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A SECOND-ORDER ABSORBING BOUNDARY CONDITION FOR TWO-DIMENSIONAL PERIDYNAMICS

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Abstract. The aim of this paper is to develop numerical analysis for the two-dimensional peridynamics which depicts nonlocal phenomena with artificial boundary conditions (ABCs). To this end, the artificial boundary conditions for the fully discretized peridynamics are proposed. Then, the numerical analysis of the fully discretized scheme is developed such that the ABCs solve the corner reflection problem with second-order accuracy. Finally numerical examples are given to verify theoretical results.

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1. Introduction

Recently the peridynamic theory introduced by Silling [10, 33] has inspired a lot of interests. Peridynamics has been developed to treat damage, cracking, deformations [33], and some other material fracture problems [16, 22, 27, 28, 38–40] by reformulating partial differential equation in terms of integro-differential equations. We refer to [10] for a more wider variety of nonlocal models in engineering and physics. Due to the difficulties for seeking analytical solutions [26, 38], numerical simulations are needed to study the behavior of general peridynamics models [7, 23–25]. In this paper, we consider the two-dimensional peridynamics equation of motion u solution to

$$\begin{cases} u_{tt}(x,y,t) = \mathcal{L}_\delta u(x,y,t) + f(x,y,t), \quad (x,y) \in \mathbb{R}^2, \quad t \in (0,T], \\ u(x,y,0) = \phi^0(x,y), \quad (x,y) \in \mathbb{R}^2, \\ u_t(x,y,0) = \phi^1(x,y), \quad (x,y) \in \mathbb{R}^2, \\ \lim_{|x|^2+|y|^2\to+\infty} u(x,y,t) = 0, \qquad t \in (0,T], \end{cases} \tag{1}$$

with f the body force and T the maximal computational time. The nonlocal operator $\mathcal{L}_\delta$ that describes long-range interactions is given by Du *et al.* [10, 13]

$$\mathcal{L}_\delta u(x,y,t) = \int_{\mathbb{R}^2} \gamma_\delta(x',y')(u(x+x',y+y',t) - u(x,y,t))\,\mathrm{d}x'\,\mathrm{d}y'. \tag{2}$$

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The interaction kernel function γ_δ appearing in (2) satisfies the following properties

- γ_δ is a radial function: $\gamma_\delta(x', y') = \gamma_\delta(r')$ with $r' = \sqrt{(x')^2 + (y')^2}$;
- γ_δ is positive: $\gamma_\delta(r') \geq 0$;
- γ_δ is compact supported (finite horizon), *i.e.* $\gamma_\delta(r') = 0$, for $r' > \delta > 0$.

The fractional Laplacian $(-\partial_x^2)^{s/2}$ is a special case of the integro-differential operator $\mathcal{L}_\delta$. We must point that the displacement u for the real peridynamics is a vector. But for convenience we deal with scalar peridynamics here. The study for the scalar cases can be extended to the vector cases just as lattice dynamics. Therefore, we will generally refer to scalar peridynamics that we treat here as peridynamics from now on.

The artificial boundary conditions (ABCs) are always applied to avoid difficulty caused by unboundedness of the spatial domain while solving problems defined on an unbounded domain. To this end, one needs to reformulate the unbounded problem into a problem defined on a bounded computational domain by applying appropriate ABCs to the domain's boundary. Inappropriate boundary conditions will lead to numerical reflection that disrupt the numerical solution in the computational domain while the waves touch the artificial boundaries. Weckner's work has indicated the difficulties in designing boundary condition [39] while studying peridynamics in an unbounded medium. An ideal boundary condition can efficiently avoid numerical reflection by annihilating the waves that touch the artificial boundaries. In this paper the ABCs are applied to the boundary of a rectangular computational domain to reformulate the unbounded problem (1) into an initial-boundary-value problem.

These years, many contributions of ABCs (see *e.g.* [2, 3, 6, 14, 15, 17–19, 35]) have been devoted to the local PDEs, such as wave equation and Schrödinger equation. Comparing to local PDEs, much less works on ABCs of nonlocal PDEs are available in the literature. For one-dimensional cases, we refer to [42, 43] for the boundary treatments of the nonlocal heat equations and to [30, 41] for the boundary treatments of nonlocal Schrödinger equations. As for one-dimensional nonlocal wave equations, we refer to [12]. In [5, 21] the PML approach is applied to study the one-dimensional heat and Schrödinger equations involving fractional operators. Furthermore, a general PML approach is introduced to study fractional PDEs in [1, 4]. Some local in-time ABCs are also introduced to treat the one-dimensional peridynamics in [34, 37]. However, for two and three dimensional cases, these algorithms have difficulties in dealing with the corner problem. In [11], ABCs for two-dimensional nonlocal PDEs are proposed, but without numerical analysis. In general, there are much less works devoted to ABCs for two-dimensional nonlocal models. A crucial difficulty for designing ABCs of two-dimensional nonlocal models is to compute the Green's functions accurately [20, 29, 31, 36]. In the present work, a numerical analysis is developed for the ABC proposed in [32]. It can be proved that the numerical ABC is of second order by computing the Green's functions accurately [32]. Thus, the ABC solves the problem of corner reflection efficiently (inappropriate boundary condition can be considered physically as an artificial obstacle setting on the boundary, which will make the waves that should have gone out of the computational domain to be reflected back at the boundary. This kind of reflection is more obvious at the corner).

The aim of the present paper is (i) to develop a numerical analysis for the numerical scheme of (1) with ABCs derived in [32] and (ii) to verify the numerical accuracy of the approach. In Section 2, a full discretization of (1) is introduced by a Crank–Nicolson time discretization and a spatial asymptotically compatible scheme. Then, the exact ABC proposed in [32] for the fully discretized version is stated. In addition, some properties of the ABCs are stated for numerical analysis. In Section 3, the numerical analysis of the scheme with ABCs is developed. In Section 4, we consider the wave equation, namely,

$$\begin{cases} u_{tt}(x,y,t) = \triangle u(x,y,t) + f(x,y,t), \quad (x,y) \in \mathbb{R}^2, & t \in (0,T], \\ u(x,y,0) = \phi^0(x,y), u_t(x,y,0) = \phi^1(x,y), \quad (x,y) \in \mathbb{R}^2, & \\ \lim_{x^2+y^2 \to +\infty} u(x,y,t) = 0, & t \in (0,T]. \end{cases} \tag{3}$$

In this section, we also develop the numerical analysis for the semi-discretized version of (3) with the exact ABCs by the same manner. In Section 5 we report some numerical examples. Finally, we draw a conclusion in Section 6.

2. Absorbing boundary conditions for the fully discretized peridynamics

In this section we give the result of the accurate absorbing boundary conditions and the full scheme for the two-dimensional peridynamics (1).

2.1. Time and space discretizations of the two-dimensional peridynamics

The square computational domain $[-x_J, x_J] \times [-x_J, x_J]$ is discretized by using $(2J+1) \times (2J+1)$ grid points $(x_j, y_\ell) = (jh, \ell h)$ for $-J \leq j, \ell \leq J$ where $h := x_J/J$ is the uniform spatial meshsize. Let us introduce the piecewise multilinear basis function $\phi_{j,\ell}(x,y)$ such that $\phi_{j,\ell}(x_m, y_k) = \delta_{j,\ell}^{m,k}$, where the $\delta_{j,\ell}^{k,m}$ is defined as Krönecker symbol, such that: $\delta_{j,\ell}^{k,m} = 0$ for $(j,\ell) \neq (k,m)$ and $\delta_{j,\ell}^{k,m} = 1$, otherwise. Then, for $(j,\ell) \in \mathbb{Z}^2$ and an integer $K \geq 1$, the AC (Asymptotic Compatible scheme) discretization of $\mathcal{L}_\delta$ derived in [12] is given by:

$$\mathcal{L}_\delta u(x_j, y_\ell) \approx \sum_{k,m\geq 0}^{K} b_{k,m}(u(x_{j+k}, y_{\ell+m}) + u(x_{j+k}, y_{\ell-m}) + u(x_{j-k}, y_{\ell+m}) + u(x_{j-k}, y_{\ell-m}) - 4u(x_j, y_\ell)), \quad (4)$$

where the real-valued coefficients $b_{k,m}$ are such that

$$b_{k,m} := \begin{cases} \dfrac{k+m}{h(k^2+m^2)} \displaystyle\iint_{B^+(0,\delta)} \phi_{k,m}(x,y)\gamma_\delta\Big(\sqrt{x^2+y^2}\Big)\frac{x^2+y^2}{x+y}\,\mathrm{d}x\,\mathrm{d}y, & (k,m) \neq (0,0), \\ 0, & (k,m) = (0,0). \end{cases} \quad (5)$$

Here $B^+(0,\delta)$ is the first quadrant of the disc centered at the origin with radius δ. It is obvious that $b_{k,m} = b_{m,k}$. Due to the disc integral horizon, we use a one-order numerical integral to treat (5) with very fine mesh.

For a sequence $u = \{u^n\}_{n=0}^\infty$ such that $\sum_{n=0}^\infty |u^n|^2 < \infty$, the operators $S^\pm$ are defined by: $S^\pm u = \{u^{n\pm1}\}_{n=0}^\infty$, setting $u^{-1} = 0$. For a uniform time step τ, let us define the average operators $E^\pm$ and difference quotient operators $D_\tau^\pm$ such that $E^\pm = (S^\pm + I)/2$ and $D_\tau^\pm = \pm(S^\pm - I)/\tau$. For convenience, the following symbols are introduced: $S^\pm u^n = (S^\pm u)^n$, $E^\pm u^n = (E^\pm u)^n$ and $D_\tau^\pm u^n = (D_\tau^\pm u)^n$.

Let us introduce the following coefficients:

$$a_{k,m} = b_{|k|,|m|}, \quad (k,m) \neq (0,0), \qquad a_{0,0} = -\sum_{-K\leq k,m\leq K,(k,m)\neq(0,0)} b_{k,m}. \quad (6)$$

Based on the AC spatial discretization (4) and the Crank–Nicolson (CN) time scheme, one obtains the fully discretized version of (1), namely,

$$D_\tau^- D_\tau^+ u_{j,\ell}^n = E^- E^+ L_{\delta,h} u_{j,\ell}^n + f_{j,\ell}^n := \sum_{k,m=-K}^{K} a_{k,m} E^- E^+ u_{j+k,\ell+m}^n + f_{j,\ell}^n, \quad \text{for } j,\ell \in \mathbb{Z},\ n \geq 1, \quad (7)$$

with $u_{j,\ell}^n$ the numerical approximation of $u(x_j, y_\ell, t_n)$ and discretized operator $L_{\delta,h} u_{j,\ell} = \sum_{k,m=-K}^{K} a_{k,m} u_{j+k,\ell+m}$ where $a_{k,m}$ is defined in (6).

2.2. The absorbing boundary conditions for the fully discretized peridynamics

In this part we will introduce the exact absorbing boundary conditions for (7) with general compactly supported initial data ϕ^0 and ϕ^1. For convenience, the notation $\mathcal{R}_L(m,n)$ is introduced to denote a rectangular box in $\mathbb{Z}^2$ such that $\mathcal{R}_L(m,n) = \{(p,q) : |p-m| \leq L, |q-n| \leq L\}$. The finite rectangular computational domain can be chosen as $\mathcal{R}_J(0,0)$ with size of $(2J+1)\times(2J+1)$ (blue square domain in Fig. 1 for the specific case $J = 4$ and $K = 2$) where $u_{j,\ell}^n$ is located at (j,ℓ), with $j = -J, \ldots, J$ and $\ell = -J, \ldots, J$. Furthermore, the white interior points are used to denote $\mathcal{R}_{J-K}(0,0)$, the yellow interior layer points are used to denote $\mathcal{R}_J(0,0)\backslash\mathcal{R}_{J-K}(0,0)$ and the black exterior layer points are used to denote $\mathcal{R}_{J+K}(0,0)\backslash\mathcal{R}_J(0,0)$.

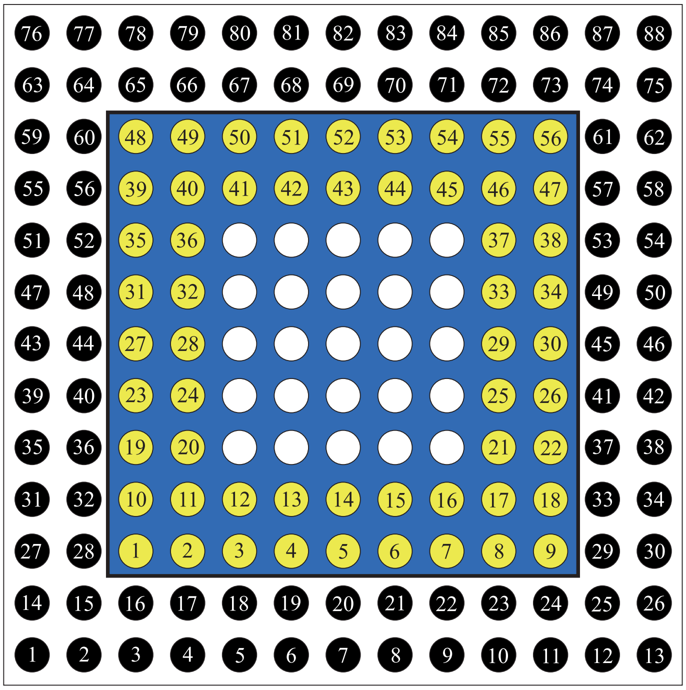

FIGURE 1. Labelling of the interior and exterior layers nodes ($J = 4$ and $K = 2$).

Then, the symbol $\nu(k)$ with $1 \leq k \leq 4K(2J+1-K)$ can be defined to denote the location of the k-th interior layer node (yellow node in Fig. 1). The notation $\mathcal{B}_{\rm Int}$ is applied to denote the set consisting of the locations of interior layer nodes such that

$$\mathcal{B}_{\rm Int} = \{\nu(k) : 1 \leq k \leq 4K(2J+1-K)\}.$$

The cardinal of $\mathcal{B}_{\rm Int}$ satisfies $|\mathcal{B}_{\rm Int}| = (2J+1)^2 - (2J+1-2K)^2 = 4K(2J+1-K)$. The interior layer vector $(\mathbf{I}[u])^n := [u^n_{\nu(1)}, \cdots, u^n_{\nu(|\mathcal{B}_{\rm Int}|)}]^T$ is defined such that $u^n_{\nu(k)}$ is located at the location $\nu(k)$ of $\mathcal{B}_{\rm Int}$ where $\mathbb{M}^T$ is the transpose of complex-valued matrix $\mathbb{M}$.

In the same manner, we use the symbol $\mu(m)$ to denote the location of the m-th exterior layer node (black node in Fig. 1). We also define: $\mathcal{B}_{\rm Ext} = \{\mu(k) : 1 \leq k \leq 4K(2J+1+K)\}$, with $|\mathcal{B}_{\rm Ext}| = 4K(2J+1+K)$. In the same manner, the exterior layer vector is defined by $(\mathbf{E}[u])^n := [u^n_{\mu(1)}, \cdots, u^n_{\mu(|\mathcal{B}_{\rm Ext}|)}]^T$. We have left the expressions of $\nu(k)$ and $\mu(m)$ in the Appendix A.

The $\mathcal{Z}$-transform of a complex-valued sequence $(f^n)_{n\in\mathbb{N}}$ is defined by

$$\widehat{f}(z) := \sum_{k=0}^{\infty} f^n z^{-n}.$$

In the same manner, the $\mathcal{Z}$-transforms for a sequence of $M \times M$ matrices $\{\mathbb{A}^n\}_{n\in\mathbb{N}}$ and for a sequence of vectors $\{\mathbf{v}^n\}_{n\in\mathbb{N}} \in \mathbb{C}^M$, are defined by:

$$\widehat{\mathbb{A}}(z) := \sum_{n=0}^{\infty} \mathbb{A}^n z^{-n} \quad \text{and} \quad \widehat{\mathbf{v}}(z) := \sum_{n=0}^{\infty} \mathbf{v}^n z^{-n},$$

respectively. In addition, the convolution $\star$ of two sequences $\{g^n\}_n$ and $\{f^n\}_n$ is defined by:

$$(f \star g)^n := \sum_{r=0}^{n} g^{n-r} f^n,$$

while the convolution of two matrices sequences $\{\mathbb{A}^n\}_n$ and $\{\mathbb{B}^n\}_n$ is defined by:

$$(\mathbb{A} \star \mathbb{B})^n := \sum_{r=0}^{n} \mathbb{A}^{n-r}\mathbb{B}^r.$$

Finally, we introduce the function

$$\rho_{\text{Int}}(z,x,y) := \left(\frac{z-1}{\tau}\right)^2 - \left(\frac{z+1}{2}\right)^2 A_{\text{Int}}(x,y), \tag{8}$$

such that

$$A_{\text{Int}}(x,y) := \sum_{k,m=-K}^{K} a_{k,m} e^{-\mathrm{i}kx} e^{-\mathrm{i}my},$$

with $\mathrm{i} = \sqrt{-1}$.

By the compact support assumption of initial data ϕ^0 and ϕ^1, the computational domain $\mathcal{R}_J(0,0)$ can be chosen such that the initial data is zero outside the selected computational domain, including its interior layer (*i.e.* compactly supported in the domain consisting of the white points in Fig. 1). In this setting, the unknowns $u^n_{\mu(j)}$ $(1 \le j \le |\mathcal{B}_{\text{Ext}}|)$ in the exterior layer (black nodes in Fig. 1) are required to complete the discretized system (7), the complete relation is discretized boundary condition. To this end, the discretized ABC (9) in Theorem 1 is proposed [32]. The discretized ABC (9) depicts the relation between the unknowns $u^n_{\mu(j)}$ in the exterior layer (black nodes in Fig. 1) and the values $u^n_{\nu(j)}$ $(1 \le j \le |\mathcal{B}_{\text{Int}}|)$ in the interior layer (yellow nodes in Fig. 1).

