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A CONVERGENT FINITE VOLUME SCHEME FOR THE STOCHASTIC BAROTROPIC COMPRESSIBLE EULER EQUATIONS

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Abstract. In this paper, we analyze a semi-discrete finite volume scheme for the three-dimensional barotropic compressible Euler equations driven by a *multiplicative* Brownian noise. We derive necessary *a priori* estimates for numerical approximations, and show that the Young measure generated by the numerical approximations converge to a *dissipative measure-valued martingale* solution to the stochastic compressible Euler system. These solutions are probabilistically weak in the sense that the driving noise and associated filtration are integral part of the solution. Moreover, we demonstrate *strong* convergence of numerical solutions to the regular solution of the limit systems at least on the lifespan of the latter, thanks to the weak (measure-valued)–strong uniqueness principle for the underlying system. To the best of our knowledge, this is the first attempt to prove the convergence of numerical approximations for the underlying system.

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1. Introduction

Most real world models involve a large number of parameters and coefficients which cannot be exactly determined. Furthermore, there is a considerable uncertainty in the source terms, initial or boundary data due to empirical approximations or measuring errors. Therefore, study of PDEs with randomness (stochastic PDEs) certainly leads to greater understanding of the actual physical phenomenon. In this paper, we are interested in a stochastic variant of the *compressible barotropic Euler system*, a set of balance laws driven by a *nonlinear multiplicative* noise for mass density ϱ and the bulk velocity $\mathbf{u}$ describing the flow of isentropic gas, where the thermal effects are neglected. The system of equations read

$$\begin{aligned} \mathrm{d}\varrho + \operatorname{div}(\varrho\mathbf{u})\,\mathrm{d}t &= 0, \\ \mathrm{d}(\varrho\mathbf{u}) + [\operatorname{div}(\varrho\mathbf{u}\otimes\mathbf{u}) + a\nabla\varrho^{\gamma}]\,\mathrm{d}t &= \Psi(\varrho,\varrho\mathbf{u})\,\mathrm{d}W. \end{aligned} \tag{1.1}$$

Here $\gamma > 1$ denotes the adiabatic exponent, $a > 0$ is the squared reciprocal of the Mach number (the ratio between average velocity and speed of sound). The driving process W is a cylindrical Wiener process defined on

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some filtered probability space $(\Omega, \mathbb{F}, \{\mathbb{F}_t\}_{t\geq 0}, \mathbb{P})$, and the noise coefficient Ψ is nonlinear and satisfies suitable growth assumptions (see Sect. 2.2 for the complete list of assumptions). Note that $(\varrho, \varrho\mathbf{u}) \mapsto \Psi(\varrho, \varrho\mathbf{u})$ is a given Hilbert space valued function signifying the *multiplicative* nature of the noise. We consider the stochastic compressible Euler equations (1.1) and (1.2) in three spatial dimensions on a periodic domain *i.e.*, on the torus $\mathbb{T}^3$. The initial conditions are random variables

$$\varrho(0, \cdot) = \varrho_0, \ \ \varrho\mathbf{u}(0, \cdot) = (\varrho\mathbf{u})_0, \tag{1.2}$$

with sufficient spatial regularity to be specified later.

1.1. Compressible Euler equations

The deterministic counterpart of the stochastic compressible Euler equations (1.1) and (1.2) have received considerable attention and, in spite of monumental efforts, satisfactory well-posedness results are still lacking. It is well-known that the smooth solution to deterministic counterpart of (1.1) and (1.2) exists only for a finite lap of time, after which singularities may develop for a generic class of initial data. Therefore, global-in-time (weak) solutions must be sought in the class of discontinuous functions. But, weak solutions may not be uniquely determined by their initial data and admissibility conditions must be imposed to single out the physically correct solution. However, the specification of such an admissibility criterion is still open. Indeed, thanks to recent phenomenal work by De Lellis and Szekelyhidi [17,18], and further investigated by Chiodaroli *et al.* [16], Feireisl [21], Buckmaster *et al.* [13,14], it is well understood that the compressible Euler equations is desparetly ill-posed, due to the lack of compactness of functions satisfying the equations. Even if the initial data is smooth, the global existence and uniqueness of solutions can fail. Moreover, a quest for the existence of global-in-time weak solutions to deterministic counterpart of (1.1) and (1.2) for *general* initial data remains elusive. Given this status quo, it is natural to seek an alternative solution paradigm for compressible Euler system. To that context, we recall the framework of *dissipative* Young measure-valued solutions in the context of compressible Navier–Stokes system, being first introduced by Neustupa [40], and subsequently revisited by Feireisl *et al.* [23]. In a nutshell, these solutions are characterized by a parametrized Young measure and a concentration Young measure in the total energy balance, and they are defined globally in time. For weak-strong uniqueness results related to the compressible fluid models, consult [29].

The study of stochastic compressible Euler equations (1.1) and (1.2) is a relatively new area of focus within the broarder field of stochastic PDEs, and a satisfactory well/ill-posedness result is largely out of reach (for a well-posedness result, see [11]). However, we want to emphasize that, to design efficient numerical schemes it is of paramount importance to have prior knowledge about the existence of global-in-time solutions for the underlying system of equations. Without such knowledge, there is no way to establish whether or not the solution produced by a numerical scheme is an approximation of the true solution. To that context, let us first mention the work by Berthelin and Vovelle [3], where the authors established the existence of a martingale solution for (1.1) and (1.2) in *one* spatial dimension. Moreover, a recent work by Breit *et al.* [10] revealed that ill-posedness issues for compressible Euler system driven by *additive* noise, in the sense of [17,21], persist even in the presense of a random forcing. We mention that for compressible Euler equations driven by a *multiplicative* noise, the existence of *dissipative measure-valued martingale* solutions was very recently established by Hofmanova *et al.* [30], and Breit *et al.* [8] (see also [15] for the incompressible case). The authors have shown that the existence can be obtained from a sequence of solutions of stochastic Navier–Stokes equations using tools from martingale theory and Young measure theory.

