1 yield that for all $\varepsilon > 0$, there exists $c_\varepsilon > 0$ such that

$$\begin{aligned}
|K_1| &\le \varepsilon\, \rho_{\varphi_{|\nabla\mathbf{u}|},\Omega}\Big(\boldsymbol{\mathcal{G}}_h^k\mathbf{u}_h + \boldsymbol{\mathcal{R}}_h^k\mathbf{u}_D^* - \nabla\mathbf{u}\Big) + c_\varepsilon\, \rho_{\varphi_{|\nabla\mathbf{u}|},\Omega}\big(\nabla_h\Pi_h^k\mathbf{u} - \nabla\mathbf{u}\big) + c_\varepsilon\, \rho_{\varphi_{|\nabla\mathbf{u}|},\Omega}\Big(\boldsymbol{\mathcal{R}}_h^k\big(\mathbf{u} - \Pi_h^k\mathbf{u}\big)\Big)\\
&\le \varepsilon\, c\left\|\mathbf{F}\Big(\boldsymbol{\mathcal{G}}_h^k\mathbf{u}_h + \boldsymbol{\mathcal{R}}_h^k\mathbf{u}_D^*\Big) - \mathbf{F}(\nabla\mathbf{u})\right\|_2^2 + c_\varepsilon\left\|\mathbf{F}\big(\nabla_h\Pi_h^k\mathbf{u}\big) - \mathbf{F}(\nabla\mathbf{u})\right\|_2^2 + c_\varepsilon\, \rho_{\varphi_{|\nabla\mathbf{u}|},\Omega}\Big(\boldsymbol{\mathcal{R}}_h^k\big(\mathbf{u} - \Pi_h^k\mathbf{u}\big)\Big)\\
&\le \varepsilon\, c\left\|\mathbf{F}\Big(\boldsymbol{\mathcal{G}}_h^k\mathbf{u}_h + \boldsymbol{\mathcal{R}}_h^k\mathbf{u}_D^*\Big) - \mathbf{F}(\nabla\mathbf{u})\right\|_2^2 + c_\varepsilon\, h^2\, \|\nabla\mathbf{F}(\nabla\mathbf{u})\|_2^2.
\end{aligned} \tag{5.12}$$

The identity $[\![\mathbf{u}_D^* \otimes \mathbf{n}]\!] = [\![\mathbf{u} \otimes \mathbf{n}]\!]$ on $\Gamma_h^i \cup \Gamma_D$, (3.2), the ε-Young inequality (2.2) with $\psi = \varphi_{\{|\Pi_h^0 \mathbf{L}_h|\}}$, a shift change in Lemma 2.7, Corollary A.8 with $\mathbf{w}_h = \Pi_h^k \mathbf{u} - \mathbf{u}$, Lemma 5.3, (2.10) and Lemma A.5 yield

$$
\begin{aligned}
|K_2| &= \Big| \Big\langle \boldsymbol{\mathcal{A}}_{\{|\Pi_h^0 \mathbf{L}_h|\}}\big(h^{-1}[\![(\mathbf{u}_h - \mathbf{u}) \otimes \mathbf{n}]\!]\big), [\![\big(\Pi_h^k \mathbf{u} - \mathbf{u}\big) \otimes \mathbf{n}]\!] \Big\rangle_{\Gamma_h^i \cup \Gamma_D} \Big| \\
&\le \varepsilon\, m_{\varphi_{\{|\Pi_h^0 \mathbf{L}_h|\}},h}(\mathbf{u}_h - \mathbf{u}) + c_\varepsilon\, m_{\varphi_{\{|\Pi_h^0 \mathbf{L}_h|\}},h}\big(\Pi_h^k \mathbf{u} - \mathbf{u}\big) \\
&\le \varepsilon\, m_{\varphi_{\{|\Pi_h^0 \mathbf{L}_h|\}},h}(\mathbf{u}_h - \mathbf{u}) + c_\kappa\, c_\varepsilon\, m_{\varphi_{|\nabla \mathbf{u}|},h}\big(\Pi_h^k \mathbf{u} - \mathbf{u}\big) \\
&\quad + \kappa\, c_\varepsilon\, h\, \rho_{\varphi_{|\nabla \mathbf{u}|},\Gamma_h^i \cup \Gamma_D}\Big(\Big||\nabla \mathbf{u}| - \Big\{\Big|\Pi_h^0\Big(\boldsymbol{\mathcal{G}}_h^k \mathbf{u}_h + \boldsymbol{\mathcal{R}}_h^k \mathbf{u}_D^*\Big)\Big|\Big\}\Big|\Big) \\
&\le \varepsilon\, m_{\varphi_{\{|\Pi_h^0 \mathbf{L}_h|\}},h}(\mathbf{u}_h - \mathbf{u}) + c_\kappa\, c_\varepsilon\, \big\|\mathbf{F}(\nabla \mathbf{u}) - \mathbf{F}\big(\nabla \Pi_h^k \mathbf{u}\big)\big\|_2^2 \\
&\quad + c_\kappa\, c_\varepsilon\, h^2 \|\nabla \mathbf{F}(\nabla \mathbf{u})\|_2^2 + \kappa\, c_\varepsilon \Big\|\mathbf{F}(\nabla \mathbf{u}) - \mathbf{F}\Big(\boldsymbol{\mathcal{G}}_h^k \mathbf{u}_h + \boldsymbol{\mathcal{R}}_h^k \mathbf{u}_D^*\Big)\Big\|_2^2 \\
&\le \varepsilon\, m_{\varphi_{\{|\Pi_h^0 \mathbf{L}_h|\}},h}(\mathbf{u}_h - \mathbf{u}) + c_\kappa\, c_\varepsilon\, h^2 \|\nabla \mathbf{F}(\nabla \mathbf{u})\|_2^2 \\
&\quad + \kappa\, c_\varepsilon \Big\|\mathbf{F}(\nabla \mathbf{u}) - \mathbf{F}\Big(\boldsymbol{\mathcal{G}}_h^k \mathbf{u}_h + \boldsymbol{\mathcal{R}}_h^k \mathbf{u}_D^*\Big)\Big\|_2^2.
\end{aligned}
\tag{5.13}
$$

To treat K_3, we use the ε-Young inequality (2.2) for $\psi = \varphi_{|\nabla \mathbf{u}|}$ to get[4]

$$
\begin{aligned}
|K_3| &\le c_\varepsilon\, h \int_{\Gamma_h^i \cup \Gamma_D} \big(\varphi_{|\nabla \mathbf{u}|}\big)^*\big(\big|\{\mathbf{A}\} - \{\Pi_h^k \mathbf{A}\}\big|\big)\, \mathrm{d}s + \varepsilon\, m_{\varphi_{|\nabla \mathbf{u}|},h}\big(\mathbf{u}_h - \Pi_h^k \mathbf{u}\big) \\
&=: c_\varepsilon \sum_{\gamma \in \Gamma_h^i \cup \Gamma_D} K_{3,1}^\gamma + \varepsilon\, K_{3,2}.
\end{aligned}
\tag{5.14}
$$

Using a shift change in Lemma 2.7, Corollary A.8 with $\mathbf{w}_h = \mathbf{u} - \Pi_h^k \mathbf{u}$, Proposition 2.8, Lemmas A.5 and 5.3, we find that

$$
\begin{aligned}
|K_{3,2}| &\le c\, m_{\varphi_{|\nabla \mathbf{u}|},h}\big(\mathbf{u} - \Pi_h^k \mathbf{u}\big) + c\, m_{\varphi_{|\nabla \mathbf{u}|},h}(\mathbf{u}_h - \mathbf{u}) \\
&\le c\, m_{\varphi_{|\nabla \mathbf{u}|},h}\big(\mathbf{u} - \Pi_h^k \mathbf{u}\big) + c\, m_{\varphi_{\{|\Pi_h^0 \mathbf{L}_h|\}},h}(\mathbf{u}_h - \mathbf{u}) + c\, h\, \rho_{\varphi_{|\nabla \mathbf{u}|},\Gamma_h^i \cup \Gamma_D}\Big(\Big||\nabla \mathbf{u}| - \Big\{\Big|\Pi_h^0\Big(\boldsymbol{\mathcal{G}}_h^k \mathbf{u}_h + \boldsymbol{\mathcal{R}}_h^k \mathbf{u}_D^*\Big)\Big|\Big\}\Big|\Big) \\
&\le c\, h^2 \|\nabla \mathbf{F}(\nabla \mathbf{u})\|_2^2 + c \Big\|\mathbf{F}(\nabla \mathbf{u}) - \mathbf{F}\Big(\boldsymbol{\mathcal{G}}_h^k \mathbf{u}_h + \boldsymbol{\mathcal{R}}_h^k \mathbf{u}_D^*\Big)\Big\|_2^2.
\end{aligned}
\tag{5.15}
$$

