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ON STRICTLY CONVEX ENTROPY FUNCTIONS FOR THE REACTIVE EULER EQUATIONS

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Abstract. This work is concerned with entropy functions of the reactive Euler equations describing inviscid compressible flow with chemical reactions. In our recent work (W. Zhao, *Math. Comput.* **91** (2022) 735–760.) we point out that for these equations as a hyperbolic system, the classical entropy function associated with the thermodynamic entropy is no longer strictly convex under the equation of state (EoS) for the ideal gas. In this work, we propose two strategies to address this issue. The first one is to correct the entropy function. Namely, we present a class of strictly convex entropy functions by adding an extra term to the classical one. Such strictly entropy functions contain that constructed in (W. Zhao, *Math. Comput.* **91** (2022) 735–760.) as a special case. The second strategy is to modify the EoS. We show that there exists a family of EoS (for the nonideal gas) such that the classical entropy function is strictly convex. Under these new EoS, the reactive Euler equations are proved to satisfy the Conservation-Dissipation Conditions for general hyperbolic relaxation systems, which guarantee the existence of zero relaxation limit. Additionally, an elegant eigen-system of the Jacobian matrix is derived for the reactive Euler equations under the proposed EoS. Numerical experiments demonstrate that the proposed EoS can also generate ZND detonations. Extension of the present results to high dimensions is direct.

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1. Introduction

In this work, we are interested in the reactive Euler equations describing inviscid compressible flow with chemical reactions, which are fundamental for modeling detonations. In one dimension (1D), these equations read as [11,13]

$$\partial_t \begin{pmatrix} \rho \\ \rho u \\ \rho E \\ \rho Y \end{pmatrix} + \partial_x \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho E u + pu \\ \rho u Y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \omega \end{pmatrix}, \tag{1}$$

where ρ is the fluid density, u is the velocity, p is the pressure, E is the total energy per unit mass and $Y \in [0,1]$ is the reactant mass fraction. The source term ω characterises the effect of chemical reactions and is nonpositive.

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The total energy is related to the internal energy ρe (e is the internal energy per unit mass) as [6,11,13]

$$\rho E = \frac{1}{2}\rho u^2 + \rho e + q\rho Y, \tag{2}$$

where $q > 0$ is the heat release of reaction. For the ideal gas , the equation of state (EoS) gives the relation of p, ρ and e as

$$p = (\gamma - 1)\rho e \tag{3}$$

with γ the specific heat ratio.

Generally, for N chemical species there should be N species conservation equations for $Y_i, i = 1, 2, \ldots, N$, with Y_i being the mass fraction of species i. The above reactive Euler equation (1) are a simplified system under the assumption that there are only two species present, reacted and unreacted species, and that the unreacted species is converted to reacted species by a one-step irreversible chemical reaction [8,11,13]. Though relatively simple in form, (1) can give complicated stable and unstable wave patterns, which have been observed in experiments [2,8]. Due to their practical importance, the reactive Euler equations have been widely investigated both theoretically and numerically in the past decades, see *e.g.* [1,2,4,6,8,12,17–19,24,30].

Equation (1) are a typical system of hyperbolic balance laws and degenerate to the Euler equations when $Y = 0$. For these equations as hyperbolic systems, the entropy-entropy flux pair, a concept introduced by Lax [22], plays a central role in their mathematical theory [9]. It is used to formulate the entropy condition that singles out the physically relevant solution among all weak solutions of hyperbolic conservation laws.

For many hyperbolic systems in applications, *e.g.*, the Euler equations for gas dynamics, the moment models of extended thermodynamics and the balance laws in isentropic motion of thermoelastic nonconductors of heat, the entropy functions are strictly convex [9]. The strict convexity of entropy not only guarantees a unique classical solution to the Cauchy problem of hyperbolic conservation laws, but also induces useful information on the boundary behavior of solutions to initial-boundary-value problems [9]. Moreover, the strict convexity is required in the proof of global existence of solutions of hyperbolic balance laws [36]. On the other hand, many numerical schemes that preserve the sign of the entropy production also rely on the strict convexity of entropy functions [3,10,14,15,28,31,32,40].

Though the strict convex entropy is crucial for the theory and computation of hyperbolic systems, it seems that there are rare discussions on that of the system (1). Since (1) are Euler-like equations, it is natural to ask whether the classical entropy function associated with the thermodynamic entropy is also a strictly convex entropy function for (1). Unfortunately, the classical entropy function is no longer strictly convex in this case, as pointed out in our recent work [40]. Therein a strictly convex entropy function is constructed by adding to the classical one an extra term ρY^2. Based on the strict convexity of this entropy function, numerical schemes preserving the sign of the entropy production are developed.

In this paper, we further discuss strictly convex entropy functions of the reactive Euler equation (1). First, we show that a class of strictly convex entropy functions for (1) under the EoS (3) can be obtained by adding a general extra term to the classical one. Such strictly convex entropy functions contain that constructed in [40] as a special case. On the other hand, we demonstrate that there exists a family of EoS such that the classical entropy function associated with the thermodynamic entropy is still strictly convex for (1). Under these new EoS, the reactive Euler equations are proved to satisfy the Conservation-Dissipation Conditions for general hyperbolic relaxation systems formulated in [37]. As a result, the existence of zero relaxation limit for initial value problems are guaranteed [34,35]. Additionally, an elegant eigen-system of the Jacobian matrix, which is crucial for computations of hyperbolic systems, is derived for the reactive Euler equations under the proposed EoS. Numerical experiments demonstrate that the proposed EoS can also generate stable ZND detonations. These nice properties imply that the reactive Euler equation (1) with the new EoS may be a promising model for detonation problems.

The rest of the paper is organized as follows. In Section 2, we propose a class of strictly convex entropy functions for (1) under the EoS (3). A family of EoS are given in Section 3 and the classical entropy function is proved to be strictly convex. Under the new EoS the reactive Euler equations are shown to satisfy Conservation-Dissipation Conditions for hyperbolic relaxation systems in Section 4. Furthermore, an eigen-system is derived in Section 5. Numerical experiments are conducted in Section 6 to support the proposed EoS. Finally, some conclusions and remarks are given in Section 7. This paper also have three appendixes which give proofs of some theorems.

2. A class of strictly convex entropy functions

We first follow [40] to show that the classical entropy function associated with the thermodynamic entropy is not strictly convex for the reactive Euler equation (1). Denote by $s = \ln(p) - \gamma\ln(\rho)$ the thermodynamic entropy and define

$$\hat{\eta} = \frac{-\rho s}{\gamma - 1}, \quad \hat{\phi} = \frac{-\rho u s}{\gamma - 1}, \tag{4}$$

which are the classical entropy and entropy flux for the Euler equations, respectively. It is direct to verify for the reactive Euler equation (1) that

$$\partial_t \hat{\eta} + \partial_x \hat{\phi} = \frac{\rho}{p} q\omega \le 0, \tag{5}$$

which implies that $(\hat{\eta}, \hat{\phi})$ may be an entropy-entropy flux pair. However, for (1) $\hat{\eta}$ is no longer strictly convex. To see this, we denote $U = (\rho, \rho u, \rho E, \rho Y)^\dagger$ with $\dagger$ the transpose and compute

$$\hat{\eta}_U := \frac{\partial \hat{\eta}}{\partial U} = \left(\frac{\gamma - s}{\gamma - 1} - \frac{\rho u^2}{2p}, \quad \frac{\rho u}{p}, \quad -\frac{\rho}{p}, \quad q\frac{\rho}{p} \right)^\dagger.$$

Note that the third and fourth entry of $\hat{\eta}_U$ are proportional. Then the third and fourth row of the Hessian matrix $\hat{\eta}_{UU}$ are proportional and thus $\hat{\eta}_{UU}$ is singular. This indicates that $\hat{\eta}$ is not strictly convex for any U.

To address the above problem, we introduce a smooth function $\zeta(Y)$ that satisfies

$$\zeta'(Y) \ge 0, \quad \zeta''(Y) > 0, \quad \forall Y \in [0, 1]. \tag{6}$$

Then we define the following entropy-entropy flux pair

$$\tilde{\eta} = \hat{\eta} + \rho\zeta(Y), \quad \tilde{\phi} = \hat{\phi} + \rho u\zeta(Y) \tag{7}$$

by adding $\rho\zeta(Y)$ and $\rho u\zeta(Y)$ to $\hat{\eta}$ and $\hat{\phi}$, respectively. Denote $\zeta = \zeta(Y)$. It follows from the first and fourth equation of (1) that

$$\begin{aligned}
&\partial_t(\rho\zeta) + \partial_x(\rho u\zeta) \\
&= \rho\partial_t\zeta + \zeta\partial_t\rho + \rho u\partial_x\zeta + \zeta\partial_x(\rho u) \\
&= \rho\partial_t\zeta + \rho u\partial_x\zeta \\
&= \zeta'(\rho\partial_t Y + \rho u\partial_x Y) \\
&= \zeta'[\partial_t(\rho Y) + \partial_x(\rho u Y)] \\
&= \zeta'\omega \le 0,
\end{aligned}$$

where the inequality is due to the first condition in (6). With this and (5), we have

$$\partial_t\tilde{\eta} + \partial_x\tilde{\phi} = (q\frac{\rho}{p} + \zeta')\omega \le 0, \tag{8}$$

which means that the entropy production is nonpositive.

To show the strict convexity of $\tilde{\eta}$, we compute the Hessian matrix as

$$\tilde{\eta}_{UU} = \begin{pmatrix} \frac{\gamma}{\gamma-1}\frac{1}{\rho} + (\gamma-1)\frac{\rho u^4}{4p^2} + \zeta''\frac{Y^2}{\rho} & -(\gamma-1)\frac{\rho u^3}{2p^2} & -\frac{1}{p} + (\gamma-1)\frac{\rho u^2}{2p^2} & q\frac{1}{p} - (\gamma-1)q\frac{\rho u^2}{2p^2} - \zeta''\frac{Y}{\rho} \\ -(\gamma-1)\frac{\rho u^3}{2p^2} & \frac{1}{p} + (\gamma-1)\frac{\rho u^2}{p^2} & -(\gamma-1)\frac{\rho u}{p^2} & (\gamma-1)q\frac{\rho u}{p^2} \\ -\frac{1}{p} + (\gamma-1)\frac{\rho u^2}{2p^2} & -(\gamma-1)\frac{\rho u}{p^2} & (\gamma-1)\frac{\rho}{p^2} & -(\gamma-1)q\frac{\rho}{p^2} \\ q\frac{1}{p} - (\gamma-1)q\frac{\rho u^2}{2p^2} - \zeta''\frac{Y}{\rho} & (\gamma-1)q\frac{\rho u}{p^2} & -(\gamma-1)q\frac{\rho}{p^2} & (\gamma-1)q^2\frac{\rho}{p^2} + \frac{\zeta''}{\rho} \end{pmatrix} \tag{9}$$

and have the following result.

