ESAIM: M2AN 57 (2023) 2681–2699
https://doi.org/10.1051/m2an/2022071

ESAIM: Mathematical Modelling and Numerical Analysis
www.esaim-m2an.org

SUPERCONVERGENCE OF DPG APPROXIMATIONS IN LINEAR ELASTICITY

Fleurianne Bertrand[1,*] and Henrik Schneider[2]

Abstract. Existing *a priori* convergence results of the discontinuous Petrov–Galerkin method to solve the problem of linear elasticity are improved. Using duality arguments, we show that higher convergence rates for the displacement can be obtained. Post-processing techniques are introduced in order to prove superconvergence and numerical experiments confirm our theory.

Mathematics Subject Classification. 65N30.

Received March 14, 2022. Accepted August 27, 2022.

1. Introduction

Finite element approximation of partial differential equations using the DPG method is a popular and effective technique introduced in [12–14]. It is a minimal residual method with broken test spaces and can therefore be seen simultaneously as a Least-Squares method and as a saddle-point method. The critical idea is the optimal test function approximation that guarantees discrete stability. This major advantage allow the use of broken test spaces consisting of functions with no continuity requirements at element interfaces. An elementary characterisation of the natural norms on these interface spaces is provided in [10]. Moreover, sufficient conditions under which stability of broken forms follows from the stability of their unbroken relatives were stated.

The inherent stability properties of the DPG method make it a promising approach in solid mechanics, especially for the determination of vibrations of elastic structure due to the possibility of obtaining a pointwise symmetric approximation for stresses in a stable way. Moreover, the simultaneous approximation of the stress-tensor σ and the displacement u, the DPG method also yields robust error bounds and the contributions [7, 11] confirm their suitability in computational mechanics. However, although the stress-tensor and the displacement are related by the strain-stress relationship $\sigma = \mathbb{C}\varepsilon(u)$ with the symmetric gradient $\varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^T)$, both functions are usually approximated simultaneously with the same order. The first aim of this paper is therefore to prove better convergence rates for the displacement variable u, either by increasing the polynomial order of the corresponding approximation space or by defining an approximation of the scalar field variable by suitable postprocessing. This issue has been raised and addressed for the Poisson problem in [16] and for the general second order case in [17]. A similar result for superconvergence for the primal DPG formulation was devoloped in [6]. It is also closely related to the refined error estimates of the Least-Squares method of [5].

Keywords and phrases. Discontinuous Petrov–Galerkin, linear elasticity.

[1] University of Twente, Drienerlolaan 5, 7522 NB Enschede, Netherlands.

[2] Fakultät für Mathematik, Universität Duisburg-Essen, Thea-Leymann-Str. 9, 45127 Essen, Germany.

*Corresponding author: fleurianne@gmail.com

As a second motivation, the refined error estimates proved in this paper also allow us to pave the way for the consideration of the DPG method for the determination of vibrations of elastic structures. Dual-mixed formulations have been considered in the solid for the elastoacoustic source problem (see, *e.g.* [18,19]) but their extension to the eigenvalue problem does not fit the existing theories for mixed eigenvalue problems. DPG-based formulations would fit this theory, because of the gained flexibility of the finite element spaces. However, similarly to the issues arising in the Least-Squares context (see [1–3]), superconvergence results are crucial to prove the convergence of eigenvalue approximation with the DPG method (see [4]), and the aim of this paper.

2. The DPG formulation

The DPG method under consideration is based on the first-order system of the linear elasticity equations in a domain Ω

$$\mathcal{A}\sigma - \varepsilon(u) = 0 \qquad \text{in } \Omega, \tag{2.1a}$$

$$-\operatorname{div}\sigma = f \qquad \text{in } \Omega, \tag{2.1b}$$

$$u = 0 \qquad \text{on } \Gamma_D, \tag{2.1c}$$

$$\sigma\cdot\nu = 0 \qquad \text{on } \Gamma_N, \tag{2.1d}$$

where $\mathcal{A} : \mathbb{S} \to \mathbb{S}$ is the compliance tensor given by $\tau \mapsto \frac{1}{2\mu}\tau - \frac{\lambda}{2\mu(2\mu+d\lambda)}\operatorname{tr}(\tau)I_{d\times d}$, with the Lam parameter λ and μ. The extension for asymmetric matrices is given by $\mathcal{A} : \mathbb{M} \to \mathbb{M}\ \tau \mapsto \mathcal{A}(\operatorname{sym}(\tau)) + \operatorname{as}(\tau)$. We assume that the domain $\Omega \subset \mathbb{R}^d =: \mathbb{V}$ is a Lipschitz domain and that the space dimension d equals 2 or 3. The boundary $\partial\Omega$ consists of two open subsets Γ_D and Γ_N, where Dirichlet and Neumann conditions are prescribed. The boundary part Γ_D, where the elastic body is clamped, is assumed to be of positive measure and Γ_N is assumed to be its complement $\partial\Omega\backslash\Gamma_D$, *i.e.* $\partial\Omega = \overline{\Gamma_D \cup \Gamma_N}$, $\Gamma_D \cap \Gamma_N = \emptyset$. Under these assumptions, problem (2.1) admits a unique solution $(\sigma, u) \in H_{\Gamma_N}(\operatorname{div}, \mathbb{S}) \times H^1_{\Gamma_D}(\Omega, \mathbb{V})$ for all $f \in L^2(\Omega; \mathbb{V})$ with

$$\begin{aligned} L^2(\Omega;\mathbb{V}) &= \left[L^2(\Omega)\right]^d,\\ H^1_{\Gamma_D}(\Omega;\mathbb{V}) &= \left[H^1_{\Gamma_D}(\Omega)\right]^d,\\ H_{\Gamma_N}(\operatorname{div},\Omega;\mathbb{S}) &= \left\{\sigma \in L^2(\Omega;\mathbb{S}) : \operatorname{div}\sigma \in L^2(\Omega,\mathbb{V}),\ \sigma\cdot\nu = 0 \text{ on } \Gamma_N\right\}. \end{aligned}$$

Throughout this paper, the domain Ω is assumed to be such that the following regularity assumption holds

$$\|u\|_{H^2(\Omega)} + \|\sigma\|_{H^1(\mathcal{T})} \le C\|f\|. \tag{2.2}$$

Remark 2.1. This regularity estimate is fulfilled for $d = 2$ if Ω is a convex polyhedral domain. If $u \in H^1_{\Gamma_D}(\Omega;\mathbb{V})$ solves

$$-\operatorname{div}(\mathbb{C}\varepsilon(u)) = f \in L^2(\Omega;\mathbb{V}),$$

then $u \in H^2(\Omega;\mathbb{V})$ and $\|u\|_{H^2(\Omega;\mathbb{V})} \le C_1\|f\|$, see [22]. Using the stress-strain relationship yields

$$\|\sigma\|_{H^1(\mathcal{T})} = \|\mathcal{A}\varepsilon(u)\|_{H^1(\mathcal{T})} \le C_2\|u\|_{H^2(\Omega)} \le C_1C_2\|f\|,$$

where C_2 is robust for $\lambda \to \infty$.

The ultra-weak formulation under consideration is derived from (2.1) by testing with broken test functions, based on a shape-regular simplicial triangulation $\mathcal{T}$. It allows for the broken test spaces

$$H^1(\mathcal{T}) := \left\{v \in L^2(\Omega) : v|_T \in H^1(T) \quad \forall T \in \mathcal{T}\right\}, \tag{2.3a}$$

$$H(\operatorname{div};\mathcal{T}) := \left\{\tau \in L^2(\Omega) : \tau|_T \in H(\operatorname{div};T) \quad \forall T \in \mathcal{T}\right\} \tag{2.3b}$$

and for the piecewise differential operators $\nabla_{\mathcal{T}} : H^1(\mathcal{T}) \to L^2(\Omega)$ and $\operatorname{div}_{\mathcal{T}} : H(\operatorname{div};\mathcal{T}) \to L^2(\Omega)$ defined on each element $T \in \mathcal{T}$ by

$$\nabla_{\mathcal{T}} v|_T := \nabla(v|_T), \quad \operatorname{div}_{\mathcal{T}} \tau|_T := \operatorname{div}(\tau|_T). \tag{2.4}$$

Using these broken spaces as test spaces has the advantage that the field variables (σ, u) can be sought in $L^2(\Omega)$ allowing for discontinuous trial functions. A weak continuity condition is imposed with the introduction of trace variables living in the following skeleton spaces:

$$H^{1/2}_{\Gamma_D}(\partial\mathcal{T}) := \Big\{\widehat{u} \in \Pi_{T\in\mathcal{T}} H^{1/2}(\partial T) : \exists w \in H^1_{\Gamma_D}(\Omega) \text{ such that } \widehat{u}|_{\partial T} = w|_{\partial T} \quad \forall T \in \mathcal{T}\Big\}, \tag{2.5a}$$

$$H^{-1/2}_{\Gamma_N}(\partial\mathcal{T}) := \Big\{\widehat{\sigma} \in \Pi_{T\in\mathcal{T}} H^{-1/2}(\partial T) : \exists \boldsymbol{q} \in \boldsymbol{H}_{\Gamma_N}(\operatorname{div};\Omega) \text{ such that } \widehat{\sigma}|_{\partial T} = (\boldsymbol{q}\cdot\boldsymbol{n}_T)|_{\partial T} \quad \forall T \in \mathcal{T}\Big\}. \tag{2.5b}$$

With the Sobolev spaces

$$L^2(\Omega;\mathbb{M}) = \big[L^2(\Omega)\big]^{d\times d}, \tag{2.6a}$$

$$L^2(\Omega;\mathbb{A}) = \big\{v \in L^2(\Omega,\mathbb{M}) : v = -v^T\big\}, \tag{2.6b}$$

$$H(\operatorname{div},\mathcal{T};\mathbb{S}) = \Big\{v \in [H(\operatorname{div},\mathcal{T})]^d : v = v^T\Big\}, \tag{2.6c}$$

our DPG formulation seeks $(\sigma, u) \in L^2(\Omega;\mathbb{M}) \times L^2(\Omega;\mathbb{V})$ as well as $(\hat\sigma_n, \hat u) \in H^{-1/2}_{\Gamma_N}(\partial\mathcal{T}) \times H^{1/2}_{\Gamma_D}(\partial\mathcal{T})$ such that

$$(\mathcal{A}\sigma,\tau) + (u, \operatorname{div}_{\mathcal{T}}\tau) + (\sigma, \nabla_{\mathcal{T}} v) + (\sigma, q) - \langle \hat u, \tau\cdot\nu\rangle_{\partial\mathcal{T}} - \langle \hat\sigma_n, v\rangle_{\partial\mathcal{T}} = (f, v) \tag{2.7}$$

holds for all $(\tau, v, q) \in H(\operatorname{div},\mathcal{T};\mathbb{S}) \times H^1(\mathcal{T};\mathbb{V}) \times L^2(\Omega;\mathbb{A})$. To simplify the notation, we also introduce the spaces $\boldsymbol{U} = L^2(\Omega;\mathbb{M}) \times L^2(\Omega;\mathbb{V}) \times H^{-1/2}_{\Gamma_N}(\partial\mathcal{T}) \times H^{1/2}_{\Gamma_D}(\partial\mathcal{T})$ and $\boldsymbol{V} = H(\operatorname{div},\mathcal{T};\mathbb{S}) \times H^1(\mathcal{T};\mathbb{V}) \times L^2(\Omega;\mathbb{A})$ as well as the bilinear forms

$$b(\boldsymbol{u},\boldsymbol{v}) = (\mathcal{A}\sigma,\tau) + (u, \operatorname{div}_{\mathcal{T}}\tau) + (\sigma, \nabla_{\mathcal{T}} v) + (\sigma, q) - \langle \hat u, \tau\cdot\nu\rangle_{\partial\mathcal{T}} - \langle \hat\sigma_n, v\rangle_{\partial\mathcal{T}}, \tag{2.8a}$$

$$l(\boldsymbol{v}) = (f, v). \tag{2.8b}$$

for $\boldsymbol{u} = (\sigma, u, \hat\sigma_n, \hat u)$ and $\boldsymbol{v} = (\tau, v, q)$. With these notations, the variational formulation (2.7) allows for the abstract form

$$b(\boldsymbol{u},\boldsymbol{v}) = l(\boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \boldsymbol{V}. \tag{2.9}$$