1.2. Numerical schemes

Parallel to mathematical efforts there has been a huge effort to derive effective numerical schemes for deterministic fluid flow equations, and there is a considerable body of literature dealing with the convergence of numerical schemes for the specific problems in fluid mechanics represented through the barotropic Euler system. In this context, we first mention the work by Karper [32] where he has established the convergence of a mixed finite element-discontinuous Galerkin scheme to compressible Navier–Stokes system under the assumption $\gamma > 3$. Subsequently, a series of works [22, 24, 25] by Feireisl and his collaborators analyzed the convergence issues for several different semi-discrete numerical schemes *via* the framework of dissipative measure-valued solutions. Note that the concept of measure-valued solutions introduced by Feireisl *et al.* [24] (and also [30]) requires natural energy bounds for approximate solutions. In [24], they showed that Lax–Friedrichs-type finite volume schemes generate a dissipative measure-valued solution to the barotropic Euler equations. We also mention that the first numerical evidence that indicated ill-posedness of the Euler system was presented by Elling [19]. Finally, we mention a series of recent works by Fjordholm *et al.* [26, 27] in the context of a general system of hyperbolic conservation laws, where they proved the convergence of a semi-discrete entropy stable finite volume scheme to a *measure-valued* solution under certain appropriate assumptions.

We remark that, despite the growing interest about the theory of stochastic PDEs and the discretization of stochastic PDEs, the specific question about numerical approximations of stochastic compressible Euler equations is virtually untouched. In fact, the challenges related to numerical aspects of (1.1) are manifold and mostly open, due to the presence of *multiplicative* noise term in (1.1).

1.3. Scope and outline of the paper

The above discussions clearly highlight the lack of effective convergent numerical schemes, for compressible fluid flow equations driven by a *multiplicative* Brownian noise, which are able to take the inherent uncertainties into account, and are equipped with modules that quantify the level of uncertainty. The challenges related to numerical aspects of the underlying problems are mostly open and the research on this frontier is still in its infancy. In fact, the main objective of this article is to lay down the foundation for a comprehensive theory related to numerical methods for (1.1) and (1.2). Although our work bears some similarities with recent wroks of Fjordholm *et al.* [26, 27] on deteministic system of conservation laws, and works of Feireisl *et al.* [22, 24, 25] on deterministic Euler systems, the main novelty of this work lies in successfully handling the *multiplicative* noise term. Our problems need to invoke ideas from numerical methods for SDE and meaningfully fuse them with available approximation methods for deterministic problems. This is easier said than done as any such attempt has to capture the noise-noise interaction (cross variation) as well. In the realm of stochastic conservation laws, noise-noise interaction terms play a fundamental role to establish well-posedness theory, for details see [4–6, 33–37].

The main contributions of this paper are listed below:

(1) We develop an appropriate mathematical framework of *dissipative measure-valued martingale* solutions to the stochastic compressible Euler system, keeping in mind that this framework would allow us to establish weak (measure-valued)–strong uniqueness principle. We remark that our solution framework requires only natural energy bounds associated to approximate solutions.
(2) We show that a Lax–Friedrichs-type numerical scheme for (1.1) and (1.2) generates a *dissipative measure-valued martingale* solution to the stochastic compressible Euler equations. With the help of the new framework based on the theory of measure-valued solutions, we adapt the concept of $\mathcal{K}$-convergence, first developed in the context of Young measures by Balder [1] (see also Feireisl *et al.* [25]), to show the pointwise convergence of arithmetic averages (Cesaro means) of numerical solutions to a *dissipative measure-valued martingale* solution of the limit system (1.1) and (1.2).
(3) When solutions of the limit continuous problem possess maximal regularity, by making use of weak (measure-valued)–strong uniqueness principle, we show *unconditional* strong L^1-convergence of numerical approximations to the regular solution of the limit systems.

A brief description of the organization of the rest of the paper is as follows: we describe all necessary mathematical/technical framework and state the main results in Section 2. Moreover, we introduce a Lax–Friedrichs-type finite volume numerical scheme for the underlying system (1.1) and (1.2). Section 3 is devoted on deriving stability properties of the scheme, while Section 4 is focused on deriving suitable formulations of the continuity and momentum equations, and exhibit consistency. In Section 5, we present a proof of convergence of numerical solutions to a dissipative measure-valued martingale solution using stochastic compactness. Section 6 is devoted on deriving the weak (measure-valued) – strong uniqueness principle by making use of a suitable relative energy inequality. Section 7 uses the concept of $\mathcal{K}$-convergence to exhibit the pointwise convergence of numerical solutions. Finally, in Section 8, we make use of weak (measure-valued)–strong uniqueness property to show the convergence of numerical approximations to the solution of stochastic compressible Euler system (1.1) and (1.2).

2. Preliminaries and main results

Here we first briefly recall some relevant mathematical tools which are used in the subsequent analysis and then we state main results of this paper. To begin, we fix an arbitrary large time horizon $T > 0$. For the sake of simplicity it will be assumed $a = 1$, since its value is not relevant in the present setting. Throughout this paper, we use the letter C to denote various generic constants that may change from line to line along the proofs. Explicit tracking of the constants could be possible but it is highly cumbersome and avoided for the sake of the reader. Let $\mathcal{M}_b(E)$ denote the space of bounded Borel measures on a metric space E whose norm is given by the total variation of measures. It is the dual space to the space of continuous functions vanishing at infinity $C_0(E)$ equipped with the supremum norm. Moreover, let $\mathcal{P}(E)$ be the space of probability measures on E.

2.1. Analytic framework

Let $\gamma \in (0,1)$, $p > 1$ be given, and Z be a separable Hilbert space. Let $W^{\gamma,p}(0,T;Z)$ denotes a Z-valued Sobolev space which is characterized by its norm

$$\|g\|^p_{W^{\gamma,p}(0,T;Z)} := \int_0^T \|g(t)\|_Z^p \,\mathrm{d}t + \int_0^T \int_0^T \frac{\|g(t)-g(s)\|_Z^p}{|t-s|^{1+\gamma p}} \,\mathrm{d}t \,\mathrm{d}s.$$

Then we have following compact embedding result from Flandoli and Gatarek ([28], Thm. 2.2).

Lemma 2.1. *If $Z \subset\subset Y$ are two Banach spaces with compact embedding, and real numbers $\gamma \in (0,1)$, $p > 1$ satisfy $\gamma p > 1$, then the following embedding*

$$W^{\gamma,p}(0,T;Z) \subset\subset C([0,T];Y)$$

is compact.

