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ANALYSIS OF LINEARIZED ELASTICITY MODELS WITH POINT SOURCES IN WEIGHTED SOBOLEV SPACES: APPLICATIONS IN TISSUE CONTRACTION

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Abstract. In order to model the contractive forces exerted by fibroblast cells in dermal tissue, we propose and analyze two modeling approaches under the assumption of linearized elasticity. The first approach introduces a collection of point forces on the boundary of the fibroblast whereas the second approach employs an isotropic stress point source in its center. We analyze the resulting partial differential equations in terms of weighted Sobolev spaces and identify the singular behavior of the respective solutions. Two finite element method approaches are proposed, one based on a direct application and another in which the singularity is subtracted and a correction field is computed. Finally, we confirm the validity of the modeling approach, demonstrate convergence of the numerical methods, and verify the analysis through the use of numerical experiments.

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1. Introduction

Injured skin undergoes a chain of biological processes that ultimately aim towards dermal healing. This chain of biological reactions can roughly be divided into four phases: (i) hemostasis, which is the coagulation (blood clotting) process that prevents further loss of blood and ingress of contaminants, (ii) inflammatory phase, which is the activation of the immune response that clears up debris and hazardous chemicals and pathogens, (iii) proliferative phase, which involves tissue regeneration and angiogenesis (regeneration of small blood vessel network), and (iv) remodeling, which replaces provisional collagen from the proliferative phase with permanent collagen with better mechanical properties. These phases in dermal healing are sequential and overlap partly. More information about the description of the underlying biological processes can be found in, among others [11,21].

During the proliferative phase in deep skin (tissue) injuries, fibroblasts, which are skin cells, migrate into the damaged area and rebuild the collagen structure. Next to the regeneration of collagen, fibroblasts exert forces on their direct environment, which reduces the volume, and hence the exposed area, of the wound.

Keywords and phrases. Point source, weighted Sobolev space, linearized elasticity.

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This process is referred to as contraction, and in many cases, this process is beneficial for the organism as it reduces the probability that contaminants and pathogens enter the body. In some cases, and in particular in severe burn injuries, the mechanism of contraction is excessive as a result of the differentiation of fibroblasts to myofibroblasts. Myofibroblasts exert larger forces on their immediate environment and produce larger amounts of collagen. The types of collagen that fibroblasts and myofibroblasts produce differ and it is known that the collagen type that is secreted by fibroblasts is more favorable than collagen secreted by myofibroblasts. The type of collagen that is deposited by myofibroblasts is replaced with fibroblasts-secreted collagen during the remodeling phase. It is known that in many wounds, myofibroblasts are subject to apoptosis (programmed cell death), however, in acute wounds, myofibroblasts have been observed to persist [28]. More information regarding the dynamics of fibroblasts and myofibroblasts can be found in, among others [7, 9].

Since the myofibroblasts exert larger forces, the wound (or scar) area suffers from a larger extent of contraction. This excessive contraction may result in the patient suffering from (local) disability. In these cases, contractions are referred to as contracture. Many biomedical studies aim at the prevention of contracture. Although there are still many open questions regarding the biological mechanisms behind contraction, it is known that local stresses in the damaged skin trigger the differentiation of fibroblasts to myofibroblasts and thereby activate the contraction reaction. These stresses can occur as a result of motion in body parts, particularly if the burn injury is near a joint, but even the act of breathing may trigger contraction. It also turns out that some patients are more prone than other patients to develop a contraction under the same conditions. Quantitative understanding of the mechanisms behind the development of contracture may help practitioners develop and optimize therapies that aim at prevention and treatment. Current treatments are based on skin grafting (skin transplantation), splinting and the application of (chemical) dressings.

Multiple mathematical formalisms have been developed to simulate the contraction process. Roughly speaking, these formalisms can be divided into agent-based and fully continuum-scale models. The fully continuum-based models are based on averaged quantities, such as cell densities, which are governed by (systems of) (nonlinear) partial differential equations. Agent-based models typically treat cells as separate entities that interact, exert forces, migrate, divide, differentiate, and expire. In this manuscript, we focus on the agent-based models where discrete cells exert forces on their immediate environment. Since the size of the cells is typically several orders of magnitude smaller than the size of the computational domain, we employ Dirac delta distributions to define the forces. This approach was used for the description of cell forces and plastic forces in [6, 23]. Although the obtained numerical results by the use of the finite element method are sensible and useful for further interpretation, some mathematical questions remain. Formally, the solutions for the displacement are not in the Sobolev space $H^1(\Omega)$, although one naively searches the finite element solution in this space. Nevertheless, in [4], among others, the convergence of finite element solutions in case of a Laplace (Poisson) equation with a Dirac delta distribution of the right-hand side is proved. The use of linear elements yields a convergence in the H^1-norm of order $\mathcal{O}(h^{1/2})$ in $\mathbb{R}^3$. Prior to this work, several studies addressed the same Laplace problem with a Dirac delta distribution. Scott [25] addressed convergence of the solution for $\mathbb{P}_1$-elements as early as 1973, which is, to the best of our knowledge, the earliest study addressing this problem for finite element methods. The main result in [25] was convergence of the finite element solution in the L^2-norm of order $\mathcal{O}(h^{2-d/2})$, where d stands for the dimensionality. Later, in 1985, Erikson [12] proved convergence for $\mathbb{P}_k$-elements of order $\mathcal{O}(h^k)$ and $\mathcal{O}(h^{k+1})$, respectively, in the $W^{1,1}$- and L^1-norm for 2D problems. Apel *et al.* [1] demonstrated convergence in the L^2-norm of the finite element solution of order $\mathcal{O}(h^2\sqrt{|\log(h)|}$ for $\mathbb{P}_1$ elements in 2D in 2011. Köppl and Wohlmuth [19] proved convergence of classical order $\mathcal{O}(h^{k+1})$ in the L^2-norm at portions of the computational domain away from the location where the Dirac delta distribution acts for $\mathbb{P}_k$ elements for 2D finite element solutions.

All these earlier studies concern the Laplace problem, whereas this manuscript is devoted to the analysis of well-posedness of the boundary value problems associated with the balance of momentum in linearized elasticity. From a physics point of view, major simplifications are made: the medium is assumed to be isotropic and subject to the linear Hooke's Law, strains are assumed to be infinitesimal, inertia is neglected, and in the finite element simulations, the displacements are assumed to be small enough such that deformation of the mesh is not taken

FIGURE 1. *Left*: contractive forces are defined on the cell boundary in the point force approach. *Right*: point stress approach employs an isotropic stress point source in the cell center.

into account. Furthermore, viscous (damping) effects are neglected. Despite reality being significantly more complicated, the current study is original as it extends parts of the study by D'Angelo [8] to linear elasticity and provides a necessary step in the development of the mathematical theory for more realistic models. The current study entails point forces and point stresses. As mentioned earlier, applications are in cells that are located in a bounded domain. Here, the size of the cells in the order of several micrometers, whereas the tissues where the model is applied could have dimensions of the order of centimeters. In this case, cells are treated as a set of point forces that are exerted on the boundary. In this setting, the cell shape is simplified to a triangle or square where on the midpoint of each boundary segment a point force is exerted. In a 3D setting, the cell could be approximated by a cube, where on all the midpoints of the faces a point force is positioned. The size (diameter) of a finite element triangle (or tetrahedron in 3D) can be of the same order as the size of a biological cell. An alternative to this modelling framework entails a point stress source, which can be used if the ratio between the size of a cell and the size of the computational domain tends to zero. This alternative strategy is also analyzed in the current paper.

A second application of the current model formulation is the case of a (biological) cell that is relatively large, moves through the domain of computation and exerts forces on its direct surroundings. This modelling case can be used to simulate a cancer cell that migrates through a narrow channel, where it has to exert forces on the channel wall to widen the channel, so that the cell is able to actually migrate through the channel.

The article is organized as follows. Section 2 introduces the two model problems of interest, referred to as the point force and the point stress problem. The former is analyzed in Section 3 and shown to be well-posed in a weighted Sobolev space, following the steps outlined in [8]. The point stress problem, on the other hand, is considered in Section 4 and we use the results of Section 3 to prove that it is well-posed. Afterward, Section 5 proposes viable finite element discretization schemes that exploit the knowledge regarding the singularity in the solution. Numerical experiments of the point stress problem are presented in Sections 6 and 7 contains the conclusions.

2. Model problems

We start by describing the approaches that lead to the two model problems that form the focus of this work. The formulations of these problems only differ in the source term, or right-hand side of the system. Both problems are thus posed on the computational domain $\Omega \subset \mathbb{R}^d$, which is assumed to be a bounded Lipschitz domain with $d \in \{1, 2, 3\}$ the dimension.

2.1. The point force problem

In the point force approach, investigated in [6, 23, 24], the forces exerted by a fibroblast cell are modeled by the use of the so-called immersed boundary method, where the cell boundary is divided into boundary segments. On each boundary segment, a point force is exerted on the environment that is normalized by the measure (area in 3D or length in 2D) of the boundary segment. Furthermore, the point force is located on the center of the boundary segment and oriented towards the interior of the cell, cf. Figure 1 (left). Integration over the entire boundary then determines the total force. Denoting the forces by $\boldsymbol{F}_i$ with i the index, this approach gives us the momentum balance equation:

$$-\nabla \cdot \sigma = \sum_i \boldsymbol{F}_i \delta(\boldsymbol{x} - \boldsymbol{x}_i), \tag{2.1a}$$

with σ the tensorial stress field. We complement this by Hooke's law using the fourth-order tensor $\mathbb{C}$, defined as $\mathbb{C}\Sigma := 2\mu\Sigma + \lambda \mathrm{Tr}(\Sigma) I$ for all $\Sigma \in \mathbb{R}^{d\times d}$:

$$\sigma = \mathbb{C}\varepsilon\boldsymbol{u} = 2\mu\varepsilon\boldsymbol{u} + \lambda(\nabla \cdot \boldsymbol{u}) I, \tag{2.1b}$$

in which μ and λ are the constant, positive, Lamé parameters, I is the identity tensor, ε is the symmetric gradient operator and thus $\varepsilon\boldsymbol{u}$ is the linearized strain tensor, which is given by $\varepsilon\boldsymbol{u} = \frac{1}{2}(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T)$. To close the system, we impose the homogeneous boundary condition

$$\boldsymbol{u} = 0, \qquad \text{on} \;\; \partial\Omega. \tag{2.1c}$$

The above boundary value problem provides a linear problem for the displacement $\mathbf{u}$. In our applications, we use this formalism with moving (and deforming) cells, and hence the positions where the point forces are acting may migrate through the domain of computation. An alternative approach is based on treating cells as holes that are migrating. The boundary of the cell is subsequently treated as a boundary condition that is moving through the domain of computation. In order to make this possible, remeshing in a region at least near to the cell boundary is necessary. In practice this is not feasible from a numerical point of view, and therefore we rely on treating cell traction forces by a series of point forces on the cell boundary. On the other hand, using Dirac delta distributions implies a singular solution in the sense that the solution is formally out of the finite element function space H^1.

Regarding the applicability, we note that we carry out simulations in multi-cell colonies, where each cell exerts traction forces that give rise to local displacements. The cells are displaced according to the displacements as a result of traction forces that are exerted by the cells. This mode of displacement is commonly referred to as *passive convection*, and this mode is commonly modelled in continuum-based models by adding a material derivative to the conservation law for the cell density. Further modes of cell displacement are random walk (diffusion) through Wiener processes, chemotaxis (migration according to the gradient of a concentration) and tensotaxis (migration according to the gradient of stiffness). For the sake of passive convection, it is crucially important to know the displacement rates on the location of cells.

In the model for cells exerting traction or pulling forces, the cell boundary is divided into nodal points, that are connected by boundary segments. In this way, a circular cell is, for instance, approximated by a polygon. On the midpoint of a boundary segment, a point force is applied. In this way, the total force follows from summation over the entire cell boundary. If the boundary segments are made infinitesimally small, that is, the number of nodal points on the boundary is sent to infinity, then one obtains the total force by integration over the cell boundary.