From Proposition 2.8, Lemma 2.9, choosing some $K \in \mathcal{T}_h$ such that $\gamma \subseteq \partial K$, using $\nabla \mathbf{u}, \mathbf{A} \in W^{1,1}(\Omega)$, the trace inequality (A.12), Poincaré's inequality on K, the stability of Π_h^k in (A.8), (2.17), (2.10), and again Poincaré's inequality on K, it follows that

$$
\begin{aligned}
\big|K_{3,1}^\gamma\big| &\le c\, h \int_\gamma \big|\mathbf{F}^*(\boldsymbol{\mathcal{A}}(\nabla \mathbf{u})) - \mathbf{F}^*\big(\Pi_h^k \boldsymbol{\mathcal{A}}(\nabla \mathbf{u})\big)\big|^2\, \mathrm{d}s \\
&\le c\, h \int_\gamma |\mathbf{F}^*(\boldsymbol{\mathcal{A}}(\nabla \mathbf{u})) - \mathbf{F}^*(\boldsymbol{\mathcal{A}}(\langle \nabla \mathbf{u}\rangle_K))|^2\, \mathrm{d}s + c\, h \int_\gamma \big|\mathbf{F}^*(\boldsymbol{\mathcal{A}}(\langle \nabla \mathbf{u}\rangle_K)) - \mathbf{F}^*\big(\Pi_h^k \boldsymbol{\mathcal{A}}(\nabla \mathbf{u})\big)\big|^2\, \mathrm{d}s \\
&\le c\, h \int_\gamma |\mathbf{F}(\nabla \mathbf{u}) - \mathbf{F}(\langle \nabla \mathbf{u}\rangle_K)|^2\, \mathrm{d}s + c\, h \int_\gamma (\varphi^*)_{|\boldsymbol{\mathcal{A}}(\langle \nabla \mathbf{u}\rangle_K)|}\big(\Pi_h^k(\boldsymbol{\mathcal{A}}(\nabla \mathbf{u}) - \boldsymbol{\mathcal{A}}(\langle \nabla \mathbf{u}\rangle_K))\big)\, \mathrm{d}s \\
&\le c \int_K |\mathbf{F}(\nabla \mathbf{u}) - \mathbf{F}(\langle \nabla \mathbf{u}\rangle_K)|^2 + h^2\, |\nabla \mathbf{F}(\nabla \mathbf{u})|^2\, \mathrm{d}x + c \int_K (\varphi^*)_{|\boldsymbol{\mathcal{A}}(\langle \nabla \mathbf{u}\rangle_K)|}\big(\Pi_h^k(\boldsymbol{\mathcal{A}}(\nabla \mathbf{u}) - \boldsymbol{\mathcal{A}}(\langle \nabla \mathbf{u}\rangle_K))\big)\, \mathrm{d}x \\
&\le c \int_K h^2\, |\nabla \mathbf{F}(\nabla \mathbf{u})|^2\, \mathrm{d}x + c \int_K (\varphi^*)_{|\boldsymbol{\mathcal{A}}(\langle \nabla \mathbf{u}\rangle_K)|}(\boldsymbol{\mathcal{A}}(\nabla \mathbf{u}) - \boldsymbol{\mathcal{A}}(\langle \nabla \mathbf{u}\rangle_K))\, \mathrm{d}x
\end{aligned}
\tag{5.16}
$$

[4]We use $\mathbf{F}(\nabla \mathbf{u}) \in W^{1,2}(\Omega)$, the embedding $W^{1,2}(\Omega) \hookrightarrow L^2(\Gamma_h^i \cup \Gamma_D)$ and (2.15) to conclude that $\nabla \mathbf{u} \in L^\varphi(\Gamma_h^i \cup \Gamma_D)$ and $\mathbf{A} \in L^{\varphi^*}(\Gamma_h^i \cup \Gamma_D)$ and, thus, have well-defined traces on $\Gamma_h^i \cup \Gamma_D$.

$$\leq c \int_K h^2 \left|\nabla \mathbf{F}(\nabla \mathbf{u})\right|^2 \mathrm{d}x + c \int_K \left|\mathbf{F}(\nabla \mathbf{u}) - \mathbf{F}(\langle \nabla \mathbf{u} \rangle_K)\right|^2 \mathrm{d}x$$
$$\leq c \int_K h^2 \left|\nabla \mathbf{F}(\nabla \mathbf{u})\right|^2 \mathrm{d}x.$$

Eventually (5.14)–(5.16) imply that

$$|K_3| \leq c_\varepsilon\, h^2 \|\nabla \mathbf{F}(\nabla \mathbf{u})\|_2^2 + \varepsilon\, c \left\| \mathbf{F}(\nabla \mathbf{u}) - \mathbf{F}\Big(\boldsymbol{\mathcal{G}}_h^k \mathbf{u}_h + \boldsymbol{\mathcal{R}}_h^k \mathbf{u}_D^*\Big) \right\|_2^2. \tag{5.17}$$

All together, choosing first $\varepsilon > 0$ and than $\kappa > 0$ small enough, and absorbing the terms with ε and κ in the left-hand side, we conclude the assertion. □

Corollary 5.5. *Let $\boldsymbol{\mathcal{A}}$ satisfy Assumption 2.3 for a balanced N-function φ. Moreover, let $\mathbf{u} \in W^{1,\varphi}(\Omega)$ be a solution of (1.1) which satisfies $\mathbf{F}(\nabla \mathbf{u}) \in W^{1,2}(\Omega)$ and let $(\mathbf{u}_h, \mathbf{L}_h, \mathbf{A}_h)^\top \in U_h^k \times X_h^k \times X_h^k$ be a solution of (3.6) for $\alpha > 0$, $h > 0$ and $k \in \mathbb{N}$. Then, it holds*

$$\|\mathbf{F}^*(\boldsymbol{\mathcal{A}}(\nabla \mathbf{u})) - \mathbf{F}^*(\mathbf{A}_h)\|_2^2 \leq c\, h^2\, \|\nabla \mathbf{F}(\nabla \mathbf{u})\|_2^2, \tag{5.18}$$

with a constant $c > 0$ depending only on the characteristics of $\boldsymbol{\mathcal{A}}$ and φ, $k \in \mathbb{N}$, the chunkiness $\omega_0 > 0$, and $\alpha^{-1} > 0$.

Proof. Lemma 5.2 with $\mathbf{Y} = \mathbf{L}_h$, $j = k$, $\mathbf{A}_h = \Pi_h^k \boldsymbol{\mathcal{A}}(\mathbf{L}_h)$, equations (2.18), (3.7) imply that

$$\begin{aligned}\|\mathbf{F}^*(\boldsymbol{\mathcal{A}}(\nabla \mathbf{u})) - \mathbf{F}^*(\mathbf{A}_h)\|_2^2 &\leq c\, \|\mathbf{F}^*(\boldsymbol{\mathcal{A}}(\nabla \mathbf{u})) - \mathbf{F}^*(\boldsymbol{\mathcal{A}}(\mathbf{L}_h))\|_2^2 + c\, h^2\, \|\nabla \mathbf{F}(\nabla \mathbf{u})\|_2^2 \\ &\leq c \left\| \mathbf{F}(\nabla \mathbf{u}) - \mathbf{F}\Big(\boldsymbol{\mathcal{G}}_h^k \mathbf{u}_h + \boldsymbol{\mathcal{R}}_h^k \mathbf{u}_D^*\Big) \right\|_2^2 + c\, h^2\, \|\nabla \mathbf{F}(\nabla \mathbf{u})\|_2^2,\end{aligned}$$

which together with Theorem 5.4 yields the assertion. Alternatively, we could use Proposition 4.9 of [19] and Theorem 5.4. □

Remark 5.6. Let us compare our results in the special case that φ possesses (p, δ)-structure and $k = 1$ with the corresponding ones in [19].

