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ANALYSIS OF COMPRESSIBLE BUBBLY FLOWS. PART I: CONSTRUCTION OF A MICROSCOPIC MODEL

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Abstract. In this note, we introduce a microscopic model for the motion of gas bubbles in a viscous fluid. By interpreting a bubble as a compressible fluid with infinite shear viscosity, we derive a pde/ode system coupling the density/velocity/pressure in the surrounding fluid with the linear/angular velocities and radii of the bubbles. We provide a 1D analogue of the system and construct an existence theory for this simplified system in a natural regularity framework. The second part of the paper is a preparatory work for the derivation of an averaged or macroscopic model.

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1. Introduction

This note is the first of two papers in which we extend the derivation of averaged compressible multiphase flows in presence of jumps at interfaces between the phases. For this purpose, we focus in these papers on the construction of 1D models describing a mixture made of a leading viscous compressible fluid transporting compressible gas bubbles. To this aim, we follow a classical scheme for deriving averaged models. Firstly, we write a so-called "microscopic" model, also refered as "local instant configuration" in the literature [8,9,14,24], where the two phases are separated and occupy disjoint domains. We prescribe equations for both phases and fix interface conditions. Secondly, we perform averaging operators on this microscopic model to derive an averaged or macroscopic model. The terms "microscopic" and "macroscopic" are borrowed from large particle systems. It should be noticed that, in our setting, the "particles" are the gas bubbles so that we keep continuum mechanics equations to describe the fluids in presence.

It is well known that the averaging method contains different severe difficulties. Beyond the writing of a relevant microscopic model, the action of mean operators on nonlinear quantities is classically problematic. In particular, *ad hoc* modelling assumptions are usually added after averaging to fix the values of some interaction terms and close the system [8,11,13,14,24]. A fully rigorous approach preventing from this problem is proposed in [3–6], see also [1,12,18] for previous tentatives. However, it is restricted to ideal interface conditions between the phases. In particular, the interface is supposed to behave as a perfect transducer: it transmits with no alteration the effort of one phase on the interface to the other phase, leading besides to a unique velocity field

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in opposition to the full Baer–Nunziato model [2]. In this series of papers, we tackle the introduction of more complex interface behavior. In particular, we shall consider different families of models for the two phases: one phase is modelled with pdes while the other one is modelled by a discrete set of odes. Surface tension effects are also considered in transmission condition at interfaces.

In this first note, we tackle the writing and analysis of the microscopic model. The second step, namely the averaging process, is the content of the next paper. This note splits then into two parts. In the next section, we propose the derivation of the 3D microscopic bubbly flow model. One originality of this model is that, focusing on the property that the bubbles remain spherical, we propose to restrict the gas equations to a set of three odes per bubble: one equation for the center of mass, one equation for the angular motion of the bubble and the last one for the radius of the bubble. One key question is then to fix the influence of the surrounding fluid in these equations. This question is now completely classical for what concerns the center of mass and angular motion of the bubbles but we found no equivalent derivation for the radius equation. For this, we propose herein to extend one method that is classically used in the case of rigid bubbles [10, 19]: we identify formally spherical compressible bubbles as a compressible viscous fluid whose shear viscosity is infinite (following [23], Lem. 1.1, Chap. 1 in the case of rigid particles). So we start from a microscopic model where the two phases are viscous compressible fluids. We write classical interface conditions: no mass transfer, continuity of velocity, jump of normal stress proportional to surface tension. In passing, we derive a global weak formulation for this set of equations. Then, we assume that the bubble remain spherical and send the shear viscosity to infinity in the gas phase and compute formally a limiting model by considering special test-functions in the weak formulation. We point out that, further than the shape dynamics of the bubbles, the above assumption restricts the possible micro-motions inside the bubble (see Prop. A.1). We are aware that this restriction borrows from droplet dynamics but we shall keep the naming bubbles throughout the paper.

With this analysis at-hand, we propose an interpretation of the different terms involved in the resulting set of equations from which we derive a 1D analogue system. This sytem reads as follows. In $\Omega = (-1, 1)$, the bubble domains are:

$$B_i = (c_i - R_i, c_i + R_i), \qquad \forall i = 1, \dots, N$$

where c_i is the center of the i-th bubble and R_i its radius. We write then the fluid equation

$$\begin{cases} \partial_t \rho_f + \partial_x(\rho_f u_f) = 0 \\ \partial_t(\rho_f u_f) + \partial_x(\rho_f u_f^2) = \partial_x \Sigma_f \end{cases} \tag{1}$$

in $\mathcal{F} := \Omega \setminus \bigcup_{i=1}^N \bar{B}_i$ and where:

$$\Sigma_f = \mu_f \partial_x u_f - p_f(\rho_f). \tag{2}$$

We denote here by $\mu_f > 0$ and $p_f : (0, \infty) \to (0, \infty)$ the viscosity and pressure respectively of the fluid phase. The fluid equations are then complemented by boundary conditions:

$$u_f(t, \pm 1) = 0 \qquad u_f(t, c_i \pm R_i) = \dot{c}_i \pm \dot{R}_i, \qquad \forall i = 1, \dots, N. \tag{3}$$

As for the gas bubbles, we obtain equations for the centers of mass c_i and radii R_i (note that in this 1D setting, there is no rotation). These equations read for $i = 1, \dots, N$:

$$\begin{cases} m_i \ddot{c}_i = \Sigma_f(t, c_i + R_i) - \Sigma_f(t, c_i - R_i) \\ \dfrac{m_i}{3} \ddot{R}_i = \Sigma_f(t, c_i + R_i) + \Sigma_f(t, c_i - R_i) - 2\Sigma_i + \kappa_i \end{cases} \tag{4}$$

where Σ_i is the gas bubble stress tensor:

$$\Sigma_i = \mu_g \frac{\dot{R}_i}{R_i} - p_g\left(\frac{m_i}{2R_i}\right). \tag{5}$$

Here we introduced the positive constants $m_i > 0$ and μ_g standing respectively for the mass (depending on i) and viscosity (independent of i) of the gas bubbles. We also introduced $\mathrm{p}_g : (0,\infty) \to (0,\infty)$ the gas pressure law. The argument of p_g is computed from the mass and radius of the bubble noting that the density is constant in each bubble. Finally the surface tension in the bubble i reads:

$$\kappa_i = \frac{\kappa}{R_i} \tag{6}$$

where $\kappa > 0$ is a given parameter.

In the last section of the paper, we construct a Cauchy theory for system (1)–(6). The above system is analogous to 1D models for the motion of point particles in a viscous compressible fluid (see [15,17,20]). However, since we consider volumic compressible bubbles, we need to address new difficulties related to the unknown time-variations of their volumes. Furthermore, having in mind the homogenization process in the companion paper, we need to address a Cauchy theory with an arbitrary number of bubbles. This motivates the following adaptation of previous results. We stick to a classical regularity framework : we have $L^\infty_t H^1_x \times (L^\infty_t H^1_x \cap L^2_t H^2_x)$ regularity for the pair (ρ_f, u_f) and H^2_t regularity of the bubble unknowns. However, the system being posed on a time-dependent unknown domain, some preliminary work is performed to design a suitable regularity framework for our solution. As in the classical case of rigid bodies moving in a viscous fluid [22], we construct our notion of solution by fixing the fluid domain with a suitable change of unknown and write the above regularity framework in this fixed-domain formulation. As classical (again) with compressible 1D equations, a good choice for fixed-domain framework is to work with time/mass lagrangian coordinates. Since there is no mass transfer through liquid/gas interfaces, this change of variable also fixes the bubble domains. Section 3 then splits into two parts. Firstly, we write the system in a fixed domain and analyse the regularity requirement in the moving frame that corresponds to a classical solution in the fixed one (see Cor. 1). In the second step, we prove local-in-time existence and uniqueness of solutions to the system in a fixed frame. We remark here that in the fixed frame (1)–(6) becomes a standard quasilinear system with non-standard boundary conditions (see (31)–(33)). We obtain then the existence/uniqueness result *via* a standard perturbation approach (see Thm. 1).

2. Derivation of microscopic model for compressible bubbly flows

In this section we provide the formal derivation of a 3D microscopic compressible model for bubbly flows. We start from a mixture of compressible fluids filling a container Ω. We assume that the mixture is made of one leading fluid – whose density/velocity/pressure are denoted (ρ_f, u_f, p_f) – that fills a subset

$$\mathcal{F} = \Omega \setminus \bigcup_{i=1}^{N} \overline{B_i}$$

which stands for the container Ω deprived from a finite number of inclusions B_i – *the bubbles*. Further on, we will assume that the B_i are balls of radius R_i. The bubbles B_i are disjoint and contain a different compressible fluid. Typically, the following model applies to a liquid (modeled by the surrounding fluid) containing gaseous drops (modeled by the bubbles). We denote (ρ_i, u_i, p_i) the respective densities/velocities/pressures of this second fluid in the bubble B_i. We impose in the model that this second phase is the same in all inclusions by assuming that the physical parameters (shear/dynamic viscosity and pressure law) do not depend on i. Namely, we write the Newtonian barotropic compressible Navier–Stokes equations for all phases, and we obtain the systems:

$$\begin{cases} \partial_t \rho_f + \mathrm{div}(\rho_f u_f) = 0 \\ \partial_t(\rho_f u_f) + \mathrm{div}(\rho_f u_f \otimes u_f) = \mathrm{div}\left[2\mu_f\left(D(u_f) - \dfrac{1}{3}\mathrm{div}\, u_f \mathbb{I}_3\right) + (\lambda_f \mathrm{div}\, u_f - p_f)\mathbb{I}_3\right] \\ p_f = p_f(\rho_f) \end{cases} \tag{7}$$

in $\mathcal{F}$, and

$$\begin{cases} \partial_t \rho_i + \mathrm{div}(\rho_i u_i) = 0 \\ \partial_t(\rho_i u_i) + \mathrm{div}(\rho_i u_i \otimes u_i) = \mathrm{div}\left[2\mu_g\left(D(u_i) - \frac{1}{3}\mathrm{div}\, u_i \mathbb{I}_3\right) + (\lambda_g \mathrm{div}\, u_i - p_i)\mathbb{I}_3\right] \\ p_i = p_g(\rho_i) \end{cases} \tag{8}$$

in B_i, for $i = 1, \ldots, N$. In these systems, we introduce (μ_f, λ_f) and (μ_g, λ_g) the respective shear and volume viscosities of the two phases. The index f stands for "fluid" and g for "gas". We use similar conventions for the pressure laws p_f and p_g. We also introduce the symbol D to denote the symmetric part of the gradient:

$$D(u) = \frac{1}{2}\big(\nabla u + \nabla^\top u\big).$$

The symbol div stands for the classical divergence of vector-fields. In case we apply the divergence to a matrix-application, it stands for the straightforward extension that one obtains by applying the vector operator row-wise.

We prescribe then the continuity of velocities through the interfaces between both fluids and a jump of stress due to a (constant) surface-tension. Precisely, we set:

$$\begin{cases} u_f - u_i = 0 \\ (\Sigma_f - \Sigma_g)n = \kappa_i n \end{cases} \quad \text{on } \partial B_i \ \text{ for } i = 1, \ldots, N, \tag{9}$$

In these conditions, we denote with n the normal to ∂B_i and we use the shortcut:

$$\Sigma_f = 2\mu_f\left(D(u_f) - \frac{1}{3}\mathrm{div}\, u_f \mathbb{I}_3\right) + (\lambda_f \mathrm{div}\, u_f - p_f)\mathbb{I}_3$$

and the corresponding definition for Σ_g. We emphasize that, with this convention, we can rewrite the momentum equation for the fluid phase:

$$\partial_t(\rho_f u_f) + \mathrm{div}(\rho_f u_f \otimes u_f) = \mathrm{div}\, \Sigma_f$$

and similarly with the gas phase. The κ_i are positive constants modelling the surface tension at the interface ∂B_i. It can be related to the state of B_i (further on we will assume that it is a function of the bubble radius) but it is constant over ∂B_i. We also prescribe that the fluid and bubble domains follow the characteristics associated with velocities u_f and u_i:

$$\begin{cases} \partial_t \mathbb{1}_{\mathcal{F}} + u_f \cdot \nabla \mathbb{1}_{\mathcal{F}} = 0 \\ \partial_t \mathbb{1}_{B_i} + u_i \cdot \nabla \mathbb{1}_{B_i} = 0 \end{cases} \quad \text{in } \Omega. \tag{10}$$

We complement the system with boundary conditions:

$$u_f = 0 \quad \text{on } \partial\Omega.$$

Since we have (9) we note that, if u_f and u_i are sufficiently smooth, we keep the property that the $(B_i)_{i=1,\ldots,N}$ together with $\mathcal{F}$ realize a partition of Ω.

We enforce now the further assumption that the B_i are balls of gas with a constant density. We denote X_i the center of B_i and R_i its radius. Since the state of the bubbles is completly fixed by their centers and radii, we propose to reduce the coupled problem (7)–(8)–(9)–(10) to a coupled system in terms of (ρ_f, u_f, p_f) and $((X_i, R_i))_{i=1,\ldots,N}$. Our derivation is based on the following remark.

Proposition 1. *Let $X \in \mathbb{R}^3$ and $R > 0$. If $u \in H^1(B(X,R))$ satisfies*

$$D(u) - \frac{1}{3}\mathrm{div}\, u \mathbb{I}_3 = 0 \quad \text{on } B(X,R)$$

then $u \in C^\infty(\bar{B}(X,R))$. *If we assume furthermore that:*

$$(u(x) - u(X)) \cdot n = cstt, \qquad \text{on } \partial B(X,R)$$

there exists $(V,\omega,\Lambda) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}$ *such that:*

$$u(x) = V + \omega \times (x - X) + \frac{\Lambda}{3}(x - X), \qquad \forall x \in B(X,R).$$

This proposition is an extension of Lemma 1.1, Chapter 1 in [23]. A proof is provided in Appendix A. We point out that we choose the normalization factor 1/3 so that:

$$\operatorname{div}\left(\frac{\Lambda}{3}(x - X)\right) = \Lambda.$$

With this proposition, we can interpret formally a bubble as a compressible fluid with infinite shear viscosity. Then, for our derivation we propose to reverse the method yielding a weak formulation for fluid/solid interaction system, see [21]. Namely, first, we write a unified weak formulation for the coupled system in terms of a (composite) density/velocity/pressure (ρ, u, p). Assuming that the bubbles remain spherical, we send then formally μ_g to ∞ and compute a reduced system in terms of $((X_i, R_i)_{i=1,\dots,N}, (\rho_f, u_f, p_f))$ with a good choice of test functions.