The well-posedness of this formulation is shown in [7] and the proof relies on the following three key properties:

(1) Uniqueness

$$\{\boldsymbol{w} \in \boldsymbol{U} : b(\boldsymbol{w},\boldsymbol{v}) = 0, \quad \forall \boldsymbol{v} \in \boldsymbol{V}\} = \{0\}, \tag{2.10}$$

(2) inf-sup condition

$$\|\boldsymbol{v}\|_{\boldsymbol{V}} \lesssim \sup_{\boldsymbol{w}\in\boldsymbol{U}\setminus\{0\}} \frac{b(\boldsymbol{w},\boldsymbol{v})}{\|\boldsymbol{w}\|_{\boldsymbol{U}}} \qquad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{2.11}$$

(3) Continuity of b

$$b(\boldsymbol{w},\boldsymbol{v}) \lesssim \|\boldsymbol{w}\|_{\boldsymbol{U}} \|\boldsymbol{v}\|_{\boldsymbol{V}} \qquad \forall \boldsymbol{w} \in \boldsymbol{U},\ \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{2.12}$$

with the norms

$$\|(\sigma, u, \hat{\sigma}_n, \hat{u})\|_{\boldsymbol{U}}^2 := \|\sigma\|^2 + \|u\|^2 + \|\hat{\sigma}_n\|_{H^{-1/2}(\partial\mathcal{T})}^2 + \|\hat{u}\|_{H^{1/2}(\partial\mathcal{T})}^2, \tag{2.13a}$$

$$\|(\tau, v, q)\|_{\boldsymbol{V}}^2 := \|\nabla_{\mathcal{T}} v\|^2 + \|v\|^2 + \|\operatorname{div}_{\mathcal{T}} \tau\|^2 + \|\tau\|^2 + \|q\|^2. \tag{2.13b}$$

In particular, the operator $B : \boldsymbol{U} \to \boldsymbol{V}'$ defined by $(B\boldsymbol{u})(\boldsymbol{v}) = b(\boldsymbol{u}, \boldsymbol{v})$ and the trial-to-test operator $\Theta : \boldsymbol{U} \to \boldsymbol{V}$ defined by

$$(\Theta \boldsymbol{w}, \boldsymbol{v})_{\boldsymbol{V}} = b(\boldsymbol{w}, \boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \boldsymbol{V} \tag{2.14}$$

allow for a constant $C > 0$ such that

$$C^{-1}\|\boldsymbol{u}\|_{\boldsymbol{U}}^2 \leq \|B\boldsymbol{u}\|_{\boldsymbol{V}'}^2 = b(\boldsymbol{u}, \Theta\boldsymbol{u}) \leq C\|\boldsymbol{u}\|_{\boldsymbol{U}}^2 \tag{2.15}$$

holds for all $\boldsymbol{u} \in \boldsymbol{U}$.

3. Finite element approximation

Let $\boldsymbol{U}_h \subset \boldsymbol{U}$ be a conforming finite-dimensional subspace of $\boldsymbol{U}$. The ideal DPG method now seeks $\boldsymbol{u}_h \in \boldsymbol{U}_h$ such that

$$b(\boldsymbol{u}_h, \Theta\boldsymbol{w}_h) = l(\Theta\boldsymbol{w}_h) \qquad \forall \boldsymbol{w}_h \in \boldsymbol{U}_h. \tag{3.1}$$

To obtain the practical DPG method, the test space is replaced by a finite element space $\boldsymbol{V}_h \subset \boldsymbol{V}$ with the following crucial compatibility condition on the spaces: there exists a Fortin operator $\Pi : \boldsymbol{V} \to \boldsymbol{V}_h$ such that there exists a constant $C_\Pi > 0$ with

$$b(\boldsymbol{u}_h, \boldsymbol{v} - \Pi\boldsymbol{v}) = 0 \text{ and } \|\Pi\boldsymbol{v}\|_{\boldsymbol{V}} \leq C_\Pi\|\boldsymbol{v}\| \qquad \forall \boldsymbol{u}_h \in \boldsymbol{U}_h, \boldsymbol{v} \in \boldsymbol{V}. \tag{3.2}$$

The discrete trial-to-test operator $\Theta_h : \boldsymbol{U}_h \to V_h$ is defined through

$$(\Theta_h \boldsymbol{w}_h, \boldsymbol{v}_h)_{\boldsymbol{V}} = b(\boldsymbol{w}_h, \boldsymbol{v}_h) \qquad \text{for all } \boldsymbol{v}_h \in \boldsymbol{V}_h$$

and the practical DPG formulation seeks $\boldsymbol{u}_h \in \boldsymbol{U}_h$ such that

$$b(\boldsymbol{u}_h, \Theta_h\boldsymbol{w}_h) = l(\Theta_h\boldsymbol{w}_h) \qquad \forall \boldsymbol{w}_h \in \boldsymbol{U}_h. \tag{3.3}$$

With respect to the simplicial triangulation $\mathcal{T}$, our focus is on approximations within the spaces of piecewise polynomials of degree at most $k \in \mathbb{N}_0$, given by

$$P^k(T) = \{v \in L^\infty(T) \mid v \text{ is polynomial on } T \text{ of degree } \leq k\}, \tag{3.4}$$

$$P^k(\mathcal{T}) = \left\{v_{\mathcal{T}} \in L^\infty(\Omega) | \forall T \in \mathcal{T}, v_{\mathcal{T}}|_T \in P^k(T)\right\}, \tag{3.5}$$

$$P^k(\mathcal{T}; \mathbb{M}) = \left(P^k(\mathcal{T})\right)^{d\times d}, \quad P^k(\mathcal{T}; \mathbb{V}) = \left(P^k(\mathcal{T})\right)^d, \tag{3.6}$$

$$P^k(\mathcal{T}; \mathbb{S}) = \left\{v \in P^k(\mathcal{T}; \mathbb{M}) : v = v^T\right\}, \quad P^k(\mathcal{T}; \mathbb{A}) = \left\{v \in P^k(\mathcal{T}; \mathbb{M}) : v = -v^T\right\}. \tag{3.7}$$

For the approximation of the trace variables, $\mathcal{E}$ denotes the set of all sides in the triangulation and we define the approximation spaces

$$P^k(\partial T) = \{v \in L^\infty(\partial T) : v_{|F} \in P^k(F) \text{ for all (d-1)-dimensional subsimplices } F \text{ of } T\}, \tag{3.8}$$

$$S^k_{\Gamma_D}(\mathcal{E}; \mathbb{V}) = \left\{v \in L^\infty(\partial\mathcal{T}; \mathbb{V}) : v|_{\partial T} \in (P_k(\partial\mathcal{T}))^d \cap C^0(\partial T)\ \forall T \in \mathcal{T}\right\} \ \cap\ H^1_{\Gamma_D}(\Omega), \tag{3.9}$$

$$P^k_{\Gamma_N}(\mathcal{E}; \mathbb{V}) = \left\{v \in L^\infty(\partial\mathcal{T}; \mathbb{V}) : v|_{\partial T} \in (P_k(\partial\mathcal{T}))^d\ \forall T \in \mathcal{T},\ \text{for all edges } E \subset \Gamma_N :\ v(E) = 0\right\}. \tag{3.10}$$

The DPG formulation (3.3) allows for a natural choice of piecewise polynomial trial spaces with

$$\boldsymbol{U}_h^{k,j} := P^k(\mathcal{T};\mathbb{M}) \times P^{k+j}(\mathcal{T};\mathbb{V}) \times P^k_{\Gamma_N}(\mathcal{E};\mathbb{V}) \times S^{k+1}_{\Gamma_D}(\mathcal{E};\mathbb{V}), \tag{3.11a}$$

$$\boldsymbol{V}_h^k := P^{k+2}(\mathcal{T};\mathbb{S}) \times P^{k+d}(\mathcal{T};\mathbb{V}) \times P^k(\mathcal{T};\mathbb{A}) \tag{3.11b}$$

where $k \geq 0$ and $j = 0, 1$. Note that the original formulation in [21] concerns only $j = 0$, *i.e.* approximations of u and the stress tensor σ in polynomial spaces of the same order. However, recalling the constitutive law $\mathcal{A}\sigma = \varepsilon(u)$, this can appear suboptimal and we will show that a better convergence rate for $j = 1$ can be obtained in Section 6. Additionally $U_h^{k,j}$ denotes the second component of $\boldsymbol{U}_h^{k,j}$. The next lemma recalls the definition of the Fortin operator Π.