(i) Theorem 4.8, Corollary 4.10(ii) of [19] provides sub-optimal convergence rates for $p \leq 2$ as well as for $p \geq 2$, while Theorem 5.4 and Corollary 5.5 prove optimal ones for all $p \in (1, \infty)$. We emphasize that in [19], the cases $p \leq 2$ and $p \geq 2$ are treated differently, while our approach provides a unified treatment. The reason for these differences are the different fluxes $\widehat{\mathbf{A}}$. Due to the shift in our new flux, it fits perfectly with the structure of the problem (*cf.* Prop. 2.8 and treatment of the term K_3 in the proof of Thm. 5.4). This is in analogy to the gradient shift in the natural distance (*cf.* Rem. 2.12).

(ii) Theorems 4.3, 4.5, Corollary 4.10(i) of [19] treat the case $\mathbf{u}_D^* = \Pi_{\mathrm{SZ}}^k \mathbf{u} \in U_h^k \cap W^{1,p}(\Omega)$, where $\mathbf{u} \in W^{1,p}(\Omega)$ is the solution of (1.1). In this case, an optimal convergence rate is proved for $p \leq 2$, while for $p \geq 2$ only sub-optimal results are provided. Inasmuch as the solution $\mathbf{u}$ is *a priori* unknown and, in general, $\Pi_{\mathrm{SZ}}^k \mathbf{u} \neq \mathbf{u} = \mathbf{u}_D$ on Γ_D, these results are of rather theoretical interest than of interest from the practical computational point of view.

(iii) The assertions of Theorem 5.4 and Corollary 5.5 also hold if we replace the extension $\mathbf{u}_D^* \in W^{1,\varphi}(\Omega)$ of the boundary datum $\mathbf{u}_D$ by the approximation $\Pi_{\mathrm{SZ}}^k \mathbf{u} \in U_h^k \cap W^{1,\varphi}(\Omega)$ of the solution $\mathbf{u} \in W^{1,\varphi}(\Omega)$, *i.e.*, we define $\mathbf{u}_D^* := \Pi_{\mathrm{SZ}}^k \mathbf{u}$. In this case, we obtain

$$\mathbf{u}_h - \mathbf{u}_D^* = (\mathbf{u}_h - \mathbf{u}) + \big(\mathbf{u} - \Pi_{\mathrm{SZ}}^k \mathbf{u}\big) \quad \text{in } U_h^k.$$

Thus, we can, once more, replace $\mathbf{u}_h - \mathbf{u}_D^* \in U_h^k$ by $\mathbf{u}_h - \mathbf{u} \in W^{1,\varphi}(\Omega)$ in the modular $m_{\varphi_{\{|\Pi_h^0 \mathbf{L}_h|\}},h}(\mathbf{u}_h - \mathbf{u}_D^*)$ if we add on the right-hand side of (5.7) the term

$$\alpha\, c\, m_{\varphi_{\{|\Pi_h^0 \mathbf{L}_h|\}},h}\big(\mathbf{u} - \Pi_{\mathrm{SZ}}^k \mathbf{u}\big). \tag{5.19}$$

Similarly, we get instead of (5.11)

$$\boldsymbol{\mathcal{G}}_h^k \Pi_h^k \mathbf{u} + \boldsymbol{\mathcal{R}}_h^k \mathbf{u}_D^* - \nabla \mathbf{u} = \big(\nabla_h \Pi_h^k \mathbf{u} - \nabla \mathbf{u}\big) + \boldsymbol{\mathcal{R}}_h^k \big(\mathbf{u} - \Pi_h^k \mathbf{u}\big) + \boldsymbol{\mathcal{R}}_h^k \big(\Pi_{\mathrm{SZ}}^k \mathbf{u} - \mathbf{u}\big),$$

which amounts in an additional term

$$\rho_{\varphi_{|\nabla \mathbf{u}|}} \Big(\boldsymbol{\mathcal{R}}_h^k \big(\Pi_{\mathrm{SZ}}^k \mathbf{u} - \mathbf{u}\big)\Big). \tag{5.20}$$

Consequently, in the case $\mathbf{u}_D^* = \Pi_{\mathrm{SZ}}^k \mathbf{u} \in U_h^k \cap W^{1,\varphi}(\Omega)$, we have to treat additionally the terms in (5.19) and (5.20). However, the Scott–Zhang interpolation operator Π_{SZ}^k has similar properties as the projection operator Π_h^k. Thus, we can handle these terms in the same way as the corresponding terms in the proof of Theorem 5.4. Thus, our approach also gives for $\mathbf{u}_D^* = \Pi_{\mathrm{SZ}}^k \mathbf{u}$ optimal convergences rates for all balanced N-functions φ.

Remark 5.7. We restrict the discussion again to the special case that φ possesses (p, δ)-structure.

(i) Theorem 5.4 and Corollary 5.5 prove linear convergence rates for all polynomial degrees $k \in \mathbb{N}$ under the natural regularity assumption $\mathbf{F}(\mathbf{Du}) \in W^{1,2}(\Omega)$. To the best of the authors knowledge there are no theoretical results, using the natural distance, showing higher order convergence rates for $k > 1$ under the regularity assumption[5] $\mathbf{F}(\mathbf{Du}) \in W^{k,2}(\Omega)$. Numerical experiments for this situation are discussed in [25] in the time-dependent situation. Roughly speaking, for $k = 2$ one obtains convergence rates slightly less than 2 for all $p \neq 2$ if $\mathbf{F}(\mathbf{Du}) \in W^{2,2}(\Omega)$ and convergence rates of order 2 if the solution is assumed to be smooth. Numerical experiments for the steady case and some particular φ-structure are discussed in Section 6. Rigorous results for higher order convergence rates in the terms of Sobolev norms under the regularity assumption $\mathbf{u} \in W^{k+1,2}(\Omega)$ are obtained in [10].

(ii) Alternatively to convergence rates with respect to the mesh size $h > 0$ for fixed polynomial degree $k \in \mathbb{N}$, one can consider convergence rates with respect to the polynomial degree $k \in \mathbb{N}$ for fixed mesh size $h > 0$. This point of view is discussed in [2] in the context of FE methods and in [14] for DG methods.

6. Numerical experiments

In this section, we apply the LDG scheme (3.6) (or (3.9)), described above, to solve numerically the system (1.1) with balanced Orlicz-structure with the nonlinear operator $\boldsymbol{\mathcal{A}} : \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}$, for every $\mathbf{P} \in \mathbb{R}^{d\times d}$ defined *via*

$$\boldsymbol{\mathcal{A}}(\mathbf{P}) := (\delta + |\mathbf{P}|)^{p-2} \ln(1 + \delta + |\mathbf{P}|)\mathbf{P},$$

where $\delta := 1\mathrm{e}{-3}$ and $p \in (1, \infty)$, *i.e.*, the operator $\mathbf{F} : \mathbb{R}^{d\times d} \to \mathbb{R}^{d\times d}$ for every $\mathbf{P} \in \mathbb{R}^{d\times d}$ is defined *via*

$$\mathbf{F}(\mathbf{P}) := (\delta + |\mathbf{P}|)^{\frac{p-2}{2}} \sqrt{\ln(1 + \delta + |\mathbf{P}|)}\, \mathbf{P}.$$

We approximate the discrete solution $\mathbf{u}_h \in U_h^k$ of the nonlinear problem (3.6) deploying the Newton line search algorithm of PETSc (version 3.16.1), *cf.* [4], with an absolute tolerance of $\tau_{\mathrm{abs}} = 1\mathrm{e}{-8}$ and a relative tolerance of $\tau_{\mathrm{rel}} = 1\mathrm{e}{-10}$. The linear system emerging in each Newton step is solved deploying PETSc's preconditioned biconjgate gradient stabilized method (BCGSTAB) with an incomplete LU factorizaton. For the numerical flux (3.4), we choose the parameter $\alpha > 0$ according to Table 1 as a function of $p \in (1, \infty)$. This choice is in accordance with the choice in Table 1 of [19].