2.1.1. Young measures, concentration defect measures

In this subsection, we first briefly recall the notion of Young measures and related results which have been used frequently in the text. For an excellent overview of applications of the Young measure theory to hyperbolic conservation laws, we refer to Balder [1]. Let us begin by assuming that $(Z, \mathcal{N}, \mu)$ is a sigma finite measure space. A Young measure from Z into $\mathbb{R}^M$ is a weakly measurable function $\mathcal{V} : Z \to \mathcal{P}(\mathbb{R}^M)$ in the sense that $x \to \mathcal{V}_x(A)$ is $\mathcal{N}$-measurable for every Borel set A in $\mathbb{R}^M$. In what follows, we make use of the following generalization of the classical result on Young measures; for details, see Section 2.8 of [9].

Lemma 2.2. *Let $N, M \in \mathbb{N}$, $\mathcal{Q} \subset \mathbb{R}^N \times (0,T)$ be connected and bounded subset and let $(\mathbf{W}_n)_{n\in\mathbb{N}}$, $\mathbf{W}_n : \Omega \times \mathcal{Q} \to \mathbb{R}^M$, be a sequence of random variables such that*

$$\mathbb{E}\Big[\|\mathbf{W}_n\|^p_{L^p(\mathcal{Q})}\Big] \le C, \text{ for a certain } p \in (1,\infty).$$

Then on the standard probability space $\big([0,1], \overline{\mathcal{B}[0,1]}, \mathcal{L}^1\big)$, there exists a new subsequence $(\widetilde{\mathbf{W}}_n)_{n\in\mathbb{N}}$ (not relabeled), and a parametrized family $\{\widetilde{\mathcal{V}}^\omega_y\}_{y\in\mathcal{Q}}$ (superscript ω emphasises the dependence on ω) of random probability measures on $\mathbb{R}^M$, regarded as a random variable taking values in $\big(L^\infty_{w^*}(\mathcal{Q};\mathcal{P}(\mathbb{R}^M)), w^*\big)$, such that $\mathbf{W}_n$ has the *same law as* $\widetilde{\mathbf{W}}_n$*, i.e.,* $\mathcal{L}(\mathbf{W}_n) = \mathcal{L}(\widetilde{\mathbf{W}}_n)$*, and the following property holds: for any Carathéodory function* $J = J(y, Z), y \in \mathcal{Q}, Z \in \mathbb{R}^M$*, such that*

$$|J(y, Z)| \leq C(1 + |Z|^q), \text{ for some } 1 \leq q < p, \text{ uniformly in } y,$$

implies $\mathcal{L}^1$*-a.s.,*

$$J\Big(\cdot, \widetilde{\mathbf{W}}_n\Big) \rightharpoonup \overline{J} \text{ in } L^{p/q}(\mathcal{Q}), \text{ where } \overline{J}(y) = \Big\langle \widetilde{\mathcal{V}}^{\omega}_{(\cdot)}; J(y, \cdot) \Big\rangle := \int_{\mathbb{R}^M} J(y, z) \, \mathrm{d}\widetilde{\mathcal{V}}^{\omega}_y(z), \text{ for a.a. } y \in \mathcal{Q}.$$

In literature, Young measure theory has been successfully exploited to extract limits of bounded continuous functions. However, for our purpose, we need to deal with typical functions F for which we only know that

$$\mathbb{E}\Big[\|F(\mathbf{W}_n)\|^p_{L^1(\mathcal{Q})}\Big] \leq C, \text{ for a certain } p \in (1, \infty), \text{ uniformly in } n.$$

In fact, using a well-known fact that $L^1(\mathcal{Q})$ is embedded in the space of bounded Radon measures $\mathcal{M}_b(\mathcal{Q})$, we can infer that $\mathbb{P}$-a.s.

$$\text{weak-* limit in } \mathcal{M}_b(\mathcal{Q}) \text{ of } F(\mathbf{W}_n) \text{ is } \Big\langle \widetilde{\mathcal{V}}^{\omega}_y; F \Big\rangle \, \mathrm{d}y + F_\infty,$$

where $F_\infty \in \mathcal{M}_b(\mathcal{Q})$, and F_∞ is called *concentration defect measure (or concentration Young measure)*. We remark that, a simple truncation analysis and Fatou's lemma reveal that $\mathbb{P}$-a.s. $\|\langle \widetilde{\mathcal{V}}^{\omega}_{(\cdot)}; F \rangle\|_{L^1(\mathcal{Q})} \leq C$ and thus $\mathbb{P}$-a.s. $\langle \widetilde{\mathcal{V}}^{\omega}_y; F \rangle$ is finite for a.e. $y \in \mathcal{Q}$. In what follows, regarding the concentration defect measure, we shall make use of the following crucial lemma. For a proof of the lemma modulo cosmetic changes, we refer to Feireisl *et al.* ([23], Lem. 2.1).

Lemma 2.3. *Let* $\{\mathbf{W}_n\}_{n>0}$*,* $\mathbf{W}_n : \Omega \times \mathcal{Q} \to \mathbb{R}^M$ *be a sequence generating a Young measure* $\{\widetilde{\mathcal{V}}^{\omega}_y\}_{y \in \mathcal{Q}}$*, where* $\mathcal{Q}$ *is a measurable set in* $\mathbb{R}^N \times (0, T)$*. Let* $G : \mathbb{R}^M \to [0, \infty)$ *be a continuous function such that*

$$\sup_{n>0} \mathbb{E}\Big[\|G(\mathbf{W}_n)\|^p_{L^1(\mathcal{Q})}\Big] < \infty, \text{ for a certain } p \in (1, \infty),$$

and let F *be continuous such that*

$$F : \mathbb{R}^M \to \mathbb{R}, \quad |F(\boldsymbol{z})| \leq G(\boldsymbol{z}), \quad \text{for all } \boldsymbol{z} \in \mathbb{R}^M.$$

Let us denote $\mathbb{P}$*-a.s.*

$$F_\infty := \widetilde{F} - \Big\langle \widetilde{\mathcal{V}}^{\omega}_y, F(\boldsymbol{v}) \Big\rangle \, \mathrm{d}y, \quad G_\infty := \widetilde{G} - \Big\langle \widetilde{\mathcal{V}}^{\omega}_y, G(\boldsymbol{v}) \Big\rangle \, \mathrm{d}y.$$

Here $\widetilde{F}, \widetilde{G} \in \mathcal{M}_b(\mathcal{Q})$ *are weak-$*$ limits of* $\{F(\mathbf{W}^n)\}_{n>0}$*,* $\{G(\mathbf{W}^n)\}_{n>0}$ *respectively in* $\mathcal{M}_b(\mathcal{Q})$*. Then* $\mathbb{P}$*-almost surely* $|F_\infty| \leq G_\infty$*.*

2.1.2. Convergence of arithmetic averages

Following Feireisl *et al.* [25], we also show that the arithmetic averages of numerical solutions converge pointwise to a generalized dissipative solution of the compressible Euler system, as introduced in Hofmanova *et al.* [30]. To that context, we have the following result.