Lemma 3.1 (Fortin operator)**.** *Let $k \in \mathbb{N}_0$, $j = 0, 1$, $\Pi_k : L^2(\Omega) \to P^k(\mathcal{T})$ denote the L^2 projection and $\Pi_k^{\mathrm{div},\mathbb{S}} : H(\mathrm{div},\mathcal{T};\mathbb{S}) \to P^k(\mathcal{T};\mathbb{S})$ denotes the symmetric divergence projection ([20, 21], Lem. 4.1, Thm. 3.6) such that for all $\tau \in H(\mathrm{div},\mathcal{T};\mathbb{S})$:*

$$\left\|\Pi_{k+2}^{\mathrm{div},\mathbb{S}}\tau\right\|_{H(\mathrm{div},\mathcal{T})} \leq C\|\tau\|_{H(\mathrm{div},\mathcal{T})}, \tag{3.12a}$$

$$\left(\tau - \Pi_{k+2}^{\mathrm{div},\mathbb{S}}\tau, \tau_h\right) = 0 \qquad \forall \tau_h \in P^k(\mathcal{T};\mathbb{S}), \tag{3.12b}$$

$$\left\langle \left(\tau - \Pi_{k+2}^{\mathrm{div},\mathbb{S}}\tau\right)\cdot n, \mu_h\right\rangle_{\partial\mathcal{T}} = 0, \qquad \forall \mu_h \in P^{k+1}(\partial\mathcal{T}), \tag{3.12c}$$

$$\text{and } \operatorname{div}_{\mathcal{T}} \Pi_{k+1}^{\mathrm{div},\mathbb{S}}\tau = \Pi_k \operatorname{div}_{\mathcal{T}} \tau. \tag{3.12d}$$

Moreover, let $\Pi_{k+d}^{\mathrm{grad}} : H^1(\mathcal{T}) \to P^{k+d}(\mathcal{T})$ be such that

$$\begin{aligned} \left\|\Pi_{k+d}^{\mathrm{grad}}\right\|_{H^1(\mathcal{T})} &\leq C\|u\|_{H^1(\mathcal{T})}, \\ \left(u - \Pi_{k+d}^{\mathrm{grad}}u, u_h\right) &= 0 \qquad \forall u_h \in P^{k-1}(\mathcal{T}), \\ \text{and } \left(u - \Pi_{k+d}^{\mathrm{grad}}u, \mu_h\right) &= 0 \qquad \forall \mu_h \in P^k(\partial\mathcal{T}) \end{aligned}$$

for all $u \in H^1(\mathcal{T})$. The operator $\Pi(\tau, v, q) := (\Pi_{k+2}^{\mathrm{div},\mathbb{S}}\tau, \Pi_{k+d}^{\mathrm{grad}}v, \Pi^k q)$ is a Fortin operator, i.e.

$$b(\boldsymbol{u}_h, \boldsymbol{v} - \Pi\boldsymbol{v}) = 0$$

and $\|\Pi\boldsymbol{v}\|_{\boldsymbol{V}} \leq C_\Pi\|\boldsymbol{v}\|$ for all $\boldsymbol{u}_h \in \boldsymbol{U}_h^{k,j}, \boldsymbol{v} \in \boldsymbol{V}$.

Proof. The case $j = 0$ is covered by Lemma 4.1 of [21]. For second case $j = 1$ we want to show that

$$\begin{aligned} b(\boldsymbol{u}_h, \boldsymbol{v} - \Pi\boldsymbol{v}) = {} & \left(\mathcal{A}\sigma_h, \tau - \Pi_{k+2}^{\mathrm{div},\mathbb{S}}\tau\right) + \left(u_h, \operatorname{div}_{\mathcal{T}}\left(\tau - \Pi_{k+2}^{\mathrm{div},\mathbb{S}}\tau\right)\right) + \left(\sigma_h, \nabla_{\mathcal{T}}\left(v - \Pi_{k+d}^{\mathrm{grad}}v\right)\right) + \left(\sigma_h, q - \Pi^k q\right) \\ & - \left\langle \hat{u}_h, \left(\tau - \Pi_{k+2}^{\mathrm{div},\mathbb{S}}\tau\right)\cdot\nu\right\rangle_{\partial\mathcal{T}} - \left\langle \hat{\sigma}_h, v - \Pi_{k+d}^{\mathrm{grad}}v\right\rangle_{\partial\mathcal{T}} \end{aligned} \tag{3.13}$$

is equal to zero. Note that only the second term is modified by the change of the space. The commutative property of $\Pi_k^{\mathrm{div},\mathbb{S}}$, the approximation property of Π_k and the fact that u_h belongs to $P^{k+1}(\mathcal{T},\mathbb{V})$ yields to

$$\left(u_h, \operatorname{div}_{\mathcal{T}}\left(\tau - \Pi_{k+2}^{\mathrm{div},\mathbb{S}}\tau\right)\right) = (u_h, \operatorname{div}_{\mathcal{T}}\tau - \Pi_{k+1}(\operatorname{div}_{\mathcal{T}}\tau)) = 0 \qquad \forall \tau \in H(\mathrm{div},\mathcal{T};\mathbb{S}).$$

□

The existence of the Fortin operator immediately leads to the quasioptimality result which we state in the next theorem.

Theorem 3.2 (Quasioptimality)**.** *Let* $\boldsymbol{u} = (\sigma, u, \hat{\sigma}_n, \hat{u}) \in \boldsymbol{U}$ *be the solution to continuous formulation* (2.7). *For* $k \geq 0$ *and* $j = 0, 1$ *let* $\boldsymbol{u}_h = (\sigma_h, u_h, \hat{\sigma}_{n,h}, \hat{u}_h) \in \boldsymbol{U}_h^{k+j}$ *be the solution to the discrete DPG formulation* (3.3). *Then, the following quasioptimality result holds*

$$\begin{aligned}\|\sigma - \sigma_h\|_{L^2(\Omega)} + \|u - u_h\|_{L^2(\Omega)} + \|\hat{\sigma}_n - \hat{\sigma}_{n,h}\|_{H^{-1/2}(\partial\mathcal{T})} + \|\hat{u} - \hat{u}_h\|_{H^{1/2}(\partial\mathcal{T})} \\ \leq C \min_{\rho_h, w_h, \hat{\rho}_h, \hat{w}_h \in \boldsymbol{U}_h} \|\sigma - \rho_h\|_{L^2(\Omega)} + \|u - w_h\|_{L^2(\Omega)} + \|\hat{\sigma}_n - \hat{\rho}_h\|_{H^{-1/2}(\partial\mathcal{T})} + \|\hat{u} - \hat{w}_h\|_{H^{1/2}(\partial\mathcal{T})}.\end{aligned}$$

The proof follows from the quasioptimality result ([21], Thm. 4.2).

4. Improved *a priori* convergence

This section aims at improving the existing *a priori* analysis from [7] to require only minimal regularity of the solution u to obtain optimal convergence rates. In fact, in the contribution [7] the best approximation error in the third component (for the traces of the stress tensor) is bounded by the best approximation error of the exact stress in the $H(\mathrm{div}, \mathbb{M})$-norm, approximated in the space

$$RT^k_{\Gamma_N}(\mathcal{T}) = \left\{\tau \in H_{\Gamma_N}(\mathrm{div}, \Omega; \mathbb{M}) : \tau_{|_T} = \tau_1 + \tau_2 \mathbf{x}^\top, \tau_1 \in P^k(\mathcal{T}; \mathbb{M}),\ \tau_2 \in P^k(\mathcal{T}; \mathbb{V})\ \forall T \in \mathcal{T}\right\}. \tag{4.1}$$

To estimate our solution we now introduce the corresponding interpolants.

Definition 4.1. Let $\Pi_k^{\mathrm{div}} : H_{\Gamma_N}(\mathrm{div}, \Omega; \mathbb{M}) \cap H^{r+1}(\mathcal{T}; \mathbb{M}) \to RT^k_{\Gamma_N}(\mathcal{T})$ denote the componentwise Raviart–Thomas interpolant such that

$$\left\|\tau - \Pi_k^{\mathrm{div}}\tau\right\| \leq C_k h^{r+1} |\tau|_{H^{r+1}(\mathcal{T})} \quad \text{for } r \in [0, k]$$

and $\mathrm{div}\, \Pi_k^{\mathrm{div}} \tau = \Pi_k \,\mathrm{div}\, \tau$ holds for all $\tau \in H(\mathrm{div}, \Omega; \mathbb{M}) \cap H^1(\mathcal{T})$ (see [24], Thm. 3 or [15], Thm. 16.4). Let $\Pi_{k+1}^{\nabla} : H^1_{\Gamma_D}(\Omega) \to S^{k+1}_{\Gamma_D}(\mathcal{T})$ denote the Scott–Zhang interpolant [25] such that

$$\left\|v - \Pi_{k+1}^{\nabla} v\right\| \leq C_{k+1} h^{k+1} \|v\|_{H^{k+2}(\Omega)}. \tag{4.2}$$

The error estimate of the $H(\mathrm{div})$ trace, therefore, relies on the term $||(1 - \Pi_k^{\mathrm{div}}) \,\mathrm{div}\, \sigma||$. For it to converge when the mesh size reduces, some regularity of $\mathrm{div}\, \sigma$ has to be assumed. Due to (2.1b) this means regularity assumptions on f. An alternative way to estimate the best approximation error corresponding to the trace component was presented in [16]. There the commutativity of Π_k^{div} and its approximation property lead directly to an estimate for $\|\hat{\sigma}_n\|_{H^{-1/2}(\partial\mathcal{T})}$ without considering the original $||(1 - \Pi_k^{\mathrm{div}}) \,\mathrm{div}\, \sigma||$ term. In the next theorem, we extend Theorem 5, Corollary 6 of [16] resp. Theorem 6 of [17] to our DPG formulation. This will be particularly useful to extend our duality argument to more general regularity assumptions. Note that the result in Theorem 5, Corollary 6 of [16] also applies in the case of reduced regularity.

Theorem 4.2. *Let* $\boldsymbol{w} = (\chi, w, \gamma_n \chi, \gamma_0 w) \in \boldsymbol{U}$ *with* $w \in H^{k+2}(\Omega; \mathbb{V})$ *and* $\chi \in H^{k+1}(\mathcal{T}; \mathbb{M}) \cap H(\mathrm{div}, \Omega; \mathbb{M})$*. For* $k \in \mathbb{N}_0$ *and* $j = 0, 1$*, the best approximation* $\boldsymbol{w}_h \in \boldsymbol{U}_h^{k,j}$ *satisfies*

$$\|\boldsymbol{w} - \boldsymbol{w}_h\|_{\boldsymbol{U}} \leq C h^{k+1} \left(\|w\|_{H^{k+2}(\Omega)} + \|\chi\|_{H^{k+1}(\mathcal{T})}\right). \tag{4.3}$$

Proof. The proof is just the componentwise application of Theorem 6 from [17]. □

5. Distance of $||u - u_h||$ to $U_h^{k,j}$

With the improved *a priori* error estimates of the previous section, we now employ a duality argument to show that the error $||u - u_h||$ is almost orthogonal to any $g \in U_h^{k,j}$. To this aim, we derive a representation of the solution to the adjoint problem in the following lemma.