2.1. Unified system

Assume that $(\rho_i, u_i, p_i)_{i=1,\dots,N}$ with (ρ_f, u_f, p_f) is a classical solution to (7)–(8)–(9)–(10). Let us define:

$$\rho := \rho_f \mathbb{1}_{\mathcal{F}} + \sum_{i=1}^N \rho_i \mathbb{1}_{B_i}, \qquad u := \mathbb{1}_{\mathcal{F}} u_f + \sum_{i=1}^N \mathbb{1}_{B_i} u_i, \qquad p := p_f \mathbb{1}_{\mathcal{F}} + \sum_{i=1}^N p_i \mathbb{1}_{B_i}.$$

Then, because of the coupling condition (9), we can formally combine (7) and (8) with (10) to derive that (ρ, u, p) satisfies:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) = \operatorname{div}\Sigma + \nabla \kappa \\ \Sigma = 2\mu\left(D(u) - \dfrac{1}{3}\operatorname{div} u \mathbb{I}_3\right) + (\lambda \operatorname{div} u - p)\mathbb{I}_3 \end{cases} \quad \text{on } \Omega \tag{11}$$

with

$$\mu = \mu_f \mathbb{1}_{\mathcal{F}} + \sum_{i=1}^N \mu_g \mathbb{1}_{B_i}, \quad \lambda = \lambda_f \mathbb{1}_{\mathcal{F}} + \sum_{i=1}^N \lambda_g \mathbb{1}_{B_i}, \quad \kappa = \sum_{i=1}^N \kappa_i \mathbb{1}_{B_i}.$$

To address the well-posedness of the derived system, we should complement our system with a pressure law:

$$p = p(\mathbb{1}_{\mathcal{F}}, \rho), \qquad p(c,\rho) = c\, p_f(\rho) + (1-c) p_g(\rho)$$

where $\mathbb{1}_{\mathcal{F}}$ and $\mathbb{1}_{B_i}$ are all solutions of the transport equation (with generic unknown $\mathbb{1}$)

$$\partial_t \mathbb{1} + u \cdot \nabla \mathbb{1} = 0.$$

In particular, assuming that there exist functions $q_f : (0,\infty) \to (0,\infty)$ and $q_g : (0,\infty) \to (0,\infty)$ such that:

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{\mathrm{q}_f(z)}{z}\right] = \frac{p_f(z)}{z^2} \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}z}\left[\frac{\mathrm{q}_g(z)}{z}\right] = \frac{\mathrm{p}_g(z)}{z^2}, \qquad \text{on } (0,\infty),$$

we derive that the "composite potential energy":

$$q := \mathrm{q}(\mathbb{1}_{\mathcal{F}}, \rho), \qquad q(c,\rho) = c\,\mathrm{q}_f(\rho) + (1-c)\mathrm{q}_g(\rho).$$

satisfies the equation:

$$\partial_t q + \operatorname{div}(qu) = p \operatorname{div} u. \tag{12}$$

Hence, multiplying the second equation of (11) with u and combining with (12), we conclude that

$$e := \mathrm{e}(\rho, u, q) = \frac{1}{2}\rho|u|^2 + q,$$

satisfies:

$$\partial_t e + \operatorname{div}\left(\rho u \frac{|u|^2}{2} - (\Sigma + \kappa \mathbb{I}_3)u\right) + 2\mu\left|D(u) - \frac{1}{3}\operatorname{div} u \mathbb{I}_3\right|^2 + \lambda|\operatorname{div} u|^2 + [\kappa \cdot \nabla]u = 0. \tag{13}$$

We underline that the state-law that we write above for the composite pressure and the composite energy is reminiscent of the idea that the indicator function plays the role of an order parameter c (see [7]).

2.2. Identification of bubbles and their mechanical properties.

In (13), the κ term can be handled by the positive dissipation. Hence, setting $\mu_g = \infty$, we conclude that

$$D(u) - \frac{1}{3}\operatorname{div} u = 0 \quad \text{ on } B_i \text{ for all } i.$$

With the further assumption that B_i remains spherical, we enforce also that

$$(u(x) - u(X_i)) \cdot (x - X_i) = 2\dot{R}_i R_i \quad \text{ on } \partial B_i$$

and thus, from Proposition 1, there exists $(V_i, \omega_i, \Lambda_i) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}$ for which

$$u(t,x) = V_i + \omega_i \times (x - X_i) + \frac{\Lambda_i}{3}(x - X_i), \quad \text{ on } B_i.$$

We recall here that X_i is chosen to be the center of B_i (and we denote by R_i its radius). In particular, we can replace u by this identity in the transport equation (10) satisfied by $\mathbb{1}_{B_i}$. Solving the characteristics problem associated with the right-hand side, we get that, at time $t > 0$, there holds $B_i = B(X_i(t), R_i(t))$ where X_i, R_i are computed by integrating the odes:

$$\dot{X}_i = V_i, \qquad \dot{R}_i = \frac{\Lambda_i}{3} R_i.$$

For later purpose, we introduce now some mechanical quantities characterizing the momentums of the bubbles. Since $\rho_i \mathbb{1}_{B_i}$ is a solution to:

$$\partial_t(\rho_i \mathbb{1}_{B_i}) + \operatorname{div}(\rho_i \mathbb{1}_{B_i} u_i) = 0$$

and we assumed ρ_i is constant on B_i, we get that $\dot{\rho}_i = -\Lambda_i \rho_i$. Next, we introduce the three important quantities associated with the kinetic energy related to the possible motion of B_i. First, we define the mass:

$$m_i = \int_{B_i} \rho_i = \frac{4\pi}{3}\rho_i R_i^3. \tag{14}$$

Given the differential equations satisfied by ρ_i and R_i, we get that m_i is a constant, independent of time-evolution (as could be expected). Second, we introduce the inertia matrix $\mathbb{J}_i \in \mathcal{M}_3(\mathbb{R})$ standing for the unique (positive) symmetric matrix such that:

$$(\mathbb{J}_i \omega) \cdot \tilde{\omega} = \int_{B_i} \rho_i (\omega \times (x - X_i)) \cdot (\tilde{\omega} \times (x - X_i))\,\mathrm{d}x, \quad \forall (\omega, \tilde{\omega}) \in \mathbb{R}^3 \times \mathbb{R}^3. \tag{15}$$

We will see below that this matrix is related to the computation of the kinetic energy coming from the bubble rotations. In the case of a homogeneous ball that we consider here, we have:

$$\mathbb{J}_i = \frac{8\pi}{15}\rho_i R_i^3 \mathbb{I}_3 = \frac{2}{5} m_i R_i^2 \mathbb{I}_3.$$

In particular, we remark that contrary to the case of a rigid ball, this quantity is time-dependent. Finally, we introduce $K_i \in \mathbb{R}$ the equivalent characteristics to $\mathbb{J}_i$ measuring the contribution of dilation to kinetic energy. It reads:

$$K_i = \int_{B_i} \frac{\rho_i}{9} |x - X_i|^2 \, \mathrm{d}x \tag{16}$$

and satisfies:

$$K_i = \frac{4\pi}{45}\rho_i R_i^5 = \frac{m_i}{15} R_i^2.$$

According to the previous formulas, we have some specific algebraic identities that will come into play in future computations. First, given $(\omega, \tilde{\omega}) \in \mathbb{R}^3 \times \mathbb{R}^3$, applying the classical formula:

$$\tilde{\omega} \cdot \omega = (\omega \cdot e)(\tilde{\omega} \cdot e) + (\omega \times e) \cdot (\tilde{\omega} \times e) \qquad \forall e \in \mathbb{S}^2$$

with $e = (x - X_i)/|x - X_i|$ and integrating in space, we obtain that

$$(\mathbb{J}_i \omega) \cdot \tilde{\omega} + \int_{B_i} \rho_i (\omega \cdot (x - X_i))(\tilde{\omega} \cdot (x - X_i)) = \int_{B_i} \rho_i \omega \cdot \tilde{\omega} |x - X_i|^2. \tag{17}$$

Second, differentiating with respect to time the explicit formulas for $\mathbb{J}_i$ and K_i and applying the differential equations for R_i (as well as the fact that m_i is constant) we deduce that:

$$\dot{K}_i = \frac{2}{3}\Lambda_i K_i, \qquad \dot{\mathbb{J}}_i = \frac{2}{3}\Lambda_i \mathbb{J}_i. \tag{18}$$

To conclude, we illustrate our definition of $(m_i, \mathbb{J}_i, K_i)$, by simply stating that, whatever the value of $(V, \omega, \Lambda) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}$, there holds:

$$\int_{B_i} \rho_i \left(V + \omega \times (x - X_i) + \frac{\Lambda}{3}(x - X_i) \right) \cdot \left(V + \omega \times (x - X_i) + \frac{\Lambda}{3}(x - X_i) \right) \mathrm{d}x = m_i |V|^2 + (\mathbb{J}_i \omega) \cdot \omega + K_i |\Lambda|^2.$$

2.3. Extraction of dynamical equations for (X_i, R_i)

We want now to understand the dynamics of the bubbly flow in the regime $\mu_g = \infty$. We first remark that, by construction, we keep the dynamical equations in the fluid domain (7) as well as the continuity of fluid velocities, with the restriction:

$$u = V_i + \omega_i \times (x - X_i) + \frac{\Lambda_i}{3}(x - X_i), \quad \text{on } \partial B_i, \quad \forall i = 1, \dots, N. \tag{19}$$

where:

$$V_i = \dot{X}_i, \quad \Lambda_i = 3\frac{\dot{R}_i}{R_i}, \qquad \forall i = 1, \dots, N. \tag{20}$$

To proceed, we still assume that the B_i are disjoint far from $\partial\Omega$, we consider a distribution of velocities $(\tilde{V}_i, \tilde{\omega}_i, \tilde{\Lambda}_i) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}$ and we construct a w that vanishes on $\partial\Omega$ and such that:

$$w(t, x) = \tilde{V}_i + \tilde{\omega}_i \times (x - X_i) + \frac{\tilde{\Lambda}_i}{3}(x - X_i), \qquad \text{on } B_i, \quad \forall i = 1, \dots, N.$$

We note that, though the distribution of velocities is time-independent, we obtain a time-dependent w because the B_i move inside the fluid domain (with time-varying radii and centers). Multiplying (11) with w and integrating by parts, we obtain that:

$$\int_\Omega (\partial_t(\rho u) + \mathrm{div}(\rho u \otimes u)) \cdot w = -\int_\Omega \Sigma : D(w) + \int_\Omega \nabla\kappa \cdot w.$$

We compute now independently the left-hand side and right-hand side of this identity. On the right-hand side, we apply the definition of Σ:

$$\begin{aligned} \mathrm{RHS} = &-\Bigg(\int_{\mathcal{F}} \Sigma_f : D(w) + \sum_{i=1}^N \int_{B_i} 2\mu_g\Big(D(u) - \frac{1}{3}\mathrm{div}\, u\mathbb{I}_3\Big) : \Big(D(w) - \frac{1}{3}\mathrm{div}\, w\mathbb{I}_3\Big) \\ &+ \int_{B_i} (\lambda_g \mathrm{div}\, u - p_g(\rho_i))\mathrm{div}\, w + \int_{B_i} \kappa_i \mathrm{div}\, w\Bigg). \end{aligned}$$

Here, we argue that $D(w) - (\mathrm{div}\, w/3)\mathbb{I}_3 = 0$ so that the term in factor of μ_g vanishes. Then, we integrate by parts the first identity and replace w by its explicit value. This yields:

$$\begin{aligned} \mathrm{RHS} = \int_{\mathcal{F}} \mathrm{div}\Sigma_f \cdot w - \sum_{i=1}^N \Bigg(&\int_{\partial B_i} \Sigma_f n \cdot \tilde{V}_i + \int_{\partial B_i} ((x - X_i) \times \Sigma_f n) \cdot \tilde{\Omega}_i \\ &+ \int_{\partial B_i} ((x - X_i) \cdot \Sigma_f n)\frac{\tilde{\Lambda}_i}{3} + (\lambda_g \Lambda_i - p_g(\rho_i) + \kappa_i)\tilde{\Lambda}_i |B_i|\Bigg) \end{aligned}$$

where n stands for the normal to $\partial\mathcal{F}$ that points outwards. As for the left-hand side, we split:

$$\mathrm{LHS} = \int_{\mathcal{F}} (\partial_t(\rho_f u_f) + \mathrm{div}(\rho_f u_f \otimes u_f)) \cdot w + \sum_{i=1}^N \mathrm{LHS}_i$$

with

$$\mathrm{LHS}_i = \int_{B_i} \partial_t(\rho_i u_i) \cdot w + \mathrm{div}(\rho_i u_i \otimes u_i) \cdot w.$$

Applying that B_i moves with the velocity-field u_i, we can integrate by parts

$$\mathrm{LHS}_i = \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{B_i} \rho_i u_i \cdot w\right] - \int_{B_i} \rho_i u_i \cdot (\partial_t w + u_i \cdot \nabla w).$$

