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ANALYSIS OF COMPRESSIBLE BUBBLY FLOWS. PART II : DERIVATION OF A MACROSCOPIC MODEL

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Abstract. This paper is the second of the series of two papers, which focuses on the derivation of an averaged 1D model for compressible bubbly flows. For this, we start from a microscopic description of the interactions between a large but finite number of small bubbles with a surrounding compressible fluid. This microscopic model has been derived and analysed in the first paper. In the present one, provided physical parameters scale according to the number of bubbles, we prove that solutions to the microscopic model exist on a timespan independent of the number of bubbles. Considering then that we have a large number of bubbles, we propose a construction of the macroscopic variables and derive the averaged system satisfied by these quantities. Our method is based on a compactness approach in a strong-solution setting. In the last section, we propose the derivation of the Williams–Boltzmann equation corresponding to our setting.

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1. Introduction

The present work represents a straight continuation of a series of articles which proposes to justify the construction of multiphase flow models. The structure of multiphase flow models can be derived formally by applying standard conservation principles [9,10,12,17]. However this procedure leaves aside key-terms that have to be related to mechanical/thermodynamical unknowns *via* state laws. To this end, a sharp description of the interactions between phases is required. Classical methods are based on averaging operators whose range of validity is still to be investigated. Furthermore, the action of these averaging operators on nonlinear quantities requires further modelling assumptions. From the analytical standpoint, the computations we provide herein follow previous analysis of the first author notably in collaboration with D. Bresch [3,4,7,14] complementing previous approaches in [1,13,20]. In these references, one-velocity Baer–Nunziato-like models are derived for multiphase fluids [2]. By this terminology, we mean that the averaged mixture is described by an additional evolution equation which governs the dynamics of the void fraction. This additional equation includes a relaxation term due to mechanical exchanges between phases. These derivations are based on the remark that, if the interfaces act as a “perfect” transducer (no mass transfer, perfect transfer of mechanical stress), combining the

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different phases equations yields a global one-fluid equation. Deriving multiphase flow models then reduces to a thorough analysis of highly-oscillatory solutions to the one-fluid equation. A particular analytical framework of mixed-regularity (smooth velocity with discontinuous densities [8, 16, 22]) is identified in [5] to make this approach fully rigorous. However, this approach is restricted to an ideal case (see [6] for further investigations in this context). The aim of this paper is to tackle the derivation of averaged models in presence of jumps at interfaces. Starting from an original microscopic model (that is derived in the first paper [15]) in which the two phases are fully separated, we derive a 1D averaged compressible bubbly-flow model by performing space averaging operators.

The averaged model reads as follows. It is set on the container $\Omega = (-1, 1)$ filled with a gas/fluid mixture. The averaged variables are the void fractions $\bar{\alpha}_{f,g} \in [0, 1]$, the mean densities $\bar{\rho}_{f,g} \in [0, \infty)$, a bubble phase covolume[1] $\bar{f}_g \in [0, \infty)$ and the mixture velocity $\bar{u} \in \mathbb{R}$. It reads:

$$\begin{cases} \partial_t\big(\bar{\alpha}_g \bar{f}_g\big) + \partial_x\big(\bar{\alpha}_g \bar{f}_g \bar{u}\big) = 0, \\ \partial_t(\bar{\alpha}_f \bar{\rho}_f) + \partial_x(\bar{\alpha}_f \bar{\rho}_f \bar{u}) = 0, \\ \partial_t(\bar{\alpha}_g \bar{\rho}_g) + \partial_x(\bar{\alpha}_g \bar{\rho}_g \bar{u}) = 0, \\ \partial_t \bar{\alpha}_f + \bar{u}\partial_x \bar{\alpha}_f = \overline{\mathrm{RT}}, \\ \partial_t(\bar{\rho}\bar{u}) + \partial_x\big(\bar{\rho}\bar{u}^2\big) = \partial_x \bar{\Sigma}, \end{cases} \qquad \text{on } (0,T)\times\Omega, \tag{1}$$

with the compatibility conditions:

$$\bar{\alpha}_f + \bar{\alpha}_g = 1, \quad \bar{\rho} = \bar{\alpha}_f \bar{\rho}_f + \bar{\alpha}_g \bar{\rho}_g \tag{2}$$

and where the mixture stress tensor writes

$$\bar{\Sigma} = \frac{\mu_g \mu_f}{\bar{\alpha}_f \mu_g + \bar{\alpha}_g \mu_f}\left[\partial_x \bar{u} - \left(\frac{\bar{\alpha}_f}{\mu_f}\mathrm{p}_f(\bar{\rho}_f) + \frac{\bar{\alpha}_g}{\mu_g}\mathrm{p}_g(\bar{\rho}_g)\right) - \bar{\gamma}_s \frac{\bar{\alpha}_g}{\mu_g}\bar{f}_g\right], \tag{3}$$

while the void fraction relaxation term reads:

$$\overline{\mathrm{RT}} = \frac{\bar{\alpha}_g \bar{\alpha}_f}{\bar{\alpha}_f \mu_g + \bar{\alpha}_g \mu_f}\big[(\mu_g - \mu_f)\partial_x \bar{u} + (\mathrm{p}_f(\bar{\rho}_f) - \mathrm{p}_g(\bar{\rho}_g)) - \bar{\gamma}_s \bar{f}_g\big]. \tag{4}$$

In these latter identities appear the constants $\mu_f, \mu_g > 0$ (resp. the functions $\mathrm{p}_f, \mathrm{p}_g$) representing the fluid and bubble viscosities (resp. the fluid and gas pressure laws). The constant $\bar{\gamma}_s > 0$ represents the surface tension.

The system (1) and (2) complemented with the state laws (3) and (4) is obtained starting from the following *microscopic* model, where the two phases are disjoint and their interactions only appear through the interfaces. Again, the two-phase flow is posed in the one-dimensional domain $\Omega = (-1, 1)$, filled by a liquid (the fluid, or the continuous phase, indexed by f) and bubbles (the gas, or the dispersed phase, indexed by g). The N bubbles are described by their centers c_k and their radii R_k, so that the kth bubble is

$$B_k = \big(x_k^-, x_k^+\big), \quad x_k^\pm = c_k \pm R_k, \quad \forall\, k = 1, \dots, N.$$

The fluid domain is

$$\mathcal{F} = \Omega \setminus \bigcup_{k=1}^{N} \overline{B_k}.$$

For later use, we also introduce the fluid intervals

$$\mathcal{F}_k = \big(x_k^+, x_{k+1}^-\big) \quad \text{for } k = 0, \dots, N \tag{5}$$

[1]The denomination covolume may be misleading here. In classical thermodynamics, the term covolume refers to the specific volume. Here the quantity $\bar{f}_g$ is linked to the volume of the gaseous phase. In 3D configurations, it would be related to the interfacial area.

setting $x_0^+ = -1$ and $x_{N+1}^- = 1$.

The fluid is supposed to be compressible and viscous, so that it is governed by the 1D compressible Navier–Stokes system, posed in $\mathcal{F}$:

$$\partial_t \rho_f + \partial_x(\rho_f u_f) = 0, \tag{6}$$

$$\partial_t(\rho_f u_f) + \partial_x(\rho_f u_f^2) = \partial_x \Sigma_f, \tag{7}$$

$$\Sigma_f = \mu_f \partial_x u_f - \mathrm{p}_f(\rho_f), \tag{8}$$

where ρ_f is the density, u_f the velocity and Σ_f the stress tensor of the fluid. Moreover, $\mu_f > 0$ is the shear viscosity and p_f is an isentropic pressure law for the fluid:

$$\mathrm{p}_f(\rho_f) = \kappa_f \rho_f^{\gamma_f},$$

where $\kappa_f > 0$ and $\gamma_f > 1$ stands for the adiabatic exponent. We assume that the fluid is present at the boundary of the domain Ω, where no-slip boundary conditions are imposed:

$$u_f(t, \pm 1) = 0. \tag{9}$$

Equations for bubble kinematics and dynamics are proposed in [15]. Therein, the derivation is based on the assumption that the bubbles are made of a compressible viscous fluid with an infinite shear viscosity (compared to the volumic viscosity) and that their spherical shapes are preserved (in three dimensions). We point out that these assumptions restricts also the possible micro-motions inside the bubbles. We are aware that this restriction borrows from droplet dynamics but we shall keep the naming bubbles throughout the paper. We have then first that the continuity of the velocity at the interfaces reads:

$$u_f(t, x_k^\pm(t)) = \dot{c}_k(t) \pm \dot{R}_k(t) \quad \text{for } k = 1, \dots, N. \tag{10}$$

In addition, imposing that the jump of the stress tensor at the interfaces is due to the surface tension, one obtains the following system for the dynamics of a bubble:

$$m_k \ddot{c}_k(t) = \Sigma_f\big(t, x_k^+\big) - \Sigma_f\big(t, x_k^-\big), \tag{11}$$

$$\frac{m_k}{3} \ddot{R}_k(t) = \Sigma_f\big(t, x_k^-\big) + \Sigma_f\big(t, x_k^+\big) - 2\Sigma_k(t), \tag{12}$$

$$\Sigma_k = \mu_g \frac{\dot{R}_k}{R_k} - \mathrm{p}_g(\rho_k) - \frac{F_s}{2}, \tag{13}$$

where $\mu_g > 0$ is the volumic viscosity of the gas, and m_k and ρ_k are the mass and the density of the bubble, linked by $m_k = 2R_k\rho_k$. As a consequence of mass conservation in bubbles, the masses m_k do not depend on time. The term F_s denotes the force due to the surface tension and writes $F_s = \gamma_s/R_k$, γ_s being the surface tension. In order to simplify the analysis, we assume an isothermal equation of state in the bubbles, so that

$$\pi_k := \mathrm{p}_g(\rho_k) + \frac{F_s}{2} = \frac{(a_g)^2 m_k + \gamma_s/2}{R_k} = \frac{\kappa_k}{R_k}, \tag{14}$$

where $a_g > 0$ is the sound speed of the gas. The last form of π_k will be used mainly for the analysis of the model, while the first form will be useful to interpret the various terms appearing in equations, notably those due to surface tension. In particular, computing surface tension effects in the microscopic system involves the quantity $1/(2R_k)$ that corresponds to the covolume of bubble B_k in our $1D$ setting. We point out that the system (6)–(13) is not integrable and, specifically, does not yield any particular value for the fluid velocity-field u_f. We are then not in the Rayleigh–Plesset regime where the bubble equations (11) and (12) reduce to ordinary differential equations in terms of (c_k, R_k) and an asymptotic pressure [23]. We refer the reader to the companion paper [15] for more details on the derivation of (6)–(13) and the analysis of the associated Cauchy problem. Yet, we shall explain in further details the construction of solutions in the next section.

The main result of this paper is to show that, starting from solutions to (6)–(13) we obtain (1)–(4) by letting the number N of bubbles go to infinity in case:

$$m_k \sim N^{-1}, \quad R_k \sim N^{-1}, \quad |\mathcal{F}_k| \sim N^{-1}, \quad \gamma_s \sim N^{-1}, \tag{15}$$

with the other parameters being fixed. This approach contains at least two severe difficulties. The first one is to prove that the scaling (15) remains valid on a timespan independent of N. The second one is that the target system (1)–(4) is highly nonlinear. Specifically, products between volume fractions and other (fluid or gas) unknowns are ubiquitous. To obtain such nonlinear terms, it appears that strong convergences of densities or gas covolume in sufficiently smooth spaces are necessary. Hence, with this approach, we face two key-difficulties:

– to prove that the scaling regime (15) holds on a timespan independent of the number N of bubbles,
– to define the macroscopic unknowns and especially, the fluid and gas densities $\bar{\rho}_f, \bar{\rho}_g$ and the gas covolume $\bar{f}_g$.

The first item in this list is the content of the next section. Therein, we consider initial data that are constructed as follows. Firstly, we fix fluid initial data $\left(\rho^0_f, u^0_f\right) \in H^1(\Omega) \times H^1_0(\Omega)$ that are thus defined globally on Ω. We assume further that they are far from vacuum. Secondly, we fix initial distributions of centers/radii $\left(c^0_k, R^0_k\right)_{k=1,\dots,N}$ such that (15) holds. We complement then the microscopic system (6)–(13) with initial conditions so that the initial bubble velocities match the velocities prescribed by the fluid on the boundaries. This reads:

$$\begin{aligned} c_k(0) &= c^0_k \quad R_k(0) = R^0_k, && \text{for } k = 1, \dots, N, && (16)\\ u(0,\cdot) &= u^0_f \quad \rho(0,\cdot) = \rho^0_f, && \text{on } \mathcal{F}^0, && (17) \end{aligned}$$

and

$$\begin{aligned} \dot{c}^0_k &= \frac{u^0_f\left(c^0_k + R^0_k\right) + u^0_f\left(c^0_k - R^0_k\right)}{2}, && \text{for } k = 1, \dots, N, && (18)\\ \dot{R}^0_k &= \frac{u^0_f\left(c^0_k + R^0_k\right) - u^0_f\left(c^0_k - R^0_k\right)}{2}, && \text{for } k = 1, \dots, N. && (19) \end{aligned}$$

The main result of Section 2 is then that there exists a classical solution to (6)–(13) on a timespan that depends only on fluid initial data and the parameters quantifying initially assumption (15). To obtain this result, we combine classical energy and regularity estimates for Navier Stokes equations. We remind that, in this strategy, one classically uses extra regularity thanks to the form of the stress tensor Σ_f. However, such regularity estimates should depend on the geometry (and then on N). In particular, we cannot rely on the methods introduced to study the piston problem as in [19, 21] since they do not consider this scaling issue. To overcome this difficulty, we propose to consider suitable extensions of Σ_f (resp. Σ_g) on the complementary gas (resp. fluid) domain. In this way, the extension is defined on a fixed domain and the regularity gain is independent of the geometry. We point out here that contrary to the classical approach in the topic of homogenization of multidimensional compressible Navier Stokes equations in perforated domains [11, 18], our construction takes advantage of the information on the moments of the fluid stress tensor on ∂B_k that are provided by the bubble equations.

The second key-difficulty of our approach is tackled in Section 3. Once solutions to (6)–(13) are constructed on a time-interval that does not depend on N, we consider the behavior of these solutions for large N. In particular, we look for definitions of the unknowns that are involved in the macroscopic system (1)–(4). Void fractions as well as global velocity-fields are obtained classically by considering indicator functions or suitably extended vector-fields (see Props. 13 and 15). However, the issue is more involved when going to density and covolume unknowns. Indeed, at the discrete level, fluid density and bubble density, for instance, are defined *a priori* on dispersed subdomains only. This cannot yield convergence with sufficient regularity. To get better convergence results, we decide to construct suitable extensions. For this we proceed in two steps. Firstly, we ensure that the initial conditions for (6)–(13) enable to define smooth extended densities and covolume (see Prop. 12). Then, we propagate this regularity with a well-chosen extended flow (see Props. 14 and 17). With this construction at-hand, the derivation of (1)–(4) is plain sailing.

In our construction, we start from initial data for the macroscopic system and define a sequence of initial conditions for the microscopic system that are compatible with the scaling (15) and enable to construct extended densities. It turns out that this requires further assumption on initial data that we explain now. We recall that initial data for the macroscopic system consists in:

– initial fluid and gas densities : $\bar{\rho}_f^0,\ \bar{\rho}_g^0$,
– initial fluid and gas void fractions $\bar{\alpha}_f^0,\ \bar{\alpha}_g^0$
– an initial velocity of the two-phase mixture $\bar{u}^0$,
– an initial gas covolume $\bar{f}_g^0$.

It is worth noting that all these functions are defined for x in Ω, since both phases are no longer separated at the macroscopic scale. We shall remain at the regularity level of classical solution and require that all these initial conditions are $H^1(\Omega)$. For our construction, we require that initial densities and void fraction satisfy:

$$\rho_{\min} \leq \min\big(\bar{\rho}_f^0, \bar{\rho}_g^0\big) \tag{20}$$

$$\alpha_{\min} \leq \min\big(\bar{\alpha}_f^0, \bar{\alpha}_g^0\big) \qquad\qquad \bar{\alpha}_g^0 + \bar{\alpha}_f^0 = 1 \tag{21}$$

for some positive constants $\rho_{\min},\ \alpha_{\min}$. The first condition means that we are away from vacuum. The second one expresses that there is a mixture of both phases everywhere in Ω. Note that the second conditions imply simultaneously

$$\max\Big(\big\|\bar{\alpha}_g^0\big\|_{L^\infty(\Omega)}, \big\|\bar{\alpha}_f^0\big\|_{L^\infty(\Omega)}\Big) \leq 1 - \alpha_{\min}. \tag{22}$$

Concerning, $\bar{f}_g^0$, we will require that:

$$f_{\min} \leq \bar{f}_g^0, \qquad \bar{\alpha}_g^0 \bar{f}_g^0 \text{ is a probability density,} \tag{23}$$

where $f_{\min}$ is a positive constant. To explain these latter conditions, we point out that in the 1D case the covolume of bubbles is proportional to the inverse radius. So $f_{\min}$ is a bound from above on the initial radius of bubbles and, since we expect $\bar{\alpha}_g^0 \bar{f}_g^0$ to be the limit of the indicator function of bubble domains multiplied by the inverse radius of bubbles, a straightforward computations yields that it is a positive function whose total mass is 1, hence a probability density.

The multiphase system we consider in this paper enters the family of spray models as studied by Williams in [24, Sect. 11]. With this standpoint, a classical tool to analyze the behavior of the dispersed phase is the so-called "Williams–Boltzmann" equation which describes the time-evolution of the particle-distribution function of the dispersed phase. In the last section of this paper, we derive what would be the equivalent equation in our setting. It is worth to mention that this is not a supplementary equation but simply a rephrasing of the bubble-gas equation that we derived previously. In particular, herein the bubble-gas velocities are correlated to their position and drag forces are at equilibrium. We do neither have collision or creation of bubbles. Hence, the only term to be taken into account is the "evaporation" term which should be understood as compression/expansion term in our compressible setting.

In brief, the outline of the paper is as follows. In the next section, we prove that solutions to the microscopic system (6)–(13) satisfying (15) do exist on a timespan independent of N if initial data are well prepared, see Theorem 1. In Sections 3 and 4, we tackle the asymptotics of these solutions when $N \to \infty$. In the last section, we discuss an alternative approach based on using particle-distribution functions for the bubbles. In appendices, we provide some technical computations involved in the construction of solutions to the microscopic model.

2. Local Cauchy theory for the microscopic system

In this section, we forget temporarily our homogenization goal. We focus on the microscopic model (6)–(13) in the scaling (15) and we address the existence of solutions with lifespan independent of the number N of bubbles provided initial data are constructed as in (16)–(19). In particular, we fix $\left(\rho_f^0, u_f^0\right) \in H^1(\Omega) \times H_0^1(\Omega)$ throughout the section. We assume these global fluid data satisfy:

$$2\underline{\rho}_\infty \leq \rho_f^0 \leq \overline{\rho}_\infty/2 \quad \text{on } \Omega \tag{24}$$

for some pair $(\underline{\rho}_\infty, \overline{\rho}_\infty) \in (0,\infty)^2$.

To make precise our main result, we start by giving a quantified version of assumption (15) that we assume to hold initially. Firstly, we fix that bubbles characteristics enjoy the property:

(IC_0) $M_\infty \leq Nm_k, N\kappa_k \leq (M_\infty)^{-1}, \quad k = 1,\dots,N,$
(IC_1) $2d_\infty \leq NR_k^0 \leq (2d_\infty)^{-1}, \quad k = 1,\dots,N,$
(IC_2) $2d_\infty \leq N|\mathcal{F}_k^0| \leq (2d_\infty)^{-1}, \quad k = 0,\dots,N.$

Here M_∞, d_∞ are strictly positive constants independent of N. We recall the convention (5) for the definition of $\mathcal{F}_k^0$ (adapted to notations for initial data). Their union constitutes the initial fluid domain $\mathcal{F}^0$. The physical parameters (μ_f, μ_g) and pressure laws are fixed independent of N. With these conventions, the main result of this section reads:

Theorem 1. *Let initial condition to (6)–(13) be constructed as in (16)–(19). Assume further that parameters $(m_k, \kappa_k)_{k=1,\dots,N}$ and initial bubble distributions $(c_k^0, R_k^0)_{k\in\{1,\dots,N\}}$ satisfy (IC_0)–(IC_2). Then, there exists $T_\infty > 0$ independent of the number of bubbles and depending only on*

$$M_\infty,\ d_\infty,\ \underline{\rho}_\infty,\ \bar{\rho}_\infty,\ \left\|u_f^0\right\|_{H^1(\Omega)},\ \left\|\rho_f^0\right\|_{H^1(\Omega)}, \tag{25}$$

such that there exists a solution to (6)–(13) on $(0, T_\infty)$.

What remains of this section is devoted to the proof of this theorem. From now on, we pick a family of physical parameters and bubble centers/radii satisfying the assumptions of Theorem 1 and we construct initial data for (6)–(13).

In the companion paper [15], we prove local-in-time existence and uniqueness of classical solutions to the Cauchy problem associated with (6)–(13). In this moving-domain setting, classical solution means broadly that:

- the motion of the bubbles is $H^2(0,T)$ (*i.e.* $(c_k, R_k) \in H^2(0,T)$),
- u is $H_t^1 L_x^2$ and $L_t^2 H_x^2$ in the fluid domain,
- ρ is $H_{t,x}^1$ in the fluid domain.