Theorem 2.1. *Assume $\rho > 0$ and $p > 0$. The Hessian matrix $\tilde{\eta}_{UU}$ in (9) is positive definite if ζ satisfies (6). Namely, $\tilde{\eta}$ is a strictly convex entropy function for any ζ satisfying (6).*

The proof of the above theorem is given in Appendix A. This and the relation (8) imply that $\tilde{\eta}$ is a strictly convex entropy function.

Remark 2.2. Theorem 2.1 indicates that there exists a class of strictly entropy functions for (1) as there are infinitely many ζ satisfying condition (6). Such ζ includes Y^2 proposed in [40] as a special case. Other choice can be $\exp(Y)$, $\tan(Y + \frac{1}{2})$, $Y \log(a + Y)$ with $a > 1$, and so on. Additionally, for any ζ_1 and ζ_2 satisfying (6), their convex combination also satisfies (6). That is, all ζ satisfying (6) constitute a convex set. Furthermore, all the entropy functions $\tilde{\eta}$ in (7) with ζ satisfying (6) constitute a convex set too.

Remark 2.3. It is direct to verify that $(\tilde{\eta}, \tilde{\phi})$ is also an entropy-entropy flux pair for multi-dimensional reactive Euler equations. The corresponding entropy potential can be computed as ρu, which is the same as that for the Euler equations.

3. A family of EoS

In this section, we provide another strategy to remedy the non-strict convexity of the entropy function $\hat{\eta}$. Since the EoS (3) is derived for the ideal gas, it seems reasonable to be modified when there are chemical reactions. Considering this point, we modify the EoS (3) as

$$p = (\gamma - 1)\left(\rho e + q\rho Y - \rho\zeta\right), \tag{10}$$

where $\zeta := \zeta(Y)$ is still a smooth function of Y satisfying (6). With this EoS, we have

$$\rho E = \frac{1}{2}\rho u^2 + \frac{p}{\gamma - 1} + \rho\zeta \tag{11}$$

and

$$\partial_t \hat{\eta} + \partial_x \hat{\phi} = \frac{\rho}{p}\zeta'\omega \tag{12}$$

(see Appendix B). The right hand side of the above equation is nonpositive under the first condition in (6), which implies the nonpositivity of the entropy production.

Next we show that $\hat{\eta}$ is a strict convex function under the EoS (10). To this end, we compute

$$\hat{\eta}_U := \frac{\partial \hat{\eta}}{\partial U} = \left(\frac{\gamma - s}{\gamma - 1} - \frac{\rho u^2}{2p} + \frac{\rho}{p}(\zeta - \zeta' Y), \quad \frac{\rho u}{p}, \quad -\frac{\rho}{p}, \quad \frac{\rho\zeta'}{p}\right)^\dagger$$

and

$$\hat{\eta}_{UU} = \begin{pmatrix} \frac{\gamma}{\gamma-1}\frac{1}{\rho} + \frac{1}{p}(2\zeta - 2\zeta' Y + \zeta'' Y^2) + (\gamma-1)\frac{\rho}{p^2}(-\zeta + \zeta' Y + \frac{u^2}{2})^2 & -(\gamma-1)\frac{\rho u}{p^2}(-\zeta + \zeta' Y + \frac{u^2}{2}) & -\frac{1}{p} + (\gamma-1)\frac{\rho}{p^2}(-\zeta + \zeta' Y + \frac{u^2}{2}) & \frac{1}{p}(\zeta' - \zeta'' Y) - (\gamma-1)\frac{\rho\zeta'}{p^2}(-\zeta + \zeta' Y + \frac{u^2}{2}) \\ -(\gamma-1)\frac{\rho u}{p^2}(-\zeta + \zeta' Y + \frac{u^2}{2}) & \frac{1}{p} + (\gamma-1)\frac{\rho u^2}{p^2} & -(\gamma-1)\frac{\rho u}{p^2} & (\gamma-1)\frac{\rho u \zeta'}{p^2} \\ -\frac{1}{p} + (\gamma-1)\frac{\rho}{p^2}(-\zeta + \zeta' Y + \frac{u^2}{2}) & -(\gamma-1)\frac{\rho u}{p^2} & (\gamma-1)\frac{\rho}{p^2} & -(\gamma-1)\frac{\rho\zeta'}{p^2} \\ \frac{1}{p}(\zeta' - \zeta'' Y) - (\gamma-1)\frac{\rho\zeta'}{p^2}(-\zeta + \zeta' Y + \frac{u^2}{2}) & (\gamma-1)\frac{\rho u \zeta'}{p^2} & -(\gamma-1)\frac{\rho\zeta'}{p^2} & \frac{\zeta''}{p} + (\gamma-1)\frac{\rho\zeta'^2}{p^2} \end{pmatrix}. \tag{13}$$

On this matrix, we have the following theorem, the proof of which is given in Appendix C.

Theorem 3.1. *Assume $\rho > 0$ and $p > 0$. The Hessian matrix $\hat{\eta}_{UU}$ in (13) is positive definite if and only if the second condition in (6) holds, i.e., $\zeta'' > 0, \forall Y \in [0,1]$.*

Remark 3.2. Theorem 3.1 and (12) indicate that $\hat{\eta}$ is a strictly convex entropy function with $\hat{\phi}$ the corresponding entropy flux for the reactive Euler equations under the EoS (10). In this case $\tilde{\eta} = \hat{\eta} + \rho\zeta$ is also a strictly convex entropy function as $(\rho\zeta)_{UU}$ is positive semidefinite for $\zeta'' > 0$.

Remark 3.3. Our modification of the EoS can also be understood as follows. If we assume that the heat release of reaction q is not a constant but a function of Y, *i.e.*, $q = q(Y)$, and choose $\zeta = qY$, then the EoS (10) degenerates to that for the ideal gas. In this case, the condition (6) becomes $q + q'Y \geq 0, 2q' + q''Y > 0$ and the strict convexity of the classical entropy function $\hat{\eta}$ is guaranteed without modifying the EoS but by imposing some appropriate conditions on the heat release q. Namely, when ζ/Y is well defined for the function ζ in our modification (*e.g.* $\zeta = q_0 Y^2$ and $\zeta = \frac{1}{2}q_0(Y^2 + Y^3)$ employed in our numerical experiments with q_0 being the constant heat release), the modification can be regarded as setting q as a function of Y, *i.e.*, $q = \zeta/Y$, while keeping the EoS unchanged. This understanding seems natural in physics [11,13].

We end this section with some discussions on the Gibbs relation under the new EoS. Denote $\hat{s} = s/(\gamma - 1)$ and define the temperature $T = p/\rho$. For the EoS (3) of the ideal gas, the classical Gibbs relation reads

$$T\mathrm{d}\hat{s} = \mathrm{d}e + p\mathrm{d}\frac{1}{\rho}\cdot \tag{14}$$

When the EoS is modified to be (10), it is direct to compute that the Gibbs relation becomes

$$T\mathrm{d}\hat{s} = \mathrm{d}e + p\mathrm{d}\frac{1}{\rho} + (q - \zeta')\mathrm{d}Y. \tag{15}$$

From the above two equations we see that for the new EoS there is an additional term related to the change of the mass fraction in the infinitesimal entropy change. Actually, the changes of mass fractions also appear in the entropy change for the multi-component Euler equations with/without chemical reactions [15]. From this point of view, our modification also seems reasonable.

4. Stability

In this section, we analyse the stability of the reactive Euler equations under the EoS (12) in the sense of hyperbolic relaxation systems. Specifically, we adopt the widely used source term in an Arrhenius form

$$\omega = -\rho Y \tilde{K} e^{-\tilde{T}/T}, \tag{16}$$

where $\tilde{K} > 0$ is a constant rate coefficient, $T = p/\rho$ is the temperature and $\tilde{T} > 0$ is the activation constant temperature. For detonation problems with fast chemical reactions, the rate coefficient $\tilde{K}$ can be quite large and the reactant mass fraction Y decreases to zero quickly. In this case, we denote $\epsilon := 1/\tilde{K}$ as a small parameter and rewrite the reactive Euler equations as

$$\partial_t U + \partial_x F(U) = \frac{1}{\epsilon} Q(U), \tag{17}$$

where $U = (\rho, \rho u, \rho E, \rho Y)^\dagger$ takes values in an open subset G of $\mathbb{R}^4$ and

$$\begin{aligned} F(U) &= \left(\rho u, \rho u^2 + p, \rho E u + p u, \rho u Y\right)^\dagger, \\ Q(U) &= (0, 0, 0, -\rho Y e^{-\tilde{T}/T})^\dagger. \end{aligned}$$

Equation (17) are a standard hyperbolic relaxation system since the convection part is hyperbolic and the source term is stiff [5, 34, 35]. For such systems, the following Conservation-Dissipation Conditions are formulated in [37] to guarantee the existence of zero relaxation limit as $\epsilon \to 0$.

(I) There is a strictly convex smooth function $\eta(U)$ such that $\eta_{UU} \partial_U F(U)$ is symmetric for all $U \in G$.
(II) There is a symmetric and non-negative definite matrix $L = L(U)$ such that

$$Q(U) = -L(U) \eta_U(U).$$

(III) The null space of $L(U)$ is independent of $U \in G$.

Here Condition (I) states the existence of a strictly convex entropy function, Condition (II) is an extension of the celebrated Onsager reciprocal relations in nonequilibrium thermodynamics, and Condition (III) expresses the fact that the physical laws of conservation hold true, no matter what state the underlying thermodynamical system is in; see [37] for more details. It is shown in [37] that if a system of PDEs satisfies Conditions (I)–(III), it satisfies all the entropy dissipation conditions in [5, 21, 25, 29, 36]. In particular, Conditions (I)–(III) are a strengthened version of the structural stability conditions in [34] for hyperbolic systems of PDEs with relaxation, which guarantee that the solution of the relaxation system converges to that of the equilibrium system in the zero relaxation limit. Moreover, it is observed in [37] that these conditions are satisfied by various models from applications, such as the Euler equations of gas dynamics in vibrational non-equilibrium [38], discrete-ordinate models for radiation hydrodynamics [26], and moment closure systems in kinetic theories [23].

Actually, it has been demonstrated in [33] that conditions (I)–(III) are not satisfied for the reactive Euler equations under the EoS (3) for the ideal gas. However, if the EoS is replaced by (10), these conditions indeed hold true.