Proposition 2.4. *Let* $(X, \mathcal{A}, \mu)$ *be a finite measure space, and* $\mathbf{U}_n \rightharpoonup \mathbf{U}$ *weakly in* $L^1(X; \mathbb{R}^M)$*. Then there exists a subsequence* $(\mathbf{U}_{n_k})_{k \geq 1}$ *of sequence* $(\mathbf{U}_n)_{n \geq 1}$ *such that*

$$\frac{1}{n} \sum_{k=1}^{n} \mathbf{U}_{n_k} \to \mathbf{U}, \quad \text{a.e. in } X.$$

Proof. Since the sequence $(\mathbf{U}_n)_{n\geq 1}$ is uniformly bounded in $L^1(X)$, thanks to Komlós theorem, there exists a subsequence $(\mathbf{U}_{n_k})_{k\geq 1}$ and $\tilde{\mathbf{U}} \in L^1(X)$ such that

$$\frac{1}{n}\sum_{k=1}^{n} \mathbf{U}_{n_k} \to \tilde{\mathbf{U}}, \quad \text{a.e. in } X.$$

Let us define $\mathbf{V}_n := \frac{1}{n}\sum_{k=1}^{n} \mathbf{U}_{n_k}$. Since $\mathbf{U}_{n_k}$ also converges weakly to $\mathbf{U}$, it implies that $\mathbf{V}_n$ converges weakly to $\mathbf{U}$ in $L^1(X)$. So the sequence V_n is uniformly integrable in $L^1(X)$. As a consequence of Vitali's convergence theorem, it implies that $\mathbf{V}_n$ converges to $\tilde{\mathbf{U}}$ strongly in $L^1(X)$. Therefore, uniqueness of weak limit implies that $\mathbf{U} = \tilde{\mathbf{U}}$ in $L^1(X)$. This concludes the proof. □

2.2. Background on stochastic framework

Here we briefly recapitulate some basics of stochastic calculus in order to define the cylindrical Wiener process W and the stochastic integral appearing in (1.1). To that context, let $(\Omega, \mathbb{F}, (\mathbb{F}_t)_{t\geq 0}, \mathbb{P})$ be a stochastic basis with a complete, right-continuous filtration. The stochastic process W is a cylindrical $(\mathbb{F}_t)$-Wiener process in a separable Hilbert space $\mathfrak{W}$. It is formally given by the expansion

$$W(t) = \sum_{k\geq 1} e_k W_k(t),$$

where $\{W_k\}_{k\geq 1}$ is a sequence of mutually independent real-valued Brownian motions relative to $(\mathbb{F}_t)_{t\geq 0}$ and $\{e_k\}_{k\geq 1}$ is an orthonormal basis of $\mathfrak{W}$. To give the precise definition of the diffusion coefficient Ψ, consider $\varrho \in L^\gamma(\mathbb{T}^3)$, $\varrho \geq 0$, and $\mathbf{u} \in L^2(\mathbb{T}^3)$ such that $\sqrt{\varrho}\mathbf{u} \in L^2(\mathbb{T}^3)$. Denote $\mathbf{m} = \varrho\mathbf{u}$ and let $\Psi(\varrho, \mathbf{m}) : \mathfrak{W} \to L^1(\mathbb{T}^3)$ be defined as follows

$$\Psi(\varrho, \mathbf{m}) e_k = \Psi_k(\cdot, \varrho(\cdot), \mathbf{m}(\cdot)).$$

The coefficients $\Psi_k : \mathbb{T}^3 \times \mathbb{R} \times \mathbb{R}^3 \to \mathbb{R}^3$ are C^1-functions that satisfy uniformly in $x \in \mathbb{T}^3$

$$\Psi_k(\cdot, 0, 0) = 0 \tag{2.1}$$

$$|\partial_\varrho \Psi_k| + |\nabla_{\mathbf{m}} \Psi_k| \leq \beta_k, \quad \sum_{k\geq 1} \beta_k < \infty. \tag{2.2}$$

As usual, we understand the stochastic integral as a process in the Hilbert space $W^{-m,2}(\mathbb{T}^3)$, $m > 3/2$. Indeed, it is easy to check that under the above assumptions on ϱ and $\mathbf{m}$, the mapping $\Psi(\varrho, \varrho\mathbf{u})$ belongs to $L_2(\mathfrak{W}; W^{-m,2}(\mathbb{T}^3))$, the space of Hilbert–Schmidt operators from $\mathfrak{W}$ to $W^{-m,2}(\mathbb{T}^3)$. Consequently, if[1]

$$\varrho \in L^\gamma\big(\Omega \times (0,T), \mathcal{P}, \mathrm{d}\mathbb{P} \otimes \mathrm{d}t; L^\gamma\big(\mathbb{T}^3\big)\big),$$
$$\sqrt{\varrho}\mathbf{u} \in L^2\big(\Omega \times (0,T), \mathcal{P}, \mathrm{d}\mathbb{P} \otimes \mathrm{d}t; L^2\big(\mathbb{T}^3\big)\big),$$

and the mean value $(\varrho(t))_{\mathbb{T}^3}$ is essentially bounded then the stochastic integral

$$\int_0^t \Psi(\varrho, \varrho\mathbf{u})\, \mathrm{d}W = \sum_{k\geq 1} \int_0^t \Psi_k(\cdot, \varrho, \varrho\mathbf{u})\, \mathrm{d}W_k$$

is a well-defined $(\mathbb{F}_t)$-martingale taking values in $W^{-m,2}(\mathbb{T}^3)$. Note that the continuity equation (1.1) implies that the mean value $(\varrho(t))_{\mathbb{T}^3}$ of the density ϱ is constant in time (but in general depends on ω). Finally, we define the auxiliary space $\mathfrak{W}_0 \supset \mathfrak{W}$ *via*

$$\mathfrak{W}_0 := \left\{ \mathbf{u} = \sum_{k\geq 1} \gamma_k e_k;\ \sum_{k\geq 1} \frac{\gamma_k^2}{k^2} < \infty \right\},$$

[1]Here $\mathcal{P}$ denotes the predictable σ-algebra associated to $(\mathbb{F}_t)$.

endowed with the norm

$$\|\mathbf{u}\|_{\mathfrak{W}_0}^2 = \sum_{k\geq 1} \frac{\gamma_k^2}{k^2}, \quad \mathbf{u} = \sum_{k\geq 1} \gamma_k e_k.$$

Note that the embedding $\mathfrak{W} \hookrightarrow \mathfrak{W}_0$ is Hilbert–Schmidt. Moreover, trajectories of W are $\mathbb{P}$-a.s. in $C([0,T];\mathfrak{W}_0)$.