Existence and uniqueness of solutions on a lifespan $(0, T_0)$ is obtained for initial data such that

- there is no overlap of the bubbles,
- initial fluid data are H^1 in the fluid domain with strictly positive density,
- initial fluid and bubble velocities match at interfaces (so that (18)–(19) hold true).

It is also worth noting that the time T_0 is uniform in data satisfying uniform bounds from below for the distance between bubbles, the minimal radius of bubbles, the minimum density and also the size of initial fluid velocity and density in H^1-spaces. We refer to [15] for more precise and quantitative statements.

So, under the assumptions of Theorem 1, the local-in-time existence result of [15] yields a solution on a time-interval $(0, T_0)$ that depends on the list of parameters (25) but also on N. To rule out this dependency, we construct T_∞ such that as long as $t < T_\infty$ the solution

$$\left(\rho_f(t,\cdot),\ u_f(t,\cdot),\ \left(c_k(t), R_k(t),\ \dot{c}_k(t),\ \dot{R}_k(t)\right)_{k\in\{1,\dots,N\}}\right)$$

yields an initial condition that is compatible with the Cauchy theory of [15] with an associated existence time independent of t. We emphasize that any classical solution does not allow overlap of the bubbles and ensures identity (18)–(19) is satisfied at any time. Controlling the existence time associated with the value of the solution at time t – considered as an initial data – reduces to obtaining uniform H^1 bound for the velocity field and for the density, uniform bound from above and from below on the fluid density, the radius of the bubbles and the length of fluid segments.

Our approach relies on a suitable combination of energy and regularity estimates for the coupled system (6)–(13). So, we recall in the next sections the classical estimates that are associated with (6)–(13). We will pay special attention to obtain estimates independent on N. This will be particularly challenging for regularity estimates. In particular, we shall study the regularity of fluid velocity-fields that can be gained through the integrability of the stress tensor by working on extensions of fluid unknowns on bubble domains and conversely. A tricky part of the proof is that we can obtain these sharp bounds under the condition that we have already *a priori* bounds. So, we implement a continuation argument. This continuation argument is explained in the last part of the section. However, the extensive proof is rather long and technical. Hence, the last subsection reduces to a roadmap of the proof that is detailed further in Appendix A.

2.1. Classical estimates

We introduce the conjugate function of the fluid pressure $\mathrm{q}_f : [0,\infty) \to [0,\infty)$ defined by

$$\mathrm{q}_f'(s)s - \mathrm{q}_f(s) = \mathrm{p}_f(s). \tag{26}$$

In other words, the function q_f represents the volumic internal energy of the fluid. Considering an isentropic pressure law, it yields

$$\mathrm{q}_f(s) = \frac{a_f s^{\gamma_f}}{\gamma_f - 1}\cdot$$

We can now state the total energy equation. In the bracket of the statement below, the first term is the total energy of the fluid, while the second and the third terms respectively are the kinetic energy and the internal energy of the bubbles.

Proposition 2. *For any reference radius $R_{\mathrm{ref}} > 0$, it holds*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Bigg[\int_{\mathcal{F}}\left(\rho_f\frac{|u_f|^2}{2} + \mathrm{q}_f(\rho_f)\right)\mathrm{d}x + \sum_{k=1}^{N} m_k\left(\frac{|\dot{c}_k|^2}{2} + \frac{\left|\dot{R}_k\right|^2}{6}\right)\\ -2\sum_{k=1}^{N}\kappa_k\ln\left(\frac{R_k}{R_{\mathrm{ref}}}\right)\Bigg] + \int_{\mathcal{F}}\mu_f|\partial_x u_f|^2\,\mathrm{d}x + 2\mu_g\sum_{k=1}^{N}\frac{\left|\dot{R}_k\right|^2}{|R_k|} = 0.\end{aligned} \tag{27}$$

Proof. First let multiply the Navier–Stokes equation (7) by the velocity u_f and integrate over the fluid domain $\mathcal{F}$. Using the mass conservation equation (6), it yields

$$\int_{\mathcal{F}}\rho_f(\partial_t u_f + u_f\partial_x u_f)u_f\,\mathrm{d}x = \int_{\mathcal{F}} u_f\partial_x\Sigma_f\,\mathrm{d}x. \tag{28}$$

Since the mass conservation (6) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{x_k^+}^{x_{k+1}^-}\rho_f\frac{|u_f|^2}{2}\,\mathrm{d}x = \int_{x_k^+}^{x_{k+1}^-}\rho_f(\partial_t u_f + u_f\partial_x u_f)u_f\,\mathrm{d}x,$$

one obtains, using an integration by part of the right-hand side,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{F}} \rho_f \frac{|u_f|^2}{2}\,\mathrm{d}x = T_1 - T_2 - T_3,$$

with

$$\begin{aligned}
T_1 &= \sum_{k=0}^{N} \Sigma_f(x_{k+1}^-)u_f(x_{k+1}^-) - \Sigma_f(x_k^+)u_f(x_k^+),\\
T_2 &= \int_{\mathcal{F}} \mu_f |\partial_x u_f|^2\,\mathrm{d}x,\\
T_3 &= -\int_{\mathcal{F}} \mathrm{p}_f(\rho_f)\partial_x u_f\,\mathrm{d}x,
\end{aligned}$$

where the terms T_2 and T_3 come from the definition (8) of the stress Σ_f.

Using the boundary conditions (9) and, after, the continuity of the velocities at the droplet interfaces (10), the term T_1 can be rewritten as

$$\begin{aligned}
T_1 &= -\sum_{k=1}^{N}\left(\Sigma_f(x_k^+)u_f(x_k^+) - \Sigma_f(x_k^-)u_f(x_k^-)\right)\\
&= -\sum_{k=1}^{N} \dot{c}_k\left(\Sigma_f(x_k^+) - \Sigma_f(x_k^-)\right) + \dot{R}_k\left(\Sigma_f(x_k^+) + \Sigma_f(x_k^-)\right).
\end{aligned}$$

Finally the droplets motion equations (11) and (12) and the definition of the droplet pressure law (14) yield (whatever the value of $R_{\text{ref}} > 0$):

$$T_1 = -\frac{\mathrm{d}}{\mathrm{d}t}\left[\left(\sum_{k=1}^{N} m_k\frac{|\dot{c}_k|^2}{2} + \frac{m_k}{3}\frac{\left|\dot{R}_k\right|^2}{2}\right) - 2\sum_{k=1}^{N}\kappa_k \ln\left(\frac{R_k}{R_{\text{ref}}}\right)\right] - 2\mu_g\sum_{k=1}^{N}\frac{\left|\dot{R}_k\right|^2}{R_k}.$$

We now turn to the term T_3. By the definition (26) of the function q_f, and by the mass conservation equation (6), it holds

$$\partial_t \mathrm{q}_f(\rho_f) + \partial_x(\mathrm{q}_f(\rho_f)u_f) = -\mathrm{p}_f(\rho_f)\partial_x u_f.$$

Because the fluid domain evolves with the velocity u_f, T_3 can be recovered

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathcal{F}} q(\rho_f)\,\mathrm{d}x = -\int_{\mathcal{F}} \mathrm{p}_f(\rho_f)\partial_x u_f\,\mathrm{d}x = T_3.$$

One deduces the final estimate (27) combining the terms T_1, T_2 and T_3. □

In the regime of initial data specified in this section, we obtain the following corollary:

Corollary 3. *If initial data are constructed as in* (16)–(19) *and satisfy* (IC_0)–(IC_1)–(IC_2)*, there exists a constant* E_0 *depending only on the list of parameters* (25) *such that any classical solution to* (6)–(13) *on some time-interval* $[0,T]$ *satisfies:*

$$\int_{\mathcal{F}}\left(\rho_f\frac{|u_f|^2}{2} + q(\rho_f)\right)\mathrm{d}x + \frac{1}{2}\sum_{k=1}^{N} m_k\left(|\dot{c}_k|^2 + \frac{1}{3}\left|\dot{R}_k\right|^2\right) - 2\sum_{k=1}^{N}\kappa_k\ln(d_\infty N R_k) \le E_0, \tag{29}$$

on $(0,T)$ *with, denoting by* $\ln_+$ *the positive part of the* $\ln$*:*

$$\int_0^T \left[\left(\int_{\mathcal{F}} \mu_f |\partial_x u_f|^2 \,\mathrm{d}x + \mu_g \sum_{k=1}^N \frac{|\dot{R}_k|^2}{R_k}\right)\right] \mathrm{d}t \leq E_0 + 2 \max_{[0,T]} \sum_{k=1}^N \kappa_k \ln_+(d_\infty N R_k). \tag{30}$$

Proof. To obtain these inequalities, we integrate (27) with $R_{\text{ref}} = 1/d_\infty N$ and remark that all the terms on the left-hand side are positive but:

$$\sum_{k=1}^N \kappa_k \ln(d_\infty N R_k).$$

We obtain then the inequalities (29) and (30) with:

$$E_0 := \int_{\mathcal{F}^0} \left(\rho_f^0 \frac{\left|u_f^0\right|^2}{2} + q_f\left(\rho_f^0\right)\right) \mathrm{d}x + \sum_{k=1}^N \left(\frac{|\dot{c}_k^0|^2}{2} + \frac{\left|\dot{R}_k^0\right|^2}{6}\right) - 2\sum_{k=1}^N \kappa_k \ln\left(R_k^0 d_\infty N\right).$$

The first term in E_0 is clearly controlled by $\left\|u_f^0\right\|_{L^2}$ and $\bar{\rho}_\infty$. As for the second term, the velocity continuity (18) and (19) gives

$$|\dot{c}_k^0| + \left|\dot{R}_k^0\right| \leq 2\|u_f^0\|_{L^\infty(\Omega)}, \quad \forall\, k = 1, \dots, N.$$

Then, with (IC_0), we obtain:

$$\sum_{k=1}^N m_k \left(|\dot{c}_k^0|^2 + \frac{1}{3}\left|\dot{R}_k^0\right|^2\right) \leq \frac{4}{M_\infty} \|u_f^0\|^2_{L^\infty(\Omega)},$$

and, with a classical Sobolev embedding, this part is again controlled by M_∞ and $\left\|u_f^0\right\|_{H_0^1(\Omega)}$. Now using the bound (IC_1) on the initial radii, it holds

$$2d_\infty^2 \leq R_k^0 N d_\infty \leq \frac{1}{2},$$

so that

$$-\sum_{k=1}^N \kappa_k \ln\left(d_\infty N R_k^0\right) \leq \frac{\left|\ln\left(2d_\infty^2\right)\right|}{M_\infty}.$$

This concludes the proof. $\square$

We proceed with a second classical regularity estimate:

Proposition 4. *The following identity holds*

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{\mathcal{F}} \left(\mu_f \frac{|\partial_x u_f|^2}{2} - \mathrm{p}_f(\rho_f)\partial_x u_f\right) \mathrm{d}x + \sum_{k=1}^N \left(\mu_g \frac{\left|\dot{R}_k\right|^2}{R_k} - 2\kappa_k \frac{\dot{R}_k}{R_k}\right)\right] \\
&\quad + \int_{\mathcal{F}} \rho_f |\partial_t u_f + u_f \partial_x u_f|^2 \mathrm{d}x + \sum_{k=1}^N m_k \left(|\ddot{c}_k|^2 + \frac{1}{3}\left|\ddot{R}_k\right|^2\right) \\
&= \int_{\mathcal{F}} \left(\mathrm{p}_f'(\rho_f)\rho_f |\partial_x u_f|^2 - \mu_f \frac{(\partial_x u_f)^3}{2}\right) \mathrm{d}x \\
&\quad + \sum_{k=1}^N \left(2\kappa_k \frac{\left|\dot{R}_k\right|^2}{R_k^2} - \mu_g \frac{\left|\dot{R}_k\right|^3}{R_k^2}\right).
\end{aligned} \tag{31}$$

Proof. Multiplying the momentum equation (7) by $\partial_t u_f + u_f \partial_x u_f$ and integrating over the fluid domain $\mathcal{F}$ yield

$$\begin{aligned}\int_{\mathcal{F}} \rho_f |\partial_t u_f + u_f \partial_x u_f|^2 \,\mathrm{d}x &= \int_{\mathcal{F}} (\partial_t u_f + u_f \partial_x u_f) \partial_x \Sigma_f \,\mathrm{d}x \\ &= T_4 - T_5,\end{aligned} \tag{32}$$

with

$$\begin{aligned} T_4 &= \sum_{k=0}^{N} \Sigma_f\big(x_{k+1}^-\big)(\partial_t u_f + u_f \partial_x u_f)\big(x_{k+1}^-\big) - \Sigma_f\big(x_k^+\big)(\partial_t u_f + u_f \partial_x u_f)\big(x_k^+\big), \\ T_5 &= \int_{\mathcal{F}} \Sigma_f \partial_x (\partial_t u_f + u_f \partial_x u_f) \,\mathrm{d}x. \end{aligned}$$

The boundary term T_4 can be simplified by using the interface conditions (10),

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\dot{c}_k \pm \dot{R}_k\Big) = \frac{\mathrm{d}}{\mathrm{d}t}\big(u_f\big(x_k^\pm\big)\big) = (\partial_t u_f + u_f \partial_x u_f)\big(x_k^\pm\big).$$

Then one obtains

$$T_4 = \sum_{k=0}^{N} \Sigma_f\big(x_{k+1}^-\big)\Big(\ddot{c}_{k+1} - \ddot{R}_{k+1}\Big) - \Sigma_f\big(x_k^+\big)\Big(\ddot{c}_k + \ddot{R}_k\Big).$$

The boundary conditions (9) allow to reorganize the sum, and using the bubble equations of motion (11) and (12) and the bubble pressure law (14), we have successively

$$\begin{aligned} T_4 &= -\sum_{k=1}^{N} \Sigma_f\big(x_k^+\big)\Big(\ddot{c}_k + \ddot{R}_k\Big) - \Sigma_f\big(x_k^-\big)\Big(\ddot{c}_k - \ddot{R}_k\Big) \\ &= -\sum_{k=1}^{N} \ddot{c}_k\big(\Sigma_f\big(x_k^+\big) - \Sigma_f\big(x_k^-\big)\big) + \ddot{R}_k\big(\Sigma_f\big(x_k^+\big) + \Sigma_f\big(x_k^-\big)\big) \\ &= -\sum_{k=1}^{N} m_k\left(|\ddot{c}_k|^2 + \frac{1}{3}\left|\ddot{R}_k\right|^2\right) + 2\ddot{R}_k\left(\mu_g \frac{\dot{R}_k}{R_k} - \frac{\kappa_k}{R_k}\right) \\ &= -\sum_{k=1}^{N}\left\{ m_k\left(|\ddot{c}_k|^2 + \frac{1}{3}|\ddot{R}_k|^2\right) + \frac{\mathrm{d}}{\mathrm{d}t}\left[\mu_g \frac{\left|\dot{R}_k\right|^2}{R_k} - 2\kappa_k \frac{\dot{R}_k}{R_k}\right]\right\} \\ &\quad + \sum_{k=1}^{N}\left(2\kappa_k \frac{\left|\dot{R}_k\right|^2}{R_k^2} - \mu_g \frac{\left(\dot{R}_k\right)^3}{R_k^2}\right). \end{aligned}$$

We now turn to the volumic term T_5. Developing the term T_5 gives

$$T_5 = T_6 + \int_{\mathcal{F}} \mu_f (\partial_x u_f)^3 \,\mathrm{d}x - T_7 - \int_{\mathcal{F}} \mathrm{p}_f(\rho_f)|\partial_x u_f|^2 \,\mathrm{d}x, \tag{33}$$

with

$$\begin{aligned} T_6 &= \int_{\mathcal{F}} \mu_f \partial_x u_f (\partial_t(\partial_x u_f) + u_f \partial_x(\partial_x u_f)) \,\mathrm{d}x, \\ T_7 &= \int_{\mathcal{F}} \mathrm{p}_f(\rho_f)(\partial_t(\partial_x u_f) + u_f \partial_x(\partial_x u_f)) \,\mathrm{d}x. \end{aligned}$$

These two terms can be handled by classical manipulations, providing

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{\mathcal{F}} \mu_f \frac{|\partial_x u_f|^2}{2}\,\mathrm{d}x\right] = T_6 + \int_{\mathcal{F}} \mu_f \frac{(\partial_x u_f)^3}{2},$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{\mathcal{F}} \mathrm{p}_f(\rho_f)\partial_x u_f\,\mathrm{d}x\right] = T_7 - \int_{\mathcal{F}} \big(\mathrm{p}_f(\rho_f) - \mathrm{p}_f'(\rho_f)\rho_f\big)|\partial_x u_f|^2\,\mathrm{d}x.$$

As a result,

$$T_5 = \frac{\mathrm{d}}{\mathrm{d}t}\left[\int_{\mathcal{F}} \mu_f \frac{|\partial_x u_f|^2}{2} - \mathrm{p}_f(\rho_f)\partial_x u_f\,\mathrm{d}x\right] + \mu_f \int_{\mathcal{F}} \frac{(\partial_x u_f)^3}{2}\,\mathrm{d}x + \int_{\mathcal{F}} \mathrm{p}_f'(\rho_f)\rho_f|\partial_x u_f|^2\,\mathrm{d}x.$$

Finally plugging the expressions of T_4 and T_5 into (32) gives the expected result. □

In the regime of initial data specified in this section, we obtain the following corollary:

Corollary 5. *If initial data are constructed as in* (16)–(19) *and satisfy* (IC_0)–(IC_1)–(IC_2)*, there exists a constant* E_1 *depending only on the list of parameters* (25) *such that any classical solution to* (6)–(13) *on some time-interval* $[0,T]$ *satisfies:*

$$\begin{aligned}
&\sup_{[0,T]}\left(\int_{\mathcal{F}} \mu_f \frac{|\partial_x u_f|^2}{2}\,\mathrm{d}x + \mu_g \sum_{k=1}^N \frac{\left|\dot{R}_k\right|^2}{R_k}\right) \\
&+ \int_0^T \left(\int_{\mathcal{F}} \rho_f |\partial_t u_f + u_f\partial_x u_f|^2\,\mathrm{d}x + \sum_{k=1}^N m_k\left(|\ddot{c}_k|^2 + \left|\ddot{R}_k\right|^2\right)\right) \\
&\le \sup_{[0,T]}\left[\left(2\sum_{k=1}^N \kappa_k \frac{\left|\dot{R}_k\right|}{R_k}\right) + \int_{\mathcal{F}} \mathrm{p}_f(\rho_f)|\partial_x u_f|\mathrm{d}x\right] + \int_0^T \int_{\mathcal{F}} \mu_f \frac{|\partial_x u_f|^3}{2}\,\mathrm{d}x \\
&+ \int_0^T \sum_{k=1}^N \left(2\kappa_k \frac{\left|\dot{R}_k\right|^2}{R_k^2} + \mu_g \frac{\left|\dot{R}_k\right|^3}{R_k^2}\right) + E_1.
\end{aligned} \tag{34}$$

Proof. Integrating identity (31) given in Proposition 4 between 0 and $t \le T$, rejecting all non-signed term on the right-hand side that we bound then by putting absolute values, it yields:

$$\begin{aligned}
&\left(\int_{\mathcal{F}} \mu_f \frac{|\partial_x u_f|^2}{2}\,\mathrm{d}x + \mu_g \sum_{k=1}^N \frac{\left|\dot{R}_k\right|^2}{R_k}\right) \\
&+ \int_0^t \left(\int_{\mathcal{F}} \rho_f |\partial_t u_f + u_f\partial_x u_f|^2\,\mathrm{d}x + \sum_{k=1}^N m_k\left(|\ddot{c}_k|^2 + \left|\ddot{R}_k\right|^2\right)\right) \\
&+ \int_0^T \int_{\mathcal{F}} \kappa_f\gamma_f\rho_f^{\gamma_f}|\partial_x u_f|^2\,\mathrm{d}x \\
&\le \left[\left(2\sum_{k=1}^N \kappa_k \frac{\left|\dot{R}_k\right|}{R_k}\right) + \int_{\mathcal{F}} \mathrm{p}_f(\rho_f)|\partial_x u_f|\,\mathrm{d}x\right] + \int_0^t \int_{\mathcal{F}} \mu_f \frac{|\partial_x u_f|^3}{2}\,\mathrm{d}x
\end{aligned}$$

$$+\int_0^t \sum_{k=1}^N \left(2\kappa_k \frac{\left|\dot R_k\right|^2}{R_k^2} + \mu_g \frac{\left|\dot R_k\right|^3}{R_k^2} \right) + \int_{\mathcal{F}^0} \mu_f \frac{\left|\partial_x u_f^0\right|^2}{2}\,\mathrm{d}x + \sum_{k=1}^N \left(\mu_g \frac{\left|\dot R_k^0\right|^2}{R_k^0} + 2\kappa_k \frac{\left|\dot R_k^0\right|}{R_k^0} \right).$$

To obtain the expected result, it remains to drop the last term in the left-hand side which is positive and to bound the last term on the right-hand side by a constant E_1 with the expected dependencies. For this, we note that the first integral in this last term clearly depends on $\left\|u_f^0\right\|_{H_0^1(\Omega)}$. Concerning the first term in the sum, the continuity of the velocity field (19) rewrites for any k :

$$\dot R_k^0 = \frac{1}{2} \int_{B_k^0} \partial_x u_f^0(s)\,\mathrm{d}s,$$

so that

$$\left|\dot R_k^0\right| \le \frac{1}{2}\sqrt{R_k^0} \left(\int_{B_k^0} \left|\partial_x u_f^0(s)\right|^2 \mathrm{d}s \right)^{1/2}.$$

As a consequence it holds

$$\sum_{k=1}^N \frac{\left|\dot R_k^0\right|^2}{R_k^0} \le \frac{1}{2} \int_{\cup B_k^0} \left|\partial_x u_f^0(s)\right|^2 \mathrm{d}s \le \left\|u_f^0\right\|^2_{H^1(\Omega)}.$$

Finally, the last term in the sum is bounded by using that κ_k scales like $1/N$. Indeed, applying (IC_0) with (IC_1) we have:

$$\frac{\kappa_k}{\sqrt{R_k^0}} \le \frac{M_\infty}{\sqrt{2d_\infty}} \frac{1}{\sqrt{N}} \qquad \forall\ k = 1, \dots, N,$$

and then, with the above bound on $\left|\dot R_k^0\right| / \sqrt{R_k^0}$, we obtain:

$$\sum_{k=1}^N \kappa_k \frac{\left|\dot R_k^0\right|}{R_k^0} \le \frac{M_\infty}{\sqrt{8d_\infty}} \left(\int_{\cup B_k^0} \left|\partial_x u_f^0\right|^2 \right)^{\frac{1}{2}}.$$

This ends the proof. □

2.2. Extended stress-tensor estimates

In order to obtain regularity estimates on the fluid velocity field, a classical way is to use the stress tensor. However Σ_f is only defined on the fluid domain $\mathcal{F}$, so that estimates on this stress tensor depend on the geometric properties of $\mathcal{F}$, in particular the number of bubbles. In order to remove this dependency, we define new stress tensors for the fluid and for the gas phase, extended to the full domain Ω:

$$\tilde\Sigma_f = \begin{cases} \Sigma_f, & \text{in } \mathcal{F}, \\ \frac{\Sigma_f(x_k^-)+\Sigma_f(x_k^+)}{2} - \frac{\Sigma_f(x_k^-)-\Sigma_f(x_k^+)}{2R_k}(x-c_k), & \text{in } B_k,\ k=1,\dots,N, \end{cases} \tag{35}$$

and

$$\tilde\Sigma_g = \begin{cases} \Sigma_k, & \text{in } B_k,\ k=1,\dots,N, \\ \Sigma_N, & \text{in } \mathcal{F}_N, \\ \Sigma_0, & \text{in } \mathcal{F}_0, \\ \Sigma_k + \frac{\Sigma_{k+1}-\Sigma_k}{x_{k+1}^- - x_k^+}(x - x_k^+), & \text{in } \mathcal{F}_k,\ k=1,\dots,N-1. \end{cases} \tag{36}$$

Observe that these two stress tensors are continuous at each interface $x_k^\pm$. We analyze here the properties of these extensions, when Σ_f obeys further the continuity properties adapted from (11) to (13). In the stationary analysis of this subsection, these latter identities may stand for definitions of $\ddot{c}_k$ and $\ddot{R}_k$. These quantities will be related to the dynamical problem afterwards.