Lemma 5.1 (Representation of the solution to the adjoint problem)**.** *Let* $g \in L^2(\Omega)$ *be and* $\boldsymbol{v} := (\tau, v, q) \in H_{\Gamma_N}(\operatorname{div}, \Omega; \mathbb{S}) \times H^1_{\Gamma_D}(\Omega; \mathbb{V}) \times L^2(\Omega;\mathbb{A})$ *is the solution to the adjoint problem*

$$\mathcal{A}\tau + \nabla v + q = 0 \qquad \text{in } \Omega, \tag{5.1a}$$
$$\operatorname{div}\tau = g \qquad \text{in } \Omega, \tag{5.1b}$$
$$v = 0 \qquad \text{on } \Gamma_D, \tag{5.1c}$$
$$\tau \cdot \nu = 0 \qquad \text{on } \Gamma_N. \tag{5.1d}$$

Then, there exits a unique element $\boldsymbol{w} = \Theta^{-1}\boldsymbol{v}$ *of* $\boldsymbol{U}$ *such that*

$$\begin{aligned}(\boldsymbol{v}, \boldsymbol{z})_{\boldsymbol{V}} &= b(\boldsymbol{w}, \boldsymbol{z}) \qquad \forall \boldsymbol{z} \in \boldsymbol{V},\\ \boldsymbol{w} &= (-q, g, -\gamma_n\tau, 0) + (\sigma^*, u^*, \gamma_n\sigma^*, \gamma_0 u^*),\end{aligned}$$

with $(\sigma^*, u^*) \in H_{\Gamma_N}(\operatorname{div}, \Omega; \mathbb{M}) \times H^1_{\Gamma_D}(\Omega; \mathbb{V})$ *the solution to the problem*

$$\mathcal{A}\sigma^* - \varepsilon(u^*) = \tau \qquad \text{in } \Omega, \tag{5.2a}$$
$$-\operatorname{div}\sigma = \operatorname{div}\mathcal{A}\tau + v \qquad \text{in } \Omega, \tag{5.2b}$$
$$u = 0 \qquad \text{on } \Gamma_D, \tag{5.2c}$$
$$\sigma \cdot \nu = 0 \qquad \text{on } \Gamma_N. \tag{5.2d}$$

Moreover, it holds

$$\|v\|_{H^2(\Omega)} + \|\tau\|_{H^1(\mathcal{T})} + \|q\| + \|u^*\|_{H^2(\Omega)} + \|\sigma^*\|_{H^1(\mathcal{T})} \le C\|g\|.$$

Proof. Since $H_{\Gamma_N}(\operatorname{div}, \Omega; \mathbb{S}) \times H^1_{\Gamma_D}(\Omega; \mathbb{V}) \times L^2(\Omega;\mathbb{A}) \subset H(\operatorname{div}, \mathcal{T}; \mathbb{S}) \times H^1(\mathcal{T}; \mathbb{V}) \times L^2(\mathcal{T};\mathbb{A}) = \boldsymbol{V}$ the solution $\boldsymbol{v} = (\tau, v, q)$ of the adjoint problem (5.1) lies in $\boldsymbol{V}$. Any test functions $(\mu, \lambda, \rho) \in \boldsymbol{V}$ allow for the product

$$((\tau, v, q), (\mu, \lambda, \rho))_{\boldsymbol{V}} = (\operatorname{div}\tau, \operatorname{div}_{\mathcal{T}}\mu) + (\tau, \mu) + (\nabla v, \nabla_{\mathcal{T}}\lambda) + (v, \lambda) + (q, \rho)$$

where the scalar product $(\cdot, \cdot)_{\boldsymbol{V}}$ induces the $\|\cdot\|_{\boldsymbol{V}}$ norm. Moreover, the adjoint problem implies $\operatorname{div}\tau = g$ and thus

$$(\operatorname{div}\tau, \operatorname{div}_{\mathcal{T}}\mu) = (g, \operatorname{div}_{\mathcal{T}}\mu) = b((0, g, 0, 0), (\mu, \lambda, \rho)).$$

The constitutive equation $\nabla v = -\mathcal{A}\tau - q$ and an integration by parts leads to

$$\begin{aligned}(\nabla v, \nabla_{\mathcal{T}}\lambda) &= -(\mathcal{A}\tau, \varepsilon_{\mathcal{T}}(\lambda)) - (q, \nabla_{\mathcal{T}}\lambda)\\ &= -\langle\gamma_n(\mathcal{A}\tau), \lambda\rangle_{\partial\mathcal{T}} + (\operatorname{div}\mathcal{A}\tau, \lambda) - (q, \nabla_{\mathcal{T}}\lambda)\\ &= -b(0, 0, \gamma_n(\mathcal{A}\tau), 0), (\mu, \lambda, \rho) + (\operatorname{div}\mathcal{A}\tau, \lambda) - (q, \nabla_{\mathcal{T}}\lambda).\end{aligned}$$

Combining the above equations, we conclude

$$\begin{aligned}((\tau, v, q), (\mu, \lambda, \rho))_{\boldsymbol{V}} &= (\operatorname{div}\tau, \operatorname{div}_{\mathcal{T}}\mu) + (\tau, \mu) + (\nabla v, \nabla_{\mathcal{T}}\lambda) + (v, \lambda) \underbrace{-(q, \nabla_{\mathcal{T}}\lambda) - (q, p) - (\mathcal{A}q, \mu)}_{-b((q,0,0,0),(\mu,\lambda,\rho))}\\ &= b((-q, g, -\gamma_n(\mathcal{A}\tau), 0), (\mu, \lambda, \rho)) + (\operatorname{div}\mathcal{A}\tau + v, \lambda) + \big(\mathcal{A}\mathcal{A}^{-1}\tau, \mu\big),\end{aligned}$$

since $(\mathcal{A}q, \mu) = 0$. For $\boldsymbol{w} := (-q, g, -\gamma_n(\mathcal{A}\tau), 0) + (\sigma^*, u^*, \gamma_n\sigma^*, \gamma_0 u^*)$ with (σ^*, u^*) solution to (5.2) we obtain

$$((\tau, v, q), (\mu, \lambda, \rho))_{\boldsymbol{V}} = b(\boldsymbol{w}, (\mu, \lambda, \rho)) \qquad \forall(\mu, \lambda, \rho) \in \boldsymbol{V}.$$

Furthermore, the assumptions on the domain leading to (2.2) also imply

$$\|u\|_{H^2(\Omega)} + \|\sigma\|_{H^1(\mathcal{T})} \leq C\big(\|\operatorname{div}\mathcal{A}\tau + v\| + \|\tau\|_{H^1(\mathcal{T})}\big) \tag{5.3}$$

and

$$\|v\|_{H^2(\Omega)} + \|\tau\|_{H^1(\mathcal{T})} + \|q\| \leq C\|g\|.$$

Therefore, we conclude

$$\|v\|_{H^2(\Omega)} + \|\tau\|_{H^1(\mathcal{T})} + \|q\| + \|u^*\|_{H^2(\Omega)} + \|\sigma^*\|_{H^1(\mathcal{T})} \leq C\|g\|.$$

□

In order to estimate the distance of the error to $U_h^{k,j}$, we will need to proof in Lemma 5.3 that the representation formula of the above lemma implies an orthogonality relation of the type

$$b(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}) = (u - u_h, g) + \hat{\rho}(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v})$$

for any $\boldsymbol{v} \in \boldsymbol{V}$ where $\hat{\rho}(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}) = -\langle \hat{u} - \hat{u}_h, \tau \cdot \nu\rangle_{\partial\mathcal{T}} - \langle \hat{\sigma}_n - \hat{\sigma}_{n,h}, v\rangle_{\partial\mathcal{T}}$ vanishes. For the convenience of the reader, we therefore recall the crucial property of the trace-spaces before proceeding with Lemma 5.3 and the estimation of the distance of the error to $U_h^{k,j}$.

Lemma 5.2 ([23], Lem. 3.1)**.** *Let Γ_D and Γ_N be relative open subsets in $\partial\Omega$, such that $\partial\Omega = \overline{\Gamma_D \cup \Gamma_N}$ and $\Gamma_D \cap \Gamma_N = \emptyset$.*

(1) *Let $v \in H^1(\mathcal{T})$. Then $v \in H^1_{\Gamma_D}(\Omega)$ iff $\langle \hat{\tau}, v\rangle_{\partial\mathcal{T}} = 0$ $\forall \hat{\tau} \in H^{-1/2}_{\Gamma_N}(\partial\mathcal{T})$.*
(2) *Let $\tau \in H(\operatorname{div}, \mathcal{T})$. Then $\tau \in H_{\Gamma_N}(\operatorname{div}, \Omega)$ iff $\langle \hat{u}, \tau \cdot \nu\rangle_{\partial\mathcal{T}} = 0$ $\forall \hat{u} \in H^{1/2}_{\Gamma_D}(\partial\mathcal{T})$.*

We are now in place to prove the crucial error estimate $|(u - u_h, g)| \leq Ch\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\boldsymbol{U}}\|g\|$ for any $g \in U_h^{k,j}$ which is the essence of the next lemma. The proof relies on the equivalence of the DPG formulation (3.3) with the following mixed formulation: find $(\boldsymbol{\varepsilon}_h, \boldsymbol{u}_h) \in V_h \times U_h$ such that

$$a((\boldsymbol{\varepsilon}_h, \boldsymbol{u}_h); (\boldsymbol{v}_h, \boldsymbol{w}_h)) = l(\boldsymbol{v}_h) \qquad \forall \boldsymbol{v}_h \in V_h \ \forall \boldsymbol{w}_h \in U_h \tag{5.4}$$

with $a((\boldsymbol{\varepsilon}_h, \boldsymbol{u}_h); (\boldsymbol{v}_h, \boldsymbol{w}_h)) = (\boldsymbol{\varepsilon}_h, \boldsymbol{v}_h)_{\boldsymbol{V}} + b(\boldsymbol{u}_h, \boldsymbol{v}_h) - b(\boldsymbol{w}_h, \boldsymbol{\varepsilon}_h)$. The continuous counterpart reads

$$a((\boldsymbol{\varepsilon}, \boldsymbol{u}); (\boldsymbol{v}, \boldsymbol{w})) = l(\boldsymbol{v}) \qquad \forall \boldsymbol{v} \in \boldsymbol{V} \forall \boldsymbol{w} \in \boldsymbol{U}, \tag{5.5}$$

where $\boldsymbol{u} \in \boldsymbol{U}$ solves (2.9) and $\boldsymbol{\varepsilon} = 0$. Recall from [9] that $\boldsymbol{\varepsilon}_h \in \boldsymbol{V}_h$ provide a reliable error estimator, *i.e.*

$$\|\boldsymbol{\varepsilon}_h\|_{\boldsymbol{V}} \lesssim \|\boldsymbol{u} - \boldsymbol{u}_h\|_{\boldsymbol{U}}. \tag{5.6}$$

Lemma 5.3 (Distance of the error to $U_h^{k,j}$)**.** *For $k \geq 0$ and $j = 0, 1$, let $\boldsymbol{u} \in \boldsymbol{U}$ be the solution to the ultra-weak formulation (2.7) and $\boldsymbol{u}_h \in \boldsymbol{U}_h^{k,j}$ be the solution of the discrete problem (3.3). Then, there exists a constant $C > 0$ independent of h, such that*

$$|(u - u_h, g)| \leq Ch\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\boldsymbol{U}}\|g\|$$

for any $g \in U_h^{k,j}$.