For the convergence of approximate solutions, it is necessary to secure strong compactness (a.s. convergence) in the ω-variable. For that purpose, we need a version of Skorokhod representation theorem, so-called Skorokhod–Jakubowski representation theorem. Note that classical Skorokhod theorem only works for Polish spaces, but in our analysis path spaces are so-called quasi-Polish spaces. In this paper, we use the following version of the Skorokhod–Jakubowski theorem, taken from Motyl ([39], Cor. 7.3).

Theorem 2.5. *Let $\mathcal{X}$ be a complete separable metric space and $\mathcal{Y}$ be a topological space such that there is a sequence of continuous functions $g_n : \mathcal{Y} \to \mathbb{R}$ that separates points of $\mathcal{Y}$. Let $(\Omega, \mathbb{F}, (\mathbb{F}_t)_{t\geq 0}, \mathbb{P})$ be a stochastic basis with a complete, right-continuous filtration and $(\xi_n)_{n\in\mathbb{N}}$ be a tight sequence of random variables in $(\mathcal{Z}, \mathcal{B}(\mathcal{X}) \otimes \mathcal{M})$, where $\mathcal{Z} = \mathcal{X} \times \mathcal{Y}$ and $\mathcal{Z}$ is equipped with the topology induced by the canonical projections $\Pi_1 : \mathcal{Z} \to \mathcal{X}$ and $\Pi_2 : \mathcal{Z} \to \mathcal{Y}$. Here $\mathcal{M}$ is the σ-algebra generated by the sequence ξ_n, $n \in \mathbb{N}$.*

Assume that there exists a random variable η in $\mathcal{X}$ such that[2] $\mathcal{L}(\Pi_1 \circ \xi_n) = \mathcal{L}(\eta)$. Then there exists a subsequence $(\xi_{n_k})_{k\in\mathbb{N}}$ and random variables $\tilde{\xi}_k, \tilde{\xi}$ in $\mathcal{Z}$ for $k \in \mathbb{N}$ on a common probability space $(\tilde{\Omega}, \tilde{\mathbb{F}}, \tilde{\mathbb{P}})$ with

(a) $\mathcal{L}(\tilde{\xi}_k) = \mathcal{L}(\xi_{n_k})$
(b) $\tilde{\xi}_k \to \tilde{\xi}$ *in $\mathcal{Z}$ almost surely for $k \to \infty$.*
(c) $\Pi_1 \circ \tilde{\xi}_k = \Pi_1 \circ \tilde{\xi}$ *almost surely.*

Finally, we mention the "Kolmogorov test" for the existence of continuous modifications of real-valued stochastic processes (for a proof, see [2]).

Lemma 2.6. *Let $X = \{X(t)\}_{t\in[0,T]}$ be a real-valued stochastic process defined on a probability space $(\Omega, \mathbb{F}, (\mathbb{F}_t)_{t\geq 0}, \mathbb{P})$. Suppose that there are constants $a > 1, b > 0$, and $C > 0$ such that for all $s, t \in [0,T]$,*

$$\mathbb{E}[|X(t) - X(s)|^a] \leq C|t - s|^{1+b}.$$

Then there exists a continuous modification of X and the paths of X are c-Hölder continuous for every $c \in [0, \frac{b}{a})$.

2.3. Stochastic compressible Euler equations

Since we aim to prove pointwise convergence of numerical solutions to the regular solution of the limit system, using the weak (measure-valued)–strong uniqueness principle for dissipative measure-valued solutions, we first recall the notion of local strong pathwise solution for stochastic compressible Euler equations, being first introduced in [7]. Such a solution is strong in both the probabilistic and PDE sense, at least locally in time. To be more precise, system (1.1) and (1.2) will be satisfied pointwise (not only in the sense of distributions) on the given stochastic basis associated to the cylindrical Wiener process W.

Definition 2.7 (Local strong pathwise solution)**.** Let $(\Omega, \mathbb{F}, (\mathbb{F}_t)_{t\geq 0}, \mathbb{P})$ be a stochastic basis with a complete right-continuous filtration. Let W be an $(\mathbb{F}_t)$-cylindrical Wiener process and $(\varrho_0, \mathbf{v}_0)$ be a $W^{m,2}(\mathbb{T}^3) \times W^{m,2}(\mathbb{T}^3)$-valued $\mathbb{F}_0$-measurable random variable, for some $m > 7/2$, and let Ψ satisfy (2.1) and (2.2). A triplet $(\varrho, \mathbf{v}, \mathfrak{t})$ is called a local strong pathwise solution to the system (1.1) and (1.2) provided

(1) $\mathfrak{t}$ is an a.s. strictly positive $(\mathbb{F}_t)$-stopping time;
(2) the density ϱ is a $W^{m,2}(\mathbb{T}^3)$-valued $(\mathbb{F}_t)$-progressively measurable process satisfying

$$\varrho(\cdot \wedge \mathfrak{t}) > 0, \ \ \varrho(\cdot \wedge \mathfrak{t}) \in C\big([0,T]; W^{m,2}(\mathbb{T}^3)\big) \quad \mathbb{P}\text{-a.s.};$$

[2]Here $\mathcal{L}(\xi)$ denotes the law of the random variable ξ.