Proposition 6. *Assume that $\Sigma_f \in H^1(\mathcal{F})$ satisfies (11) and (12) with Σ_k defined by (13). Then $\tilde{\Sigma}_f \in H^1(\Omega)$ and there exists a constant $C_0 > 0$ such that*

$$\left\|\tilde{\Sigma}_f\right\|_{H^1(\Omega)} \leq C_0 \left[\|\Sigma_f\|^2_{H^1(\mathcal{F})} + \sum_{k=1}^{N} (m_k)^2 \left(\left|\ddot{R}_k\right|^2 + \frac{|\ddot{c}_k|^2}{R_k}\right) + \sum_{k=1}^{N} \left(\mu_g^2 \frac{\left|\dot{R}_k\right|^2}{R_k} + \frac{\kappa_k^2}{R_k}\right)\right]^{\frac{1}{2}}. \tag{37}$$

Proof. By continuity of $\tilde{\Sigma}_f$ at the interfaces,

$$\left\|\tilde{\Sigma}_f\right\|^2_{H^1(\Omega)} = \|\Sigma_f\|^2_{H^1(\mathcal{F})} + \sum_{k=1}^{N} \left\|\tilde{\Sigma}_f\right\|^2_{H^1(B_k)}.$$

We just have to study the H^1 norm of $\tilde{\Sigma}_f$ on a bubble B_k. The L^2 norm of Σ_f can be bounded as follows:

$$\begin{aligned}
\left\|\tilde{\Sigma}_f\right\|^2_{L^2(B_k)} &= \int_{B_k} \left|\frac{\Sigma_f(x_k^-) + \Sigma_f(x_k^+)}{2}\right|^2 + \left|\frac{\Sigma_f(x_k^-) - \Sigma_f(x_k^+)}{2R_k}\right|^2 |x - c_k|^2 \,\mathrm{d}x \\
&= \left|\frac{\Sigma_f(x_k^-) + \Sigma_f(x_k^+)}{2}\right|^2 2R_k + \left|\frac{\Sigma_f(x_k^-) - \Sigma_f(x_k^+)}{2R_k}\right|^2 \frac{2R_k^3}{3} \\
&= \left|\Sigma_f(x_k^-) + \Sigma_f(x_k^+)\right|^2 \frac{R_k}{2} + \left|\Sigma_f(x_k^-) - \Sigma_f(x_k^+)\right|^2 \frac{R_k}{6}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\left\|\partial_x \tilde{\Sigma}_f\right\|^2_{L^2(B_k)} &= \int_{B_k} \left|\frac{\Sigma_f(x_k^-) - \Sigma_f(x_k^+)}{2R_k}\right|^2 \mathrm{d}x \\
&= \left|\frac{\Sigma_f(x_k^-) - \Sigma_f(x_k^+)}{2R_k}\right|^2 2R_k = \frac{\left|\Sigma_f(x_k^-) - \Sigma_f(x_k^+)\right|^2}{2R_k}.
\end{aligned}$$

We now gather the two estimates, and obtain

$$\begin{aligned}
\left\|\tilde{\Sigma}_f\right\|^2_{H^1(B_k)} &= \frac{\left|\Sigma_f(x_k^-) - \Sigma_f(x_k^+)\right|^2}{2R_k} + \left|\Sigma_f(x_k^-) + \Sigma_f(x_k^+)\right|^2 \frac{R_k}{2} \\
&\quad + \left|\Sigma_f(x_k^-) - \Sigma_f(x_k^+)\right|^2 \frac{R_k}{6}.
\end{aligned}$$

Using the equations of motion of the bubbles (11) and the definition (13) of the stress tensor Σ_k, one gets

$$\begin{aligned}
\left\|\tilde{\Sigma}_f\right\|^2_{H^1(B_k)} &\leq m_k^2 |\ddot{c}_k|^2 \left(\frac{1}{2R_k} + \frac{R_k}{6}\right) + \left(\frac{m_k}{3}\ddot{R}_k + 2\left(\mu_g \frac{\dot{R}_k}{R_k} - \frac{\kappa_k}{R_k}\right)\right)^2 \frac{R_k}{2} \\
&\leq m_k^2 |\ddot{c}_k|^2 \left(\frac{1}{2R_k} + \frac{R_k}{6}\right) + \frac{2}{9} m_k^2 \left|\ddot{R}_k\right|^2 R_k + 8\mu_g^2 \frac{\left|\dot{R}_k\right|^2}{R_k} + 8\frac{\kappa_k^2}{R_k}
\end{aligned}$$

Finally, this gives the estimate

$$\left\|\tilde{\Sigma}_f\right\|^2_{H^1(\Omega)} \leq 8\left[\|\Sigma_f\|^2_{H^1(\mathcal{F})} + \sum_{k=1}^N (m_k)^2\left(\left|\ddot{R}_k\right|^2 R_k + |\ddot{c}_k|^2\left(\frac{1}{R_k} + R_k\right)\right) + \sum_{k=1}^N\left(\mu_g^2\frac{\left|\dot{R}_k\right|^2}{R_k} + \frac{\kappa_k^2}{R_k}\right)\right]^{\frac{1}{2}},$$

which leads to the desired result since $R_k < 1$. □

From the above inequality we deduce the following L^∞-bound in case Σ_f is a viscous stress tensor:

Proposition 7. *Assume that $\Sigma_f \in H^1(\mathcal{F})$ satisfies (11) and (12) with Σ_k defined by (13). Assume further that Σ_f is related to $(\rho_f, u_f) \in H^1(\mathcal{F}) \times H^2(\mathcal{F})$ via (8). Then, there exists $C_1 > 0$ such that*

$$\|\partial_x u_f\|_{L^\infty(\mathcal{F})} \leq \frac{C_1}{\mu_f}\left(\left\|\tilde{\Sigma}_f\right\|_{H^1(\Omega)} + \|\mathrm{p}_f(\rho_f)\|_{L^\infty(\mathcal{F})}\right). \tag{38}$$

Proof. In the fluid domain, the stress tensor writes $\Sigma_f = \mu_f \partial_x u_f - \mathrm{p}_f$, which gives

$$\partial_x u_f = \frac{1}{\mu_f}(\Sigma_f - \mathrm{p}_f(\rho_f)).$$

Hence one has

$$\|\partial_x u_f\|_{L^\infty(\mathcal{F})} \leq \frac{1}{\mu_f}\left(\|\Sigma_f\|_{L^\infty(\mathcal{F})} + \|\mathrm{p}_f(\rho_f)\|_{L^\infty(\mathcal{F})}\right).$$

The definition of global tensor $\tilde{\Sigma}_f$ gives then

$$\|\Sigma_f\|_{L^\infty(\mathcal{F})} \leq \left\|\tilde{\Sigma}_f\right\|_{L^\infty(\Omega)}.$$

The $H^1(\Omega) \subset L^\infty(\Omega)$ embedding allows to conclude the proof. □

One can note here the gain of working with an extended stress tensor. Indeed, the constant C_1 we obtain in the previous proposition is independent of the position of the particles and their radius. This would not be *a priori* the case if we wanted to control $\partial_x u$ by Σ_f only. Nevertheless, in (37) we introduced on the right-hand side negative powers of R_k that we shall control independently. To this end, we performed a symmetric construction with the bubble stress-tensor Σ_g and we provide now a corresponding proposition:

Proposition 8. *There holds $\tilde{\Sigma}_g \in H^1(\Omega)$ and there exists a constant $C_2 > 0$ such that*

$$\left\|\tilde{\Sigma}_g\right\|_{H^1(\Omega)} \leq C_2\left[\left\|\tilde{\Sigma}_f\right\|^2_{H^1(\Omega)} + \frac{1}{\min\limits_{k\in\{0,\ldots,N\}}|\mathcal{F}_k|}\sum_{k=1}^N (m_k)^2\left(\left|\ddot{R}_k\right|^2 + |\ddot{c}_k|^2\right)\right]^{1/2}. \tag{39}$$

Proof. By straightforward calculations, the definition of $\tilde{\Sigma}_g$ yields

$$\begin{aligned}\left\|\tilde{\Sigma}_g\right\|^2_{H^1(\Omega)} \leq &\sum_{k=1}^N 2R_k|\Sigma_k|^2 + |\Sigma_0|^2\left|x_1^- - x_0^+\right| + |\Sigma_N|^2\left|x_{N+1}^- - x_N^+\right| \\ &+ \sum_{k=1}^{N-1}\left(\frac{|\Sigma_{k+1} - \Sigma_k|^2}{\left|x_{k+1}^- - x_k^+\right|} + 2|\Sigma_k|^2\left|x_{k+1}^- - x_k^+\right|\right. \\ &\left.+\frac{2}{3}|\Sigma_{k+1} - \Sigma_k|^2\left|x_{k+1}^- - x_k^+\right|\right)\end{aligned}$$

$$\leq C\left(\sum_{k=1}^{N} R_k|\Sigma_k|^2 + \sum_{k=0}^{N}|\mathcal{F}_k||\Sigma_k|^2 + \sum_{k=1}^{N}\frac{|\Sigma_{k+1}-\Sigma_k|^2}{|x_{k+1}^- - x_k^+|}\right)$$

where C is a positive constant, since the length of the bubbles and of the fluid parts are bounded. Summing equations (11) and (12) leads to

$$\Sigma_k = \Sigma_f\left(x_k^+\right) - \frac{m_k}{2}\left(\ddot{c}_k + \frac{\ddot{R}_k}{3}\right). \tag{40}$$

We deduce the following estimates, with some constant $C' > 0$,

$$\begin{aligned}|\Sigma_k| &\leq \left\|\tilde{\Sigma}_f\right\|_{L^\infty(B_k)} + m_k C'\left(|\ddot{c}_k| + \left|\ddot{R}_k\right|\right),\\ |\Sigma_{k+1}-\Sigma_k| &\leq \left|x_{k+1}^+ - x_k^+\right|^{1/2}\left\|\partial_x\tilde{\Sigma}_f\right\|_{L^2\left(x_k^+,x_{k+1}^+\right)}\\ &\quad + \frac{m_k}{2}\left(\left|\ddot{R}_k\right| + |\ddot{c}_k|\right) + \frac{m_{k+1}}{2}\left(\left|\ddot{R}_{k+1}\right| + |\ddot{c}_{k+1}|\right).\end{aligned}$$

One can now go back to the estimate on $\tilde{\Sigma}_g$. Noting the relation:

$$\sum_{k=0}^{N}|\mathcal{F}_k| + \sum_{k=1}^{N}2R_k = |\Omega|,$$

the embedding $H^1(\Omega)\subset L^\infty(\Omega)$ implies the expected result. □

As for the fluid stress tensor, we deduce from the previous computation a control on the $(\Sigma_k)_{k=1,\ldots,N}$ by applying again the embedding $H^1(\Omega)\subset L^\infty(\Omega)$:

Corollary 9. *Under the same assumptions as in Proposition 8, there holds:*

$$\max_{k=1,\ldots,N-1}\left|\mu_g\frac{\dot{R}_k}{R_k} - \frac{\kappa_k}{R_k}\right| \leq C_2\left[\left\|\tilde{\Sigma}_f\right\|^2_{H^1(\Omega)} + \frac{1}{\min\limits_{k\in\{0,\ldots,N\}}|\mathcal{F}_k|}\sum_{k=1}^{N}(m_k)^2\left(\left|\ddot{R}_k\right|^2 + |\ddot{c}_k|^2\right)\right]^{1/2}. \tag{41}$$

This latter corollary shall enable to control the radius of the bubble from below, preventing from collapse.

2.3. Proof of Theorem 1

We combine now the computations of the previous section to construct a solution on a time-interval independent of the number N of bubbles. For this, we show that the following bounds can be continued:

(Q_1) $d_\infty \leq NR_k \leq (d_\infty)^{-1}$, $k=1,\ldots,N$,
(Q_2) $d_\infty \leq N|\mathcal{F}_k| \leq (d_\infty)^{-1}$, $k=1,\ldots,N$,
(Q_3) $\underline{\rho}_\infty \leq \rho_f \leq \bar{\rho}_\infty$ on $\mathcal{F}(t)$

and, introducing a sufficiently large $K>0$:

(Q_4) $\left[\int_{\mathcal{F}}\mu_f\frac{|\partial_x u_f|^2}{2}\,\mathrm{d}x + \mu_g\sum_{k=1}^{N}\frac{|\dot{R}_k|^2}{R_k}\right] \leq K,$

(Q_5) $\int_0^t\left[\left\|\tilde{\Sigma}_f\right\|^2_{H^1(\Omega)} + \left\|\tilde{\Sigma}_g\right\|^2_{H^1(\Omega)} + \sum_{k=1}^{N}m_k\left(\left|\ddot{R}_k\right|^2 + |\ddot{c}_k|^2\right)\right]\mathrm{d}s \leq K.$

We keep the convention here that tildas represent extended stress tensors as constructed in the previous subsection. We prove that, if K is chosen sufficiently large wrt the list of parameters (25), then we have such estimates on a time interval $(0,T)$ that depends only on the same list of parameters (25) (possibly *via* K).

Technically, we apply a continuation argument based on the *a priori* assumption that the solution exists. The precise statement is the following proposition in which we denote $(\mathscr{Q}_i)_{i=1,\dots,5}$ the estimates corresponding to the above $(Q_i)_{i=1,\dots,5}$ where large inequalities are replaced with strict inequalities. Tacitly, all constants that are introduced in the following proposition may depend on the list of parameters (25).

Proposition 10. *There exists $K_\infty > 0$ such that, for any $K > K_\infty$ there exists $T_\infty[K] > 0$ for which the following statement holds: if $T \leq T_\infty[K]$ and $((\rho_f, u_f), (c_k, R_k)_{k=1,\dots,N})$ is a classical solution to* (6)*–*(13) *on $(0,T)$ satisfying (Q_1)–(Q_5) then it satisfies also $(\mathscr{Q}_1)$–$(\mathscr{Q}_5)$.*

The proof of Proposition 10 is the content of Appendix A. We explain here how it implies Theorem 1. For this, given $K > 0$ we introduce:

$$\mathcal{I} := \{T \in (0,\infty) \text{ s.t. the unique classical solution exists on } (0,T) \text{ and satisfies } (Q_1) - (Q_5)\}.$$

Firstly, thanks to the local-in-time existence result, there exists T_0 depending on N such that we have a classical solution on $(0,T_0)$. Indeed, for such a solution the radius R_k and c_k are continuous in time. Since we assume initially (IC_1)–(IC_2) (resp. (24)) we have that, up to restrict T_0, this solution satisfies (Q_1)–(Q_2) (resp. (Q_3)) on $[0,T_0]$. Similarly, we remark that the quantities on the left-hand side of (Q_4)–(Q_5) are continuous time-dependent functions of the classical solution. Since the left-hand side of (Q_4) is controlled initially by $\left\| u_f^0 \right\|_{H_0^1(\Omega)}$ and parameters involved in (25) (see the proof of Coro. 5), there exists K_0 sufficiently large depending only on the list of parameters (25) such that we can enforce (Q_4)–(Q_5) on $[0,T_0]$ also whatever the value of $K > K_0$.

Let fix now $K = \max(K_0, K_\infty)$ with K_∞ given by Proposition 10 and denote $T_\infty = T_\infty[K]$. By the previous arguments, we have that $[0,T_0] \subset \mathcal{I}$. We show now that $[0,T_\infty] \subset \mathcal{I}$ which shall end the proof. By restriction, $\mathcal{I} \cap [0,T_\infty]$ is a closed subinterval of $[0,T_\infty]$ containing $[0,T_0]$. Let us prove that $\mathcal{I} \cap [0,T_\infty]$ is open (in $[0,T_\infty]$). Indeed, assume $[0,T]$ is a strict subinterval of $[0,T_\infty]$ in $\mathcal{I}$, then we can apply Proposition 10 and the solution satisfies $(\mathscr{Q}_1)$–$(\mathscr{Q}_5)$ on $[0,T]$. It remains to show that we can continue the solution beyond $[0,T]$. The inequalities $(\mathscr{Q}_1)$-$(\mathscr{Q}_5)$ being strict, the large inequalities (Q_1)–(Q_5) shall be satisfied on a slightly longer interval by continuity. To extend the solution, we note that (Q_1)–(Q_2) (resp. (Q_3)) entail "a minimum distance between" and "a minimum radius of" bubbles (resp. strictly positive distance to vacuum) on $[0,T]$. Inequality (Q_4) also ensures a (uniform) bound from above for $\|u_f\|_{H^1(\mathcal{F})}$ on $[0,T]$. By Proposition 29 of Appendix B we have also a uniform bound for $\|\rho_f\|_{H^1(\mathcal{F})}$ (up to take T_∞ smaller). We can then apply the local-in-time existence result with initial data $\Big((\rho_f(T',\cdot), u_f(T',\cdot)), (c_k(T'), R_k(T'))_{k=1,\dots,N}\Big)$ for T' arbitrary close to T. This yields a solution on some time-interval ΔT (independent of T', given the uniform bound above). By concatenation, we obtain a solution on $(0, T' + \Delta T)$ where $T' + \Delta T > T$ for a well-chosen T'.

To conclude this section, we mention that the proof above entails that we have the following corollary to Theorem 1:

Corollary 11. *The unique classical solution to* (6)*–*(13) *on $[0,T_\infty]$ satisfies the bounds (Q_1)–(Q_2) (resp. (Q_3)) with d_∞ corresponding to (IC_1)–(IC_2) (res. $\underline{\rho}_\infty$, $\bar{\rho}_\infty$ corresponding to* (24)*) and (Q_4)–(Q_5) with K_∞ depending on the list of parameters* (25)*.*

3. Construction of macroscopic unknowns

In this section, we detail the construction of the unknowns for the macroscopic model starting from a sequence of solutions to the microscopic model with increasing number of gas bubbles. The full justification of the system (1)–(4) is postponed to the next section. From now on, we fix initial data $\left(\bar{\rho}_f^0,\, \bar{\rho}_g^0,\, \bar{u}^0,\, \bar{\alpha}_f^0,\, \bar{\alpha}_g^0,\, \bar{f}_g^0\right)$ for the macroscopic model. All these quantities are $H^1(\Omega)$ functions. We assume further that they fulfill conditions (20), (21) and (23).