Theorem 1 ([32]). *Under the above setting, let us assume that $\tau^2 \le |a_{0,0}|^{-1}/4$, with $a_{0,0}$ given by (6). The interior layer nodes satisfy $u^0_{\nu(j)} = 0$ and $u^1_{\nu(j)} = 0$ with $1 \le j \le |\mathcal{B}_{\text{Int}}|$. Then, the ABC for the discretized peridynamics (7) is*

$$u^n_{\mu(j)} = \sum_{k=1}^{|\mathcal{B}_{\text{Int}}|} (\mathbb{C}_{j,k} \star u_{\nu(k)})^n, \quad \text{for } 1 \le j \le |\mathcal{B}_{\text{Ext}}|, \tag{9}$$

for $n \in \mathbb{N}$. Here, the sequence of boundary kernel matrices $\{\mathbb{C}^n\}_n$ of size $|\mathcal{B}_{\text{Ext}}| \times |\mathcal{B}_{\text{Int}}|$ is given by $\widehat{\mathbb{C}}(z) = \widehat{\mathbb{B}}(z)\widehat{\mathbb{A}}^{-1}(z)$. The matrix $\mathbb{B}^n$ is given by $\mathbb{B}^n_{j,k} = f^n_{|\mu(j)-\nu(k)|}$, for $1 \le j \le |\mathcal{B}_{\text{Ext}}|$, $1 \le k \le |\mathcal{B}_{\text{Int}}|$; the matrix $\mathbb{A}^n$ is given by $\mathbb{A}^n_{j,k} = f^n_{|\nu(j)-\nu(k)|}$, for $1 \le j,k \le |\mathcal{B}_{\text{Int}}|$. Here the sequence $\{f^n_{j,\ell}\}_n$ is given by

$$\hat{f}_{j,\ell}(z) = \frac{\hat{g}_{j,\ell}(z)}{\hat{g}_{0,0}(z)}, \tag{10}$$

with

$$\hat{g}_{j,\ell}(z) = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} (\rho_{\text{Int}}(z,x,y))^{-1} e^{-\mathrm{i}jx - \mathrm{i}\ell y} \,\mathrm{d}x\,\mathrm{d}y, \tag{11}$$

where the notation $|(m,n)|$ represents $(|m|,|n|)$.

More specifically, one can find $|\mathcal{B}_{\text{Int}}|$ sequences $\{v^n_{\nu(k)}\}_n$, with $1 \le k \le |\mathcal{B}_{\text{Int}}|$, satisfying

$$u^n_{\nu(j)} = \sum_{k=1}^{|\mathcal{B}_{\text{Int}}|} (f_{\nu(j)-\nu(k)} \star v_{\nu(k)})^n, \quad 1 \le j \le |\mathcal{B}_{\text{Int}}|. \tag{12}$$

We remarked that it is not easy to obtained an exact boundary condition for the continuous equation (1). Thus, a discretized exact boundary condition for the according discretized system (7) is required.

In fact, sequence $\{g_{j,\ell}^n\}_n$, the discrete Z-inversion for $\hat{g}_{j,\ell}(z)$ in (11), is the discretized Green's function for the fully discretized peridynamics (7) due to the following Lemma 1. Therefore, the discretized boundary kernel function $\{\mathbb{C}^n\}_n$ is generated by the discretized Green's function $\{g_{j,\ell}^n\}_n$ from $\widehat{\mathbb{C}}(z) = \widehat{\mathbb{B}}(z)\widehat{\mathbb{A}}^{-1}(z)$. This is similar to the continuous cases. To implement the ABC (9), one needs to compute the discretized Green's function $\{g_{j,\ell}^n\}_n$ accurately by Lemma 1. To state Lemma 1, let us define $\Delta_{k,m}^x g_{j,\ell}^n = g_{j+k,\ell+m}^n - g_{j-k,\ell+m}^n$.

Lemma 1 ([32])**.** *The sequences $\{g_{j,\ell}^n\}_n$, for $(j,\ell)\in\mathbb{Z}^2$ and $n\geq 2$, satisfy the following relations:*

$$D_\tau^- D_\tau^+ g_{j,\ell}^n = E^- E^+ L_{\delta,h} g_{j,\ell}^n + \delta_{j,\ell}^{0,0}, \tag{13}$$

$$g_{j,\ell}^0 = 0, g_{j,\ell}^1 = 0, \tag{14}$$

and

$$\begin{aligned} &n\sum_{k,m=1}^K a_{k,m}\left(\Delta_{k,m}^x g_{j,\ell}^{n+2} + \Delta_{k,-m}^x g_{j,\ell}^{n+2}\right) + (3n-1)\sum_{k,m=1}^K k a_{k,m}\left(\Delta_{k,m}^x g_{j,\ell}^{n+1} + \Delta_{k,-m}^x g_{j,\ell}^{n+1}\right)\\ &\quad + (3n-2)\sum_{k,m=1}^K k a_{k,m}\left(\Delta_{k,m}^x g_{j,\ell}^{n} + \Delta_{k,-m}^x g_{j,\ell}^{n}\right) + (n-1)\sum_{k,m=1}^K k a_{k,m}\left(\Delta_{k,m}^x g_{j,\ell}^{n-1} + \Delta_{k,-m}^x g_{j,\ell}^{n-1}\right)\\ &\quad + (3n-2)\sum_{k=1}^K k a_{k,0}\Delta_{k,0}^x g_{j,\ell}^n + (n-1)\sum_{k=1}^K k a_{k,0}\Delta_{k,0}^x g_{j,\ell}^{n-1} + n\sum_{k=1}^K k a_{k,0}\Delta_{k,0}^x g_{j,\ell}^{n+2}\\ &\quad + (3n-1)\sum_{k=1}^K k a_{k,0}\Delta_{k,0}^x g_{j,\ell}^{n+1} = \frac{16j}{\tau^2} g_{j,\ell}^n - \frac{16j}{\tau^2} g_{j,\ell}^{n+1}. \end{aligned} \tag{15}$$

We use $r_{j,\ell}$ to denote the relation (15) for the indices (j,ℓ). By setting $L=2J+K$ we give the numerical algorithm to compute $g_{j,\ell}$ for $(i,j)\in\Omega_1=\{(j,\ell): 0\leq\ell\leq j\leq L\}$. In order to presenting the indices clearly, we report an particular example in Figure 2 with $J=3$, $K=2$ and $L=8$. The $(1+L)(2+L)/2$ relations (13) with $(i,j)\in\Omega_1$ (the light gray domain in Fig. 2) is used to compute the sequences $\{g_{j,\ell}^n\}_n$ with $(i,j)\in\Omega_1$. To do this, the values of sequences $\{g_{j,\ell}^n\}_n$ with $(j,\ell)\in\Omega_3=\{(j,\ell):\ell\leq j, L+1\leq j\leq L+K, 0\leq\ell\leq L+K\}$ (the white domain in Fig. 2), and $(j,\ell)\in\Omega_2=\{(j,\ell):\ell\leq j,-K\leq j\leq L+K,-K\leq\ell\leq -1\}$ (the red domain in Fig. 2), are needed to close (13) by $g_{\ell,j}^n=g_{j,\ell}^n$. On the one hand, the $g_{j,\ell}^n$, for $\ell<0$, will be derived by $g_{j,\ell}^n=g_{j,-\ell}^n$, and $g_{j,\ell}^n$, for $j<\ell$, can be obtained from $g_{j,\ell}^n=g_{\ell,j}^n$. Therefore, we can obtain the $\{g_{j,\ell}^n\}_n$ with $(j,\ell)\in\Omega_2$ from $\{g_{j,\ell}^n\}_n$ with $(j,\ell)\in\Omega_1\cup\Omega_3$. On the other hand, we only need to find another $K(L+1)+(K+1)K/2$ relations to close the $(1+L)(2+L)/2$ relations (13), for $(i,j)\in\Omega_1$. To this end, we can choose the relations $r_{j,\ell}$ (the blue domain in Fig. 2), for $(j,\ell)\in\Omega_4=\{(j,\ell): L-K+1\leq j\leq L, 0\leq\ell\leq L\}\cup\{(j,\ell):\ell\leq j, L-2K+1\leq j\leq L-K, L-K+1\leq\ell\leq L\}$, to close the $(1+L)(2+L)/2$ relations (13).

The previous numerical algorithm is applied to solve $g_{j,\ell}^n$, for $n\geq 3$. Thus, we still need to evaluate $g_{j,\ell}^2$, for $0\leq\ell\leq j\leq L+K$. This can be implemented by a direct numerical quadrature through

$$g_{j,\ell}^2 = \frac{\tau^2}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}\frac{1}{(1-\tau^2 A_{\mathrm{Int}}(x,y)/4)}e^{-\mathrm{i}jx-\mathrm{i}\ell y}\,\mathrm{d}x\,\mathrm{d}y.$$

After getting the value of $\{g_{j,\ell}^n\}_n$ for $(j,\ell)\in\Omega_1$, we obtain the discretized boundary kernel $\mathbb{C}^n$ by $\mathbb{B}^n = (\mathbb{C}\star\mathbb{A})^n = \sum_{k=0}^n \mathbb{C}^k\cdot\mathbb{A}^{n-k} = \mathbb{C}^n\mathbb{A}^0 + \sum_{k=0}^{n-1}\mathbb{C}^k\mathbb{A}^{n-k}$, or equivalently

$$\mathbb{C}^n = \left(\mathbb{B}^n - \sum_{k=0}^{n-1}\mathbb{C}^k\mathbb{A}^{n-k}\right)\left(\mathbb{A}^0\right)^{-1}.$$

In the end of this section, we give a lemma for numerical analysis.

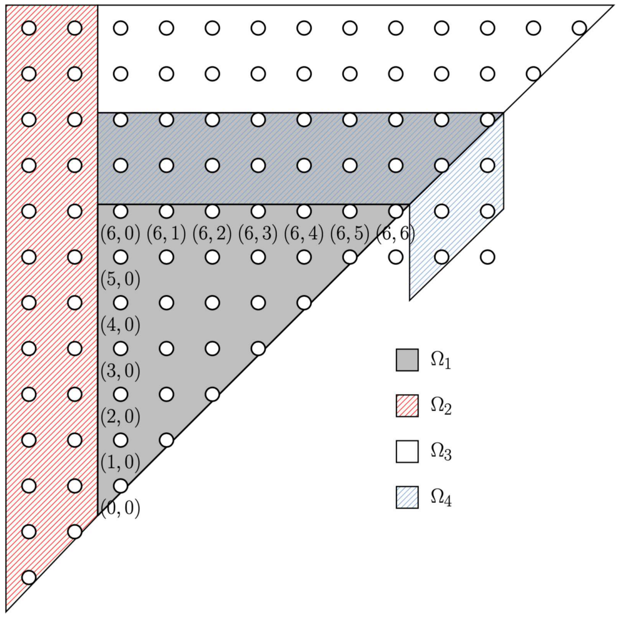

FIGURE 2. Indices for closing the discretized system (13) ($J = 3$, $K = 2$ and $L = 8$).

Lemma 2 ([32]). *Under the setting of Theorem 1, if one defines*

$$u^n_{j,\ell} = \sum_{k=1}^{|\mathcal{B}_{\rm Int}|} \big(f_{(j,\ell)-\nu(k)} \star v_{\nu(k)}\big)^n,$$

for $(j,\ell) \in \mathbb{Z}^2 \backslash \mathcal{R}_J(0,0)$, *then* $u^n_{j,\ell}$ *satisfies the discretized system* (7).

Furthermore, we will obtain

$$\begin{aligned} \big|u^n_{j,\ell}\big| &\le \frac{C|\mathcal{B}_{\rm Int}|4^N}{\big||j|-J\big|}, && \text{for} \quad |j| > J, \\ \big|u^n_{j,\ell}\big| &\le \frac{C|\mathcal{B}_{\rm Int}|4^N}{\big||\ell|-J\big|}, && \text{for} \quad |\ell| > J, \\ \big|u^n_{j,\ell}\big| &\le \frac{C|\mathcal{B}_{\rm Int}|4^N}{\big||\ell|-J\big|\big||j|-J\big|}, && \text{for} \quad |\ell| > J, |j| > J. \end{aligned} \tag{16}$$

2.3. Numerical algorithm for the fully discretized peridynamics with ABC

In this part we propose the corresponding numerical algorithm for the reformulated fully discretized peridynamics on a bounded computational domain $\mathcal{R}_J(0,0)$. Following Section 2.2, one can solve $u^n_{j,\ell}$, for $n \ge 2$ and $(j,\ell) \in \mathcal{R}_J(0,0)$, by (7) and (9). Since (7) is a three layers scheme, we still need the values of $u^1_{j,\ell}$ to compute $u^n_{j,\ell}$, for $n \ge 2$. To determine value $u^1_{j,\ell}$, one can apply the Runge–Kutta 4 (RK4) scheme to the equivalent form of (1): $\dot u_t = v$, $\dot v_t = \mathcal{L}_\delta u + f$, leading to

$$u^1_{j,\ell} = u^0_{j,\ell} + \big(F^1_{j,\ell} + 2F^2_{j,\ell} + 2F^3_{j,\ell} + F^4_{j,\ell}\big)\tau/6, \quad v^1_{j,\ell} = v^0_{j,\ell} + \big(G^1_{j,\ell} + 2G^2_{j,\ell} + 2G^3_{j,\ell} + G^4_{j,\ell}\big)\tau/6,$$

$$
\begin{aligned}
F^1_{j,\ell} &= v^0_{j,\ell}, & G^1_{j,\ell} &= L_{\delta,h} u^0_{j,\ell} + f(jh,\ell h,0),\\
F^2_{j,\ell} &= v^0_{j,\ell} + G^1_{j,\ell}\tau/2, & G^2_{j,\ell} &= L_{\delta,h}\big(u^0_{j,\ell} + F^1_{j,\ell}\tau/2\big) + f(jh,\ell h,\tau/2),\\
F^3_{j,\ell} &= v^0_{j,\ell} + G^2_{j,\ell}\tau/2, & G^3_{j,\ell} &= L_{\delta,h}\big(u^0_{j,\ell} + F^2_{j,\ell}\tau/2\big) + f(jh,\ell h,\tau/2),\\
F^4_{j,\ell} &= v^0_{j,\ell} + G^3_{j,\ell}\tau, & G^4_{j,\ell} &= L_{\delta,h}\big(u^0_{j,\ell} + F^3_{j,\ell}\tau\big) + f(jh,\ell h,\tau),
\end{aligned} \tag{17}
$$

by setting $u^0_{j,\ell} = \phi^0(x_j, y_\ell)$ and $v^0_{j,\ell} = \phi^1(x_j, y_\ell)$. Thus we can use (17) to compute $u^1_{j,\ell}$ from $u^0_{j,\ell}$ in a large enough domain since the initial data are compactly supported.

We remarked the main computing cost comes from (9). For total computational step N, the computing cost for every convolution in (9) is $\mathcal{O}(N^2)$. Thus, for fixed j, the computing cost of (9) is $\mathcal{O}(N^2)|\mathcal{B}_{\mathrm{Int}}| = \mathcal{O}(N^2 J)$. Then, we know that the total computing cost of the boundary condition is $\mathcal{O}(N^2 J)|\mathcal{B}_{\mathrm{Ext}}| = \mathcal{O}(N^2 J^2)$.

3. Errors analysis for the two dimensional peridynamics

According to the assumption in Introduction, we recall that γ_δ is homogeneous in the exterior domain. It may be *a priori* inhomogeneous in the computational domain. However, from now on we assume that γ_δ is also homogeneous inside the computational domain to simplify the presentation of the proofs. The γ_δ has a singularity behaviour $\gamma_\delta(a) \sim |a|^{-2-2\alpha}$ with $\alpha < 1$ around $|a| = 0$. In addition, if γ_δ is non homogeneous in the domain of computation, the proofs can be adapted but at the price of more complexity with the similar singularity types.

3.1. Estimates for the solution u of system (1)

We first propose some useful estimates for the solution u to the continuous problem (1). Firstly we introduce $\mathbf{T}_{x',y'}u$ to denote $u(x+x', y+y')$. In this setting we introduce a notation $\Delta_{x',y'}$ such that $\triangle_{x',y'}u = \mathbf{T}_{x',0}u + \mathbf{T}_{-x',0}u + \mathbf{T}_{0,y'}u + \mathbf{T}_{0,-y'}u - 4u$. The notations $\nabla^{\pm}_{x',0}$ and $\nabla^{\pm}_{0,y'}$ can be defined that $\nabla^{+}_{x',0}u = \mathbf{T}_{x',0}u - u$, $\nabla^{-}_{x',0}u = u - \mathbf{T}_{-x',0}u$, $\nabla^{+}_{0,y'}u = \mathbf{T}_{0,y'}u - u$ and $\nabla^{-}_{0,y'}u = u - \mathbf{T}_{0,-y'}u$. It is obvious that $\nabla^{-}_{x',0}\nabla^{+}_{x',0}u = \nabla^{+}_{x',0}\nabla^{-}_{x',0}u = \triangle_{x',0}u$ and $\nabla^{-}_{0,y'}\nabla^{+}_{0,y'}u = \nabla^{+}_{0,y'}\nabla^{-}_{0,y'}u = \triangle_{0,y'}u$.

Lemma 3. *Let us assume that p, q and ℓ are all nonnegative integers. The initial data ϕ^0 and ϕ^1 belong to $H^{p+q+\ell+2}(\mathbb{R}^2)$ and the source term f belongs to $C^l([0,T], H^{p+q+4}(\mathbb{R}^2))$. Then for the solution u to the continuous problem (1), the following bounds hold,*

$$
\begin{aligned}
\left\|\partial_x^p\partial_y^q\partial_t^\ell u(\cdot,\cdot,T)\right\|_{L^2} \le C\Bigg(&\left\|\phi^0\right\|_{H^{p+q+\ell}} + \left\|\phi^1\right\|_{H^{p+q+\ell}} + \int_0^t \left\|\partial_t^{\ell-1} f(\cdot,\cdot,s)\right\|^2_{H^{p+q}} \mathrm{d}s\\
&+ \sum_{0\le c\le \ell} \|\partial_t^c f(\cdot,\cdot,0)\|^2_{H^{p+q+2}}\Bigg)
\end{aligned} \tag{18}
$$

$$
\begin{aligned}
\left\|\partial_x^p\partial_y^q\partial_t^\ell u(\cdot,\cdot,T)\right\|_{L^\infty} \le C\Bigg(&\left\|\phi^0\right\|_{H^{p+q+\ell+2}} + \left\|\phi^1\right\|_{H^{p+q+\ell+2}} + \int_0^t \left\|\partial_t^{\ell-1} f(\cdot,\cdot,s)\right\|^2_{H^{p+q+2}} \mathrm{d}s\\
&+ \sum_{0\le c\le \ell} \|\partial_t^c f(\cdot,\cdot,0)\|^2_{H^{p+q+4}}\Bigg),
\end{aligned} \tag{19}
$$

where $C > 0$ is a constant depends on T, p, q, ℓ, γ_δ and initial data ϕ^0, ϕ^1.