In this identity, we use the explicit formulas:

$$\begin{cases} u_i = V_i + \omega_i \times (x - X_i) + \dfrac{\Lambda_i}{3}(x - X_i), \\ w = \tilde{V}_i + \tilde{\omega}_i \times (x - X_i) + \dfrac{\tilde{\Lambda}_i}{3}(x - X_i), \\ \partial_t w + u_i \cdot \nabla w = \tilde{\omega}_i \times \Big(\omega_i \times (x - X_i) + \dfrac{\Lambda_i}{3}(x - X_i)\Big) + \dfrac{\tilde{\Lambda}_i}{3}\Big(\omega_i \times (x - X_i) + \dfrac{\Lambda_i}{3}(x - X_i)\Big), \end{cases}$$

and, after tedious but straightforward computations, we obtain

$$\int_{B_i} \rho_i u_i \cdot w = m_i V_i \cdot \tilde{V}_i + (\mathbb{J}_i \omega_i) \cdot \tilde{\omega}_i + K_i \Lambda_i \tilde{\Lambda}_i,$$

and

$$\int_{B_i} \rho_i u_i \cdot (\partial_t w + u_i \cdot \nabla w) = T_1 + T_2 + T_3,$$

where

$$T_1 = \int_{B_i} \rho_i V_i \cdot (\partial_t w + u_i \cdot \nabla w) = 0,$$

because X_i is the center of B_i, and

$$\begin{aligned} T_2 &= \frac{\Lambda_i}{3}(\mathbb{J}_i \omega_i) \cdot \tilde{\omega}_i + \frac{\tilde{\Lambda}_i}{3}(\mathbb{J}_i \omega_i) \cdot \omega_i, \\ T_3 &= \frac{\Lambda_i}{3} \int_{B_i} \rho_i \tilde{\omega}_i \cdot (x - X_i)\, \omega_i \cdot (x - X_i) - \int_{B_i} \rho_i \tilde{\omega}_i \cdot \omega_i |x - X_i|^2 + \frac{\tilde{\Lambda}_i}{3} K_i |\Lambda_i|^2. \end{aligned}$$

Here, we apply (17) to yield that:

$$T_2 + T_3 = \frac{\tilde{\Lambda}_i}{3}\Big((\mathbb{J}_i \omega_i) \cdot \omega_i + K_i |\Lambda_i|^2 \Big).$$

Finally, this entails that:

$$\mathrm{LHS}_i = \frac{\mathrm{d}}{\mathrm{d}t}\Big[m_i V_i \cdot \tilde{V}_i + (\mathbb{J}_i \omega_i) \cdot \tilde{\omega}_i + K_i \Lambda_i \tilde{\Lambda}_i \Big] - \frac{\tilde{\Lambda}_i}{3}\Big[(\mathbb{J}_i \omega_i) \cdot \omega_i + K_i |\Lambda_i|^2 \Big].$$

Combining the previous computations for LHS and RHS and recalling that (ρ_f, u_f, p_f) satisfies the Navier–Stokes equations on $\mathcal{F}$, we conclude that:

$$\begin{aligned} &\sum_{i=1}^N \frac{\mathrm{d}}{\mathrm{d}t}\Big[m_i V_i \cdot \tilde{V}_i + (\mathbb{J}_i \omega_i) \cdot \tilde{\omega}_i + K_i \Lambda_i \tilde{\Lambda}_i \Big] - \frac{\tilde{\Lambda}_i}{3}\big[(\mathbb{J}_i \omega_i) \cdot \omega_i + K_i |\Lambda_i|^2 \big] \\ &\qquad = -\sum_{i=1}^N \Bigg(\int_{\partial B_i} \Sigma_f n \cdot \tilde{V}_i + \int_{\partial B_i} ((x - X_i) \times \Sigma_f n) \cdot \tilde{\omega}_i \\ &\qquad\qquad + \int_{\partial B_i} ((x - X_i) \cdot \Sigma_f n) \frac{\tilde{\Lambda}_i}{3} + (\lambda_g \Lambda_i - p_g(\rho_i) + \kappa_i) \tilde{\Lambda}_i |B_i| \Bigg). \end{aligned}$$

Choosing sequentially that only $\tilde{V}_i$ or $\tilde{\omega}_i$ or $\tilde{\Lambda}_i$ does not vanish, we end up with the system:

$$\left\{ \begin{aligned} & m_i \dot{V}_i = -\int_{\partial B_i} \Sigma_f n, \\ & \frac{\mathrm{d}}{\mathrm{d}t}[\mathbb{J}_i \omega_i] = -\int_{\partial B_i} (x - X_i) \times (\Sigma_f n), \end{aligned} \right. \tag{21}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}[K_i \Lambda_i] - \frac{1}{3}\Big[(\mathbb{J}_i \omega_i) \cdot \omega_i + K_i |\Lambda_i|^2 \Big] = -\Bigg(\frac{1}{3} \int_{\partial B_i} (x - X_i) \cdot \Sigma_f n + (\lambda_g \Lambda_i - p_g(\rho_i) + \kappa_i) |B_i| \Bigg). \tag{22}$$

The two first equations are the classical Newton laws of solid dynamics. The latter one is new to our knowledge. We point out that, in the last identity, all the quantities can be computed in terms of R_i and $\dot{R}_i$. In particular, the second term on the left-hand side is a geometrical term that is induced by the fact that $\mathbb{J}_i$ and K_i are time-dependant because of the time evolution of R_i.

2.4. Conclusion

We conclude with a reformulation of our system in terms of the only unknown (ρ_f, u_f, p_f) for the fluid and $(X_i, R_i)_{i=1,\ldots,N}$ for the bubbles. Concerning the fluid, we have:

$$\begin{cases} \partial_t \rho_f + \operatorname{div}(\rho_f u_f) = 0, \\ \partial_t(\rho_f u_f) + \operatorname{div}(\rho_f u_f \otimes u_f) = \operatorname{div} \Sigma_f, \end{cases}$$

in the fluid domain

$$\mathcal{F}(t) = \Omega \setminus \bigcup_{i=1}^{N} B_i,$$

where $B_i = B(X_i, R_i)$. This system is completed with boundary conditions:

$$\begin{cases} u_f = \dot{X}_i + \omega_i \times (x - X_i) + \dfrac{\dot{R}_i}{R_i}(x - X_i), & \text{on } \partial B_i, \\ u_f = 0, & \text{on } \partial\Omega. \end{cases} \tag{23}$$

It is also coupled with the dynamical equations for the bubbles:

$$\begin{cases} m_i \ddot{X}_i = -\displaystyle\int_{\partial B_i} \Sigma_f n, \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\left[\dfrac{2m_i}{5} R_i^2 \omega_i\right] = -\displaystyle\int_{\partial B_i} (x - X_i) \times (\Sigma_f n), \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\left[\dfrac{m_i}{5}\dot{R}_i R_i\right] - \dfrac{m_i R_i^2}{3}\left[\dfrac{2}{5}\omega_i^2 + \dfrac{3}{5}\left|\dfrac{\dot{R}_i}{R_i}\right|^2\right] = -\dfrac{1}{3}\displaystyle\int_{\partial B_i} ((x - X_i) \cdot \Sigma_f n) \\ \qquad\qquad - \left(3\lambda_g \dfrac{\dot{R}_i}{R_i} - p_g\left(\dfrac{3m_i}{4\pi R_i^3}\right) + \kappa_i\right)\dfrac{4}{3}\pi R_i^3 \end{cases} \tag{24}$$

where m_i is the mass of bubble B_i, λ_g is its volumic viscosity, p_g its pressure law and Σ_f is the fluid stress tensor given by

$$\Sigma_f = 2\mu_f\left(D(u_f) - \frac{1}{3}\operatorname{div} u_f \mathbb{I}_3\right) + (\lambda_f \operatorname{div} u_f - p_f)\mathbb{I}_3, \quad p_f = p_f(\rho_f). \tag{25}$$

We mention here that, in these latter Newton-like equations, the geometric term on the left-hand side of the last equation handles the fact that the inertia parameters of B_i depend on R_i. As for the right-hand side, it measures the different actions of the bubbles and fluid. We note in particular that, in the last equation, the last term can be rewritten:

$$-\frac{1}{3}\int_{\partial B_i} (x - X_i) \cdot (\Sigma_f - \Sigma_g - \kappa_i) n \,\mathrm{d}\sigma.$$

The factor $1/3$ is here a dimensional artefact due to the algebra relating the dependencies of u_i and $\operatorname{div} u_i$ in Λ_i. In particular, the right-hand side is not properly the surface force applied on ∂B_i (because the $(x - X_i)$ term appearing in the latter integral is homogeneous to a length).

2.5. 1D analogue

We propose now a 1D analogue to the system derived previously. So, we consider a mixture that fills the 1D container $\Omega = (-1, 1)$. We denote the bubbles

$$B_i = (c_i - R_i, c_i + R_i), \qquad \forall i = 1, \ldots, N,$$

where $c_i \in \Omega$, $R_i > 0$ for all i, and

$$\mathcal{F} = \Omega \setminus \bigcup_{i=1}^{N} \overline{B_i}.$$

Following the previous notations, we introduce (ρ_f, u_f) the fluid density/velocity which solve the 1D compressible Navier–Stokes system:

$$\begin{cases} \partial_t \rho_f + \partial_x(\rho_f u_f) = 0 \\ \partial_t(\rho_f u_f) + \partial_x\left(\rho_f u_f^2\right) = \partial_x \Sigma_f \\ \Sigma_f = \mu_f \partial_f u_f - p_f(\rho_f). \end{cases}$$

The motion of the bubbles is given by:

$$u_i(t,x) = \dot{c}_i + \frac{\dot{R}_i}{R_i}(x - c_i), \qquad \text{on } B_i$$

so that we add the no-slip boundary conditions:

$$\begin{cases} u(t, c_i \pm R_i) = \dot{c}_i \pm \dot{R}_i \qquad \text{for } i = 1, \ldots, N \\ u(t, \pm 1) = 0. \end{cases}$$

For the bubble dynamics, we introduce ρ_i the density and m_i the mass of the bubble B_i. Requiring again that m_i is constant, we obtain that the kinetic energy of the bubble B_i is given by:

$$\frac{1}{2}\int_{B_i} \rho_i \left(\dot{c}_i + \frac{\dot{R}_i}{R_i}(x - c_i) \right)^2 \mathrm{d}x = \frac{1}{2} m_i |\dot{c}_i|^2 + \frac{1}{2}\frac{m_i R_i^2}{3}\left|\frac{\dot{R}_i}{R_i}\right|^2.$$

So, following the computations in the 3D case, we propose the following extended Newton laws (note that there is no rotation here):

$$\begin{cases} m_i \ddot{c}_i = \Sigma_f(c_i + R_i) - \Sigma_f(c_i - R_i), \\ \dfrac{m_i}{3}\dfrac{\mathrm{d}}{\mathrm{d}t}\left[\dot{R}_i R_i\right] - \dfrac{m_i}{3}\left|\dot{R}_i\right|^2 = R_i[(\Sigma_f(c_i + R_i) + \Sigma_f(c_i - R_i)) - 2\Sigma_i + \kappa_i], \end{cases}$$

where the bubble stress tensor is given by:

$$\Sigma_i = \mu_g \frac{\dot{R}_i}{R_i} - \mathrm{p}_g\left(\frac{m_i}{2R_i}\right), \qquad \text{on } B_i.$$

We note that the second equation simplifies:

$$\frac{m_i}{3}\ddot{R}_i = [(\Sigma_f(c_i + R_i) + \Sigma_f(c_i - R_i)) - 2\Sigma_i + \kappa_i].$$

Below, we assume that κ_i is computed for bubbles thanks to a Laplace–Young law ([16], Chap. VII):

$$\kappa_i = \frac{\kappa}{R_i} \qquad \text{for some paramater } \kappa > 0.$$

Hence, from now on, we drop κ_i in the system and we incorporate the surface-tension effects in the pressure law p_g.

3. Cauchy theory for 1D compressible bubbly flows

In this section, we address the existence and uniqueness of classical solutions to the 1D compressible bubbly-flow model that we derived in the previous section. We focus on the particular case of polytropic pressure laws (in the fluid and in the bubbles). Precisely, we consider the sytem with unknowns (ρ, u) and $(c_i, R_i)_{i=1,\dots,N}$ given by the 1D Navier–Stokes equations:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2) = \partial_x(\mu_f \partial_x u - p_f(\rho)), \end{cases} \quad \text{on } (-1,1) \setminus \bigcup_{i=1}^{N} [c_i - R_i, c_i + R_i], \tag{26}$$

with the boundary conditions:

$$\begin{cases} u(t, c_i \pm R_i) = \dot{c}_i \pm \dot{R}_i, & \text{for } i = 1, \dots, N, \\ u(t, \pm 1) = 0, \end{cases} \tag{27}$$

and complemented with generalized Newton laws:

$$\begin{cases} m_i \ddot{c}_i = [\mu_f \partial_x u - p_f(\rho)]_{c_i \pm R_i}, \\ \dfrac{m_i}{3} \ddot{R}_i = \{\mu_f \partial_x u - \mathrm{p}_f(\rho)\}_{c_i \pm R_i} - 2\left(\mu_g \dfrac{\dot{R}_i}{R_i} - p_g(R_i)\right), \end{cases} \quad \forall i = 1, \dots, N. \tag{28}$$

In this system, we introduced the physical parameters: μ_f, p_f (resp. m_i, μ_g, p_g) characterizing the fluid properties (resp. the bubble properties). We emphasize that m_i stands for the mass of the bubble i and can vary between the bubbles. We also constructed the pressure law with respect to the radius R_i since, the mass being conserved and the density constant in the bubbles, the density of each bubble is directly related to its radius. This enables to incorporate also surface tension effects thanks to a Laplace–Young law for instance. Finally, for a function f defined on the fluid domain, we denoted:

$$[f]_{c_i \pm R_i} = (f(c_i + R_i) - f(c_i - R_i)), \qquad \{f\}_{c_i \pm R_i} = (f(c_i + R_i) + f(c_i - R_i)), \quad \forall i = 1, \dots, N.$$

We complete the system with initial conditions:

$$\begin{aligned} &\begin{cases} c_i(0) = c_i^0, & \dot{c}_i(0) = \dot{c}_i^0, \\ R_i(0) = R_i^0, & \dot{R}_i(0) = \dot{R}_i^0, \end{cases} \quad \forall i = 1, \dots, N, \\ &\begin{cases} \rho(0,x) = \rho_0(x), \\ u(0,x) = u_0(x), \end{cases} \quad \forall x \in (-1,1) \setminus \bigcup_{i=1}^{N} \left(c_i^0 - R_i^0, c_i^0 + R_i^0\right). \end{aligned} \tag{29}$$

We state this system formally, but we obviously need that the bubble domains $(c_i - R_i, c_i + R_i)$, $i = 1, \dots, N$, remain well separated initially and with time-evolution so that all these equations are meaningful.