The framework identified in the previous section must be adapted for homogenization purpose. For instance, given a N-bubble solution the gas unknowns at-hand are *a priori* the discrete set of center/radius/mass $(c_k, R_k, m_k)_{k=1,\dots,N}$. From them, we can reconstruct a (functional) density and a covolume by defining:

$$f_g^{(N)} := \sum_{k=1}^{N} \frac{1}{2NR_k} \mathbb{1}_{B_k} \qquad \rho_g^{(N)} := \sum_{k=1}^{N} \frac{m_k}{2R_k} \mathbb{1}_{B_k}. \tag{42}$$

However, these reconstructed functions experience $O(1)$ jumps through bubble/fluid interfaces and might not have sufficient regularity to perform the homogenization process. To gain regularity, we shall propagate an initial regularity through a well-chosen evolution equation (which extends the one satisfied by $f_g^{(N)}$, $\rho_g^{(N)}$ on the B_k). However, this requires to be able to construct regular initial covolume and density (with uniform bounds in terms of N). This is obtained with the following proposition:

Proposition 12. *Under the assumption that the initial data fulfill the conditions* (20)–(21)–(23)*, there exist sequences of initial bubble center/radii* $\left(\left(c_k^{(N),0}, R_k^{(N),0}\right)_{k=1,\dots,N}\right)_{N\in\mathbb{N}}$ *and masses* $\left(m_k^{(N)}\right)_{k=1,\dots,N}$ *so that:*

(i) (IC_0)–(IC_1)–(IC_2) *are satisfied with* M_∞ *and* d_∞ *independent of* N*,*

(ii) *there exist* $H^1(\Omega)$ *extensions* $\left(\tilde{f}_g^{(N),0}, \tilde{\rho}_g^{(N),0}\right)$ *of the associated reconstructed covolumes and densities such that:*
- $\left(\tilde{f}_g^{(N),0}, \tilde{\rho}_g^{(N),0}\right)$ *is bounded in* $H^1(\Omega)$
- *for arbitrary* $\beta \in C^1([0,\infty) \times [0,\infty))$ *there holds:*

$$\beta\left(\tilde{\rho}_g^{(N),0}, \tilde{f}_g^{(N),0}\right) \mathbb{1}_{\Omega \setminus \bar{\mathcal{F}}^{(N),0}} \rightharpoonup \bar{\alpha}_g^0 \beta\left(\bar{\rho}_g^0, \bar{f}_g^0\right) \quad \text{in } \mathcal{D}'(\Omega).$$

Proof. Up to a localizing argument, we give a proof in the case:

$$(1 - \alpha_{\min}) \left\| \bar{f}_g^0 \right\|_{L^\infty(\Omega)} < f_{\min} := \inf_{\Omega} \bar{f}_g^0. \tag{43}$$

To construct our gas bubble, we note that $\bar{\alpha}_g^0 \bar{f}_g^0$ is a probability density on Ω. Then, we might construct the associated cumulative distribution function:

$$F_g(x) = \int_{-1}^{x} \bar{\alpha}_g^0(x) \bar{f}_g^0(x) \, \mathrm{d}x.$$

With assumptions (20)–(21)–(23), this is a C^1 one-to-one mapping $\bar{\Omega} \to [0,1]$ with $F_g' \geq \alpha_{\min} f_{\min}$ on Ω. We set then:

$$c_k^0 := F_g^{-1}\left(\frac{k}{N+1}\right), \quad R_k^0 := \frac{1}{2N} \left[\bar{f}_g(c_k^0)\right]^{-1} \quad m_k := 2R_k^0 \bar{\rho}_g^0(c_k^0) \quad \text{for } k = 1, \dots, N. \tag{44}$$

Considering the bounds from above and from below for F_g', we obtain that:

$$\frac{1}{N+1} \frac{1}{(1 - \alpha_{\min}) \left\| \bar{f}_g^0 \right\|_{L^\infty(\Omega)}} \leq c_{k+1}^0 - c_k^0 \leq \frac{1}{N+1} \frac{1}{\alpha_{\min} f_{\min}}$$

while

$$\frac{1}{2N \left\| \bar{f}_g^0 \right\|_{L^\infty(\Omega)}} \leq R_k^0 \leq \frac{1}{2N} \frac{1}{f_{\min}}.$$

In particular

$$\begin{aligned}\left|\mathcal{F}_k^0\right| = \left(c_{k+1}^0 - R_{k+1}^0\right) - \left(c_k^0 + R_k^0\right) &\geq \frac{1}{N}\left(\frac{N/(N+1)}{(1-\alpha_{\min})\left\|\bar{f}_g^0\right\|_{L^\infty(\Omega)}} - \frac{1}{f_{\min}}\right)\\ &\leq c_{k+1}^0 - c_k^0 \leq \frac{1}{N}\frac{1}{\alpha_{\min} f_{\min}}\end{aligned}$$

where $N/((N+1)(1-\alpha_{\min}))\left\|\bar{f}_g^0\right\|_{L^\infty(\Omega)} - 1/f_{\min} > 0$ by (43) for N large. Finally, we have:

$$\frac{1}{N}\frac{\rho_{\min}}{\left\|\bar{f}_g^0\right\|_{L^\infty(\Omega)}} \leq m_k \leq \frac{1}{N}\frac{\left\|\bar{\rho}_g^0\right\|_{L^\infty(\Omega)}}{f_{\min}}.$$

Item (i) is satisfied.

For item (ii), we remark that the reconstructed densities and covolumes read:

$$f_g^{(N),0} := \sum_{k=1}^N \frac{1}{2NR_k^0}\mathbb{1}_{B_k^0} \qquad \rho_g^{(N),0} := \sum_{k=1}^N \frac{m_k}{2R_k^0}\mathbb{1}_{B_k^0}.$$

We recall that we denote $B_k^0 = \left(x_k^-, x_k^+\right)$ where $x_k^\pm = c_k^0 \pm R_k^0$ (and $x_0^+ = -1$, $x_{N+1}^- =1$). At this point, we note that by item (i), we have:

$$\min_{k\in\{0,\ldots,N\}}\left|x_{k+1}^- - x_k^+\right| \geq \frac{1}{2d_\infty N}.$$

Consequently, for $k = 2,\ldots,N-1$. we can construct a piecewise affine function ψ_k^0 with satisfies $\psi_k^0 = 1$ on B_k^0, that vanishes in x_{k+1}^- and x_{k-1}^+ and further away from B_k^0. For $k=1$ and $k=N$ we define similarly ψ_1^0 and ψ_N^0 up to the condition that ψ_1^0 is constant equal to 1 between -1 and B_1^0 (resp. ψ_N^0 is constant equal to 1 between B_N^0 and 1). Then, we set:

$$\tilde{f}_g^{(N),0} := \sum_{k=1}^N \frac{1}{2NR_k^0}\psi_k^0 \qquad \tilde{\rho}_g^{(N),0} := \sum_{k=1}^N \frac{m_k}{2R_k^0}\psi_k^0.$$

By standard computations, we have for instance:

$$\begin{aligned}\left\|\tilde{f}_g^{(N),0}\right\|_{L^2(\Omega)}^2 &\leq \sum_{k=1}^N \frac{1}{N^2|R_k^0|^2}\left\|\psi_k^0\right\|_{L^2(\Omega)}^2\\ &\lesssim \frac{1}{N}\sum_{k=1}^N \frac{1}{N^2|R_k^0|^2} \lesssim \left\|\bar{f}_g^0\right\|_{L^\infty(\Omega)}^2\end{aligned}$$

where the first inequality on the second line involves a constant depending on d_∞. We also derive using that $\psi_{k+1} = 1 - \psi_k$ on $\mathrm{Supp}(\psi_k') \cap \mathrm{Supp}\left(\psi_{k+1}'\right)$:

$$\begin{aligned}\left\|\partial_x \tilde{f}_g^{(N),0}\right\|_{L^2(\Omega)}^2 &\lesssim \sum_{k=1}^{N-1}\left[\frac{1}{NR_{k+1}^0} - \frac{1}{NR_k^0}\right]^2 N\\ &\lesssim \sum_{k=1}^{N-1} N\left|\int_{c_k^0}^{c_{k+1}^0} \partial_x \bar{f}_g^0(z)\,\mathrm{d}z\right|^2 \lesssim \left\|\partial_x \bar{f}_g^0\right\|_{L^2(\Omega)}^2.\end{aligned}$$

In these computations, we use extensively the definitions (44) and also that $|B_k^0|$ and $|\mathcal{F}_k^0|$ are both of size $O(1/N)$. Similar arguments yield that:

$$\left\|\tilde{\rho}_g^{(N),0}\right\|_{H^1(\Omega)}^2 \lesssim \left\|\bar{\rho}_g^0\right\|_{H^1(\Omega)}^2.$$

Finally, for arbitrary $\beta \in C^1([0,\infty)\times[0,\infty))$ and $\varphi \in C_c^\infty(\Omega)$, we have:

$$\begin{aligned}\int_\Omega \beta\Big(\tilde\rho_g^{(N),0}, \tilde f_g^{(N),0}\Big)\mathbb{1}_{\Omega\setminus\tilde{\mathcal F}^{(N),0}}\varphi \,\mathrm{d}x &= \sum_{k=1}^N \int_{B_k^0} \beta\big(\bar\rho_g(c_k^0), \bar f_g(c_k^0)\big)\varphi(x)\,\mathrm{d}x \\ &= \sum_{k=1}^N 2R_k^0\beta\big(\bar\rho_g(c_k^0), \bar f_g(c_k^0)\big)\varphi(c_k^0) + O(1/N)\|\partial_x\varphi\|_{L^\infty(\Omega)} \\ &= \frac{1}{N}\sum_{k=1}^N \frac{\beta\big(\bar\rho_g(c_k^0), \bar f_g(c_k^0)\big)}{\bar f_g^0(c_k^0)}\varphi(c_k^0) + O(1/N)\|\partial_x\varphi\|_{L^\infty(\Omega)}.\end{aligned}$$

At this point, we remark that, by construction, we have that

$$\frac{1}{N}\sum_{k=1}^N \delta_{c_k^0} \rightharpoonup \bar\alpha_g^0 \bar f_g^0 \quad \text{in } \mathbb{P}(\Omega).$$

Since $t \mapsto \beta\big(\bar\rho_g^0(t), \bar f_g^0(t)\big)/\bar f_g^0(t)$ is continuous on $\bar\Omega$ we infer that:

$$\lim_{N\to\infty}\int_\Omega \beta\Big(\tilde\rho_g^{(N),0}, \tilde f_g^{(N),0}\Big)\mathbb{1}_{\Omega\setminus\tilde{\mathcal F}^{(N),0}}\varphi\,\mathrm{d}x = \int_\Omega \beta\big(\bar\rho_g, \bar f_g^0\big)\bar\alpha_g^0\varphi\,\mathrm{d}x.$$

This concludes the proof. □

Below, we pick a sequence of initial bubble distribution $\Big(c_k^{(N),0}, R_k^{(N),0}\Big)_{k=1,\dots,N}$ and masses $\Big(m_k^{(N)}\Big)_{k=1,\dots,N}$ given by Proposition 12. For any $N \in \mathbb{N}$, assuming the fluid initial data is associated with $\bar\rho_f^0, \bar u^0$, we construct initial data for the microscopic system like in (16)–(19). We have then that the initial data match the assumptions of Theorem 1 and we obtain a solution

$$\left(\rho_f^{(N)}, u_f^{(N)}, \Big(c_k^{(N)}, R_k^{(N)}\Big)_{k\in\{1,\dots,N\}}\right)$$

that is defined on a time-span $[0,T]$ which does not depend on N. This creates a sequence of solutions indexed by N whose asymptotic behavior (when $N\to\infty$) is analyzed in the remaining sections.

Firstly, Corollary 11 entails that we have uniform bounds on $[0,T]$ in the form of (A.1) and (A.2) with a right-hand side E_0 independent of N, and that (Q_1)–(Q_5) hold also with a constant K independent of N. In passing, we point out that all the bounds that are derived in Appendices A and B are available since they are obtained under the sole assumptions that initial data are of the form (16)–(19) and that the bounds (Q_1)–(Q_5) hold true. Below we denote $\tilde u^{(N)}$ the "mixture" velocity-field meaning that

$$\tilde u^{(N)} = \begin{cases} u_f^{(N)}, & \text{on } \mathcal{F}^{(N)}, \\ \dfrac{u_f\Big(x_k^{-,(N)}\Big)+u_f\Big(x_k^{+,(N)}\Big)}{2} \\ \qquad -\dfrac{u_f\Big(x_k^{-,(N)}\Big)-u_f\Big(x_k^{+,(N)}\Big)}{2R_k^{(N)}}(x-c_k), & \text{on } B_k^{(N)}, \quad k=1,\dots,N.\end{cases} \tag{45}$$

Note that the restriction of $\tilde u^{(N)}$ on the bubbles boils down to

$$\tilde u^{(N)}(\cdot,x) = \dot c_k^{(N)} + \frac{\dot R_k^{(N)}}{R_k^{(N)}}\Big(x - c_k^{(N)}\Big) \quad \text{on } B_k^{(N)}. \tag{46}$$

In what remains of this section, we introduce functions describing the different species and the mixture and we analyse their possible convergences. Since we use mostly compactness argument below, all convergence results must be understood "up to the extraction of a subsequence that we do not relabel."

3.1. Fluid unknowns

In (1), the fluid behavior is encoded through its "void fraction" $\bar{\alpha}_f$ and its density $\bar{\rho}_f$. We recover such quantities from microscopic counterparts. We start with the following construction of the void fraction:

Proposition 13. *Let $\chi^{(N)} = \mathbb{1}_{\mathcal{F}^{(N)}}$. It satisfies*

$$\begin{cases} \partial_t \chi^{(N)} + \tilde{u}^{(N)} \partial_x \chi^{(N)} = 0, & \text{on}(0,T)\times\Omega, \\ \chi^{(N)}(0,.) = \mathbb{1}_{\mathcal{F}^{(N),0}}. \end{cases} \tag{47}$$

Moreover, there exists $\bar{\alpha}_f \in L^\infty((0,T)\times\Omega)$, called the void fraction of the fluid*, such that, up to the extraction of a subsequence,*

$$\chi^{(N)} \rightharpoonup \bar{\alpha}_f \quad \text{in } L^\infty((0,T)\times\Omega) - w^* \quad \text{and} \quad 0 \le \bar{\alpha}_f \le 1 - 2d_\infty^2/3 \ a.e. \tag{48}$$

Proof. Since the fluid domain $\mathcal{F}^{(N)}$ is transported by the velocity field $\tilde{u}^{(N)}$, (47) holds. The convergence result is straightforward since the sequence $\chi^{(N)}$ is nonnegative and bounded in $L^\infty((0,T)\times\Omega)$. The limit is obviously positive. The only crucial information is the bound from above. For this, we remark that under (Q_1)–(Q_2), any sequence of bubble + fluid intervals has at most length $3/(d_\infty N)$. Hence, for large N, any segment in Ω of length ℓ contains at least $\ell N d_\infty/3 - 2$ such sequences in which the volumic proportion of gas-bubbles is at least $2\ell d_\infty^2/3 + O(1/N)$. The fluid part of this segment is then asymptotically less than $\ell\big(1 - 2d_\infty^2/3\big)$. □

We point out that a strictly positive bound from below for $\bar{\alpha}_f$ is also true with similar arguments. We dot not state this bound here since it will not help in the sequel. For constructing the macroscopic density, we choose to extend at first the microscopic fluid density by "filling" the bubbles in a sufficiently smooth manner. To this end, we take advantage of the fact that ρ_f^0 is initially defined (and sufficiently regular) on the whole Ω. So, we introduce $\tilde{\rho}_f^{(N)}$ as the unique solution to:

$$\begin{cases} \partial_t \tilde{\rho}_f^{(N)} + \tilde{u}^{(N)} \partial_x \tilde{\rho}_f^{(N)} = -\frac{\tilde{\rho}_f^{(N)}}{\mu_f}\left(\tilde{\Sigma}_f^{(N)} + \mathrm{p}_f\left(\tilde{\rho}_f^{(N)}\right)\right), & \text{on } (0,T)\times\Omega, \\ \tilde{\rho}_f^{(N)}(0,.) = \bar{\rho}_f^0, & \text{on } \Omega, \end{cases} \tag{49}$$

where $\tilde{\Sigma}_f^{(N)}$ is defined from Σ_f by (35).

Proposition 14. *There exists a time $T_0 < T$, independent of N, such that the Cauchy problem (49) admits a unique solution $\tilde{\rho}_f^{(N)} \in C([0,T_0]\times\Omega)$.*

Moreover, there exists $\bar{\rho}_f \in L^2((0,T_0)\times\Omega)$ called the density of the fluid *such that, up to the extraction of a subsequence,*

$$\tilde{\rho}_f^{(N)} \longrightarrow \bar{\rho}_f \quad \text{in } L^2((0,T_0)\times\Omega) \quad \text{when} \quad N \to +\infty.$$

Proof. The well-posedness of the Cauchy problem (49) is guaranteed by the method of characteristics, since $\tilde{u}^{(N)}$ belongs to $L^2((0,T);W^{1,\infty}(\Omega))$.

The result of convergence is an application of the Aubin–Lions lemma. One has to check:

- $\left(\tilde{\rho}_f^{(N)}\right)_N$ bounded in $L^2\big((0,T);H^1(\Omega)\big)$,
- $\left(\partial_t\tilde{\rho}_f^{(N)}\right)_N$ bounded in $L^2\big((0,T);L^2(\Omega)\big)$.

For the first item, we apply Proposition 29 in Appendix B which yields that, up to restrict to some time-interval $[0, T_0] \subset [0, T]$ we have that $\tilde{\rho}_f^{(N)}$ satisfies a uniform bound in $L^\infty((0,T); H^1(\Omega))$. As for the second item, using directly equation (49), a uniform estimate can be obtained:

$$\left\|\partial_t \tilde{\rho}_f^{(N)}\right\|_{L^2((0,T)\times\Omega)} \leq C_0 \left[\left\|\tilde{u}^{(N)}\right\|_{L^2((0,T);H^1(\Omega))} \left\|\partial_x \tilde{\rho}_f^{(N)}\right\|_{L^\infty((0,T);L^2((\Omega))} \right.$$
$$\left. + \frac{\left\|\tilde{\rho}_f^{(N)}\right\|_{L^\infty((0,T)\times\Omega)}}{\mu_f} \left(\left\|\tilde{\Sigma}_f^{(N)}\right\|_{L^2((0,T);H^1(\Omega))} + \sqrt{T} \left\|\tilde{\rho}_f^{(N)}\right\|^\gamma_{L^\infty((0,T)\times\Omega)} \right) \right],$$

where C_0 depends only on the parameters of the problem independent of N. Here again, the right-hand side is uniformly bounded with respect to N, so that the Aubin–Lions lemma can be applied to deduce the existence of the limit $\bar{\rho}_f$ stated in the proposition. □

To illustrate again that our choice for $\tilde{\rho}_f^{(N)}$ is rigorously adapted, we mention that, on the fluid domain $\mathcal{F}^{(N)}$, the definition of the fluid tensor (8) gives

$$\frac{1}{\mu_f}\left(\tilde{\Sigma}_f^{(N)} + \mathrm{p}_f\left(\tilde{\rho}_f^{(N)}\right)\right) = \partial_x u_f^{(N)}.$$

Moreover, $u_f^{(N)}$ and $\tilde{u}^{(N)}$ coincide on $\mathcal{F}^{(N)}$, and the density $\tilde{\rho}_f^{(N)}$ is also solution of

$$\begin{cases} \partial_t \tilde{\rho}_f^{(N)} + \partial_x\left(\tilde{\rho}_f^{(N)} u_f^{(N)}\right) = 0, \quad \text{on } (0,T)\times\mathcal{F}^N, \\ \tilde{\rho}_f^{(N)}(0,.) = \bar{\rho}_f^0. \end{cases} \tag{50}$$

As a consequence, the fluid density $\rho_f^{(N)}$ on the fluid domain $\mathcal{F}^{(N)}$ is the restriction of the global microscopic density $\tilde{\rho}_f^{(N)}$:

$$\rho_f^{(N)} = \tilde{\rho}_f^{(N)}, \quad \text{on } (0,T)\times\mathcal{F}^{(N)}. \tag{51}$$

3.2. Mixture unknowns

We proceed with the construction of unknowns that are involved in composite equations: a mixture velocity, a mixture density and a mixture stress tensor.