Proof. It is clearly that

$$
\int_{-L}^{L}\left[\int_{-L}^{L} \nabla^{+}_{x',0}u\cdot v\,\mathrm{d}x\right]\mathrm{d}y = \int_{-L}^{L}\left[\int_{-L}^{L} (\mathbf{T}_{x',0}u - u)v\,\mathrm{d}x\right]\mathrm{d}y
$$

$$
\begin{aligned}
&= \int_{-L}^{L}\left[\int_{-L+x'}^{L+x'} u\cdot \mathbf{T}_{-x',0}v\,\mathrm{d}x - \int_{-L}^{L} uv\,\mathrm{d}x\right]\mathrm{d}y\\
&= -\int_{-L}^{L}\left[\int_{-L+x'}^{L} u\nabla^{-}_{x',0}v\,\mathrm{d}x\right]\mathrm{d}y\\
&\quad + \int_{-L}^{L}\left[\int_{L}^{L+x'} u\cdot \mathbf{T}_{-x',0}v\,\mathrm{d}x - \int_{-L}^{-L+x'} u\cdot v\,\mathrm{d}x\right]\mathrm{d}y. \qquad (20)
\end{aligned}
$$

The term in the last line of (20) is no more than $\iint_{\mathbb{R}^2\backslash[-L+x',L-x']^2}(u^2+v^2)\,\mathrm{d}x\,\mathrm{d}y$. This term tends to 0 as $L\to\infty$. Thus, by taking $L\to\infty$ in (20) one has

$$
\iint_{\mathbb{R}^2} \nabla^{+}_{x',0}u\cdot v\,\mathrm{d}x\,\mathrm{d}y = -\iint_{\mathbb{R}^2} u\cdot \nabla^{-}_{x',0}v\,\mathrm{d}x\,\mathrm{d}y,
$$

and

$$
\iint_{\mathbb{R}^2} \nabla^{+}_{0,y'}u\cdot v\,\mathrm{d}x\,\mathrm{d}y = -\iint_{\mathbb{R}^2} u\cdot \nabla^{-}_{0,y'}v\,\mathrm{d}x\,\mathrm{d}y.
$$

By the above two equalities one has

$$
\begin{aligned}
&\iint_{\mathbb{R}^2} \triangle_{x',0}u\cdot v\,\mathrm{d}x\,\mathrm{d}y = \iint_{\mathbb{R}^2}\nabla^{+}_{x',0}\nabla^{-}_{x',0}u\cdot v\,\mathrm{d}x\,\mathrm{d}y = -\iint_{\mathbb{R}^2}\nabla^{-}_{x',0}u\cdot\nabla^{-}_{x',0}v\,\mathrm{d}x\,\mathrm{d}y.\\
&\iint_{\mathbb{R}^2} \triangle_{0,y'}u\cdot u\,\mathrm{d}x\,\mathrm{d}y = -\iint_{\mathbb{R}^2}\nabla^{-}_{0,y'}u\cdot\nabla^{-}_{0,y'}v\,\mathrm{d}x\,\mathrm{d}y.\\
&\iint_{\mathbb{R}^2} \triangle_{x',y'}u\cdot v\,\mathrm{d}x\,\mathrm{d}y = \iint_{\mathbb{R}^2}(\triangle_{x',0}+\triangle_{0,y'})u\cdot v\,\mathrm{d}x\,\mathrm{d}y\\
&\qquad = -\iint_{\mathbb{R}^2}\left(\nabla^{-}_{x',0}u\nabla^{-}_{x',0}v+\nabla^{-}_{0,y'}u\nabla^{-}_{0,y'}v\right)\mathrm{d}x\,\mathrm{d}y. \qquad (21)
\end{aligned}
$$

Thus, taking v by u_t in (21) we have

$$
\begin{aligned}
\iint_{\mathbb{R}^2} \triangle_{x',y'}u\cdot u_t\,\mathrm{d}x\,\mathrm{d}y &= -\iint_{\mathbb{R}^2}\left(\nabla^{-}_{x',0}u\nabla^{-}_{x',0}u_t+\nabla^{-}_{0,y'}u\nabla^{-}_{0,y'}u_t\right)\mathrm{d}x\,\mathrm{d}y\\
&= -\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\iint_{\mathbb{R}^2}\left(\left|\nabla^{-}_{x',0}u\right|^2+\left|\nabla^{-}_{0,y'}u\right|^2\right)\mathrm{d}x\,\mathrm{d}y.
\end{aligned}
$$

Now let us consider the governing equation

$$
\begin{aligned}
\partial_{tt}u(x,y,t) &= \iint_{B_\delta(0)}\gamma_\delta(x',y')\Big(u(x+x',y+y',t)-u(x,y,t)\Big)\,\mathrm{d}x'\,\mathrm{d}y' + f(x,y,t)\\
&= \iint_{B^{+}_\delta(0)}\gamma_\delta(x',y')\triangle_{x',y'}u(x,y,t)\,\mathrm{d}x'\,\mathrm{d}y' + f(x,y,t). \qquad (22)
\end{aligned}
$$

By

$$
\begin{aligned}
u(x-h)-2u(x)+u(x+h) &= h^2\int_0^1\int_0^{\xi_1}\left(\frac{\mathrm{d}^2u}{\mathrm{d}x^2}(x+h\xi_2)+\frac{\mathrm{d}^2u}{\mathrm{d}x^2}(x-h\xi_2)\right)\mathrm{d}\xi_2\,\mathrm{d}\xi_1\\
&= h^2\int_0^1\int_0^{\xi_1}\left[2\frac{\mathrm{d}^2u}{\mathrm{d}x^2}(x)+h^2\xi_2^2\int_0^1\int_0^{\xi_3}\left(\frac{\mathrm{d}^4u}{\mathrm{d}x^4}(x+h\xi_2\xi_4)+\frac{\mathrm{d}^4u}{\mathrm{d}x^4}(x-h\xi_2\xi_4)\right)\mathrm{d}\xi_4\,\mathrm{d}\xi_3\right]\mathrm{d}\xi_2\,\mathrm{d}\xi_1, \qquad (23)
\end{aligned}
$$

one has

$$\left|\iint_{\mathbb{R}^2}\left[\iint_{B_\delta^+(0)}\gamma_\delta(x',y')\triangle_{x',0}u\cdot\partial_t u\,\mathrm{d}x'\,\mathrm{d}y'\right]\mathrm{d}x\,\mathrm{d}y\right|\le\iint_{B_\delta^+(0)}\gamma_\delta(x',y')\Big(\iint_{\mathbb{R}^2}|\triangle_{x',0}u\cdot\partial_t u|\,\mathrm{d}x\,\mathrm{d}y\Big)$$
$$\le\iint_{B_\delta^+(0)}\gamma_\delta(x',y')\Big(\iint_{\mathbb{R}^2}|\triangle_{x',0}u|^2\,\mathrm{d}x\,\mathrm{d}y\Big)^{\frac12}\Big(\iint_{\mathbb{R}^2}\partial_t u^2\,\mathrm{d}x\,\mathrm{d}y\Big)^{\frac12}$$
$$\le\iint_{B_\delta^+(0)}\gamma_\delta(x',y')(x')^2\,\mathrm{d}x'\,\mathrm{d}y'\Big(\|\partial_{xx}u(\cdot,\cdot,t)\|_{L^2}^2+\|\partial_{xxxx}u(\cdot,\cdot,t)\|_{L^2}^2\Big)^{\frac12}\|\partial_t u(\cdot,\cdot,t)\|_{L^2}$$
$$\le C\|u(\cdot,\cdot,t)\|_{H^4}\|\partial_t u(\cdot,\cdot,t)\|_{L^2}. \tag{24}$$

Summing up (22) multiplied by $\partial_t u$, and integrating the equality on $\mathbb{R}^2$, by (24) and dominated convergence theorem we obtain

$$\frac12\partial_t\iint_{\mathbb{R}^2}|\partial_t u|^2\,\mathrm{d}x\,\mathrm{d}y=\iint_{B_\delta^+(0)}\gamma_\delta(x',y')\,\mathrm{d}x'\,\mathrm{d}y'\left[\iint_{\mathbb{R}^2}\triangle_{x',y'}u\cdot\partial_t u\,\mathrm{d}x\,\mathrm{d}y\right]+\iint_{\mathbb{R}^2}f\partial_t u\,\mathrm{d}x\,\mathrm{d}y$$
$$=-\iint_{B_\delta^+(0)}\gamma_\delta(x',y')\,\mathrm{d}x'\,\mathrm{d}y'\iint_{\mathbb{R}^2}\Big(\nabla^-_{x',0}u\cdot\nabla^-_{x',0}\partial_t u+\nabla^-_{0,y'}u\cdot\nabla^-_{0,y'}\partial_t u\Big)\,\mathrm{d}x\,\mathrm{d}y+\iint_{\mathbb{R}^2}f\partial_t u\,\mathrm{d}x\,\mathrm{d}y$$
$$=-\frac12\partial_t\left[\iint_{B_\delta^+(0)}\gamma_\delta(x',y')\,\mathrm{d}x'\,\mathrm{d}y'\Big(\Big\|\nabla^-_{x',0}u(\cdot,\cdot,t)\Big\|_{L^2}^2+\Big\|\nabla^-_{0,y'}u(\cdot,\cdot,t)\Big\|_{L^2}^2\Big)\right]+\iint_{\mathbb{R}^2}f\partial_t u\,\mathrm{d}x\,\mathrm{d}y,$$

which leads to

$$\frac12\partial_t\left[\iint_{\mathbb{R}^2}|\partial_t u|^2\,\mathrm{d}x\,\mathrm{d}y+\iint_{B_\delta^+(0)}\gamma_\delta(x',y')\,\mathrm{d}x'\,\mathrm{d}y'\Big(\Big\|\nabla^-_{x',0}u(\cdot,\cdot,t)\Big\|_{L^2}^2+\Big\|\nabla^-_{0,y'}u(\cdot,\cdot,t)\Big\|_{L^2}^2\Big)\right]$$
$$\le\iint_{\mathbb{R}^2}|f|^2\,\mathrm{d}x\,\mathrm{d}y+\iint_{\mathbb{R}^2}|\partial_t u|^2\,\mathrm{d}x\,\mathrm{d}y.$$

By integral with respect to variable t, the above inequality will give us

$$\iint_{\mathbb{R}^2}|\partial_t u(\cdot,\cdot,t)|^2\,\mathrm{d}x\,\mathrm{d}y$$
$$\le\left[\iint_{\mathbb{R}^2}|\partial_t u(\cdot,\cdot,0)|^2\,\mathrm{d}x\,\mathrm{d}y+\iint_{B_\delta^+(0)}\gamma_\delta(x',y')\,\mathrm{d}x'\,\mathrm{d}y'\Big(\|\nabla_{x',0}u(\cdot,\cdot,0)\|_{L^2}^2+\|\nabla_{0,y'}u(\cdot,\cdot,0)\|_{L^2}^2\Big)\right]$$
$$+2\int_0^t\left[\iint_{\mathbb{R}^2}|f(\cdot,\cdot,s)|^2\,\mathrm{d}x\,\mathrm{d}y\right]\mathrm{d}s+2\int_0^t\left[\iint_{\mathbb{R}^2}|\partial_t u(\cdot,\cdot,s)|^2\,\mathrm{d}x\,\mathrm{d}y\right]\mathrm{d}s.$$

Then, by Gronwell's inequality one has

$$\iint_{\mathbb{R}^2}|\partial_t u(\cdot,\cdot,t)|^2\,\mathrm{d}x\,\mathrm{d}y$$
$$\le C\left[\iint_{\mathbb{R}^2}|\partial_t u(\cdot,\cdot,0)|^2\,\mathrm{d}x\,\mathrm{d}y+\iint_{B_\delta^+(0)}\gamma_\delta(x',y')\,\mathrm{d}x'\,\mathrm{d}y'\Big(\|\nabla_{x',0}u(\cdot,\cdot,0)\|_{L^2}^2+\|\nabla_{0,y'}u(\cdot,\cdot,0)\|_{L^2}^2\Big)\right.$$
$$\left.+\int_0^t\iint_{\mathbb{R}^2}|f|^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\right].$$

This gives

$$\iint_{\mathbb{R}^2}|\partial_t u|^2\,\mathrm{d}x\,\mathrm{d}y+\iint_{B_\delta^+(0)}\gamma_\delta(x',y')\,\mathrm{d}x'\,\mathrm{d}y'\Big(\Big\|\nabla^-_{x',0}u(\cdot,\cdot,t)\Big\|_{L^2}^2+\Big\|\nabla^-_{0,y'}u(\cdot,\cdot,t)\Big\|_{L^2}^2\Big) \tag{25}$$

$$
\begin{aligned}
&\le C \iint_{\mathbb{R}^2} |\partial_t u(\cdot,\cdot,0)|^2 \,\mathrm{d}x\,\mathrm{d}y + C\int_0^t \iint_{\mathbb{R}^2} |f|^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s \\
&\quad + C\iint_{B_\delta^+(0)} \gamma_\delta(x',y')\,\mathrm{d}x'\,\mathrm{d}y'\Big(\big\|\nabla^-_{x',0}u(\cdot,\cdot,0)\big\|^2_{L^2} + \big\|\nabla^-_{0,y'}u(\cdot,\cdot,0)\big\|^2_{L^2}\Big) \\
&\le C \iint_{\mathbb{R}^2} |\partial_t u(\cdot,\cdot,0)|^2 \,\mathrm{d}x\,\mathrm{d}y + C\int_0^t \iint_{\mathbb{R}^2} |f|^2 \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s \\
&\quad + C\iint_{B_\delta^+(0)} \gamma_\delta(x',y')\big((x')^2+(y')^2\big)\,\mathrm{d}x'\,\mathrm{d}y'\Big(\|\partial_x u(\cdot,\cdot,0)\|^2_{L^2} + \|\partial_y u(\cdot,\cdot,0)\|^2_{L^2}\Big) \\
&\le C\Big(\big\|\phi^1\big\|^2_{L^2} + \big\|\phi^0\big\|^2_{H^1} + \int_0^t \iint_{\mathbb{R}^2} |f|^2\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}s\Big).
\end{aligned}
$$

By the same manner, taking $\partial_t^{\ell-1}\partial_x^p\partial_y^q$ on the both sides of (22) one has

$$
\partial_{tt}\big(\partial_t^{\ell-1}\partial_x^p\partial_y^q u(x,y,t)\big) = \iint_{B_\delta^+(0)} \gamma_\delta(x',y')\triangle_{x',y'}\partial_t^{\ell-1}\partial_x^p\partial_y^q u(x,y,t)\,\mathrm{d}x'\,\mathrm{d}y' + \partial_t^{\ell-1}\partial_x^p\partial_y^q f(x,y,t). \tag{26}
$$

Multiplying (26) by $\partial_t^\ell\partial_x^p\partial_y^q u$ and integrating it on $\mathbb{R}^2$, one has

$$
\begin{aligned}
&\big\|\partial_t^\ell\partial_x^p\partial_y^q u(\cdot,\cdot,T)\big\|^2_{L^2} + \iint_{B_\delta^+(0)} \gamma_\delta(x',y')\,\mathrm{d}x'\,\mathrm{d}y'\Big(\big\|\nabla^-_{x',0}\partial_t^{\ell-1}\partial_x^p\partial_y^q u(\cdot,\cdot,T)\big\|^2_{L^2} + \big\|\nabla^-_{0,y'}\partial_t^{\ell-1}\partial_x^p\partial_y^q u(\cdot,\cdot,T)\big\|^2_{L^2}\Big) \\
&\qquad \le C\big\|\partial_t^\ell\partial_x^p\partial_y^q u(\cdot,\cdot,0)\big\|^2_{L^2} + C\int_0^t \big\|\partial_t^{\ell-1}\partial_x^p\partial_y^q f(\cdot,\cdot,s)\big\|^2_{L^2}\,\mathrm{d}s \\
&\qquad\quad + C\Big(\big\|\partial_t^{\ell-1}\partial_x^{p+1}\partial_y^q u(\cdot,\cdot,0)\big\|^2_{L^2} + \big\|\partial_t^{\ell-1}\partial_x^p\partial_y^{q+1}u(\cdot,\cdot,0)\big\|^2_{L^2}\Big).
\end{aligned} \tag{27}
$$

Further more, taking $\partial_t^{\ell-2}\partial_x^p\partial_y^q$ on the both sides of (22) for $\ell \ge 2$, we derive

$$
\big\|\partial_t^\ell\partial_x^p\partial_y^q u(\cdot,\cdot,0)\big\|^2_{L^2} \le C\Big(\big\|\partial_t^{\ell-2}\partial_x^{p+2}\partial_y^q u(\cdot,\cdot,0)\big\|^2_{L^2} + \big\|\partial_t^{\ell-2}\partial_x^p\partial_y^{q+2}u(\cdot,\cdot,0)\big\|^2_{L^2}\Big) + C\big\|\partial_t^{\ell-2}\partial_x^p\partial_y^q f(\cdot,\cdot,0)\big\|^2_{L^2}.
$$

By induction one has

$$
\big\|\partial_t^\ell\partial_x^p\partial_y^q u(\cdot,\cdot,0)\big\|^2_{L^2} \le C\sum_{0\le c\le p+\ell,\,0\le d\le q+\ell}\Big(\big\|\partial_x^c\partial_y^d u^0\big\|^2_{L^2} + \big\|\partial_x^c\partial_y^d u^1\big\|^2_{L^2}\Big) + C\sum_{0\le c\le \ell}\big\|\partial_t^c\partial_x^p\partial_y^q f(\cdot,\cdot,0)\big\|^2_{L^2}.
$$

Taking the above estimate into (27), one derives

$$
\begin{aligned}
&\big\|\partial_t^\ell\partial_x^p\partial_y^q u(\cdot,\cdot,T)\big\|^2_{L^2} + \iint_{B_\delta^+(0)} \gamma_\delta(x',y')\,\mathrm{d}x'\,\mathrm{d}y'\Big(\big\|\nabla^-_{x',0}\partial_t^{\ell-1}\partial_x^p\partial_y^q u(\cdot,\cdot,T)\big\|^2_{L^2} + \big\|\nabla^-_{0,y'}\partial_t^{\ell-1}\partial_x^p\partial_y^q u(\cdot,\cdot,T)\big\|^2_{L^2}\Big) \\
&\qquad \le C\sum_{0\le c\le p+\ell,\,0\le d\le q+\ell}\Big(\big\|\partial_x^c\partial_y^d u^0\big\|^2_{L^2} + \big\|\partial_x^c\partial_y^d u^1\big\|^2_{L^2}\Big) + C\int_0^t \big\|\partial_t^{\ell-1}\partial_x^p\partial_y^q f(\cdot,\cdot,s)\big\|^2_{L^2}\,\mathrm{d}s \\
&\qquad\quad + C\sum_{0\le c\le \ell,\,0\le d\le p+1,\,0\le e\le q+1}\big\|\partial_t^c\partial_x^d\partial_y^e f(\cdot,\cdot,0)\big\|^2_{L^2},
\end{aligned}
$$

which gives (18).