Proof. For $g \in U_h^{k,j} \subset L^2(\Omega;\mathbb{V})$, let $\boldsymbol{v} = (\tau, v, q) \in \boldsymbol{V}$ denote the solution to the adjoint problem (5.1) with data g, *i.e.*

$$b(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{v}) = (\sigma-\sigma_h,\mathcal{A}\tau) + (\sigma-\sigma_h,\nabla v) + (\sigma-\sigma_h,q) + (u-u_h,\operatorname{div}\tau) + \hat\rho(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{v}).$$

Since Lemma 5.2 implies that $\hat\rho(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{v})$ vanishes, the equations $\operatorname{div}\tau = g$ and the constitutive equation $\mathcal{A}\tau + \nabla v + q = 0$ from the adjoint problem (5.1) lead to

$$b(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{v}) = (u-u_h,g).$$

The property $b(\boldsymbol{w},\tilde{\boldsymbol{v}}) = (\boldsymbol{v},\tilde{\boldsymbol{v}})_{\boldsymbol{V}} = (\tilde{\boldsymbol{v}},\boldsymbol{v})_{\boldsymbol{V}}$ for all $\tilde{\boldsymbol{v}} \in \boldsymbol{V}$ and $\boldsymbol{w} = \Theta^{-1}\boldsymbol{v}$ from Lemma 5.1, imply

$$\begin{aligned}(u-u_h,g) &= b(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{v})\\ &= (\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}_h,\boldsymbol{v})_{\boldsymbol{V}} + b(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{v}) - (\boldsymbol{v},\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}_h)_{\boldsymbol{V}}\\ &= a((\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}_h),(\boldsymbol{w},\boldsymbol{v})).\end{aligned}$$

The Galerkin orthogonality shows that any $(\boldsymbol{w}_h,\boldsymbol{v}_h) \in \boldsymbol{U}_h^{k,j} \times V_h$ statisfies

$$(u-u_h,g) = a((\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{\varepsilon}-\boldsymbol{\varepsilon}_h),(\boldsymbol{w}-\boldsymbol{w}_h,\boldsymbol{v}-\boldsymbol{v}_h)).$$

Consequently, the boundedness of a and (5.6) imply

$$(u-u_h,g) \lesssim \|\boldsymbol{u}-\boldsymbol{u}_h\|_{\boldsymbol{U}}(\|\boldsymbol{w}-\boldsymbol{w}_h\|_{\boldsymbol{U}} + \|\boldsymbol{v}-\boldsymbol{v}_h\|_{\boldsymbol{V}}).$$

We now estimated the two terms in the bracket on the right-hand side separately.

– **Estimation of $\|\boldsymbol{v}-\boldsymbol{v}_h\|_{\boldsymbol{V}}, j = 0$**

For $\boldsymbol{v}_h = (\Pi_{k+2}^{\operatorname{div},\mathbb{S}}\tau, \Pi_1^{\nabla} v, \Pi_0 q) \in \boldsymbol{V}_h$, the approximation properties of the projections (3.12), (4.2) and Lemma 5.1 imply

$$\begin{aligned}\|\boldsymbol{v}-\boldsymbol{v}_h\|_{\boldsymbol{V}} &\le \left\|v-\Pi_1^{\nabla}v\right\|_{H^1(\Omega)} + \left\|\tau-\Pi_{k+2}^{\operatorname{div},\mathbb{S}}\tau\right\|_{H(\operatorname{div},\Omega)} + \|q-\Pi_0 q\|\\ &\lesssim h\|g\| + \left\|\operatorname{div}\left(\tau-\Pi_{k+2}^{\operatorname{div},\mathbb{S}}\tau\right)\right\| + \|q-\Pi_0 q\|.\end{aligned}$$

Moreover, the commutativity property (3.12d) of $\Pi_{k+2}^{\operatorname{div},\mathbb{S}}$, the adjoint problem and $g \in P^k(\mathcal{T})$ lead to

$$\left\|\operatorname{div}\left(\tau-\Pi_{k+2}^{\operatorname{div},\mathbb{S}}\tau\right)\right\| = \|(1-\Pi_{k+1})\operatorname{div}\tau\| = \|(1-\Pi_{k+1})g\| = 0.$$

For the last term, we infer

$$\begin{aligned}\|\Pi_0 q - q\| &= \|\Pi_0\nabla v - \nabla v + \Pi_0\mathcal{A}\tau - \mathcal{A}\tau\|\\ &\le \|\Pi_0\nabla v - \nabla v\| + \|\Pi_0\mathcal{A}\tau - \mathcal{A}\tau\|\\ &\lesssim h\|v\|_{H^2(\Omega)} + h\|\tau\|_{H^1(\mathcal{T})} \lesssim h\|g\|.\end{aligned}$$

Overall, we obtain

$$\|\boldsymbol{v}-\boldsymbol{v}_h\|_{\boldsymbol{V}} \lesssim h\|g\|. \tag{5.7}$$

– **Estimation of $\|\boldsymbol{v}-\boldsymbol{v}_h\|_{\boldsymbol{V}}, j = 1$**

We follow the lines of the proof of the case $j=0$ choosing $\boldsymbol{v}_h = \left(\Pi_{k+2}^{\operatorname{div},\mathbb{S}}\tau, \Pi_1^{\nabla} v, \Pi_0 q\right) \in \boldsymbol{V}_h$. This leads to

$$\|\boldsymbol{v}-\boldsymbol{v}_h\|_{\boldsymbol{V}} \lesssim h\|g\|. \tag{5.8}$$

– **Estimation of $\|\boldsymbol{w}-\boldsymbol{w_h}\|_{\boldsymbol{U}}$**
Considering the representation $\boldsymbol{w} = (-q, g, -\gamma_n\tau, 0) + \tilde{\boldsymbol{w}}$ from Lemma 5.1, we choose

$$\boldsymbol{w}_h = \big(-\Pi_0 q, g, -\gamma_n \Pi_0^{\mathrm{div}}\tau, 0\big) + \tilde{\boldsymbol{w}}_h,$$

whereby $\tilde{\boldsymbol{w}}_h = \big(\Pi_k w, \Pi_k \chi, \gamma_0\big(\Pi_{k+1}^{\nabla} w\big), \gamma_n\big(\Pi_k^{\mathrm{div}}\chi\big)\big)$ denotes best approximation and lead to

$$\|\boldsymbol{w}-\boldsymbol{w}_h\|_{\boldsymbol{U}} \le \|q-\Pi_0 q\| + \big\|\gamma_n\big(\tau - \Pi_0^{\mathrm{div}}\tau\big)\big\|_{H^{-1/2}(\partial\mathcal{T})} + \|\tilde{\boldsymbol{w}}-\tilde{\boldsymbol{w}}_h\|_{\boldsymbol{U}}. \tag{5.9}$$

The first two terms can be estimate using the approximation property of $\Pi_0 q$ in $L^2(\Omega)$ and $\gamma_n\Pi_k^{\mathrm{div}}$ in $H^{-1/2}(\partial\mathcal{T})$. Lemma 5.1 and Theorem 4.2 imply

$$\|\tilde{\boldsymbol{w}}-\tilde{\boldsymbol{w}}_h\|_{\boldsymbol{U}} \lesssim h\Big(\|u^*\|_{H^2(\Omega)} + \|\sigma^*\|_{H^1(\mathcal{T})}\Big) \le h\|g\| \tag{5.10}$$

and thus

$$\|\boldsymbol{w}-\boldsymbol{w}_h\|_{\boldsymbol{U}} \lesssim h\|g\|. \tag{5.11}$$

The estimates (5.7) (respectively (5.8)) and (5.11) lead to

$$|(u-u_h, g)| \le Ch\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\boldsymbol{U}}\|g\|$$

in the case $j=0$ (respectively $j=1$). □

6. Superconvergence through increased polynomial degree and postprocessing

The previous section shows that the error $||u-u_h||$ is almost orthogonal to any $g \in \boldsymbol{U}_h^{k,0}$. As an auxiliary result, we also obtain the supercloseness of u_h to the L^2 projection $\Pi_k u$, as stated in the following theorem. This is a vector valued generalisation from the scalar result in Theorem 3 of [17] and is included here for the convenience of the reader.

Theorem 6.1 (Supercloseness to L^2 projection)**.** *Let $\boldsymbol{u} = (\sigma, u, \hat{\sigma}_n, \hat{u}) \in \boldsymbol{U}$ the solution to the ultra-weak formulation (2.7) and assume that $u \in H^{k+2}(\Omega;\mathbb{V})$ and $\sigma \in H^{k+1}(\mathcal{T};\mathbb{M})$. Let $\boldsymbol{u}_h = (\sigma_h, u_h, \hat{\sigma}_n, \hat{u}) \in \boldsymbol{U}_h^{k,0}$ the solution to the ultra-weak formulation (3.3), then it holds*

$$\|u_h - \Pi_k u\| \le Ch^{k+2}\big(\|u\|_{H^{k+2}(\Omega)} + \|\sigma\|_{H^{k+1}(\mathcal{T})}\big),$$
$$\|u-\Pi_k u\| \le \|u-u_h\| \le \|u-\Pi_k u\| + Ch^{k+2}\big(\|u\|_{H^{k+2}(\Omega)} + \|\sigma\|_{H^{k+1}(\mathcal{T})}\big).$$

Proof. The triangle inequality and the properties of Π_k lead to

$$\|u-\Pi_k u\| \le \|u-u_h\| \le \|u-\Pi_k u\| + \|\Pi_k(u-u_h)\|.$$

Choosing $g := \Pi_k u - u_h \in P^k(\mathcal{T},\mathbb{V}) = U_h^{k,0}$ leads to

$$\|g\|^2 = (g,g) = (\Pi_k(u-u_h), g) = (u-u_h, g).$$

Lemma 5.3 implies

$$\|g\|^2 = (u-u_h, g) \lesssim h\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\boldsymbol{U}}\|g\| \lesssim h^{k+2}\big(\|u\|_{H^{k+2}(\Omega)} + \|\sigma\|_{H^{k+1}(\mathcal{T})}\big)\|g\|.$$

Dividing by $\|g\|$ concludes the proof. □

The *a priori* estimates of Section 4 prove the same order of convergence in all variables, *i.e.* for the stress and the displacements. However, the point of using an augmented trial space (*i.e.* the finite element space $\boldsymbol{U}_h^{k,j}$ with $j = 1$) is to obtain an improved convergence rate for the displacement field. The aim of this section is therefore to prove that in the case $j = 1$ the error in the displacement field $\|u - u_h\|$ converges at a higher rate than the total error. To achieve this, we use the improved convergence rate from Section 4 and combine it with the result of our duality argument from Lemma 5.3. The following proofs use similar arguments as the proof of Theorem 6.1.

Theorem 6.2 (Improved convergence rate for $j = 1$)**.** *For $f \in L^2(\Omega; \mathbb{V})$, let $\boldsymbol{u} = (\sigma, u, \hat{\sigma}_n, \hat{u}) \in \boldsymbol{U}$ be the solution to the ultra-weak formulation (5.5). Let $\boldsymbol{u}_h \in \boldsymbol{U}_h^{k,1}$ the solution to the corresponding discrete problem (5.4). If $u \in H^{k+2}(\Omega; \mathbb{V})$ and $\sigma \in H^{k+1}(\mathcal{T}; \mathbb{M})$, then*

$$\|u - u_h\| \leq Ch^{k+2}\big(\|u\|_{H^{k+2}(\Omega)} + \|\sigma\|_{H^{k+1}(\mathcal{T})}\big)$$

holds.