The mixture velocity is deduced from the reconstructed velocity $\tilde{u}^{(N)}$ defined by (45):

Proposition 15. *There exists $\bar{u} \in L^2((0,T); L^2(\Omega))$ such that, up to the extraction of a subsequence,*

$$\tilde{u}^{(N)} \to \bar{u} \quad \text{in } L^2((0,T);L^2(\Omega)) \quad \text{when } N \to +\infty.$$

Proof. This result is an application of the Aubin–Lions lemma again. From Corollary 3 and (Q_1), the sequence $(\tilde{u}^{(N)})$ is bounded in $L^2((0,T);H^1(\Omega))$. It remains to prove a uniform bound for $\left(\partial_t \tilde{u}^{(N)}\right)_N$ in $L^2((0,T);L^2(\Omega))$. By (7) and (46), the time derivative of the velocity reads:

$$\partial_t \tilde{u}^{(N)} = \begin{cases} -u_f^{(N)} \partial_x u_f^{(N)} - \frac{1}{\rho_f^{(N)}} \partial_x \Sigma_f^{(N)} & \text{on } \mathcal{F}, \\ \ddot{c}_k + \frac{\ddot{R}_k R_k - (\dot{R}_k)^2}{R_k^2}(x - c_k) - \dot{c}_k \frac{\dot{R}_k}{R_k} & \text{on } B_k, \end{cases}$$

(note that some exponents (N) have been removed to lighten the notations). Since the velocity $\tilde{u}^{(N)}$ is continuous through the interfaces $c_k \pm R_k$, one has, in $\mathcal{D}'((0,T)\times\Omega)$,

$$\partial_t \tilde{u}^{(N)} = \left(-u_f^{(N)}\partial_x u_f^{(N)} - \frac{1}{\rho_f^{(N)}}\partial_x \Sigma_f^{(N)}\right)\mathbb{1}_{\mathcal{F}} + \sum_{k=1}^{N}\left[\ddot{c}_k + \left(\frac{\ddot{R}_k}{R_k} - \frac{(\dot{R}_k)^2}{R_k^2}\right)(x-c_k) - \dot{c}_k\frac{\dot{R}_k}{R_k}\right]\mathbb{1}_{B_k}.$$

We now take the L^2 norm:

$$\begin{aligned}\left\|\partial_t \tilde{u}^{(N)}\right\|_{L^2(\Omega)}^2 &\leq \left\|\tilde{u}^{(N)}\right\|_{L^\infty(\Omega)}^2 \left\|\partial_x \tilde{u}^{(N)}\right\|_{L^2(\Omega)}^2 + \frac{1}{\left|\underline{\rho}_\infty\right|^2}\left\|\partial_x \tilde{\Sigma}_f^{(N)}\right\|_{L^2(\Omega)}^2 \\ &\quad + 2\sum_{k=1}^{N}\left[R_k(\ddot{c}_k)^2 + R_k\left(\ddot{R}_k\right)^2 + \frac{\left(\dot{R}_k\right)^4}{R_k} + \frac{\left(\dot{c}_k\dot{R}_k\right)^2}{R_k}\right] \\ &\leq C\left\|\tilde{u}^{(N)}\right\|_{H^1(\Omega)}^4 + \frac{1}{\left|\underline{\rho}_\infty\right|^2}\left\|\tilde{\Sigma}_f^{(N)}\right\|_{H^1(\Omega)}^2 \\ &\quad + 2\frac{1}{d_\infty M_\infty}\sum_{k=1}^{N} m_k\left((\ddot{c}_k)^2 + \left(\ddot{R}_k\right)^2\right) + 2\sum_{k=1}^{N}\frac{1}{R_k}\left(\left(\dot{R}_k\right)^4 + \left(\dot{c}_k\dot{R}_k\right)^2\right)\end{aligned}$$

by (IC_0) and (Q_1). Time-integrals of the two first terms on the right-hand side are bounded by (Q_4) and (Q_5) respectively. The third is controlled using (Q_5). Moreover, by (IC_0), (Q_1), and then by (A.1), the last term can be bounded this way:

$$\begin{aligned}&\int_0^T \sum_{k=1}^{N}\frac{\left(\dot{R}_k\right)^2}{R_k}\left(\left(\dot{R}_k\right)^2 + (\dot{c}_k)^2\right)\,\mathrm{d}t \\ &\qquad \leq \frac{1}{d_\infty M_\infty}\int_0^T \max_{k=1,\dots,N}\frac{\left(\dot{R}_k\right)^2}{R_k^2}\sum_{k=1}^{N} m_k\left(\left(\dot{R}_k\right)^2 + (\dot{c}_k)^2\right)\,\mathrm{d}t \\ &\qquad \leq \frac{2E_0}{d_\infty M_\infty}\int_0^T \max_{k=1,\dots,N}\frac{\left(\dot{R}_k\right)^2}{R_k^2}\,\mathrm{d}t.\end{aligned}$$

The last right-hand side is finally bounded by using Lemma 27. This concludes the proof of the assumptions of the Aubin–Lions lemma, leading to the convergence of the sequence $(\tilde{u}^{(N)})_N$ in $L^2\big((0,T);L^2(\Omega)\big)$. $\square$

We focus now on the mixture density. For this, we construct the global density $\rho^{(N)}$:

$$\rho^{(N)} = \rho_f^{(N)}\mathbb{1}_{\mathcal{F}^{(N)}} + \sum_{k=1}^{N}\rho_k^{(N)}\mathbb{1}_{B_k}, \tag{52}$$

where $\rho_k^{(N)} = m_k^{(N)}/(2R_k^{(N)})$ is the bubble density that we reconstruct from the bubble mass and radius. Notice that the global density $\rho^{(N)}$ belongs to $L^\infty((0,T)\times\Omega)$, and satisfies a classical mass conservation law (the proof is left to the reader):

$$\partial_t\rho^{(N)} + \partial_x\left(\rho^{(N)}\tilde{u}_f^{(N)}\right) = 0, \quad \text{in } (0,T)\times\Omega. \tag{53}$$

To conclude, we address the asymptotic behavior of extended stresses. This is the content of the following proposition:

Proposition 16. *There exist $\tilde{\Sigma}_f$ and $\tilde{\Sigma}_g$ in $L^2\big((0,T);H^1(\Omega)\big)$ such that, up to the extraction of a subsequence,*

$$\begin{matrix} \tilde{\Sigma}_f^{(N)} \rightharpoonup \bar{\Sigma}_f \\ \tilde{\Sigma}_g^{(N)} \rightharpoonup \bar{\Sigma}_g \end{matrix} \quad \text{in } L^2\big((0,T);H^1(\Omega)\big) \quad \text{when } N \to +\infty.$$

Proof. The estimate (Q_5) ensures that the sequences $\tilde{\Sigma}_f^{(N)}$ and $\tilde{\Sigma}_g^{(N)}$ are both bounded in the space $L^2((0,T);H^1(\Omega))$. Hence they are relatively compact in $L^2\big((0,T);H^1(\Omega)\big)$ endowed with the weak topology, and the result follows. □

3.3. Bubble unknowns

We mention first that the indicator of the bubble domains reads $1-\chi^{(N)}$. Similarly to Proposition 13 we obtain that it converges weakly to some $\bar{\alpha}_g$ satisfying also $0 \le \bar{\alpha}_g \le 1$ a.e.. Since $1 = \bar{\alpha}_g + \bar{\alpha}_f$, Proposition 13 entails further that $\bar{\alpha}_g \ge 2d_\infty^2/3$.

For our analysis, we need a sufficiently strong (pointwise) convergence of bubble density $\rho_g^{(N)}$ and covolume $f_g^{(N)}$ as defined in (42). Yet, these quantities are defined only on subsets depending on N. To overcome this difficulty, we note that both quantities satisfy the same continuity equation:

$$\begin{cases} \partial_t \rho_g^{(N)} + \partial_x(\rho_g^{(N)}\tilde{u}^{(N)}) = 0, \\ \partial_t f_g^{(N)} + \partial_x(f_g^{(N)}\tilde{u}^{(N)}) = 0, \end{cases} \quad \text{in } \mathcal{D}'((0,T)\times\Omega). \tag{54}$$

We used here in particular that $m_k^{(N)}$ is time-independent and that the bubbles follow the flow associated with the extended velocity. We propose then to reproduce the same method we used in the case of fluid unknowns (see Prop. 14). We remark that, on B_k, there holds:

$$\partial_x \tilde{u}^{(N)} = \frac{1}{\mu_g}\Sigma_k^{(N)} + \frac{\kappa_k}{R_k^{(N)}}\cdot \tag{55}$$

We recall that, on the right-hand side, the first term is the restriction to B_k of the extended stress tensor $\tilde{\Sigma}_g^{(N)}$. As for the last term, we wish to extract the contribution of the pressure and the contribution of the surface tension for modelling reason (even though keeping the current form would not change the remark in progress). So we rewrite:

$$\frac{\kappa_k}{R_k^{(N)}} = \mathrm{p}_g\Big(\rho_k^{(N)}\Big) + \frac{\bar{\gamma}_S}{2NR_k^{(N)}}\cdot$$

Here, the second term could be related artificially to a density, but it is usually related to a "covolume" and is treated independently. Actually, this is the reason motivating the introduction of the unknown $f_g^{(N)}$. We use now this novel writing of the term $\partial_x\tilde{u}^{(N)}$ to see that $\Big(\rho_g^{(N)}, f_g^{(N)}\Big)$ is the restriction of a pair $\Big(\tilde{\rho}_g^{(N)}, \tilde{f}_g^{(N)}\Big)$ solution to:

$$\partial_t \begin{pmatrix} \tilde{\rho}_g^{(N)} \\ \tilde{f}_g^{(N)} \end{pmatrix} + \tilde{u}^{(N)}\partial_x \begin{pmatrix} \tilde{\rho}_g^{(N)} \\ \tilde{f}_g^{(N)} \end{pmatrix} = -\frac{1}{\mu_g}\begin{pmatrix} \tilde{\rho}_g^{(N)} \\ \tilde{f}_g^{(N)} \end{pmatrix}\Big(\tilde{\Sigma}_g^{(N)} + \mathrm{p}_g(\tilde{\rho}_g^{(N)}) + \bar{\gamma}_s\tilde{f}_g^{(N)}\Big), \quad \text{on } (0,T)\times\Omega. \tag{56}$$

We can then use the stability properties of this latter equation to yield the following proposition:

Proposition 17. *There exists a time $T_0 < T$ (independent of N) and sequences $\Big(\tilde{\rho}_g^{(N)}, \tilde{f}_g^{(N)}\Big) \in C\big([0,T_0];H^1(\Omega)\big)$ satisfying the following properties:*

– *there holds* $\rho_g^{(N)} = \tilde{\rho}_g^{(N)}$ *and* $f_g^{(N)} = \tilde{f}_g^{(N)}$ *on* B_k *for all* $k = 1, \ldots, N$,
– *there exists* $(\bar{\rho}_g, \bar{f}_g) \in L^2((0,T_0)\times\Omega)^2$ *such that, up to the extraction of a subsequence,*

$$\left(\tilde{\rho}_g^{(N)}, \tilde{f}_g^{(N)}\right) \longrightarrow \left(\bar{\rho}_g, \bar{f}_g\right) \quad \text{in } L^2((0,T_0)\times\Omega)^2 \quad \text{when} \quad N \to +\infty.$$

Proof. We recall that the initial bubble distribution $\left(c_k^{(N)}, R_k^{(N)}\right)_{k=1,\ldots,N}$ is obtained by applying Proposition 12 so that they are associated with a sequence of initial density/covolume $\tilde{\rho}_g^{(N),0}$, $\tilde{f}_g^{(N),0}$ which extend initially $\rho_g^{(N)}$ and $f_g^{(N)}$ and that converge weakly in $H^1(\Omega)$. Hence, we complement (56) with initial condition

$$\tilde{\rho}_g^{(N)}(0,\cdot) = \rho_g^{(N),0} \qquad \tilde{f}_g^{(N)}(0,\cdot) = f_g^{(N),0} \quad \text{on } \Omega. \tag{57}$$

The result is then proved following exactly the same steps as in the proof of Proposition 14, since (Q_5) involves similar controls on $\tilde{\Sigma}_f^{(N)}$ and $\tilde{\Sigma}_g^{(N)}$. $\square$

3.4. Two technical lemmas

We close this section by providing two crucial results which allow to pass to the limit in some nonlinear terms. The procedure we apply here is similar to the construction in [6].

Let $b \in C^1([0,1]\times\mathbb{R}^+\times\mathbb{R}^+)$ and consider the sequence

$$b^{(N)}(t,x) = b\Big(\chi^{(N)}(t,x), \rho^{(N)}(t,x), f_g^{(N)}(t,x)\Big), \quad \forall\,(t,x) \in (0,T)\times\Omega, \tag{58}$$

where $\rho^{(N)}$ is defined by (52) and $f_g^{(N)}$ by (42).

Proposition 18. *There exists* $\bar{b} \in L^\infty((0,T)\times\Omega)$ *such that, up to extraction of a subsequence,*

$$b^{(N)} \rightharpoonup \bar{b}, \quad \text{in } L^\infty((0,T)\times\Omega) - w^\star \text{ when } N \to +\infty.$$

This limit verifies the following identity, for almost every $(t,x) \in (0,T)\times\Omega$,

$$\bar{b} = b(1,\bar{\rho}_f,0)\bar{\alpha}_f + b\big(0,\bar{\rho}_g,\bar{f}_g\big)\bar{\alpha}_g. \tag{59}$$

Proof. By definition, we have:

$$b^{(N)} = b\Big(1,\tilde{\rho}_f^{(N)},0\Big)\chi^{(N)} + b\Big(0,\tilde{\rho}_g^{(N)},\tilde{f}_g^{(N)}\Big)\Big(1-\chi^{(N)}\Big)$$

The strong convergence of $\tilde{\rho}_f^{(N)}$ (resp. $\tilde{\rho}_g^{(N)}$ and $\tilde{f}_g^{(N)}$), see Proposition 14 (resp. Prop. 17) and the weak convergence of $\chi^{(N)}$ (Prop. 13) ensure that the first term converges weakly towards $b(1,\bar{\rho}_f,0)\bar{\alpha}_f$ and the second one to $b\big(0,\bar{\rho}_g,\bar{f}_g\big)\bar{\alpha}_g$. $\square$

In the following result, the term $\tilde{\Sigma}^{(N)}$ denotes either $\tilde{\Sigma}_f^{(N)}$ or $\tilde{\Sigma}_g^{(N)}$.

Proposition 19. *Assume that* $\tilde{\Sigma}^{(N)}$ *converges weakly in* $L^2((0,T);H^1(\Omega))$, *and denote by* $\bar{\Sigma}$ *its limit. Then for all* $b \in C^1([0,1]\times\mathbb{R}^+\times\mathbb{R}^+)$, *it holds*

$$\tilde{\Sigma}^{(N)} b^{(N)} \rightharpoonup \bar{\Sigma}\bar{b}, \quad \text{in } \mathcal{D}'((0,T)\times\Omega) \quad \text{when } N \to +\infty.$$

Proof. This result is a variant of so-called "compensated compactness" lemma. We can reproduce here the proof of [4, Lemma 10] up to adapt the definition of the operator ∂_x^{-1} on mean free functions. $\square$

4. Derivation of a macroscopic model

Thanks to the results of the previous section, we are now in position to address the limit $N \to +\infty$ for the microscopic model (6)–(13). Based on the previous definitions of macroscopic unknowns, we derive successively the various equations of (1). This is the content of the following theorem.

Theorem 20. *Let $\bar\rho_f, \bar\alpha_f, \bar\alpha_g, \bar\rho_g, \bar u$ be as constructed in the previous section. Then, we have that $\big(\bar\alpha_f, \bar\rho_f, \bar\alpha_g, \bar\rho_g, \bar f_g, \bar u\big)$ is a solution to (1)–(2)–(3)–(4) on $(0,T)$ with initial condition on Ω:*

$$\begin{aligned} \bar\alpha_f(0,\cdot) &= \bar\alpha_f^0 & \bar\alpha_g(0,\cdot) &= \bar\alpha_f^0 \\ \bar\rho_f(0,\cdot) &= \bar\rho_f^0 & \bar\alpha_g\bar\rho_g(0,\cdot) &= \bar\alpha_g^0\bar\rho_g^0 \\ \bar u(0,\cdot) &= \bar u^0 & \bar\alpha_g\bar f_g(0,\cdot) &= \bar\alpha_g^0\bar f_g^0 \end{aligned}$$

What remains of this section is devoted to the proof of this theorem. Our first result provides the limit equation for the limit $\bar b$ associated with an abstract choice of b.

Proposition 21. *Let $b \in C^1([0,1]\times\mathbb{R}^+\times\mathbb{R}^+)$ and define*

$$\begin{aligned} b_{1,f}(z,\xi,\nu) &= (\partial_2 b(z,\xi,\nu)\xi + \partial_3 b(z,\xi,\nu)\nu - b(z,\xi,\nu))z, \\ b_{1,g}(z,\xi,\nu) &= (\partial_2 b(z,\xi,\nu)\xi + \partial_3 b(z,\xi,\nu)\nu - b(z,\xi,\nu))(1-z), \\ b_{2,f}(z,\xi,\nu) &= (\partial_2 b(z,\xi,\nu)\xi + \partial_3 b(z,\xi,\nu)\nu - b(z,\xi,\nu))z\mathrm{p}_f(\xi), \\ b_{2,g}(z,\xi,\nu) &= (\partial_2 b(z,\xi,\nu)\xi + \partial_3 b(z,\xi,\nu)\nu - b(z,\xi,\nu))(1-z)(\mathrm{p}_g(\xi)+\bar\gamma_s\nu). \end{aligned}$$

Then, the limit $\bar b$ defined in Proposition 18 satisfies the equation

$$\begin{cases} \partial_t\bar b + \partial_x(\bar u\bar b) + \dfrac{1}{\mu_f}\big(\bar b_{1,f}\bar\Sigma_f + \bar b_{2,f}\big) + \dfrac{1}{\mu_g}\big(\bar b_{1,g}\bar\Sigma_g + \bar b_{2,g}\big) = 0 \\ \bar b(0,\cdot) = \bar\alpha_f^0 b\big(1,\bar\rho_f^0,0\big) + \bar\alpha_g^0 b\big(0,\bar\rho_g^0,\bar f_g^0\big) \end{cases} \tag{60}$$

Proof. Let us compute for arbitrary $N\in\mathbb{N}$

$$\begin{aligned} \partial_t b\Big(\chi^{(N)},\rho^{(N)},f_g^{(N)}\Big) &= \partial_1 b^{(N)}\partial_t\chi^{(N)} + \partial_2 b^{(N)}\partial_t\rho^{(N)} + \partial_3 b^{(N)}\partial_t f_g^{(N)} \\ &= -\partial_1 b^{(N)}\tilde u^{(N)}\partial_x\chi^{(N)} - \partial_2 b^{(N)}\partial_x\Big(\rho^{(N)}\tilde u^{(N)}\Big) - \partial_3 b^{(N)}\partial_x\Big(f_g^{(N)}\tilde u^{(N)}\Big) \end{aligned}$$

by (47), (53) and (54). As a result, we obtain:

$$\begin{aligned} \partial_t b\Big(\chi^{(N)},\rho^{(N)},f_g^{(N)}\Big) &+ \partial_x\Big(b\Big(\chi^{(N)},\rho^{(N)},f_g^{(N)}\Big)\tilde u^{(N)}\Big) \\ &+ \Big(\partial_2 b\Big(\chi^{(N)},\rho^{(N)},f_g^{(N)}\Big)\rho^{(N)} + \partial_3 b\Big(\chi^{(N)},\rho^{(N)},f_g^{(N)}\Big)f_g^{(N)} \\ &- b\Big(\chi^{(N)},\rho^{(N)},f_g^{(N)}\Big)\Big)\partial_x\tilde u^{(N)} = 0, \end{aligned} \tag{61}$$

in $\mathcal{D}'((0,T)\times\Omega)$. In this equation, due to the weak convergence of $b^{(N)}$ and the strong convergence of $\tilde u_f^{(N)}$, respectively stated in Propositions 18 and 15, it holds that:

$$\begin{cases} b^{(N)} \rightharpoonup \bar b, \\ u_f^{(N)} b\Big(\chi^{(N)},\rho^{(N)},f_g^{(N)}\Big) \rightharpoonup \bar u\,\bar b, \end{cases} \qquad \text{in } \mathcal{D}'((0,T)\times\Omega).$$

Then, we rewrite:

$$\partial_2 b\Big(\chi^{(N)},\rho^{(N)},f_g^{(N)}\Big)\rho^{(N)} + \partial_3 b\Big(\chi^{(N)},\rho^{(N)},f_g^{(N)}\Big)f_g^{(N)} - b^{(N)}\partial_x u_f^{(N)}$$

$$= \frac{1}{\mu_f}\Big(b_{1,f}^{(N)}\tilde{\Sigma}_f^{(N)} + b_{2,f}^{(N)}\Big) + \frac{1}{\mu_g}\Big(b_{1,g}^{(N)}\tilde{\Sigma}_g^{(N)} + b_{2,g}^{(N)}\Big).$$

The weak convergence stated in Proposition 19 allows to pass to the limit the right-hand side, leading to

$$\partial_2 b\Big(\chi^{(N)},\rho^{(N)},f_g^{(N)}\Big)\rho^{(N)} + \partial_3 b\Big(\chi^{(N)},\rho^{(N)},f_g^{(N)}\Big)f_g^{(N)} - b^{(N)}\partial_x u_f^{(N)} \rightharpoonup \frac{1}{\mu_f}\big(\bar{b}_{1,f}\bar{\Sigma}_f + \bar{b}_{2,f}\big) + \frac{1}{\mu_g}\big(\bar{b}_{1,g}\bar{\Sigma}_g + \bar{b}_{2,g}\big),$$

where the terms $\bar{b}_{1,f}$, $\bar{b}_{1,g}$, $\bar{b}_{2,f}$, and $\bar{b}_{2,g}$ are defined as in Proposition 18. This provides equation (60) for $\bar{b}$.

Finally, we have initially

$$b^{(N)}(0,\cdot) = \chi^{(N),0}b\big(1,\rho_f^0,0\big) + \Big(1-\chi^{(N),0}\Big)b\Big(0,\tilde{\rho}_g^{(N),0},\tilde{f}_g^{(N),0}\Big)$$

and we are in position to apply Proposition 12 to pass to the limit in this identity when $N\to\infty$. □

Let us recall that the link between the limit $\bar{b}$ and the function b is provided in Proposition 18. According to the choice of b, different relevant macroscopic equations can be obtained.