All experiments were carried out using the finite element software package FEniCS (version 2019.1.0), *cf.* [26]. All graphics are generated using the Matplotlib library (version 3.5.1), *cf.* [24].

[5]It is a widely open question if this regularity can be proved under reasonable assumptions.

TABLE 1. Choice of the stabilization parameter $\alpha > 0$.

	p									
–	1.25	4/3	1.5	5/3	1.8	2	2.25	2.5	3	4
α	0.06	0.1	0.2	0.5	1	2	2	2.5	2.5	2.5

TABLE 2. Experimental order of convergence: $\texttt{EOC}_i(e_{\mathbf{L},i})$, $i = 1, \ldots, 5$, in the case of $k = 1$, $\Gamma_D = \partial\Omega$, and $\mathbf{F}(\nabla\mathbf{u}) \in W^{1,2}(\Omega)$.

	p									
$\frac{h_0}{2^i}$	1.25	4/3	1.5	5/3	1.8	2	2.25	2.5	3	4
$i = 1$	0.93	0.94	0.94	0.93	0.92	0.91	0.91	0.91	0.92	0.93
$i = 2$	0.96	0.96	0.96	0.96	0.95	0.93	0.94	0.93	0.94	0.96
$i = 3$	0.97	0.97	0.97	0.97	0.96	0.95	0.95	0.95	0.95	0.96
$i = 4$	0.97	0.97	0.97	0.97	0.96	0.95	0.95	0.95	0.96	0.97
$i = 5$	0.97	0.97	0.98	0.97	0.97	0.96	0.96	0.96	0.96	0.97

For our numerical experiments, we choose $\Omega = (-2, 2)^2$, $\Gamma_D = \partial\Omega$, $\Gamma_N = \emptyset$, and employ both linear and quadratic elements, *i.e.*, $k \in \{1, 2\}$. For $\beta \in \{0.01, \frac{2}{p} + 0.01\}$, we choose $\mathbf{g} \in L^{p'}(\Omega)$ and $\mathbf{u}_D \in W^{1-\frac{1}{p},p}(\Gamma_D)$ such that $\mathbf{u} \in W^{1,p}(\Omega)$, for every $x := (x_1, x_2)^\top \in \Omega$ defined *via*

$$\mathbf{u}(x) := |x|^\beta (x_2, -x_1)^\top, \tag{6.1}$$

is a solution of (1.1) and satisfies $\mathbf{F}(\nabla\mathbf{u}) \in W^{1,2}(\Omega)$ if $\beta = 0.01$ and $\mathbf{F}(\nabla\mathbf{u}) \in W^{2,2}(\Omega)$ if $\beta = \frac{2}{p} + 0.01$. We construct a starting triangulation $\mathcal{T}_{h_0}$ with $h_0 = 1$ by subdividing a rectangular cartesian grid into regular triangles with different orientations. Finer triangulations $\mathcal{T}_{h_i}$, $i = 1, \ldots, 5$, with $h_{i+1} = \frac{h_i}{2}$ for all $i = 1, \ldots, 5$, are obtained by regular subdivision of the previous grid: Each triangle is subdivided into four equal triangles by connecting the midpoints of the edges.

Then, for the resulting series of triangulations $\mathcal{T}_{h_i}$, $i = 1, \ldots, 5$, we apply the above Newton scheme to compute the corresponding numerical solutions $(\mathbf{u}_i, \mathbf{L}_i, \mathbf{A}_i)^\top := (\mathbf{u}_{h_i}, \mathbf{L}_{h_i}, \mathbf{A}_{h_i})^\top \in U^k_{h_i} \times X^k_{h_i} \times X^k_{h_i}$, $i = 1, \ldots, 5$, and the error quantities

$$\left.\begin{aligned} e_{\mathbf{L},i} &:= \|\mathbf{F}(\mathbf{L}_i) - \mathbf{F}(\mathbf{L})\|_2, \\ e_{[\![\,]\!],i} &:= m_{\varphi_{\{|\mathbf{L}_i|\}},h_i}(\mathbf{u}_i - \mathbf{u})^{\frac{1}{2}}, \end{aligned}\right\} \quad i = 1, \ldots, 5.$$

As estimation of the convergence rates, the experimental order of convergence (EOC)

$$\texttt{EOC}_i(e_i) := \frac{\log(e_i/e_{i-1})}{\log(h_i/h_{i-1})}, \quad i = 1, \ldots, 5,$$

where for every $i = 1, \ldots, 5$, we denote by e_i either $e_{\mathbf{L},i}$ or $e_{[\![\,]\!],i}$, resp., is recorded. For different values of $p \in \{1.25, 4/3, 1.5, 5/3, 1.8, 2, 2.25, 2.5, 3, 4\}$ and for a series of triangulations $\mathcal{T}_{h_i}$, $i = 1, \ldots, 5$, obtained by regular, global refinement as described above with $h_0 = 1$, the EOC is computed and presented in Tables 2, 3, 4, 5, 6 and 7, resp. For $k \in \{1, 2\}$ and $\beta = 0.01$, we observe a convergence ratio of $\texttt{EOC}_i(e_i) \approx 1$, $i = 1, \ldots, 5$, as predicted by Theorem 5.4. Thus, the results of Theorem 5.4 are optimal in these cases.

In the case $k = 2$ and $\beta = \frac{2}{p} + 0.01$, which is not covered by Theorem 5.4, we observe a convergence ratio of $\texttt{EOC}_i(e_i) \approx 2$, $i = 1, \ldots, 5$, as expected for quadratic ansatz functions.

TABLE 3. Experimental order of convergence: $\texttt{EOC}_i(e_{[\![]\!],i})$, $i = 1, \dots, 5$, in the case of $k = 1$, $\Gamma_D = \partial\Omega$, and $\mathbf{F}(\nabla\mathbf{u}) \in W^{1,2}(\Omega)$.

	p									
$\frac{h_0}{2^i}$	1.25	4/3	1.5	5/3	1.8	2	2.25	2.5	3	4
$i = 1$	1.12	1.11	1.08	1.05	1.04	1.03	1.03	1.03	1.04	1.04
$i = 2$	1.04	1.04	1.03	1.03	1.03	1.03	1.03	1.03	1.04	1.04
$i = 3$	1.02	1.02	1.02	1.02	1.02	1.02	1.03	1.03	1.03	1.03
$i = 4$	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.02	1.03
$i = 5$	1.01	1.01	1.01	1.02	1.02	1.02	1.02	1.02	1.02	1.03

TABLE 4. Experimental order of convergence: $\texttt{EOC}_i(e_{\mathbf{L},i})$, $i = 1, \dots, 5$, in the case of $k = 2$, $\Gamma_D = \partial\Omega$, and $\mathbf{F}(\nabla\mathbf{u}) \in W^{1,2}(\Omega)$.

	p									
$\frac{h_0}{2^i}$	1.25	4/3	1.5	5/3	1.8	2	2.25	2.5	3	4
$i = 1$	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.01	1.01
$i = 2$	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.02	1.02
$i = 3$	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.02	1.02	1.02
$i = 4$	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.02	1.02	1.02
$i = 5$	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.02	1.02	1.02

TABLE 5. Experimental order of convergence: $\texttt{EOC}_i(e_{[\![]\!],i})$, $i = 1, \dots, 5$, in the case of $k = 2$, $\Gamma_D = \partial\Omega$, and $\mathbf{F}(\nabla\mathbf{u}) \in W^{1,2}(\Omega)$.

	p									
$\frac{h_0}{2^i}$	1.25	4/3	1.5	5/3	1.8	2	2.25	2.5	3	4
$i = 1$	1.43	1.23	1.08	1.03	1.02	1.01	1.01	1.02	1.02	1.03
$i = 2$	1.10	1.04	1.01	1.01	1.01	1.01	1.01	1.01	1.02	1.02
$i = 3$	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.02	1.02
$i = 4$	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.02	1.02	1.02
$i = 5$	1.01	1.01	1.01	1.01	1.01	1.01	1.01	1.02	1.02	1.02