Theorem 4.1. *If ζ satisfies (6) and $\zeta'(0) = 0$, i.e.,*

$$\zeta'(0) = 0, \quad \zeta''(Y) > 0, \quad \forall Y \in [0, 1], \tag{18}$$

then the reactive Euler equation (17) with the EoS (10) satisfy the Conservation-Dissipation Conditions (I)–(III).

Proof. It has been shown in the previous section that $\eta = \hat{\eta}$ is a strictly convex entropy function for the EoS (10) with ζ satisfying (6). Thus we take $\eta = \hat{\eta}$ and Condition (I) holds.

To verify the second condition, we introduce a diagonal matrix

$$L(U) = \mathrm{diag}(0, 0, 0, \frac{Ype^{-\tilde{T}/T}}{\zeta'}),$$

which satisfies $Q(U) = -L(U)\eta_U(U)$. Then we only need to show that $L(U)$ is non-negative definite. Note that $\zeta'(0) = 0$ and $\zeta'' > 0$. We have $\zeta'(Y) > 0, \forall Y \in (0, 1]$ and thereby $L(U)$ is non-negative definite for $Y \in (0, 1]$. When $Y = 0$, the numerator and denominator of $\frac{Ype^{-\tilde{T}/T}}{\zeta'}$ are both zero as $\zeta'(0) = 0$. Then it follows that

$$\left.\frac{Ype^{-\tilde{T}/T}}{\zeta'}\right|_{Y=0} = \left.\frac{pe^{-\tilde{T}/T}}{\zeta''}\right|_{Y=0},$$

which is positive due to $\zeta'' > 0$. Thus Condition (II) holds.

The third condition is obviously true with the above $L(U)$ and the proof is complete. ☐

Remark 4.2. With conditions (1)-(III) satisfied for the system (17) under the EoS (10), we can directly apply Theorems 6.1 and 6.2 in [34] to obtain: For smooth initial data, there is a finite and ϵ-independent time interval $[0, \mathcal{T}]$ such that the initial value problem of (10) has a unique smooth solution $U^\epsilon = U^\epsilon(t, x)$ defined for $t \in [0, \mathcal{T}]$ and satisfying

$$U^\epsilon = \begin{pmatrix} U^{(eq)} \\ 0 \end{pmatrix} + O(\epsilon)$$

in a certain Sobolev space, as ϵ goes to zero. Here $U^{(eq)}$ solves the corresponding initial value problem of the equilibrium system of (17), *i.e.*, the Euler equations.

Remark 4.3. Note that the condition (18) is stronger than (6). Nevertheless, there are still infinitely many ζ satisfying (18). An interesting class of ζ is $\zeta = q[\alpha Y^2 + (1-\alpha)Y^n]$, where q is the constant heat release of reaction in (2), $\alpha \in (0, 1]$ and $n \geq 2$. Such ζ obviously satisfies (18) and the corresponding EoS becomes

$$p = (\gamma - 1)\left[\rho e + q\rho Y(1 - \alpha Y - (1-\alpha)Y^{n-1})\right]. \tag{19}$$

The above EoS degenerates to that for the ideal gas when $Y = 0$ and $Y = 1$, which seems reasonable in physics as there are no chemical reactions at these states.

5. Eigen-system of the Jacobian matrix

In this section, we further show that under the EoS (10) the Jacobian matrix of the flux

$$F(U) = \left(\rho u, \rho u^2 + p, \rho E u + pu, \rho u Y\right)^\dagger$$

of the reactive Euler equation (1) has an elegant eigen-system, though the equations and the EoS are quite complicated. The eigen-system, including eigenvalues and eigenvectors, is of practical importance for computations of hyperbolic systems. They are used to formulate the numerical fluxes in various schemes [3, 7, 10, 14–16, 20, 27, 28, 31, 32, 40].

With usual notations $H = E + \frac{p}{\rho}$ and $c = \sqrt{\gamma\frac{p}{\rho}}$, we directly compute the Jacobian matrix under the EoS (10) as

$$\mathsf{A} = \partial_U F(U) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{\gamma-3}{2}u^2 - (\gamma-1)(\zeta - \zeta' Y) & (3-\gamma)u & \gamma - 1 & -(\gamma-1)\zeta' \\ \frac{\gamma-1}{2}u^3 - Hu - (\gamma-1)(\zeta - \zeta' Y)u & H - (\gamma-1)u^2 & \gamma u & -(\gamma-1)\zeta' u \\ -uY & Y & 0 & u \end{pmatrix}.$$

In the following we extend the eigen-decomposition of the Jacobian matrix under the EoS (2) in [6] to the above matrix.

Theorem 5.1. *If ζ satisfies* (18) *and $\zeta(0)=0$, i.e.,*

$$\zeta(0)=0, \quad \zeta'(0)=0, \quad \zeta''(Y)>0, \quad \forall Y\in[0,1], \tag{20}$$

then the Jacobian matrix A *has an eigen-decomposition*

$$\mathsf{A}=\mathsf{R}diag(u,u,u+c,u-c)\mathsf{L}$$

with

$$\mathsf{R}=\begin{pmatrix} 1 & 0 & 1 & 1 \\ u & 0 & u+c & u-c \\ \frac{1}{2}u^2 & \zeta' & H+uc & H-uc \\ -\frac{\zeta}{\zeta'}+Y & 1 & Y & Y \end{pmatrix},$$

$$\mathsf{L}=\mathsf{R}^{-1}=\begin{pmatrix} 1-\frac{\gamma-1}{2}\frac{u^2}{c^2}+(\zeta-\zeta'Y)\frac{\gamma-1}{c^2} & (\gamma-1)\frac{u}{c^2} & -\frac{\gamma-1}{c^2} & \zeta'\frac{\gamma-1}{c^2} \\ -Y+\frac{\zeta}{\zeta'}[1-(\gamma-1)\frac{u^2}{2c^2}+(\zeta-\zeta'Y)\frac{\gamma-1}{c^2}] & (\gamma-1)\frac{\zeta}{\zeta'}\frac{u}{c^2} & -\frac{\gamma-1}{c^2}\frac{\zeta}{\zeta'} & 1+\frac{\gamma-1}{c^2}\zeta \\ \frac{\gamma-1}{4}\frac{u^2}{c^2}-\frac{1}{2}\frac{u}{c}-(\zeta-\zeta'Y)\frac{\gamma-1}{2c^2} & -\frac{\gamma-1}{2}\frac{u}{c^2}+\frac{1}{2c} & \frac{\gamma-1}{2c^2} & -\zeta'\frac{\gamma-1}{2c^2} \\ \frac{\gamma-1}{4}\frac{u^2}{c^2}+\frac{1}{2}\frac{u}{c}-(\zeta-\zeta'Y)\frac{\gamma-1}{2c^2} & -\frac{\gamma-1}{2}\frac{u}{c^2}-\frac{1}{2c} & \frac{\gamma-1}{2c^2} & -\zeta'\frac{\gamma-1}{2c^2} \end{pmatrix}.$$

Proof. We first point out that $\frac{\zeta}{\zeta'}$ is well defined when $Y=0$ under the condition (20), noting that

$$\left.\frac{\zeta}{\zeta'}\right|_{Y=0}=\left.\frac{\zeta'}{\zeta''}\right|_{Y=0}=0.$$

Thus the terms containing $\frac{\zeta}{\zeta'}$ in L and R are well defined.

To verify the above decomposition, we denote $\mathsf{A}=(A_1,A_2,A_3,A_4)^\dagger$ with A_i the i-th row of A and $\mathsf{R}=(R_1,R_2,R_3,R_4)$ with R_i the i-th column of R, respectively. Then we turn to verify

$$\mathsf{A}R_1=uR_1, \quad \mathsf{A}R_2=uR_2, \quad \mathsf{A}R_3=(u+c)R_3, \quad \mathsf{A}R_4=(u-c)R_4.$$

The first two equalities in the above equation can be easily verified. For the third one, it is obvious that $A_1R_3=u+c$ and $A_4R_3=(u+c)Y$. On the other hand, it is direct to compute

$$\begin{aligned} A_2R_3 &= \frac{\gamma-3}{2}u^2-(\gamma-1)(\zeta-\zeta'Y)+(3-\gamma)u(u+c)+(\gamma-1)(H+uc)-(\gamma-1)\zeta'Y \\ &= \frac{\gamma-3}{2}u^2-(\gamma-1)\zeta+(3-\gamma)(u^2+uc)+(\gamma-1)(\frac{1}{2}u^2+\frac{\gamma}{\gamma-1}\frac{p}{\rho}+\zeta)+(\gamma-1)uc \\ &= u^2+\gamma\frac{p}{\rho}+2uc \\ &= (u+c)^2 \end{aligned}$$

and

$$\begin{aligned} A_3R_3 &= \frac{\gamma-1}{2}u^3-Hu-(\gamma-1)(\zeta-\zeta'Y)u+(H-(\gamma-1)u^2)(u+c) \\ &\quad +\gamma u(H+uc)-(\gamma-1)\zeta'uY \\ &= \frac{\gamma-1}{2}u^3-Hu-(\gamma-1)(H-\frac{\gamma}{\gamma-1}\frac{p}{\rho}-\frac{1}{2}u^2)u \\ &\quad +(H-(\gamma-1)u^2)(u+c)+\gamma u(H+uc) \\ &= H(u+c)+u^2c+c^2u \\ &= (H+uc)(u+c), \end{aligned}$$

where the fact $\zeta = H - \frac{\gamma}{\gamma-1}\frac{p}{\rho} - \frac{1}{2}u^2$ has been used for the second equality in the second equation. Thus we obtain $\mathsf{A}R_3 = (u+c)R_3$. Similarly, one can find that $\mathsf{A}R_4 = (u-c)R_4$. Furthermore, it is straightforward to verify that LR is equal to the identity matrix. Then the proof is complete. $\square$

Remark 5.2. The constraint $\zeta(0) = 0$ in (20) seems natural as it guarantees that when $Y = 0$ (no chemical reactions) the EoS (10) degenerates to that for the ideal gas. In addition, the choice of ζ in Remark 4.3 satisfies this constraint. Actually, when $\zeta(0) \neq 0$, the eigen-decomposition can be easily obtained by replacing the last entry in the first column of R with $-\zeta + \zeta' Y$ and slightly modifying matrix L.

6. Numerical experiments

In this section, we conduct several numerical experiments to demonstrate the proposed EoS (10). According to Remark 4.3, we test two ζ, $\zeta = qY^2$ and $\zeta = \frac{1}{2}q(Y^2 + Y^3)$, which both satisfy (20) (and thereby (6) and (18)). The corresponding EoS are

$$p = (\gamma - 1)\left[\rho e + q\rho Y(1 - Y)\right] \tag{21}$$

and

$$p = (\gamma - 1)\left[\rho e + q\rho Y(1 - \frac{1}{2}Y - \frac{1}{2}Y^2)\right], \tag{22}$$

respectively, and they degenerate to the EoS for the ideal gas when $Y = 0$ and $Y = 1$. The source term is taken as the Arrhenius form in (16).