This section splits into two parts. Firstly, we provide a functional framework for solving (26)–(27)–(28)–(29). To this aim, we use a Lagrangian formulation with time/mass coordinates and provide a unified formulation of the fluid+bubble system. We prove the equivalence between the two formulations and we state our main result. The last subsection is devoted to the proof of our main result.

3.1. Functional setting and main result

We assume that the full system has unit mass. This property does not restrict the generality up to a scaling of the viscosity and pressure laws.

As classical in the 1D setting, we look for a solution to (26)–(27)–(28) in the time/mass coordinates:

$$\left(t, m = \int_{-1}^{x} \bar{\rho}(t,z)\,\mathrm{d}z\right) \tag{30}$$

where $\bar{\rho}$ is the extension of ρ by $\rho_i = m_i/2R_i$ in the bubble domains. Denoting

$$m_i^{\pm} = \int_{-1}^{c_i \pm R_i} \bar{\rho}(t,z)\,\mathrm{d}z, \qquad \forall i = 1,\dots,N,$$

and introducing the specific volume $v = 1/\rho$, our system reads in these new coordinates:

$$\begin{cases} \partial_t v = \partial_m u, \\ \partial_t u = \partial_m \left(\dfrac{\partial_m u}{v} - \pi_f(v) \right), \end{cases} \quad \text{on } \mathcal{F}_0 := (0,1) \setminus \bigcup_{i=1}^{N} [m_i^-, m_i^+], \tag{31}$$

with the boundary conditions:

$$\begin{cases} u(m_i^{\pm}) = \dot{c}_i \pm \dot{R}_i, & \text{for } i = 1,\dots,N, \\ u(0) = u(1) = 0, \end{cases} \tag{32}$$

and complemented with generalized Newton laws:

$$\begin{cases} m_i \ddot{c}_i = \left[\dfrac{\mu_f}{v} \partial_m u - \pi_f(v) \right]_{m_i^-}^{m_i^+}, \\ \dfrac{m_i}{3} \ddot{R}_i = \left\{ \dfrac{\mu_f}{v} \partial_m u - \pi_f(v) \right\}_{m_i^-}^{m_i^+} - 2\left(\mu_g \dfrac{\dot{R}_i}{R_i} - p_g(R_i) \right), \end{cases} \quad \forall i = 1,\dots,N, \tag{33}$$

with previous notations for brackets. We also introduced in this system the specific volume pressure law $\pi_f(v) = p_f(1/v)$ for $v > 0$. We point out that, by construction, m_i^+ and m_i^- are constant and that $m_i^+ = m_i^- + m_i$ for all i.

In order to handle the previous Lagrangian formulation of our problem, we work with extended unknowns. First, we set:

$$\bar{v}(t,m) = \begin{cases} v(t,m), & \text{in } \mathcal{F}_0, \\ \dfrac{2R_i}{m_i^+ - m_i^-}, & \text{in } (m_i^-, m_i^+) \text{ for } i = 1,\dots,N, \end{cases}$$

and

$$\bar{u}(t,m) = \begin{cases} u(t,m), & \text{in } \mathcal{F}_0, \\ \left(\dot{c}_i - \dot{R}_i\right) + 2\dot{R}_i \dfrac{m - m_i^-}{m_i^+ - m_i^-}, & \text{in } (m_i^-, m_i^+) \text{ for } i = 1,\dots,N. \end{cases}$$

In the classical regularity setting for Navier–Stokes equations, we look for a solution such that the velocity-field u has $C_t H_x^1 \cap L_t^2 H_x^2$ regularity in the fluid domain. Correspondingly, we must have that v is $W_t^{1,\infty} L_x^2 \cap H_t^1 H_x^1$. On the other hand, the bubbles prescribe that v is constant and u affine in the intervals (m_i^-, m_i^+). Combining both remarks leads to the following construction of function spaces. We introduce the symbol $\mathbf{m}$ which encodes the list of interval $((m_i^-, m_i^+))_{i=1,\dots,N}$ and we denote, for $p \in [1,\infty]$:

$$L^p_{\mathbf{m}} := \left\{ v \in L^p((0,1)) \text{ s.t. } v_{|_{(m_i^-, m_i^+)}} \text{ is constant } \forall i \right\},$$

as well as, for $k \geq 1$:

$$\mathbb{H}^k_{\mathbf{m}} := \left\{ u \in H_0^k((0,1)) \text{ s.t. } u_{|_{(m_i^-, m_i^+)}} \text{ is affine } \forall i \right\}.$$

We emphasize that, *a priori* $L^2_{\mathbf{m}}$ and $\mathbb{H}^0_{\mathbf{m}}$ do not coincide. In particular, contrary to the classical setting, the differential operator ∂_m maps $\mathbb{H}^1_{\mathbf{m}}$ into $L^2_{\mathbf{m}}$ (and not $\mathbb{H}^0_{\mathbf{m}}$). With the above remarks, we will require that a solution satisfies

$$\begin{aligned} &\bar{v} \in H^1\big(0,T;H^1(\mathcal{F}_0)\big) \cap C\big([0,T];L^2_{\mathbf{m}}\big), \\ &\bar{u} \in H^1\big(0,T;L^2(0,1)\big) \cap C\big([0,T];\mathbb{H}^1_{\mathbf{m}}\big) \cap L^2\big(0,T;H^2(\mathcal{F}_0)\big). \end{aligned} \tag{34}$$

We encode then the full system (31)–(32)–(33) into one system in terms of $\bar{v},\bar{u}$. Indeed, if $((u,v),(c_i,R_i)_{i=1,\dots,N})$ is a solution to (31)–(32)–(33) we first remark that we can compute explicitly $\partial_t\bar{v}$ and $\partial_m\bar{u}$ on the bubbles (m_i^-,m_i^+). This entails that we have the transport equation:

$$\partial_t\bar{v} = \partial_m\bar{u} \tag{35}$$

on the whole interval $(0,1)$. Second, given $\bar{w} \in \mathbb{H}^1_{\mathbf{m}}$ we multiply the momentum equation in (31) with $\bar{w}$. After integration by parts and application of the continuity condition (32) and extended Newton laws (33), we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^1 \bar{u}\bar{w}\right] + \int_0^1 \Big(\frac{\mu}{\bar{v}}\partial_m\bar{u} - \pi(m,\bar{v})\Big)\partial_m\bar{w} = 0 \tag{36}$$

where:

$$\mu = \mu_f\mathbb{1}_{\mathcal{F}_0} + \mu_g(1-\mathbb{1}_{\mathcal{F}_0}) \quad \pi(m,v) = \mathbb{1}_{\mathcal{F}_0}\pi_f(v) + \sum_{i=1}^N \mathbb{1}_{(m_i^-,m_i^+)}\pi_i(v) \tag{37}$$

with $\pi_i(v) = \mathrm{p_g}(m_iv/2)$ for $i = 1,\dots,N$. We point out that, contrary to the Eulerian setting, we cannot write the pressure law in terms of the indicator functions $\mathbb{1}_{\mathcal{F}_0}$ only, since the quantity m_i is involved in the computation.

In conclusion, the solutions we are looking for are pairs $(\bar{u},\bar{v})$ with the regularity (34) such that $\bar{v} > 0$ on $(0,T)\times(0,1)$ and that satisfy simultaneously (35) on $(0,T)\times(0,1)$ and (36) on $(0,T)$ for any $\bar{w} \in \mathbb{H}^1_{\mathbf{m}}$. We perform our construction with viscosity and pressure law of the form (37). In that respect, our main result reads:

Theorem 1. *Consider viscosities* $(\mu_f,\mu_g) \in (0,\infty)$ *and pressure laws* $(\pi_f,(\pi_i)_{i=1,\dots,N})$ *that are* C^1 *on* $(0,\infty)$. *Then, given* $\bar{v}_0 \in L^\infty_{\mathbf{m}} \cap H^1(\mathcal{F}_0)$ *such that* $\inf_{(0,1)}\bar{v}_0 > 0$ *and* $\bar{u}_0 \in \mathbb{H}^1_{\mathbf{m}}$, *there exists* $T_0 > 0$ *(depending only on* $\inf_{(0,1)}\bar{v}_0, \|\bar{v}_0\|_{L^\infty((0,1))}, \|\bar{v}_0\|_{H^1(\mathcal{F}_0)}, \|\bar{u}_0\|_{H^1((0,1))}$*) such that, for arbitrary* $T \in (0,T_0)$, *there is a unique pair* $(\bar{u},\bar{v})$ *satisfying*

(i) *condition* (34) *with* $\inf_{(0,T)\times(0,1)}\bar{v} \geq \min\bar{v}_0/2$,
(ii) *equations* (35) *on* $(0,T)\times(0,1)$, *and* (36) *on* $(0,T)$ *for any* $\bar{w} \in \mathbb{H}^1_{\mathbf{m}}$,
(iii) *initial condition* $\bar{v}(0,\cdot) = \bar{v}_0$ *and* $\bar{u}(0,\cdot) = \bar{u}_0$ *on* $(0,1)$.

We provide a proof of this result in the next section. Before going to this content, we note that it implies existence and uniqueness of solutions to our initial system. Namely, we have the following corollary:

Corollary 1. *Consider viscosities* $(\mu_f,\mu_g) \in (0,\infty)$, *strictly positive masses* $(m_i)_{i=1,\dots,N}$ *and pressure laws* $(p_f,\mathrm{p_g})$ *that are* C^1 *on* $(0,\infty)$. *Then, assume that initial data* $(c_i^0,R_i^0)_{i=1,\dots,N}$ *and* $(\dot{c}_i^0,\dot{R}_i^0)_{i=1,\dots,N}$ *ensure the non-overlap condition:*

$$\begin{aligned} &\big(c_i^0 - R_i^0, c_i^0 + R_i^0\big) \cap \big(c_j^0 - R_j^0, c_j^0 + R_j^0\big) = \emptyset, && \forall i \neq j \\ &R_i^0 > 0, \quad \big(c_i^0 - R_i^0, c_i^0 + R_i^0\big) \Subset (-1,1), && \forall i. \end{aligned}$$

Denoting $\mathcal{F}^0 := (-1,1) \setminus \cup_{i=1}^N(c_i^0 - R_i^0, c_i^0 + R_i^0)$, *assume also that initial data* $(\rho_0,u_0) \in H^1(\mathcal{F}^0) \times H^1(\mathcal{F}^0)$ *satisfy the compatibility conditions:*

$$\int_{\mathcal{F}^0} \rho_0 + \sum_{i=1}^N m_i = 1, \qquad \min\rho_0 > 0,$$

$$u_0\big(c_i^0 \pm R_i^0\big) = \dot{c}_i^0 \pm \dot{R}_i^0, \qquad \forall i = 1, \dots, N, \quad u_0(\pm 1) = 0.$$

Then, there exists $T_0 > 0$ (depending only on $\inf_{(0,1)} \rho_0, \min R_i, \|\rho_0\|_{H^1(\mathcal{F}^0)}, \|u_0\|_{H^1(\mathcal{F}^0)}$ and the minimal distance between bubbles and between bubbles and container boundaries) such that, for any $T < T_0$ there exists a unique $((\rho, u), (c_i, R_i)_{i=1,\dots,N})$ satisfying:

(a) *$(c_i, R_i) \in H^2(0,T)$ with the non-overlap condition on $(0,T)$:*

$$\begin{aligned} &(c_i - R_i, c_i + R_i) \cap (c_j - R_j, c_j + R_j) = \emptyset, && \forall i \neq j, \\ &R_i > 0, \quad (c_i - R_i, c_i + R_i) \Subset (-1, 1), && \forall i, \end{aligned}$$

(b) *the pair (ρ, u) has the regularity:*

$$\begin{aligned} (\rho, u) &\in H^1\left(\bigcup_{t \in (0,T)} \{t\} \times \left((-1,1) \setminus \bigcup_{i=1}^N (c_i - R_i, c_i + R_i) \right) \right) \\ \partial_{xx} u &\in L^2\left(\bigcup_{t \in (0,T)} \{t\} \times \left((-1,1) \setminus \bigcup_{i=1}^N (c_i - R_i, c_i + R_i) \right) \right), \end{aligned}$$

(c) *(ρ, u) satisfies equations (26)–(27)–(28) almost everywhere on their respective set of definitions and initial condition (29).*

Proof. Consider initial data as in the previous statement and construct

$$\bar{\rho}_0 = \left(1 - \sum_{i=1}^N \mathbb{1}_{(c_i^0 - R_i^0, c_i^0 + R_i^0)} \right) \rho_0 + \sum_{i=1}^N \frac{m_i}{2R_i^0} \mathbb{1}_{(c_i^0 - R_i^0, c_i^0 + R_i^0)}.$$

Then set:

$$m_i^{\pm} = \int_{-1}^{c_i^0 \pm R_i^0} \bar{\rho}_0(z)\,\mathrm{d}z, \quad m_0(x) = \int_{-1}^{x} \bar{\rho}_0(z)\,\mathrm{d}z.$$

We note that, by construction m_0 is continuous piecewise C^1 with $\inf \partial_x m_0 \geq C_0 > 0$. This shows that m_0 realizes a one-to-one mapping between $(-1,1)$ and $(0,1)$ with an inverse continuous piecewise C^1 mapping. We can then fix:

$$\bar{v}_0(m) = \frac{1}{\bar{\rho}_0(m_0^{-1}(m))} \qquad \forall m \in (0,1).$$

Under the assumption that ρ_0 has H^1-regularity outside the bubbles, we get an initial data $\bar{v}_0 \in L^\infty_{\mathbf{m}} \cap H^1(\mathcal{F}_0)$. Similarly, we set:

$$\bar{u}_0(m) = u_0\big(m_0^{-1}(m)\big) \qquad \forall m \in (0,1) \setminus \bigcup_{i=1}^N \big(m_i^-, m_i^+\big).$$

With the regularity of u_0 and m_0^{-1} there holds:

$$\bar{u}_0 \in H^1\left((0,1) \setminus \bigcup_{i=1}^N \big(m_i^-, m_i^+\big) \right).$$

Furthermore, since initial data are assumed to satisfy the no-slip condition on fluid/bubble interfaces, we have:

$$\bar{u}_0\big(m_i^{\pm}\big) = u_0\big(c_i^0 \pm R_i^0\big) = \dot{c}_i^0 \pm \dot{R}_i^0.$$

Consequently, extending $\bar{u}_0$ with:

$$\bar{u}_0(m) = \left(\dot{c}_i^0 - \dot{R}_i^0\right) + \frac{2\left(m - m_i^-\right)}{m_i^+ - m_i^-} R_i^0 \qquad \forall m \in \left(m_i^-, m_i^+\right)$$

we obtain $\bar{u}_0 \in \mathbb{H}^1_{\mathbf{m}}$. We have constructed initial data $(\bar{v}_0, \bar{u}_0)$ that match the regularity assumptions of Theorem 1. We have then at-hand $T > 0$ and $(\bar{u}, \bar{v})$ that satisfy items (i)–(ii)–(iii). We show now that there is a correspondence between these solutions and a $((\rho, u), (c_i, R_i)_{i=1,\dots,N})$ satisfying the items (a)–(b)–(c) of our corollary.