Proof. Using the triangle inequality we obtain

$$\|u - u_h\| \leq \|u - \Pi_{k+1}u\| + \|\Pi_{k+1}u - u_h\| \leq \|u - \Pi_{k+1}u\| + \|g\| \tag{6.1}$$

with $g := \Pi_{k+1}u - u_h \in P^{k+1}(\mathcal{T}) = U_h^{k,1}$. The first term can be estimated using the approximation property of Π_{k+1}, *i.e.*

$$\|u - \Pi_{k+1}u\| \lesssim h^{k+2}\|u\|_{H^{k+2}(\Omega)}. \tag{6.2}$$

Moreover, since

$$\|g\|^2 = (g, g) = (\Pi_{k+1}(u - u_h), g) = (u - u_h, g),$$

Lemma 5.3 and Theorem 4.2 lead to

$$\|g\|^2 = (u - u_h, g) \lesssim h\|\boldsymbol{u} - \boldsymbol{u}_h\|_U\|g\| \lesssim hh^{k+1}\big(\|u\|_{H^{k+2}(\Omega)} + \|\sigma\|_{H^{k+1}(\mathcal{T})}\big)\|g\|$$

and thus to

$$\|g\| \lesssim h^{k+2}\big(\|u\|_{H^{k+2}(\Omega)} + \|\sigma\|_{H^{k+1}(\mathcal{T})}\big). \tag{6.3}$$

Inserting (6.3) and (6.2) in (6.1) finishes the proof. □

Another possibility to achieve higher convergence rates is to postprocess the part of the solution u_h. To this aim, we introduce the space of rigid body motion $\mathrm{RM}(\mathcal{T})$, defined as the kernel of the symmetric gradient

$$\mathrm{RM}(\mathcal{T}) = \big\{v \in H^1(\mathcal{T}; \mathbb{V}) \mid \varepsilon_T(v) = 0\ \forall T \in \mathcal{T}\big\} \subset P^1(\mathcal{T})$$

and let Π_{rm} denotes the L^2-projection onto $\mathrm{RM}(\mathcal{T})$. For the convenience of the reader, we state the well-known approximation property of this projection in the next remark.

Lemma 6.3. *The projection Π_{rm} fullfils the following approximation property*

$$\|\Pi_{\mathrm{rm}}v - v\| \leq Ch\|\varepsilon_{\mathcal{T}}(v)\| \tag{6.4}$$

for all $v \in H^1(\mathcal{T})$.

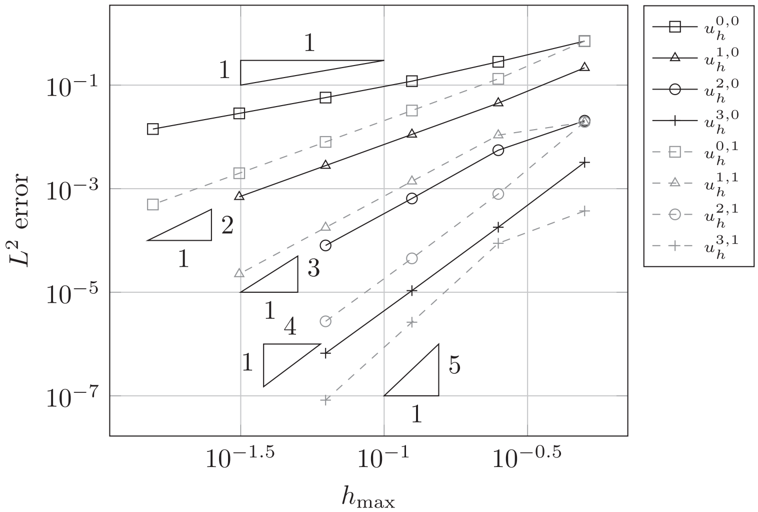

FIGURE 1. Convergence of the DPG solution for augmented trial spaces.

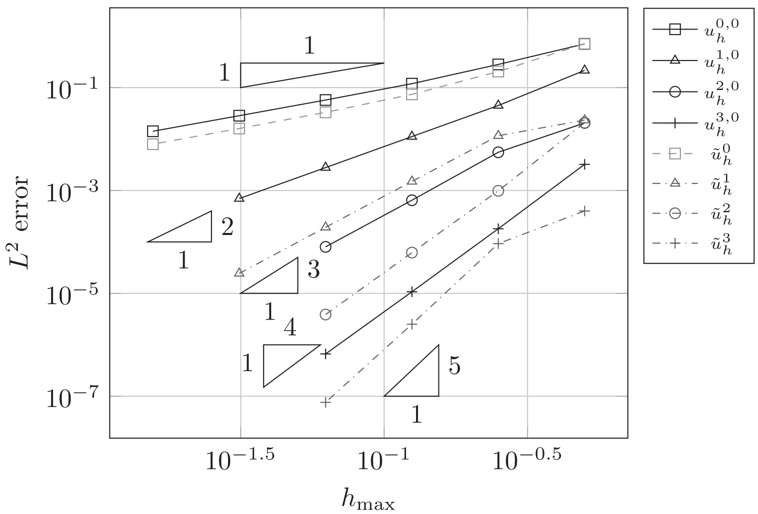

FIGURE 2. Convergence of the post-processing.

Proof. Let us first note that the left-hand side vanishes if the right-hand side is zero. In fact, if $\varepsilon_{\mathcal{T}}(v) = 0$, $\Pi_{\mathrm{rm}} v = 0$ and the inequality becomes trivial. Consider now $v \in H^1(\mathcal{T})/\mathrm{RM}(\mathcal{T})$. Then,

$$(\Pi_{\mathrm{rm}} v, w) = (\Pi_0 v, w) \qquad \forall w \in P_0(\mathcal{T}).$$

Therefore, Π_{rm} inherits the approximation property of Π_0 and together with the Korn inequality, it holds

$$\|\Pi_{\mathrm{rm}} v - v\| \le Ch \|\nabla_{\mathcal{T}} v\| \le Ch \|\varepsilon_{\mathcal{T}}(v)\|.$$

□

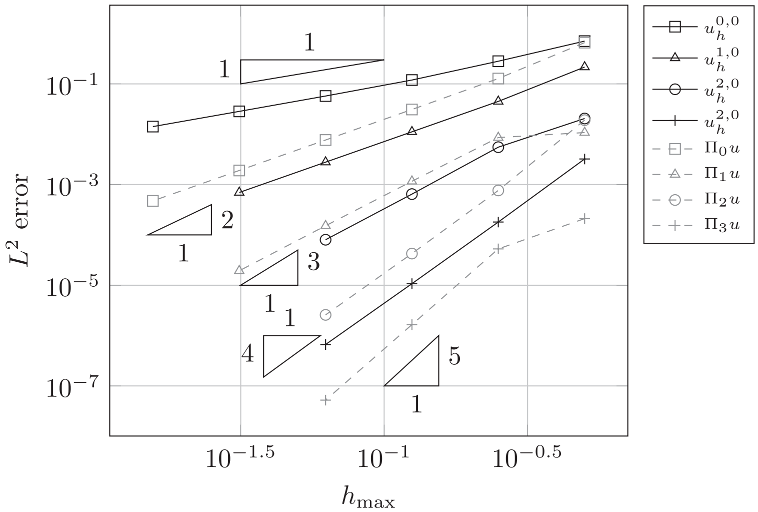

FIGURE 3. Superconvergence of the L^2 projection.

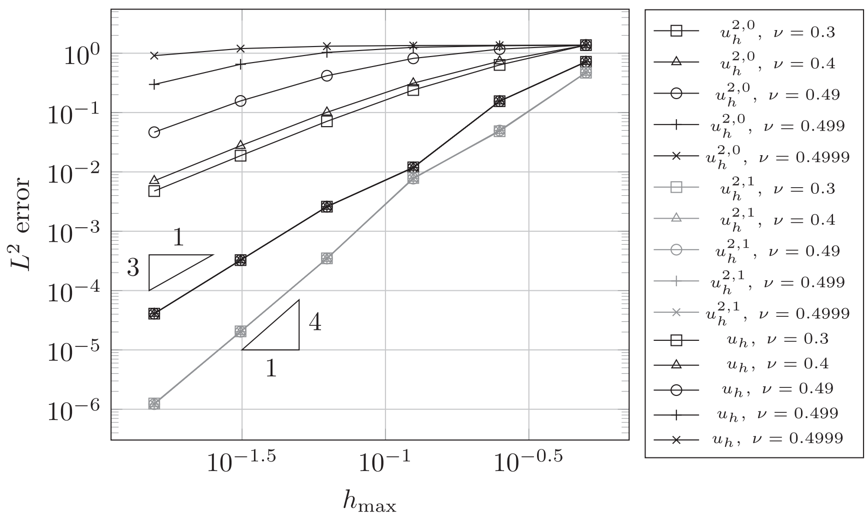

FIGURE 4. Robustness of the approximation in augmented trial spaces.

In order to post-process the part u_h of the solution $\boldsymbol{u}_h = (\sigma_h, u_h, \hat{\sigma}_n, \hat{u}) \in U_h$ of the ultra-weak formulation, we consider a local Neumann problem in the spirit of [26]:

Find $\tilde{u}_h \in P^{k+1}(\mathcal{T};\mathbb{V})$ such that

$$\begin{cases} \Pi_{\mathrm{rm}}\tilde{u}_h = \Pi_{\mathrm{rm}} u_h, \\ (\varepsilon(\tilde{u}_h), \varepsilon(v_h)) = (\mathcal{A}\sigma_h, \varepsilon(v_h)) \qquad \forall v_h \in \big\{ v \in P^{k+1}(\mathcal{T};\mathbb{V}) \mid (v,w)_T = 0 \ \forall T \in \mathcal{T},\ \forall w \in \mathrm{RM}(\mathcal{T}) \big\}. \end{cases} \tag{6.5}$$

Theorem 6.4 (Convergence rate of the post-processing)**.** *Let $\boldsymbol{u} = (\sigma, u, \hat{\sigma}_n, \hat{u}) \in \boldsymbol{U}$ be the solution to the ultra-weak formulation (5.5) for some $f \in L^2(\Omega;\mathbb{V})$, and assume that $u \in H^{k+2}(\Omega;\mathbb{V})$ and $\sigma \in H^{k+1}(\mathcal{T};\mathbb{M})$. Let $\boldsymbol{u}_h = (\sigma_h, u_h, \hat{\sigma}_n, \hat{u}) \in \boldsymbol{U}_h^{k,0}$ the solution to the ultra-weak formulation and $\tilde{u}_h \in P^{k+1}(\mathcal{T};\mathbb{V})$ the post-processing*

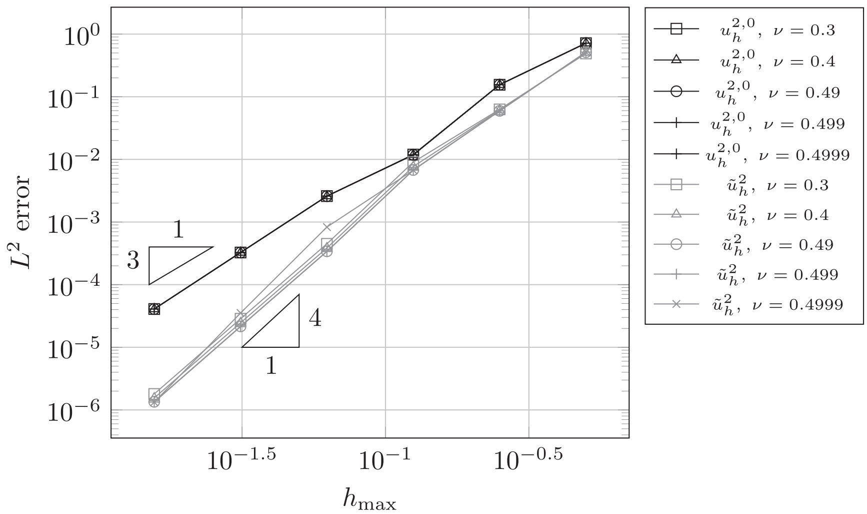

FIGURE 5. Robustness of the post-processing.