(3) the velocity $\mathbf{v}$ is a $W^{m,2}(\mathbb{T}^3)$-valued $(\mathbb{F}_t)$-progressively measurable process satisfying

$$\mathbf{v}(\cdot \wedge \mathfrak{t}) \in C\big([0,T]; W^{m,2}(\mathbb{T}^3)\big) \quad \mathbb{P}\text{-a.s.};$$

(4) there holds $\mathbb{P}$-a.s.

$$\varrho(t \wedge \mathfrak{t}) = \varrho_0 - \int_0^{t\wedge\mathfrak{t}} \operatorname{div}(\varrho\mathbf{v})\,\mathrm{d}s,$$
$$(\varrho\mathbf{v})(t\wedge\mathfrak{t}) = \varrho_0\mathbf{v}_0 - \int_0^{t\wedge\mathfrak{t}} \operatorname{div}(\varrho\mathbf{v}\otimes\mathbf{v})\,\mathrm{d}s - \int_0^{t\wedge\mathfrak{t}} a\nabla\varrho^\gamma\,\mathrm{d}s + \int_0^{t\wedge\mathfrak{t}} \Psi(\varrho,\varrho\mathbf{v})\,\mathrm{d}W,$$

for all $t \in [0,T]$.

Note that classical solutions require spatial derivatives of $\mathbf{v}$ and ϱ to be continuous $\mathbb{P}$-a.s. This motivates the following definition.

Definition 2.8 (Maximal strong pathwise solution)**.** Fix a stochastic basis with a cylindrical Wiener process and an initial condition as in Definition 2.7. A quadruplet

$$(\varrho, \mathbf{v}, (\mathfrak{t}_R)_{R\in\mathbb{N}}, \mathfrak{t})$$

is a maximal strong pathwise solution to system (1.1) and (1.2) provided

(1) $\mathfrak{t}$ is an a.s. strictly positive $(\mathbb{F}_t)$-stopping time;
(2) $(\mathfrak{t}_R)_{R\in\mathbb{N}}$ is an increasing sequence of $(\mathbb{F}_t)$-stopping times such that $\mathfrak{t}_R < \mathfrak{t}$ on the set $[\mathfrak{t} < T]$, $\lim_{R\to\infty}\mathfrak{t}_R = \mathfrak{t}$ a.s. and

$$\sup_{t\in[0,\mathfrak{t}_R]} \|\mathbf{v}(t)\|_{1,\infty} \geq R \quad \text{on} \quad [\mathfrak{t} < T];$$

(3) each triplet $(\varrho, \mathbf{v}, \mathfrak{t}_R)$, $R \in \mathbb{N}$, is a local strong pathwise solution in the sense of Definition 2.7.

There are quite a few results available in the literature concerning the existence of maximal pathwise solutions for various SPDE or SDE models, see for instance [12,20]. For compressible Euler equations, a specific work can be found in Breit and Mensah ([7], Thm. 2.4).

Theorem 2.9. *Let $m > 7/2$ and the coefficients Ψ_k satisfy hypotheses (2.1), (2.2) and let $(\varrho_0, \mathbf{v}_0)$ be an $\mathbb{F}_0$-measurable, $W^{m,2}(\mathbb{T}^3)\times W^{m,2}(\mathbb{T}^3)$-valued random variable such that $\varrho_0 > 0$, $\mathbb{P}$-a.s. Then there exists a unique maximal strong pathwise solution, in the sense of Definition 2.8, $(\varrho, \mathbf{v}, (\mathfrak{t}_R)_{R\in\mathbb{N}}, \mathfrak{t})$ to problem (1.1) and (1.2) with the initial condition $(\varrho_0, \mathbf{v}_0)$.*

2.4. Measure-valued solutions

For the introduction of measure-valued solutions, it is convenient to work with the following reformulation of the problem (1.1) and (1.2) in the *conservative* variables ϱ and $\mathbf{m} = \varrho\mathbf{u}$:

$$\mathrm{d}\varrho + \operatorname{div}\mathbf{m}\,\mathrm{d}t = 0, \tag{2.3}$$
$$\mathrm{d}\mathbf{m} + \left[\operatorname{div}\left(\frac{\mathbf{m}\otimes\mathbf{m}}{\varrho}\right) + \nabla p(\varrho)\right]\mathrm{d}t = \Psi(\varrho,\mathbf{m})\,\mathrm{d}W \tag{2.4}$$
$$(\varrho(0), \mathbf{m}(0)) = (\varrho_0, \mathbf{m}_0 = (\varrho\mathbf{u})_0), \tag{2.5}$$

where $p(\varrho) = a\varrho^\gamma$. For later purpose, we define P with the following relation

$$P(\varrho) := \varrho\int_1^\varrho \frac{p(z)}{z^2}\,\mathrm{d}z = \frac{a\varrho^\gamma}{\gamma-1}\cdot \tag{2.6}$$

Note that, in general any uniformly bounded sequence in $L^1(\mathbb{T}^3)$ does not immediately imply weak convergence of it due to the presence of oscillations and concentration effects. To overcome such a problem, two kinds of tools are used:

(a) Young measures: these are probability measures on the phase space and accounts for the persistence of oscillations in the solution;
(b) Concentration defect measures: these are measures on physical space-time, accounts for blow up type collapse due to possible concentration points.

2.4.1. Dissipative measure-valued martingale solutions

Keeping in mind the previous discussion, we now introduce the concept of *dissipative measure-valued martingale solution* to the stochastic compressible Euler system. In what follows, let

$$\mathcal{M} = \Big\{ [\varrho, \mathbf{m}] \ \Big| \ \varrho > 0, \ \mathbf{m} \in \mathbb{R}^3 \Big\}$$

be the phase space associated to the Euler system.