Let consider $(\bar{v}, \bar{u})$ the solution constructed *via* Theorem 1. Thanks to the transport equation satisfied by $\bar{v}$ and homogeneous boundary conditions for $\bar{u}$ we have that the mass of $\bar{v}$ is constant on $(0, T)$. We can then construct:

$$x(t, m) = -1 + \int_0^m \bar{v}(t, \zeta)\,\mathrm{d}\zeta.$$

With this definition, x is piecewise C^1 on $(0, T) \times (0, 1)$ with:

$$\partial_t x = \bar{u} \quad \partial_m x = \bar{v}.$$

In particular, for arbitrary $t \in (0, 1)$, we have that x realizes a $W^{1,\infty}$-change of variables with $W^{1,\infty}$-inverse mapping. Actually, for $t = 0$ we can replace $\partial_m x(0, \cdot)$ with the initial change of unknowns to realize that $x(0, \cdot) = m_0^{-1}$. Since the mass of $\bar{v}$ does not depend on time, we have that $x(t, \cdot)$ realizes an homeomorphism between $(0, 1)$ and $(-1, 1)$ for all $t \in (0, T)$.

At this point, for $i = 1, \dots, N$, we fix:

$$R_i = \frac{1}{2} \int_{m_i^-}^{m_i^+} \bar{v}(t, m)\,\mathrm{d}m \quad c_i(t) = c_i^0 + \int_0^t \frac{1}{m_i} \int_{m_i^-}^{m_i^+} \bar{u}(s, m)\,\mathrm{d}m\,\mathrm{d}s.$$

We note here that $\bar{v}$ is constant on (m_i^-, m_i^+) as well as $\partial_m \bar{u}$. In particular, the $H^1(0, T; L^2(m_i^-, m_i^+))$-regularity of $\bar{u}$ implies that $R_i \in H^2(0, T)$. Similarly, we obtain that:

$$\dot{c}_i = \frac{1}{m_i} \int_{m_i^-}^{m_i^+} \bar{u}(t, m)\,\mathrm{d}m,$$

inherits the $H^1(0, T)$-regularity of $\bar{u}$ so that $c_i \in H^2(0, T)$.

We prove now that the non-overlap condition holds on $(0, T)$. First, since $\bar{v}$ is bounded from above and by below, we have that $R_i > 0$. Second, we fix $B_i = x((m_i^-, m_i^+))$. Since x is an homeomorphism between $(0, 1)$ and $(-1, 1)$ we obtain that B_i is an interval and that $B_i \cap B_j = \emptyset$ for $i \neq j$ and $B_i \Subset (-1, 1)$ for all i. It remains to prove that $B_i = (c_i - R_i, c_i + R_i)$ on $(0, T)$. To this end, let denote x_i^-, x_i^+ the endpoints of B_i. We emphasize that these are continuous time-dependent functions. We have:

$$x_i^+ - x_i^- = \int_{m_i^-}^{m_i^+} \bar{v}(t, \zeta)\,\mathrm{d}\zeta = 2R_i.$$

Then, applying the definition of the change of variable, the transport equation for $\bar{v}$ together with the fact that it is constant on (m_i^-, m_i^+) (equal to $2R_i/m_i$) we infer that:

$$\frac{1}{2R_i} \int_{x_i^-}^{x_i^+} z\,\mathrm{d}z = \frac{1}{2R_i} \int_{m_i^-}^{m_i^+} x(t, \zeta)\bar{v}(t, \zeta)\,\mathrm{d}\zeta = \frac{1}{m_i} \int_{m_i^-}^{m_i^+} x(t, \zeta)\,\mathrm{d}\zeta.$$

We can thus differentiate wrt time to yield that:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{2R_i}\int_{x_i^-}^{x_i^+} z\,\mathrm{d}z\right] = \frac{1}{m_i}\int_{m_i^-}^{m_i^+} \partial_t x(t,\zeta)\,\mathrm{d}\zeta = \frac{1}{m_i}\int_{m_i^-}^{m_i^+} \bar{u}(t,\zeta)\,\mathrm{d}\zeta = \dot{c}_i.$$

Since we have initially that $B_i = (c_i^0 - R_i^0, c_i^0 + R_i^0)$ we conclude that:

$$\frac{1}{2R_i}\int_{x_i^-}^{x_i^+} z\,\mathrm{d}z = c_i \qquad \text{on } (0,T),$$

and $B_i = (c_i - R_i, c_i + R_i)$. This concludes the proof of (a).

To obtain (b), we simply argue by change of variables. For $t \in (0,T)$ and $y \in (-1,1)$, we fix:

$$u(t,y) = \bar{u}(t, x^{-1}(t,y)) \qquad \rho(t,y) = \frac{1}{\bar{v}(t, x^{-1}(t,y))}.$$

By classical change of variable arguments (since $x(t,\cdot)$ and $x^{-1}(t,\cdot)$ are $L^\infty_t W^{1,\infty}_x$ with norms bounded by $\sup \bar{v}$ and $\inf \bar{v}$) we obtain then that u, ρ enjoy the regularity of (b). In particular, we have that:

$$\partial_x u(t, x(t,m)) = \frac{\partial_m \bar{u}(t,m)}{\bar{v}(t,m)} \qquad \partial_t u(t,x(t,m)) + u(t,x(t,m))\partial_x u(t,x(t,m)) = \partial_t \bar{u}(t,m).$$

To conclude with (c), we first reproduce the change of variables above to yield that outside the $(c_i - R_i, c_i + R_i)$ there holds

$$\frac{1}{\rho(t,y)^2}(\partial_t \rho(t,x) + u(t,x)\partial_x \rho(t,x)) = -\partial_t \bar{v}(t, x^{-1}(t,y)) \quad \frac{\partial_x u(t,y)}{\rho(t,y)} = \partial_m \bar{u}(t, x^{-1}(t,y)),$$

and thus we have:

$$\partial_t \rho + \partial_x(\rho u) = 0 \quad \text{ on } (-1,1) \setminus \bigcup (c_i - R_i, c_i + R_i).$$

Finally, we can plug in (36) any $\bar{w} \in C^\infty_c(\mathcal{F}_0)$. Equation (36) then implies that we have:

$$\partial_t \bar{u} = \partial_m\left(\frac{\mu_f}{\bar{v}}\partial_m \bar{u} - \pi_f(\bar{v})\right) \text{ on } (0,1) \setminus \bigcup (m_i^-, m_i^+)$$

and, by change of variable:

$$\partial_t(\rho u) + \partial_x(\rho u^2) = \partial_x(\mu_f \partial_x u - p_f(\rho_f)) \text{ on } (-1,1) \setminus \bigcup (c_i - R_i, c_i + R_i).$$

Furthermore, by taking arbitrary $\bar{w}$ that is affine on (m_i^-, m_i^+) we get (33) and, after change of variable (28). Thanks to the existence part of Theorem 1 we have finally the existence part of our corollary.

The proof of uniqueness follows the same line. From a candidate solution $((\rho,u),(c_i,R_i)_{i=1,\ldots,N})$ we construct the extension:

$$\bar{\rho}(t,x) = \rho(t,x)\left(1 - \sum_{i=1}^N \mathbb{1}_{(c_i - R_i, c_i + R_i)}\right) + \sum_{i=1}^N \frac{m_i}{2R_i}\mathbb{1}_{(c_i - R_i, c_i + R_i)}$$

and construct the change of variable m as in the formula (30). A similar analysis as previously yields that m is a $W^{1,\infty}$-homeomorphism between $(-1,1)$ and $(0,1)$ with $W^{1,\infty}$ inverse. We construct then:

$$\bar{v}(t,\zeta) = \frac{1}{\bar{\rho}(t, m^{-1}(t,\zeta))}$$

and

$$\bar{u}(t,\zeta)=\begin{cases} u\big(t,m^{-1}(t,\zeta)\big), & \text{if } \zeta\in(0,1)\setminus\bigcup(m_i^-,m_i^+),\\ \Big(\dot{c}_i-\dot{R}_i\Big)+\dfrac{2\dot{R}_i(\zeta-m_i^-)}{(m_i^+-m_i^-)}, & \text{in } \big(m_i^-,m_i^+\big).\end{cases}$$

With the regularity of (b) we obtain with similar computations as in the previous part of the proof that

$$\bar{u}\in H^1\left((0,T)\times\left((0,1)\setminus\bigcup_{i=1}^N(m_i^-,m_i^+)\right)\right)\quad \partial_{mm}\bar{u}\in L^2\left((0,T)\times\left((0,1)\setminus\bigcup_{i=1}^N(m_i^-,m_i^+)\right)\right).$$

In particular, there holds:

$$\bar{u}\in H^1(0,T;L^2(0,1))\cap L^2(0,T;H^2(\mathcal{F}_0)),$$

and thus $\bar{u}\in C([0,T];H^1(0,1)-w)$. To obtain continuity for the strong topology, we prove now that $\|\bar{u}\|_{H^1((0,1))}$ is continuous. For this, we remark that we already have continuity of the L^2-norm and of the H^1-norm of the restriction to the bubbles. So, we focus on the $\|\partial_x\bar{u}\|_{H^1((m_i^+,m_{i+1}^-))}$, for $i\in\{1,\dots,N-1\}$ (computations on $(0,m_1^-)$ and $(m_N^+,1)$ are similar).

We fix i and $0<t_1<t_2<T$. By a multiplier argument and computation of the traces of $\partial_t\bar{u}$ on m_i^+,m_{i+1}^-, we infer:

$$\begin{aligned}\frac{1}{2}&\left[\int_{m_i^+}^{m_{i+1}^-}|\partial_m\bar{u}(t_2,\zeta)|^2-\int_{m_i^+}^{m_{i+1}^-}|\partial_m\bar{u}(t_1,\zeta)|^2\right]\\ &=\int_{t_1}^{t_2}[\partial_t\bar{u}(t,\cdot)\partial_m\bar{u}(t,\cdot)]_{m_i^+}^{m_{i+1}^-}-\int_{t_1}^{t_2}\int_{m_i^-}^{m_i^+}\partial_t\bar{u}(t,\zeta)\partial_{mm}\bar{u}(t,\zeta)\,\mathrm{d}\zeta\\ &=\int_{t_1}^{t_2}\Big[\Big(\ddot{c}_{i+1}-\ddot{R}_{i+1}\Big)\partial_m\bar{u}\big(t,m_{i+1}^-\big)-\Big(\ddot{c}_i+\ddot{R}_i\Big)\partial_m\bar{u}\big(t,m_i^+\big)\Big]-\int_{t_1}^{t_2}\int_{m_i^-}^{m_i^+}\partial_t\bar{u}(t,\zeta)\partial_{mm}\bar{u}(t,\zeta)\,\mathrm{d}\zeta.\end{aligned}$$

Hence, introducing:

$$\phi_i(t,m)=\Big(\ddot{c}_i+\ddot{R}_i\Big)+\frac{(\ddot{c}_{i+1}-\ddot{c}_i)-\Big(\ddot{R}_{i+1}+\ddot{R}_i\Big)}{m_{i+1}^--m_i^+}\big(m-m_i^+\big)\qquad\forall m\in\big[m_i^-,m_i^+\big]$$

we obtain finally that:

$$\begin{aligned}\frac{1}{2}\left[\int_{m_i^+}^{m_{i+1}^-}|\partial_m\bar{u}(t_2,\zeta)|^2-\int_{m_i^+}^{m_{i+1}^-}|\partial_m\bar{u}(t_1,\zeta)|^2\right]&=\int_{t_1}^{t_2}\int_{m_i^-}^{m_i^+}\partial_m\phi_i(t,\zeta)\partial_m\bar{u}(t,\zeta)+\int_{t_1}^{t_2}\int_{m_i^-}^{m_i^+}\phi_i(t,\zeta)\partial_{mm}\bar{u}(t,\zeta)\\ &\quad-\int_{t_1}^{t_2}\int_{m_i^-}^{m_i^+}\partial_t\bar{u}(t,\zeta)\partial_{mm}\bar{u}(t,\zeta)\,\mathrm{d}\zeta.\end{aligned}$$

At this point we argue that $\phi_i\in L^2(0,T;H^1(m_i^+,m_{i+1}^-))$ because of the time-regularity of the c_i and R_i. Since we also have

$$\partial_t\bar{u},\partial_m\bar{u},\partial_{mm}\bar{u}\in L^2\big((0,T)\times\big(m_i^+,m_{i+1}^-\big)\big),$$

the latter identity entails that:

$$\lim_{t_1-t_2\to 0}\left[\int_{m_i^+}^{m_{i+1}^-}|\partial_m\bar{u}(t_2,\zeta)|^2\,\mathrm{d}\zeta-\int_{m_i^+}^{m_{i+1}^-}|\partial_m\bar{u}(t_1,\zeta)|^2\,\mathrm{d}\zeta\right]=0.$$