In earlier work [6], we considered a line source for the production of a chemical. This approach was possible since the production of a chemical involves a scalar quantity. In the current problem, we are dealing with forces, which represent a vectorial quantity. In order to generalize the direction of the force (being contractive or repulsive), this approach is not feasible since over each boundary point the force direction changes. This is true in particular if the cell shape differs much from circular of spherical geometries. This approach is suitable for cases where the cell size is at least of the same order as the finite element mesh size.

We refer to (2.1) as the *point force problem.* This system can easily be discretized using standard finite elements method since the right-hand side then corresponds to a point evaluation of piecewise polynomial test functions. The disadvantage, however, is that the problem depends on the number of segments in which the cell boundary is decomposed and, strictly speaking, produces a different solution for each choice. Although these differences are not detectable at coarse discretization levels if each $\mathbf{F}_i$ is properly scaled [24], the solution to the continuous problem is affected and therewith the convergence of the discretization. For this reason, we are interested in an alternative approach that admits a unique solution, independent of cell boundary decompositions.

2.2. The point stress problem

An alternative approach is to model the contractive effect of fibroblasts by employing concentrated, isotropic stress fields. By letting the cell radius tend to zero within the model, we obtain point sources similar to those found in seismic imaging inversion, see *e.g.* [26]. In contrast to that application, however, we are currently interested in the forward model in mechanical equilibrium, where elastic wave propagation is neglected.

This approach is based on representing the force field generated by a fibroblast by $\boldsymbol{F} = \nabla \cdot (\delta_\epsilon I)$ with δ_ϵ a scalar function and I the identity tensor. We assume that the support of δ_ϵ is given by a ball of radius ϵ centered at the fibroblast center, which is assumed to be at the origin. To avoid confusion, we emphasize that ϵ represents the radius of a ball, whereas ε represents the symmetric gradient operator (strain operator). In order to generate an inward oriented force field, we define δ_ϵ as a positive function that is decreasing from the cell center to the boundary. Moreover, we assume that δ_ϵ is normalized in the sense that its integral is one. A smooth example is the Gaussian probability density function of a normal distribution with standard deviation ϵ and mean the position of the cell center.

In models of dermal tissue, the fibroblasts are typically mobile in the computational domain and it becomes an involved task to integrate moving, compactly supported source functions with respect to a stationary, underlying mesh. To address this, we model the stress field generated by a fibroblast using the Dirac delta measure δ that assigns unit mass to the fibroblast center. This leads us to the following system of equations:

$$\sigma - \mathbb{C}\varepsilon \boldsymbol{u} = 0, \qquad \text{in } \Omega, \tag{2.2a}$$

$$-\nabla \cdot \sigma = \nabla \cdot (\delta I), \qquad \text{in } \Omega, \tag{2.2b}$$

$$\boldsymbol{u} = 0, \qquad \text{on } \partial\Omega. \tag{2.2c}$$

In order to make sense of the right-hand side, we briefly revert to the regularization $\delta_\epsilon \in \mathring{H}^1(\Omega)$, *i.e.* a regularization that is square integrable, has square integrable weak derivative, and vanishes on $\partial\Omega$. For $\boldsymbol{v}$ a sufficiently smooth, vector-valued function, we derive

$$(\nabla \cdot (\delta_\epsilon I), \boldsymbol{v})_\Omega = (\nabla \delta_\epsilon, \boldsymbol{v})_\Omega = (\delta_\epsilon, \boldsymbol{n} \cdot \boldsymbol{v})_{\partial\Omega} - (\delta_\epsilon, \nabla \cdot \boldsymbol{v})_\Omega = (\delta_\epsilon, -\nabla \cdot \boldsymbol{v})_\Omega, \tag{2.3}$$

with $(\cdot,\cdot)_\Omega$ denoting the L^2-inner product on Ω. Letting $\epsilon \downarrow 0$, we define the following functional for sufficiently smooth functions $\boldsymbol{v}$:

$$\langle \nabla \cdot (\delta I), \boldsymbol{v} \rangle := \langle \delta, -\nabla \cdot \boldsymbol{v} \rangle, \tag{2.4}$$

with angled brackets indicating the duality pairing. In other words, $\nabla \cdot (\delta I)$ is the functional that performs a point evaluation of the negated divergence. The domain of $\nabla \cdot (\delta I)$ therefore contains all measurable vector-functions that have a well-defined, continuous divergence at the origin.

Remark 2.1. This approach is reminiscent of Schwarzschild black holes from general relativity theory in which the energy-momentum tensor is assumed to be a Dirac delta distribution. Moreover, the use of a concentrated stress is comparable to modeling magnetic dipoles in electromagnetism [16].

3. Analysis of the point force problem in weighted spaces

We now continue by analyzing the model problems, starting with the point force problem (2.1) with a single point force located at the origin. Due to the radial nature of the problem, we introduce r as the distance from the origin and $\hat{\boldsymbol{r}}$ the corresponding unit vector:

$$r(\boldsymbol{x}) = |\boldsymbol{x}|, \qquad \hat{\boldsymbol{r}}(\boldsymbol{x}) = \frac{\boldsymbol{x}}{|\boldsymbol{x}|}\cdot \tag{3.1}$$

We note that, in 1D, we have $\hat{r} = \mathrm{sgn}(x)$, which is a scalar function and therefore denoted without bold face.

In this section, we identify the singular behavior of the solution, introduce the weighted function spaces, and show that the weak formulation of (2.1) in this setting is well-posed.

3.1. Identifying the singularity

Given a vector $\boldsymbol{F} \in \mathbb{R}^d$, let $\boldsymbol{g_F}$ be the solution to the point force problem:

$$-\nabla \cdot (\mathbb{C}\varepsilon \boldsymbol{g_F}) = \boldsymbol{F}\delta.$$

The explicit formula for this function is known to be of the form $\boldsymbol{g_F} = G\boldsymbol{F}$, see *e.g.* [2, 17, 24, 27], with $G : \Omega \to \mathbb{R}^{d\times d}$ the tensor field given by

$$G := \begin{cases} -\frac{1}{2\mu+\lambda}\frac{r}{2}, & d = 1, \\ \frac{1}{8\pi\mu(1-\nu)}(-(3-4\nu)\log(r)I + \hat{\boldsymbol{r}} \otimes \hat{\boldsymbol{r}}), & d = 2, \\ \frac{1}{16\pi\mu(1-\nu)}\frac{1}{r}((3-4\nu)I + \hat{\boldsymbol{r}} \otimes \hat{\boldsymbol{r}}), & d = 3. \end{cases} \tag{3.2}$$

Here, ν is the Poisson ration given by $\frac{\lambda}{2(\lambda+\mu)}$. We note that a rigid body motion can be added to the fundamental solution but, in our case of a bounded domain, this mode is determined by the boundary conditions. One can demonstrate that the tensor field G is not in H^1. In particular, the solution behaves "badly" at the point of action of the force ($r = 0$). If point forces located on the cell boundary are moving, then passive convection requires to displace the cells according to the time derivative of the displacement vector that is obtained from the point force problem. In the case that a cell is very close to another cell that exerts a force, this may give inaccurate results due to the singular nature of the solution. However, in general, our simulation studies deal with large number of cells so that the eventual displacement vector is determined on the bases of forces that are exerted by many cells that are located further away. Hence this inaccuracy is averaged out. As an alternative to cope with this issue, one may apply the singularity removal method (cf. Sect. 5.2), which implies that the finite element method "searches" a solution in the right solution space. However, in order to reconstruct the displacement field, one eventually has to add, as a post-processing step, the fundamental solution to the obtained finite element solution, which contains the singularity.

3.2. Functional setting

From (3.2), we note that the solution to (2.1) contains a singularity that does not have a square integrable weak derivative, *i.e.* we have $\boldsymbol{u} \notin \boldsymbol{H}^1(\Omega)$ for $d \geq 2$. In order to analyze the problem in the appropriate setting, we therefore turn to weighted function spaces. Following [8], we introduce the following weighted norm for $\alpha \in (-1, 1)$:

$$\|\boldsymbol{u}\|^2_{\boldsymbol{L}^2_\alpha(\Omega)} := \int_\Omega r^{2\alpha}|\boldsymbol{u}|^2 \mathrm{d}\Omega, \tag{3.3}$$

with r the distance to the origin. Let $\boldsymbol{L}^2_\alpha(\Omega)$ be the space of measurable functions that have finite $\boldsymbol{L}^2_\alpha$-norm:

$$\boldsymbol{L}^2_\alpha(\Omega) := \Big\{\boldsymbol{u} : \ \|\boldsymbol{u}\|_{\boldsymbol{L}^2_\alpha(\Omega)} < \infty\Big\}. \tag{3.4}$$

The scalar and tensorial analogues of $\boldsymbol{L}^2_\alpha(\Omega)$ are denoted by $L^2_\alpha(\Omega)$ and $\mathbb{L}^2_\alpha(\Omega)$, respectively. More precisely, we define for tensor fields $\Sigma : \Omega \to \mathbb{R}^{d\times d}$:

$$\|\Sigma\|^2_{\mathbb{L}^2_\alpha} := \int_\Omega r^{2\alpha}\|\Sigma\|^2 \,\mathrm{d}\Omega, \qquad \mathbb{L}^2_\alpha(\Omega) := \Big\{\Sigma : \|\Sigma\|^2_{\mathbb{L}^2_\alpha} < \infty\Big\}, \tag{3.5}$$

with $\|\Sigma\|$ the Frobenius norm. Note that $\boldsymbol{L}^2_\alpha(\Omega)$ is naturally endowed with the inner product

$$(\boldsymbol{u}, \boldsymbol{v})_{\boldsymbol{L}^2_\alpha(\Omega)} = \int_\Omega r^{2\alpha} \boldsymbol{u} \cdot \boldsymbol{v} \, \mathrm{d}\Omega = (r^\alpha \boldsymbol{u}, r^\alpha \boldsymbol{v})_\Omega, \tag{3.6}$$

and we directly have $(\boldsymbol{u}, \boldsymbol{u})_{\boldsymbol{L}^2_\alpha(\Omega)} = \|\boldsymbol{u}\|^2_{\boldsymbol{L}^2_\alpha(\Omega)}$. In addition to $\boldsymbol{L}^2_\alpha(\Omega)$, we define

$$\boldsymbol{H}^1_\alpha(\Omega) := \big\{\boldsymbol{v} \in \boldsymbol{L}^2_\alpha(\Omega) : \nabla \boldsymbol{v} \in \mathbb{L}^2_\alpha(\Omega)\big\}, \tag{3.7a}$$

$$\boldsymbol{W}_\alpha(\Omega) := \big\{\boldsymbol{v} \in \boldsymbol{L}^2_\alpha(\Omega) : \varepsilon \boldsymbol{v} \in \mathbb{L}^2_\alpha(\Omega), \boldsymbol{v}|_{\partial\Omega} = 0\big\}. \tag{3.7b}$$

We endow $\boldsymbol{W}_\alpha(\Omega)$ with the norm

$$\|\boldsymbol{v}\|_{\boldsymbol{W}_\alpha(\Omega)} := \|\varepsilon \boldsymbol{v}\|_{\mathbb{L}^2_\alpha(\Omega)}. \tag{3.8}$$

In the following, we omit the reference to the domain Ω when no confusion arises.

As is typical in the analysis of linearized elasticity systems, we require a type of Korn's inequality, posed in the weighted spaces of our setting. Since this inequality is not available in the literature, and its proof is outside the scope of this work, we formulate it as a conjecture.

Conjecture 3.1 (Weighted Korn's inequality)**.** Let $\boldsymbol{e}$ be a rigid body motion on Ω. Then, for all $\boldsymbol{v} \in \boldsymbol{H}^1_\alpha(\Omega)$ with $(\boldsymbol{v}, \boldsymbol{e})_{\partial\Omega} = 0$, the following inequality holds

$$\|\nabla \boldsymbol{v}\|_{\mathbb{L}^2_\alpha} \lesssim \|\boldsymbol{v}\|_{\boldsymbol{W}_\alpha}. \tag{3.9}$$

Here, and in the following, the relation $x \lesssim y$ implies that a constant $C > 0$ exists such that $x \le Cy$. Computational evidence supporting Conjecture 3.1 is given in Appendix A, and it is used as a working hypothesis in some parts of the analysis.