TABLE 6. Experimental order of convergence: $\texttt{EOC}_i(e_{\mathbf{L},i})$, $i = 1, \dots, 5$, in the case of $k = 2$, $\Gamma_D = \partial\Omega$, and $\mathbf{F}(\nabla\mathbf{u}) \in W^{2,2}(\Omega)$.

	p									
$\frac{h_0}{2^i}$	1.25	4/3	1.5	5/3	1.8	2	2.25	2.5	3	4
$i = 1$	2.08	2.07	2.04	2.01	1.99	1.97	1.94	1.92	1.88	1.84
$i = 2$	2.04	2.04	2.03	2.01	2.00	1.99	1.96	1.95	1.92	1.89
$i = 3$	2.01	2.01	2.00	2.00	1.99	1.99	1.98	1.97	1.95	1.93
$i = 4$	2.00	2.00	2.00	1.99	1.99	1.99	1.98	1.98	1.97	1.95
$i = 5$	2.00	2.00	2.00	2.00	1.99	1.99	1.99	1.99	1.98	1.97

TABLE 7. Experimental order of convergence: $\texttt{EOC}_i(e_{[\![]\!],i})$, $i = 1, \ldots, 5$, in the case of $k = 2$, $\Gamma_D = \partial\Omega$, and $\mathbf{F}(\nabla\mathbf{u}) \in W^{2,2}(\Omega)$.

	p									
$\frac{h_0}{2^i}$	1.25	4/3	1.5	5/3	1.8	2	2.25	2.5	3	4
$i = 1$	1.98	1.97	1.97	1.95	1.93	1.92	1.91	1.90	1.89	1.88
$i = 2$	1.99	1.99	1.98	1.96	1.95	1.93	1.92	1.91	1.90	1.89
$i = 3$	2.00	2.00	1.99	1.98	1.97	1.96	1.95	1.94	1.93	1.92
$i = 4$	2.00	2.00	1.99	1.99	1.99	1.98	1.97	1.96	1.95	1.94
$i = 5$	2.00	2.00	2.00	2.00	1.99	1.99	1.98	1.98	1.97	1.96

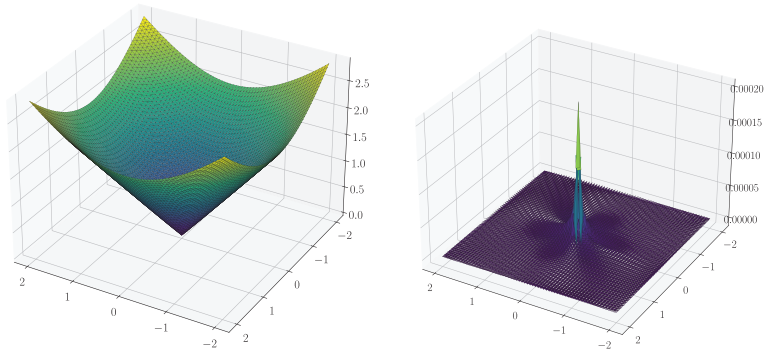

FIGURE 1. Plots of $\Pi^1_{h_4}(|\mathbf{u}_4|)$ (*left*) and $\Pi^1_{h_4}(|\mathbf{u}_4 - \mathbf{u}|)$ (*right*) for $p = 4$, $\alpha = 2.5$.

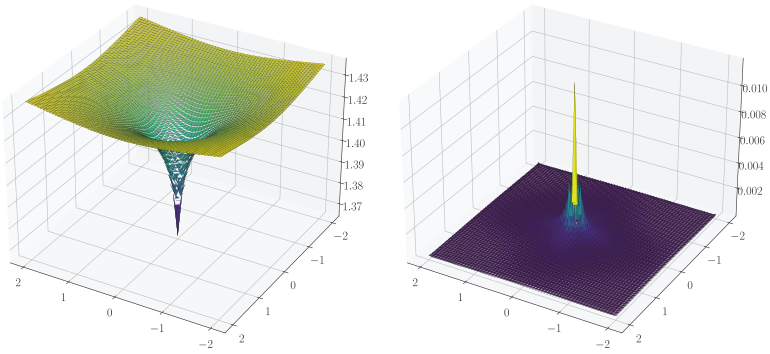

FIGURE 2. Plots of $\Pi^1_{h_4}(|\mathbf{L}_4|)$ (*left*) and $\Pi^1_{h_4}(|\mathbf{L}_4 - \mathbf{L}|)$ (*right*) for $p = 4$, $\alpha = 2.5$.

In the Figures 1–3, for $p = 4$ and $\alpha = 2.5$, we display $\Pi^1_{h_4}(|\mathbf{u}_4|)$, $\Pi^1_{h_4}(|\mathbf{u}_4 - \mathbf{u}|)$, $\Pi^1_{h_4}(|\mathbf{L}_4|)$, $\Pi^1_{h_4}(|\mathbf{L}_4 - \mathbf{L}|)$, and $\Pi^1_{h_4}(|\mathbf{R}^1_{h_4}(\mathbf{u}_4 - \mathbf{u})|)$. In it, we clearly observe that the major proportion of the error is located near the singularity of the exact solution $\mathbf{u} \in W^{1,4}(\Omega)$. For all other values $p \in \{1.25, 4/3, 1.5, 5/3, 1.8, 2, 2.25, 2.5, 3\}$ with associated $\alpha > 0$, according to Table 1, one get similar pictures.

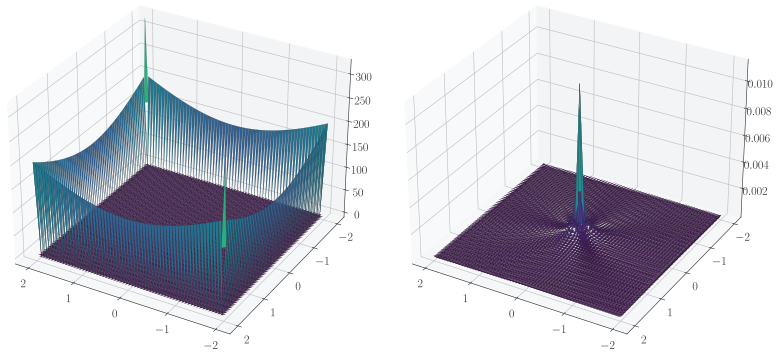

FIGURE 3. Plots of $\Pi^1_{h_4}(|\mathbf{R}^1_{h_4}\mathbf{u}_4|)$ (*left*) and $\Pi^1_{h_4}(|\mathbf{R}^1_{h_4}(\mathbf{u}_4-\mathbf{u})|)$ (*right*) for $p=4$, $\alpha=2.5$.

APPENDIX A.

This appendix collects known results, used in the paper, and proves new results in the DG Orlicz-setting.

Lemma A.1. *Let $\psi:\mathbb{R}^{\geq 0}\to\mathbb{R}^{\geq 0}$ be an N-function satisfying the Δ_2-condition and $k\in\mathbb{N}_0$. Then, for every $\mathbf{w}_h\in W^{1,\psi}(\mathcal{T}_h)$, we have that*

$$\begin{aligned}\int_{S_\gamma}\psi\Big(\boldsymbol{\mathcal{R}}^k_{h,\gamma}\mathbf{w}_h\Big)\,\mathrm{d}x &\leq c\,h\int_\gamma \psi(h^{-1}|[\![\mathbf{w}_h\otimes\mathbf{n}]\!]|)\,\mathrm{d}s,\\ \rho_{\psi,\Omega}\Big(\boldsymbol{\mathcal{R}}^k_h\mathbf{w}_h\Big) &\leq c\,m_{\psi,h}(\mathbf{w}_h)\end{aligned}\tag{A.1}$$

with a constant $c>0$ depending only on $k\in\mathbb{N}_0$, $\Delta_2(\psi)>0$, and $\omega_0>0$.