In addition, we have the following equality,

$$
\partial_x^p\partial_y^q u(x,y,t) = \int\int_{\mathbb{R}^2} e^{2\pi i x\xi_1 + 2\pi i y\xi_2}\,\widehat{\partial_x^p\partial_y^q u}(\xi_1,\xi_2,t)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2,
$$

where $\widehat{u}(\xi_1, \xi_2, t)$ is the Fourier transform of $u(x, y, t)$ with respect to the variables x and y such that

$$\widehat{u}(\xi_1, \xi_2, t) = \int\int_{\mathbb{R}^2} e^{-2\pi i x\xi_1 - 2\pi i y\xi_2} u(x, y, t)\,\mathrm{d}x\,\mathrm{d}y.$$

This gives us

$$\begin{aligned}\left\|\partial_x^p\partial_y^q u(\cdot,\cdot,t)\right\|_{L^\infty} &\leq \iint_{\mathbb{R}^2}\left|\widehat{\partial_x^p\partial_y^q u}(\xi_1,\xi_2,t)\right|\mathrm{d}\xi_1\,\mathrm{d}\xi_2\\ &= \iint_{\mathbb{R}^2}\left(1+|\xi_1|^2+|\xi_2|^2\right)^{-1}\left(1+|\xi_1|^2+|\xi_2|^2\right)\left|\widehat{\partial_x^p\partial_y^q u}(\xi_1,\xi_2,t)\right|\mathrm{d}\xi_1\,\mathrm{d}\xi_2\\ &\leq C\Big(\left\|\partial_x^p\partial_y^q u(\cdot,\cdot,t)\right\|_{L^2}+\left\|\partial_x^{p+2}\partial_y^q u(\cdot,\cdot,t)\right\|_{L^2}+\left\|\partial_x^p\partial_y^{q+2}u(\cdot,\cdot,t)\right\|_{L^2}\Big),\end{aligned}$$

which proves (19). □

3.2. Truncated error estimates

Let us introduce $e_{j,\ell}^n$ such that

$$e_{j,\ell}^n = D_\tau^- D_\tau^+ u_{j,\ell}(t_n) - L_{\delta,h}E^-E^+u_{j,\ell}(t_n) - f_{j,\ell}(t_n), \tag{28}$$

for $(j,\ell)\in\mathbb{Z}^2$ with $u_{j,\ell}(t) = u(jh,\ell h,t)$ where $u(x,y,t)$ designates the solution to (1). The operator $L_{\delta,h}$ is introduced in (7). For a two-dimensional sequence $\{u_{j,\ell}(t)\}_{j,\ell}$, we can define the discretized forward derivative operator D_x^+ such that $D_x^+u_{j,\ell}(t) := (u_{j+1,\ell}(t) - u_{j,\ell}(t))/h$ and the discretized backward derivative operator D_x^- can be defined as $D_x^-u_{j,\ell}(t) := (u_{j,\ell}(t) - u_{j-1,\ell}(t))/h$. In the same manner, the operators D_y^- and D_y^+ can be also defined. The operator D_x^+ can also act on numerical solution $u_{j,\ell}^n$ such that $D_x^+u_{j,\ell}^n = (u_{j+1,\ell}^n - u_{j,\ell}^n)/h$. The notations $D_x^-u_{j,\ell}^n$, $D_y^+u_{j,\ell}^n$ and $D_y^-u_{j,\ell}^n$ work in the same way. Then, the following technical lemma holds.

Lemma 4. *Let us assume that $\|u(\cdot,\cdot,t)\|_{H^P}\leq C$ with $t\leq T$ and P large enough. For $N\tau\leq T$, we have*

$$h^2\tau\sum_{n=0}^{N-1}\sum_{(j,\ell)\in\mathbb{Z}^2}\left|e_{j,\ell}^n\right|^2\leq C\left(h^2+\tau^2\right)^2, \tag{29}$$

$$h^2\tau\sum_{n=0}^{N-1}\sum_{(j,\ell)\in\mathbb{Z}^2}\left|D_x^\pm e_{j,\ell}^n\right|^2\leq C\left(h^2+\tau^2\right)^2, \tag{30}$$

$$h^2\tau\sum_{n=0}^{N-1}\sum_{(j,\ell)\in\mathbb{Z}^2}\left|D_y^\pm e_{j,\ell}^n\right|^2\leq C\left(h^2+\tau^2\right)^2, \tag{31}$$

$$h^2\tau\sum_{n=0}^{N-1}\sum_{(j,\ell)\in\mathbb{Z}^2}\left|D_x^\pm D_y^\pm e_{j,\ell}^n\right|^2\leq C\left(h^2+\tau^2\right)^2, \tag{32}$$

where C depends only on T, γ_δ and initial data ϕ^0, ϕ^1.

Proof. It is well known that Taylor's expansions

$$\begin{aligned}u(x,y,t+\tau) &= u(x,y,t)+\partial_t u(x,y,t)\tau+\int_0^\tau\partial_t^2u(x,y,t+\varsigma)(\tau-\varsigma)\,\mathrm{d}\varsigma\\ &= u(x,y,t)+\partial_t u(x,y,t)\tau+\tau^2\int_0^1\partial_t^2u(x,y,t+\tau\varsigma)(1-\varsigma)\,\mathrm{d}\varsigma,\end{aligned}$$

$$u(x,y,t-\tau) = u(x,y,t) - \partial_t u(x,y,t)\tau + \tau^2 \int_0^1 \partial_t^2 u(x,y,t-\tau\varsigma)(1-\varsigma)\,\mathrm{d}\varsigma, \tag{33}$$

and

$$\begin{aligned}
u(x,y,t+\tau) &= u(x,y,t) + \partial_t u(x,y,t)\tau + \frac{1}{2}\partial_t^2 u(x,y,t)\tau^2 + \frac{1}{6}\partial_t^3 u(x,y,t)\tau^3 + \frac{\tau^4}{6}\int_0^1 \partial_t^4 u(x,y,t+\tau\varsigma)(1-\varsigma)^3\,\mathrm{d}\varsigma,\\
u(x,y,t-\tau) &= u(x,y,t) - \partial_t u(x,y,t)\tau + \frac{1}{2}\partial_t^2 u(x,y,t)\tau^2 - \frac{1}{6}\partial_t^3 u(x,y,t)\tau^3\\
&\quad + \frac{\tau^4}{6}\int_0^1 \partial_t^4 u(x,y,t-\tau\varsigma)(1-\varsigma)^3\,\mathrm{d}\varsigma.
\end{aligned} \tag{34}$$

Applying (33) with $t = t_n = n\tau$, we have

$$\begin{aligned}
E^- E^+ u(x,y,t_n) &= \frac{u(x,y,t_n-\tau) + 2u(x,y,t_n) + u(x,y,t_n+\tau)}{4}\\
&= u(x,y,t_n) + \tau^2 U_1\big(x,y,t_{n+1/2}\big),
\end{aligned} \tag{35}$$

with

$$U_1(x,y,t) = \frac{1}{4}\int_0^1 \big(\partial_t^2 u(x,y,t-\tau\varsigma) + \partial_t^2 u(x,y,t+\tau\varsigma)\big)(1-\varsigma)\,\mathrm{d}\varsigma.$$

In the same way by using (34) one has

$$\begin{aligned}
D_\tau^- D_\tau^+ u(x,y,t_n) &= \frac{u(x,y,t_n-\tau) - 2u(x,y,t_n) + u(x,y,t_n+\tau)}{\tau^2}\\
&= \partial_{tt} u(x,y,t_n) + \tau^2 U_2(x,y,t_n),
\end{aligned} \tag{36}$$

with

$$U_2(x,y,t) = \frac{1}{6}\int_0^1 \big(\partial_t^4 u(x,y,t+\tau\varsigma) + \partial_t^4 u(x,y,t-\tau\varsigma)\big)(1-\varsigma)^3\,\mathrm{d}\varsigma.$$

Recalling [13]

$$|\mathcal{L}_\delta u(x,y,t) - L_{\delta,h} u(x,y,t))| \le Ch^2 \|u(\cdot,\cdot,t)\|_{C^4(\mathcal{R}_\delta(x,y))},$$

we have

$$\begin{aligned}
\big|e_{j,\ell}^n\big| &= \big|D_\tau^- D_\tau^+ u_{j,\ell}(t_n) - L_{\delta,h} E^- E^+ \psi_{j,\ell}(t_n) - f_{j,\ell}(t^n)\big|\\
&= \big|\partial_{tt} u(x_j,y_\ell,t_n) - L_{\delta,h} u(x_j,y_\ell,t_n) - f_{j,\ell}(t_n) + \tau^2 \partial_{tt} U_1(x_j,y_\ell,t_n) - \tau^2 L_{\delta,h} U_2(x_j,y_\ell,t_n)\big|\\
&\le |\partial_{tt} u(x_j,y_\ell,t_n) - L_\delta u(x_j,y_\ell,t_n) - f(x_j,y_\ell,t_n)| + |\mathcal{L}_{\delta,h} u(x_j,y_\ell,t_n) - \mathcal{L}_\delta u(x_j,y_\ell,t_n)|\\
&\quad + C\tau^2 \|U_1(\cdot,\cdot,t_n)\|_{C^1(\mathcal{R}_h(x_j,y_\ell))} + \tau^2 |\mathcal{L}_{\delta,h} U_2(x_j,y_\ell,t_n) - \mathcal{L}_\delta U_2(x_j,y_\ell,t_n)|\\
&\quad + C\tau^2 \|U_2(\cdot,\cdot,t_n)\|_{C^2(\mathcal{R}_h(x_j,y_\ell))}\\
&\le Ch^2 \|u(\cdot,\cdot,t_n)\|_{C^4(\mathcal{R}_h(x_j,y_\ell))} + C\tau^2 \|u\|_{C^6(\mathcal{R}_h(x_j,y_\ell)\times[t_{n-1},t_{n+1}])} + Ch^2\tau^2 \|U_2(\cdot,\cdot,t_n)\|_{C^4(\mathcal{R}_h(x_j,y_\ell))}\\
&\le C\big(h^2+\tau^2\big)\|u\|_{C^8(\mathcal{R}_\delta(x_j,y_\ell)\times[t_{n-1},t_{n+1}])}.
\end{aligned} \tag{37}$$

For any function $q(x,y,t)$ smooth enough, there exists $(\tilde{x},\tilde{y},\tilde{t}) \in \mathcal{R}_\delta(x_j,y_\ell)\times[t_n,t_{n+1}]$ (recalling $\mathcal{R}_\delta(x_j,y_\ell)$ defined in Sect. 2.2) such that

$$\begin{aligned}
\|q(x,y,t)\|_{C(\mathcal{R}_\delta(x_j,y_\ell)\times[t_n,t_{n+1}])} &= \big|q\big(\tilde{x},\tilde{y},\tilde{t}\big)\big|\\
&= \Bigg| q\big(x_j,y_\ell,t_{n+1/2}\big) + \int_{x_j}^{\tilde{x}} \partial_x q\big(v,y_\ell,t_{n+1/2}\big)\,\mathrm{d}v + \int_{y_\ell}^{\tilde{y}} \partial_y q\big(\tilde{x},w,t_{n+1/2}\big)\,\mathrm{d}w
\end{aligned}$$

$$+\int_{t_{n+1/2}}^{\tilde t}\partial_t q(\tilde x,\tilde y,t_1)\,\mathrm{d}t_1\Big|. \tag{38}$$

On the right hand side of (38), the term $\int_{y_\ell}^{\tilde y}\partial_y q(\tilde x,w,t_{n+1/2})\,\mathrm{d}w$ can be written as

$$\int_{y_\ell}^{\tilde y}\partial_y q\big(\tilde x,w,t_{n+1/2}\big)\,\mathrm{d}w=\int_{y_\ell}^{\tilde y}\partial_y q\big(x_j,w,t_{n+1/2}\big)\,\mathrm{d}w+\int_{x_j}^{\tilde x}\int_{y_\ell}^{\tilde y}\partial_x\partial_y q\big(v,w,t_{n+1/2}\big)\,\mathrm{d}v\,\mathrm{d}w,$$

and the term $\int_{t_{n+1/2}}^{\tilde t}\partial_t q(\tilde x,\tilde y,t_1)\,\mathrm{d}t_1$ can be written as

$$\begin{aligned}\int_{t_{n+1/2}}^{\tilde t}\partial_t q(\tilde x,\tilde y,t_1)\,\mathrm{d}t_1&=\int_{t_{n+1/2}}^{\tilde t}\partial_t q(x_j,\tilde y,t_1)\,\mathrm{d}t_1+\int_{t_{n+1/2}}^{\tilde t}\int_{x_j}^{\tilde x}\partial_t\partial_x q(v,\tilde y,t_1)\,\mathrm{d}v\,\mathrm{d}t_1\\&=\int_{t_{n+1/2}}^{\tilde t}\partial_t q(x_j,y_\ell,t_1)\,\mathrm{d}t_1+\int_{t_{n+1/2}}^{\tilde t}\int_{y_\ell}^{\tilde y}\partial_t\partial_y q(x_j,w,t_1)\,\mathrm{d}t_1\,\mathrm{d}w\\&\quad+\int_{t_{n+1/2}}^{\tilde t}\int_{x_j}^{\tilde x}\partial_t\partial_x q(v,y_\ell,t_1)\,\mathrm{d}v\,\mathrm{d}t_1+\int_{t_{n+1/2}}^{\tilde t}\int_{x_j}^{\tilde x}\int_{y_\ell}^{\tilde y}\partial_t\partial_x\partial_y q(v,w,t_1)\,\mathrm{d}v\,\mathrm{d}w\,\mathrm{d}t_1.\end{aligned}$$

Thus, summing up the square of the first two terms in the right hand side of (38) with $(j,\ell)\in\mathbb{Z}^2$ we obtain

$$\begin{aligned}&h^2\sum_{(j,\ell)\in\mathbb{Z}^2}\left|q\big(x_j,y_\ell,t_{n+1/2}\big)+\int_{x_j}^{\tilde x}\partial_x q\big(v,y_\ell,t_{n+1/2}\big)\,\mathrm{d}v\right|^2\\&\le Ch^2\sum_{(j,\ell)\in\mathbb{Z}^2}\big|q\big(x_j,y_\ell,t_{n+1/2}\big)\big|^2+Ch\sum_{j\in\mathbb{Z}}\int_{x_j-\delta}^{x_j+\delta}\mathrm{d}v\left(h\sum_{\ell\in\mathbb{Z}}\big|\partial_x q\big(v,y_\ell,t_{n+1/2}\big)\big|^2\right)\\&\le C\iint_{\mathbb{R}^2}\Big|q\Big(\cdot,\cdot,t_{n+\frac12}\Big)\Big|^2\,\mathrm{d}x\,\mathrm{d}y+Ch\sum_{j\in\mathbb{Z}}\int_{x_j-\delta}^{x_j+\delta}\mathrm{d}v\left(\int_{-\infty}^{\infty}\big|\partial_x q\big(v,\cdot,t_{n+1/2}\big)\big|^2\,\mathrm{d}y\right)\\&\le C\iint_{\mathbb{R}^2}\Big|q\Big(\cdot,\cdot,t_{n+\frac12}\Big)\Big|^2\,\mathrm{d}x\,\mathrm{d}y+C\int_{-\infty}^{\infty}\left(\int_{-\delta}^{\delta}h\sum_{j\in\mathbb{Z}}\big|\partial_x q\big(x_j+v,\cdot,t_{n+1/2}\big)\big|^2\,\mathrm{d}v\right)\mathrm{d}y\\&\le C\iint_{\mathbb{R}^2}\left(\Big|q\Big(\cdot,\cdot,t_{n+\frac12}\Big)\Big|^2+\Big|\partial_x q\Big(\cdot,\cdot,t_{n+\frac12}\Big)\Big|^2\right)\mathrm{d}x\,\mathrm{d}y.\end{aligned} \tag{39}$$

In the same manner, the square summation of the third term in the right hand side of (38) with $(j,\ell)\in\mathbb{Z}^2$ satisfies the following estimate,

$$\begin{aligned}h^2\sum_{(j,\ell)\in\mathbb{Z}^2}\left|\int_{y_\ell}^{\tilde y}\partial_y q\big(\tilde x,w,t_{n+1/2}\big)\,\mathrm{d}w\right|^2&\le Ch^2\sum_{(j,\ell)\in\mathbb{Z}^2}\int_{x_j-\delta}^{x_j+\delta}\int_{y_\ell-\delta}^{y_\ell+\delta}\big|\partial_x\partial_y q\big(v,w,t_{n+1/2}\big)\big|^2\,\mathrm{d}v\,\mathrm{d}w\\&\quad+Ch\sum_{\ell\in\mathbb{Z}}\int_{y_\ell-\delta}^{y_\ell+\delta}\mathrm{d}w\left(h\sum_{j\in\mathbb{Z}}\big|\partial_y q\big(x_j,w,t_{n+1/2}\big)\big|^2\right)\\&\le C\int_{-\delta}^{\delta}\int_{-\delta}^{\delta}\left(h^2\sum_{(j,\ell)\in\mathbb{Z}^2}\big|\partial_x\partial_y q\big(x_j+v,y_\ell+w,t_{n+1/2}\big)\big|^2\right)\mathrm{d}v\,\mathrm{d}w\end{aligned}$$

$$+C\int_{-\delta}^{\delta} \mathrm{d}w \int_{-\mathbb{R}}^{\mathbb{R}} \mathrm{d}v \left(h \sum_{\ell\in\mathbb{Z}} \left| \partial_y q\big(v, y_\ell + w, t_{n+1/2}\big) \right|^2 \right)$$
$$\leq C \iint_{\mathbb{R}^2} \left(\left| \partial_y q\Big(\cdot,\cdot,t_{n+\frac{1}{2}}\Big)\right|^2 + \left| \partial_x\partial_y q\Big(\cdot,\cdot,t_{n+\frac{1}{2}}\Big)\right|^2 \right) \mathrm{d}x\,\mathrm{d}y, \tag{40}$$

and the square summation of the fourth term in the right hand side of (38) with $(j,\ell) \in \mathbb{Z}^2$ satisfies the following estimate,

$$h^2 \sum_{(j,\ell)\in\mathbb{Z}^2} \left| \int_{t_{n+1/2}}^{\tilde{t}} \partial_t q(\tilde{x},\tilde{y},t_1)\,\mathrm{d}t_1 \right|^2 \leq \iint_{\mathbb{R}^2} \left(|\partial_t q|^2 + |\partial_t\partial_x q|^2 + |\partial_t\partial_y q|^2 + |\partial_t\partial_x\partial_y q|^2 \right) \mathrm{d}x\,\mathrm{d}y. \tag{41}$$

Combining the estimates (39)–(41) one derives

$$h^2 \sum_{(j,\ell)\in\mathbb{Z}^2} \|q(x,y,t)\|^2_{C(\mathcal{R}_\delta(x_j,y_\ell)\times[t_n,t_{n+1}])} \leq C \iint_{\mathbb{R}^2} \sum_{\alpha_1+\alpha_2\leq 2} \left|\partial_x^{\alpha_1}\partial_y^{\alpha_2} q(\cdot,\cdot,t_{n+1/2})\right|^2 \mathrm{d}x\,\mathrm{d}y$$
$$+ C\int_{t_n}^{t_{n+1}} \left(\iint_{\mathbb{R}^2} \sum_{\alpha_1+\alpha_2+\alpha_3\leq 3} \left|\partial_x^{\alpha_1}\partial_y^{\alpha_2}\partial_t^{\alpha_3} q\right|^2 \mathrm{d}x\,\mathrm{d}y \right) \mathrm{d}t. \tag{42}$$

By (37) and (42), finally we get

$$h^2\tau \sum_{n=0}^{N-1} \sum_{(j,\ell)\in\mathbb{Z}^2} \left|e^n_{j,\ell}\right|^2 \leq C\big(h^2+\tau^2\big)^2 \tau \sum_{n=0}^{N-1} \left(h^2 \sum_{(j,\ell)\in\mathbb{Z}^2} \|u\|^2_{C^8(\mathcal{R}_\delta(x_j,y_\ell)\times[t_n,t_{n+1}])} \right)$$
$$\leq C\big(h^2+\tau^2\big)^2 \tau \sum_{n=0}^{N-1} \left(\iint_{\mathbb{R}^2} \sum_{\alpha_1+\alpha_2\leq 10} \left|\partial_x^{\alpha_1}\partial_y^{\alpha_2} u(\cdot,\cdot,t_{n+1/2})\right|^2 \mathrm{d}x\,\mathrm{d}y \right.$$
$$\left. + \int_{t_n}^{t_{n+1}} \left(\iint_{\mathbb{R}^2} \sum_{\alpha_1+\alpha_2+\alpha_3\leq 11} \left|\partial_x^{\alpha_1}\partial_y^{\alpha_2}\partial_t^{\alpha_3} u\right|^2 \mathrm{d}x\,\mathrm{d}y \right) \mathrm{d}t \right)$$
$$\leq C\big(h^2+\tau^2\big)^2 \max_{t\in[0,T]} \sum_{\alpha_1+\alpha_2+\alpha_3\leq 11} \iint_{\mathbb{R}^2} \left|\partial_x^{\alpha_1}\partial_y^{\alpha_2}\partial_t^{\alpha_3} u(\cdot,\cdot,t)\right|^2 \mathrm{d}x\,\mathrm{d}y.$$

Due to Lemma 3, above estimate implies (29).