We conclude this subsection with the Poincaré inequality [20]:

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^2_\alpha} \lesssim \|\nabla \boldsymbol{u}\|_{\mathbb{L}^2_\alpha}, \qquad \forall \boldsymbol{u} \in \mathring{\boldsymbol{H}}^1_\alpha. \tag{3.10}$$

Combined with Weighted Korn's inequality (3.9), we note that this implies that $\|\cdot\|_{\boldsymbol{W}_\alpha}$ forms a proper norm on $\boldsymbol{W}_\alpha$.

3.3. Well-posedness analysis

With the functional setting in place, we formulate and analyze two variational formulations of the point force problem (2.1). These two formulations, which we refer to as the primal and solid pressure formulations, lead to different discretization schemes in Section 5. Let us start with the *primal formulation*: Given $\boldsymbol{F} \in \mathbb{R}^d$, find $\boldsymbol{u} \in \boldsymbol{W}_\alpha$ such that

$$(\mathbb{C}\varepsilon \boldsymbol{u}, \varepsilon \boldsymbol{v})_\Omega = \langle \boldsymbol{F}\delta, \boldsymbol{v} \rangle, \qquad \forall \boldsymbol{v} \in \boldsymbol{W}_{-\alpha}. \tag{3.11}$$

This problem can be cast in the abstract form:

$$\mathcal{A}(\boldsymbol{u}, \boldsymbol{v}) = \langle \boldsymbol{f}, \boldsymbol{v} \rangle, \qquad \forall \boldsymbol{v} \in \boldsymbol{W}_{-\alpha} \tag{3.12}$$

with $\mathcal{A} : \boldsymbol{W}_\alpha \times \boldsymbol{W}_{-\alpha} \to \mathbb{R}$ and $\boldsymbol{f} \in \boldsymbol{W}'_{-\alpha}$ naturally defined as

$$\mathcal{A}(\boldsymbol{u}, \boldsymbol{v}) := (\mathbb{C}\varepsilon \boldsymbol{u}, \varepsilon \boldsymbol{v})_\Omega, \quad \langle \boldsymbol{f}, \boldsymbol{v} \rangle := \langle \boldsymbol{F}\delta, \boldsymbol{v} \rangle, \qquad \forall \boldsymbol{u} \in \boldsymbol{W}_\alpha, \boldsymbol{v} \in \boldsymbol{W}_{-\alpha}.$$

From formula (3.2), it is clear that for $d = 1$, standard H^1, that is, $\alpha = 0$, allows the point trace and that for $d = 2$, it is necessary to set $\alpha > 0$ in order to incorporate the point trace, whereas $\alpha > \frac{1}{2}$ seems necessary for

$d = 3$. Hence it is important to emphasize that $\boldsymbol{f} \in \boldsymbol{W}'_{-\alpha}$ for $d \leq 2$ and $\alpha > 0$ [8]. Although we believe that some of the analysis in [15] can be extended to 3D, we are aware that we may lose some of the embedding properties that warrant solutions that satisfy certain smoothness requirements. We therefore restrict the analysis to $d \leq 2$.

Before we present the main result, we make two useful observations. First, recalling that $\mathbb{C} := 2\mu + \lambda I \mathrm{Tr}$, we note the bounds

$$(\mathbb{C}\varepsilon \boldsymbol{u}, \varepsilon \boldsymbol{v})_\Omega \leq \left(2\mu + d^2\lambda\right) \|\varepsilon \boldsymbol{u}\|_{\mathbb{L}^2} \|\varepsilon \boldsymbol{v}\|_{\mathbb{L}^2}, \qquad \forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{L}^2, \tag{3.13a}$$

$$(\mathbb{C}\varepsilon \boldsymbol{u}, \varepsilon \boldsymbol{u})_\Omega \geq 2\mu \|\varepsilon \boldsymbol{u}\|_{\mathbb{L}^2}^2 \qquad \forall \boldsymbol{u} \in \boldsymbol{L}^2. \tag{3.13b}$$

Secondly, we derive a decomposition of the space $\mathbb{L}_s^2$ similar to Lemma 2.1 of [8], that forms a key tool in the subsequent analysis.

Lemma 3.2. *Let $s \in (-1, 1)$. For each $Q \in \mathbb{L}_s^2$, there exists a unique pair $(\Sigma, \boldsymbol{z}) \in \mathbb{L}_s^2 \times \boldsymbol{W}_s$ such that*

$$Q = \Sigma + \varepsilon \boldsymbol{z}, \qquad (\Sigma, \mathbb{C}\varepsilon \tilde{\boldsymbol{z}})_\Omega = 0, \qquad \forall \tilde{\boldsymbol{z}} \in \boldsymbol{W}_{-s}, \tag{3.14a}$$

$$\|\varepsilon \boldsymbol{z}\|_{\mathbb{L}_s^2} \leq 2\|Q\|_{\mathbb{L}_s^2}, \quad \|\Sigma\|_{\mathbb{L}_s^2} \leq \|Q\|_{\mathbb{L}_s^2}. \tag{3.14b}$$

Proof. Following Lemma 2.1 of [8], we set up and analyze the following variational problem: Find $(\Sigma, \boldsymbol{z}) \in \mathbb{L}_s^2 \times \boldsymbol{W}_s$ such that

$$a\left(\Sigma, \tilde{\Sigma}\right) + b_1\left(\boldsymbol{z}, \tilde{\Sigma}\right) = \left(Q, \tilde{\Sigma}\right)_\Omega, \qquad \forall \tilde{\Sigma} \in \mathbb{L}_{-s}^2, \tag{3.15a}$$

$$b_2(\tilde{\boldsymbol{z}}, \Sigma) = 0, \qquad \forall \tilde{\boldsymbol{z}} \in \boldsymbol{W}_{-s} \tag{3.15b}$$

with

$$a\left(\Sigma, \tilde{\Sigma}\right) := \left(\Sigma, \tilde{\Sigma}\right)_\Omega, \quad b_1\left(\boldsymbol{z}, \tilde{\Sigma}\right) := \left(\varepsilon \boldsymbol{z}, \tilde{\Sigma}\right)_\Omega, \quad b_2(\tilde{\boldsymbol{z}}, \Sigma) := \left(\mathbb{C}\varepsilon \boldsymbol{z}, \tilde{\Sigma}\right)_\Omega.$$

Obtaining (3.14) now amounts to proving that (3.15) is well-posed. To do this, we verify the hypotheses of the Banach–Nečas–Babuška (BNB) theorem for generalized saddle-point problems (cf. [3], Cor. 2.1, [13], Ex. 2.14, [5], Rem. 4.2.7).

- *Continuity.* Using the Cauchy–Schwarz inequality, we obtain:

$$a\left(\Sigma, \tilde{\Sigma}\right) = \left(r^s \Sigma, r^{-s} \tilde{\Sigma}\right)_\Omega \leq \|\Sigma\|_{\mathbb{L}_s^2} \left\|\tilde{\Sigma}\right\|_{\mathbb{L}_{-s}^2},$$

$$b_1\left(\boldsymbol{z}, \tilde{\Sigma}\right) = \left(r^s \varepsilon \boldsymbol{z}, r^{-s} \tilde{\Sigma}\right)_\Omega \leq \|\varepsilon \boldsymbol{z}\|_{\mathbb{L}_s^2} \left\|\tilde{\Sigma}\right\|_{\mathbb{L}_{-s}^2},$$

$$b_2(\tilde{\boldsymbol{z}}, \Sigma) = \left(r^s \mathbb{C}\varepsilon \tilde{\boldsymbol{z}}, r^{-s} \Sigma\right)_\Omega \leq \left(2\mu + d^2\lambda\right) \|\varepsilon \tilde{\boldsymbol{z}}\|_{\mathbb{L}_s^2} \|\Sigma\|_{\mathbb{L}_{-s}^2}.$$

  The final inequality follows by (3.13a).
- *Inf-sup of a.* Let $\Sigma \in \mathbb{L}_s^2$ be given and let the test function be chosen as $\tilde{\Sigma} := r^{2s}\Sigma \in \mathbb{L}_{-s}^2$. We then derive

$$a\left(\Sigma, \tilde{\Sigma}\right) = (r^s \Sigma, r^s \Sigma)_\Omega = \|\Sigma\|_{\mathbb{L}_s^2}^2 = \|\Sigma\|_{\mathbb{L}_s^2} \left\|r^{2s}\Sigma\right\|_{\mathbb{L}_{-s}^2} = \|\Sigma\|_{\mathbb{L}_s^2} \left\|\tilde{\Sigma}\right\|_{\mathbb{L}_{-s}^2}.$$

  Vice versa, for $\tilde{\Sigma} \in \mathbb{L}_{-s}^2$, we set $\Sigma := r^{-2s}\tilde{\Sigma} \in \mathbb{L}_s^2$ and obtain the analogous result with the roles interchanged. Together, these imply the conditions:

$$\inf_{\Sigma \in \mathbb{L}_s^2} \sup_{\tilde{\Sigma} \in \mathbb{L}_{-s}^2} \frac{a\left(\Sigma, \tilde{\Sigma}\right)}{\|\Sigma\|_{\mathbb{L}_s^2} \left\|\tilde{\Sigma}\right\|_{\mathbb{L}_{-s}^2}} \geq 1, \quad \inf_{\tilde{\Sigma} \in \mathbb{L}_{-s}^2} \sup_{\Sigma \in \mathbb{L}_s^2} \frac{a\left(\Sigma, \tilde{\Sigma}\right)}{\|\Sigma\|_{\mathbb{L}_s^2} \left\|\tilde{\Sigma}\right\|_{\mathbb{L}_{-s}^2}} \geq 1.$$

– *Inf-sup of* b_i. Let $\boldsymbol{z} \in \boldsymbol{W}_s$ be given and let $\tilde{\Sigma} := r^{2s}\varepsilon\boldsymbol{z} \in \mathbb{L}^2_{-s}$. We then derive

$$b_1\left(\boldsymbol{z}, \tilde{\Sigma}\right) = \|\varepsilon\boldsymbol{z}\|^2_{\mathbb{L}^2_s} = \|\varepsilon\boldsymbol{z}\|_{\mathbb{L}^2_s}\left\|r^{2s}\varepsilon\boldsymbol{z}\right\|_{\mathbb{L}^2_{-s}} = \|\varepsilon\boldsymbol{z}\|_{\mathbb{L}^2_s}\left\|\tilde{\Sigma}\right\|_{\mathbb{L}^2_{-s}}.$$

On the other hand, let $\tilde{\boldsymbol{z}} \in \boldsymbol{W}_{-s}$ be given and let $\Sigma := r^{-2s}\varepsilon\tilde{\boldsymbol{z}} \in \mathbb{L}^2_s$. Similar to the above, we obtain

$$b_2(\tilde{\boldsymbol{z}}, \Sigma) = \left(\mathbb{C}r^{-s}\varepsilon\tilde{\boldsymbol{z}}, r^{-s}\varepsilon\tilde{\boldsymbol{z}}\right)_\Omega \geq 2\mu\|\varepsilon\tilde{\boldsymbol{z}}\|^2_{\mathbb{L}^2_{-s}} = 2\mu\|\varepsilon\tilde{\boldsymbol{z}}\|_{\mathbb{L}^2_{-s}}\|\Sigma\|_{\mathbb{L}^2_s}.$$

Together, these give us the inf-sup conditions

$$\inf_{\boldsymbol{z}\in\boldsymbol{W}_s}\sup_{\tilde{\Sigma}\in\mathbb{L}^2_{-s}}\frac{b_1\left(\boldsymbol{z}, \tilde{\Sigma}\right)}{\|\varepsilon\boldsymbol{z}\|_{\mathbb{L}^2_s}\left\|\tilde{\Sigma}\right\|_{\mathbb{L}^2_{-s}}} \geq 1, \quad \inf_{\tilde{\boldsymbol{z}}\in\boldsymbol{W}_{-s}}\sup_{\Sigma\in\mathbb{L}^2_s}\frac{b_2(\tilde{\boldsymbol{z}}, \Sigma)}{\|\varepsilon\tilde{\boldsymbol{z}}\|_{\mathbb{L}^2_{-s}}\|\Sigma\|_{\mathbb{L}^2_s}} \geq 2\mu.$$

The BNB-theorem for generalized saddle-point problems ([3], Cor. 2.1) now gives us the well-posedness and the stability estimates. □

3.3.1. The primal formulation

Lemma 3.2 provides the key to proving well-posedness of problem (3.11), as shown in the following theorem.