We conclude that $\bar{u} \in C([0,T];\mathbb{H}^1_{\mathbf{m}})$. Hence, up to restrict T, we have that $(\bar{u},\bar{v})$ satisfies item (i) of Theorem 1. We can then reproduce the computations in the introduction of this section to yield that $(\bar{v},\bar{u})$ satisfies also the item (ii) and that it satisfies the item (iii). We apply now the uniqueness part of Theorem 1 to yield that $(\bar{v},\bar{u})$ is the solution provided by Theorem 1. This concludes the proof. $\square$

3.2. Proof of Theorem 1

We conclude this section by providing a proof of Theorem 1 on the basis of a standard perturbation approach. In the whole section we fix a N-uplet $m_1,\dots,m_N$ of strictly positive masses and

$$\bar{v}_0 \in L^\infty_{\mathbf{m}} \cap H^1(\mathcal{F}_0), \quad \bar{u}_0 \in \mathbb{H}^1_{\mathbf{m}},$$

such that $\inf_{(0,1)} \bar{v}_0 > 0$. We construct solutions as a fixed-point of a mapping $\mathcal{C} : (\tilde{v},\tilde{u}) \mapsto (\bar{v},\bar{u})$ on some sufficiently small time-interval $(0,T)$. Precisely, we fix $T>0$. Given $K \in (0,\infty)$ we denote by $\bar{S}_T[K]$ the set of pairs $(\bar{u},\bar{v})$ satisfying (34) with $\bar{u}(0,\cdot)=\bar{u}_0$, $\bar{v}(0,\cdot)=v_0$ and

$$\frac{1}{2}\inf_{(0,1)} \bar{v}_0 \le \inf_{(0,T)\times(0,1)} \bar{v} \le \sup_{(0,T)\times(0,1)} \bar{v} \le 2\sup_{(0,1)} \bar{v}_0$$

$$\sup_{(0,T)} \|\bar{v}\|_{L^2((0,1))} + \sup_{(0,T)} \|\bar{v}\|_{H^1(\mathcal{F}_0)} \le 2\Big(\|\bar{v}_0\|_{L^2((0,1))} + \|\bar{v}_0\|_{H^1(\mathcal{F}_0)}\Big)$$

$$\int_0^T \|\partial_t \bar{v}\|^2_{L^\infty((0,1))} \le 2K^2$$

$$\sup_{(0,T)} \|\bar{u}\|^2_{H^1((0,1))} + \int_0^T \|\partial_{mm}\bar{u}\|^2_{L^2(\mathcal{F}_0)} \le K^2.$$

With standard arguments, we have that, whatever the values of $T>0$ and $K \in (0,\infty)$, the set $\bar{S}_T[K]$ is a convex complete metric space when endowed with the distance:

$$d_S((\bar{v}_1,\bar{u}_1),(\bar{v}_2,\bar{u}_2)) = \sup_{(0,T)}\Big(\|\bar{v}_1-\bar{v}_2\|_{L^2((0,1))} + \|\bar{u}_1-\bar{u}_2\|_{L^2((0,1))}\Big) + \left(\int_0^T \|\partial_m(\bar{u}_2-\bar{u}_1)\|^2_{L^2(0,1)}\right)^{\frac{1}{2}}.$$

Below, we endow $\bar{S}_T[K]$ with this topology. Now for fixed $K \in (0,\infty)$ we define $\mathcal{C}$ on $\bar{S}_T[K]$ as follows. Given $(\tilde{v},\tilde{u}) \in \bar{S}_T[K]$, we compute $(\bar{v},\bar{u}) := \mathcal{C}[(\tilde{v},\tilde{u})]$ in two steps. First $\bar{v}$ is the solution to

$$\begin{cases} \partial_t \bar{v} = \partial_m \tilde{u} & \text{on } (0,T)\times(0,1) \\ \bar{v}(0,\cdot) = \bar{v}_0 & \text{on } (0,1). \end{cases} \tag{38}$$

Then, we compute $\bar{u}$ by solving the weak formulation:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left[\displaystyle\int_0^1 \bar{u}\bar{w}\right] + \displaystyle\int_0^1 \Big(\frac{\mu}{\tilde{v}}\partial_m\bar{u} - \pi(m,\tilde{v})\Big)\partial_m\bar{w} = 0 & \text{for all } w \in \mathbb{H}^1_{\mathbf{m}}, \\ \bar{u}(0,\cdot) = \bar{u}_0 & \text{on } (0,1). \end{cases} \tag{39}$$

We recall here that μ and π are defined by (37).

With these conventions, our proof reduces to the following proposition:

Proposition 2. *There exists a constant C^0_∞ depending only on $\sup_{(0,1)} \bar{v}_0, \inf_{(0,1)} \bar{v}_0$ and the physical parameters of the system such that, defining*

$$K^0 = \Big[C^0_\infty\Big(\|\bar{u}_0\|^2_{H^1((0,1))} + 1\Big)\Big]^{\frac{1}{2}} \tag{40}$$

there exists $T_0>0$ such that $\mathcal{C}$ realizes a contraction on $\bar{S}_{T_0}[K^0]$.

We split the proof of this proposition into two lemmas whose proofs are detailed below.

Lemma 1. *Let $K^0 > 0$ be fixed. There exists $T_0 > 0$ (depending only on K^0, $\inf_{(0,1)} \bar{v}_0$, $\|\bar{v}_0\|_{L^2((0,1))}$, $\|\bar{v}_0\|_{H^1(\mathcal{F}_0)}$ and $\|\bar{u}_0\|_{H^1((0,1))}$) such that, given $T < T_0$ the following statements hold true.*

(a) *For any $(\tilde{v}, \tilde{u}) \in \bar{S}_T[K^0]$, there exists a unique*

$$\bar{v} \in C([0,T]; L^2_{\mathbf{m}}) \cap H^1\big(0,T; H^1(\mathcal{F}_0)\big).$$

solution to (38).

(b) *Moreover the solution $\bar{v}$ satisfies:*
   – *the uniform bounds:*

$$\frac{1}{2}\inf_{(0,1)} \bar{v}_0 \leq \inf_{(0,T)\times(0,1)} \bar{v} \leq \sup_{(0,T)\times(0,1)} \bar{v} \leq 2\sup_{(0,1)} \bar{v}_0,$$
$$\sup_{(0,T)} \|\bar{v}\|_{L^2((0,1))} + \sup_{(0,T)} \|\bar{v}\|_{H^1(\mathcal{F}_0)} \leq 2\Big(\|\bar{v}_0\|_{L^2((0,1))} + \|\bar{v}_0\|_{H^1(\mathcal{F}_0)}\Big),$$

   – *the control from above:*

$$\int_0^T \|\partial_t \bar{v}\|^2_{L^\infty((0,1))} \leq 2\big|K^0\big|^2.$$

(c) *Furthermore, for any pairs $(\tilde{v}_1, \tilde{u}_1) \in \bar{S}_T[K^0]$ and $(\tilde{v}_2, \tilde{u}_2) \in \bar{S}_T[K^0]$ we have (with obvious notations)*

$$\sup_{(0,T)} \|\bar{v}_2 - \bar{v}_1\|_{L^2((0,1))} \leq \frac{1}{4}\left(\int_0^T \|\partial_m(\tilde{u}_2 - \tilde{u}_1)\|^2_{L^2((0,1))}\right)^{\frac{1}{2}}.$$

Lemma 2. *There exists a constant C^0_∞ (depending only on $\sup_{(0,1)} \bar{v}_0, \inf_{(0,1)} \bar{v}_0$ and physical parameters) and $T_0 > 0$ (depending only on $\inf_{(0,1)} \bar{v}_0$, $\|\bar{v}_0\|_{L^2((0,1))}$, $\|\bar{v}_0\|_{H^1(\mathcal{F}_0)}$ and $\|\bar{u}_0\|_{H^1((0,1))}$) for which, fixing K^0 by (40) and $T < T_0$, the following statements hold true.*

(a) *For any $(\tilde{v}, \tilde{u}) \in \bar{S}_T[K^0]$ there exists a unique*

$$\bar{u} \in H^1\big(0,T; L^2((0,1))\big) \cap C\big([0,T]; \mathbb{H}^1_{\mathbf{m}}\big) \cap L^2\big(0,T; H^2(\mathcal{F}_0)\big)$$

solution to (39).

(b) *Moreover the solution $\bar{u}$ satisfies:*

$$\sup_{(0,T)} \|\bar{u}\|^2_{H^1((0,1))} + \left(\int_0^T \|\partial_{mm}\bar{u}\|^2_{L^2((0,1))}\right) \leq C^0_\infty\Big(\|\bar{u}_0\|^2_{H^1((0,1))} + 1\Big).$$

(c) *Furthermore, for all pairs $(\tilde{v}_1, \tilde{u}_1) \in \bar{S}_T[K^0]$ and $(\tilde{v}_2, \tilde{u}_2) \in \bar{S}_T[K^0]$ we have (with obvious notations)*

$$\sup_{(0,T)} \|\bar{u}_2 - \bar{u}_1\|_{L^2((0,1))} + \left(\int_0^T \|\partial_m(\bar{u}_2 - \bar{u}_1)\|^2_{L^2((0,1))}\right)^{\frac{1}{2}} \leq \frac{1}{4}\sup_{(0,T)} \|\bar{v}_2 - \bar{v}_1\|_{L^2((0,1))}.$$

Proposition 2 yields as a straightforward combination of these two lemmas. So, we end up this section with the proof of these lemmas.

Proof of Lemma 1. To begin with, we consider Lemma 1. We pick $T > 0$. We shall comment on the smallness of T which will fix T_0 in order that (a), (b) and (c) hold simultaneously.

We start with (a) and (b). For this, we fix $(\tilde{v}, \tilde{u})$ in $\bar{S}_T[K_0]$. Equation (38) is integrated straightforwardly:

$$\bar{v}(t,m) = \bar{v}_0 + \int_0^t \partial_m \bar{u}(t,\zeta)\,\mathrm{d}\zeta.$$

Since $\bar{v}_0 \in L^\infty_{\mathbf{m}} \subset L^2_{\mathbf{m}}$ and, by differentiation, $\partial_m \bar{u} \in C([0,T]; L^2_{\mathbf{m}})$, there holds $\bar{v} \in C([0,T]; L^2_{\mathbf{m}})$. Furthermore, we have $\bar{v}_0 \in H^1(\mathcal{F}_0)$ and, by differentiation again, $\partial_m \bar{u} \in L^2(0,T; H^1(\mathcal{F}_0))$, so that $\bar{v} \in H^1(0,T; H^1(\mathcal{F}_0))$. In conclusion, $\bar{v}$ satisfies the expected regularity. This completes the proof of (a).

We proceed with estimates. Below we introduce C_∞ that depends on the physical parameters of the system only. It may vary between lines. First, since $\bar{v}$ is constant on the (m_i^-, m_i^+) and H^1 on the complement, we have that $\bar{v}(t,\cdot)$ is piecewise continuous for all $t > 0$ and thus bounded from below. Then, we compute $\inf \bar{v}(t,\cdot)$ by considering differently $m \in (m_i^-, m_i^+)$ and $m \in \mathcal{F}_0$. First, when $m \in (m_i^-, m_i^+)$, since $\bar{v}(t,\cdot)$ is constant, there holds:

$$\inf_{(0,1)} \bar{v}_0 - \frac{1}{\sqrt{m_i}} \int_0^T \|\partial_t \bar{v}\|_{L^2((0,1))} \leq \bar{v}(t,m) = \frac{1}{m_i} \int_{m_i^-}^{m_i^+} \bar{v}(t,\zeta)\,\mathrm{d}\zeta \leq \sup_{(0,1)} \bar{v}_0 + \frac{1}{\sqrt{m_i}} \int_0^T \|\partial_t \bar{v}\|_{L^2((0,1))}.$$

Replacing $\partial_t \bar{v}$ with the time-evolution equation, we infer that:

$$\inf_{(0,1)} \bar{v}_0 - C_\infty T K^0 \leq \bar{v}(t,m) \leq \sup_{(0,1)} \bar{v}_0 + T C_\infty K^0.$$

On $\mathcal{F}_0$, we apply the embedding $H^1 \hookrightarrow L^\infty$ to yield:

$$|\bar{v}(t,m) - \bar{v}_0(t,m)| \leq C_\infty \int_0^T \Big(\|\tilde{u}\|_{H^1((0,1))} + \|\partial_{mm}\tilde{u}\|_{L^2(\mathcal{F}_0)} \Big) \leq \sqrt{T} C_\infty K^0,$$

up to assume that $T < 1$. Consequently, for T sufficiently small wrt K^0 we obtain that:

$$\frac{1}{2} \inf_{(0,1)} \bar{v}_0 \leq \inf_{(0,T)\times(0,1)} \bar{v} \leq \sup_{(0,T)\times(0,1)} \bar{v} \leq 2 \sup_{(0,1)} \bar{v}_0.$$

Then, we have:

$$\|\bar{v}(t,\cdot)\|_{L^2((0,1))} \leq \|\bar{v}_0\|_{L^2((0,1))} + \int_0^t \|\bar{u}\|_{\mathbb{H}^1_{\mathbf{m}}} \leq \|\bar{v}_0\|_{L^2((0,1))} + T K^0$$
$$\|\bar{v}(t,\cdot)\|_{H^1(\mathcal{F}_0)} \leq \|\bar{v}_0\|_{H^1(\mathcal{F}_0)} + \int_0^t \|\partial_m \bar{u}\|_{H^1(\mathcal{F}_0)} \leq \|\bar{v}_0\|_{H^1(\mathcal{F}_0)} + \sqrt{T} K^0.$$

Hence, fixing T sufficiently small wrt K^0 we obtain:

$$\|\bar{v}(t,\cdot)\|_{L^2((0,1))} + \|\bar{v}(t,\cdot)\|_{H^1(\mathcal{F}_0)} \leq 2\Big(\|\bar{v}_0\|_{L^2((0,1))} + \|\bar{v}_0\|_{H^1(\mathcal{F}_0)} \Big) \qquad \forall t \in (0,T).$$

Then, with similar computation as in the previous derivation of L^∞-bounds, we use that $\bar{v}$ is constant on any (m_i^-, m_i^+) to bound as follows:

$$\int_0^T \|\partial_t \bar{v}\|^2_{L^\infty((m_i^-, m_i^+))} \leq \frac{1}{\sqrt{m_i}} \int_0^T \|\tilde{u}\|^2_{H^1((0,1))} \leq C_\infty T \big|K^0\big|^2$$
$$\int_0^T \|\partial_t \bar{v}\|^2_{L^\infty(\mathcal{F}_0)} \leq \int_0^T \|\tilde{u}\|^2_{H^2(\mathcal{F}_0)} \leq (1+T)\big|K^0\big|^2.$$

This entails finally that, up to choose again T sufficiently small wrt K^0 we have:

$$\int_0^t \|\partial_t \bar{v}\|^2_{L^\infty((0,1))} \leq 2|K^0|^2.$$

This completes the proof of (b).