By the same procedure, one has

$$\left|D_x^+ e^n_{j,\ell}\right| \leq C\big(h^2+\tau^2\big)\|u\|_{C^9(\mathcal{R}_\delta(x_j,y_\ell)\times[t_n,t_{n+1}])},$$
$$\left|D_y^+ e^n_{j,\ell}\right| \leq C\big(h^2+\tau^2\big)\|u\|_{C^9(\mathcal{R}_\delta(x_j,y_\ell)\times[t_n,t_{n+1}])},$$
$$\left|D_x^+ D_y^+ e^n_{j,\ell}\right| \leq C\big(h^2+\tau^2\big)\|u\|_{C^{10}(\mathcal{R}_\delta(x_j,y_\ell)\times[t_n,t_{n+1}])}.$$

Therefore, we obtain

$$\sum_{n=0}^{N-1} \sum_{(j,\ell)\in\mathbb{Z}^2} \left|D_x^+ e^n_{j,\ell}\right|^2 h^2\tau \leq C\big(h^2+\tau^2\big)^2 \max_{t\in[0,T]} \iint_{\mathbb{R}^2} \sum_{\alpha_1+\alpha_2+\alpha_3\leq 12} \left|\partial_x^{\alpha_1}\partial_y^{\alpha_2}\partial_t^{\alpha_3} u(\cdot,\cdot,t)\right|^2 \mathrm{d}x\,\mathrm{d}y,$$
$$\sum_{n=0}^{N-1} \sum_{(j,\ell)\in\mathbb{Z}^2} \left|D_y^+ e^n_{j,\ell}\right|^2 h^2\tau \leq C\big(h^2+\tau^2\big)^2 \max_{t\in[0,T]} \iint_{\mathbb{R}^2} \sum_{\alpha_1+\alpha_2+\alpha_3\leq 12} \left|\partial_x^{\alpha_1}\partial_y^{\alpha_2}\partial_t^{\alpha_3} u(\cdot,\cdot,t)\right|^2 \mathrm{d}x\,\mathrm{d}y,$$

$$\sum_{n=0}^{N-1} \sum_{(j,\ell)\in\mathbb{Z}^2} \left|D_x^+ D_y^+ e_{j,\ell}^n\right|^2 h^2\tau \le C\left(h^2+\tau^2\right)^2 \max_{t\in[0,T]} \iint_{\mathbb{R}^2} \sum_{\alpha_1+\alpha_2+\alpha_3\le 13} \left|\partial_x^{\alpha_1}\partial_y^{\alpha_2}\partial_t^{\alpha_3} u(\cdot,\cdot,t)\right|^2 \mathrm{d}x\,\mathrm{d}y,$$

which leads to (30)–(32) by Lemma 3. □

3.3. Error estimates of the scheme (7) with boundary condition (9)

Let us now give error estimates for the proposed scheme based on the solution of (7) and boundary condition (9). For the sequence $\{u_{\nu(j)}^n\}_n$ such that $u_{\nu(j)}^0 = 0$ and $u_{\nu(j)}^1 = 0$ with $1 \le j \le |\mathcal{B}_{\mathrm{Int}}|$, by Theorem 1 one can find $|\mathcal{B}_{\mathrm{Int}}|$ sequences $\{v_k^n\}_n$, with $1 \le k \le |\mathcal{B}_{\mathrm{Int}}|$, such that

$$u_{\nu(j)}^n = \sum_{k=1}^{|\mathcal{B}_{\mathrm{Int}}|} \left(f_{\nu(j)-\nu(k)} \star v_{\nu(k)}\right)^n, \quad 1 \le j \le |\mathcal{B}_{\mathrm{Int}}|, \tag{43}$$

which is an equivalent form of boundary condition (9). On the other hand, from Lemma 2 the solution $u_{j,\ell}^n$ of the infinite discretized system (7) with $(j,\ell) \in \mathbb{Z}^2 \backslash \mathcal{R}_J(0,0)$, can be defined by

$$u_{j,l}^n = \sum_{k=1}^{|\mathcal{B}_{\mathrm{Int}}|} \left(f_{(j,l)-\nu(k)} \star v_{\nu(k)}\right)^n. \tag{44}$$

Therefore, the $u_{j,\ell}^n$ solved by (7) with the boundary condition (9) and the $u_{j,\ell}^n$ represented by (44) satisfy $D_\tau^- D_\tau^+ u_{j,\ell}^n = L_{\delta,h} E^- E^+ u_{j,\ell}^n + f_{j,\ell}^n$ with $(j,\ell) \in \mathbb{Z}^2$. Now, let us consider $w_{j,\ell}^n = u_{j,\ell}^n - u(x_j, y_\ell, t_n)$ where $u(x_j, y_\ell, t_n)$ is the solution to peridynamics (1) evaluated at (x_j, y_ℓ, t_n). Thus, for $(j,\ell) \in \mathbb{Z}^2$, we have

$$D_\tau^- D_\tau^+ w_{j,\ell}^n = L_{\delta,h} E^- E^+ w_{j,\ell}^n + e_{j,\ell}^n, \tag{45}$$

where $e_{j,\ell}^n$ has been introduced in (28). In addition, let us define the norm: $\|\mathbf{w}^n\|_{\ell^\infty} := \max_{(j,\ell)\in\mathbb{Z}^2} |w_{j,\ell}^n|$ and $\|\mathbf{w}^n\|_{\ell^2} := (h^2 \sum_{(j,\ell)\in\mathbb{Z}^2} |w_{j,\ell}^n|^2)^{1/2}$.

Theorem 2. *Let us assume that the solution* u *to* (1) *satisfies* $\|u(\cdot,\cdot,t)\|_{H^P} \le C$ *for* $t \le T$ *with* P *large enough, and a singularity behaviour* $\gamma_\delta(a) \sim |a|^{-2-2\alpha}$ *around* $|a| = 0$, $0 < \alpha < 1$. *We also assume* $u_{j,\ell}^n$ *is solution to* (7) *with boundary conditions* (9). *For small* $\tau > 0$ *and* $N\tau \le T$, *we have the error bound*

$$\left\|\mathbf{w}^N\right\|_{\ell^\infty} \le C\left(\tau^2 + h^2\right), \tag{46}$$

for constant $C > 0$ *that only depends on* γ_δ, δ, T, *and initial data* ϕ^0, ϕ^1.

Proof. Let us remark that: $D_\tau^+ D_\tau^- = (D_\tau^+ - D_\tau^-)/\tau$, $(E^+ - E^-)/\tau = (D_\tau^+ + D_\tau^-)/2$ and $E^+E^- = (E^+ + E^-)/2$. By (45), $\{w_{j,\ell}^n\}_n$, for $(j,\ell) \in \mathbb{Z}^2$, satisfies the following fully discretized peridynamics system

$$\begin{aligned}\left(D_\tau^+ - D_\tau^-\right) w_{j,\ell}^n/\tau = D_\tau^+ D_\tau^- w_{j,\ell}^n &= \sum_{(k,m)\in\mathcal{R}_K(0,0)} a_{k,m} E^+ E^- w_{j+k,\ell+m}^n + e_{j,\ell}^n \\ &= \sum_{(k,m)\in\mathcal{R}_K(0,0)} a_{k,m}\left(E^+ + E^-\right) w_{j+k,\ell+m}^n/2 + e_{j,\ell}^n.\end{aligned}$$

Multiplying the above equality by $(D_\tau^+ + D_\tau^-) w_{j,\ell}^n$, and summing up all the terms for $(j,\ell) \in \mathcal{R}_M(0,0)$ where M is an integer such that $M > J$, we get

$$\begin{aligned}
&\sum_{\mathcal{R}_M(0,0)} \big((D_\tau^+ - D_\tau^-)w_{j,\ell}^n\big)\big(D_\tau^+ + D_\tau^-\big)w_{j,\ell}^n/\tau \\
&\quad = \sum_{\mathcal{R}_M(0,0)} \sum_{(k,m)\in\mathcal{R}_K(0,0)} a_{k,m}\big((E^+ + E^-)w_{j+k,\ell+m}^n\big)\big(D_\tau^+ + D_\tau^-\big)w_{j,\ell}^n/2 + \sum_{\mathcal{R}_M(0,0)} T_{j,\ell}^n \\
&\quad = \sum_{\mathcal{R}_M(0,0)} \sum_{(k,m)\in\mathcal{R}_K(0,0)} a_{k,m}\big((E^+ + E^-)w_{j+k,\ell+m}^n\big)\big(E^+ - E^-\big)w_{j,\ell}^n/(2\tau) + \sum_{\mathcal{R}_M(0,0)} T_{j,\ell}^n, \quad (47)
\end{aligned}$$

with $T_{j,\ell}^n = e_{j,\ell}^n\Big((D_\tau^+ + D_\tau^-)w_{j,\ell}^n\Big)$. Thus, equation (47) can be written as

$$\begin{aligned}
&2\sum_{\mathcal{R}_M(0,0)} \frac{\Big|(D_\tau^+)w_{j,\ell}^n\Big|^2 - \Big|(D_\tau^-)w_{j,\ell}^n\Big|^2}{\tau} \\
&= \sum_{(k,m)\in\mathcal{R}_K(0,0)} a_{k,m} \sum_{\mathcal{R}_M(0,0)} \big((E^+ + E^-)w_{j+k,\ell+m}^n\big)\big(E^+ - E^-\big)w_{j,\ell}^n/\tau + \sum_{\mathcal{R}_M(0,0)} T_{j,\ell}^n \\
&= \sum_{(k,m)\in\mathcal{R}_K(0,0)} a_{k,m} \sum_{\mathcal{R}_M(0,0)} \big(E^+ w_{j+k,\ell+m}^n E^+ w_{j,\ell}^n - E^- w_{j+k,\ell+m}^n E^- w_{j,\ell}^n\big)/\tau \\
&\quad + \sum_{(k,m)\in\mathcal{R}_K(0,0)} a_{k,m} \sum_{\mathcal{R}_M(0,0)} \big(E^- w_{j+k,\ell+m}^n E^+ w_{j,\ell}^n - E^+ w_{j+k,\ell+m}^n E^- w_{j,\ell}^n\big)/\tau + \sum_{\mathcal{R}_M(0,0)} T_{j,\ell}^n \\
&= \sum_{(k,m)\in\mathcal{R}_K(0,0)} a_{k,m} \sum_{\mathcal{R}_M(0,0)} \big(E^+ w_{j+k,\ell+m}^n E^+ w_{j,\ell}^n - E^- w_{j+k,\ell+m}^n E^- w_{j,\ell}^n\big)/\tau \\
&\quad + \sum_{(k,m)\in R_K(0,0)} a_{k,m} \left(\sum_{\mathcal{R}_M(0,0)} E^- w_{j+k,\ell+m}^n E^+ w_{j,\ell}^n - \sum_{\mathcal{R}_M(-k,-m)} E^- w_{j+k,\ell+m}^n E^+ w_{j,\ell}^n\right)/\tau + \sum_{\mathcal{R}_M(0,0)} T_{j,\ell}^n \\
&= \sum_{(k,m)\in\mathcal{R}_K(0,0)} a_{k,m} \sum_{\mathcal{R}_M(0,0)} \big(E^+ w_{j+k,\ell+m}^n E^+ w_{j,\ell}^n - E^- w_{j+k,\ell+m}^n E^- w_{j,\ell}^n\big)/\tau \\
&\quad + \sum_{(k,m)\in\mathcal{R}_K(0,0)} a_{k,m} \left(\sum_{\mathcal{R}_M(0,0)\setminus\mathcal{R}_M(-k,-m)} E^- w_{j+k,\ell+m}^n E^+ w_{j,\ell}^n \right. \\
&\quad \left. - \sum_{\mathcal{R}_M(-k,-m)\setminus\mathcal{R}_M(0,0)} E^- w_{j+k,\ell+m}^n E^+ w_{j,\ell}^n\right)/\tau + \sum_{\mathcal{R}_M(0,0)} T_{j,\ell}^n,
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&2\sum_{\mathcal{R}_M(0,0)} \frac{\Big|(D_\tau^+)w_{j,\ell}^n\Big|^2 - \Big|(D_\tau^-)w_{j,\ell}^n\Big|^2}{\tau} && (48)\\
&\quad = \sum_{(k,m)\in\mathcal{R}_K(0,0)} a_{k,m} \sum_{\mathcal{R}_M(0,0)} \big(E^+ w_{j+k,\ell+m}^n E^+ w_{j,\ell}^n - E^- w_{j+k,\ell+m}^n E^- w_{j,\ell}^n\big)/\tau && (49)\\
&\qquad - \frac{1}{\tau} \sum_{(j,\ell)\in\mathcal{B}_{\text{Int}}(M)} \sum_{(k,m)\in\mathcal{B}_{\text{Ext}}(M)} a_{k-j,m-\ell}\big(E^+ w_{k,m}^n E^- w_{j,\ell}^n - E^- w_{k,m}^n E^+ w_{j,\ell}^n\big) && (50)\\
&\qquad + \sum_{\mathcal{R}_M(0,0)} T_{j,\ell}^n.
\end{aligned}$$

Let us now set $\nabla^{E^+}_{k,m} w^n_{j,\ell} := E^+ w^n_{j+k,\ell+m} - E^+ w^n_{j,\ell}$. Then, the first term of (49) can be written as

$$\begin{aligned}
I_1^n &:= \sum_{(k,m)\in\mathcal{R}_K(0,0)} a_{k,m} \sum_{\mathcal{R}_M(0,0)} \left(E^+ w^n_{j+k,\ell+m} E^+ w^n_{j,\ell}\right)/\tau \\
&= \sum_{(k,m)\neq(0,0)} a_{k,m} \sum_{\mathcal{R}_M(0,0)} \left(\nabla^{E^+}_{k,m} w^n_{j,\ell}\right) E^+ w^n_{j,\ell}/\tau \qquad (51)\\
&= -\sum_{(k,m)\neq(0,0)} \frac{a_{k,m}}{\tau} \sum_{\mathcal{R}_M(0,0)} \left(\nabla^{E^+}_{k,m} w^n_{j,\ell}\right)^2 + \sum_{(k,m)\neq(0,0)} \frac{a_{k,m}}{\tau} \sum_{\mathcal{R}_M(0,0)} \left(\nabla^{E^+}_{k,m} w^n_{j,\ell}\right) E^+ w^n_{j+k,\ell+m} \\
&= -\sum_{(k,m)\neq(0,0)} \frac{a_{k,m}}{\tau} \sum_{\mathcal{R}_M(0,0)} \left(\nabla^{E^+}_{k,m} w^n_{j,\ell}\right)^2 - \sum_{(k,m)\neq(0,0)} \frac{a_{k,m}}{\tau} \sum_{\mathcal{R}_M(-k,-m)} \left(\nabla^{E^+}_{k,m} w^n_{j,\ell}\right) E^+ w^n_{j,\ell}. \qquad (52)
\end{aligned}$$

Thus, the term I_1^n can be written as the average of (51) and (52), *i.e.*

$$\begin{aligned}
I_1^n = &\sum_{(k,m)\neq(0,0)} a_{k,m} \left[\sum_{\mathcal{R}_M(0,0)} \left(\nabla^{E^+}_{k,m} w^n_{j,\ell}\right) E^+ w^n_{j,\ell} - \sum_{\mathcal{R}_M(-k,-m)} \left(\nabla^{E^+}_{k,m} w^n_{j,\ell}\right) E^+ w^n_{j,\ell}\right]/(2\tau) \\
&- \sum_{(k,m)\neq(0,0)} a_{k,m} \sum_{\mathcal{R}_M(0,0)} \left(\nabla^{E^+}_{k,m} w^n_{j,\ell}\right)^2/(2\tau).
\end{aligned}$$

The above equality is equivalent to

$$\begin{aligned}
I_1^n = &\sum_{(k,m)\neq(0,0)} a_{k,m} \left[\sum_{\mathcal{R}_M(0,0)\setminus\mathcal{R}_M(-k,-m)} \left(\nabla^{E^+}_{k,m} w^n_{j,\ell}\right)\left(-\nabla^{E^+}_{k,m} w^n_{j,\ell}\right)\right. \\
&\left. + \sum_{\mathcal{R}_M(0,0)\setminus\mathcal{R}_M(-k,-m)} \nabla^{E^+}_{k,m} w^n_{j,\ell} E^+ w^n_{j+k,\ell+m} - \sum_{\mathcal{R}_M(-k,-m)\setminus\mathcal{R}_M(0,0)} \left(\nabla^{E^+}_{k,m} w^n_{j,\ell}\right) E^+ w^n_{j,\ell}\right]/(2\tau) \\
&- \sum_{(k,m)\neq(0,0)} a_{k,m} \sum_{\mathcal{R}_M(0,0)} \left(\nabla^{E^+}_{k,m} w^n_{j,\ell}\right)^2/(2\tau).
\end{aligned}$$