Corollary 22. *The void fractions satisfy the following equations*

$$\begin{cases} \partial_t\bar{\alpha}_f + \partial_x(\bar{\alpha}_f\bar{u}) = \dfrac{\bar{\alpha}_f}{\mu_f}\big(\bar{\Sigma}_f + \mathrm{p}_f(\bar{\rho}_f)\big), \qquad \bar{\alpha}_f(0,\cdot) = \bar{\alpha}_f^0 \\ \bar{\alpha}_f + \bar{\alpha}_g = 1 \end{cases} \tag{62}$$

The covolume unknown $\bar{f}_g$ satisfies the conservation equation:

$$\partial_t\big(\bar{\alpha}_g\bar{f}_g\big) + \partial_x\big(\bar{\alpha}_g\bar{f}_g\bar{u}\big) = 0, \qquad \bar{\alpha}_g(0,\cdot)\bar{f}_g(0,\cdot) = \bar{\alpha}_g^0\bar{f}_g^0.$$

The mass conservation laws of both phases read

$$\partial_t(\bar{\alpha}_f\bar{\rho}_f) + \partial_x(\bar{\alpha}_f\bar{\rho}_f\bar{u}) = 0, \qquad \bar{\alpha}_f(0,\cdot)\bar{\rho}_f(0,\cdot) = \bar{\alpha}_f^0\bar{\rho}_f^0 \tag{63}$$

$$\partial_t(\bar{\alpha}_g\bar{\rho}_g) + \partial_x(\bar{\alpha}_g\bar{\rho}_g\bar{u}) = 0, \qquad \bar{\alpha}_g(0,\cdot)\bar{\rho}_g(0,\cdot) = \bar{\alpha}_g^0\bar{\rho}_g^0. \tag{64}$$

Proof. By Proposition 21, it suffices to compute the different terms of equation (60). In the first case, we consider $b(z,\xi,\nu) = z$. It yields $\bar{b} = \bar{\alpha}_f$ and

$$\begin{aligned} &b_{1,f}(1,r) = -1, \quad &&b_{1,g}(1,r) = 0, \quad &&b_{2,f}(1,r) = -\mathrm{p}_f(r), \quad &&b_{2,g}(1,r) = 0,\\ &b_{1,f}(0,r) = 0, &&b_{1,g}(0,r) = 0, &&b_{2,f}(0,r) = 0, &&b_{2,g}(0,r) = 0. \end{aligned}$$

Computing the associated limits, one recovers the first equation of (62). The second equation is true by construction. The equation on $\bar{f}_g$ is obtained in the same way, taking $b(z,\xi,\nu) = \nu$. Finally the phasic mass conservation laws are derived using $b(z,\xi,\nu) = z\xi$ and $b(z,\xi,\nu) = (1-z)\xi$ respectively. □

4.1. Momentum equation and closure laws

We proceed with the derivation of the momentum equation.

Proposition 23. *Let $\bar{\rho} = \bar{\alpha}_f\bar{\rho}_f + \bar{\alpha}_g\bar{\rho}_g$ be the mixture density. The mixture momentum equation reads*

$$\partial_t(\bar{\rho}\bar{u}) + \partial_x\big(\bar{\rho}\bar{u}^2\big) = \partial_x\big(\bar{\alpha}_f\bar{\Sigma}_f + \bar{\alpha}_g\bar{\Sigma}_g\big), \tag{65}$$

with

$$\partial_x\bar{u} = \frac{\bar{\alpha}_f}{\mu_f}\big[\bar{\Sigma}_f + \mathrm{p}_f(\bar{\rho}_f)\big] + \frac{\bar{\alpha}_g}{\mu_g}\big[\bar{\Sigma}_g + \mathrm{p}_g(\bar{\rho}_g) + \bar{\gamma}_s\bar{f}_g\big], \tag{66}$$

and

$$\bar{\Sigma}_f = \bar{\Sigma}_g. \tag{67}$$

Proof. Let us consider the momentum equation in the fluid domain and multiply it by a test function $w \in C_c^\infty((0,T)\times\Omega)$. It yields

$$\int_0^T \int_{\mathcal{F}^{(N)}(t)} \left(\partial_t\left(\rho_f^{(N)} u_f^{(N)}\right) + \partial_x\left(\rho_f^{(N)}\left|u_f^{(N)}\right|^2\right)\right) w \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_{\mathcal{F}^{(N)}(t)} \partial_x \Sigma_f^{(N)} w \,\mathrm{d}x\,\mathrm{d}t.$$

Since the fluid domain $\mathcal{F}^{(N)}(t)$ is transported with the velocity $u_f^{(N)}$, an integration by part in time of the left-hand side gives

$$\begin{aligned}\int_0^T \int_{\mathcal{F}^{(N)}(t)} &\left(\partial_t\left(\rho_f^{(N)} u_f^{(N)}\right) + \partial_x\left(\rho_f^{(N)}\left|u_f^{(N)}\right|^2\right)\right) w \,\mathrm{d}x\,\mathrm{d}t \\ &= -\int_0^T \int_{\mathcal{F}^{(N)}(t)} \rho_f^{(N)} u_f^{(N)}\left(\partial_t w + u_f^{(N)}\partial_x w\right) \mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

The right-hand side is handled by an integration by part in space. Reorganising the boundary terms yields (we omit time dependencies for simplicity):

$$\begin{aligned}\int_0^T \int_{\mathcal{F}^{(N)}(t)} \partial_x \Sigma_f^{(N)} w \,\mathrm{d}x\,\mathrm{d}t = &-\int_0^T \sum_{k=1}^N \left(\Sigma_f^{(N)}(x_k^+)w(x_k^+) - \Sigma_f^{(N)}(x_k^-)w(x_k^-)\right) \mathrm{d}t \\ &-\int_0^T \int_{\mathcal{F}^{(N)}} \Sigma_f^{(N)} \partial_x w \,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

We now focus on the boundary terms. For $k = 1, \dots, N$, one has

$$\begin{aligned}\Sigma_f^{(N)}(x_k^+)w(x_k^+) - \Sigma_f^{(N)}(x_k^-)w(x_k^-) = &\left(\Sigma_f^{(N)}(x_k^+) - \Sigma_f^{(N)}(x_k^-)\right) w(c_k) \\ &+ \left(\Sigma_f^{(N)}(x_k^+) + \Sigma_f^{(N)}(x_k^-)\right) R_k \partial_x w(c_k) \\ &+ O\left(\left\|\Sigma_f^{(N)}\right\|_{L^\infty(\Omega)} R_k^2 \|w\|_{C^2}\right).\end{aligned}$$

From the bubbles equations (11) and (12), one deduces

$$\begin{aligned}\Sigma_f^{(N)}(x_k^+)w(x_k^+) - \Sigma_f^{(N)}(x_k^-)w(x_k^-) = &\, m_k \ddot{c}_k w(t,c_k) + \left(\frac{m_k}{3}\ddot{R}_k + 2\Sigma_k\right) R_k \partial_x w(t,c_k) \\ &+ O\left(\left\|\Sigma_f^{(N)}\right\|_{L^\infty(\Omega)} R_k^2 \|w\|_{C^2}\right).\end{aligned}$$

The term involving the stress tensor can be rewritten as follows

$$2\Sigma_k R_k \partial_x w(t,c_k) = \int_{B_k} \Sigma_g^{(N)} \partial_x w \,\mathrm{d}x + O\left(\left\|\Sigma_g^{(N)}\right\|_{L^\infty(\Omega)} R_k^2 \|w\|_{C^2}\right).$$

Therefore, one has

$$\begin{aligned}-&\int_0^T \sum_{k=1}^N \left(\Sigma_f^{(N)}(x_k^+)w(x_k^+) - \Sigma_f^{(N)}(x_k^-)w(x_k^-)\right) \mathrm{d}t \\ &= -\int_0^T \sum_{k=1}^N \Bigg(m_k \ddot{c}_k w(t,c_k) + \frac{m_k}{3}\ddot{R}_k R_k \partial_x w(t,c_k) + \int_{B_k} \Sigma_g^{(N)} \partial_x w \,\mathrm{d}x\end{aligned}$$

$$+O\Bigg(\bigg(\Big\|\Sigma_f^{(N)}\Big\|_{L^\infty(\Omega)}+\Big\|\Sigma_g^{(N)}\Big\|_{L^\infty(\Omega)}\bigg)R_k^2\|w\|_{C^2}\Bigg)\Bigg)\ \mathrm{d}t.$$

An integration by part in time gives

$$\begin{aligned}
&-\int_0^T\sum_{k=1}^N\Big(\Sigma_f^{(N)}(x_k^+)w(x_k^+)-\Sigma_f^{(N)}(x_k^-)w(x_k^-)\Big)\ \mathrm{d}t\\
&=\int_0^T\sum_{k=1}^N m_k\bigg(|\dot c_k|^2\partial_x w(t,c_k)+\frac{1}{3}\Big|\dot R_k\Big|^2\partial_x w(t,c_k)+\frac{1}{3}\dot R_k R_k\dot c_k\partial_{xx}w(t,c_k)\bigg)\ \mathrm{d}t\\
&\quad+\int_0^T\sum_{k=1}^N m_k\bigg(\dot c_k\partial_t w(t,c_k)+\frac{1}{3}\dot R_k R_k\partial_{xt}w(t,c_k)\bigg)\ \mathrm{d}t\\
&\quad-\int_0^T\int_{\Omega\setminus\mathcal{F}^{(N)}}\Sigma_g^{(N)}\partial_x w\ \mathrm{d}x\ \mathrm{d}t\\
&\quad+O\Bigg(\bigg(\Big\|\Sigma_f^{(N)}\Big\|_{L^2((0,T),H^1(\Omega))}+\Big\|\Sigma_g^{(N)}\Big\|_{L^2((0,T),H^1(\Omega))}\bigg)\sqrt{T}\|w\|_{C^2}\max_{[0,T]}\sum_{k=1}^N R_k^2\Bigg)\\
&=\int_0^T\sum_{k=1}^N m_k\bigg(|\dot c_k|^2\partial_x w(t,c_k)+\frac{1}{3}\Big|\dot R_k\Big|^2\partial_x w(t,c_k)+\dot c_k\partial_t w(t,c_k)\bigg)\ \mathrm{d}t\\
&\quad-\int_0^T\int_{\Omega\setminus\mathcal{F}^{(N)}}\Sigma_g^{(N)}\partial_x w\ \mathrm{d}x\ \mathrm{d}t\\
&\quad+O\Bigg(\bigg(\Big\|\Sigma_f^{(N)}\Big\|_{L^2((0,T),H^1(\Omega))}+\Big\|\Sigma_g^{(N)}\Big\|_{L^2((0,T),H^1(\Omega))}\bigg)\sqrt{T}\|w\|_{C^2}(d_\infty N)^{-1}\\
&\qquad\quad+(M_\infty N)^{-\frac{1}{2}}\|w\|_{C^2}T\sqrt{E_0}\Bigg).
\end{aligned}$$

where we applied (IC_0) and (A.1) with (Q_1) to yield the last term in the last inequality. On the bubble B_k, it holds

$$\begin{aligned}
&\int_{B_k}\rho_k\tilde u^{(N)}\Big(\partial_t w+\tilde u^{(N)}\partial_x w\Big)\ \mathrm{d}x\\
&\quad=\int_{B_k}\frac{m_k}{2R_k}\tilde u^{(N)}\Big(\partial_t w+\tilde u^{(N)}\partial_x w\Big)\ \mathrm{d}x\\
&\quad=m_k\dot c_k\partial_t w(c_k)+m_k\bigg(|\dot c_k|^2+\frac{1}{3}\Big|\dot R_k\Big|^2\bigg)\partial_x w(c_k)\\
&\qquad+O\Big(m_k\|w\|_{C^2}\Big(1+|\dot c_k|+\Big|\dot R_k\Big|\Big)\Big(|\dot c_k|+|\dot R_k|\Big)|R_k|\Big).
\end{aligned}$$

Gathering the fluid and gas expressions yields

$$\begin{aligned}
&-\int_0^T\int_\Omega\rho^{(N)}\tilde u^{(N)}\Big(\partial_t w+\tilde u^{(N)}\partial_x w\Big)\ \mathrm{d}x\ \mathrm{d}t\\
&\quad=-\int_0^T\int_\Omega\Big(\chi^{(N)}\tilde\Sigma_f^{(N)}+\Big(1-\chi^{(N)}\Big)\tilde\Sigma_g^{(N)}\Big)\partial_x w\ \mathrm{d}x\ \mathrm{d}t+O\Big(N^{-1/2}\Big).
\end{aligned}$$

Using the strong convergence of $\tilde{u}^{(N)}$ and the weak convergence of $\rho^{(N)}$, obtained by Proposition 18 with $b(z,\xi,\nu)=\xi$, the left-hand side tends to

$$-\int_0^T\int_\Omega \bar{\rho}\bar{u}(\partial_t w+\bar{u}\partial_x w)\,\mathrm{d}x\,\mathrm{d}t.$$

The limit of the right-hand side is deduced from Proposition 19. One ends up with the desired momentum equation (65).

It remains to close the system by determining relations between the tensors $\bar{\Sigma}_f$ and $\bar{\Sigma}_g$ and the other quantities. To do so, we prove that $\bar{\Sigma}_f$ and $\bar{\Sigma}_g$ are solutions of a 2×2 system.

First observe that

$$\partial_x u^{(N)} = \chi^{(N)}\frac{\tilde{\Sigma}_f^{(N)}+\mathrm{p}_f\left(\rho_f^{(N)}\right)}{\mu_f}+\left(1-\chi^{(N)}\right)\frac{\tilde{\Sigma}_g^{(N)}+\mathrm{p}_g\left(\rho_g^{(N)}\right)+F_s/2}{\mu_g}.$$

The different results of convergence given in Section 3, especially Proposition 19, allow to pass to the limit in both sides of the equation. In particular, in the right-hand side, the definition of the surface tension yields

$$\left(1-\chi^{(N)}\right)\frac{F_s}{2}=\sum_{k=1}^N\frac{\bar{\gamma}_s}{2NR_k}\mathbb{1}_{B_k}=\bar{\gamma}_s\tilde{f}_g^{(N)}\left(1-\chi^{(N)}\right)\rightharpoonup\bar{\gamma}_s\bar{\alpha}_g\bar{f}_g.$$

Eventually, it holds

$$\partial_x\bar{u}=\frac{\bar{\alpha}_f}{\mu_f}\left[\bar{\Sigma}_f+\mathrm{p}_f(\bar{\rho}_f)\right]+\frac{\bar{\alpha}_g}{\mu_g}\left[\bar{\Sigma}_g+\mathrm{p}_g(\bar{\rho}_g)+\bar{\gamma}_s\bar{f}_g\right].$$

The second equation is obtained while studying the difference $\bar{\Sigma}_f-\bar{\Sigma}_g$. Using the definition (35) of the extended tensor $\tilde{\Sigma}_f$ and the Newton laws (11) and (12) for the bubbles, it holds

$$\begin{aligned}\tilde{\Sigma}_f^{(N)}&=\frac{\Sigma_f\left(x_k^-\right)+\Sigma_f\left(x_k^+\right)}{2}-\frac{\Sigma_f\left(x_k^-\right)-\Sigma_f\left(x_k^+\right)}{2R_k^{(N)}}\left(c-c_k^{(N)}\right)\\&=\frac{m_k}{6}\ddot{R}_k+\Sigma_k+\frac{m_k\ddot{c}_k}{2R_k}(x-c_k).\end{aligned}$$

Since $\tilde{\Sigma}_g^{(N)}=\Sigma_k$ on the bubbles domain B_k, one has

$$\left(1-\chi^{(N)}\right)\left(\tilde{\Sigma}_f^{(N)}-\tilde{\Sigma}_g^{(N)}\right)=\sum_{k=1}^N\left(\frac{m_k}{6}\ddot{R}_k+\frac{m_k\ddot{c}_k}{2R_k}(x-c_k)\right)\mathbb{1}_{B_k}.$$

Proposition 19 applies to the left-hand side:

$$\left(1-\chi^{(N)}\right)\left(\tilde{\Sigma}_f^{(N)}-\tilde{\Sigma}_g^{(N)}\right)\rightharpoonup(1-\bar{\alpha}_f)(\bar{\Sigma}_f-\bar{\Sigma}_g),$$

in the sense of distributions. The right-hand side can be proved to tend to zero in $L^2((0,T)\times\Omega)$ since

$$\begin{aligned}\left\|\sum_{k=1}^N\left(\frac{m_k}{6}\ddot{R}_k+\frac{m_k\ddot{c}_k}{2R_k}(x-c_k)\right)\mathbb{1}_{B_k}\right\|^2_{L^2(\Omega)}&\le\left\|\frac{1}{2}\sum_{k=1}^N m_k\left(\left|\ddot{R}_k\right|+|\ddot{c}_k|\right)\mathbb{1}_{B_k}\right\|^2_{L^2(\Omega)}\\&\le\sum_{k=1}^N\int_{B_k}m_k^2\left(\left|\ddot{R}_k\right|^2+|\ddot{c}_k|^2\right)\mathrm{d}x\end{aligned}$$

$$\leq \frac{2}{N^2 d_\infty M_\infty} \sum_{k=1}^N m_k \left(\left| \ddot{R}_k \right|^2 + |\ddot{c}_k|^2 \right)$$
$$\leq \frac{2K_\infty}{N^2 d_\infty M_\infty},$$

thanks to (IC_0), (Q_1) and (Q_5). Recalling the second part of (48), one recovers (67). □

5. An alternative description of the bubble dynamics

In order to describe the dynamics of the bubbles, an alternative approach is to introduce the distribution function in position and (scaled) radius

$$S_t^{(N)} = \frac{1}{N} \sum_{k=1}^N \delta_{c_k(t), NR_k(t)}, \tag{68}$$

which is a measure on $\Omega \times (0,\infty)$. According to (Q_1), one has

$$\operatorname{supp}\left(S_t^{(N)} \right) \subset \bar{\Omega} \times [d_\infty, 1/d_\infty], \quad \forall\, t \in (0,T).$$

Proposition 24. *For all $\beta \in C(\bar{\Omega} \times [d_\infty, 1/d_\infty])$, the distribution function $S_t^{(N)}$ satisfies*

$$\partial_t \left\langle S_t^{(N)}, \beta \right\rangle - \left\langle S_t^{(N)}, \tilde{u}^{(N)}(x) \partial_x \beta \right\rangle - \frac{1}{\mu_g} \left\langle S_t^{(N)}, \left(\left(\tilde{\Sigma}_g^{(N)}(x) + \mathrm{p}_g\left(\tilde{\rho}_g^{(N)}\right) \right) r + \bar{\gamma}_s/2 \right) \partial_r \beta \right\rangle = 0. \tag{69}$$

Moreover, the sequence of applications $t \mapsto S_t^{(N)}$ is compact in $C\big([0,T]; \mathbb{P}\big(\bar{\Omega} \times [d_\infty, 1/d_\infty]\big)\big)$. As a consequence, there exists $\bar{S}_g \in C\big([0,T]; \mathbb{P}\big(\bar{\Omega} \times [d_\infty, 1/d_\infty]\big)\big)$ such that, up to the extraction of a subsequence,

$$\left\langle S^{(N)}, \beta \right\rangle \to \langle \bar{S}_g, \beta \rangle, \quad \text{in } C([0,T]),$$

for any $\beta \in C\big(\bar{\Omega} \times [d_\infty, 1/d_\infty]\big)$.

Proof. Let us first prove that, for any $\beta \in C^1\big(\bar{\Omega} \times [d_\infty, 1/d_\infty]\big)$, the sequence $\beta^{(N)} : t \mapsto \left\langle S_t^{(N)}, \beta \right\rangle$ is uniformly equicontinuous. The definition of $\beta^{(N)}$ and (68) enable to write

$$\beta^{(N)}(t) = \frac{1}{N} \sum_{k=1}^N \beta(c_k(t), NR_k(t)).$$

For legibility, we drop the exponent (N) in c_k and R_k here and in what remains of the proof. By construction, c_k and R_k belong to $H^2(0,T)$ and thus are in $C^1([0,T])$. It follows that $\beta^{(N)} \in C^1([0,T])$, and

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{dt}} \beta^{(N)}(t) &= \frac{1}{N} \sum_{k=1}^N \left(\dot{c}_k \partial_x \beta(c_k, NR_k) + N\dot{R}_k \partial_r \beta(c_k, NR_k) \right) \\ &= \frac{1}{N} \sum_{k=1}^N \left(\tilde{u}_f^{(N)}(c_k) \partial_x \beta(c_k, NR_k) + \frac{\dot{R}_k}{R_k} NR_k \partial_r \beta(c_k, NR_k) \right). \end{aligned} \tag{70}$$

Recall that, by Corollary 3, $\tilde{u}_f^{(N)}$ is bounded in $L^2((0,T); H^1(\Omega))$ and then in $L^2((0,T); C(\bar{\Omega}))$. Moreover (Q_1) ensures that NR_k is bounded by d_∞. From Lemma 27, $\frac{\mathrm{d}}{\mathrm{dt}} \beta^{(N)}$ is bounded in $L^2(0,T)$. Thus $\beta^{(N)}$ is bounded in $H^1(0,T)$ and then uniformly equicontinuous. The compactness result and the existence of $\bar{S}_g$ is then straightforward.