In our computations, we employ the fifth-order WENO scheme and the third-order Runge-Kuta scheme for the space and time discretizations, respectively. For the WENO scheme, the eigen-decomposition in Theorem 5.1 is used to compute the characteristic field [20]. Additionally, the mesh size and the CFL number are fixed as $\Delta x = 1/20$ and CFL $= 0.5$, respectively. Under these settings, we simulate the one-dimensional stable and unstable detonations widely studied in the literature (see *e.g.* [1, 2, 8, 17, 24, 39]).

6.1. Stable detonation

In the stable detonation problem, the spatial domain is $[0, 100]$ and the initial condition is given by

$$(\rho, u, p, Y) = \begin{cases} (2.5706, 6.86089, 80.02055, 0), & x < 2, \\ (1, 0, 1, 1), & x > 2. \end{cases} \tag{23}$$

The parameters are chosen as $\gamma = 1.4, q = 50, \tilde{T} = 50$ and $K = 145.68913$ [24, 39]. The computational solutions of the EoS for the ideal gas and the proposed EoS with $\zeta = qY^2$ and $\zeta = \frac{1}{2}q(Y^2 + Y^3)$ at $t = 7.5$ are plotted in Figure 1. It can be seen that for the three different EoS the solutions are similar and the ZND structure can be clearly observed, *i.e.*, there is a thin reaction zone following a nonreactive shock. In particular, the positions of the shock for the three EoS are almost the same while the pressure at the shock front are different. These demonstrate that, like the EoS for the ideal gas, the proposed EoS can also generate ZND detonations.

6.2. Unstable detonations

In the unstable detonation problem, the spatial domain is $[0, 1000]$ and the initial condition is

$$(\rho, u, p, Y) = \begin{cases} (3.64280, 6.24889, 54.82406, 0), & x < 20, \\ (1, 0, 1, 1), & x > 20. \end{cases} \tag{24}$$

The parameters are chosen as $\gamma = 1.2, q = 50, \tilde{T} = 50$ and $K = 230.75$ [24, 39].

For the three different EoS, the solutions are still similar, as shown in Figure 2. Figure 3 compares the velocity and the pressure behind the reaction zone. We can see that both the velocity and the pressure fluctuate steadily with the shock propagation and the frequency of the fluctuation is different for the three EoS.

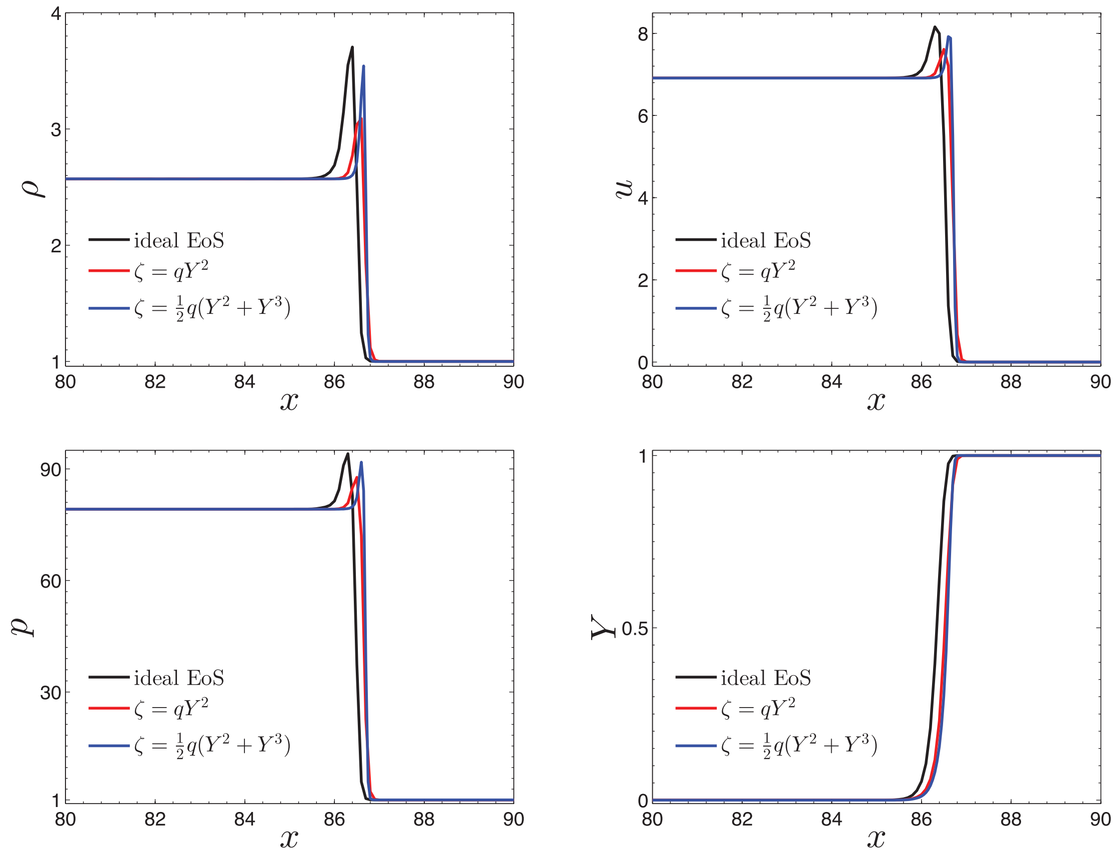

FIGURE 1. Stable detonation. Solutions at $t = 7.5$.

An important physical quantity for the unstable detonation is the maximum pressure in the reaction zone, the time history of which is plotted in Figure 4. Like that of the EoS for the ideal gas, the maximum pressure for the EoS with $\zeta = qY^2$ and $\zeta = \frac{1}{2}q(Y^2 + Y^3)$ becomes periodic after a certain time. Interestingly, the periods corresponding to the EoS with $\zeta = qY^2$ and $\zeta = \frac{1}{2}q(Y^2 + Y^3)$ are smaller than that of the EoS for the ideal gas ($\zeta = qY$). It seems that the period of the maximum pressure is closely related to ζ. These show that ζ has a significant influence on the structure of unstable detonations. It may provide a freedom to fit experiment data and thus allow the reactive Euler equations to produce more accurate results.

7. Conclusions and remarks

In this paper, we propose two strategies to obtain strictly convex entropy functions of the reactive Euler equation (1). First, we construct a class of strictly convex entropy functions for (1) under the EoS (3) by adding a general extra term to the classical one. Such strictly convex entropy functions contain that in [40] as a special case. On the other hand, we show that there exists a family of EoS such that the classical entropy function associated with the thermodynamic entropy is still strictly convex for (1). Under these new EoS, the reactive Euler equations are proved to satisfy the Conservation-Dissipation Conditions for general hyperbolic relaxation systems formulated in [37]. As a result, the existence of zero relaxation limit for initial value problems are guaranteed [34, 35]. Additionally, an elegant eigen-system of the Jacobian matrix is derived for the reactive Euler equations under the proposed EoS. Numerical experiments demonstrate that the proposed EoS can also

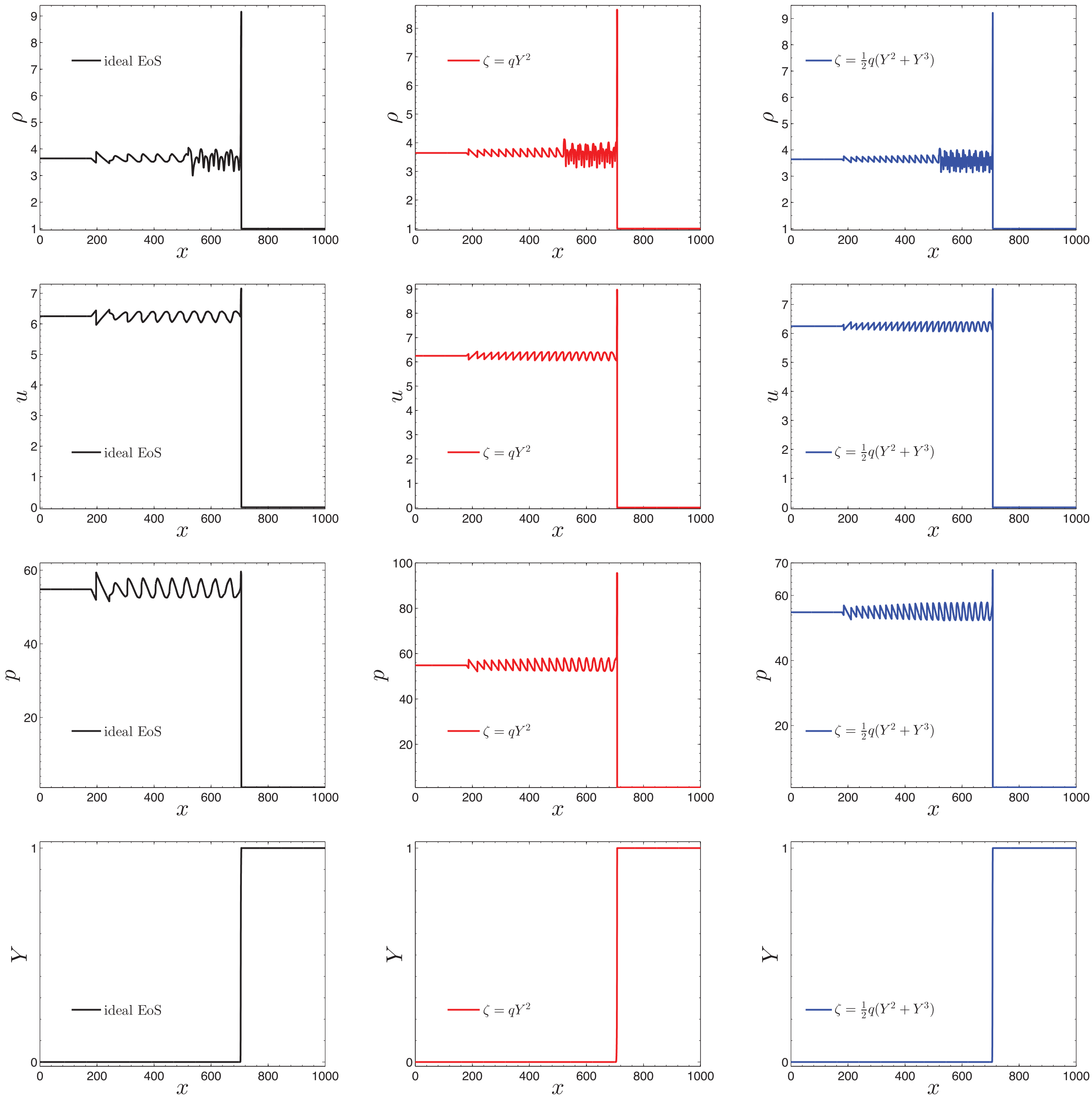

FIGURE 2. Unstable detonation. Solutions at $t = 80$.

generate stable ZND detonations. These nice properties imply that the reactive Euler Equation (1) with the new EoS may be a promising model for detonation problems.