As a consequence, we have

$$\begin{aligned}
I_1^n = &-\sum_{(k,m)\neq(0,0)} a_{k,m} \sum_{\mathcal{R}_M(0,0)\setminus\mathcal{R}_M(-k,-m)} \left(\nabla^{E^+}_{k,m} w^n_{j,\ell}\right)^2/(2\tau) \qquad (53)\\
&+ \sum_{(j,\ell)\in\mathcal{B}_{\mathrm{Int}}(M)} \sum_{(k,m)\in\mathcal{B}_{\mathrm{Ext}}(M)} \frac{a_{k-j,m-\ell}}{\tau} E^+ w^n_{k,m}\left(E^+ w^n_{k,m} - E^+ w^n_{j,\ell}\right) \\
&- \sum_{(k,m)\neq(0,0)} \frac{a_{k,m}}{2\tau} \sum_{\mathcal{R}_M(0,0)} \left(\nabla^{E^+}_{k,m} w^n_{j,\ell}\right)^2, \qquad (54)
\end{aligned}$$

while the term (54) can be written as

$$-\sum_{(k,m)\neq(0,0)} \frac{a_{k,m}}{2\tau} \sum_{\mathcal{R}_M(0,0)\setminus\mathcal{R}_M(-k,-m)} \left(\nabla^{E^+}_{k,m} w^n_{j,\ell}\right)^2 - \sum_{(k,m)\neq(0,0)} \frac{a_{k,m}}{2\tau} \sum_{\mathcal{R}_M(0,0)\bigcap\mathcal{R}_M(-k,-m)} \left(\nabla^{E^+}_{k,m} w^n_{j,\ell}\right)^2.$$

By the above equality, equations (53) and (54), we deduce

$$
\begin{aligned}
I_1^n &= \frac{1}{\tau} \sum_{(j,\ell)\in\mathcal{B}_{\mathrm{Int}}(M)} \sum_{(k,m)\in\mathcal{B}_{\mathrm{Ext}}(M)} a_{k-j,m-\ell}\Big[E^+w_{k,m}^n\big(E^+w_{k,m}^n - E^+w_{j,\ell}^n\big) - \big(E^+w_{k,m}^n - E^+w_{j,\ell}^n\big)^2\Big] \\
&\quad - \frac{1}{2\tau} \sum_{(k,m)\neq(0,0)} a_{k,m} \sum_{\mathcal{R}_M(0,0)\bigcap\mathcal{R}_M(-k,-m)} \Big(\nabla_{k,m}^{E^+} w_{j,\ell}^n\Big)^2 \\
&= \frac{1}{\tau} \sum_{(j,\ell)\in\mathcal{B}_{\mathrm{Int}}(M)} \sum_{(k,m)\in\mathcal{B}_{\mathrm{Ext}}(M)} a_{k-j,m-\ell} E^+w_{j,\ell}^n\big(E^+w_{k,m}^n - E^+w_{j,\ell}^n\big) \\
&\quad - \frac{1}{2\tau} \sum_{(k,m)\neq(0,0)} a_{k,m} \sum_{\mathcal{R}_M(0,0)\bigcap\mathcal{R}_M(-k,-m)} \Big(\nabla_{k,m}^{E^+} w_{j,\ell}^n\Big)^2 \\
&= \frac{1}{\tau} \sum_{(j,\ell)\in\mathcal{B}_{\mathrm{Int}}(M)} \sum_{(k,m)\in\mathcal{B}_{\mathrm{Ext}}(M)} a_{k-j,m-\ell} E^+w_{k,m}^n E^+w_{j,\ell}^n - H^n,
\end{aligned}
$$

with

$$
H^n = \frac{1}{2\tau} \sum_{(k,m)\neq(0,0)} a_{k,m} \sum_{\mathcal{R}_M(0,0)\bigcap\mathcal{R}_M(-k,-m)} \Big(\nabla_{k,m}^{E^+} w_{j,\ell}^n\Big)^2 + \frac{1}{\tau} \sum_{(j,\ell)\in\mathcal{B}_{\mathrm{Int}}(M)} \sum_{(k,m)\in\mathcal{B}_{\mathrm{Ext}}(M)} a_{k-j,m-\ell} |E^+w_{j,\ell}^n|^2.
$$

It is clear that (49) can be written as $I_1^n - I_1^{n-1}$. Thus, we can rewrite (48) as

$$
\begin{aligned}
&2\sum_{\mathcal{R}_M(0,0)} \frac{\Big|(D_\tau^+)w_{j,\ell}^n\Big|^2 - \Big|(D_\tau^-)w_{j,\ell}^n\Big|^2}{\tau} \\
&\quad = \frac{1}{\tau} \sum_{(j,\ell)\in\mathcal{B}_{\mathrm{Int}}(M)} \sum_{(k,m)\in\mathcal{B}_{\mathrm{Ext}}(M)} a_{k-j,m-\ell}\big(E^+w_{k,m}^n E^+w_{j,\ell}^n - E^-w_{k,m}^n E^-w_{j,\ell}^n\big) \\
&\qquad - \frac{1}{\tau} \sum_{(j,\ell)\in\mathcal{B}_{\mathrm{Int}}(M)} \sum_{(k,m)\in\mathcal{B}_{\mathrm{Ext}}(M)} a_{k-j,m-\ell}\big(E^+w_{k,m}^n E^-w_{j,\ell}^n - E^-w_{k,m}^n E^+w_{j,\ell}^n\big) - H^n + H^{n-1} + \sum_{\mathcal{R}_M(0,0)} T_{j,\ell}^n \\
&\quad = \frac{1}{\tau} \sum_{(j,\ell)\in\mathcal{B}_{\mathrm{Int}}(M)} \sum_{(k,m)\in\mathcal{B}_{\mathrm{Ext}}(M)} a_{k-j,m-\ell}\big(E^+w_{k,m}^n + E^-w_{k,m}^n\big)\big(E^+w_{j,\ell}^n - E^-w_{j,\ell}^n\big) \\
&\qquad + \sum_{\mathcal{R}_M(0,0)} T_{j,\ell}^n - H^n + H^{n-1}.
\end{aligned} \tag{55}
$$

Multiplying (55) by h^2 and summing up it from $n = 1$ to $N - 1$, one has

$$
\begin{aligned}
&2h^2 \sum_{\mathcal{R}_M(0,0)} \frac{\Big|(D_\tau^+)w_{j,\ell}^{N-1}\Big|^2 - \Big|(D_\tau^+)w_{j,\ell}^0\Big|^2}{\tau} = 2h^2 \sum_{n=1}^{N-1} \sum_{\mathcal{R}_M(0,0)} \frac{\Big|(D_\tau^+)w_{j,\ell}^n\Big|^2 - \Big|(D_\tau^-)w_{j,\ell}^n\Big|^2}{\tau} \\
&\quad = \frac{h^2}{\tau} \sum_{n=1}^{N-1} \sum_{(j,\ell)\in\mathcal{B}_{\mathrm{Int}}(M)} \sum_{(k,m)\in\mathcal{B}_{\mathrm{Ext}}(M)} a_{k-j,m-\ell}\big(E^+w_{k,m}^n - E^-w_{k,m}^n\big)\big(E^+w_{j,\ell}^n + E^-w_{j,\ell}^n\big) \\
&\qquad - h^2 H^{N-1} + h^2 H^0 + h^2 \sum_{n=1}^{N-1} \sum_{\mathcal{R}_M(0,0)} T_{j,\ell}^n.
\end{aligned} \tag{56}
$$

The first term for R.H.S of (56) can be evaluated as

$$\left|\frac{h^2}{\tau}\sum_{n=1}^{N-1}\sum_{(j,\ell)\in\mathcal{B}_{\rm Int}(M)}\sum_{(k,m)\in\mathcal{B}_{\rm Ext}(M)} a_{k-j,m-\ell}\big(E^+w^n_{k,m}-E^-w^n_{k,m}\big)\big(E^+w^n_{j,\ell}+E^-w^n_{j,\ell}\big)\right|$$
$$\le\frac{Ch^2}{\tau}\sum_{n=1}^{N-1}\sum_{\mathbb{Z}^2\setminus\mathcal{R}_{M-K}(0,0)}\left(\left|u^{n+1}_{j,\ell}\right|^2+\left|u^{n}_{j,\ell}\right|^2+\left|u^{n-1}_{j,\ell}\right|^2+|u_{j,\ell}(t_{n+1})|^2+|u_{j,\ell}(t_n)|^2+|u_{j,\ell}(t_{n-1})|^2\right). \quad (57)$$

Firstly we estimate the terms on the right hand side of (57) with respect to the numerical solution $u^n_{j,\ell}$, namely,

$$\begin{aligned}h^2&\sum_{n=1}^{N-1}\sum_{\mathbb{Z}^2\setminus\mathcal{R}_{M-K}(0,0)}\left(\left|u^{n+1}_{j,\ell}\right|^2+\left|u^{n}_{j,\ell}\right|^2+\left|u^{n-1}_{j,\ell}\right|^2\right).\\&\le Ch^2\sum_{n=1}^{N-1}\left(\sum_{|j|\le J,|\ell|>J,(j,\ell)\in\mathbb{Z}^2\setminus R_{M-K}(0,0)}\left|u^n_{j,\ell}\right|^2+\sum_{|j|>J,|\ell|\le J,(j,\ell)\in\mathbb{Z}^2\setminus R_{M-K}(0,0)}\left|u^n_{j,\ell}\right|^2\right.\\&\qquad\left.+\sum_{|j|>J,|\ell|>J,(j,\ell)\in\mathbb{Z}^2\setminus R_{M-K}(0,0)}\left|u^n_{j,\ell}\right|^2\right).\end{aligned}$$

Consequently, by (16) of Theorem 1, as $M\to\infty$, above estimate satisfies

$$\begin{aligned}h^2&\sum_{n=1}^{N-1}\sum_{\mathbb{Z}^2\setminus\mathcal{R}_{M-K}(0,0)}\left(\left|u^{n+1}_{j,\ell}\right|^2+\left|u^{n}_{j,\ell}\right|^2+\left|u^{n-1}_{j,\ell}\right|^2\right).\\&\le Ch^2N|\mathcal{B}_{\rm Int}(J)|^2(16)^N\sum_{\ell=M-K}^{\infty}\frac{2J}{(\ell-J)^2}+Ch^2N|\mathcal{B}_{\rm Int}(J)|^2(16)^N\sum_{j=M-K}^{\infty}\frac{2J}{(j-J)^2}\\&\qquad+Ch^2N|\mathcal{B}_{\rm Int}(J)|^2(16)^N\left(\sum_{\ell=J+1}^{\infty}\sum_{j=M-K}^{\infty}+\sum_{j=J+1}^{\infty}\sum_{\ell=M-K}^{\infty}\right)\frac{1}{(j-J)^2(\ell-J)^2}\to 0.\end{aligned} \quad (58)$$

On the other hand, as $M\to\infty$, the terms on the right hand side of (57) with respect to the exact solution $u_{j,\ell}(t)$ satisfy

$$\begin{aligned}h^2&\sum_{|j|\le J,|\ell|>J,(j,\ell)\in\mathbb{Z}^2\setminus R_{M-K}(0,0)}|u_{j,\ell}(t_n)|^2=h^2\left(\sum_{j=-J}^{J}\sum_{\ell=M-K}^{\infty}+\sum_{j=-J}^{J}\sum_{\ell=-\infty}^{-(M-K)}\right)|u_{j,\ell}(t_n)|^2\\&\qquad\le C\left(\int_{-Jh}^{Jh}\int_{(M-K)h}^{\infty}+\int_{-Jh}^{Jh}\int_{-\infty}^{-(M-K)h}\right)|u(x,y,t_n)|^2\,\mathrm{d}x\,\mathrm{d}y\to 0,\\h^2&\sum_{|\ell|\le J,|j|>J,(j,\ell)\in\mathbb{Z}^2\setminus R_{M-K}(0,0)}|u_{j,\ell}(t_n)|^2\to 0,\\h^2&\sum_{|\ell|>J,|j|>J,(j,\ell)\in\mathbb{Z}^2\setminus R_{M-K}(0,0)}|u_{j,\ell}(t_n)|^2\to 0.\end{aligned} \quad (59)$$

Thus, as $M\to\infty$ one derives

$$\frac{h^2}{\tau}\sum_{n=1}^{N-1}\sum_{(j,\ell)\in\mathcal{B}_{\rm Int}(M)}\sum_{(k,m)\in\mathcal{B}_{\rm Ext}(M)} a_{k-j,m-\ell}\big(E^+w^n_{k,m}-E^-w^n_{k,m}\big)\big(E^+w^n_{j,\ell}+E^-w^n_{j,\ell}\big)\to 0, \quad (60)$$

due to (57)–(59). Consequently, by taking $M \to \infty$ in (55), one has

$$h^2 \sum_{(j,\ell)\in\mathbb{Z}^2} \left|(D_\tau^+) w_{j,\ell}^{N-1}\right|^2 + \frac{h^2}{4} \sum_{(k,m)\neq(0,0)} a_{k,m} \sum_{(j,\ell)\in\mathbb{Z}^2} \left(\nabla_{k,m}^{E^+} w_{j,\ell}^N\right)^2$$
$$= h^2 \sum_{(j,\ell)\in\mathbb{Z}^2} \left|(D_\tau^+) w_{j,\ell}^0\right|^2 + \frac{h^2}{4} \sum_{(k,m)\neq(0,0)} a_{k,m} \sum_{(j,\ell)\in\mathbb{Z}^2} \left(\nabla_{k,m}^{E^+} w_{j,\ell}^0\right)^2 + \tau h^2 \sum_{n=1}^{N-1} \sum_{(j,\ell)\in\mathbb{Z}^2} T_{j,\ell}^n. \tag{61}$$

Recalling (17) with $u_{j,\ell}^0 = \phi^0(jh, \ell h)$ and $v_{j,\ell}^0 = \phi^1(jh, \ell h)$ we derive that

$$\begin{aligned}
u_{j,\ell}^1 &= u_{j,\ell}^0 + \left(F_{j,\ell}^1 + 2F_{j,\ell}^2 + 2F_{j,\ell}^3 + F_{j,\ell}^4\right)\tau/6 \\
&= u_{j,\ell}^0 + v_{j,\ell}^0\tau + \left(G_{j,\ell}^1 + G_{j,\ell}^2 + G_{j,\ell}^3\right)\tau^2/6 \\
&= u_{j,\ell}^0 + v_{j,\ell}^0\tau + L_{\delta,h} u_{j,\ell}^0 \tau^2/2 + (f(jh,\ell h,0) + 2f(jh,\ell h,\tau/2))\tau^2/6 + \left(L_{\delta,h}F_{j,\ell}^1 + L_{\delta,h}F_{j,\ell}^2\right)\tau^3/12 \\
&= \phi^0(jh,\ell h) + \phi^1(jh,\ell h)\tau + \left(L_{\delta,h}\phi_{h,\ell}^0 + f(jh,\ell h,0)\right)\tau^2/2 \\
&\quad + (f(jh,\ell h,\tau/2) - f(jh,\ell h,0))\tau^2/3 + \left(L_{\delta,h}F_{j,\ell}^1 + L_{\delta,h}F_{j,\ell}^2\right)\tau^3/12 \\
&= \phi^0(jh,\ell h) + \phi^1(jh,\ell h)\tau + \left(\left(\mathcal{L}_\delta\phi^0\right)(jh,\ell h) + f(jh,\ell h,0)\right)\tau^2/2 + \tau\Phi_1(jh,\ell h,\tau)
\end{aligned}$$

with $\Phi_1(x,y,t)$ a function smooth enough and $|\Phi_1(jh,\ell h,\tau)| \le C(h^2+\tau^2)\|u\|_{C^4_{\mathcal{R}_\delta(jh,\ell h)\times[0,\tau]}}$. By Taylor expansion,

$$\begin{aligned}
u(jh,\ell h,\tau) &= \phi^0(jh,\ell h) + \phi^1(jh,\ell h)\tau + u_{tt}(jh,\ell h,0)\tau^2/2 + \tau\Phi_2(jh,\ell h,\tau) \\
&= \phi^0(jh,\ell h) + \phi^1(jh,\ell h)\tau + \left(\left(\mathcal{L}_\delta\phi^0\right)(jh,\ell h) + f(jh,\ell h,0)\right)\tau^2/2 + \tau\Phi_2(jh,\ell h,\tau),
\end{aligned}$$

with $\Phi_2(x,y,t)$ a function smooth enough and $|\Phi_2(jh,\ell h,\tau)| \le C(h^2+\tau^2)\|u\|_{C^3_{(\mathcal{R}_\delta(jh,\ell h)\times[0,\tau])}}$. Combining the estimates of $u(jh,\ell h,\tau)$ and $u_{j,\ell}^1$, we have

$$\begin{aligned}
\left|D_\tau^+ w_{j,\ell}^0\right| &= |\Phi_1(jh,\ell h,\tau) - \Phi_2(jh,\ell h,\tau)| \le C\left(h^2+\tau^2\right)\|u\|_{C^4_{(\mathcal{R}_\delta(jh,\ell h)\times[0,\tau])}}, \\
\left|E^+ w_{j,\ell}^0\right| &= |\tau(\Phi_1(jh,\ell h,\tau) + \Phi_2(jh,\ell h,\tau))/2| \le C\tau\left(h^2+\tau^2\right)\|u\|_{C^4_{(\mathcal{R}_\delta(jh,\ell h)\times[0,\tau])}}, \\
\left|L_{\delta,h}E^+ w_{j,\ell}^0\right| &= |\tau(L_{\delta,h}\Phi_1(jh,\ell h,\tau) + L_{\delta,h}\Phi_2(jh,\ell h,\tau))/2| \le C\tau\left(h^2+\tau^2\right)\|u\|_{C^6_{(\mathcal{R}_\delta(jh,\ell h)\times[0,\tau])}}.
\end{aligned} \tag{62}$$

Let $\triangle_{k,m}^{E^+} w_{j,\ell}^0 = w_{j+k,\ell+m}^0 - 2w_{j,\ell}^0 + w_{j-k,\ell-m}^0$, it is obvious that

$$\iint_{\mathbb{R}^2} \triangle_{k,m}^{E^+} u \cdot v \,\mathrm{d}x\,\mathrm{d}y = -\iint_{\mathbb{R}^2} \nabla_{k,m}^{E^+} u \nabla_{k,m}^{E^+} v \,\mathrm{d}x\,\mathrm{d}y.$$