Finally, consider $(\tilde{v}_1, \tilde{u}_1)$ and $(\tilde{v}_2, \tilde{u}_2)$ in $\bar{S}_T[K^0]$. We have then:

$$\bar{v}_1(t,\zeta) = \bar{v}_0(\zeta) + \int_0^t \partial_m \tilde{u}_1(s,\zeta)\,\mathrm{d}s \qquad \bar{v}_2(t,\zeta) = \bar{v}_0(\zeta) + \int_0^t \partial_m \tilde{u}_2(s,\zeta)\,\mathrm{d}s$$

so that:

$$\sup_{(0,T)} \|\bar{v}_2 - \bar{v}_1\|_{L^2((0,1))} = \sqrt{T}\left(\int_0^T \|\partial_m(\tilde{u}_2 - \tilde{u}_1)\|^2_{L^2((0,1))}\right).$$

Hence, we get the expected property up to take $\sqrt{T}$ sufficiently small again. This completes the proof of (c). □

Proof of Lemma 2. The proof of Lemma 2 deserves a little more details since the mapping $(\tilde{v}, \tilde{u}) \to \bar{u}$ is non-linear.

First, we obtain existence of (a) and (b) *via* a Galerkin method. Indeed, we remark that $\mathbb{H}^1_{\mathbf{m}}$ is a closed subspace of $H^1_0((0,1))$ and, as such, is a separable Hilbert space. We can then introduce a linearly independent family $(\bar{w}_k)_{k\in\mathbb{N}}$ that is total in $\mathbb{H}^1_{\mathbf{m}}$. Without restriction, we can assume that $\bar{w}_k$ is smooth for arbitrary k.

Given $P \in \mathbb{N}$, we say that $\bar{u}_P$ is a P-approximate solution, if $\bar{u}_P \in C([0,T]; \langle \bar{w}_1, \dots, \bar{w}_P\rangle)$ satisfies:

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left[\displaystyle\int_0^1 \bar{u}_P \bar{w}\right] + \displaystyle\int_0^1 \Big(\frac{\mu}{\tilde{v}}\partial_m \bar{u}_P - \pi(m,\tilde{v})\Big)\partial_m \bar{w} = 0 \qquad \text{for all } \bar{w} \in \langle \bar{w}_1, \dots, \bar{w}_P\rangle, \\ \bar{u}_P(0,\cdot) = \mathbb{P}_P[\bar{u}_0]. \end{cases}$$

In this system, we denote $\mathbb{P}_P$ the projection (for the $H^1_0(0,1)$ scalar product) on $\langle \bar{w}_1, \dots, \bar{w}_P\rangle$. Decomposing $\bar{u}_P$ on the basis $\bar{w}_1, \dots, \bar{w}_P$ we remark that the construction of $\bar{u}_P$ reduces to a finite-dimensional (linear) differential system. We have then existence and uniqueness of a P-approximate solution for arbitrary $P \in \mathbb{N}$.

We prove now estimates satisfied by the P-approximate solutions for arbitrary $P \in \mathbb{N}$. We introduce below the symbol C^0_∞ for a constant that depends only on the initial quantities $\sup \bar{v}_0, \inf \bar{v}_0$ and the physical parameters of the system. It may vary between lines.

First, we remark that the system solved by $\bar{u}_P$ can be rewritten:

$$\partial_t \bar{u}_P - \partial_m \mathbb{Q}_P\Big[\frac{\mu}{\tilde{v}}\partial_m \bar{u}_P - \pi(m,\tilde{v})\Big] = 0 \tag{41}$$

where $\mathbb{Q}_p : L^2((0,1)) \to \langle w_1, \dots, w_P\rangle$ is the (continuous) linear mapping defined by the duality formula:

$$\int_0^1 \mathbb{Q}_p[\tilde{w}]\partial_m \bar{w} = \int_0^1 \tilde{w}\partial_m \bar{w} \qquad \forall(\tilde{w}, \bar{w}) \in L^2((0,1)) \times \langle \bar{w}_1, \dots, \bar{w}_P\rangle.$$

In particular, we can multiply (41) with $\bar{u}_P$. This entails that:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^1 |\bar{u}_P|^2\right] + \int_0^1 \frac{\mu}{\tilde{v}}|\partial_m \bar{u}_P|^2 = \int_0^1 \pi(m,\tilde{v})\partial_m \bar{u}_P.$$

By a standard Cauchy–Schwarz inequality, we conclude that:

$$\frac{1}{2}\sup_{(0,T)} \int_0^1 |\bar{u}_P|^2 + \frac{\min(\mu_f, \mu_g)}{2\|\tilde{v}\|_{L^\infty((0,1))}} \int_0^T \int_0^1 |\partial_m \bar{u}_P|^2 \leq \|\bar{u}_0\|^2_{H^1((0,1))} + \int_0^T \int_0^1 \frac{\tilde{v}|\pi(m,\tilde{v})|^2}{\mu}.$$

Here we argue that π is continuous on $(0,1)\times(0,\infty)$. Consequently, since $\tilde{v}$ is bounded from above and below by a constant depending only on initial data, $\tilde{v}(m), \pi(m,\tilde{v}(m))$ is also bounded on $(0,T)\times(0,1)$ by a constant depending only on initial data. We obtain then that:

$$\frac{1}{2}\sup_{(0,T)}\int_0^1 |\bar{u}_P|^2 + \frac{\min(\mu_f,\mu_g)}{4\|\bar{v}_0\|_{L^\infty((0,1))}}\int_0^T\int_0^1 |\partial_m \bar{u}_P|^2 \leq \frac{1}{2}\|\bar{u}_0\|^2_{H^1((0,1))} + TC^0_\infty. \tag{42}$$

We multiply now (41) with $\partial_t \bar{u}_P \in C([0,T];\langle \bar{w}_1,\ldots,\bar{w}_P\rangle)$. We infer that:

$$\int_0^1 |\partial_t \bar{u}_P|^2 + \int_0^1 \frac{\mu}{\tilde{v}}\partial_m \bar{u}_P \partial_m \partial_t \bar{u}_P = \int_0^1 \pi(m,\tilde{v}(m))\partial_{mt}\bar{u}_P.$$

On the left-hand side, we have:

$$\int_0^1 \frac{\mu}{\tilde{v}}\partial_m \bar{u}_P \partial_m \partial_t \bar{u}_P = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^1 \frac{\mu}{\tilde{v}}|\partial_m \bar{u}_P|^2\right] + \int_0^1 \frac{\mu}{2\tilde{v}^2}\partial_t \tilde{v}|\partial_m \bar{u}_P|^2$$

whilst we rewrite the right-hand side:

$$\int_0^1 \pi(m,\tilde{v}(m))\partial_{mt}\bar{u}_P = \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^1 \pi(m,\tilde{v})\partial_m \bar{u}_P\right] - \int_0^1 \partial_2\pi(m,\tilde{v})\partial_t\tilde{v}\partial_m \bar{u}_P.$$

We conclude thus that:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_1[\bar{u}_P] + \int_0^1 |\partial_t \bar{u}_P|^2 = -\int_0^1 \partial_t\tilde{v}\Big(\partial_2\pi(m,\tilde{v})\partial_m \bar{u}_P + \frac{\mu}{2\tilde{v}^2}|\partial_m \bar{u}_P|^2\Big) \tag{43}$$

where:

$$\mathcal{E}_1[\bar{u}_P] := \int_0^1 \frac{\mu}{2\tilde{v}}|\partial_m \bar{u}_P|^2 + \int_0^1 \pi(m,\tilde{v})\partial_m \bar{u}_P.$$

In this latter quantity, we can again use an L^∞-bound for $\tilde{v}\pi(m,\tilde{v})$. Introducing a standard Minkowski inequality, we derive that:

$$\mathcal{E}_1[\bar{u}_P] \geq \int_0^1 \frac{\mu}{4\tilde{v}}|\partial_m \bar{u}_P|^2 - \frac{\sup_{(0,1)}\tilde{v}|\pi(m,\tilde{v})|^2}{\min(\mu_f,\mu_g)} \geq \int_0^1 \frac{\mu}{4\tilde{v}}|\partial_m \bar{u}_P|^2 - C^0_\infty.$$

At this point, we bound the right-hand side RHS of (43) as follows:

$$\begin{aligned}\mathrm{RHS} &\leq \|\partial_t\tilde{v}\|_{L^\infty((0,1))}\left(\int_0^1 \frac{\tilde{v}|\partial_2\pi(m,\tilde{v})|^2}{2\mu} + \left(1+\left\|\frac{1}{\tilde{v}}\right\|_{L^\infty((0,1))}\right)\int_0^1 \frac{\mu}{2\tilde{v}}|\partial_m \bar{u}_P|^2\right)\\ &\leq C^0_\infty\|\partial_t\tilde{v}\|_{L^\infty((0,1))}(1+\mathcal{E}_1[\bar{u}_P]).\end{aligned}$$

This entails that:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}[1+\mathcal{E}_1(\bar{u}_P)] &\leq C^0_\infty\|\partial_t\tilde{v}\|_{L^\infty((0,1))}(1+\mathcal{E}_1[\bar{u}_P])\\ \int_0^1 |\partial_t \bar{u}_P|^2 &\leq C^0_\infty\|\partial_t\tilde{v}\|_{L^\infty((0,1))}(1+\mathcal{E}_1[\bar{u}_P]).\end{aligned} \tag{44}$$

Integrating the first inequality with a standard Gronwall lemma, we obtain, with the control on $\partial_t\tilde{v}$,

$$\mathcal{E}_1[\bar{u}_P] \leq (\mathcal{E}_1[\mathbb{P}_P(u_0)]+1)\exp\left(C^0_\infty\int_0^T \|\partial_t\tilde{v}\|_{L^\infty((0,1))}\right)$$

$$\leq (\mathcal{E}_1[\mathbb{P}_P(u_0)] + 1)\exp\Big(C^0_\infty \sqrt{T} K^0\Big).$$

Hence taking T sufficiently small wrt K^0, we can bound the exponential in the latter right-hand side by 2. Recalling the above bound for $\mathcal{E}_1[\bar{u}_P]$, we conclude that:

$$\frac{1}{2}\int_0^1 |\partial_m \bar{u}_P|^2 \leq C^0_\infty\Big(1 + \|\bar{u}_0\|^2_{H^1((0,1))}\Big).$$

Integrating now the second inequality of (44) we obtain that:

$$\int_0^T \int_0^1 |\partial_t \bar{u}_P|^2 \leq C^0_\infty\Big(1 + \|\bar{u}_0\|^2_{H^1((0,1))}\Big).$$

Combining (42) with the two latter inequalities, we obtain that, for T sufficiently small (depending only on the norm of initial data), we have:

$$\sup_{(0,T)} \|\bar{u}_P\|^2_{H^1((0,1))} + \int_0^T \Big(\|\partial_m \bar{u}_P\|^2_{L^2((0,1))} + \|\partial_t \bar{u}_P\|^2_{L^2((0,1))}\Big) \leq C^0_\infty\Big(\|\bar{u}_0\|^2_{H^1(\mathcal{F}_0)} + 1\Big).$$

The sequence $\bar{u}_P$ is then bounded in

$$H^1\big((0,T);L^2((0,1))\big) \cap L^\infty\big(0,T;H^1((0,1))\big) \cap L^\infty\big((0,T);\mathbb{H}^1_{\mathbf{m}}\big).$$

We can thus extract a weak converging sequence. The limit $\bar{u}$ enjoys then the inequality:

$$\sup_{(0,T)} \|\bar{u}\|^2_{H^1((0,1))} + \int_0^T \|\partial_t \bar{u}\|^2_{L^2((0,1))} \leq C^0_\infty\Big(\|\bar{u}_0\|^2_{H^1((0,1))} + 1\Big)$$

and, by standard argument (since the problem is linear in $\bar{u}$) is a solution to (39). In particular, extending the weak formulation to time-dependent $\bar{w}$, which have compact support in $\mathcal{F}_0$, we obtain that $\bar{u}$ satisfies

$$\partial_t \bar{u} = \partial_m\Big(\frac{\mu_f}{\tilde{v}}\partial_m \bar{u} - \pi_f(\tilde{v})\Big) \quad \text{on } (0,T)\times\mathcal{F}_0.$$

Consequently, we have:

$$\frac{\mu_f}{\tilde{v}}\partial_m \bar{u} - \pi_f(\tilde{v}) \in L^2\big(0,T;H^1(\mathcal{F}_0)\big) \text{ and thus } \partial_m \bar{u} \in L^2(0,T;L^\infty(\mathcal{F}_0))$$

with (because $\tilde{v}$ is bounded by initial data):

$$\int_0^T \|\partial_m \bar{u}\|^2_{L^2((0,1))} \leq C^0_\infty\Big(\|\bar{u}_0\|^2_{H^1(\mathcal{F}_0)} + 1\Big).$$

We rewrite then the pde on $\mathcal{F}_0$ as:

$$\partial_{mm}\bar{u} = \frac{\tilde{v}}{\mu_f}\Big(\partial_t \bar{u} + \pi_f'(\tilde{v})\partial_m \tilde{v} + \frac{\mu_f}{\tilde{v}^2}\partial_m \tilde{v}\partial_m \bar{u}\Big) \text{ on } (0,T)\times\mathcal{F}_0.$$

With the above regularity of $\bar{u}$ and the assumed regularity of $\tilde{v}$ we conclude that $\bar{u} \in L^2(0,T;H^2(\mathcal{F}_0))$ with

$$\int_0^T \|\partial_{mm}\bar{u}\|^2_{L^2(\mathcal{F}_0)} \leq C^0_\infty\left(\int_0^T \|\partial_t \bar{u}\|^2_{L^2((0,1))} + T\sup_{(0,T)}\|\tilde{v}\|^2_{H^1(\mathcal{F}_0)} + \sup_{(0,T)}\|\tilde{v}\|_{H^1(\mathcal{F}_0)}\int_0^T \|\partial_m \bar{u}\|^2_{L^\infty(\mathcal{F}_0)}\right)$$

and we have finally:

$$\int_0^T \|\partial_{mm}\bar{u}\|^2_{L^2(\mathcal{F}_0)} \leq C^0_\infty \Big(\|\bar{u}_0\|^2_{H^1(\mathcal{F}_0)} + 1\Big),$$

when $T < 1$. This concludes the proof of the existence part of (a) and (b). Uniqueness in (a) will follow from the contraction estimate below.