Theorem 3.3. *Problem* (3.12) *is well-posed for right-hand sides* $\boldsymbol{f} \in \boldsymbol{W}'_{-\alpha}$*, and its unique solution satisfies*

$$\|\boldsymbol{u}\|_{\boldsymbol{W}_\alpha} \leq \mu^{-1}\|\boldsymbol{f}\|_{\boldsymbol{W}'_{-\alpha}}. \tag{3.16}$$

Proof. We verify the hypotheses of the BNB theorem for linear problems ([13], Thm. 2.6).

– *Continuity.* This follows directly from (3.13a):

$$\mathcal{A}(\boldsymbol{u}, \boldsymbol{v}) = \left(\mathbb{C}r^\alpha\varepsilon\boldsymbol{u}, r^{-\alpha}\varepsilon\boldsymbol{v}\right)_\Omega \leq \left(2\mu + d^2\lambda\right)\|\varepsilon\boldsymbol{u}\|_{\mathbb{L}^2_\alpha}\|\varepsilon\boldsymbol{v}\|_{\mathbb{L}^2_{-\alpha}}, \qquad \forall\boldsymbol{u} \in \boldsymbol{W}_\alpha, \boldsymbol{v} \in \boldsymbol{W}_{-\alpha}.$$

– *Inf-sup of* $\mathcal{A}$. Let $\boldsymbol{u} \in \boldsymbol{W}_\alpha$ be given and let $Q := r^{2\alpha}\varepsilon\boldsymbol{u} \in \mathbb{L}^2_{-\alpha}$. Using Lemma 3.2 with $s = -\alpha$, we construct Σ and $\boldsymbol{z}$ with properties (3.14). Setting $\boldsymbol{v} = \boldsymbol{z} \in \boldsymbol{W}_{-\alpha}$, we use these properties and (3.13b) to derive

$$\begin{aligned}\mathcal{A}(\boldsymbol{u}, \boldsymbol{v}) &= \left(\mathbb{C}\varepsilon\boldsymbol{u}, r^{2\alpha}\varepsilon\boldsymbol{u} - \Sigma\right)_\Omega = (\mathbb{C}r^\alpha\varepsilon\boldsymbol{u}, r^\alpha\varepsilon\boldsymbol{u})_\Omega \geq 2\mu\|\varepsilon\boldsymbol{u}\|^2_{\mathbb{L}^2_\alpha} = 2\mu\|\boldsymbol{u}\|^2_{\boldsymbol{W}_\alpha}\\ \|\boldsymbol{v}\|_{\boldsymbol{W}_{-\alpha}} &= \|\varepsilon\boldsymbol{v}\|_{\mathbb{L}^2_{-\alpha}} \leq 2\left\|r^{2\alpha}\varepsilon\boldsymbol{u}\right\|_{\mathbb{L}^2_{-\alpha}} = 2\|\varepsilon\boldsymbol{u}\|_{\mathbb{L}^2_\alpha}. = 2\|\boldsymbol{u}\|_{\boldsymbol{W}_\alpha}.\end{aligned}$$

Thus, since $\boldsymbol{u} \in \boldsymbol{W}_\alpha$ was arbitrary, we have shown that

$$\inf_{\boldsymbol{u}\in\boldsymbol{W}_\alpha}\sup_{\boldsymbol{v}\in\boldsymbol{W}_{-\alpha}}\frac{\mathcal{A}(\boldsymbol{u}, \boldsymbol{v})}{\|\boldsymbol{u}\|_{\boldsymbol{W}_\alpha}\|\boldsymbol{v}\|_{\boldsymbol{W}_{-\alpha}}} \geq \mu.$$

Reversing the roles of $\boldsymbol{u}$ and $\boldsymbol{v}$ in the previous step and changing α to $-\alpha$, we obtain the analogous result:

$$\inf_{\boldsymbol{v}\in\boldsymbol{W}_{-\alpha}}\sup_{\boldsymbol{u}\in\boldsymbol{W}_\alpha}\frac{\mathcal{A}(\boldsymbol{u}, \boldsymbol{v})}{\|\boldsymbol{u}\|_{\boldsymbol{W}_\alpha}\|\boldsymbol{v}\|_{\boldsymbol{W}_{-\alpha}}} \geq \mu.$$

These conditions suffice to invoke the BNB theorem, providing well-posedness and the stability bound. □

3.3.2. The solid pressure formulation

With the idea of constructing a viable discretization scheme in the future, we note that using lowest order elements in the primal formulation is prone to the "locking" phenomenon. To avoid this, we aim for a mixed finite element formulation based on the solid pressure, or Herrmann formulation [18]. With this purpose in mind, we introduce the solid pressure $p = \lambda\nabla\cdot\boldsymbol{u}$ and reformulate the problem: Find $(\boldsymbol{u}, p)$ such that

$$-\nabla\cdot(2\mu\varepsilon\boldsymbol{u} + pI) = \boldsymbol{F}\delta \tag{3.17a}$$

$$-\nabla\cdot\boldsymbol{u} + \lambda^{-1}p = 0. \tag{3.17b}$$

By posing the variational formulation of (3.17) in weighted spaces, we obtain the *solid pressure* formulation: Find $(\boldsymbol{u}, p) \in \boldsymbol{W}_\alpha \times L^2_\alpha$ such that

$$(2\mu\varepsilon\boldsymbol{u}, \varepsilon\boldsymbol{v})_\Omega + (p, \nabla\cdot\boldsymbol{v})_\Omega = \langle\boldsymbol{F}\delta, \boldsymbol{v}\rangle \qquad \forall\boldsymbol{v}\in\boldsymbol{W}_{-\alpha} \tag{3.18a}$$

$$-(\nabla\cdot\boldsymbol{u}, q)_\Omega + \left(\lambda^{-1}p, q\right)_\Omega = 0, \qquad \forall q\in L^2_{-\alpha}. \tag{3.18b}$$

To analyze (3.18), we recognize its structure as a generalized, perturbed saddle-point problem on the product space $X_\alpha := \boldsymbol{W}_\alpha \times L^2_\alpha$. In particular, let us define $\mathcal{A}_{He} : X_\alpha \times X_{-\alpha} \to \mathbb{R}$ and $f \in X'_{-\alpha}$ such that (3.18) takes the abstract form

$$\mathcal{A}_{He}((\boldsymbol{u}, p); (\boldsymbol{v}, q)) := a(\boldsymbol{u}, \boldsymbol{v}) - b_1(\boldsymbol{v}, p) + b_2(\boldsymbol{u}, q) + c(p, q) = \langle f; (\boldsymbol{v}, q)\rangle \tag{3.19}$$

in which the bilinear forms are given by

$$a : \boldsymbol{W}_\alpha \times \boldsymbol{W}_{-\alpha} \to \mathbb{R}, \qquad a(\boldsymbol{u}, \boldsymbol{v}) = (2\mu\varepsilon\boldsymbol{u}, \varepsilon\boldsymbol{v})_\Omega, \tag{3.20a}$$

$$b_1 : \boldsymbol{W}_{-\alpha} \times L^2_\alpha \to \mathbb{R}, \qquad b_1(\boldsymbol{v}, p) = -(\nabla\cdot\boldsymbol{v}, p)_\Omega, \tag{3.20b}$$

$$b_2 : \boldsymbol{W}_\alpha \times L^2_{-\alpha} \to \mathbb{R}, \qquad b_2(\boldsymbol{u}, q) = -(\nabla\cdot\boldsymbol{u}, q)_\Omega, \tag{3.20c}$$

$$c : L^2_\alpha \times L^2_{-\alpha} \to \mathbb{R}, \qquad c(p, q) = \left(\lambda^{-1}p, q\right)_\Omega. \tag{3.20d}$$

We endow X_α with the weighted norm

$$\|(\boldsymbol{u}, p)\|^2_{X_\alpha} = 2\mu\|\varepsilon\boldsymbol{u}\|^2_{\mathbb{L}^2_\alpha} + \lambda^{-1}\|p\|^2_{L^2_\alpha}. \tag{3.21}$$

Then the right-hand side f is naturally given by $\langle f, (\boldsymbol{v}, p)\rangle = \langle\boldsymbol{F}\delta, \boldsymbol{v}\rangle$ for all $(\boldsymbol{v}, p) \in X_{-\alpha}$. We emphasize that $f \in X'_{-\alpha}$ since the Dirac delta functional is well-defined on $\boldsymbol{W}_{-\alpha}$ for $\alpha > 0$.

It turns out that the key to analyzing (3.19) lies once again in the decomposition from Lemma 3.2. We present this formally in the following theorem.

Theorem 3.4. *Problem* (3.19) *is well-posed for* $f \in X'_{-\alpha}$, *i.e. it admits a unique solution* $(\boldsymbol{u}, p) \in X_\alpha$ *that satisfies*

$$\|(\boldsymbol{u}, p)\|_{X_\alpha} \le C_{He}\|f\|_{X'_{-\alpha}}$$

with $C_{He} := \sqrt{2 + d^2\frac{\lambda}{\mu}}$.

Proof. We follow the hypotheses of the BNB theorem [13].

– *Continuity.* Using the Cauchy–Schwarz inequality, we obtain:

$$a(\boldsymbol{u}, \boldsymbol{v}) \le 2\mu\|\varepsilon\boldsymbol{u}\|_{\mathbb{L}^2_\alpha}\|\varepsilon\boldsymbol{v}\|_{\mathbb{L}^2_{-\alpha}}, \qquad b_1(\boldsymbol{v}, p) \le \|\nabla\cdot\boldsymbol{v}\|_{L^2_{-\alpha}}\|p\|_{L^2_\alpha} \le d\|\varepsilon\boldsymbol{v}\|_{\mathbb{L}^2_{-\alpha}}\|p\|_{L^2_\alpha},$$

$$c(p, q) \le \lambda^{-1}\|p\|_{L^2_\alpha}\|q\|_{L^2_{-\alpha}}, \qquad b_2(\boldsymbol{u}, q) \le \|\nabla\cdot\boldsymbol{u}\|_{L^2_\alpha}\|q\|_{L^2_{-\alpha}} \le d\|\varepsilon\boldsymbol{u}\|_{\mathbb{L}^2_\alpha}\|q\|_{L^2_{-\alpha}}.$$

An application of the Cauchy–Schwarz inequality on the sum then gives us

$$\begin{aligned}\mathcal{A}_{He}((\boldsymbol{u},p);(\boldsymbol{v},q)) &\leq \Big(\big(2\mu+d^2\lambda\big)\|\varepsilon\boldsymbol{u}\|^2_{\mathbb{L}^2_\alpha} + 2\lambda^{-1}\|p\|^2_{L^2_\alpha}\Big)^{\frac{1}{2}}\Big(\big(2\mu+d^2\lambda\big)\|\varepsilon\boldsymbol{v}\|^2_{\mathbb{L}^2_{-\alpha}} + 2\lambda^{-1}\|q\|^2_{L^2_{-\alpha}}\Big)^{\frac{1}{2}}\\ &\leq \max\left\{\frac{C^2_{He}}{2}, 2\right\}\|(\boldsymbol{u},p)\|_{X_\alpha}\|(\boldsymbol{v},q)\|_{X_{-\alpha}}.\end{aligned}$$