We would like to point out that based on the present strictly convex entropy functions, entropy stable schemes can be directly designed with the framework in [14, 28]. Moreover, since the new EoS contains a function to be determined, it could be used to fit the solutions of the original reactive Euler equations with detailed chemical reactions. In this way, more accurate models than the conventional one with the EoS for the ideal gas are expected

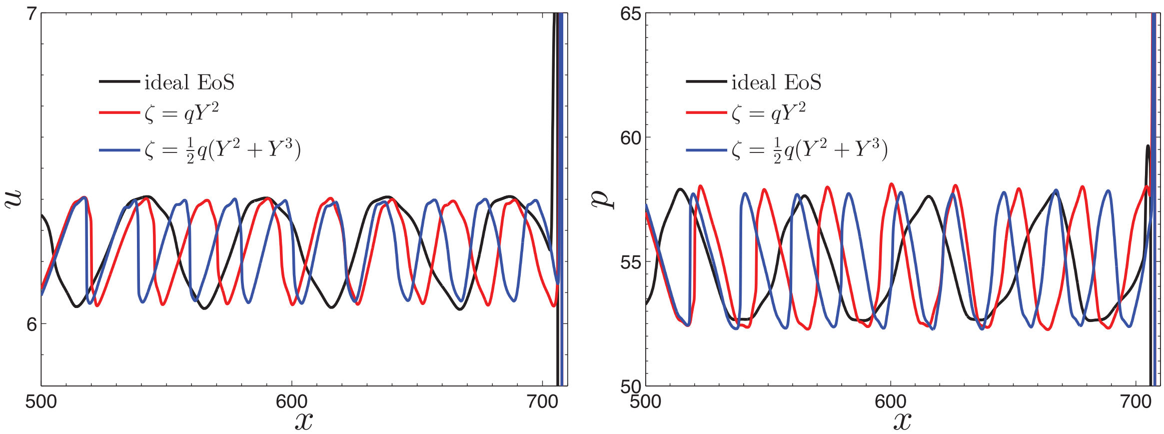

FIGURE 3. Unstable detonation. The velocity and the pressure behind the reaction zone at $t = 80$.

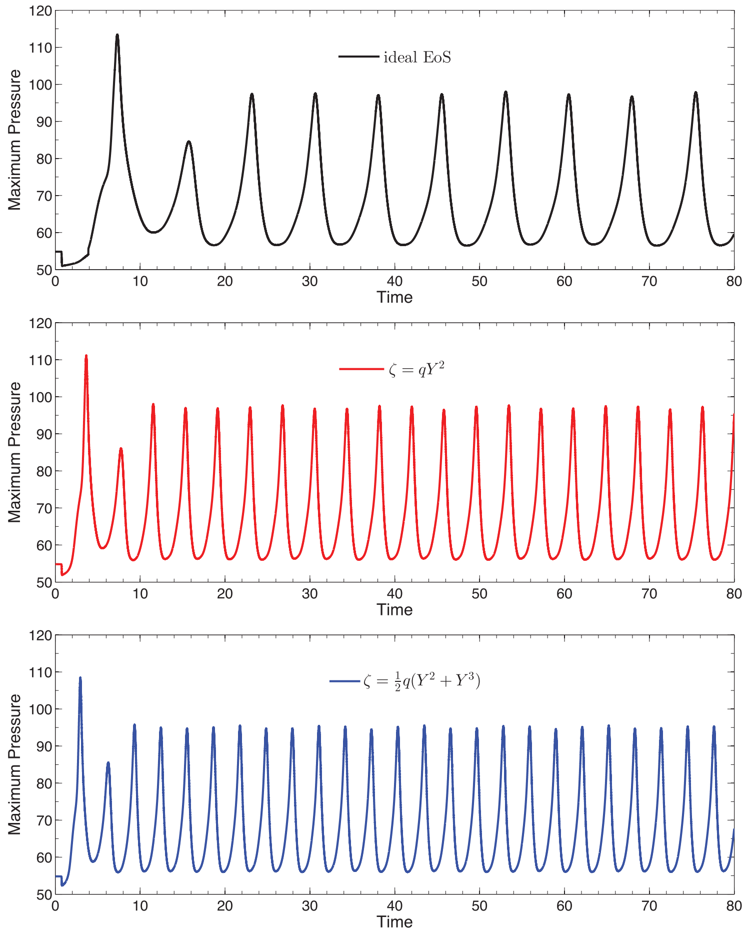

FIGURE 4. Unstable detonation. Time history of the the maximum pressure in the reaction zone.

to be obtained. Additionally, with the Conservation-Dissipation Conditions (I)–(III) satisfied, relaxation shock profiles can be analysed as in [41]. These interesting topics will be investigated in the next step.

Appendix A. Proof of Theorem 2.1

We prove Theorem 2.1 by showing that all the leading principal minors of $\tilde{\eta}_{UU}$ are positive. Note that the leading principle 3×3 submatrix of $\tilde{\eta}_{UU}$ is actually $A_3 := A_{\text{Euler}} + \operatorname{diag}(\zeta''\frac{Y^2}{\rho}, 0, 0)$, where A_{Euler} is the Hessian matrix of the classical entropy function $\hat{\eta}$ for the Euler equations. Since A_{Euler} is positive definite and $\zeta'' > 0$, A_3 is positive definite and the first three leading principal minors of $\tilde{\eta}_{UU}$ are positive. Next we only need to show that the fourth leading principal minor, *i.e.*, the determinant of $\tilde{\eta}_{UU}$, is positive. To do this, we compute

$$
\begin{aligned}
&\det(\tilde{\eta}_{UU})\\
&= \det\begin{pmatrix}
\frac{\gamma}{\gamma-1}\frac{1}{\rho} + (\gamma-1)\frac{\rho u^4}{4p^2} + \zeta''\frac{Y^2}{\rho} & -(\gamma-1)\frac{\rho u^3}{2p^2} & -\frac{1}{p} + (\gamma-1)\frac{\rho u^2}{2p^2} & q\frac{1}{p} - (\gamma-1)q\frac{\rho u^2}{2p^2} - \zeta''\frac{Y}{\rho}\\
-(\gamma-1)\frac{\rho u^3}{2p^2} & \frac{1}{p} + (\gamma-1)\frac{\rho u^2}{p^2} & -(\gamma-1)\frac{\rho u}{p^2} & (\gamma-1)q\frac{\rho u}{p^2}\\
-\frac{1}{p} + (\gamma-1)\frac{\rho u^2}{2p^2} & -(\gamma-1)\frac{\rho u}{p^2} & (\gamma-1)\frac{\rho}{p^2} & -(\gamma-1)q\frac{\rho}{p^2}\\
q\frac{1}{p} - (\gamma-1)q\frac{\rho u^2}{2p^2} & (\gamma-1)q\frac{\rho u}{p^2} & -(\gamma-1)q\frac{\rho}{p^2} & (\gamma-1)q^2\frac{\rho}{p^2}
\end{pmatrix}\\
&+ \det\begin{pmatrix}
\frac{\gamma}{\gamma-1}\frac{1}{\rho} + (\gamma-1)\frac{\rho u^4}{4p^2} + \zeta''\frac{Y^2}{\rho} & -(\gamma-1)\frac{\rho u^3}{2p^2} & -\frac{1}{p} + (\gamma-1)\frac{\rho u^2}{2p^2} & q\frac{1}{p} - (\gamma-1)q\frac{\rho u^2}{2p^2} - \zeta''\frac{Y}{\rho}\\
-(\gamma-1)\frac{\rho u^3}{2p^2} & \frac{1}{p} + (\gamma-1)\frac{\rho u^2}{p^2} & -(\gamma-1)\frac{\rho u}{p^2} & (\gamma-1)q\frac{\rho u}{p^2}\\
-\frac{1}{p} + (\gamma-1)\frac{\rho u^2}{2p^2} & -(\gamma-1)\frac{\rho u}{p^2} & (\gamma-1)\frac{\rho}{p^2} & -(\gamma-1)q\frac{\rho}{p^2}\\
-\zeta''\frac{Y}{\rho} & 0 & 0 & \frac{\zeta''}{\rho}
\end{pmatrix}\\
&= \det\begin{pmatrix}
\frac{\gamma}{\gamma-1}\frac{1}{\rho} + (\gamma-1)\frac{\rho u^4}{4p^2} + \zeta''\frac{Y^2}{\rho} & -(\gamma-1)\frac{\rho u^3}{2p^2} & -\frac{1}{p} + (\gamma-1)\frac{\rho u^2}{2p^2} & q\frac{1}{p} - (\gamma-1)q\frac{\rho u^2}{2p^2} - \zeta''\frac{Y}{\rho}\\
-(\gamma-1)\frac{\rho u^3}{2p^2} & \frac{1}{p} + (\gamma-1)\frac{\rho u^2}{p^2} & -(\gamma-1)\frac{\rho u}{p^2} & (\gamma-1)q\frac{\rho u}{p^2}\\
-\frac{1}{p} + (\gamma-1)\frac{\rho u^2}{2p^2} & -(\gamma-1)\frac{\rho u}{p^2} & (\gamma-1)\frac{\rho}{p^2} & -(\gamma-1)q\frac{\rho}{p^2}\\
-\zeta''\frac{Y}{\rho} & 0 & 0 & \frac{\zeta''}{\rho}
\end{pmatrix}\\
&= \zeta''\frac{Y}{\rho}\det\begin{pmatrix}
-(\gamma-1)\frac{\rho u^3}{2p^2} & -\frac{1}{p} + (\gamma-1)\frac{\rho u^2}{2p^2} & q\frac{1}{p} - (\gamma-1)q\frac{\rho u^2}{2p^2} - \zeta''\frac{Y}{\rho}\\
\frac{1}{p} + (\gamma-1)\frac{\rho u^2}{p^2} & -(\gamma-1)\frac{\rho u}{p^2} & (\gamma-1)q\frac{\rho u}{p^2}\\
-(\gamma-1)\frac{\rho u}{p^2} & (\gamma-1)\frac{\rho}{p^2} & -(\gamma-1)q\frac{\rho}{p^2}
\end{pmatrix} + \frac{\zeta''}{\rho}\det(A_3).
\end{aligned}
$$