Proof. The assertions are proved in equations (A.23), (A.25) of [19]. □

Lemma A.2. *Let $\psi:\mathbb{R}^{\geq 0}\to\mathbb{R}^{\geq 0}$ be an N-function satisfying the Δ_2-condition and $k\in\mathbb{N}_0$. Then, for every $\mathbf{w}_h\in W^{1,\psi}(\mathcal{T}_h)$, we have that*

$$c^{-1}\,M_{\psi,h}(\mathbf{w}_h)\leq \rho_{\psi,\Omega}\Big(\boldsymbol{\mathcal{G}}^k_h\mathbf{w}_h\Big)+m_{\psi,h}(\mathbf{w}_h)\leq c\,M_{\psi,h}(\mathbf{w}_h)\tag{A.2}$$

with a constant $c>0$ depending only on $k\in\mathbb{N}_0$, $\Delta_2(\psi)>0$, and $\omega_0>0$.

Proof. The assertion follows from equations (A.26), (A.28) of [19] and Lemma A.1. □

Lemma A.3. *Let $\psi:\mathbb{R}^{\geq 0}\to\mathbb{R}^{\geq 0}$ be an N-function satisfying the Δ_2-condition and $k\in\mathbb{N}_0$. Then, for every $K\in\mathcal{T}_h$, $\mathbf{u}\in W^{\ell,\psi}(K)$ and $0\leq j\leq \ell\leq k+1$, we have that*

$$\fint_K \psi\Big(h_K^j\big|\nabla^j\big(\mathbf{u}-\Pi^k_h\mathbf{u}\big)\big|\Big)\,\mathrm{d}x\leq c\fint_K\psi\big(h_K^\ell|\nabla^\ell\mathbf{u}|\big)\,\mathrm{d}x,\tag{A.3}$$

$$\fint_K \psi\Big(h_K^j\big|\nabla^j\Pi^k_h\mathbf{u}\big|\Big)\,\mathrm{d}x\leq c\fint_K\psi\big(h_K^\ell|\nabla^\ell\mathbf{u}|\big)\,\mathrm{d}x\tag{A.4}$$

with a constant $c>0$ depending only on $\ell,k\in\mathbb{N}_0$, $\Delta_2(\psi)>0$, and $\omega_0>0$.

Proof. The first assertion is shown in equation (A.7) from [19], and the second one follows from the triangle inequality. □

Corollary A.4. *Let $\psi : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ be an N-function satisfying the Δ_2-condition and $k \in \mathbb{N}_0$. Then, for every $\mathbf{u} \in L^\psi(\Omega)$ and $\mathbf{w}_h \in W^{1,\psi}(\mathcal{T}_h)$, we have that*

$$\rho_{\psi,\Omega}\big(\mathbf{u} - \Pi_h^k \mathbf{u}\big) \leq c\, \rho_{\psi,\Omega}(\mathbf{u}), \tag{A.5}$$

$$\rho_{\psi,\Omega}\big(\mathbf{w}_h - \Pi_h^k \mathbf{w}_h\big) \leq c\, \rho_{\psi,\Omega}(h\, \nabla_h \mathbf{w}_h), \tag{A.6}$$

$$\rho_{\psi,\Omega}\big(\nabla_h \mathbf{w}_h - \nabla_h \Pi_h^k \mathbf{w}_h\big) \leq c\, \rho_{\psi,\Omega}(\nabla_h \mathbf{w}_h) \tag{A.7}$$

with a constant $c > 0$ depending only on $k \in \mathbb{N}_0$, $\Delta_2(\psi) > 0$, and $\omega_0 > 0$. In particular, this implies for every $k \in \mathbb{N}_0$, $\mathbf{u} \in L^\psi(\Omega)$ and $\mathbf{w}_h \in W^{1,\psi}(\mathcal{T}_h)$ that

$$\rho_{\psi,\Omega}\big(\Pi_h^k \mathbf{u}\big) \leq c\, \rho_{\psi,\Omega}(\mathbf{u}), \tag{A.8}$$

$$\rho_{\psi,\Omega}\big(\nabla_h \Pi_h^k \mathbf{w}_h\big) \leq c\, \rho_{\psi,\Omega}(\nabla_h \mathbf{w}_h). \tag{A.9}$$

For every $k \in \mathbb{N}$ and $\mathbf{w}_h \in W^{2,\psi}(\mathcal{T}_h)$, we have that

$$\rho_{\psi,\Omega}\big(\nabla_h \mathbf{w}_h - \nabla_h \Pi_h^k \mathbf{w}_h\big) \leq c\, \rho_{\psi,\Omega}\big(h \nabla_h^2 \mathbf{w}_h\big) \tag{A.10}$$

with a constant $c > 0$ depending only on $k \in \mathbb{N}$, $\Delta_2(\psi) > 0$, and $\omega_0 > 0$.

Lemma A.5. *Let $\boldsymbol{\mathcal{A}}$ satisfy Assumption 2.3 for a balanced N-function φ and $k \in \mathbb{N}$. Let $\mathbf{F}(\nabla \mathbf{u}) \in W^{1,2}(\Omega)$, then*

$$\big\|\mathbf{F}\big(\nabla_h \Pi_h^k \mathbf{u}\big) - \mathbf{F}(\nabla \mathbf{u})\big\|_2^2 \leq c\, h^2\, \|\nabla \mathbf{F}(\nabla \mathbf{u})\|_2^2$$

with $c > 0$ depending only on the characteristics of $\boldsymbol{\mathcal{A}}$ and φ, and the chunkiness ω_0. The same assertion also holds for the Scott–Zhang interpolation operator Π_{SZ}^k.

Proof. The assertion is proved in Corollary 5.8 of [17]. □

Lemma A.6. *Let ψ be an N-function satisfying the Δ_2-condition and $k \in \mathbb{N}_0$. Let $K \in \mathcal{T}_h$ and γ be a face of K. Then, for every $\mathbf{u} \in W^{1,\psi}(K)$ and $\mathbf{u}_h \in \mathcal{P}_k(K)$, it holds*

$$\fint_\gamma \psi(|\mathbf{u}|)\, \mathrm{d}s \leq c \fint_K \psi(|\mathbf{u}|)\, \mathrm{d}x + c \fint_K \psi(h|\nabla \mathbf{u}|)\, \mathrm{d}x, \tag{A.11}$$

$$\fint_\gamma \psi(|\mathbf{u}_h|)\, \mathrm{d}s \leq c \fint_K \psi(|\mathbf{u}_h|)\, \mathrm{d}x, \tag{A.12}$$

with constants $c > 0$ depending only on $k \in \mathbb{N}_0$, $\Delta_2(\psi) > 0$, and $\omega_0 > 0$.

Proof. The assertions are proved in equations (A.13), (A.14) of [19]. □

Corollary A.7. *Let $\psi : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ be an N-function satisfying the Δ_2-condition and $k \in \mathbb{N}_0$. Let $K \in \mathcal{T}_h$ and γ be a face of K. Then, for every $\mathbf{u} \in W^{1,\psi}(K)$, $\mathbf{w}_h \in W^{1,\psi}(\mathcal{T}_h)$ and $\mathbf{u}_h \in U_h^k$, we have that*

$$h \int_\gamma \psi\big(h^{-1}\big|\mathbf{u} - \Pi_h^k \mathbf{u}\big|\big)\, \mathrm{d}s \leq c \int_K \psi(|\nabla \mathbf{u}|)\, \mathrm{d}x, \tag{A.13}$$

$$m_{\psi,h}\big(\mathbf{w}_h - \Pi_h^k \mathbf{w}_h\big) \leq c\, \rho_{\psi,\Omega}(\nabla_h \mathbf{w}_h), \tag{A.14}$$

$$m_{\psi,h}(\mathbf{w}_h) \leq c\, \rho_{\psi,\Omega}\big(h^{-1} \mathbf{w}_h\big) + c\, \rho_{\psi,\Omega}(\nabla_h \mathbf{w}_h), \tag{A.15}$$

$$h\, \rho_{\psi,\Gamma_h^i \cup \Gamma_D}(\{\mathbf{u}_h\}) \leq c\, \rho_{\psi,\Omega}(\mathbf{u}_h), \tag{A.16}$$

$$h\, \rho_{\psi,\Gamma_h^i \cup \Gamma_D}\big(\{\mathbf{w}_h - \Pi_h^k \mathbf{w}_h\}\big) \leq c\, \rho_{\psi,\Omega}(h \nabla_h \mathbf{w}_h), \tag{A.17}$$

with constants $c > 0$ depending only on $k \in \mathbb{N}_0$, $\Delta_2(\psi)$, and $\omega_0 > 0$.