– *Inf-sup of $\mathcal{A}_{He}$.* Let $(\boldsymbol{u},p)\in X_\alpha$ be given and let $Q := r^{2\alpha}\varepsilon\boldsymbol{u}\in\mathbb{L}^2_{-\alpha}$. Using Lemma 3.2 with $s=-\alpha$, we construct Σ and $\boldsymbol{z}$ with properties (3.14). Setting the test functions $\boldsymbol{v}=\boldsymbol{z}\in\boldsymbol{W}_{-\alpha}$ and $q = r^{2\alpha}p + \lambda\mathrm{Tr}(\Sigma)\in L^2_{-\alpha}$, we derive

$$\begin{aligned}a(\boldsymbol{u},\boldsymbol{v}) = \big(2\mu\varepsilon\boldsymbol{u}, r^{2\alpha}\varepsilon\boldsymbol{u}-\Sigma\big)_\Omega &= 2\mu\|\varepsilon\boldsymbol{u}\|^2_{\mathbb{L}^2_\alpha} - (\mathbb{C}\varepsilon\boldsymbol{u},\Sigma)_\Omega + (\lambda\nabla\cdot\boldsymbol{u},\mathrm{Tr}(\Sigma))_\Omega\\ &= 2\mu\|\varepsilon\boldsymbol{u}\|^2_{\mathbb{L}^2_\alpha} + (\lambda\nabla\cdot\boldsymbol{u},\mathrm{Tr}(\Sigma))_\Omega \end{aligned}\tag{3.22a}$$

where we have used $(\mathbb{C}\varepsilon\boldsymbol{u},\Sigma)_\Omega = 0$ due to (3.14). For the coupling operators, we note that

$$\nabla\cdot\boldsymbol{v} = \mathrm{Tr}\varepsilon\boldsymbol{v} = \mathrm{Tr}Q - \mathrm{Tr}\Sigma = r^{2\alpha}\nabla\cdot\boldsymbol{u} - \mathrm{Tr}\Sigma.$$

By substitution, we obtain

$$\begin{aligned}-b_1(\boldsymbol{v},p) + b_2(\boldsymbol{u},q) &= \big(r^{2\alpha}\nabla\cdot\boldsymbol{u} - \mathrm{Tr}(\Sigma), p\big)_\Omega - \big(\nabla\cdot\boldsymbol{u}, r^{2\alpha}p + \lambda\mathrm{Tr}(\Sigma)\big)_\Omega\\ &= -(\mathrm{Tr}(\Sigma),p)_\Omega - (\nabla\cdot\boldsymbol{u},\lambda\mathrm{Tr}(\Sigma))_\Omega.\end{aligned}\tag{3.22b}$$

The final operator becomes

$$c(p,q) = \big(\lambda^{-1}p, r^{2\alpha}p + \lambda\mathrm{Tr}(\Sigma)\big)_\Omega = \lambda^{-1}\|p\|^2_{L^2_\alpha} + (p,\mathrm{Tr}(\Sigma))_\Omega.\tag{3.22c}$$

Summing (3.22), the cross terms conveniently cancel and we obtain

$$\mathcal{A}_{He}((\boldsymbol{u},p);(\boldsymbol{v},q)) = 2\mu\|\varepsilon\boldsymbol{u}\|^2_{\mathbb{L}^2_\alpha} + \lambda^{-1}\|p\|^2_{L^2_\alpha} = \|(\boldsymbol{u},p)\|^2_{X_\alpha}.\tag{3.23a}$$

On the other hand, the test functions are bounded in the following way:

$$\begin{aligned}\|(\boldsymbol{v},q)\|^2_{X_{-\alpha}} = 2\mu\|\varepsilon\boldsymbol{v}\|^2_{\mathbb{L}^2_{-\alpha}} + \lambda^{-1}\|q\|^2_{\boldsymbol{L}^2_{-\alpha}} &\leq 4\mu\|Q\|^2_{\mathbb{L}^2_{-\alpha}} + 2\lambda^{-1}\big\|r^{2\alpha}p\big\|^2_{L^2_{-\alpha}} + 2\lambda\|\mathrm{Tr}(\Sigma)\|^2_{L^2_{-\alpha}}\\ &\leq 2\big(2\mu+d^2\lambda\big)\big\|r^{2\alpha}\varepsilon\boldsymbol{u}\big\|^2_{\mathbb{L}^2_{-\alpha}} + 2\lambda^{-1}\big\|r^{2\alpha}p\big\|^2_{L^2_{-\alpha}}\\ &\leq \frac{2\mu+d^2\lambda}{\mu}\Big(2\mu\|\varepsilon\boldsymbol{u}\|^2_{\mathbb{L}^2_\alpha} + \lambda^{-1}\|p\|^2_{L^2_\alpha}\Big)\\ &= C^2_{He}\|(\boldsymbol{u},p)\|^2_{X_\alpha},\end{aligned}\tag{3.23b}$$

where we used (3.14) and Young's inequality in the first step. The second step follows by the bound on Σ in (3.14) and the remaining steps follow from the definitions of C_{He} and $\|\cdot\|_{X_\alpha}$.
Combining (3.23), we have the first inf-sup condition

$$\inf_{(\boldsymbol{u},p)\in X_\alpha}\sup_{(\boldsymbol{v},q)\in X_{-\alpha}}\frac{\mathcal{A}_{He}((\boldsymbol{u},p);(\boldsymbol{v},q))}{\|(\boldsymbol{u},p)\|_{X_\alpha}\|(\boldsymbol{v},q)\|_{X_{-\alpha}}} \geq C^{-1}_{He}.$$

Again, reversing the roles of $(\boldsymbol{u},p)$ and $(\boldsymbol{v},q)$ and negating α gives us the second condition:

$$\inf_{(\boldsymbol{v},q)\in X_{-\alpha}}\sup_{(\boldsymbol{u},p)\in X_\alpha}\frac{\mathcal{A}_{He}((\boldsymbol{u},p);(\boldsymbol{v},q))}{\|(\boldsymbol{u},p)\|_{X_\alpha}\|(\boldsymbol{v},q)\|_{X_{-\alpha}}} \geq C^{-1}_{He}.$$

To conclude, we have shown that the assumptions of the BNB theorem are satisfied and thus the result follows. ∎

4. Analysis of the point stress problem in weighted spaces

For the point force problem, we obtained the fundamental solutions from potential theory in Section 3.1. In the case of the point stress problem (2.2), the source term is given by the unconventional $\nabla \cdot (\delta I)$ and we therefore require a more involved approach. As in the previous section, we first identify the singular behavior of the solution and then show well-posedness in appropriately weighted function spaces.

4.1. Identifying the singularity

We start by considering a simpler but closely related problem, namely the negative Laplace equation. Let g be its fundamental solution, *i.e.* the function that solves

$$-\Delta g = \delta. \tag{4.1}$$

Since g is explicitly known in the literature ([14], Sect. 2.2), we can calculate ∇g for $\boldsymbol{x} \neq 0$. We obtain

$$\nabla g = \begin{cases} -\frac{\hat{r}}{2}, & d = 1, \\ -\frac{\hat{\boldsymbol{r}}}{2\pi r}, & d = 2, \\ -\frac{\hat{\boldsymbol{r}}}{4\pi r^2}, & d = 3. \end{cases} \tag{4.2}$$

The function ∇g is closely related to the solution to (2.2) and we dedicate the following lemma to highlight this relationship.

Lemma 4.1. *Given g the fundamental solution to the negative Laplace equation (4.1). Then $\boldsymbol{g} := (2\mu+\lambda)^{-1}\nabla g$ satisfies*

$$-\nabla \cdot \mathbb{C}\varepsilon \boldsymbol{g} = \nabla \cdot (\delta I). \tag{4.3}$$

Proof. Recalling that $\mathbb{C} = 2\mu + \lambda I \mathrm{Tr}$, we start with the first term. Substituting ∇g and using the Einstein summation convention, we derive

$$\begin{aligned} -\nabla \cdot (2\mu\varepsilon\nabla g) &= -2\mu\partial_j\left(\frac{1}{2}(\partial_j\partial_i g + \partial_i\partial_j g)\right) = -2\mu\partial_j(\partial_j\partial_i g) \\ &= -2\mu\partial_i(\partial_j\partial_j g) = 2\mu\nabla(-\Delta g) = 2\mu\nabla \cdot (\delta I). \end{aligned}$$

The same substitution in the second term leads us to

$$-\nabla \cdot (\lambda(\nabla \cdot \nabla g)I) = \nabla \cdot (\lambda(-\Delta g)I) = \lambda\nabla \cdot (\delta I).$$

The result follows by linearity. □

Lemma 4.2. *For $\boldsymbol{g}$ defined in Lemma 4.1, we have $\boldsymbol{g} \in \boldsymbol{L}^2_\alpha(\Omega)$ for $\alpha > \frac{d-2}{2}$.*

Proof. By the previous lemma, we have that $\boldsymbol{g} = (2\mu+\lambda)^{-1}\nabla g$ with ∇g given by (4.2). Letting c_d be the surface area of the unit $(d-1)$-sphere, a direct computation gives us

$$\|\nabla g\|_{\boldsymbol{L}^2_\alpha(\Omega)} = \int_\Omega r^{2\alpha}\left|r^{1-d}c_d^{-1}\hat{\boldsymbol{r}}\right|^2 \mathrm{d}\Omega = c_d^{-2}\int_\Omega r^{2\alpha-(d-2)-d}\,\mathrm{d}\Omega = c_d^{-2}\int_\Omega r^{\epsilon_\alpha - d}\,\mathrm{d}\Omega < \infty$$

for $\epsilon_\alpha = 2\alpha - (d-2) > 0$. In the final inequality, we use that a transition to radial coordinates (spherical in 3D) introduces a scaling with r^{d-1}. In turn, the integrability of $r^{\epsilon_\alpha - 1}$ on any bounded subset of $\mathbb{R}$ gives the result. □

FIGURE 2. Numbering and directions of point forces in 1D and 2D. By considering the point forces with $k \neq 0$ and the limit $|\boldsymbol{x}_0 - \boldsymbol{x}_k| \downarrow 0$, we obtain the point stress problem.

4.2. Well-posedness analysis

The point stress problem (2.2) is formed by changing the right hand side of (3.11) to $-\langle \nabla \cdot (\delta I), \boldsymbol{v} \rangle$. However, now Theorem 3.3 is not applicable because $\nabla \cdot (\delta I) \notin \boldsymbol{W}'_{-\alpha}$ for $\alpha \geq 0$. Moreover, the solid pressure formulation from Section 5.1.2 does not provide the desired result either. In particular, if we change variables to $\tilde{p} = p + \delta$, we are led to the system: Find $(\boldsymbol{u}, \tilde{p}) \in X_\alpha$ such that

$$(2\mu\varepsilon\boldsymbol{u}, \varepsilon\boldsymbol{v})_\Omega + (\tilde{p}, \nabla \cdot \boldsymbol{v})_\Omega = 0 \qquad \forall \boldsymbol{v} \in \boldsymbol{W}_{-\alpha} \tag{4.4a}$$

$$-(\nabla \cdot \boldsymbol{u}, q)_\Omega + \left(\lambda^{-1}\tilde{p}, q\right)_\Omega = \left\langle \lambda^{-1}\delta, q \right\rangle, \qquad \forall q \in L^2_{-\alpha}. \tag{4.4b}$$

But again we note that $\lambda^{-1}\delta \notin (L^2_{-\alpha})'$ for $\alpha \leq 1$ [15]. We therefore require a different approach and we proceed by considering the point stress problem as the limit of a sequence of point force problems.

During this construction, we manipulate the solutions of separate point force problem using translations and rotations. For simplicity, we assume that the domain $\Omega \subset \mathbb{R}^d$ is the unit d-ball centered at the origin, with $d \leq 2$. Let us define the following key building blocks:

$$R := -1, \qquad \hat{\boldsymbol{e}}_0 := 1, \qquad \text{if } d = 1,$$

$$R := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \hat{\boldsymbol{e}}_0 := \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \text{if } d = 2.$$

The rotation ρ (reflection in 1D) and the unit vector $\hat{\boldsymbol{e}}_k$ are then given by

$$(\rho\boldsymbol{u})(\boldsymbol{x}) = R\boldsymbol{u}\left(R^{-1}\boldsymbol{x}\right), \quad \hat{\boldsymbol{e}}_k = R^k\hat{\boldsymbol{e}}_0, \quad 1 \leq k \leq 2^d.$$

Next, we consider translations. Let $\boldsymbol{x}_k := -\epsilon\hat{\boldsymbol{e}}_k$ for given $\epsilon > 0$ (cf. Fig. 2) and $1 \leq k \leq 2^d$. If we directly define a translation with respect to these points, it may occur that a function needs to be evaluated outside of its domain Ω. We avoid this by introducing a natural extension to the exterior of Ω and utilizing the translation of the extension:

$$(\mathcal{R}\boldsymbol{v})(x) = \begin{cases} \boldsymbol{v}(x), & x \in \Omega, \\ -\boldsymbol{v}(\Pi x), & x \notin \Omega, \end{cases} \qquad \tau_k\boldsymbol{v}(x) := (\mathcal{R}\boldsymbol{v})(\boldsymbol{x} - \boldsymbol{x}_k) \tag{4.5}$$

in which Π returns the reflection in the boundary $\partial\Omega$. This reflection can easily be obtained by the assumed shape of Ω.