Furthermore, we have

$$
\begin{aligned}
&\zeta''\frac{Y}{\rho}\det\begin{pmatrix}
-(\gamma-1)\frac{\rho u^3}{2p^2} & -\frac{1}{p} + (\gamma-1)\frac{\rho u^2}{2p^2} & q\frac{1}{p} - (\gamma-1)q\frac{\rho u^2}{2p^2} - \zeta''\frac{Y}{\rho}\\
\frac{1}{p} + (\gamma-1)\frac{\rho u^2}{p^2} & -(\gamma-1)\frac{\rho u}{p^2} & (\gamma-1)q\frac{\rho u}{p^2}\\
-(\gamma-1)\frac{\rho u}{p^2} & (\gamma-1)\frac{\rho}{p^2} & -(\gamma-1)q\frac{\rho}{p^2}
\end{pmatrix}\\
&= \zeta''\frac{Y}{\rho}\det\begin{pmatrix}
-(\gamma-1)\frac{\rho u^3}{2p^2} & -\frac{1}{p} + (\gamma-1)\frac{\rho u^2}{2p^2} & q\frac{1}{p} - (\gamma-1)q\frac{\rho u^2}{2p^2}\\
\frac{1}{p} + (\gamma-1)\frac{\rho u^2}{p^2} & -(\gamma-1)\frac{\rho u}{p^2} & (\gamma-1)q\frac{\rho u}{p^2}\\
-(\gamma-1)\frac{\rho u}{p^2} & (\gamma-1)\frac{\rho}{p^2} & -(\gamma-1)q\frac{\rho}{p^2}
\end{pmatrix}
\end{aligned}
$$

$$+\zeta''\frac{Y}{\rho}\det\begin{pmatrix} -(\gamma-1)\frac{\rho u^3}{2p^2} & -\frac{1}{p}+(\gamma-1)\frac{\rho u^2}{2p^2} & -\zeta''\frac{Y}{\rho} \\ \frac{1}{p}+(\gamma-1)\frac{\rho u^2}{p^2} & -(\gamma-1)\frac{\rho u}{p^2} & 0 \\ -(\gamma-1)\frac{\rho u}{p^2} & (\gamma-1)\frac{\rho}{p^2} & 0 \end{pmatrix}$$
$$= -\left(\zeta''\frac{Y}{\rho}\right)^2 \det\begin{pmatrix} \frac{1}{p}+(\gamma-1)\frac{\rho u^2}{p^2} & -(\gamma-1)\frac{\rho u}{p^2} \\ -(\gamma-1)\frac{\rho u}{p^2} & (\gamma-1)\frac{\rho}{p^2} \end{pmatrix}$$

and

$$\begin{aligned}
&\frac{\zeta''}{\rho}\det(A_3)\\
&=\frac{\zeta''}{\rho}\det(A_{\text{Euler}})+\frac{\zeta''}{\rho}\det\begin{pmatrix} \zeta''\frac{Y^2}{\rho} & -(\gamma-1)\frac{\rho u^3}{2p^2} & -\frac{1}{p}+(\gamma-1)\frac{\rho u^2}{2p^2} \\ 0 & \frac{1}{p}+(\gamma-1)\frac{\rho u^2}{p^2} & -(\gamma-1)\frac{\rho u}{p^2} \\ 0 & -(\gamma-1)\frac{\rho u}{p^2} & (\gamma-1)\frac{\rho}{p^2} \end{pmatrix}\\
&=\frac{\zeta''}{\rho}\det(A_{\text{Euler}})+\left(\zeta''\frac{Y}{\rho}\right)^2\det\begin{pmatrix} \frac{1}{p}+(\gamma-1)\frac{\rho u^2}{p^2} & -(\gamma-1)\frac{\rho u}{p^2} \\ -(\gamma-1)\frac{\rho u}{p^2} & (\gamma-1)\frac{\rho}{p^2} \end{pmatrix}.
\end{aligned}$$

Combining the above equations, we arrive at $\det(\tilde{\eta}_{UU}) = \frac{\zeta''}{\rho}\det(A_{\text{Euler}})$, which is positive due to the second condition in (6). Therefore, all the leading principal minors of $\tilde{\eta}_{UU}$ are positive and $\tilde{\eta}_{UU}$ is positive definite. Thus $\tilde{\eta}$ is a strictly convex entropy function for any ζ satisfying (6).

Appendix B. Derivation of (12)

In this appendix, we give the derivation of (12). Using the equations (1) and (11), we directly compute

$$\begin{aligned}
&\partial_t\hat{\eta}+\partial_x\hat{\phi}\\
&=-\frac{1}{\gamma-1}[\partial_t(\rho s)+\partial_x(\rho u s)]\\
&=-\frac{1}{\gamma-1}[s\partial_t\rho+\rho\partial_t s+s\partial_x(\rho u)+\rho u\partial_x s]\\
&=-\frac{1}{\gamma-1}[\rho\partial_t s+\rho u\partial_x s]\\
&=-\frac{1}{\gamma-1}[\frac{\rho}{p}\partial_t p-\gamma\partial_t\rho+\frac{\rho u}{p}\partial_x p-\gamma u\partial_x\rho]\\
&=-\frac{1}{\gamma-1}[\frac{\rho}{p}\partial_t p+\frac{\rho u}{p}\partial_x p]-\frac{\gamma}{\gamma-1}\rho\partial_x u\\
&=-\frac{\rho}{p}(\partial_t+u\partial_x)(\rho E-\frac{1}{2}\rho u^2-\rho\zeta)-\frac{\gamma}{\gamma-1}\rho\partial_x u\\
&=\frac{\rho}{p}[\rho E\partial_x u+\partial_x(pu)]+\frac{1}{2}\frac{\rho}{p}(\partial_t+u\partial_x)(\rho u^2)+\frac{\rho}{p}(\partial_t+u\partial_x)(\rho\zeta)-\frac{\gamma}{\gamma-1}\rho\partial_x u\\
&=\frac{\rho}{p}[\rho E\partial_x u+\partial_x(pu)]+\frac{1}{2}\frac{\rho}{p}\partial_t\frac{(\rho u)^2}{\rho}+\frac{1}{2}\frac{\rho u}{p}\partial_x(\rho u^2)+\frac{\rho}{p}(\partial_t+u\partial_x)(\rho\zeta)-\frac{\gamma}{\gamma-1}\rho\partial_x u\\
&=\frac{\rho}{p}[\rho E\partial_x u+\partial_x(pu)]+\frac{1}{2}\frac{\rho}{p}(2u\partial_t(\rho u)-u^2\partial_t\rho)+\frac{1}{2}\frac{\rho u}{p}\partial_x(\rho u^2)\\
&\quad+\frac{\rho}{p}(\partial_t+u\partial_x)(\rho\zeta)-\frac{\gamma}{\gamma-1}\rho\partial_x u
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\rho}{p}[\rho E\partial_x u + \partial_x(pu)] + [\frac{\rho u^2}{2p}\partial_x(\rho u) + \frac{1}{2}\frac{\rho u}{p}\partial_x(\rho u^2) + \frac{\rho u}{p}\partial_t(\rho u)] \\
&\quad + \frac{\rho}{p}(\partial_t + u\partial_x)(\rho\zeta) - \frac{\gamma}{\gamma-1}\rho\partial_x u \\
&= \frac{\rho}{p}[\rho E\partial_x u + \partial_x(pu)] - \frac{\rho^2 u^2}{2p}\partial_x u - \frac{\rho u}{p}\partial_x p + \frac{\rho}{p}(\partial_t + u\partial_x)(\rho\zeta) - \frac{\gamma}{\gamma-1}\rho\partial_x u.
\end{aligned}
$$

Note that

$$
\begin{aligned}
&(\partial_t + u\partial_x)(\rho\zeta) \\
&= \partial_t(\rho\zeta) + \partial_x(\rho u\zeta) - \rho\zeta\partial_x u \\
&= \rho\partial_t\zeta + \zeta\partial_t\rho + \rho u\partial_x\zeta + \zeta\partial_x(\rho u) - \rho\zeta\partial_x u \\
&= \rho\partial_t\zeta + \rho u\partial_x\zeta - \rho\zeta\partial_x u \\
&= \zeta'(\rho\partial_t Y + \rho u\partial_x Y) - \rho\zeta\partial_x u \\
&= \zeta'[\partial_t(\rho Y) + \partial_x(\rho u Y)] - \rho\zeta\partial_x u \\
&= \zeta'\omega - \rho\zeta\partial_x u.
\end{aligned}
$$

Combining the above two equations, we further compute

$$
\begin{aligned}
&\partial_t\hat{\eta} + \partial_x\hat{\phi} \\
&= \frac{\rho}{p}[\rho E\partial_x u + \partial_x(pu)] - \frac{\rho^2 u^2}{2p}\partial_x u - \frac{\rho u}{p}\partial_x p + \frac{\rho}{p}(\partial_t + u\partial_x)(\rho\zeta) - \frac{\gamma}{\gamma-1}\rho\partial_x u \\
&= \frac{\rho}{p}[\rho E\partial_x u + \partial_x(pu)] - \frac{\rho^2 u^2}{2p}\partial_x u - \frac{\rho u}{p}\partial_x p + \frac{\rho}{p}(\zeta'\omega - \rho\zeta\partial_x u) - \frac{\gamma}{\gamma-1}\rho\partial_x u \\
&= \frac{\rho}{p}[\rho E - \frac{1}{2}\rho u^2 - \rho\zeta]\partial_x u + \rho\partial_x u - \frac{\gamma}{\gamma-1}\rho\partial_x u + \frac{\rho}{p}\zeta'\omega \\
&= \frac{1}{\gamma-1}\rho\partial_x u + \rho\partial_x u - \frac{\gamma}{\gamma-1}\rho\partial_x u + \frac{\rho}{p}\zeta'\omega \\
&= \frac{\rho}{p}\zeta'\omega.
\end{aligned}
$$

Appendix C. Proof of Theorem 3.1

We prove Theorem 3.1 by showing that all the leading principal minors of $\hat{\eta}_{UU}$ are positive. We start with the first leading principal minor

$$
D_1 = \frac{\gamma}{\gamma-1}\frac{1}{\rho} + \frac{1}{p}(2\zeta - 2\zeta' Y + \zeta'' Y^2) + (\gamma-1)\frac{\rho}{p^2}(-\zeta + \zeta' Y + \frac{u^2}{2})^2.
$$