It remains to check that $S_t^{(N)}$ verifies equation (68). This comes directly from (70) where the term $\dot{R}_k/R_k$ is replaced using (13):

$$\begin{aligned}\frac{\dot{R}_k}{R_k} &= \frac{1}{\mu_g}\left(\Sigma_k + \mathrm{p}_g(\rho_k) + \frac{\bar{\gamma}_s}{2}\frac{1}{NR_k}\right)\\ &= \frac{1}{\mu_g}\left(\tilde{\Sigma}_g(c_k) + \mathrm{p}_g(\tilde{\rho}_g(c_k)) + \frac{\bar{\gamma}_s}{2}\frac{1}{NR_k}\right).\end{aligned}$$

□

Actually, the dependence of the measures $\bar{S}_{g,t}$ with respect to the space variable x can be precised:

Proposition 25. *For any $\beta \in C^\infty(\mathbb{R}^+)$, there exists $\bar{S}_\beta \in L^\infty((0,T);L^\infty(\Omega))$ such that, for all $\Phi \in C_c^\infty((0,T)\times\Omega)$ and all $t \in (0,T)$,*

$$\left\langle \bar{S}_{g,t}, \Phi(t,\cdot)\otimes\beta\right\rangle = \int_\Omega \bar{S}_\beta(t,x)\Phi(t,x)\,\mathrm{d}x. \tag{71}$$

In other words, we have:

$$\bar{S}_\beta(t,\cdot) = \int_{\mathbb{R}^+}\beta(r)\bar{S}_{g,t}(\cdot,\mathrm{d}r) \in L^\infty((0,T)\times\Omega).$$

Proof. Let $\beta \in C^\infty(\mathbb{R}^+)$ and $\phi \in C_c^\infty((0,T)\times\Omega)$. One has for every $t \in (0,T)$

$$\begin{aligned}\left\langle S_{g,t}^{(N)}, \phi(t,\cdot)\otimes\beta\right\rangle &= \frac{1}{N}\sum_{k=1}^N \beta(NR_k(t))\phi(t,c_k(t))\\ &= \sum_{k=1}^N \int_{B_k}\frac{1}{2NR_k}\beta(NR_k)\phi(t,x)\,\mathrm{d}x\\ &\quad - \sum_{k=1}^N \int_{B_k}\frac{1}{2NR_k}\beta(NR_k)(\phi(t,x)-\phi(t,c_k)).\end{aligned}$$

The second term of the right-hand side can be bounded by

$$\left(\max_{k=1,\dots,N} R_k\right)\|\beta\|_{L^\infty([d_\infty,1/d_\infty])}\|\partial_x\phi\|_{L^\infty((0,T);L^\infty(\Omega))},$$

and then tends to 0 when $N\to+\infty$ (see (Q_1)). The first term can be written as

$$\int_\Omega S_\beta^{(N)}(t,x)\phi(t,x)\,\mathrm{d}x$$

with

$$S_\beta^{(N)}(t,x) = \sum_{k=1}^N \frac{1}{2NR_k(t)}\beta(NR_k(t))\mathbb{1}_{B_k(t)}(x)$$

which provides a bounded sequence in $L^\infty((0,T)\times\Omega)$, by (Q_1). Therefore, there exists $\bar{S}_\beta \in L^\infty((0,T)\times\Omega)$ such that, up to the extraction of a subsequence,

$$S_\beta^{(N)} \rightharpoonup \bar{S}_\beta, \quad \text{in } L^\infty((0,T)\times\Omega) - w^\star \quad \text{when } N\to+\infty.$$

Then, letting $N\to\infty$ in the previous equality yields (71). □

We obtain then the following limiting equation for $\bar{S}_{g,t}$:

Proposition 26. *The limit $\bar{S}_{g,t}$ defined in Proposition 24 satisfies the equation*

$$\partial_t \bar{S}_{g,t} + \partial_x\big(\bar{S}_{g,t}\bar{u}\big) + \frac{1}{\mu_g}\partial_r\Big(\big(r\big(\bar{\Sigma}_g + \mathrm{p}_g(\bar{\rho}_g)\big) + \bar{\gamma}_s/2\big)\bar{S}_{g,t}\Big) = 0. \tag{72}$$

Proof. To obtain a time-evolution PDE for $\bar{S}_{g,t}$, we go back to equation (69) with a tensorised test function $\beta(x,r) = \beta_x(x)\beta_r(r)$, which writes

$$\begin{aligned}\partial_t\Big\langle S_t^{(N)}, \beta\Big\rangle = & \Big\langle S_t^{(N)}, \tilde{u}^{(N)}(x)\beta_x' \otimes \beta_r\Big\rangle \\ & + \frac{1}{\mu_g}\Big\langle S_t^{(N)}, r\tilde{\Sigma}_g^{(N)}(x)\beta_x \otimes \beta_r'\Big\rangle \\ & + \frac{1}{\mu_g}\Big\langle S_t^{(N)}, r\mathrm{p}_g\Big(\tilde{\rho}_g^{(N)}\Big)\beta_x \otimes \beta_r'\Big\rangle \\ & + \frac{\bar{\gamma}_s}{2\mu_g}\Big\langle S_t^{(N)}, \beta_x \otimes \beta_r'\Big\rangle.\end{aligned}$$

The first term of the right-hand side can be dealt using the strong convergence of $(\tilde{u}^{(N)})_N$ in $L^2\big((0,T), L^2(\Omega)\big)$. Since it is bounded in $L^2\big((0,T), H^1(\Omega)\big)$, the sequence $\big(\tilde{u}^{(N)}\big)_N$ converges also in $L^2\big((0,T), C\big(\bar{\Omega}\big)\big)$ by interpolation. The weak convergence of $\Big(S_t^{(N)}\Big)_N$ in $C\big([0,T], \mathbb{P}\big(\bar{\Omega} \times \mathbb{R}^+\big)\big)$ together with this strong convergence gives

$$\Big\langle S_t^{(N)}, \tilde{u}^{(N)}\beta_x' \otimes \beta_r\Big\rangle \longrightarrow \big\langle \bar{S}_{g,t}, \bar{u}\beta_x' \otimes \beta_r\big\rangle.$$

Since $\tilde{\Sigma}_g$ is uniformly bounded in $L^2((0,T), H^1(\Omega)) \subset L^2((0,T), C^{0,1/2}(\bar{\Omega}))$, the second term writes

$$\begin{aligned}\Big\langle S_t^{(N)}, r\tilde{\Sigma}_g^{(N)}(x)\beta_x \otimes \beta_r'\Big\rangle &= \frac{1}{N}\sum_{k=1}^N \tilde{\Sigma}_g^{(N)}(c_k) N R_k \beta_x(c_k)\beta_r'(NR_k) \\ &= \frac{1}{2}\sum_{k=1}^N \int_{B_k} \tilde{\Sigma}_g^{(N)}(x)\beta_x(x)\beta_r'(NR_k)\,\mathrm{d}x + \frac{C_\beta\Big\|\tilde{\Sigma}_g\Big\|_{H^1(\Omega)}}{\sqrt{N}} \\ &= \frac{1}{2}\int_\Omega \tilde{\Sigma}_g^{(N)}(x)\beta_x(x)b^{(N)}(x)\,\mathrm{d}x + \frac{C_\beta\Big\|\tilde{\Sigma}_g\Big\|_{H^1(\Omega)}}{\sqrt{N}}\end{aligned}$$

where $b^{(N)}$ is defined by equation (58), with

$$b(1,\cdot,\cdot) = 0, \quad b(0,\cdot,\nu) = \beta_r'(1/(2\nu)).$$

Indeed, this provides

$$b^{(N)} = \begin{cases} 0 & \text{in } \mathcal{F}^{(N)}, \\ \beta_r'(1/(2f_k)) & \text{in } B_k. \end{cases}$$

Using successively Propositions 19, 18 and 25, we obtain

$$\begin{aligned}\Big\langle S_t^{(N)}, r\tilde{\Sigma}_g^{(N)}(x)\beta_x \otimes \beta_r'\Big\rangle \longrightarrow & \frac{1}{2}\int_\Omega \bar{\Sigma}_g(x)\bar{b}(x)\beta_x(x)\,\mathrm{d}x \\ & = \frac{1}{2}\int_\Omega \bar{\Sigma}_g(x)\Big[b(1,\bar{\rho}_g,0)\bar{\alpha}_f\end{aligned}$$

$$
\begin{aligned}
&+\int_{\mathbb{R}+}(2r)b(0,\bar\rho_g,1/(2r))\bar S_{g,t}(\mathrm{d}r)\Big]\beta_x(x)\,\mathrm{d}x\\
&=\frac12\int_\Omega\bar\Sigma_g(x)\left[\int_{\mathbb{R}+}(2r)\beta_r'(r)\bar S_{g,t}\,(\mathrm{d}r)\right]\beta_x(x)\,\mathrm{d}x\\
&=\left\langle\bar S_{g,t},r\bar\Sigma_g\beta_x\otimes\beta_r'\right\rangle.
\end{aligned}
$$

For the third term, we proceed similarly, defining

$$b(1,\cdot,\cdot)=0,\quad b(0,\xi,\nu)=\mathrm{p}_g(\xi)\beta_r'(1/(2\nu)),$$

so that

$$\left\langle S_t^{(N)},r\mathrm{p}_g\left(\tilde\rho_g^{(N)}\right)\beta_x\otimes\beta_r'\right\rangle\longrightarrow\left\langle\bar S_{g,t},r\mathrm{p}_g(\bar\rho_g)\beta_x\otimes\beta_r'\right\rangle.$$

The convergence of the last term is nothing else but the convergence of $S_t^{(N)}$. □

Observe that $\bar\alpha_g\bar f_g$ and $\bar\alpha_g$ are respectively the zeroth and first moments of $\bar S_g$. Their PDE's, see Corollary 22, can be deduced from the equation (72).

Appendix A. Proof of Proposition 10

In the whole section, we consider $T>0$ and $\left(\rho_f,u_f,(c_k,R_k)_{k=1,\dots,N}\right)$ is a classical solution to (6)–(13) on $(0,T)$, satisfying (Q_1)–(Q_5).

To start with, we recall that Corollary 3 applies. With (Q_1), these estimates yield:

$$
\begin{aligned}
\int_{\mathcal F}\left(\rho_f\frac{|u_f|^2}{2}+q(\rho_f)\right)\mathrm{d}x+\frac12\sum_{k=1}^N m_k\left(|\dot c_k|^2+\frac13\left|\dot R_k\right|^2\right)&\\
-\sum_{k=1}^N\kappa_k\ln(d_\infty NR_k)\le E_0,&
\end{aligned}
\tag{A.1}
$$

$$\int_0^T\left[\left(\int_{\mathcal F}\mu_f|\partial_xu_f|^2\,\mathrm{d}x+\mu_g\sum_{k=1}^N\frac{|\dot R_k|^2}{R_k}\right)\right]\mathrm{d}t\le E_0,\tag{A.2}$$

with a constant E_0 depending only on the list of parameters (25).

Strict version $(\mathscr{Q}_1)$ of (Q_1)

Since $|a|-|b|\le|a-b|$ and $(\alpha^2+\beta^2+\gamma^2)^{1/2}\le\alpha+(\beta^2+\gamma^2)^{1/2}$ as soon as the α,β and γ are nonnegative, it follows from Corollary 9, (IC_0) and the bounds (Q_1) on R_k, (Q_2) on $|\mathcal F_k|$ that

$$\left|\frac{\dot R_k}{R_k}\right|\le\frac{1}{\mu_g}\frac{1}{M_\infty NR_k}+\frac{C_1}{\mu_g}\left\|\tilde\Sigma_g\right\|_{H^1(\Omega)}+\frac{C_1}{\mu_g}\left(\frac{1}{\min_k|\mathcal F_k|}\sum_{k=1}^N(m_k)^2\left(\left|\ddot R_k\right|^2+|\ddot c_k|^2\right)\right)^{1/2}$$

and then:

$$\left|\frac{\dot R_k}{R_k}\right|\le\frac{1}{\mu_g}\frac{1}{M_\infty d_\infty}+\frac{C_1}{\mu_g}\left\|\tilde\Sigma_g\right\|_{H^1(\Omega)}+\frac{C_1}{\mu_g}\left(\frac{1}{M_\infty d_\infty}\sum_{k=1}^N m_k\left(\left|\ddot R_k\right|^2+|\ddot c_k|^2\right)\right)^{1/2}.\tag{A.3}$$

The last term can be bounded by $\sqrt{K_\infty}$ according to (Q_5). Integrating on the time interval $(0,t)$, $t<T$, it yields

$$\int_0^t\left|\frac{\dot R_k}{R_k}\right|\mathrm{d}t\le\frac{1}{\mu_gM_\infty d_\infty}T+\frac{C_2}{\mu_g}\left(1+\frac{1}{\sqrt{M_\infty d_\infty}}\right)\sqrt{KT}.$$

Considering a smaller time T, only depending on μ_g, d_∞, M_∞, C_0 and K_∞, it holds

$$\int_0^t \left| \frac{\dot{R}_k}{R_k} \right| \mathrm{d}t < \frac{1}{2},$$

which gives

$$\frac{R_k^0}{2} < e^{-1/2} R_k^0 < R_k < e^{1/2} R_k^0 < 2R_k^0.$$

Finally the Assumption (IC_1) on the initial radii leads to the desired estimate $(\mathscr{Q}_1)$. We note in passing that we obtained the following lemma:

Lemma 27. *There exists a constant $\tilde{K}$ depending on μ_g, d_∞, M_∞, C_0 and K, such that:*

$$\int_0^T \left(\max_{k=1,\dots,N} \left| \frac{\dot{R}_k(t)}{R_k(t)} \right| \right)^2 \mathrm{d}t \le \tilde{K}.$$

The proof of this lemma is a straightforward application of (A.3) and is left to the reader.

Strict version $(\mathscr{Q}_2)$ of (Q_2)

First, we remark that we can also adapt the previous proof to yield the following lemma:

Lemma 28. *There exists a constant C', depending only on K and the list of parameters (25), such that, for $T < 1$, there holds*

$$\int_0^T \|\partial_x u_f\|_{L^\infty(\mathcal{F})} \,\mathrm{d}t \le C' \sqrt{T}.$$

Proof. We use the L^∞ bound on $\partial_x u_f$, see (38), and the bound (Q_3) on the density ρ_f. It holds, by integrating on $(0,T)$,

$$\int_0^T \|\partial_x u_f\|_{L^\infty(\mathcal{F})} \,\mathrm{d}t \le \frac{C}{\mu_f} \int_0^T \left\| \tilde{\Sigma}_f \right\|_{H^1(\Omega)} \mathrm{d}t + T \max_{\underline{\rho}_\infty/2 \le r \le 2\bar{\rho}_\infty} \mathrm{p}_f(r).$$

Inequality (Q_5) on the stress tensor leads to

$$\int_0^T \|\partial_x u_f\|_{L^\infty(\mathcal{F})} \,\mathrm{d}t \le \frac{C}{\mu_f} \sqrt{TK} + T \max_{\underline{\rho}_\infty/2 \le r \le 2\bar{\rho}_\infty} \mathrm{p}_f(r),$$

which gives the expected bound for $T < 1$. $\square$

The continuity of the velocities (10) implies then that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(x_{k+1}^- - x_k^+\right) &= u\left(x_{k+1}^-\right) - u\left(x_k^+\right) \\ &\le \|\partial_x u_f\|_{L^\infty(\mathcal{F})} \left| x_{k+1}^- - x_k^+ \right|. \end{aligned}$$

From Lemma 28, we can choose T small (depending only on C') such that there holds:

$$\frac{\left|\mathcal{F}_k^0\right|}{2} < |\mathcal{F}_k| < 2\left|\mathcal{F}_k^0\right|,$$

on $(0,T)$, which leads to the desired estimate.

Strict version ($\mathscr{Q}_3$) of (Q_3)

Since the fluid density ρ_f satisfies a continuity equation associated with the velocity u_f on the fluid domains $\mathcal{F}_k$ which are transported by the same velocity field u_f, a classical estimate on $(0,T)$ provides

$$\left(\min_{x\in\mathcal{F}^0}\rho_f^0\right)\times\exp\left(-\int_0^T\|\partial_x u_f\|_{L^\infty(\mathcal{F})}\,\mathrm{d}t\right) \le \rho_f(t,x) \le \left(\max_{x\in\mathcal{F}^0}\rho_f^0\right)\times\exp\left(\int_0^T\|\partial_x u_f\|_{L^\infty(\mathcal{F})}\,\mathrm{d}t\right), \tag{A.4}$$

Lemma 28 allows to bound the exponential terms in (A.4) for small time. Namely, for T small (depending on C') it holds, on $[0,T]$,

$$\rho_f(t,x)\in\left(\frac{1}{2}\min_{x\in\mathcal{F}^0}\rho_f^0,\, 2\max_{x\in\mathcal{F}^0}\rho_f^0\right).$$

Then the assumption (24) on the initial fluid density allows to deduce a strict version of estimate (Q_3).

Strict version ($\mathscr{Q}_4$) of (Q_4)

Applying (34), we obtain:

$$\begin{aligned}
&\sup_{[0,T]}\left(\int_{\mathcal{F}}\mu_f\frac{|\partial_x u_f|^2}{2}\,\mathrm{d}x+\mu_g\sum_{k=1}^N\frac{\left|\dot R_k\right|^2}{R_k}\right)\\
&\quad+\int_0^T\left(\int_{\mathcal{F}}\rho_f|\partial_t u_f+u_f\partial_x u_f|^2\,\mathrm{d}x+\sum_{k=1}^N m_k\left(|\ddot c_k|^2+\left|\ddot R_k\right|^2\right)\right)\\
&\quad\le\sup_{[0,T]}\left[\left(2\sum_{k=1}^N\kappa_k\frac{\left|\dot R_k\right|}{R_k}\right)+\int_{\mathcal{F}}\mathrm{p}_f(\rho_f)|\partial_x u_f|\,\mathrm{d}x\right]\\
&\quad+\int_0^T\int_{\mathcal{F}}\mu_f\frac{|\partial_x u_f|^3}{2}\,\mathrm{d}x\\
&\quad+\int_0^T\sum_{k=1}^N\left(2\kappa_k\frac{\left|\dot R_k\right|^2}{R_k^2}+\mu_g\frac{\left|\dot R_k\right|^3}{R_k^2}\right)+E_1,
\end{aligned}$$

with a constant E_1 depending only on the list of parameters (25). To proceed, we detail now the controls of the five remaining terms on the right-hand side.