Proof. For the assertions (A.13) and (A.14), we refer to equations (A.15), (A.16) from [19]. The assertions (A.15) and (A.16) follow from (A.11) and (A.12), resp., by summation, while (A.17) follows from (A.13) by summation. □

Corollary A.8. *Let $\boldsymbol{\mathcal{A}}$ satisfy Assumption 2.3 for a balanced N-function φ and $k \in \mathbb{N}_0$. Moreover, let $\mathbf{u} \in W^{1,\varphi}(\Omega)$ satisfy $\mathbf{F}(\nabla \mathbf{u}) \in W^{1,2}(\Omega)$. Then, for every $\mathbf{w}_h \in W^{1,\varphi}(\mathcal{T}_h)$, it holds*

$$m_{\varphi_{|\nabla \mathbf{u}|},h}\big(\mathbf{w}_h - \Pi_h^k \mathbf{w}_h\big) \leq c\, \rho_{\varphi_{|\nabla \mathbf{u}|},\Omega}(\nabla_h \mathbf{w}_h) + c\, h^2 \, \|\nabla \mathbf{F}(\nabla \mathbf{u})\|_2^2, \tag{A.18}$$

with constants $c > 0$ depending only on $k \in \mathbb{N}_0$, the characteristics of $\boldsymbol{\mathcal{A}}$ and φ, and $\omega_0 > 0$.

Proof. Using the convexity of $\varphi_{|\nabla \mathbf{u}|}$, twice a change of shift (*cf.* Lem. 2.7), Proposition 2.8, the local trace inequalities (A.13) and (A.11) and Poincaré's inequality on K together with Lemma A.12 of [7] we find that

$$\begin{aligned}
&h \int_\gamma \varphi_{|\nabla \mathbf{u}|}\big(h^{-1} \big|[\![(\mathbf{w}_h - \Pi_h^k \mathbf{w}_h) \otimes \mathbf{n}]\!]\big|\big)\, \mathrm{d}s \\
&\quad \leq h \sum_{K \in \mathcal{T}_h; K \subseteq S_\gamma} \int_\gamma \varphi_{|\langle \nabla \mathbf{u} \rangle_K|}\big(h^{-1} \big|\mathbf{w}_h - (\Pi_h^k \mathbf{w}_h)|_K\big|\big) + \big|\mathbf{F}(\nabla \mathbf{u}) - \mathbf{F}(\langle \nabla \mathbf{u} \rangle_K)\big|^2 \, \mathrm{d}s \\
&\quad \leq \sum_{K \in \mathcal{T}_h; K \subseteq S_\gamma} \int_K \varphi_{|\langle \nabla \mathbf{u} \rangle_K|}(|\nabla \mathbf{w}_h|)\, \mathrm{d}s + h^2 \sum_{K \in \mathcal{T}_h; K \subseteq S_\gamma} \int_K |\nabla \mathbf{F}(\nabla \mathbf{u})|^2 \, \mathrm{d}s \\
&\quad \leq \int_{S_\gamma} \varphi_{|\nabla \mathbf{u}|}(|\nabla_h \mathbf{w}_h|)\, \mathrm{d}s + h^2 \int_{S_\gamma} |\nabla \mathbf{F}(\nabla \mathbf{u})|^2 \, \mathrm{d}s.
\end{aligned}$$

Then, the assertion follows by summing with respect to $\gamma \in \Gamma_h^i \cup \Gamma_D$. □

Lemma A.9. *Let $\psi : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ be an N-function satisfying the Δ_2-condition and $k \in \mathbb{N}$. Then, for every $\mathbf{u} \in W^{1,\psi}(\Omega)$, we have that*

$$\rho_{\psi,\Omega}\big(\nabla_h \big(\Pi_h^k \mathbf{u} - \mathbf{u}\big)\big) \to 0 \quad (h \to 0), \tag{A.19}$$

$$m_{\psi,\Omega}\big(\Pi_h^k \mathbf{u} - \mathbf{u}\big) \to 0 \quad (h \to 0), \tag{A.20}$$

$$\rho_{\psi,\Omega}\Big(\boldsymbol{\mathcal{R}}_h^k \big(\Pi_h^k \mathbf{u} - \mathbf{u}\big)\Big) \to 0 \quad (h \to 0), \tag{A.21}$$

$$\rho_{\psi,\Omega}\Big(\boldsymbol{\mathcal{G}}_h^k \big(\Pi_h^k \mathbf{u} - \mathbf{u}\big)\Big) \to 0 \quad (h \to 0). \tag{A.22}$$

Proof. For every $\mathbf{u} \in W^{1,\psi}(\Omega)$, there exists a sequence $(\mathbf{u}^n)_{n \in \mathbb{N}} \subseteq C^\infty(\overline{\Omega})$ such that

$$\rho_{\psi,\Omega}(\nabla(\mathbf{u}^n - \mathbf{u})) \to 0 \quad (n \to \infty). \tag{A.23}$$

Thus, using (A.9), (A.10) and the properties of the N-function ψ, we find that

$$\begin{aligned}
\rho_{\psi,\Omega}\big(\nabla_h \big(\Pi_h^k \mathbf{u} - \mathbf{u}\big)\big) &\leq c\, \rho_{\psi,\Omega}\big(\nabla_h \Pi_h^k (\mathbf{u} - \mathbf{u}^n)\big) + c\, \rho_{\psi,\Omega}\big(\nabla_h \big(\Pi_h^k \mathbf{u}^n - \mathbf{u}^n\big)\big) + c\, \rho_{\psi,\Omega}(\nabla(\mathbf{u}^n - \mathbf{u})) \\
&\leq c\, \rho_{\psi,\Omega}(\nabla(\mathbf{u}^n - \mathbf{u})) + c\, \rho_{\psi,\Omega}\big(h\, \nabla^2 \mathbf{u}^n\big).
\end{aligned} \tag{A.24}$$

Due to (A.23), for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $c\rho_{\psi,\Omega}(\nabla(\mathbf{u}^n - \mathbf{u})) \leq \varepsilon$ for all $n \in \mathbb{N}$ with $n \geq n_0$. Therefore, choosing $n = n_0 \in \mathbb{N}$ in (A.24), we conclude that $\lim_{h \to 0} \rho_{\psi,\Omega}(\nabla_h \big(\Pi_h^k \mathbf{u} - \mathbf{u}\big)) \leq \varepsilon$, which yields (A.19), since $\varepsilon > 0$ was chosen arbitrarily.

Choosing $\Pi_h^k \mathbf{u} - \mathbf{u} = \Pi_h^k \mathbf{u} - \mathbf{u} - \Pi_h^k(\Pi_h^k \mathbf{u} - \mathbf{u})$ in (A.14), since $W^{1,\psi}(\Omega) \subseteq W^{1,\psi}(\mathcal{T}_h)$, also using (A.19), we find that $m_{\psi,\Omega}(\Pi_h^k \mathbf{u} - \mathbf{u}) \leq c\, \rho_{\psi,\Omega}(\nabla_h(\Pi_h^k \mathbf{u} - \mathbf{u})) \to 0 (h \to 0)$, which is (A.20).

Next, equations (A.1) and (A.20) yield $\rho_{\psi,\Omega}(\boldsymbol{\mathcal{R}}_h^k(\Pi_h^k \mathbf{u} - \mathbf{u})) \leq c\, m_{\psi,\Omega}(\Pi_h^k \mathbf{u} - \mathbf{u}) \to 0$ $(h \to 0)$, which is (A.21).