For $1 \leq k \leq 2^d$, let point sources be defined as $\delta_k := \delta_0(\boldsymbol{x} - \boldsymbol{x}_k)$ with δ_0 the Dirac delta distribution centered at the origin. We are interested in the following central finite difference limit for sufficiently smooth $\boldsymbol{v}$, illustrated in Figure 2:

$$\langle \delta, -\nabla \cdot \boldsymbol{v} \rangle = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \sum_{k=1}^{2^d} \langle \delta_k \hat{\boldsymbol{e}}_k, \boldsymbol{v} \rangle. \tag{4.6}$$

Our aim is to use this observation to obtain existence and stability of the solution to the point stress problem (2.2). The first step is to consider each point source problem separately. Thus, for $0 \leq k \leq 2^d$, let $\boldsymbol{u}_k \in \boldsymbol{W}_{k,\alpha}$ solve the weak formulation:

$$\mathcal{A}(\boldsymbol{u}_k, \boldsymbol{v}_k)_\Omega = \langle \delta_k \hat{\boldsymbol{e}}_k, \boldsymbol{v} \rangle, \qquad \forall \boldsymbol{v}_k \in \boldsymbol{W}_{k,-\alpha}. \tag{4.7}$$

Here, the weight in $\boldsymbol{W}_{k,\alpha}$ is given by $r_k^\alpha(\boldsymbol{x}) := |\boldsymbol{x} - \boldsymbol{x}_k|^\alpha$. We note that problem (4.7) is well-posed by Theorem 3.3. Moreover, we have a characterization of the solution, shown in the following lemma.

Lemma 4.3. *Let $\boldsymbol{w}_k \in \boldsymbol{H}^1(\Omega)$ satisfy*

$$\begin{aligned} \mathcal{A}(\boldsymbol{w}_k, \boldsymbol{v}) &= 0, && \forall \boldsymbol{v} \in \mathring{\boldsymbol{H}}^1(\Omega) \\ \boldsymbol{w}_k &= -\tau_k \rho^k \boldsymbol{u}_0, && \text{on } \partial\Omega. \end{aligned}$$

Then the solution $\boldsymbol{u}_k$ to (4.7) can be characterized as

$$\boldsymbol{u}_k = \tau_k \rho^k \boldsymbol{u}_0 + \boldsymbol{w}_k.$$

Proof. We first note that $\delta_k \hat{\boldsymbol{e}}_k = \tau_k \rho^k (\delta_0 \hat{\boldsymbol{e}}_0)$. By translational and rotational invariance, we see that $\tau_k \rho^k \boldsymbol{u}_0$ satisfies (4.7) except for the boundary conditions. Thus, we let $\boldsymbol{w}_k$ form a correction from the kernel of $\mathcal{A}$. In turn, $\tau_k \rho^k \boldsymbol{u}_0 + \boldsymbol{w}_k$ solves (4.7) and the characterization follows by the uniqueness of the solution shown in Theorem 3.3. □

Next, we aim to formulate a combined problem with multiple sources. For that, we cannot use a simple superposition argument for the weighted function spaces since $\boldsymbol{W}_{k_1,\alpha} \neq \boldsymbol{W}_{k_2,\alpha}$ for $k_1 \neq k_2$. Instead, we need test functions that are sufficiently smooth at all source locations. Let us therefore define the test space $\boldsymbol{V}_{-\alpha}$ and its associated norm as

$$\boldsymbol{V}_{-\alpha} := \bigcap_{k=1}^{2^d} \boldsymbol{W}_{k,-\alpha}, \quad \|\boldsymbol{v}\|_{\boldsymbol{V}_{-\alpha}} := \left\| r_{\min}^{-\alpha} \partial_x \boldsymbol{v} \right\|_\Omega,$$

with $r_{\min}(\boldsymbol{x}) = \min_{1 \leq k \leq 2^d} r_k(\boldsymbol{x})$ for $\boldsymbol{x} \in \Omega$. The trial space is then defined as

$$\boldsymbol{V}_\alpha := \left\{ \boldsymbol{v} : \|\boldsymbol{v}\|_{\boldsymbol{V}_\alpha} < \infty \right\}, \quad \|\boldsymbol{v}\|_{\boldsymbol{V}_\alpha} := \| r_{\min}^\alpha \partial_x \boldsymbol{v} \|_\Omega.$$

The fact that $\| \cdot \|_{\boldsymbol{V}_{-\alpha}}$ is a proper norm on $\boldsymbol{V}_{-\alpha}$ is shown below.

Lemma 4.4. *The following norm equivalence holds*

$$\|\boldsymbol{v}\|_{\boldsymbol{V}_{-\alpha}}^2 \eqsim \sum_{k=1}^{2^d} \|\boldsymbol{v}\|_{\boldsymbol{W}_{k,-\alpha}}^2.$$

Proof. By denoting the subdomain $\Omega_k := \{x \in \Omega : \ r_k(x) = r_{\min}(x)\}$, we derive

$$\left\| r_{\min}^{-\alpha} \partial_x \boldsymbol{v} \right\|_\Omega^2 = \sum_{k=1}^{2^d} \left\| r_k^{-\alpha} \partial_x \boldsymbol{v} \right\|_{\Omega_k}^2 \leq \sum_{k=1}^{2^d} \left\| r_k^{-\alpha} \partial_x \boldsymbol{v} \right\|_\Omega^2 \leq 2^d \left\| r_{\min}^{-\alpha} \partial_x \boldsymbol{v} \right\|_\Omega^2.$$

Here, the first step uses the definition of Ω_k, the second step is due to the fact that $\Omega_k \subseteq \Omega$, and the final inequality follows from the definition of $r_{\min}$. □

With the spaces $\boldsymbol{V}_{\pm\alpha}$ defined and a given ϵ, we are ready to formulate the limiting problem with multiple point forces: Find $\boldsymbol{u}_\epsilon \in \boldsymbol{V}_\alpha$ such that

$$\mathcal{A}(\boldsymbol{u}_\epsilon, \boldsymbol{v}) = \frac{1}{2\epsilon} \sum_{k=1}^{2^d} \langle \delta_k \hat{\boldsymbol{e}}_k, \boldsymbol{v} \rangle, \qquad \forall \boldsymbol{v} \in \boldsymbol{V}_{-\alpha}. \tag{4.8}$$

Lemma 4.5. *The combined problem* (4.8) *admits a unique solution* $\boldsymbol{u}_\epsilon \in \boldsymbol{V}_\alpha$*. Furthermore, we have the characterization*

$$\boldsymbol{u}_\epsilon = \frac{1}{2\epsilon} \sum_{k=1}^{2^d} \boldsymbol{u}_k \tag{4.9}$$

with $\boldsymbol{u}_k \in \boldsymbol{W}_{k,\alpha}$ *the solution to* (4.7).

Proof. First, we use the arguments from Theorem 3.3, with $r_{\min}$ substituted for r, to derive that (4.8) admits a unique solution in the space $\boldsymbol{V}_\alpha$.

Next, we note that $\boldsymbol{V}_{-\alpha} \subseteq \boldsymbol{W}_{k,-\alpha}$ for each k. In turn, the linearity of the problem implies that

$$\mathcal{A}\left(\frac{1}{2\epsilon} \sum_{k=1}^{2^d} \boldsymbol{u}_k, \boldsymbol{v} \right) = \frac{1}{2\epsilon} \sum_{k=1}^{2^d} \mathcal{A}(\boldsymbol{u}_k, \boldsymbol{v}) = \frac{1}{2\epsilon} \sum_{k=1}^{2^d} \langle \delta_k \hat{\boldsymbol{e}}_k, \boldsymbol{v} \rangle, \qquad \forall \boldsymbol{v} \in \boldsymbol{V}_{-\alpha}.$$

By observing that $\boldsymbol{W}_{k,\alpha} \subseteq \boldsymbol{V}_\alpha$ for each k, we obtain the explicit representation (4.9) of the (unique) solution $\boldsymbol{u}_\epsilon$. ☐

With problem (4.8) well-defined, we continue by considering the limit $\epsilon \downarrow 0$.

Lemma 4.6. *The limit solution can be decomposed as*

$$\lim_{\epsilon \downarrow 0} \boldsymbol{u}_\epsilon = \tilde{\boldsymbol{u}} + \tilde{\boldsymbol{w}}, \tag{4.10}$$

in which $\tilde{\boldsymbol{u}} \in \boldsymbol{L}^2_\alpha$ *is given by*

$$\tilde{\boldsymbol{u}} := \begin{cases} \partial_x \boldsymbol{u}_0, & d = 1, \\ (\partial_x I + \partial_y \rho) \boldsymbol{u}_0, & d = 2, \end{cases}$$

and $\tilde{\boldsymbol{w}} \in \boldsymbol{H}^1(\Omega)$ *satisfies*

$$\mathcal{A}(\tilde{\boldsymbol{w}}, \boldsymbol{v}) = 0 \qquad \forall \boldsymbol{v} \in \mathring{\boldsymbol{H}}^1(\Omega), \quad \tilde{\boldsymbol{w}} = -\tilde{\boldsymbol{u}} \qquad \text{on } \partial\Omega.$$

Proof. We first use Lemmas 4.5 and 4.3 to decompose the limit as

$$\lim_{\epsilon \downarrow 0} \boldsymbol{u}_\epsilon = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \sum_{k=1}^{2^d} \boldsymbol{u}_k = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \sum_{k=1}^{2^d} (\tau_k \rho^k \boldsymbol{u}_0 + \boldsymbol{w}_k).$$

Let us consider the two terms on the right-hand side separately. For the first term, we note that the solution possesses the symmetry $\boldsymbol{u}_0(-\boldsymbol{x}) = \boldsymbol{u}_0(\boldsymbol{x})$ due to the assumed geometry Ω. This can be summarized for both values of d by $\rho^d \boldsymbol{u}_0 = -\boldsymbol{u}_0$.