Definition 2.10 (Dissipative measure-valued martingale solution)**.** Let Λ be a Borel probability measure on $L^{\gamma}(\mathbb{T}^3) \times L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^3)$. Then $[(\Omega, \mathbb{F}, (\mathbb{F}_t)_{t\geq 0}, \mathbb{P}); \mathcal{V}^{\omega}_{t,x}, W]$ is a dissipative measure-valued martingale solution of (2.3) and (2.4), with initial condition $\mathcal{V}^{\omega}_{0,x} \in L^{\infty}_{w^*}(\mathbb{T}^3; \mathcal{P}(\mathcal{M})), \mathbb{P}$-almost surely; if

(1) $\mathcal{V}^{\omega}$ is a random variable taking values in the space of Young measures on $L^{\infty}_{w^*}\big([0,T] \times \mathbb{T}^3; \mathcal{P}(\mathcal{M})\big)$. In other words, $\mathbb{P}$-a.s. $\mathcal{V}^{\omega}_{t,x} : (t,x) \in [0,T] \times \mathbb{T}^3 \to \mathcal{P}(\mathcal{M})$ is a parametrized family of probability measures on $\mathcal{M}$,
(2) $\big(\Omega, \mathbb{F}, (\mathbb{F}_t)_{t\geq 0}, \mathbb{P}\big)$ is a stochastic basis with a complete right-continuous filtration,
(3) W is a $(\mathbb{F}_t)$-cylindrical Wiener process,
(4) the average density $\langle \mathcal{V}^{\omega}_{t,x}; \varrho \rangle$ satisfies $t \mapsto \langle \langle \mathcal{V}^{\omega}_{t,x}; \varrho \rangle (t,\cdot), \varphi \rangle \in C[0,T]$ for any $\varphi \in C^{\infty}(\mathbb{T}^3)$ $\mathbb{P}$-a.s., the function $t \mapsto \langle \langle \mathcal{V}^{\omega}_{t,x}; \varrho \rangle (t,\cdot), \varphi \rangle$ is progressively measurable and

$$\mathbb{E}\left[\sup_{t\in(0,T)} \big\| \big\langle \mathcal{V}^{\omega}_{t,x}; \varrho \big\rangle (t,\cdot) \big\|^p_{L^{\gamma}(\mathbb{T}^3)} \right] < \infty \tag{2.7}$$

for all $1 \leq p < \infty$,
(5) the average momentum $\langle \mathcal{V}^{\omega}_{t,x}; \mathbf{m} \rangle$ satisfies $t \mapsto \langle \langle \mathcal{V}^{\omega}_{t,x}; \mathbf{m} \rangle (t,\cdot), \boldsymbol{\varphi} \rangle \in C[0,T]$ for any $\boldsymbol{\varphi} \in C^{\infty}(\mathbb{T}^3)$ $\mathbb{P}$-a.s., the function $t \mapsto \langle \langle \mathcal{V}^{\omega}_{t,x}; \mathbf{m} \rangle (t,\cdot), \boldsymbol{\varphi} \rangle$ is progressively measurable and

$$\mathbb{E}\left[\sup_{t\in(0,T)} \big\| \big\langle \mathcal{V}^{\omega}_{t,x}; \mathbf{m} \big\rangle (t,\cdot) \big\|^p_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^3)} \right] < \infty \tag{2.8}$$

for all $1 \leq p < \infty$,
(6) $\Lambda = \mathcal{L}[\mathcal{V}^{\omega}_{0,x}]$,
(7) the integral identity

$$\int_{\mathbb{T}^3} \big\langle \mathcal{V}^{\omega}_{\tau,x}; \varrho \big\rangle \, \varphi \,\mathrm{d}x - \int_{\mathbb{T}^3} \big\langle \mathcal{V}^{\omega}_{0,x}; \varrho \big\rangle \, \varphi \,\mathrm{d}x = \int_0^{\tau} \int_{\mathbb{T}^3} \big\langle \mathcal{V}^{\omega}_{t,x}; \mathbf{m} \big\rangle \cdot \nabla_x \varphi \,\mathrm{d}x \,\mathrm{d}t \tag{2.9}$$

holds $\mathbb{P}$-a.s., for all $\tau \in [0,T)$, and for all $\varphi \in C^{\infty}(\mathbb{T}^3)$,
(8) the integral identity

$$\begin{aligned} &\int_{\mathbb{T}^3} \big\langle \mathcal{V}^{\omega}_{\tau,x}; \mathbf{m} \big\rangle \cdot \boldsymbol{\varphi} \,\mathrm{d}x - \int_{\mathbb{T}^3} \big\langle \mathcal{V}^{\omega}_{0,x}; \mathbf{m} \big\rangle \cdot \boldsymbol{\varphi} \,\mathrm{d}x \\ &\quad = \int_0^{\tau} \int_{\mathbb{T}^3} \left[\left\langle \mathcal{V}^{\omega}_{t,x}; \frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} \right\rangle : \nabla_x \boldsymbol{\varphi} + \big\langle \mathcal{V}^{\omega}_{t,x}; a\varrho^{\gamma} \big\rangle \operatorname{div}_x \boldsymbol{\varphi} \right] \mathrm{d}x \,\mathrm{d}t \\ &\qquad + \int_0^{\tau} \int_{\mathbb{T}^3} \big\langle \mathcal{V}^{\omega}_{t,x}; \Psi(\varrho, \mathbf{m}) \big\rangle \cdot \boldsymbol{\varphi} \,\mathrm{d}x \,\mathrm{d}W + \int_0^{\tau} \int_{\mathbb{T}^3} \nabla_x \boldsymbol{\varphi} : \mathrm{d}\mu_m, \end{aligned} \tag{2.10}$$

holds $\mathbb{P}$-a.s., for all $\tau \in [0,T)$, and for all $\boldsymbol{\varphi} \in C^{\infty}(\mathbb{T}^3; \mathbb{R}^3)$, where $\mu_m \in L^{\infty}_{w^*}([0,T]; \mathcal{M}_b(\mathbb{T}^3))$, $\mathbb{P}$-a.s., is a tensor–valued measure,

(9) there exists a real-valued martingale M_E, such that the following energy inequality

$$\mathrm{E}(t+) \leq \mathrm{E}(s-) + \frac{1}{2}\int_s^t \left(\int_{\mathbb{T}^3} \sum_{k=1}^{\infty} \left\langle \mathcal{V}^{\omega}_{\tau,x}; \varrho^{-1} |\Psi_k(\varrho, \mathbf{m})|^2 \right\rangle \right) \mathrm{d}\tau + \frac{1}{2}\int_s^t \int_{\mathbb{T}^3} \mathrm{d}\mu_e + \int_s^t \mathrm{d}M_E \tag{2.11}$$

holds $\mathbb{P}$-a.s., for all $0 \leq s < t$ in $(0,T)$ with

$$\mathrm{E}(t-) := \liminf_{\tau\to 0+} \frac{1}{\tau}\int_{t-\tau}^t \left(\int_{\mathbb{T}^3} \left\langle \mathcal{V}^{\omega}_{s,x}; \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right\rangle \mathrm{d}x + \mathcal{D}(s) \right) \mathrm{d}s$$
$$\mathrm{E}(t+) := \liminf_{\tau\to 0+} \frac{1}{\tau}\int_{t}^{t+\tau} \left(\int_{\mathbb{T}^3} \left\langle \mathcal{V}^{\omega}_{s,x}; \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right\rangle \mathrm{d}x + \mathcal{D}(s) \right) \mathrm{d}s.$$