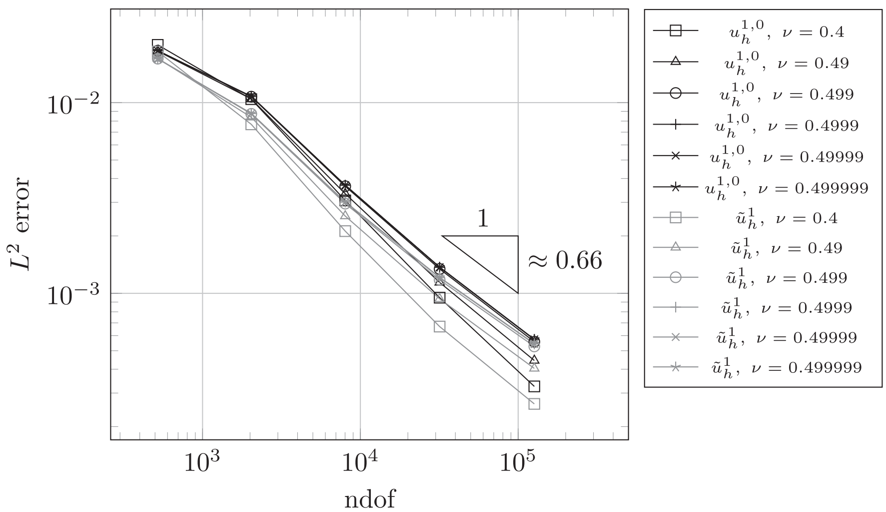

FIGURE 6. Robustness uniform L-shape for the postprocessing.

satisfying (6.5)*. Then, for* $k \geq 1$

$$\|u_h - \tilde{u}_h\| \leq Ch^{k+2}\big(\|u\|_{H^{k+2}(\Omega)} + \|\sigma\|_{H^{k+1}(\mathcal{T})}\big)$$

holds.

Proof. $\Pi_{\mathrm{rm}}\tilde{u}_h = \Pi_{\mathrm{rm}}u_h$ and the approximation properties (6.4) of Π_{rm} lead to

$$\|u - \tilde{u}_h\| \leq \|(1 - \Pi_{\mathrm{rm}})(u - \tilde{u}_h)\| + \|\Pi_{\mathrm{rm}}(u - \tilde{u}_h)\| \lesssim h\|\varepsilon_{\mathcal{T}}(u - \tilde{u}_h)\| + \|\Pi_{\mathrm{rm}}(u - u_h)\|.$$

Moreover, $g := \Pi_{\mathrm{rm}}(u - u_h) \in \mathrm{RM}(\mathcal{T}) \subset P^1(\mathcal{T})$ Lemma 5.3 and Theorem 4.2 imply

$$\|g\|^2 = (\Pi_{\mathrm{rm}}(u - u_h), g) = (u - u_h, g) \lesssim h\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\boldsymbol{U}}\|g\| \lesssim h^{k+2}(\|u\|_{H^{k+2}(\Omega)} + \|\sigma\|_{H^{k+1}(\mathcal{T})})\|g\|.$$

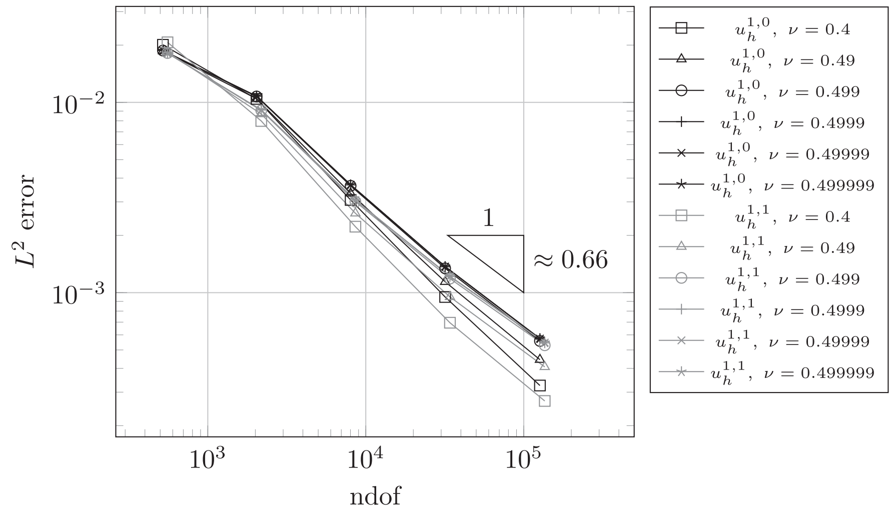

FIGURE 7. Robustness uniform L-shape for the augmented trial spaces.

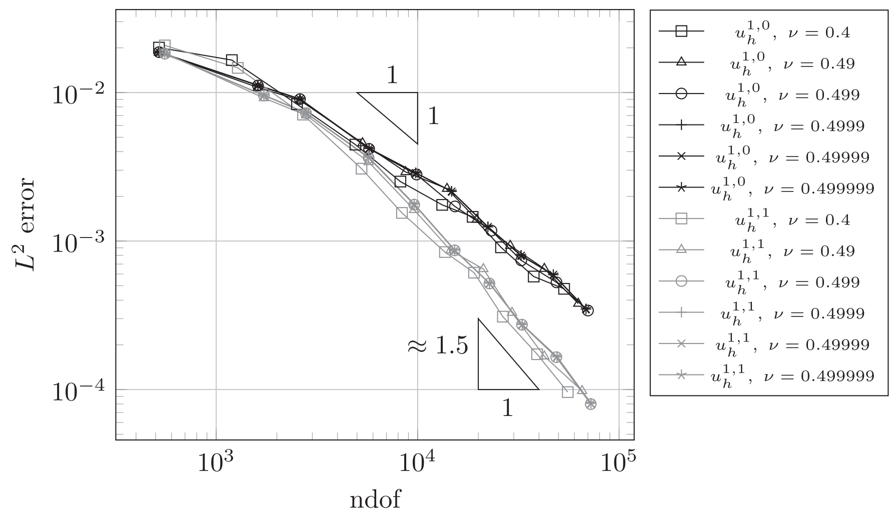

FIGURE 8. Robustness adaptive L-shape for the augmented trial spaces.

It remains to estimate $\|\varepsilon_{\mathcal{T}}(u - \tilde{u}_h)\|$. Since $\bar{u}_h \in P^{k+1}(\mathcal{T}; \mathbb{V})$ satisfies

$$(\varepsilon(\bar{u}_h), \varepsilon(v_h)) = (\mathcal{A}\sigma, \varepsilon(v_h))_T \qquad \forall v_h \in \{v \in P^{k+1}(\mathcal{T}; \mathbb{V}) \mid (v, w)_T = 0 \ \forall T \in \mathcal{T}, \ \forall w \in \mathrm{RM}(\mathcal{T})\},$$

it holds

$$\begin{aligned}\|\varepsilon_{\mathcal{T}}(\bar{u}_h - \tilde{u}_h)\|^2 &= (-\mathcal{A}(\sigma - \sigma_h), \varepsilon_{\mathcal{T}}(\bar{u}_h - \tilde{u}_h)) \\ &\lesssim \|\boldsymbol{u} - \boldsymbol{u}_h\|_{\boldsymbol{U}} \|\varepsilon_{\mathcal{T}}(\bar{u}_h - \tilde{u}_h)\| \lesssim h^{k+1}\big(\|u\|_{H^{k+2}(\Omega)} + \|\sigma\|_{H^{k+1}(\mathcal{T})}\big)\|\varepsilon_{\mathcal{T}}(\bar{u}_h - \tilde{u}_h)\|.\end{aligned}$$

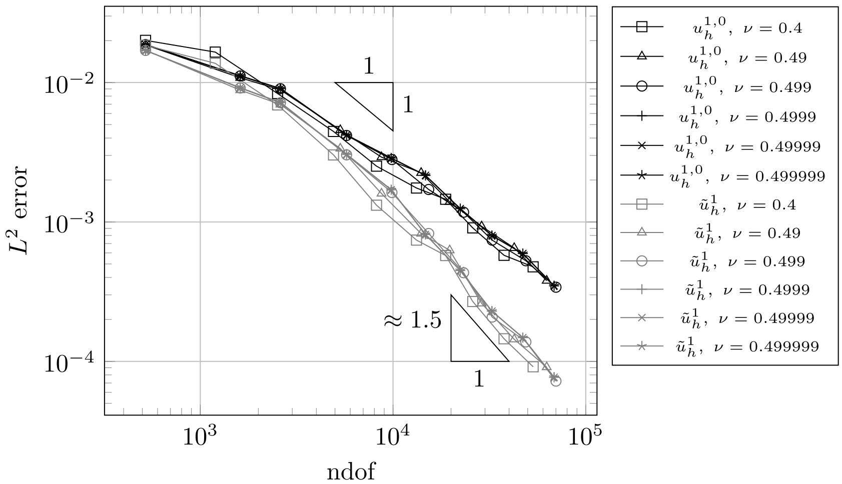

FIGURE 9. Robustness adaptive L-shape for the postprocessing.

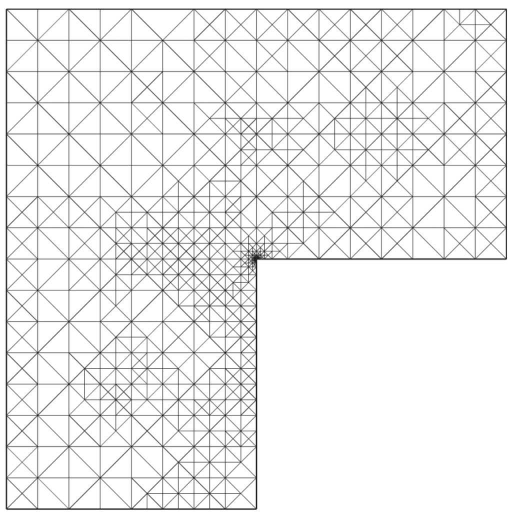

FIGURE 10. L-shape adaptive shape after 10 steps.