Therefore we have

$$\begin{aligned}
\sum_{\mathbb{Z}^2} L_{\delta,h}E^+ w_{j,\ell}^0 \cdot E^+ w_{j,\ell}^0 &= \sum_{\mathbb{Z}^2} \sum_{(k,m)\in\mathcal{R}_K(0,0)} a_{k,m} E^+ w_{j+k,\ell+m}^0 E^+ w_{j,\ell}^0 \\
&= 1/2 \sum_{(k,m)\in\mathcal{R}_K(0,0)\setminus(0,0)} a_{k,m} \sum_{\mathbb{Z}^2} \triangle_{k,m}^{E^+} w_{j,\ell}^0 E^+ w_{j,\ell}^0 \\
&= -1/2 \sum_{(k,m)\in\mathcal{R}_K(0,0)\setminus(0,0)} a_{k,m} \sum_{\mathbb{Z}^2} \left|\nabla_{k,m}^{E^+} w_{j,\ell}^0\right|^2.
\end{aligned}$$

By (62), above equality gives

$$h^2 \sum_{(j,\ell)\in\mathbb{Z}^2} \big|(D_\tau^+) w_{j,\ell}^0\big|^2 + \frac{h^2}{4} \sum_{(k,m)\neq(0,0)} a_{k,m} \sum_{(j,\ell)\in\mathbb{Z}^2} \Big(\nabla_{k,m}^{E^+} w_{j,\ell}^0\Big)^2$$
$$= h^2 \sum_{(j,\ell)\in\mathbb{Z}^2} \big|(D_\tau^+) w_{j,\ell}^0\big|^2 - \frac{h^2}{2} \sum_{(j,\ell)\in\mathbb{Z}^2} L_{\delta,h} E^+ w_{j,\ell}^0 \cdot E^+ w_{j,\ell}^0$$
$$\leq h^2 \sum_{(j,\ell)\in\mathbb{Z}^2} |\Phi(jh,\ell h,\tau)|^2 + Ch^2 \sum_{(j,\ell)\in\mathbb{Z}^2} \big|L_{\delta,h} E^+ w_{j,\ell}^0\big|^2 + Ch^2 \sum_{(j,\ell)\in\mathbb{Z}^2} \big|E^+ w_{j,\ell}^0\big|^2 \leq C\big(\tau^2+h^2\big).$$

By Lemma 4, equation (61) and above estimate one gets

$$h^2 \sum_{(j,\ell)\in\mathbb{Z}^2} \Big|(D_\tau^+) w_{j,\ell}^{N-1}\Big|^2 + \frac{h^2}{4} \sum_{(k,m)\neq(0,0)} a_{k,m} \sum_{(j,\ell)\in\mathbb{Z}^2} \Big(\Delta_{k,m}^{E^+} w_{j,\ell}^N\Big)^2$$
$$\leq C\big(\tau^2+h^2\big)^2 + Ch^2\tau \sum_{n=0}^{N-1} \sum_{(j,\ell)\in\mathbb{Z}^2} \big|e_{j,\ell}^n\big|^2 + Ch^2\tau \sum_{n=0}^{N-1} \sum_{(j,\ell)\in\mathbb{Z}^2} \big((D_\tau^+ + D_\tau^-) w_{j,\ell}^n\big)^2$$
$$\leq h^2\tau \sum_{n=0}^{N-1} \sum_{(j,\ell)\in\mathbb{Z}^2} \big|D_\tau^+ w_{j,\ell}^n\big|^2 + C\big(\tau^2+h^2\big)^2,$$

which is equivalent to

$$\big\|D_\tau^+ \mathbf{w}^{N-1}\big\|_{\ell^2}^2 \leq C\big(\tau^2+h^2\big)^2 + C\tau \sum_{n=0}^{N} \big\|D_\tau^+ \mathbf{w}^n\big\|_{\ell^2}^2,$$

by which a Gronwall inequality [8] yields: $\|D_\tau^+ \mathbf{w}^n\|_{\ell^2} \leq Ce^{CN\tau}(\tau^2+h^2) \leq Ce^{CT}(\tau^2+h^2)$ with $n \leq N$. Thus one has

$$h^2 \sum_{(j,\ell)\in\mathbb{Z}^2} \big|w_{j,\ell}^N\big|^2 = h^2 \sum_{(j,\ell)\in\mathbb{Z}^2} \left|w_{j,\ell}^0 + \tau \sum_{n=0}^{N-1} (D_\tau^+) w_{j,\ell}^n\right|^2 \leq 2h^2 \sum_{(j,\ell)\in\mathbb{Z}^2} \big|w_{j,\ell}^0\big|^2 + 2h^2 \sum_{(j,\ell)\in\mathbb{Z}^2} \left|\tau \sum_{n=0}^{N-1} (D_\tau^+) w_{j,\ell}^n\right|^2$$
$$\leq 2h^2 \sum_{(j,\ell)\in\mathbb{Z}^2} \big|w_{j,\ell}^0\big|^2 + 2N\tau^2 h^2 \sum_{(j,\ell)\in\mathbb{Z}^2} \sum_{n=0}^{N-1} \big|(D_\tau^+) w_{j,\ell}^n\big|^2$$
$$\leq 2h^2 \sum_{(j,\ell)\in\mathbb{Z}^2} \big|w_{j,\ell}^0\big|^2 + 2T\tau \sum_{n=0}^{N-1} \big\|D_\tau^+ \mathbf{w}^{N-1}\big\|_{\ell^2}^2 \leq C\big(\tau^2+h^2\big)^2.$$

Now, let us consider $D_x^+ w_{j,\ell}^n = D_x^+ u_{j,\ell}^n - D_x^+ u(x_j, y_\ell, t_n)$. For $(j,\ell) \in \mathbb{Z}^2$, we have

$$\big(D_\tau^+ - D_\tau^-\big) D_x^+ w_{j,\ell}^n / \tau = D_\tau^+ D_\tau^- D_x^+ w_{j,\ell}^n = \sum_{(k,m)\in\mathcal{R}_K(0,0)} a_{k,m} E^+ E^- D_x^+ w_{j+k,\ell+m}^n + D_x^+ e_{j,\ell}^n.$$

Multiplying the above equality by $h^2(D_\tau^+ + D_\tau^-) D_x^+ w_{j,\ell}^n$, and summing up all the terms for $(j,\ell) \in \mathcal{R}_M(0,0)$, we get

$$h^2 \sum_{\mathcal{R}_M(0,0)} \big((D_\tau^+ - D_\tau^-) D_x^+ w_{j,\ell}^n\big)\big((D_\tau^+ + D_\tau^-) D_x^+ w_{j,\ell}^n\big)/\tau = h^2 \sum_{\mathcal{R}_M(0,0)} D_x^+ w_{j,\ell}^n \big((D_\tau^+ + D_\tau^-) D_x^+ w_{j,\ell}^n\big)$$
$$+ h^2 \sum_{\mathcal{R}_M(0,0)} \sum_{(k,m)\in\mathcal{R}_K(0,0)} a_{k,m} \big((E^+ + E^-) D_x^+ w_{j+k,\ell+m}^n\big)\big((E^+ - E^-) D_x^+ w_{j,\ell}^n\big)/(2\tau).$$

After the same manipulation as previous, as $M \to \infty$ one has

$$h^2 \sum_{(j,\ell)\in\mathbb{Z}^2} \left|\left(D_\tau^+\right) D_x^+ w_{j,\ell}^{N-1}\right|^2 + \frac{h^2}{4} \sum_{(k,m)\neq(0,0)} a_{k,m} \sum_{(j,\ell)\in\mathbb{Z}^2} \left(\Delta_{k,m}^{E^+} D_x^+ w_{j,\ell}^N\right)^2$$
$$= h^2 \sum_{(j,\ell)\in\mathbb{Z}^2} \left|\left(D_\tau^+\right) D_x^+ w_{j,\ell}^1\right|^2 + \frac{h^2}{4} \sum_{(k,m)\neq(0,0)} a_{k,m} \sum_{(j,\ell)\in\mathbb{Z}^2} \left(\Delta_{k,m}^{E^+} D_x^+ w_{j,\ell}^1\right)^2$$
$$+ h^2 \sum_{(j,\ell)\in\mathbb{Z}^2} D_x^+ e_{j,\ell}^n \left(\left(D_\tau^+ + D_\tau^-\right) D_x^+ w_{j,\ell}^n\right),$$

which leads to

$$\left\|D_\tau^+ \nabla_x^+ \mathbf{w}^{N-1}\right\|_{\ell^2}^2 - \left\|D_\tau^+ \nabla_x^+ \mathbf{w}^0\right\|_{\ell^2}^2 \le C\left(\tau^2 + h^2\right)^2 + C\tau \sum_{n=0}^{N} \left\|D_\tau^+ \nabla_x^+ \mathbf{w}^n\right\|_{\ell^2}^2.$$

Thus, we derive that $\|D_\tau^+ D_x^+ \mathbf{w}^n\|_{\ell^2} \le Ce^{CN\tau}(\tau^2 + h^2) \le C(\tau^2 + h^2)$ with $n \le N$, this implies $\|D_x^+ \mathbf{w}^n\|_{\ell^2} \le Ce^{CN\tau}(\tau^2 + h^2) \le C(\tau^2 + h^2)$ with $n \le N$. By a similar technique, we can prove

$$\left\|D_y^+ \mathbf{w}^N\right\|_{\ell^2} \le Ce^{CT}\left(\tau^2 + h^2\right),$$

and

$$\left\|D_x^+ D_y^+ \mathbf{w}^N\right\|_{\ell^2} \le Ce^{CT}\left(\tau^2 + h^2\right).$$

Finally, for any $(j,\ell) \in \mathbb{Z}^2$ we have

$$\left|w_{j,\ell}^N\right| \le \max_{\ell\in\mathbb{Z}} \left|w_{j,\ell}^N\right| \le C\left(h \sum_{\ell\in\mathbb{Z}} \left|w_{j,\ell}^N\right|^2\right)^{\frac{1}{2}} + C\left(h \sum_{\ell\in\mathbb{Z}} \left|D_y^+ w_{j,\ell}^N\right|^2\right)^{\frac{1}{2}}$$
$$\le C\left(h \sum_{\ell\in\mathbb{Z}} \left|\max_{j\in\mathbb{Z}} w_{j,\ell}^N\right|^2\right)^{\frac{1}{2}} + C\left(h \sum_{\ell\in\mathbb{Z}} \left|\max_{j\in\mathbb{Z}} D_y^+ w_{j,\ell}^N\right|^2\right)^{\frac{1}{2}}$$
$$\le C\left(h^2 \sum_{\ell\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \left|w_{j,\ell}^N\right|^2 + h^2 \sum_{\ell\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \left|D_x^+ w_{j,\ell}^N\right|^2\right)^{\frac{1}{2}}$$
$$+ C\left(h^2 \sum_{\ell\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \left|D_y^+ w_{j,\ell}^N\right|^2 + h^2 \sum_{\ell\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \left|D_x^+ D_y^+ w_{j,\ell}^N\right|^2\right)^{\frac{1}{2}}$$
$$\le \left\|\mathbf{w}^N\right\|_{\ell^2} + \left\|D_y^+ \mathbf{w}^N\right\|_{\ell^2} + \left\|D_x^+ \mathbf{w}^N\right\|_{\ell^2} + \left\|D_x^+ D_y^+ \mathbf{w}^N\right\|_{\ell^2} \le C\left(h^2 + \tau^2\right),$$

which gives (65). □

We remark that the main difficulty in numerical analysis is the complicated estimate on the boundary, but the exact discretized boundary conditions help us to avoid the difficulty.

We also point that frog-leap scheme will be more suitable for wave equation due to the good CFL condition. It will be interesting to analysis the stability and error by using frog-leap scheme.

4. Accurate boundary conditions for wave equation on rectangular domain

The initial data $\phi^0(x,y)$, $\phi^1(x,y)$ and the source $f(x,y,t)$ of peridynamics (3) are assumed to be compactly supported in $[-x_J, x_J] \times [-x_J, x_J]$. Under the settings of Section 2.2, based on CN time scheme at time $t_n = n\tau$

we obtain the fully discretized version for (3), namely,

$$D_\tau^+ D_\tau^- u_{j,\ell}^n = E^+E^-L_h u_{j,\ell}^n + f_{j,\ell}^n := \sum_{k=-1}^{1}\sum_{m=-1}^{1} a_{k,m}E^+E^-u_{j+k,\ell+m}^n + f_{j,\ell}^n$$
$$= \frac{E^+E^-f_{j+1,\ell}^n + E^+E^-u_{j-1,\ell}^n + E^+E^-u_{j,\ell+1}^n + E^+E^-u_{j+1,\ell}^n - 4E^+E^-u_{j,\ell}^n}{h^2}$$
$$+ f_{j,\ell}^n, \qquad \text{for } j,\ell \in \mathbb{Z}, \tag{63}$$

with

$$a_{0,1} = a_{1,0} = a_{0,-1} = a_{-1,0} = \frac{1}{h^2},$$
$$a_{1,1} = a_{1,-1} = a_{-1,1} = a_{-1,-1} = 0,$$
$$a_{0,0} = -\frac{4}{h^2},$$
$$a_{j,\ell} = 0, \qquad \text{for } |j| \geq 2 \text{ or } |\ell| \geq 2, \tag{64}$$

which is exactly a special case of semi-discretized peridynamics (7). Thus, the boundary conditions can be obtained by Theorem 1 for the special $a_{j,\ell}$ proposed in (64).

For u solution to (3), we set $u_{j,\ell}(t) := u(x_j, y_\ell, t)$, for $(j,\ell) \in \mathbb{Z}^2$. The truncation error $e_{j,\ell}^n$ for scheme (63) is then

$$e_{j,\ell}^n = \frac{1}{\tau^2}\Big(u_{j,\ell}(t_{n+1}) - 2u_{j,\ell}(t_n) + u_{j,\ell}(t_{n-1})\Big) - \frac{1}{4}L_h\Big(u_{j,\ell}(t_{n+1}) + 2u_{j,\ell}(t_n) + u_{j,\ell}(t_{n-1})\Big) - f_{j,\ell}^n,$$

where $0 \leq n \leq N-1$. We set $D_y^{\pm}e_{j,\ell}^n$ and $D_x^{\pm}e_{j,\ell}^n$ as we did in Section 3.2. Therefore, Lemma 3 also holds for (3) by the same technique. Then, we have a local version of Lemma 4.

Lemma 5. *Let us assume that the solution u to (3) satisfies $\|u(\cdot,\cdot,t)\|_{H^P} \leq C$ for $t \leq T$ with P large enough. For $N\tau \leq T$, the truncated error estimates (29)–(32) hold. The constant C depends only on the initial data ϕ^0, ϕ^1 and source f.*

Proof. By (23) it is clear that

$$|\triangle u(x,y,t) - L_h u(x,y,t))| \leq Ch^2\|u(\cdot,\cdot,t)\|_{C^4(\mathcal{R}_h(x,y))}.$$

Combining above estimate, equations (33), (34) and (63), one has

$$\begin{aligned}
\left|e_{j,\ell}^n\right| &= \left|D_\tau^- D_\tau^+ u_{j,\ell}(t_n) - L_h E^-E^+u_{j,\ell}(t_n) - f_{j,\ell}(t^n)\right| \\
&= \left|\partial_{tt}u(x_j,y_\ell,t_n) - L_h u(x_j,y_\ell,t_n) - f_{j,\ell}(t_n) + \tau^2\partial_{tt}U_1(x_j,y_\ell,t_n) - \tau^2 L_h U_2(x_j,y_\ell,t_n)\right| \\
&\leq |\partial_{tt}u(x_j,y_\ell,t_n) - \partial_{xx}u(x_j,y_\ell,t_n) - f(x_j,y_\ell,t_n)| + |L_h u(x_j,y_\ell,t_n) - \partial_{xx}u(x_j,y_\ell,t_n)| \\
&\quad + C\tau^2\|U_1(\cdot,\cdot,t_n)\|_{C^1(\mathcal{R}_h(x_j,y_\ell))} + \tau^2|L_h U_2(x_j,y_\ell,t_n) - \partial_{xx}U_2(x_j,y_\ell,t_n)| \\
&\quad + C\tau^2\|U_2(\cdot,\cdot,t_n)\|_{C^2(\mathcal{R}_h(x_j,y_\ell))} \\
&\leq Ch^2\|u(\cdot,\cdot,t_n)\|_{C^4(\mathcal{R}_h(x_j,y_\ell))} + C\tau^2\|u\|_{C^6(\mathcal{R}_h(x_j,y_\ell)\times[t_{n-1},t_{n+1}])} + Ch^2\tau^2\|U_2(\cdot,\cdot,t_n)\|_{C^4(\mathcal{R}_h(x_j,y_\ell))} \\
&\leq C\big(h^2+\tau^2\big)\|u\|_{C^8(\mathcal{R}_\delta(x_j,y_\ell)\times[t_{n-1},t_{n+1}])}.
\end{aligned}$$

Thus, the technique in Lemma 4 leads to (29)–(32). □

Finally, for numerical error $w_{j,\ell}^n = u_{j,\ell}^n - u(x_j,y_\ell,t_n)$, the truncated error estimates give a parallel conclusion to Theorem 2.

Theorem 3. *Assume that the solution u to (3) satisfies $\|u(\cdot,\cdot,t)\|_{H^P} \leq C$ for $t \leq T$ with P large enough. For $\tau > 0$ and $N\tau \leq T$, we have the error bounds*

$$\left\|\mathbf{w}^N\right\|_{\ell^\infty} \leq C\big(\tau^2 + h^2\big), \tag{65}$$

for constant $C > 0$ that only depends on T, initial data ϕ^0, ϕ^1 and source $f(x,y,t)$.

Remark 1. For the 4-th spatial order fully discretized version of wave equation (3), namely,

$$\begin{aligned} D_\tau^+ D_\tau^- u_{j,\ell}^n = &\frac{1}{12h^2}\big(-E^+E^-u_{j-2,\ell}^n + 16E^+E^-u_{j-1,\ell}^n - 30E^+E^-u_{j,\ell}^n + 16E^+E^-u_{j+1,\ell}^n - E^+E^-u_{j+2,\ell}^n\big) \\ &+ \frac{1}{12h^2}\big(-E^+E^-u_{j,\ell-2}^n + 16E^+E^-u_{j,\ell-1}^n - 30E^+E^-u_{j,\ell}^n + 16E^+E^-u_{j,\ell+1}^n - E^+E^-u_{j,\ell+2}^n\big) + f_{j,\ell}^n, \end{aligned}$$

the estimate (65) is replaced by

$$\left\|\mathbf{w}^N\right\|_{\ell^\infty} \leq C\big(\tau^2 + h^4\big). \tag{66}$$

5. Numerical example

5.1. Numerical example of peidynamics

Consider the peridynamics (1) such that the interaction kernel function γ_δ is given by

$$\gamma_\delta(r) = \frac{4}{r^2\pi\delta^2}, \quad 0 \leq r := \sqrt{x^2+y^2} \leq \delta,$$

with initial data

$$\phi^0(x,y) = 650xy\frac{e^{-4r^2}}{r}J_2\Big(\frac{r}{2}\Big), \quad \phi^1(x,y) = 0,$$

where J_m is the Bessel's function of order m. We set the computational domain as $[-2,2]^2$ and the total computational time as $T = 5$. We compute the reference solutions u^{ref} on a larger spatial domain $[-16,16]^2$ in order to avoid any numerical reflection with $h = 2^{-9}$ and $\tau = 2^{-12}$. The L^∞-norm error $e_{\infty,T}^{h,\tau}$ is defined by

$$e_{\infty,T}^{h,\tau} = \max_{-J\leq j,\ell\leq J}\left|u^{\mathrm{ref}}(x_j,y_\ell,T) - u_{j,\ell}^N\right|,$$

by refining h, fixing $\tau = 2^{-12}$ and $T = 5$ (with $N\tau = T$). For $\delta = 0.125$ and $\delta = 0.25$, the numerical errors are ploted in the Figure 3. We observe that the scheme is around two-order in space.