For $d = 1$, this gives us

$$\lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \sum_{k=1}^{2} \tau_k \rho^k \boldsymbol{u}_0 = \lim_{\epsilon \downarrow 0} \frac{\tau_2 \boldsymbol{u}_0 - \tau_1 \boldsymbol{u}_0}{2\epsilon} = \partial_x \boldsymbol{u}_0 = \tilde{\boldsymbol{u}}.$$

For $d = 2$, on the other hand, we have

$$\lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \sum_{k=1}^{4} \tau_k \rho^k \boldsymbol{u}_0 = \lim_{\epsilon \downarrow 0} \left(\frac{\tau_4 \boldsymbol{u}_0 - \tau_2 \boldsymbol{u}_0}{2\epsilon} + \frac{\tau_1 \rho \boldsymbol{u}_0 - \tau_3 \rho \boldsymbol{u}_0}{2\epsilon} \right) = (\partial_x I + \partial_y \rho) \boldsymbol{u}_0 = \tilde{\boldsymbol{u}}.$$

For the second term in the limit, we recall that each $\boldsymbol{w}_k$ satisfies $\mathcal{A}(\boldsymbol{w}_k, \boldsymbol{v}) = 0$ for all $\boldsymbol{v} \in \mathring{\boldsymbol{H}}^1$. In turn, since the kernel of the corresponding operator is closed, the limit $\tilde{\boldsymbol{w}}$ will also be in the kernel of $\mathcal{A}$. Such functions are completely determined by the boundary values, and we therefore derive

$$\tilde{\boldsymbol{w}} = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \sum_{k=1}^{2^d} \boldsymbol{w}_k = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \sum_{k=1}^{2^d} \left(-\tau_k \rho^k \boldsymbol{u}_0 \right) = -\tilde{\boldsymbol{u}}, \qquad \text{on } \partial\Omega.$$

□

Lemma 4.7. *For $d \leq 2$, the function $\tilde{\boldsymbol{w}}$ from Lemma 4.6 satisfies*

$$\|\tilde{\boldsymbol{w}}\|_{\boldsymbol{L}^2_\alpha} \lesssim \|\tilde{\boldsymbol{u}}\|_{\boldsymbol{L}^2_\alpha}. \tag{4.11}$$

Proof. For $d = 1$, $\tilde{\boldsymbol{u}} = \partial_x \boldsymbol{u}_0$ is piecewise constant by (3.2) and attains the values $\pm c$ for some $c > 0$. Since $\tilde{\boldsymbol{w}}$ is a harmonic extension, we have $|\tilde{\boldsymbol{w}}| \leq c$ almost everywhere and thus

$$\|\tilde{\boldsymbol{w}}\|_{\boldsymbol{L}^2_\alpha} \leq \|c\|_{\boldsymbol{L}^2_\alpha} = \|\tilde{\boldsymbol{u}}\|_{\boldsymbol{L}^2_\alpha}.$$

We continue with $d = 2$ and recall that (3.2) gives us

$$\boldsymbol{u}_0 = \frac{1}{8\pi\mu(1-\nu)} \left(-(3 - 4\nu) \log(r) I + \hat{\boldsymbol{r}} \otimes \hat{\boldsymbol{r}} \right) \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{4.12}$$

A straightforward computation leads us to

$$\tilde{\boldsymbol{u}} = (\partial_x I + \partial_y \rho) \boldsymbol{u}_0 = \frac{2\nu - 1}{4\pi\mu(1-\nu)} \frac{\hat{\boldsymbol{r}}}{r} = -\frac{1}{2\mu + \lambda} \frac{\hat{\boldsymbol{r}}}{2\pi r}, \tag{4.13}$$

in accordance with Lemma 4.1. Let $c := (2\pi(2\mu + \lambda))^{-1}$, then we have

$$\|\tilde{\boldsymbol{w}}\|^2_{\boldsymbol{L}^2_\alpha} \lesssim \|\tilde{\boldsymbol{w}}\|^2_{\boldsymbol{W}_\alpha} \leq \frac{1}{2\mu} \mathcal{A}(\tilde{\boldsymbol{w}}, \tilde{\boldsymbol{w}}) \leq \frac{1}{2\mu} \mathcal{A}(c\boldsymbol{x}, c\boldsymbol{x}) \leq \frac{\mu + 2\lambda}{\mu} \|\varepsilon(c\boldsymbol{x})\|^2_{\mathbb{L}^2}. \tag{4.14a}$$

Here, we used Korn's inequality (3.9), the coercivity (3.13b), and the fact that $\tilde{\boldsymbol{w}}$ solves a minimization problem. The final inequality is the continuity (3.13a). We continue as follows for $\alpha \leq 1$:

$$\|\varepsilon(c\boldsymbol{x})\|_{\mathbb{L}^2} = \|cI\|_{\mathbb{L}^2} = 2\|c\hat{\boldsymbol{r}}\|_{\boldsymbol{L}^2} \leq 2\left\|cr^{\alpha-1}\hat{\boldsymbol{r}}\right\|_{\boldsymbol{L}^2} = 2\|\tilde{\boldsymbol{u}}\|_{\boldsymbol{L}^2_\alpha}. \tag{4.14b}$$

Together, equations (4.14) provide the result. □

Theorem 4.8. *For $d \leq 2$, the point stress problem admits at least one solution $\boldsymbol{u} \in \boldsymbol{W}_\alpha$ that satisfies*

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^2_\alpha} \lesssim \|\hat{e}_0 \delta\|_{\boldsymbol{W}'_{-\alpha}}. \tag{4.15}$$

Proof. Combining the results from Lemmas 4.6 and 4.7, we derive

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^2_\alpha} = \lim_{\epsilon \downarrow 0} \|r^\alpha_{\min} \boldsymbol{u}_\epsilon\|_\Omega = \|\tilde{\boldsymbol{u}} + \tilde{\boldsymbol{w}}\|_{\boldsymbol{L}^2_\alpha} \leq \|\tilde{\boldsymbol{u}}\|_{\boldsymbol{L}^2_\alpha} + \|\tilde{\boldsymbol{w}}\|_{\boldsymbol{L}^2_\alpha} \lesssim \|\tilde{\boldsymbol{u}}\|_{\boldsymbol{L}^2_\alpha} \lesssim \|\boldsymbol{u}_0\|_{\boldsymbol{W}_\alpha} \lesssim \|\hat{e}_0 \delta\|_{\boldsymbol{W}'_{-\alpha}}.$$

Here, the final step is due to Theorem 3.3. □

5. Finite element approximation of the point stress problem

With the well-posedness of the systems shown, we continue with the discretization methods. For this, we choose conforming finite elements of lowest order in order to have a straightforward exposition. Moreover, we limit this presentation to the point stress problem as it is the less conventional of the two modeling approaches. Two discretization strategies are presented. The first is a direct discretization of (2.2) while the second exploits the results from Section 4 to effectively remove the known singularity.

5.1. Direct discretization methods

We first consider a direct application of the finite element method. In this context, the primal formulation and the solid pressure formulation lead to different schemes and we therefore present these separately.

5.1.1. *The primal formulation*

Let us discretize the point stress problem (2.2) with the use of linear, continuous Lagrange elements, *i.e.* $\boldsymbol{V}_h := \mathbb{P}_1^d$ and $\mathring{\boldsymbol{V}}_h = \boldsymbol{V}_h \cap \mathring{\boldsymbol{H}}^1(\Omega)$. This leads to the discrete problem: Find $\boldsymbol{u}_h \in \mathring{\boldsymbol{V}}_h$ such that

$$(2\mu\varepsilon\boldsymbol{u}_h, \varepsilon\boldsymbol{v}_h)_\Omega + (\lambda\nabla\cdot\boldsymbol{u}_h, \nabla\cdot\boldsymbol{v}_h)_\Omega = -\langle\delta, \nabla\cdot\boldsymbol{v}_h\rangle, \qquad \forall\boldsymbol{v}_h \in \mathring{\boldsymbol{V}}_h. \tag{5.1}$$

The integral on the right-hand side is an evaluation of $\nabla\cdot\boldsymbol{v}_h$ at the fibroblast's center. This is a very simple computation since $\nabla\cdot\boldsymbol{v}_h$ is a piecewise constant function in the lowest order case. Numerically, this amounts to merely finding the element in which the cell center resides.

5.1.2. *The solid pressure formulation*

For nearly incompressible materials, the Lagrange elements of lowest order suffer from the "locking" phenomenon. To avoid this, we discretize (4.4) and obtain the discrete problem: Find $(\boldsymbol{u}_h, \tilde{p}_h) \in \mathring{\boldsymbol{V}}_h \times Q_h$ such that

$$\begin{aligned}
(2\mu\varepsilon\boldsymbol{u}_h, \varepsilon\boldsymbol{v}_h)_\Omega + (\tilde{p}_h, \nabla\cdot\boldsymbol{v}_h)_\Omega &= 0, && \forall\boldsymbol{v}_h \in \mathring{\boldsymbol{V}}_h, && (5.2\text{a})\\
-(\nabla\cdot\boldsymbol{u}_h, q_h)_\Omega + \left(\lambda^{-1}\tilde{p}_h, q_h\right)_\Omega &= \left\langle\lambda^{-1}\delta, q_h\right\rangle, && \forall q_h \in Q_h. && (5.2\text{b})
\end{aligned}$$

Any Stokes-stable pair of mixed finite elements can be used for $\mathring{\boldsymbol{V}}_h \times Q_h$ and we focus on the pair $\mathbb{P}_2^2 \times \mathbb{P}_0$ in 2D and Taylor-Hood $\mathbb{P}_2^3 \times \mathbb{P}_1$ in 3D. We emphasize that, due to the change of variables, the right-hand side in the second equation corresponds to a point evaluation of the test function. Moreover, in the incompressible limit $\lambda \to \infty$, the influence of the fibroblast naturally decays as the medium does not allow for volumetric changes.

5.2. A singularity removal method

Following the ideas of [10, 15], we formulate a singularity removal method to solve (2.2). The strategy is to decompose the solution $\boldsymbol{u} = \boldsymbol{g} + \boldsymbol{w}$ with $\boldsymbol{g}$ the derived solution from Lemma 4.1 and $\boldsymbol{w} \in \boldsymbol{H}^1(\Omega)$ a smooth correction.

We approximate $\boldsymbol{w}$ by introducing a finite element space $\boldsymbol{V}_h \subset \boldsymbol{H}^1(\Omega)$ and solve the following problem: Find $\boldsymbol{w}_h \in \boldsymbol{V}_h$ such that

$$\begin{aligned}
(2\mu\varepsilon\boldsymbol{w}_h, \varepsilon\boldsymbol{v}_h)_\Omega + (\lambda\nabla\cdot\boldsymbol{w}_h, \nabla\cdot\boldsymbol{v}_h)_\Omega &= 0, && \forall\boldsymbol{v}_h \in \mathring{\boldsymbol{V}}_h, && (5.3\text{a})\\
\boldsymbol{w}_h &= -\Pi_h\boldsymbol{g}, && \text{on } \partial\Omega. && (5.3\text{b})
\end{aligned}$$

Here, $\mathring{\boldsymbol{V}}_h = \boldsymbol{V}_h \cap \mathring{\boldsymbol{H}}^1$ and Π_h is a suitable projection onto the boundary degrees of freedom of $\boldsymbol{V}_h$.

Problem (5.3) is a conventional boundary value problem. In turn, due to standard arguments for finite element methods, $\boldsymbol{w}_h$ is expected to be a good approximation of $\boldsymbol{w}$, particularly if the boundary data is smooth. This is the case in our setting if the singularity is located sufficiently far away from the boundary. The error then depends on $\|(I-\Pi_h)\boldsymbol{g}\|_{\partial\Omega}$ and convergence depends solely on the approximation properties of the finite element space and the shape-regularity of the mesh. We remark that the correction $\boldsymbol{w}_h$ can alternatively be obtained using the solid pressure formulation discussed in Section 5.1.2.

A disadvantage of the singularity removal method is that it is only applicable for linear problems, and hence it is not easily extendable from linearized elasticity to more involved models. The incorporation of plasticity, morphoelasticity, finite strain measures, or material anisotropy, for example, complicates the derivation of the analytical solution, often making it unavailable in practice. The direct discretization schemes of Section 5.1 are therefore more amenable to such extensions. The problems regarding the singularity are expected to persist in viscoelasticity due to lack of an analytic solution. However, due to the presence of the viscosity term in the time derivative, viscoelasticity discretizations possibly do not suffer from locking problems in case of Lagrangian finite element methods. These issues are reserved for future analyses.

6. Numerical results

Let $\Omega = [-1,1]^d$ with $d = 1,2,3$ and let Ω_h be a regular, structured, and simplicial grid with mesh size h. We include a stress source at $\boldsymbol{x}_0 = \{-1/6\}^d$ and choose h so that the point source is always located in the interior of a single element of the mesh. For simplicity, we set the Lamé parameters to unity, *i.e.* $\mu = \lambda = 1$.

Two test cases are presented that differ only in the imposed boundary conditions. The first is designed to verify the analysis of Section 4, in particular the validity of the derived solution $\boldsymbol{g}$. On the other hand, a second test case is presented that demonstrates the advantages of the singularity removal method. The numerical experiments were implemented in FEniCS [22] with the finite element spaces chosen as in Section 5.1.

Our main interest lies in the convergence of the displacement variable for the different formulations and the different test cases in all dimensions. We therefore compute the relative error in a variety of norms for several refinements of the grid and compute the convergence rates accordingly.

6.1. Approximation of the singular solution

In this first test case, we set the boundary conditions to

$$\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{g}(\boldsymbol{x} - \boldsymbol{x}_0), \quad \boldsymbol{x} \in \partial\Omega. \tag{6.1}$$

The analysis of Section 3 shows that $\boldsymbol{g}(\boldsymbol{x} - \boldsymbol{x}_0)$ is the solution to this problem. In turn, the singularity removal method of Section 5.2 would lead us to the trivial correction $\boldsymbol{w}_h = 0$. It is therefore more interesting to directly apply the primal and total pressure finite element formulations of Section 5.1 to this problem.