Concerning the first line in the right-hand side, the first term can be rewritten with (IC_0):

$$\sum_{k=0}^N\kappa_k\frac{\left|\dot R_k\right|}{R_k}\le\frac{1}{M_\infty}\sum_{k=0}^N\left(\frac{\left|\dot R_k\right|}{\sqrt{R_k}}\frac{1}{N\sqrt{R_k}}\right).$$

The Cauchy–Schwarz inequality gives then

$$\sum_{k=0}^N\kappa_k\frac{\left|\dot R_k\right|}{R_k}\le\frac{1}{M_\infty\sqrt{\mu_g}}\left(\mu_g\sum_{k=0}^N\frac{\left|\dot R_k\right|^2}{R_k}\right)^{1/2}\left(\sum_{k=0}^N\frac{1}{N^2R_k}\right)^{1/2}.$$

The first parenthesis can be bounded by K using (Q_4) and the second one by $1/\sqrt{d_\infty}$ thanks to (Q_1) so that

$$\sum_{k=0}^{N} \kappa_k \frac{\left|\dot{R}_k\right|}{R_k} \leq \frac{1}{M_\infty \sqrt{\mu_g}} \frac{\sqrt{K}}{\sqrt{d_\infty}}. \tag{A.5}$$

The control of the second pressure term relies on (Q_3)–(Q_4) and a Cauchy–Schwarz inequality:

$$\begin{aligned}\int_{\mathcal{F}} \mathrm{p}_f(\rho_f)|\partial_x u_f| \, \mathrm{d}x &\leq \frac{\sqrt{2}}{\sqrt{\mu_f}} \left(\int_{\mathcal{F}} \mu_f \frac{|\partial_x u_f|^2}{2} \, \mathrm{d}x \right)^{1/2} \left[\max_{[\underline{\rho}_\infty/2, 2\bar{\rho}_\infty]} \mathrm{p}_f \right] \sqrt{2'} \\ &\leq \frac{2}{\sqrt{\mu_f}} \left[\max_{[\underline{\rho}_\infty/2, 2\bar{\rho}_\infty]} \mathrm{p}_f \right] \sqrt{K}. \end{aligned} \tag{A.6}$$

As for the term on the second line in the right-hand side of (34), we decompose as follows

$$\int_0^T \int_{\mathcal{F}} \mu_f \frac{|\partial_x u_f|^3}{2} \, \mathrm{d}x \leq \left(\sup_{[0,T]} \int_{\mathcal{F}} \mu_f \frac{|\partial_x u_f|^2}{2} \right) \left(\int_0^T \|\partial_x u_f\|_{L^\infty(\mathcal{F})} \right),$$

where the first term can be bounded by K according to (Q_4). The second one is bounded using Lemma 28. It follows that

$$\int_0^T \int_{\mathcal{F}} \mu_f \frac{|\partial_x u_f|^3}{2} \, \mathrm{d}x \leq C' \sqrt{T} K. \tag{A.7}$$

We now turn to the first term on the third line. Applying a standard $L^\infty - L^1$ Hölder inequality allows to bound this term by

$$\int_0^T \sum_{k=1}^{N} \kappa_k \frac{\left|\dot{R}_k\right|^2}{R_k^2} \leq \frac{1}{M_\infty \mu_g N} \max_{k \in \{1,\ldots,N\}} \left\| \frac{1}{R_k} \right\|_{L^\infty(0,T)} \int_0^T \mu_g \sum_{k=1}^{N} \frac{\left|\dot{R}_k\right|^2}{R_k}.$$

The L^∞ norm can be handled by the bound (Q_1) and the integral term by (A.2). It follows that

$$\int_0^T \sum_{k=1}^{N} \kappa_k \frac{\left|\dot{R}_k\right|^2}{R_k^2} \leq \frac{E_0}{M_\infty \mu_g d_\infty}. \tag{A.8}$$

It remains to bound the second term on the third line. In this respect, we decompose the nonlinear term

$$\frac{\left|\dot{R}_k\right|^3}{R_k^2} = \frac{\left|\dot{R}_k\right|}{R_k} \frac{\left|\dot{R}_k\right|^{3/2}}{R_k^{3/4}} \frac{\left|\dot{R}_k\right|^{1/2}}{R_k^{1/4}}$$

and apply a $L^\infty - L^{4/3} - L^4$ Hölder inequality to yield:

$$\sum_{k=1}^{N} \frac{\left|\dot{R}_k\right|^3}{R_k^2} \leq \max_{k \in \{1,\ldots,N\}} \frac{\left|\dot{R}_k\right|}{R_k} \left(\sum_{k=1}^{N} \frac{\left|\dot{R}_k\right|^2}{R_k} \right)^{3/4} \left(\sum_{k=1}^{N} \frac{\left|\dot{R}_k\right|^2}{R_k} \right)^{1/4}.$$

Integrating over $(0,T)$ we obtain again with a $L^2 - L^4 - L^4$ Hölder inequality that:

$$\int_0^T \sum_{k=1}^{N} \frac{\left|\dot{R}_k\right|^3}{R_k^2} \leq \left(\int_0^T \left| \max_{k \in \{1,\ldots,N\}} \frac{\left|\dot{R}_k\right|}{R_k} \right|^2 \right)^{1/2} \left(\int_0^T \left| \sum_{k=1}^{N} \frac{\left|\dot{R}_k\right|^2}{R_k} \right|^3 \right)^{1/4} \left(\int_0^T \sum_{k=1}^{N} \frac{\left|\dot{R}_k\right|^2}{R_k} \right)^{1/4}.$$

Corollary 9 and inequality (Q_1) allow to control the first term on the right-hand side. Indeed,

$$\mu_g \max_{k\in\{1,\ldots,N\}} \frac{\left|\dot R_k\right|}{R_k} \le \frac{1}{M_\infty d_\infty} + C_1\left(\left\|\tilde\Sigma_f\right\|^2_{H^1(\Omega)} + \frac{1}{M_\infty d_\infty}\sum_{k=1}^N m_k\left(\left|\ddot R_k\right|^2 + |\ddot c_k|^2\right)\right)^{1/2}.$$

Taking the L^2-norm in time and applying a triangular inequality and (Q_5) provides

$$\left(\int_0^T \left|\max_{k\in\{1,\ldots,N\}} \frac{\left|\dot R_k\right|}{R_k}\right|^2\right)^{1/2} \le \frac{1}{\mu_g}\left(\sqrt{T}\frac{1}{M_\infty d_\infty} + C_1\sqrt{K\left(1+\frac{1}{M_\infty d_\infty}\right)}\right).$$

As the second term is concerned, it holds

$$\left(\int_0^T \left|\sum_{k=1}^N \frac{\left|\dot R_k\right|^2}{R_k}\right|^3\right)^{1/4} \le T^{1/4}\left(\sup_{[0,T]}\sum_{k=1}^N \frac{\left|\dot R_k\right|^2}{R_k}\right)^{3/4},$$

which can be handled thanks to (Q_4), leading to

$$\left(\int_0^T \left|\sum_{k=1}^N \frac{\left|\dot R_k\right|^2}{R_k}\right|^3\right)^{1/4} \le T^{1/4}\left(\frac{K}{\mu_g}\right)^{3/4}.$$

Now the bound (A.2) gives

$$\left(\int_0^T \sum_{k=1}^N \frac{\left|\dot R_k\right|^2}{R_k}\right)^{1/4} \le \left(\frac{E_0}{\mu_g}\right)^{1/4}.$$

To sum up, it finally yields

$$\int_0^T \sum_{k=1}^N \frac{\left|\dot R_k\right|^3}{R_k^2} \le \frac{1}{\mu_g^2}T^{1/4}K^{3/4}E_0^{1/4}\left(\sqrt{T}\frac{1}{M_\infty d_\infty} + C_1\sqrt{K\left(1+\frac{1}{M_\infty d_\infty}\right)}\right). \tag{A.9}$$

Plugging (A.5)–(A.9) into (34), it yields

$$\begin{aligned}
&\sup_{[0,T]}\left(\int_{\mathcal F} \frac{|\partial_x u_f|^2}{2}\,\mathrm{d}x + \sum_{k=1}^N \mu_g \frac{\left|\dot R_k^2\right|}{R_k}\right) \qquad\qquad (A.10)\\
&\le 2\left(\frac{\sqrt K}{M_\infty\sqrt{\mu_g d_\infty}} + \frac{1}{\sqrt{\mu_f}}\max_{[\underline\rho_\infty/2,2\bar\rho_\infty]} \mathrm{p}_f(r)\sqrt K\right)\\
&+ C'\sqrt T K + \frac{E_0}{M_\infty \mu_g d_\infty}\\
&+ \frac{2}{\mu_g^2}T^{1/4}K^{3/4}E_0^{1/4}\left(\sqrt T\frac{1}{M_\infty d_\infty} + C_1\sqrt{K\left(1+\frac{1}{M_\infty d_\infty}\right)}\right) + E_1.
\end{aligned}$$

A key remark here is that, on the right hand side, we have two types of quantities: some quantities are constants multiplied by powers of K less than 1 (these can be made arbitary smaller than K for sufficiently large K) either we have some constant depending on K multiplied by a positive power of T (these can be made arbitrary smaller than K for small T). If one considers $K > 1$ and $T < 1$, defining

$$C_1' = 2\left(\frac{1}{M_\infty\sqrt{\mu_g d_\infty}} + \frac{1}{\sqrt{\mu_f}} \max_{[\underline{\rho}_\infty/2, 2\bar{\rho}_\infty]} \mathrm{p}_f(r)\right),$$
$$C_2' = \frac{2}{\mu_g^2} E_0^{1/4}\left(\frac{1}{M_\infty d_\infty} + C_1\sqrt{1 + \frac{1}{M_\infty d_\infty}}\right),$$

the previous inequality writes

$$\sup_{[0,T]}\left(\int_{\mathcal{F}} \frac{|\partial_x u_f|^2}{2}\,\mathrm{d}x + \sum_{k=1}^N \mu_g \frac{\left|\dot{R}_k^2\right|}{R_k}\right) \le C_1'\sqrt{K} + \frac{E_0}{M_\infty \mu_g d_\infty} + C'\sqrt{T}K + T^{1/4}K^{5/2}C_2' + E_1. \tag{A.11}$$

Introduce $\lambda \in (0, 1/2)$ to be fixed later on and set:

$$K_\infty := \frac{4|C_1'|^2}{\lambda^2} + \frac{1}{M_\infty \mu_g d_\infty}\frac{E_0}{2\lambda} + \frac{E_1}{1-2\lambda}\cdot$$

When $K > K_\infty$, the sum of the two first terms on the right-hand side of (A.11) are bounded by λK. Now taking T small enough, for instance

$$T = \min\left\{(\lambda/2|C'|)^2, \left(\lambda\left(K^{3/2}C_2'\right)^{-1}/2\right)^4\right\},$$

the sum of the third and fourth term can be bounded by λK as well. Finally, the right-hand side is bounded according to

$$\sup_{[0,T]}\left(\int_{\mathcal{F}} \frac{|\partial_x u_f|^2}{2}\,\mathrm{d}x + \sum_{k=1}^N \mu_g \frac{\left|\dot{R}_k^2\right|}{R_k}\right) \le E_1 + 2\lambda K < K$$

since $\lambda < 1/2$ and $C_1' > 0$.

Strict version $(\mathscr{Q}_5)$ of (Q_5)

In order to prove this estimate, we can adjust with the parameter λ. First, thanks to Proposition 6 and to the bounds (Q_1) and (IC_0), there exists $C > 0$, depending in particular on M_∞ and d_∞, such that

$$\int_0^T \left\|\tilde{\Sigma}_f\right\|_{H^1(\Omega)}^2 \mathrm{d}t \le C_0 \int_0^T \Bigg[\|\Sigma_f\|_{H^1(\mathcal{F})}^2 + \sum_{k=1}^N m_k\left(\left|\ddot{R}_k\right|^2 + |\ddot{c}_k|^2\right) + \sum_{k=1}^N\left(\mu_g^2\frac{\left|\dot{R}_k\right|^2}{R_k} + \frac{\kappa_k^2}{R_k}\right)\Bigg]\,\mathrm{d}t. \tag{A.12}$$

The second term of the right-hand side can be bounded with the help of (34) by bounding the right-hand side of (34) as in the previous analysis on (Q_4). This entails:

$$\int_0^T \sum_{k=1}^N m_k\left(\left|\ddot{R}_k\right|^2 + |\ddot{c}_k|^2\right)\mathrm{d}t \le E_1 + 2\lambda K. \tag{A.13}$$

The third term is controlled using (A.2). The last term can be bounded by $T/((M_\infty)^2 d_\infty)$. Therefore, this inequality becomes

$$\int_0^T \left\|\tilde{\Sigma}_f\right\|^2_{H^1(\Omega)} \mathrm{d}t \leq C_0\left[\int_0^T \|\Sigma_f\|^2_{H^1(\mathcal{F})} \mathrm{d}t + (E_1 + 2\lambda K) + \mu_g E_0 + \frac{T}{(M_\infty)^2 d_\infty}\right]. \tag{A.14}$$

Let us now focus on the first term. We use the definition (8) of Σ_f and the momentum equation (7) to write

$$\int_0^T \|\Sigma_f\|^2_{H^1(\mathcal{F})} \mathrm{d}t = \int_0^T \|\mu_f \partial_x u_f - \mathrm{p}_f(\rho_f)\|^2_{L^2(\mathcal{F})} \mathrm{d}t \\ + \int_0^T \|\rho_f(\partial_t u_f + u_f \partial_x u_f)\|^2_{L^2(\mathcal{F})} \mathrm{d}t.$$

Thanks to (A.2) and (Q_3), the first term can be bounded. For the second term, one has

$$\int_0^T \|\rho_f(\partial_t u_f + u_f \partial_x u_f)\|^2_{L^2(\mathcal{F})} \mathrm{d}t \leq \bar{\rho}_\infty \int_0^T \int_{\mathcal{F}} \rho_f |\partial_t u_f + u_f \partial_x u_f|^2 \mathrm{d}t,$$

and this right-hand side actually appears in (34) and thus, the previous estimate obtained to prove (Q_4) can be used. This provides

$$\int_0^T \|\Sigma_f\|^2_{H^1(\mathcal{F})} \mathrm{d}t \leq 2\mu_f E_0 + 2T\left[\max_{[\underline{\rho}_\infty, \bar{\rho}_\infty]} \mathrm{p}_f\right]^2 + \bar{\rho}_\infty(E_1 + 2\lambda K).$$

Gathering the previous estimates and after rearrangement,

$$\int_0^T \left\|\tilde{\Sigma}_f\right\|^2_{H^1(\Omega)} \mathrm{d}t \leq C_0\Bigg[(2\mu_f E_0 + (\bar{\rho}_\infty + 1)E_1 + \mu_g E_0) \\ + \left(2\left[\max_{[\underline{\rho}_\infty, \bar{\rho}_\infty]} \mathrm{p}_f\right]^2 + \frac{1}{(M_\infty)^2 d_\infty}\right)T + 2(\bar{\rho}_\infty + 1)\lambda K\Bigg]$$

holds. Now starting from Proposition 8, a similar estimate can be proved for $\tilde{\Sigma}_g$. Then, using again (A.13), one finally have

$$\int_0^T \left[\left\|\tilde{\Sigma}_f\right\|^2_{H^1(\Omega)} + \left\|\tilde{\Sigma}_g\right\|^2_{H^1(\Omega)} + m_k\left(\left|\ddot{R}_k\right|^2 + |\ddot{c}_k|^2\right)\right] \mathrm{d}t \leq C_1 + C_2 T + C_3 \lambda K,$$

where C_1, C_2 and C_3 are positive and independent of N, T, λ and K. To conclude, it suffices to choose λ sufficiently small so that $\lambda \leq (4C_3)^{-1}$ and $K \geq 4C_1$. We can then take T smaller if necessary so that $C_2 T \leq K/4$.

Appendix B. Analysis of the density equation

This section is devoted to the proof of the following proposition:

Proposition 29. *Assume that $T > 0$ and $\left(\rho_f, u_f, (c_k, R_k)_{k=1,\dots,N}\right)$ is a classical solution to (6)–(13) on $(0,T)$ – complemented with initial conditions constructed as in (16)–(19) – that satisfies (Q_1)–(Q_5). Then, there exists strictly positive constants K_1 and T_1 depending only on the list of parameters (25) and K such that*

$$\|\rho_f(t)\|_{H^1(\mathcal{F}(t))} \leq K_1 \quad \text{on } (0, T_1).$$

Again, the main difficulty in obtaining this proposition is to make the constant K_1 independent of the parameter N. For this, we proceed as in Section 4 and interpret ρ_f on $\mathcal{F}(t)$ as the trace of some global density defined on Ω. We notice here that, by assumption, we already have this property initially since we set

$$\rho_f(0,\cdot) = \rho_f^0 \quad \text{on } \mathcal{F}^0$$

with $\rho_f^0 \in H^1(\Omega)$. To extend this property, we construct again extensions $\tilde{u}_f$ of fluid velocity-field u_f and $\tilde{\Sigma}_f$ of stress tensor Σ_f with the same formula as in (45) and (35) respectively. We then construct $\tilde{\rho}_f$

$$\begin{cases} \partial_t \tilde{\rho}_f + \tilde{u}_f \partial_x \tilde{\rho}_f = -\frac{\tilde{\rho}_f}{\mu_f}\Big(\tilde{\Sigma}_f + \mathrm{p}_f(\tilde{\rho}_f)\Big), & \text{on } (0,T)\times\Omega, \\ \tilde{\rho}_f(0,.) = \rho_f^0, & \text{on } \Omega, \end{cases} \tag{B.15}$$

By (Q_5) with Proposition 38 for the fluid part and (Q_1)–(Q_5) with Corollary 9 for the bubble part, we obtain that $\tilde{u}_f \in L^2\big(0,T;W^{1,\infty}(\Omega)\big)$ with $\tilde{\Sigma}_f \in L^2\big(0,T;H^1(\Omega)\big)$. Consequently, we have a unique solution (B.15) which solves (6) on $\mathcal{F}$. By uniqueness of the solution to (6) in the regularity class of classical solutions (see [15]), we have thus $\tilde{\rho}_f = \rho_f$ on $\mathcal{F}(t)$ for $t \in (0,T)$. So, our proof reduces to computing bounds for $\tilde{\rho}_f$.

First, we prove that there exists $T_0 \le T$ such that we can control $\|\tilde{\rho}_f\|_{L^\infty(\Omega)}$ explicitly on $(0,T_0)$. By the method of characteristics and the explicit value of p_f :

$$\|\tilde{\rho}_f(t,.)\|_{L^\infty(\Omega)} \le \big\|\rho_f^0\big\|_{L^\infty(\Omega)} \exp\left(\frac{1}{\mu_f}\int_0^T \left(\Big\|\tilde{\Sigma}_f\Big\|_{L^\infty(\Omega)} + a_f\|\tilde{\rho}_f(t,.)\|_{L^\infty(\Omega)}^{\gamma_f}\right)\mathrm{d}t\right).$$

The bound (Q_5) coupled with the embedding of $H^1(\Omega)$ in $L^\infty(\Omega)$ allows to control the stress tensor norm by K. If $\|\tilde{\rho}_f(t,.)\|_{L^\infty(\Omega)} \le 2\Big\|\rho_f^0(t,.)\Big\|_{L^\infty(\Omega)}$, it yields

$$\|\tilde{\rho}_f(t,.)\|_{L^\infty(\Omega)} \le \big\|\rho_f^0\big\|_{L^\infty(\Omega)} \exp\left(\frac{1}{\mu_f}\sqrt{TK} + 2a_f T\big\|\rho_f^0(t,.)\big\|_{L^\infty(\Omega)}^{\gamma_f}\right).$$

By a standard continuation argument, we construct then a time-interval $(0,T_0)$ depending only on K, a_f, γ_f and $\|\rho_f^0(t,.)\|_{L^\infty(\Omega)}$ so that:

$$\|\tilde{\rho}_f(t,.)\|_{L^\infty(\Omega)} \le 2\big\|\rho_f^0\big\|_{L^\infty(\Omega)}$$

for $t < T_0$.

We focus now on $\partial_x\tilde{\rho}_f$. For this, we apply a space derivative to (B.15):

$$\begin{cases} \partial_t(\partial_x\tilde{\rho}_f) + \partial_x(\tilde{u}\partial_x\tilde{\rho}_f) = -\frac{\tilde{\rho}_f}{\mu_f}\partial_x\tilde{\Sigma}_f \\ \qquad\qquad\qquad\qquad\qquad\quad -\frac{1}{\mu_f}\Big(\tilde{\Sigma}_f + \mathrm{p}_f(\tilde{\rho}_f) + \tilde{\rho}_f\mathrm{p}_f'(\tilde{\rho}_f)\Big)\partial_x\tilde{\rho}_f \\ (\partial_x\tilde{\rho}_f)(0,\cdot) = \partial_x\rho_f^0. \end{cases}$$

For simplicity, we denote from now on $Y := \partial_x\tilde{\rho}_f$. We multiply the previous equation by $2Y$, leading to

$$\partial_t(Y^2) + \partial_x(\tilde{u}Y^2) = -2Y\frac{\tilde{\rho}_f}{\mu_f}\partial_x\tilde{\Sigma}_f - Y^2 A$$

where A denotes $\partial_x\tilde{u} + \frac{2}{\mu_f}\Big(\tilde{\Sigma}_f + \kappa_f(\gamma_f+1)(\tilde{\rho}_f)^{\gamma_f}\Big)$. Let first bound the right-hand side by a standard Cauchy-Schwarz/Minkowski inequality:

$$\int_\Omega\left(-2Y\frac{\tilde{\rho}_f}{\mu_f}\partial_x\tilde{\Sigma}_f - Y^2A\right)\mathrm{d}x \le \frac{1}{\mu_f}\Big\|\tilde{\rho}_f\partial_x\tilde{\Sigma}_f\Big\|_{L^2(\Omega)}^2 + \left(\frac{1}{\mu_f} + \|A\|_{L^\infty(\Omega)}\right)\|Y\|_{L^2(\Omega)}^2.$$

Going back to the PDE for Y^2, the L^2 norm of $\partial_x \tilde{\rho}_f$ can be bounded as

$$\|\partial_x \tilde{\rho}_f\|^2_{L^2(\Omega)} \leq \left(\left\|\partial_x \rho_f^0\right\|^2_{L^2(\Omega)} + \frac{1}{\mu_f} \int_0^T \left\|\tilde{\rho}_f \partial_x \tilde{\Sigma}_f\right\|^2_{L^2(\Omega)} \mathrm{d}t \right) \times \exp\left(\frac{T}{\mu_f} + \int_0^T \left(\|\partial_x \tilde{u}\|_{L^\infty(\Omega)} + \frac{2}{\mu_f} \left(\left\|\tilde{\Sigma}_f\right\|_{L^\infty(\Omega)} + \kappa_f(\gamma_f + 1)\|\tilde{\rho}_f\|^{\gamma_f}_{L^\infty(\Omega)} \right) \right) \right).$$

All the terms can be controlled using (Q_3) and (Q_5), except $\int_0^T \|\partial_x \tilde{u}\|_{L^\infty(\Omega)}\,\mathrm{d}t$. This latter term can be bounded using lemmas 28 and 27 (corresponding respectively to the contributions of $\|\partial_x \tilde{u}\|_{L^\infty(\mathcal{F})}$ and $\|\partial_x \tilde{u}\|_{L^\infty(\Omega\setminus\mathcal{F})}$). Then, for a sufficiently small time $T_1 \leq T_0$,

$$\int_0^{T_1} \|\partial_x \tilde{u}\|_{L^\infty(\Omega)} \mathrm{d}t < \frac{1}{2},$$

so that on $(0, T_1)$:

$$\|\partial_x \tilde{\rho}_f\|^2_{L^2(\Omega)} \leq \left(\left\|\partial_x \rho_f^0\right\|^2_{L^2(\Omega)} + \frac{2}{\mu_f} \|\rho_f^0\|^2_{L^\infty(\Omega)} K \right) \times \exp\left(\frac{T_1}{\mu_f} + \frac{1}{2} + \frac{2}{\mu_f}\sqrt{T_1 K} + T_1 \kappa_f(\gamma_f + 1) 2^{\gamma_f} \left\|\bar{\rho}_f^0\right\|^{\gamma_f}_{L^\infty(\Omega)} \right).$$

This completes the proof.

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