In the same manner, by refining τ and fixed $h = 2^{-7}$, for $T = 5$ (with $N\tau = T$) the numerical errors are ploted in the Figure 4 with $\delta = 0.125$ and $\delta = 0.25$. We observe that the scheme is around two-order in time.

In addition, we show the behaviour of the numerical solution with initial data ϕ^0 and $\phi^1 = 0$ at different times for $\delta = 0.25$ in Figure 5. We observe that the wave packet goes through the corners without reflection.

5.2. Numerical example of wave equation

The wave equation (3) is considered with the initial data ϕ^0 and ϕ^1 as the followings,

$$\begin{aligned} &\phi^0(x,y) = \frac{e^{-4r^2}J_0(0.5r) - e^{-4R^2}J_0(0.5R)}{1 - e^{-4R^2}J_0(0.5R)}, && 0 \leq r \leq R = 2.25, \\ &\phi^0(x,y) = 0, && r > 2.25, \end{aligned}$$

where $r = \sqrt{x^2+y^2}$ and

$$\phi^1(x,y) = 0.$$

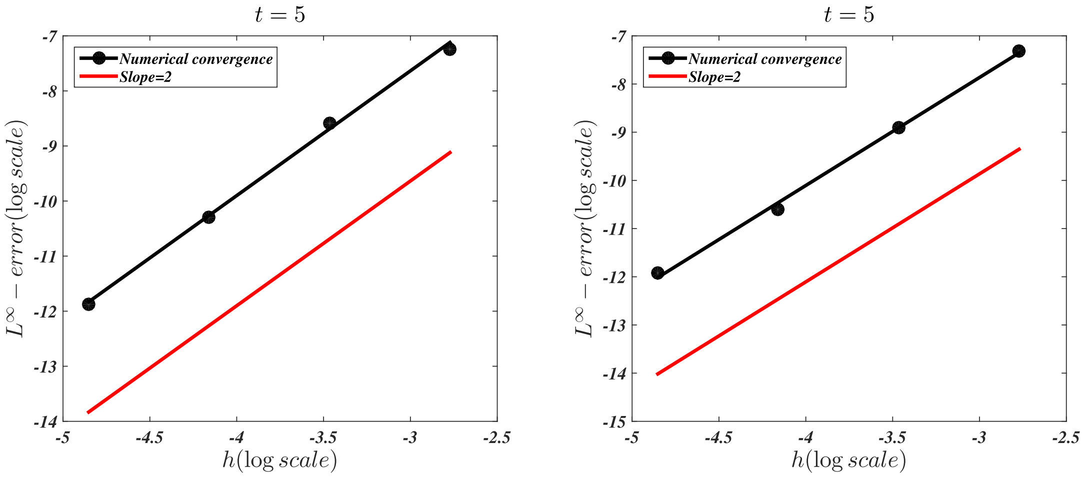

FIGURE 3. The space convergence order for $u(x, y, t)$ at $t = 5$ for $\delta = 0.125$ and $\delta = 0.25$, respectively.

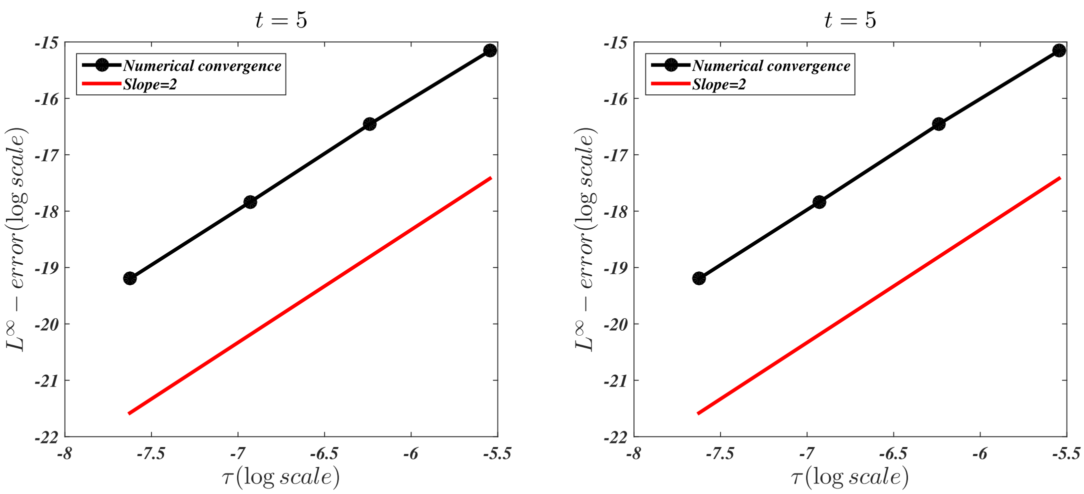

FIGURE 4. The time convergence order for $u(x, y, t)$ at $t = 5$ for $\delta = 0.125$ and $\delta = 0.25$, respectively.

We employ the 4-th spatial order scheme in Remark 1 and the corresponding ABCs to implement the computation. The computational domain is set as $[-2.5, 2.5]^2$ and the total computational time is set as $T = 4$. We still compute the reference solutions u^{ref} on a larger spatial domain $[-16, 16]^2$ with $h = 2^{-8}$ and $\tau = 2^{-16}$. The L^∞-norm error $e_{\infty,T}^{h,\tau}$ is defined by the same way as it was in Section 5.1 by refining h, fixing $\tau = 2^{-16}$ and $T = 4$ (with $N\tau = T$). The numerical errors are ploted in Figure 6. We observe that the scheme is around four-order in space.

By refining τ and fixing $h = 2^{-8}$, the numerical errors are ploted in Figure 7 with $T = 4$ ($N\tau = T$). We observe that the scheme is around two-order in time.

In the Figure 8 we shows the behaviour of the numerical solution with initial data ϕ^0 (and $\phi^1 = 0$) at different times. We observe that the wave packet goes through the corners without reflection.

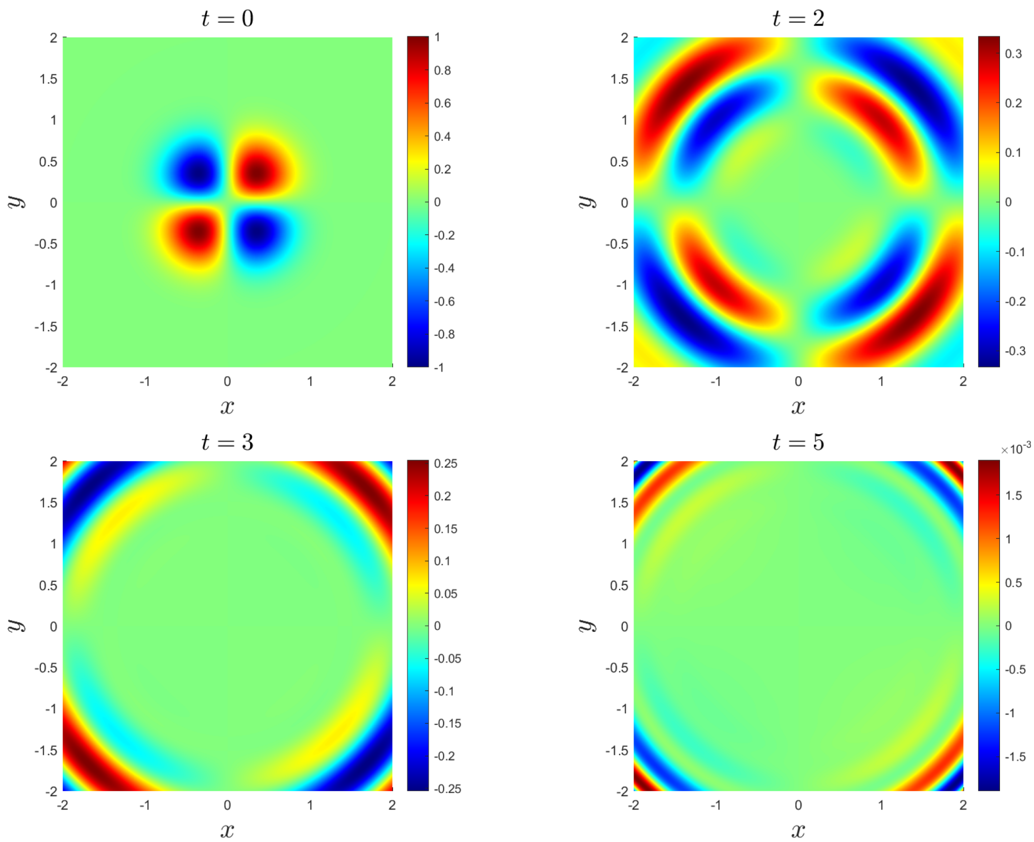

FIGURE 5. Snapshots of the numerical solution u at $t = 0, 2, 3, 5$ for $\delta = 0.25$.

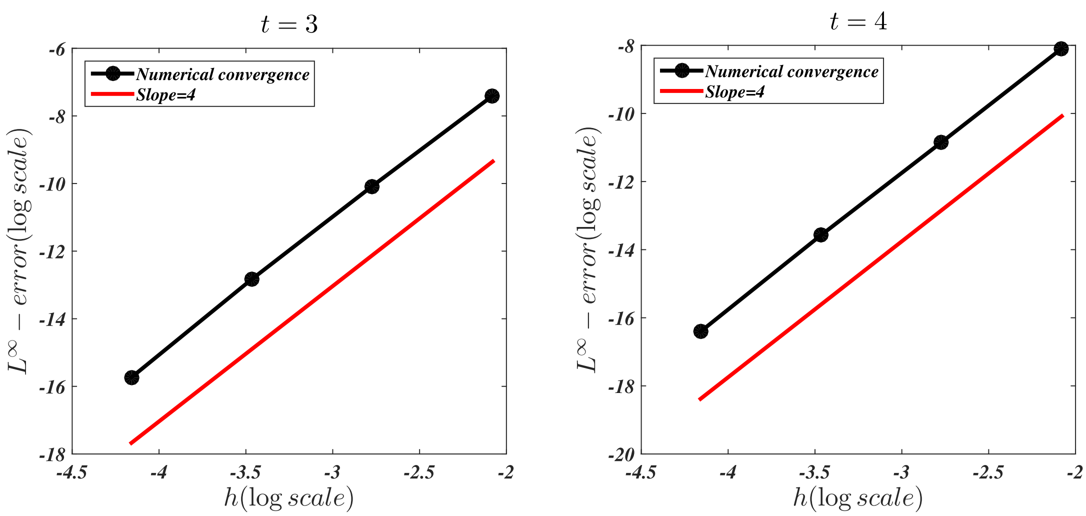

FIGURE 6. The space convergence order for $u(x, y, t)$ at $t = 3$ and $t = 4$, respectively.

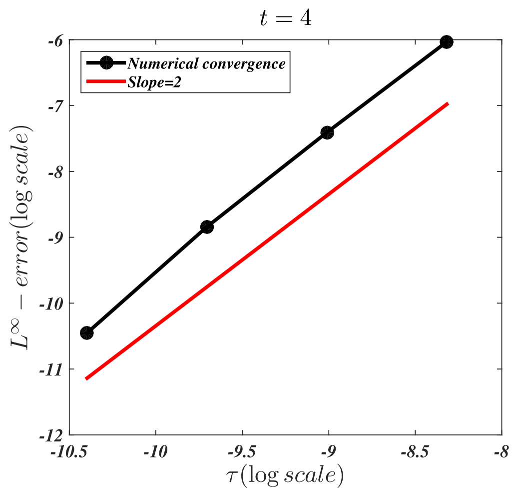

FIGURE 7. The time convergence order for $u(x, y, t)$ at $t = 4$.

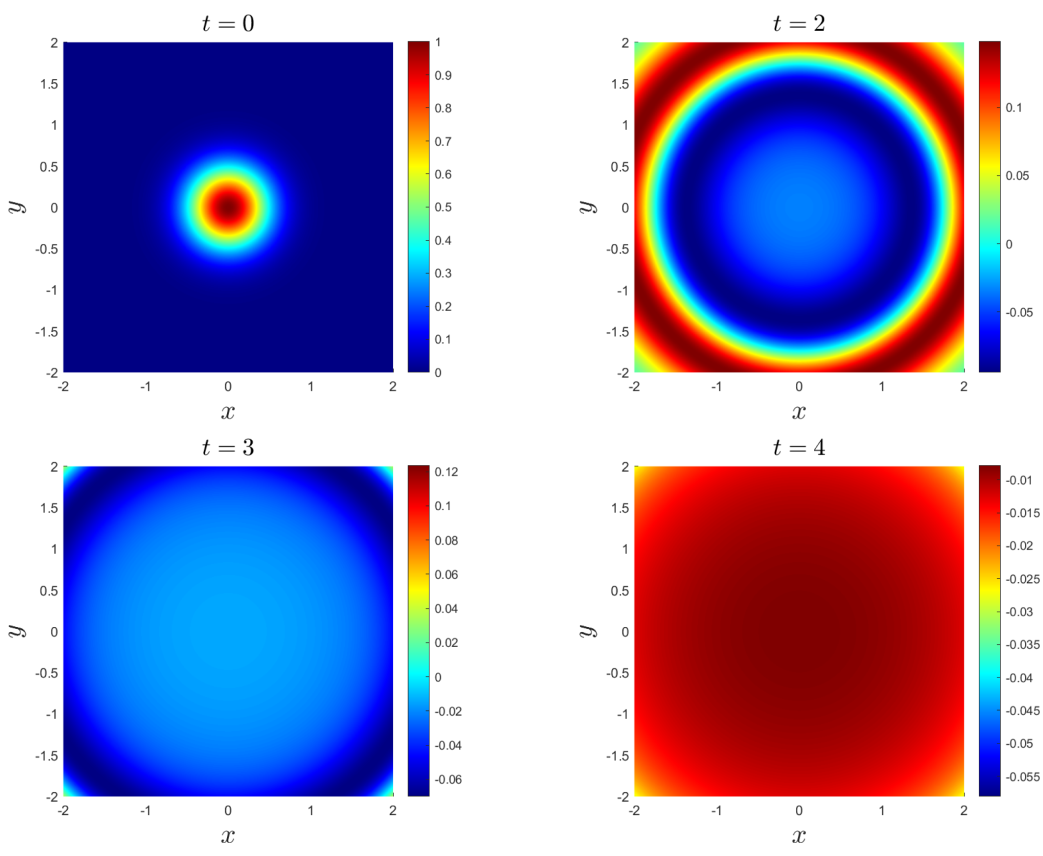

FIGURE 8. Snapshots of the numerical solution u at $t = 0, 2, 3, 4$.

6. Conclusion

The construction of ABCs is proposed for the two-dimensional peridynamics which is fully discretized by an Crank–Nicolson scheme in time and an asymptotically compatible scheme in space. The numerical analysis for the full scheme with the ABCs is developed. Numerical examples confirm the theoretical results. We verify that there is no numerical reflection at the corners of the square domain. Further works include the study of fast algorithm as well as nonlinearities. It will be also interesting to analysis the stability and error by using other scheme such as frog-leap scheme.

Appendix A.

Here we give the expressions of the symbol $\nu(k)$ that denote the location of the k-th interior layer node (yellow node in Fig. 1). The notation $\mathrm{mod}(a,b)$ is introduced to denote the modulo function which provides the remainder of dividing a by b. Then, the symbol $\nu(k)$ with $1 \le k \le 4K(2J+1-K)$, can be defined as

– For $1 \le k \le (2J+1)K$:

(1) If $\mathrm{mod}(k,2J+1) \neq 0$:

$$\nu(k) = \left(-J + \frac{k - \mathrm{mod}(k,2J+1)}{2J+1}, -J + \mathrm{mod}(k,2J+1) - 1\right).$$

(2) If $\mathrm{mod}(k,2J+1) = 0$:

$$\nu(k) = \left(-J + \frac{k - \mathrm{mod}(k,2J+1)}{2J+1} - 1, J\right).$$

– For $(2J+1)K + 1 \le k \le (2J+1)K + (2J+1-2K)2K$:

Set: $k_1 = k - (2J+1)K - 1$:

(1) If $\mathrm{mod}(k_1,2K) \neq 0$:

(a) If $\mathrm{mod}(k_1,2K) \le K$:

$$\nu(k) = \left(-J + K + \frac{k - \mathrm{mod}(k,2K)}{2K}, -J + \mathrm{mod}(k,2K) - 1\right).$$

(b) If $\mathrm{mod}(k_1,2K) \ge K+1$:

$$\nu(k) = \left(-J + K + \frac{k - \mathrm{mod}(k,2K)}{2K}, J - (2K - \mathrm{mod}(k,2K)) + 1\right).$$

(2) If $\mathrm{mod}(k_1,2K) = 0$:

$$\nu(k) = \left(-J + K + \frac{k_1 - \mathrm{mod}(k_1,4K)}{4K} - 1, J\right).$$

– For $(2J+1)K + (2J+1-4K)2K + 1 \le k \le J^2 - (J-2K)^2$:

Set $k_1 = k - (2J+1)K - (2J+1-2K)2K$

(1) If $\mathrm{mod}(k_1,2J+1) \neq 0$:

$$\nu(k) = \left(J - K + 1 + \frac{k_1 - \mathrm{mod}(k_1,2J+1)}{2J+1}, -J + \mathrm{mod}(k_1,2J+1) - 1\right).$$

(2) If $\mathrm{mod}(k_1,2J+1) = 0$:

$$\nu(k) = \left(J - K + 1 + \frac{k_1 - \mathrm{mod}(k_1,2J+1)}{2J+1} - 1, J\right).$$

In the same manner, the symbol $\mu(m)$ can be used to denote the location of the m-th exterior layer node (black node in Fig. 1).

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