If $\zeta - \zeta' Y \geq 0$, it is obvious that

$$
\begin{aligned}
D_1 &\geq \frac{\gamma}{\gamma-1}\frac{1}{\rho} + \frac{1}{p}\zeta'' Y^2 + (\gamma-1)\frac{\rho}{p^2}(-\zeta + \zeta' Y + \frac{u^2}{2})^2 \\
&> \frac{1}{p}\zeta'' Y^2 \geq 0,
\end{aligned}
$$

where the last equality is due to $\zeta'' > 0$. If $\zeta - \zeta' Y < 0$, we have $-\zeta + \zeta' Y + \frac{u^2}{2} \geq -\zeta + \zeta' Y > 0$ and thereby

$$
\begin{aligned}
D_1 &\geq \frac{\gamma}{\gamma-1}\frac{1}{\rho} + \frac{1}{p}(2\zeta - 2\zeta' Y + \zeta'' Y^2) + (\gamma-1)\frac{\rho}{p^2}(\zeta - \zeta' Y)^2 \\
&= \frac{\gamma}{\gamma-1}\frac{1}{\rho} + (\gamma-1)\frac{\rho}{p^2}(\zeta - \zeta' Y)^2 + \frac{1}{p}(2\zeta - 2\zeta' Y) + \frac{1}{p}\zeta'' Y^2 \\
&\geq 2\sqrt{\gamma}\frac{1}{p}|\zeta - \zeta' Y| + \frac{1}{p}(2\zeta - 2\zeta' Y) + \frac{1}{p}\zeta'' Y^2 \\
&= 2(\sqrt{\gamma} - 1)\frac{1}{p}(\zeta' Y - \zeta) + \frac{1}{p}\zeta'' Y^2.
\end{aligned}
$$

Since $\gamma > 1$, we obtain $D_1 > \frac{1}{p}\zeta'' Y^2 \geq 0$. The two cases imply $D_1 > 0$.

The second leading principal minor is

$$
\begin{aligned}
&D_2 = \\
&\det\begin{pmatrix} \frac{\gamma}{\gamma-1}\frac{1}{\rho} + \frac{1}{p}(2\zeta - 2\zeta' Y + \zeta'' Y^2) + (\gamma-1)\frac{\rho}{p^2}(-\zeta + \zeta' Y + \frac{u^2}{2})^2 & -(\gamma-1)\frac{\rho u}{p^2}(-\zeta + \zeta' Y + \frac{u^2}{2}) \\ -(\gamma-1)\frac{\rho u}{p^2}(-\zeta + \zeta' Y + \frac{u^2}{2}) & \frac{1}{p} + (\gamma-1)\frac{\rho u^2}{p^2} \end{pmatrix} \\
&= \frac{1}{p}\left[\frac{\gamma}{\gamma-1}\frac{1}{\rho} + \frac{1}{p}(2\zeta - 2\zeta' Y + \zeta'' Y^2) + (\gamma-1)\frac{\rho}{p^2}(-\zeta + \zeta' Y + \frac{u^2}{2})^2\right] \\
&\quad + (\gamma-1)\frac{\rho u^2}{p^2}\left[\frac{\gamma}{\gamma-1}\frac{1}{\rho} + \frac{1}{p}(2\zeta - 2\zeta' Y + \zeta'' Y^2)\right] \\
&= \frac{1}{p}\left[\frac{\gamma}{\gamma-1}\frac{1}{\rho} + \frac{2}{p}(\zeta - \zeta' Y) + (\gamma-1)\frac{\rho}{p^2}(-\zeta + \zeta' Y + \frac{u^2}{2})^2\right] + \frac{1}{p^2}\zeta'' Y^2 \\
&\quad + \gamma\frac{u^2}{p^2} + (\gamma-1)\frac{2\rho u^2}{p^3}(\zeta - \zeta' Y) + (\gamma-1)\frac{\rho u^2}{p^3}\zeta'' Y^2 \\
&= \frac{1}{p}\left[\frac{\gamma}{\gamma-1}\frac{1}{\rho} + (\gamma-1)\frac{\rho}{p^2}(\zeta - \zeta' Y + \frac{u^2}{2})^2 + \frac{2}{p}(\zeta - \zeta' Y) + \gamma\frac{u^2}{p}\right] + \left(\frac{1}{p^2} + (\gamma-1)\frac{\rho u^2}{p^3}\right)\zeta'' Y^2 \\
&\geq \frac{1}{p}\left[\frac{\gamma}{\gamma-1}\frac{1}{\rho} + (\gamma-1)\frac{\rho}{p^2}(\zeta - \zeta' Y + \frac{u^2}{2})^2 + \frac{2}{p}(\zeta - \zeta' Y) + \gamma\frac{u^2}{p}\right] \\
&\geq \frac{1}{p}\left[2\sqrt{\gamma}\frac{1}{p}|\zeta - \zeta' Y + \frac{u^2}{2}| + \frac{2}{p}(\zeta - \zeta' Y) + \gamma\frac{u^2}{p}\right] \\
&\geq \frac{1}{p}\left[\frac{2}{p}|\zeta - \zeta' Y + \frac{u^2}{2}| + \frac{2}{p}(\zeta - \zeta' Y + \frac{u^2}{2})\right] \geq 0.
\end{aligned}
$$

In the above equation, $D_2 = 0$ holds only if

$$
Y = 0, \quad \frac{\gamma}{\gamma-1}\frac{1}{\rho} = (\gamma-1)\frac{\rho}{p^2}(\zeta - \zeta' Y + \frac{u^2}{2})^2, \quad \zeta - \zeta' Y + \frac{u^2}{2} = 0, \quad u = 0,
$$

which is impossible as $\rho > 0$. Thus $D_2 > 0$.

For the third leading principal minor, we compute it with the expansion along the first column as

$$
\begin{aligned}
D_3 &= \\
&\det\begin{pmatrix}
\frac{\gamma}{\gamma-1}\frac{1}{\rho}+\frac{1}{p}(2\zeta-2\zeta'Y+\zeta''Y^2)+(\gamma-1)\frac{\rho}{p^2}(-\zeta+\zeta'Y+\frac{u^2}{2})^2 & -(\gamma-1)\frac{\rho u}{p^2}(-\zeta+\zeta'Y+\frac{u^2}{2}) & -\frac{1}{p}+(\gamma-1)\frac{\rho}{p^2}(-\zeta+\zeta'Y+\frac{u^2}{2}) \\
-(\gamma-1)\frac{\rho u}{p^2}(-\zeta+\zeta'Y+\frac{u^2}{2}) & \frac{1}{p}+(\gamma-1)\frac{\rho u^2}{p^2} & -(\gamma-1)\frac{\rho u}{p^2} \\
-\frac{1}{p}+(\gamma-1)\frac{\rho}{p^2}(-\zeta+\zeta'Y+\frac{u^2}{2}) & -(\gamma-1)\frac{\rho u}{p^2} & (\gamma-1)\frac{\rho}{p^2}
\end{pmatrix} \\
&= \left[\frac{\gamma}{\gamma-1}\frac{1}{\rho}+\frac{1}{p}(2\zeta-2\zeta'Y+\zeta''Y^2)+(\gamma-1)\frac{\rho}{p^2}(-\zeta+\zeta'Y+\frac{u^2}{2})^2\right](\gamma-1)\frac{\rho}{p^3} \\
&\quad+(\gamma-1)\frac{\rho u}{p^2}(-\zeta+\zeta'Y+\frac{u^2}{2})\left(-(\gamma-1)\frac{\rho u}{p^3}\right) \\
&\quad+\left[-\frac{1}{p}+(\gamma-1)\frac{\rho}{p^2}(-\zeta+\zeta'Y+\frac{u^2}{2})\right]\left[\frac{1}{p^2}+(\gamma-1)\frac{\rho u^2}{p^3}-(\gamma-1)\frac{\rho}{p^3}(-\zeta+\zeta'Y+\frac{u^2}{2})\right] \\
&= (\gamma-1)\frac{1}{p^3}+(\gamma-1)\frac{\rho}{p^4}(2\zeta-2\zeta'Y+\zeta''Y^2)-(\gamma-1)\frac{\rho u^2}{p^4}+(\gamma-1)\frac{2\rho}{p^4}(-\zeta+\zeta'Y+\frac{u^2}{2}) \\
&= (\gamma-1)\frac{1}{p^3}+(\gamma-1)\frac{\rho}{p^4}\zeta''Y^2>0,
\end{aligned}
$$

where $\zeta''>0$ has been used for the last equality.

For the fourth leading principle minor, *i.e.*, $\det(\hat{\eta}_{UU})$, we divide it into two determinants according to the last row:

$$
\begin{aligned}
&\det(\hat{\eta}_{UU}) \\
&= \det\begin{pmatrix}
\frac{\gamma}{\gamma-1}\frac{1}{\rho}+\frac{1}{p}(2\zeta-2\zeta'Y+\zeta''Y^2)+(\gamma-1)\frac{\rho}{p^2}(-\zeta+\zeta'Y+\frac{u^2}{2})^2 & -(\gamma-1)\frac{\rho u}{p^2}(-\zeta+\zeta'Y+\frac{u^2}{2}) & -\frac{1}{p}+(\gamma-1)\frac{\rho}{p^2}(-\zeta+\zeta'Y+\frac{u^2}{2}) & \frac{1}{p}(\zeta'-\zeta''Y)-(\gamma-1)\frac{\rho\zeta'}{p^2}(-\zeta+\zeta'Y+\frac{u^2}{2}) \\
-(\gamma-1)\frac{\rho u}{p^2}(-\zeta+\zeta'Y+\frac{u^2}{2}) & \frac{1}{p}+(\gamma-1)\frac{\rho u^2}{p^2} & -(\gamma-1)\frac{\rho u}{p^2} & (\gamma-1)\frac{\rho u\zeta'}{p^2} \\
-\frac{1}{p}+(\gamma-1)\frac{\rho}{p^2}(-\zeta+\zeta'Y+\frac{u^2}{2}) & -(\gamma-1)\frac{\rho u}{p^2} & (\gamma-1)\frac{\rho}{p^2} & -(\gamma-1)\frac{\rho\zeta'}{p^2} \\
\frac{1}{p}(\zeta'-\zeta''Y)-(\gamma-1)\frac{\rho\zeta'}{p^2}(-\zeta+\zeta'Y+\frac{u^2}{2}) & (\gamma-1)\frac{\rho u\zeta'}{p^2} & -(\gamma-1)\frac{\rho\zeta'}{p^2} & (\gamma-1)\frac{\rho\zeta'^2}{p^2}
\end{pmatrix} \\
&+\det\begin{pmatrix}
\frac{\gamma}{\gamma-1}\frac{1}{\rho}+\frac{1}{p}(2\zeta-2\zeta'Y+\zeta''Y^2)+(\gamma-1)\frac{\rho}{p^2}(-\zeta+\zeta'Y+\frac{u^2}{2})^2 & -(\gamma-1)\frac{\rho u}{p^2}(-\zeta+\zeta'Y+\frac{u^2}{2}) \\
-(\gamma-1)\frac{\rho u}{p^2}(-\zeta+\zeta'Y+\frac{u^2}{2}) & \frac{1}{p}+(\gamma-1)\frac{\rho u^2}{p^2} \\
-\frac{1}{p}+(\gamma-1)\frac{\rho}{p^2}(-\zeta+\zeta'Y+\frac{u^2}{2}) & -(\gamma-1)\frac{\rho u}{p^2} \\
0 & 0
\end{pmatrix}
\end{aligned}
$$