To obtain uniqueness and (c), we consider $(\tilde{v}_1, \tilde{u}_1)$ and $(\tilde{v}_2, \tilde{u}_2)$ in $\bar{S}_T[K^0]$ and $\bar{u}_1, \bar{u}_2$ associated solutions to (39). In particular, taking the difference between the weak-formulations for $\bar{u}_1$ and $\bar{u}_2$ we have that $\bar{u} = \bar{u}_2 - \bar{u}_1$ satisfies $\bar{u}(0,\cdot) = 0$ with

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^1 \bar{u}\bar{w}\right] + \int_0^1 \frac{\mu}{\tilde{v}_1}\partial_m\bar{u}\partial_m\bar{w} + \int_0^1 \left(\mu\left(\frac{1}{\tilde{v}_1} - \frac{1}{\tilde{v}_2}\right)\partial_m\bar{u}_2 - (\pi(m,\tilde{v}_1) - \pi(m,\tilde{v}_2))\right)\partial_m\bar{w} = 0 \tag{45}$$

for all $\bar{w} \in \mathbb{H}^1_{\mathbf{m}}$. By a standard approximation procedure, we extend this weak formulation to

$$\bar{w} \in C\big([0,T]; H^1((0,1))\big) \cap H^1\big(0,T; L^2((0,1))\big)$$

so that we can test with $\bar{w} = \bar{u}$. This entails:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_0^1 |\bar{u}|^2\right] + \int_0^1 \frac{\mu}{\tilde{v}_1}|\partial_m\bar{u}|^2 = \int_0^1 \left(\mu\left(\frac{1}{\tilde{v}_2} - \frac{1}{\tilde{v}_1}\right)\partial_m\bar{u}_2 - (\pi(m,\tilde{v}_2) - \pi(m,\tilde{v}_1))\right)\partial_m\bar{u}.$$

At this point, we note that, if $\tilde{v}_1 = \tilde{v}_2$, the right-hand side vanishes which entails that $\bar{u} = 0$. This proves the uniqueness part of (a).

For the proof of the contraction estimate, we use again that $\tilde{v}_1$ and $\tilde{v}_2$ are bounded from above and by below by a constant that depends on initial data only. We can then bound the right-hand side:

$$\begin{aligned}
\mathrm{RHS} &\leq \frac{1}{2}\int_0^1 \frac{\mu}{\tilde{v}_1}|\partial_m\bar{u}|^2 + 2\left(\int_0^1 \mu\frac{\tilde{v}_1(\tilde{v}_2 - \tilde{v}_1)^2}{(\tilde{v}_2\tilde{v}_1)^2}|\partial_m\bar{u}_2|^2 + \int_0^1 \frac{\tilde{v}_1}{\mu}|\pi(m,\tilde{v}_2) - \pi(m,\tilde{v}_1)|^2\right) \\
&\leq \frac{1}{2}\int_0^1 \frac{\mu}{\tilde{v}_1}|\partial_m\bar{u}|^2 + 2C^\infty_0\Big(1 + \|\partial_m\bar{u}_2\|^2_{L^\infty((0,1))}\Big)\|\tilde{v}_2 - \tilde{v}_1\|^2_{L^2((0,1))}
\end{aligned}$$

where we applied that $\pi(m,\cdot) \in C^1$ for all $m \in (0,1)$ with:

$$\sup_{(0,1)} \|\partial_2\pi(m,\cdot)\|_{L^\infty\left(\inf_{(0,1)} v_0/2, 2\sup_{(0,1)} v_0\right)} \leq C^\infty_0 < \infty.$$

Consequently, we obtain:

$$\frac{1}{2}\sup_{(0,T)}\|\bar{u}\|^2_{L^2((0,1))} + \frac{\min(\mu_f,\mu_g)}{4\sup_{(0,1)}\bar{v}_0}\int_0^T\int_0^1 |\partial_m\bar{u}|^2 \leq C^\infty_0 \int_0^T \|\partial_m\bar{u}_2\|^2_{L^\infty((0,1))} \sup_{(0,T)}\|\tilde{v}_2 - \tilde{v}_1\|.$$

To conclude, it is sufficient to prove that we can make

$$\int_0^T \|\partial_m\bar{u}_2\|^2_{L^\infty((0,1))}$$

as small as we want by taking T sufficiently small. For this, we remark first that, on any (m_i^-, m_i^+), since $\bar{u}_2$ is affine, we have:

$$|\partial_m\bar{u}_2(t,m)| \leq \frac{1}{m_i}\int_{m_i^-}^{m_i^+} |\partial_m\bar{u}_2| \leq C^\infty_0\|\partial_m\bar{u}_2(t,\cdot)\|_{L^2((0,1))}.$$

While, on $\mathcal{F}_0$ we apply a refined Sobolev estimate which guarantees again that:

$$|\partial_m \bar{u}_2(t,m)|^2 \leq C^0_\infty \|\partial_m \bar{u}_2\|^2_{L^2(\mathcal{F}_0)} + \|\partial_m \bar{u}_2\|_{L^2(\mathcal{F}_0)} \|\partial_{mm} \bar{u}_2\|_{L^2(\mathcal{F}_0)}.$$

Consequently, we have, for any $t \in (0,T)$:

$$\|\partial_m \bar{u}_2(t,\cdot)\|^2_{L^\infty((0,1))} \leq C^0_\infty \|\partial_m \bar{u}_2\|^2_{L^2((0,1))} + \|\partial_m \bar{u}_2\|_{L^2((0,1))} \|\partial_{mm} \bar{u}_2\|_{L^2(\mathcal{F}_0)}$$

and thus, with the control from above yielding from (b) of this lemma (applied to $\bar{u}_2$), we conclude:

$$\begin{aligned}\int_0^T \|\partial_m \bar{u}_2(t,\cdot)\|^2_{L^\infty((0,1))} &\leq C^0_\infty \left(T \sup_{(0,T)} \|\partial_m \bar{u}_2\|^2_{L^2((0,1))} + \sqrt{T} \sup_{(0,T)} \|\partial_m \bar{u}_2\|_{L^2((0,1))} \left(\int_0^T \|\partial_{mm} \bar{u}_2\|^2_{L^2(\mathcal{F}_0)} \right)^{\frac{1}{2}} \right) \\ &\leq C^0_\infty \sqrt{T} \Big(1 + \|\bar{u}_0\|^2_{H^1((0,1))} \Big),\end{aligned}$$

when $T < 1$. We can thus make the right-hand side of this inequality as small as we want by taking T sufficiently small. This concludes the proof. □

Appendix A. Proof of Proposition 1

In this appendix we provide a proof of the proposition:

Proposition A.1. *Let $X \in \mathbb{R}^3$ and $R > 0$. If $u \in H^1(B(X,R))$ satisfies*

$$D(u) - \frac{1}{3} \text{div}\, u \mathbb{I}_3 = 0 \quad \text{ on } B(X,R) \tag{A.1}$$

then $u \in C^\infty(\bar{B}(X,R))$. If we assume furthermore that:

$$(u(x) - u(X)) \cdot n = cstt, \quad \text{ on } \partial B(X,R)$$

there exists $(V, \omega, \Lambda) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}$ such that:

$$u(x) = V + \omega \times (x - X) + \frac{\Lambda}{3}(x - X), \qquad \forall x \in B(X,R).$$

Proof. Without restriction, we assume that $X = 0$ and $R = 1$ so that $B(X,R) = B(0,1) =: B$. Furthermore, up to a convolution argument that we sketch below, we first consider that $u \in C^\infty(B)$.

Under the assumption of our theorem, we have that:

$$\nabla u(x) = \begin{pmatrix} \lambda & \lambda_{12} & \lambda_{13} \\ -\lambda_{12} & \lambda & \lambda_{23} \\ -\lambda_{13} & -\lambda_{23} & \lambda \end{pmatrix}$$

where $\lambda = 1/3 \, \text{div}\, u$ and $\lambda_{i,j} = \partial_i u_j$. Let focus on λ_{12} to sart with. Following the method of Lemma 1.1, Chapter 1 from [23], we have that:

$$\partial_1 \lambda_{12} = \partial_{12} u_1 = \partial_2 \lambda \partial_2 \lambda_{12} = \partial_{22} u_1 = -\partial_{12} u_2 = -\partial_1 \lambda$$

and:

$$\begin{aligned}\partial_3 \lambda_{12} &= \partial_{32} u_1 \\ &= \frac{1}{2}(\partial_{32} u_1 - \partial_{31} u_2)\end{aligned}$$

$$= \frac{1}{2}(\partial_2(\partial_3 u_1 + \partial_1 u_3) - \partial_1(\partial_3 u_2 + \partial_2 u_3))$$
$$= 0.$$

Eventually λ_{12} does not depend on x_3 and its perpendicular gradient is the gradient of λ. We have thus that $\partial_1\lambda, \partial_2\lambda$ do not depend on x_3 and satisfy:

$$\partial_{11}\lambda + \partial_{22}\lambda = 0.$$

Arguing similarly with λ_{13} and λ_{12} we infer that there exists 3 functions $\lambda_1, \lambda_2, \lambda_3$ such that:

$$\partial_i\lambda(x) = \lambda_i(x_i) + cstt$$

that solve:

$$\partial_i\lambda_i(x_i) + \partial_j\lambda_j(x_j) = 0 \qquad \forall i \neq j.$$

Then, there exists three constants (a_1, a_2, a_3) for which:

$$\lambda(x) = a_1x_1 + a_2x_2 + a_3x_3 + cstt \qquad \forall i.$$

Eventually, we obtain that :

$$\nabla u(x) = \begin{pmatrix} a_1x_1 + a_2x_2 + a_3x_3 & a_2x_1 - a_1x_2 & a_3x_1 - a_1x_3 \\ a_1x_2 - a_2x_1 & a_1x_1 + a_2x_2 + a_3x_3 & a_3x_2 - a_2x_3 \\ a_1x_3 - a_3x_1 & a_2x_3 - a_3x_2 & a_1x_1 + a_2x_2 + a_3x_3 \end{pmatrix} + cstt$$

and thus, with $a = (a_1, a_2, a_3)$ we have:

$$u(x) = u_a(x) + u_{\text{aff}}(x), \qquad \text{where} \qquad u_a(x) = (a \cdot x)x - a\frac{|x|^2}{2}$$

and u_{aff} is an affine mapping. In particular, we have that $u \in C^\infty(\overline{B})$.

Let now make precise the convolution argument. If $u \in H^1(B)$ satisfies (A.1) and $\varepsilon < 1/2$ a convolution u_ε of u with an approximation of identity having support in $B(0,\varepsilon)$ will satisfy (A.1) on $B(0, 1-\varepsilon)$ and be smooth on B. Reproducing the previous arguments, we construct $a^{(\varepsilon)} \in \mathbb{R}^3$ and an affine mapping $u_{\text{aff}}^{(\varepsilon)}$ so that $u_\varepsilon = u_{a^{(\varepsilon)}} + u_{\text{aff}}^{(\varepsilon)}$ on $B(0, 1-\varepsilon)$. However, we see that there exists a constant C for which

$$|a^\varepsilon| = C \int_{B(0,1/2)} |\text{div}\, u_\varepsilon|^2$$

while $u_{\text{aff}}^{(\varepsilon)}$ is controlled by the skew-symmetric part of u_ε and the mean of u_ε on $B(0, 1/2)$. Eventually, we obtain that, when $\varepsilon \to 0$ we have $a^{(\varepsilon)} \to a$ in $\mathbb{R}^3$ and $u_{\text{aff}}^{(\varepsilon)} \to u_{\text{aff}}$ in the set of affine mappings with

$$u = u_a + u_{\text{aff}} \quad \text{ on } B.$$

Next, we realise that

$$D(u_a) - \frac{1}{3}\text{div}(u_a)\mathbb{I}_3 = 0$$

so that the same property holds for u_{aff}. Combining this information with the fact that u_{aff} is affine, we obtain the existence of V, ω, Λ so that:

$$u_{\text{aff}}(x) = V + \omega \times (x - X) + \frac{\Lambda}{3}(x - X).$$

At this point, we note that, on ∂B, there holds:

$$(u(x) - u(0)) \cdot n = \Lambda + u_a(x) \cdot x = \Lambda + \frac{a \cdot x}{2}$$

which can be constant if and only if $a = 0$. This ends the proof. □

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