Finally, equation (A.22) follows from the definition of $\boldsymbol{\mathcal{G}}_h^k$ in (2.28), (A.19) and (A.21). □

Lemma A.10. *Let $\psi : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ be an N-function such that ψ and ψ^* satisfy the Δ_2-condition. Then, for every $\mathbf{w}_h \in W^{1,\psi}(\mathcal{T}_h)$, we have that*

$$\rho_{\psi,\Omega}(\mathbf{w}_h) \leq c\, M_{\psi,h}(\mathrm{diam}(\Omega)\, \mathbf{w}_h), \tag{A.25}$$

$$\rho_{\psi,\Gamma_N}(\mathbf{w}_h) \leq c\ \mathrm{diam}(\Omega)^{-1} M_{\psi,h}(\mathrm{diam}(\Omega)\, \mathbf{w}_h), \tag{A.26}$$

where $c > 0$ only depends on $\Omega, \Omega' \subseteq \mathbb{R}^n$, $n \geq 2$, and $\Delta_2(\psi), \Delta_2(\psi^*), \omega_0 > 0$.

Proof. This is proved in Lemmas A.9, A.10 of [19]. □

Lemma A.11. *Let $\psi : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ be an N-function such that ψ and ψ^* satisfy the Δ_2-condition and $k \in \mathbb{N}$. Let $\mathbf{w}_h \in W^{1,\psi}(\mathcal{T}_h)$, $h > 0$, be such that $\sup_{h>0} M_{\psi,h}(\mathbf{w}_h) \leq c$. Then, for the sequence $\mathbf{w}_n := \mathbf{w}_{h_n} \in W^{1,\psi}(\mathcal{T}_{h_n})$, where $h_n \to 0$ $(n \to \infty)$, there exists a function $\mathbf{w} \in W^{1,\psi}_{\Gamma_D}(\Omega)$ such that, up to subsequences, we have that*

$$\mathbf{w}_n \rightharpoonup \mathbf{w} \quad \text{in } L^\psi(\Omega) \quad (n \to \infty), \tag{A.27}$$

$$\boldsymbol{\mathcal{G}}^k_{h_n} \mathbf{w}_n \rightharpoonup \nabla \mathbf{w} \quad \text{in } L^\psi(\Omega) \quad (n \to \infty), \tag{A.28}$$

$$\mathbf{w}_n \rightharpoonup \mathbf{w} \quad \text{in } L^\psi(\Gamma_N) \quad (n \to \infty). \tag{A.29}$$

Proof. The proof is a straightforward adaptation of the proof of Theorem 5.7 from [15]. In fact, from Poincaré's inequality (A.25) and the reflexivity of $L^\psi(\Omega)$, it follows that there exists $\mathbf{w} \in L^\psi(\Omega)$ such that for a not relabeled subsequence, it holds (A.27). We extend both $\mathbf{w}$ and $\mathbf{w}_n$ by zero to $\Omega' \setminus \Omega$ and denote the extensions again by $\mathbf{w}$ and $\mathbf{w}_n$, respectively. Moreover, we extend $\boldsymbol{\mathcal{R}}^k_{h_n} \mathbf{w}_n$ by zero to $\Omega' \setminus \Omega$ and denote the extension again by $\boldsymbol{\mathcal{R}}^k_{h_n} \mathbf{w}_n$. Using these extensions and (A.2), we obtain a not relabeled sub-sequence and a function $\mathbf{G} \in L^\psi(\Omega')$ such that

$$\boldsymbol{\mathcal{G}}^k_{h_n} \mathbf{w}_n \rightharpoonup \mathbf{H} \quad \text{in } L^\psi(\Omega') \quad (n \to \infty). \tag{A.30}$$

We have to show that $\mathbf{H} = \nabla \mathbf{w}$ holds in $L^\psi(\Omega')$. To this end, we observe that for every $\mathbf{X} \in C_0^\infty(\Omega')$, there holds

$$\begin{aligned} \left(\boldsymbol{\mathcal{G}}^k_{h_n} \mathbf{w}_n, \mathbf{X}\right)_{\Omega'} &= (\nabla_{h_n} \mathbf{w}_n, \mathbf{X})_{\Omega'} - \left(\boldsymbol{\mathcal{R}}^k_{h_n} \mathbf{w}_n, \Pi^k_{h_n} \mathbf{X}\right)_{\Omega'} \\ &= -\left(\mathbf{w}_n, \mathrm{div}\mathbf{X}\right)_{\Omega'} + \left\langle [\![\mathbf{w}_n \otimes \mathbf{n}]\!], \left\{\mathbf{X} - \Pi^k_{h_n} \mathbf{X}\right\}\right\rangle_{\Gamma^i_h{}' \cup \Gamma'_D}. \end{aligned} \tag{A.31}$$

Using Young's inequality, $\sup_{n\in\mathbb{N}} m_{\psi,h_n}(\mathbf{w}_n) \leq \sup_{n\in\mathbb{N}} M_{\psi,h_n}(\mathbf{w}_n) < \infty$ and (A.14) for ψ^*, by passing for $n \to \infty$, for every $\mathbf{X} \in C_0^\infty(\Omega')$, we find that

$$\left\langle [\![\mathbf{w}_n \otimes \mathbf{n}]\!], \left\{\mathbf{X} - \Pi^k_{h_n} \mathbf{X}\right\}\right\rangle_{\Gamma^i_h{}' \cup \Gamma'_D} \leq h_n c\big(m_{\psi^*,h_n}\big(\mathbf{X} - \Pi^k_{h_n} \mathbf{X}\big) + m_{\psi,h_n}(\mathbf{w}_n)\big) \to 0.$$

Thus, by passing for $n \to \infty$ in (A.31), using (A.30) and (A.27), for any $\mathbf{X} \in C_0^\infty(\Omega')$, we arrive at $(\mathbf{H}, \mathbf{X})_{\Omega'} = -(\mathbf{w}, \mathrm{div}\mathbf{X})_{\Omega'}$, *i.e.*, $\mathbf{H} = \nabla \mathbf{w}$ in Ω' and, thus, $\mathbf{w} \in W^{1,\psi}(\Omega')$. Since $\mathbf{w} = \mathbf{0}$ in $\Omega' \setminus \Omega$, we get $\mathbf{w} \in W^{1,\psi}_{\Gamma_D}(\Omega)$.

Inequality (A.26) and the reflexivity of $L^\psi(\Gamma_N)$ yield a not relabeled subsequence and a function $\mathbf{g} \in L^\psi(\Gamma_N)$ such that

$$\mathbf{w}_n \rightharpoonup \mathbf{h} \quad \text{in } L^\psi(\Gamma_N) \quad (n \to \infty). \tag{A.32}$$

Similar arguments as above yield that for every $\mathbf{X} \in C^\infty(\overline{\Omega})$, we have that

$$\left(\boldsymbol{\mathcal{G}}^k_{h_n} \mathbf{w}_n, \mathbf{X}\right)_{\Omega} = -(\mathbf{w}_n, \mathrm{div}\mathbf{X})_{\Omega} + \left\langle [\![\mathbf{w}_n \otimes \mathbf{n}]\!], \left\{\mathbf{X} - \Pi^k_{h_n} \mathbf{X}\right\}\right\rangle_{\Gamma^i_h \cup \Gamma_D} + \langle \mathbf{w}_n \otimes \mathbf{n}, \mathbf{X}\rangle_{\Gamma_N}.$$

Taking the limit with respect to $n \to \infty$ in this equality, we find that

$$\begin{aligned} (\nabla \mathbf{w}, \mathbf{X})_{\Omega} &= -(\mathbf{w}, \mathrm{div}\mathbf{X})_{\Omega} + \langle \mathbf{h} \otimes \mathbf{n}, \mathbf{X}\rangle_{\Gamma_N} \\ &= (\nabla \mathbf{w}, \mathbf{X})_{\Omega} + \langle (\mathbf{h} - \mathbf{w}) \otimes \mathbf{n}, \mathbf{X}\rangle_{\Gamma_N}. \end{aligned}$$

Choosing $\mathbf{X} = \mathbf{z} \otimes \mathbf{n}$ for arbitrary $\mathbf{z} \in C_0^\infty(\Gamma_N)$, we conclude that $\mathbf{h} = \mathbf{w}$ in $L^\psi(\Gamma_N)$, which together with (A.32) proves (A.29). □

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