$$
\left.\begin{matrix}
-\frac{1}{p}+(\gamma-1)\frac{\rho}{p^2}(-\zeta+\zeta'Y+\frac{u^2}{2}) & \frac{1}{p}(\zeta'-\zeta''Y)-(\gamma-1)\frac{\rho\zeta'}{p^2}(-\zeta+\zeta'Y+\frac{u^2}{2}) \\
-(\gamma-1)\frac{\rho u}{p^2} & (\gamma-1)\frac{\rho u\zeta'}{p^2} \\
(\gamma-1)\frac{\rho}{p^2} & -(\gamma-1)\frac{\rho\zeta'}{p^2} \\
0 & \frac{\zeta''}{p}
\end{matrix}\right)
$$

$$
:= d_1 + d_2.
$$

The second determinant d_2 is actually equal to $\frac{\zeta''}{p}D_3$. The first determinant d_1 can be further divided into two parts according to the first column:

$$
\begin{aligned}
d_1 = &\det\begin{pmatrix}
0 & -(\gamma-1)\frac{\rho u}{p^2}(-\zeta+\zeta'Y+\frac{u^2}{2}) & -\frac{1}{p}+(\gamma-1)\frac{\rho}{p^2}(-\zeta+\zeta'Y+\frac{u^2}{2}) & \frac{1}{p}(\zeta'-\zeta''Y)-(\gamma-1)\frac{\rho\zeta'}{p^2}(-\zeta+\zeta'Y+\frac{u^2}{2}) \\
0 & \frac{1}{p}+(\gamma-1)\frac{\rho u^2}{p^2} & -(\gamma-1)\frac{\rho u}{p^2} & (\gamma-1)\frac{\rho u\zeta'}{p^2} \\
-\frac{1}{p} & -(\gamma-1)\frac{\rho u}{p^2} & (\gamma-1)\frac{\rho}{p^2} & -(\gamma-1)\frac{\rho\zeta'}{p^2} \\
\frac{1}{p}(\zeta'-\zeta''Y) & (\gamma-1)\frac{\rho u\zeta'}{p^2} & -(\gamma-1)\frac{\rho\zeta'}{p^2} & (\gamma-1)\frac{\rho\zeta'^2}{p^2}
\end{pmatrix} \\
&+\det\begin{pmatrix}
\frac{\gamma}{\gamma-1}\frac{1}{\rho}+\frac{1}{p}(2\zeta-2\zeta'Y+\zeta''Y^2)+(\gamma-1)\frac{\rho}{p^2}(-\zeta+\zeta'Y+\frac{u^2}{2})^2 & -(\gamma-1)\frac{\rho u}{p^2}(-\zeta+\zeta'Y+\frac{u^2}{2}) & -\frac{1}{p}+(\gamma-1)\frac{\rho}{p^2}(-\zeta+\zeta'Y+\frac{u^2}{2}) & \frac{1}{p}(\zeta'-\zeta''Y)-(\gamma-1)\frac{\rho\zeta'}{p^2}(-\zeta+\zeta'Y+\frac{u^2}{2}) \\
-(\gamma-1)\frac{\rho u}{p^2}(-\zeta+\zeta'Y+\frac{u^2}{2}) & \frac{1}{p}+(\gamma-1)\frac{\rho u^2}{p^2} & -(\gamma-1)\frac{\rho u}{p^2} & (\gamma-1)\frac{\rho u\zeta'}{p^2} \\
(\gamma-1)\frac{\rho}{p^2}(-\zeta+\zeta'Y+\frac{u^2}{2}) & -(\gamma-1)\frac{\rho u}{p^2} & (\gamma-1)\frac{\rho}{p^2} & -(\gamma-1)\frac{\rho\zeta'}{p^2} \\
-(\gamma-1)\frac{\rho\zeta'}{p^2}(-\zeta+\zeta'Y+\frac{u^2}{2}) & (\gamma-1)\frac{\rho u\zeta'}{p^2} & -(\gamma-1)\frac{\rho\zeta'}{p^2} & (\gamma-1)\frac{\rho\zeta'^2}{p^2}
\end{pmatrix}.
\end{aligned}
$$

Note that the second determinant in the above equation is equal to zero as the third and the fourth rows are proportional. Then we further compute D_4 as

$$
\begin{aligned}
D_4 &= d_1 + \frac{\zeta''}{p}D_3 \\
&= \det\left(\begin{matrix}
0 & -(\gamma-1)\frac{\rho u}{p^2}(-\zeta+\zeta'Y+\frac{u^2}{2}) \\
0 & \frac{1}{p}+(\gamma-1)\frac{\rho u^2}{p^2} \\
-\frac{1}{p} & -(\gamma-1)\frac{\rho u}{p^2} \\
\frac{1}{p}(\zeta'-\zeta''Y) & (\gamma-1)\frac{\rho u\zeta'}{p^2}
\end{matrix}\right.
\end{aligned}
$$

$$\begin{pmatrix} -\frac{1}{p}+(\gamma-1)\frac{\rho}{p^2}(-\zeta+\zeta'Y+\frac{u^2}{2}) & \frac{1}{p}(\zeta'-\zeta''Y)-(\gamma-1)\frac{\rho\zeta'}{p^2}(-\zeta+\zeta'Y+\frac{u^2}{2}) \\ -(\gamma-1)\frac{\rho u}{p^2} & (\gamma-1)\frac{\rho u\zeta'}{p^2} \\ (\gamma-1)\frac{\rho}{p^2} & -(\gamma-1)\frac{\rho\zeta'}{p^2} \\ -(\gamma-1)\frac{\rho\zeta'}{p^2} & (\gamma-1)\frac{\rho\zeta'^2}{p^2} \end{pmatrix}+\frac{\zeta''}{p}D_3$$

$$=-\frac{1}{p}\det\begin{pmatrix} -(\gamma-1)\frac{\rho u}{p^2}(-\zeta+\zeta'Y+\frac{u^2}{2}) & -\frac{1}{p}+(\gamma-1)\frac{\rho}{p^2}(-\zeta+\zeta'Y+\frac{u^2}{2}) & \frac{1}{p}(\zeta'-\zeta''Y)-(\gamma-1)\frac{\rho\zeta'}{p^2}(-\zeta+\zeta'Y+\frac{u^2}{2}) \\ \frac{1}{p}+(\gamma-1)\frac{\rho u^2}{p^2} & -(\gamma-1)\frac{\rho u}{p^2} & (\gamma-1)\frac{\rho u\zeta'}{p^2} \\ (\gamma-1)\frac{\rho u\zeta'}{p^2} & -(\gamma-1)\frac{\rho\zeta'}{p^2} & (\gamma-1)\frac{\rho\zeta'^2}{p^2} \end{pmatrix}$$

$$-\frac{1}{p}(\zeta'-\zeta''Y)\times$$

$$\det\begin{pmatrix} -(\gamma-1)\frac{\rho u}{p^2}(-\zeta+\zeta'Y+\frac{u^2}{2}) & -\frac{1}{p}+(\gamma-1)\frac{\rho}{p^2}(-\zeta+\zeta'Y+\frac{u^2}{2}) & \frac{1}{p}(\zeta'-\zeta''Y)-(\gamma-1)\frac{\rho\zeta'}{p^2}(-\zeta+\zeta'Y+\frac{u^2}{2}) \\ \frac{1}{p}+(\gamma-1)\frac{\rho u^2}{p^2} & -(\gamma-1)\frac{\rho u}{p^2} & (\gamma-1)\frac{\rho u\zeta'}{p^2} \\ -(\gamma-1)\frac{\rho u}{p^2} & (\gamma-1)\frac{\rho}{p^2} & -(\gamma-1)\frac{\rho\zeta'}{p^2} \end{pmatrix}$$

$$+\frac{\zeta''}{p}D_3$$

$$\begin{aligned}
&=\frac{1}{p^2}\det\begin{pmatrix} -\frac{1}{p}+(\gamma-1)\frac{\rho}{p^2}(-\zeta+\zeta'Y+\frac{u^2}{2}) & \frac{1}{p}(\zeta'-\zeta''Y)-(\gamma-1)\frac{\rho\zeta'}{p^2}(-\zeta+\zeta'Y+\frac{u^2}{2}) \\ -(\gamma-1)\frac{\rho\zeta'}{p^2} & (\gamma-1)\frac{\rho\zeta'^2}{p^2} \end{pmatrix} \\
&\quad+\frac{1}{p^2}(\zeta'-\zeta''Y)\det\begin{pmatrix} -\frac{1}{p}+(\gamma-1)\frac{\rho}{p^2}(-\zeta+\zeta'Y+\frac{u^2}{2}) & \frac{1}{p}(\zeta'-\zeta''Y)-(\gamma-1)\frac{\rho\zeta'}{p^2}(-\zeta+\zeta'Y+\frac{u^2}{2}) \\ (\gamma-1)\frac{\rho}{p^2} & -(\gamma-1)\frac{\rho\zeta'}{p^2} \end{pmatrix} \\
&\quad+\frac{\zeta''}{p}D_3 \\
&=-(\gamma-1)\frac{\rho\zeta''Y}{p^4}\det\begin{pmatrix} -\frac{1}{p}+(\gamma-1)\frac{\rho}{p^2}(-\zeta+\zeta'Y+\frac{u^2}{2}) & \frac{1}{p}(\zeta'-\zeta''Y)-(\gamma-1)\frac{\rho\zeta'}{p^2}(-\zeta+\zeta'Y+\frac{u^2}{2}) \\ 1 & -\zeta' \end{pmatrix} \\
&\quad+\frac{\zeta''}{p}D_3 \\
&=-(\gamma-1)\frac{\rho(\zeta''Y)^2}{p^5}+\frac{\zeta''}{p}\left[(\gamma-1)\frac{1}{p^3}+(\gamma-1)\frac{\rho}{p^4}\zeta''Y^2\right] \\
&=(\gamma-1)\frac{\zeta''}{p^4}.
\end{aligned}$$

From the final expression of D_4 in the above equation we see that $D_4>0$ if and only if $\zeta''>0$. This and the positivity of the first three leading principal minors yield the conclusion of the theorem. The proof is complete.

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