Table 1 shows the relative errors and convergence rates for the one-dimensional test case. Here, we observe that the primal formulation converges to the solution given in (4.2). The discontinuity in the solution implies that $u \notin H^1(\Omega)$ and this impacts the convergence in $L^2(\Omega)$, reducing the rate to approximately 0.5. However, let $\mathcal{B}_{\{\boldsymbol{x}_0, 0.01\}}$ be a ball of radius 0.01 centered at $\boldsymbol{x}_0$. If $\mathcal{B}_{\{\boldsymbol{x}_0, 0.01\}}$ is removed from the domain of integration, we see that the solution is obtained for sufficiently fine meshes (up to machine precision). On the other hand, first-order convergence can moreover be recovered by using a weighted norm with $\alpha = 0.5$, as shown in the final columns.

The results for the two-dimensional case discretized by the primal and solid pressure formulations are shown in Table 2. First-order convergence is observed away from the singularity. Furthermore, increasing α from 0.5 up to 1 increases the order of convergence in L^2-norms over the entire domain Ω. Finally, Table 3 shows the three-dimensional results using the primal formulation. Also in this case, first-order convergence is observed away from the singularity. In all cases, convergence is obtained if the error is evaluated by either excluding a ball of positive radius around the singularity or using a properly weighted norm. Here, we have used weighting parameters $\alpha = 0.5 + \frac{d-2}{2}$ and $\alpha = 1 + \frac{d-2}{2}$, in accordance with Lemma 4.2.

We note that the problem in 3D is not covered by the presented theory, mainly because we cannot directly assume that the functional δ is well-defined on $\boldsymbol{W}_{-\alpha}$. Nevertheless, it is easily implementable and the results show the validity the fundamental solution suggested by Lemma 4.1.

TABLE 1. Convergence results of test case 1 in 1D using the primal formulation.

	$L^2(\Omega \setminus \mathcal{B}_{\{\boldsymbol{x}_0,0.01\}})$		$L^2(\Omega)$		$L^2_{0.5}(\Omega)$	
$\log_2(1/h)$	Error	Rate	Error	Rate	Error	Rate
5	7.58E−02		1.18E−01		1.63E−02	
6	3.02E−02	1.33	8.35E−02	0.50	8.15E−03	1.00
7	5.67E−04	5.74	5.89E−02	0.50	4.08E−03	1.00
8	4.70E−14	–	4.21E−02	0.49	2.04E−03	1.00
9	4.21E−14	–	2.95E−02	0.51	1.02E−03	1.00
10	2.31E−13	–	2.17E−02	0.45	5.12E−04	0.99
11	5.68E−13	–	1.48E−02	0.55	2.54E−04	1.01
12	3.01E−12	–	1.20E−02	0.31	1.41E−04	0.85

TABLE 2. Convergence results of test case 1 in 2D.

	Primal						Solid pressure					
	$\boldsymbol{L}^2(\Omega \setminus \mathcal{B}_{\{\boldsymbol{x}_0,0.1\}})$		$\boldsymbol{L}^2_{0.5}(\Omega)$		$\boldsymbol{L}^2_{1.0}(\Omega)$		$\boldsymbol{L}^2(\Omega \setminus \mathcal{B}_{\{\boldsymbol{x}_0,0.1\}})$		$\boldsymbol{L}^2_{0.5}(\Omega)$		$\boldsymbol{L}^2_{1.0}(\Omega)$	
$\log_2(1/h)$	Error	Rate	Error	Rate	Error	Rate	Error	Rate	Error	Rate	Error	Rate
2	5.48E−01		5.10E−01		3.68E−01		3.44E−01		3.36E−01		1.38E−01	
3	4.50E−01	0.28	4.35E−01	0.23	2.97E−01	0.31	1.38E−01	1.32	2.36E−01	0.51	6.97E−02	0.99
4	3.94E−01	0.19	3.43E−01	0.34	2.01E−01	0.56	5.66E−02	1.29	1.65E−01	0.52	3.49E−02	1.00
5	2.70E−01	0.55	2.56E−01	0.42	1.24E−01	0.70	1.46E−02	1.95	1.13E−01	0.54	1.75E−02	1.00
6	1.52E−01	0.83	1.86E−01	0.46	7.16E−02	0.79	2.20E−03	2.73	7.54E−02	0.59	8.75E−03	1.00
7	7.77E−02	0.96	1.34E−01	0.47	4.01E−02	0.84	5.45E−04	2.01	4.45E−02	0.76	4.22E−03	1.05
8	3.91E−02	0.99	9.71E−02	0.47	2.20E−02	0.87						
9	1.96E−02	1.00	7.09E−02	0.45	1.19E−02	0.88						

TABLE 3. Convergence results of test case 1 in 3D using the primal formulation.

	$\boldsymbol{L}^2(\Omega \setminus \mathcal{B}_{\{\boldsymbol{x}_0,0.3\}})$		$\boldsymbol{L}^2_{1.0}(\Omega)$		$\boldsymbol{L}^2_{1.5}(\Omega)$	
$\log_2(1/h)$	Error	Rate	Error	Rate	Error	Rate
2	9.02E−01		9.13E−01		8.61E−01	
3	1.13E00	−0.32	8.78E−01	0.06	7.25E−01	0.25
4	7.05E−01	0.68	6.99E−01	0.33	4.71E−01	0.62
5	3.58E−01	0.98	5.24E−01	0.42	2.85E−01	0.72
6	1.79E−01	1.00	3.79E−01	0.47	1.62E−01	0.81
7	8.98E−02	0.99	2.68E−01	0.50	8.98E−02	0.85

6.2. The Dirichlet problem

As a second test case, we consider homogeneous Dirichlet boundary conditions, *i.e.* we set

$$\boldsymbol{u}(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in \partial\Omega. \tag{6.2}$$

In this case, the direct approaches suffer from the same suboptimal convergence observed in Section 6.1. We therefore use this test case to explore the singularity removal method of Section 5.2. For 1D, we note that the correction w_h is given by a linear distribution which can easily be represented exactly by linear finite elements. The numerical experiments therefore only consider the cases $d \in \{2, 3\}$. Since the true correction $\boldsymbol{w}$ is not available to us, we use the correction $\boldsymbol{w}_h$ obtained on a finer grid as a surrogate.

Table 4. Convergence of the correction in the singularity removal method.

	2D Primal				2D Solid pressure				3D Primal			
	$\boldsymbol{L}^2(\Omega)$		$\boldsymbol{H}^1(\Omega)$		$\boldsymbol{L}^2(\Omega)$		$\boldsymbol{H}^1(\Omega)$		$\boldsymbol{L}^2(\Omega)$		$\boldsymbol{H}^1(\Omega)$	
$\log_2(1/h)$	Error	Rate	Error	Rate	Error	Rate	Error	Rate	Error	Rate		
2	9.15E−02		3.32E−01		1.89E−02		1.11E−01		1.72E−01		5.66E−01	
3	3.38E−02	1.44	1.92E−01	0.79	3.70E−03	2.35	3.98E−02	1.49	6.47E−02	1.41	3.30E−01	0.78
4	1.02E−02	1.73	1.01E−01	0.93	8.02E−04	2.21	1.53E−02	1.38	1.91E−02	1.76	1.74E−01	0.92
5	2.75E−03	1.89	5.12E−02	0.98	1.85E−04	2.12	6.62E−03	1.21	5.02E−03	1.93	8.85E−02	0.98
6	7.02E−04	1.97	2.58E−02	0.99	3.98E−05	2.22	2.83E−03	1.22	1.20E−03	2.07	4.45E−02	0.99
7	1.71E−04	2.04	1.29E−02	1.00								
8	3.91E−05	2.13	6.56E−03	0.98								

Table 4 shows that the solution exhibits second order convergence in the $\boldsymbol{L}^2$ norm and first order in $\boldsymbol{H}^1$, as expected for Lagrange elements of lowest order applied to the boundary value problem problem (5.3). Importantly, this demonstrates that the singularity removal method is a viable solution technique that converges with optimal rate, namely with rates of 2 and 1 in the L^2 and H^1-norms, respectively.

7. Conclusions and Generalization

We have analyzed the well-posedness, in terms of existence and uniqueness, of linear elasticity problems with point sources in weighted Sobolev spaces. We have presented fundamental solutions and assessed singularities of various mathematical nature that arise in the case of point forces and point stresses. The results have been validated using computational examples.

To generalize this work, we are often interested in multiple moment point sources with different locations and strengths. Thus, indexing each of these sources with i, we formulate the generalized problem as

$$\sigma - \mathbb{C}\varepsilon \boldsymbol{u} = 0 \tag{7.1a}$$

$$-\nabla\cdot\sigma = \sum_i \nabla\cdot(\delta_i M_i), \qquad \text{in } \Omega, \tag{7.1b}$$

$$\boldsymbol{u} = 0, \qquad \text{on } \partial\Omega, \tag{7.1c}$$

in which δ_i is the Dirac delta distribution centered at $\boldsymbol{x}_i$ and M_i is the corresponding stress tensor. Note that M_i need not be isotropic in general, allowing for the modeling of torque, and shear moments. Following a derivation analogous to (2.3), this gives us the functional:

$$\langle \nabla\cdot(\delta_i M_i), \boldsymbol{v}\rangle := \left\langle \delta_i, -\nabla\cdot\left(M_i^T \boldsymbol{v}\right)\right\rangle. \tag{7.2}$$

The theoretical and numerical investigation of such models in the appropriate functional setting forms a topic for future investigation.

Appendix A. A numerical investigation of Korn's inequality in weighted Sobolev spaces

In order to show that Conjecture 3.1 is a reasonable hypothesis, we herein present a numerical experiment on a unit square that suggests its validity in the discrete setting. For that, we first note that the one-dimensional version is trivial, since the symmetric and conventional gradient coincide in that case. We therefore focus on the two-dimensional case.

TABLE A.1. Korn's constant in weighted norms for the Lagrange elements of first and second order.

	Linear elements ($\mathbb{P}_1$)			Quadratic elements ($\mathbb{P}_2$)		
$\log_2(1/h)$	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 1.0$	$\alpha = 0.1$	$\alpha = 0.5$	$\alpha = 1.0$
2	1.88	1.84	1.79	2.03	2.05	2.05
3	1.98	1.99	1.97	2.06	2.14	2.15
4	2.01	2.09	2.08	2.08	2.23	2.22
5	2.04	2.19	2.17	2.09	2.30	2.27
6	2.06	2.27	2.23	2.10	2.36	2.30
7	2.08	2.33	2.27	2.10	2.41	2.32
8	2.09	2.38	2.31	2.11	2.44	2.36

We aim to show that a constant $\lambda_{\max} < \infty$ exists such that

$$\|r^\alpha \nabla \boldsymbol{v}\|_\Omega^2 \leq \lambda_{\max} \|r^\alpha \varepsilon \boldsymbol{v}\|_\Omega^2, \qquad \forall \boldsymbol{v} \in \boldsymbol{W}_\alpha. \tag{A.1}$$

We do this by setting up a generalized eigenvalue problem. For given finite element space $\mathring{\boldsymbol{V}}_h$, we have the following problem: Find $\boldsymbol{u} \in \boldsymbol{V}_h$ and $\lambda_{\max} \in \mathbb{R}$ such that

$$(r^\alpha \nabla \boldsymbol{u}, r^\alpha \nabla \boldsymbol{v})_\Omega = \lambda_{\max} (r^\alpha \varepsilon \boldsymbol{u}, r^\alpha \varepsilon \boldsymbol{v})_\Omega, \qquad \forall \boldsymbol{v} \in \mathring{\boldsymbol{V}}_h. \tag{A.2}$$

The eigenfunction corresponding to the largest (generalized) eigenvalue satisfies (A.1) with equality. In turn, this eigenvalue gives us insight into Korn's constant and we therefore investigate whether it remains bounded with respect to mesh size and polynomial degree. This is indeed the case, as shown in Table A.1.

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