Moreover, the Galerkin orthogonality $(\varepsilon_\mathcal{T}(u - \bar{u}_h), \varepsilon_\mathcal{T}(v_h)) = 0$ for all $v_h \in P^{k+1}(\mathcal{T}; \mathbb{V})$ leads to

$$\|\varepsilon_\mathcal{T}(u - \bar{u}_h)\| = \min_{v_h \in P^{k+1}(\mathcal{T})} \|\varepsilon_\mathcal{T}(u - v_h)\| \lesssim h^{k+1} \|u\|_{H^{k+2}(\Omega)}.$$

Combining the previous inequalities finishes the proof:

$$\begin{aligned}
\|u - \tilde{u}_h\| &\lesssim h\|\varepsilon_\mathcal{T}(u - \tilde{u}_h)\| + \|g\| \\
&\lesssim h(\|\varepsilon_\mathcal{T}(u - \bar{u}_h)\| + \|\varepsilon_\mathcal{T}(\bar{u}_h - \tilde{u}_h)\|) + h^{k+2}\big(\|u\|_{H^{k+2}(\Omega)} + \|\sigma\|_{H^{k+1}(\mathcal{T})}\big) \\
&\lesssim h^{k+2}\big(\|u\|_{H^{k+2}(\Omega)} + \|\sigma\|_{H^{k+1}(\mathcal{T})}\big).
\end{aligned}$$

$\square$

Remark 6.5 (Regularity of the solution)**.** For the purpose of the exposition, we assumed previously $u \in H^2(\Omega)$. This can be not expected in general. For example in general domains, let the solution $u \in H^{1+s}(\Omega)$ for some $s \in (1/2, k+1]$ and the adjoint solution $v \in H^{1+s'}(\Omega)$ for some $s' \in (1/2, 1]$. In the presence of this different regularity assumption, Theorems 6.1, 6.2 and 6.4 remain valid by replacing h^{k+2} by $h^{1+s+s'}$.

7. Numerical results

In this section, we illustrate the behaviour of the proposed improved approximations on a particular test problem, similarly to [7]. It considers the unit square domain $\Omega = (0,1)^2$ with a smooth solution and the exact displacement $u(x,y) = (\sin(\pi x)\sin(\pi y), \sin(\pi x)\sin(\pi y))$ and $\Gamma_D = \partial\Omega$. The Lam parameter are chosen as $\lambda = \mu = 1$.

Following the structure of the paper, we start with the validation of the improved *a priori* convergence of Section 4. As the solution is smooth, we expect that the error $\|u - u_h^{k,0}\|$ decrease with decrease of the maximum mesh-size (respectively increase of degrees of freedom) with a rate $k+1$, *i.e.* that $\|u - u_h^{k,0}\|$ is $\mathcal{O}(h^{k+1})$. This is observed in Figure 1.

We then investigate the effect of using the augmented trial space (*i.e.* the finite element space $\boldsymbol{U}_h^{k,j}$ with $j = 1$). We want to validate the results of Theorem 6.2 where improved convergence rate for the displacement field are obtained. We expect that the error $\|u - u_h^{k,1}\|$ converges at a higher rate than the total error, *i.e.* is $\mathcal{O}(h^{k+2})$. This is indeed observed in Figure 1.

The two possibilities to recover an higher-order approximation from a lower-order finite element space $\boldsymbol{U}_h^{k,j}$ are validated in Figures 2 and 3. Our theory stated (Thms. 6.1 and 6.4) that both $\|u - \tilde{u}_h^k\|$ and $\|u^{0,k} - \Pi_k u\|$ are $\mathcal{O}(h^{k+2})$ and this is numerically observed.

In order to illustrate the robustness of the higher-order approximations, we consider a further example similarly to [8]. Also here, the domain $\Omega = (0,1)^2$ is the unit square, and Dirichlet boundary condition are set on the hole boundary. The smooth solution and data read:

$$\begin{aligned} f_x &= 2\pi^3\mu(2\cos(2\pi x) - 1)\sin(\pi y)\cos(\pi y) \\ f_y &= 2\pi^3\mu(1 - 2\cos(2\pi y))\sin(\pi x)\cos(\pi x) \\ u_x &= \pi\cos(\pi y)\sin(\pi x)^2\sin(\pi y) \\ u_y &= -\pi\cos(\pi y)\sin(\pi x)\sin(\pi y)^2. \end{aligned}$$

Note that the data depends only on the parameter μ, which is not critical after re-scaling. We therefore consider the following parameter $E = 10^5$ and $\nu \in \{0.3, 0.4.0.49, 0.499, 0.4999\}$. The relation to the Lam parameter is given by $\lambda = \frac{E\nu}{(1-2\nu)(1+\nu)}$ and $\mu = \frac{E}{2(1+\nu)}$. We observe in Figures 4 and 5 that the error bounds for the different values ν are robust. In order to emphasize that our scheme is looking free, we added in Figure 4 the error for the solution u_h of standard displacement formulation with continuous piecewise linear polynomials. The L^2-projection is independent of the Lam-parameter such that the corresponding lines distance in Figure 5 the same since it is independent on them.

A second numerical test is performed on the L-shape domain $\Omega = (-1,1)^2 \setminus [0,1] \times [0,-1]$, see [11]. We set $u = 0$ on the whole boundary $\partial\Omega$, $E = 1$ and consider the parameter $\nu \in \{0.4, 0.49, 0.499, 0.4999, 0.49999\}$. The right-hand side f is defined by

$$f = \begin{cases} (1,0)^T & \text{if } xy \geq 0,\ \max\{|x|,|y|\} \leq 0.5, \\ (0,0)^T & \text{else} \end{cases}.$$

A uniform refinement strategy leads to robust approximations with respect to the incompressibility parameter, but as expected achieve improved convergence rates neither for the augmented trial space nor for the post-processing, see Figures 6 and 7.

In orter to retain optimal convergence rates, the classical AFEM loop with Drfler-Marking and bulk parameter $\theta = 0.5$ is used to obtain the results for the augmented trial spaces and for the post-processing presented in Figures 8 and 9. The error estimator is the residual ε, it is proven to be reliable and efficient [9]. As expected, optimal convergence rates for the trial space $u_h^{1,0}$ is recovered, as well as for $u_h^{1,1}$ and $\tilde{u}_h^1$. Not surprisingly, the mesh refinement is concentrated in the re-entrant corner, see Figure 10. All results show that the numerical experiment is robust for $\nu \to 0.5$.

Acknowledgements. Fleurianne Bertrand gratefully acknowledge support by the DFG in the Priority Program SPP 1748 *Reliable simulation techniques in solid mechanics, Development of non standard discretization methods, mechanical and mathematical analysis* under the project number BE 6511/1-1. Henrik Schneider gratefully acknowledges support by DFG in the Priority Programme SPP 1962 *Non-smooth and Complementarity-based Distributed Parameter Systems* under grant number STA 402/13-2.

References

[1] L. Alzaben, F. Bertrand and D. Boffi, Computation of eigenvalues in linear elasticity with least-squares finite elements: dealing with the mixed system, in WCCM-ECCOMAS2020. SCIPEDIA (2021).

[2] F. Bertrand and D. Boffi, First order least-squares formulations for eigenvalue problems. *IMA J. Numer. Anal.* **42** (2022) 1339–1363.

[3] F. Bertrand and D. Boffi, Least-squares formulations for eigenvalue problems associated with linear elasticity. *Comput. Math. App.* **95** (2021) 19–27.

[4] F. Bertrand, D. Boffi and H. Schneider, Discontinuous Petrov–Galerkin approximation of eigenvalue problems. *Comput. Methods Appl. Math.* (2022). DOI: 10.1515/cmam-2022-0069.

[5] F. Bertrand and H. Schneider, Least-squares methods for linear elasticity: refined error estimates, in WCCM-ECCOMAS2020. SCIPEDIA (2021).

[6] T. Bouma, J. Gopalakrishnan and A. Harb, Convergence rates of the DPG method with reduced test space degree. *Comput. Math. Appl.* **68** (2014) 1550–1561.

[7] J. Bramwell, L. Demkowicz, J. Gopalakrishnan and W. Qiu, A locking-free *hp* DPG method for linear elasticity with symmetric stresses. *Numer. Math.* **122** (2012) 671–707.

[8] C. Carstensen, M. Eigel and J. Gedicke, Computational competition of symmetric mixed FEM in linear elasticity. *Comput. Methods Appl. Mech. Eng.* **200** (2011) 2903–2915.

[9] C. Carstensen, L. Demkowicz and J. Gopalakrishnan, A posteriori error control for DPG methods. *SIAM J. Numer. Anal.* **52** (2014) 1335–1353.

[10] C. Carstensen, L. Demkowicz and J. Gopalakrishnan, Breaking spaces and forms for the DPG method and applications including Maxwell equations. *Comput. Math. Appl.* **72** (2016) 494–522.

[11] C. Carstensen and F. Hellwig, Low-order discontinuous Petrov–Galerkin finite element methods for linear elasticity. *SIAM J. Numer. Anal.* **54** (2016) 3388–3410.

[12] L. Demkowicz and J. Gopalakrishnan, A class of discontinuous Petrov–Galerkin methods. Part I: the transport equation. *Comput. Methods Appl. Mech. Eng.* **199** (2010) 1558–1572.

[13] L. Demkowicz and J. Gopalakrishnan, A class of discontinuous Petrov–Galerkin methods. II. Optimal test functions. *Numer. Methods Part. Differ. Equ.* **27** (2011) 70–105.

[14] L. Demkowicz, J. Gopalakrishnan and A.H. Niemi, A class of discontinuous Petrov–Galerkin methods. Part III: adaptivity. *Appl. Numer. Math.* **62** (2012) 396–427.

[15] A. Ern and J.-L. Guermond, Finite Elements I – Approximation and Interpolation. Vol. 72 of *Texts in Applied Mathematics.* Springer, Cham (2021).

[16] T. Führer, Superconvergence in a DPG method for an ultra-weak formulation. *Comput. Math. Appl.* **75** (2018) 1705–1718.

[17] T. Führer, Superconvergent dpg methods for second-order elliptic problems. *Comput. Methods Appl. Math.* **19** (2019) 483–502.

[18] G. Gatica, A. Márquez and S. Meddahi, Analysis of the coupling of primal and dual-mixed finite element methods for a two-dimensional fluid-solid interaction problem. *SIAM J. Numer. Anal.* **45** (2007) 2072–2097.

[19] G. Gatica, A. Márquez and S. Meddahi, Analysis of the coupling of lagrange and arnold-falk-winther finite elements for a fluid-solid interaction problem in three dimensions. *SIAM J. Numer. Anal.* **50** (2012) 1648–1674.

[20] J. Gopalakrishnan and J. Guzmán, Symmetric nonconforming mixed finite elements for linear elasticity. *SIAM J. Numer. Anal.* **49** (2011) 1504–1520.

[21] J. Gopalakrishnan and W. Qiu, An analysis of the practical DPG method. *Math. Comp.* **83** (2014) 537–552.

[22] P. Grisvard, Elliptic Problems in Nonsmooth Domains. Vol. 24 of *Monographs and Studies in Mathematics.* Pitman (Advanced Publishing Program), Boston, MA (1985).

[23] B. Keith, F. Fuentes and L. Demkowicz, The DPG methodology applied to different variational formulations of linear elasticity. *Comput. Methods Appl. Mech. Eng.* **309** (2016) 579–609.

[24] P.-A. Raviart and J.M. Thomas, A mixed finite element method for 2nd order elliptic problems, in Mathematical Aspects of Finite Element Methods (Proc. Conf., Consiglio Naz. delle Ricerche (C.N.R.), Rome, 1975). *Lecture Notes in Math.*. Vol. 606. Springer, Berlin (1977) 292–315.

[25] L.R. Scott and S. Zhang, Finite element interpolation of nonsmooth functions satisfying boundary conditions. *Math. Comp.* **54** (1990) 483–493.

[26] R. Stenberg, A family of mixed finite elements for the elasticity problem. *Numer. Math.* **53** (1988) 513–538.