Here $P(\varrho) = \frac{a\varrho^\gamma}{\gamma-1}$, $\mu_e \in L^\infty_{w^*}\big([0,T]; \mathcal{M}_b(\mathbb{T}^3)\big)$, $\mathbb{P}$-a.s., $\mathcal{D} \in L^\infty(0,T)$, $\mathcal{D} \geq 0$, $\mathbb{P}$-a.s., with initial energy.

$$\mathrm{E}(0-) = \int_{\mathbb{T}^3} \left\langle \mathcal{V}^{\omega}_{0,x}; \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right\rangle \mathrm{d}x,$$

(10) there exists a constant $C > 0$ such that

$$\int_0^\tau \int_{\mathbb{T}^3} \mathrm{d}|\mu_m| + \int_0^\tau \int_{\mathbb{T}^3} \mathrm{d}|\mu_e| \leq C \int_0^\tau \mathcal{D}(t)\, \mathrm{d}t, \tag{2.12}$$

holds $\mathbb{P}$-a.s., for every $\tau \in (0,T)$.

Remark 2.11. We remark that, in light of a standard Lebesgue point argument applied to (2.11), energy inequality holds for a.e. $0 \leq s < t$ in $(0,T)$:

$$\begin{aligned}
&\int_{\mathbb{T}^3} \left\langle \mathcal{V}^{\omega}_{t,x}; \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right\rangle \mathrm{d}x + \mathcal{D}(t) \\
&\quad \leq \int_{\mathbb{T}^3} \left\langle \mathcal{V}^{\omega}_{s,x}; \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho) \right\rangle \mathrm{d}x + \mathcal{D}(s) + \frac{1}{2}\int_s^t \left(\int_{\mathbb{T}^3} \sum_{k=1}^{\infty} \left\langle \mathcal{V}^{\omega}_{\tau,x}; \varrho^{-1} |\Psi_k(\varrho, \mathbf{m})|^2 \right\rangle \right) \mathrm{d}\tau \\
&\qquad + \frac{1}{2}\int_s^t \int_{\mathbb{T}^3} \mathrm{d}\mu_e + \int_s^t \mathrm{d}M_E^2, \ \mathbb{P}\text{-a.s.}
\end{aligned} \tag{2.13}$$

However, to establish weak (measure-valued)–strong uniqueness principle, we require energy inequality to hold for *all* $s, t \in (0,T)$. This can be achieved following the argument depicted in Section 5.

Remark 2.12. Note that the above solution concept slightly differs from the dissipative measure-valued martingale solution concept introduced by Hofmanová *et al.* [30]. Indeed, the main difference lies in the successful identification of the martingale term present in (2.10), thanks to the weak continuity of Itô integral.

Following [25] we finally introduce the concept of dissipative martingale solution to the Euler system.

Definition 2.13 (Dissipative martingale solution)**.** If there exists a dissipative measure-valued martingale solution $[(\Omega, \mathbb{F}, (\mathbb{F}_t)_{t\geq 0}, \mathbb{P}); \mathcal{V}^{\omega}_{t,x}, W]$ in the sense of Definition 2.10 and we define

$$\varrho_b(x,t) = \langle \mathcal{V}^{\omega}_{t,x}; \varrho \rangle, \quad \mathbf{m}_b(x,t) = \langle \mathcal{V}^{\omega}_{t,x}; \mathbf{m} \rangle, \ \text{ for a.a. } (t,x) \in (0,T)\times\mathbb{T}^3, \tag{2.14}$$

then $(\varrho_b, \mathbf{m}_b)$ is called a dissipative martingale solution to Euler system (2.3) and (2.4).

2.5. Numerical scheme

It is well known that standard finite difference, finite volume and finite element methods have been very successful in computing solutions to system of hyperbolic conservation laws, including deterministic compressible fluid flow equations. Here we consider a semi-discrete finite volume scheme for the stochastic compressible Euler equations (1.1) and (1.2). In what follows, drawing preliminary motivation from the analysis depicted in [22,24,25], we describe the finite volume numerical scheme which is later shown to converge in appropriate sense. More precisely, we show that the sequence of numerical solutions generate the Young measure that represents the dissipative measure-valued martingale solution.

2.5.1. Spatial discretization

We begin by introducing some notation needed to define the semi-discrete finite volume scheme. Throughout this paper, we reserve the parameter h to denote small positive numbers that represent the spatial discretizations parameter of the numerical scheme. Note that, since we are working in a periodic domain in $\mathbb{R}^3$, the relevant domain for the space discretization is $[0,\ell]^3$, $\ell > 0$ with periodic boundary condition. To this end, we introduce the space discretization by finite volumes (control volumes). For that we need to recall the definition of so called admissible meshes for finite volume scheme.

Definition 2.14 (Admissible mesh)**.** An admissible mesh $\mathcal{T}$ of $[0,\ell]^3$ is a family of disjoint regular quadrilateral connected subset of $[0,\ell]^3$ satisfying the following:

(i) $[0,\ell]^3$ is the union of the closure of the elements (called control volume K) of $\mathcal{T}$, *i.e.*, $[0,\ell]^3 = \cup_{K\in\mathcal{T}}\bar{K}$.
(ii) The common interface of any two elements of $\mathcal{T}$ is included in a hyperplane of $[0,\ell]^3$.
(iii) The mesh is Cartesian and control volumes are cubes of side lengths h.

In the sequel, we denote the followings:

- E_K: the set of interfaces of the control volume K.
- $\mathcal{N}(K)$: the set of control volumes neighbours of the control volume K.
- $\sigma_{K,L}$: the common interface between K and L, for any $L \in \mathcal{N}(K)$.
- E: the set of all the interfaces of the mesh $\mathcal{T}$.
- $\overrightarrow{\mathbf{n}}_{K,L}$: the unit normal vector to interface $\sigma_{K,L}$, oriented from K to L, for any $L \in \mathcal{N}(K)$.
- $\mathbf{e}_p$: the unit basis vector in the p-th space direction, $p = 1,2,3$. Note that in our case the mesh is Cartesian, and thus $\overrightarrow{\mathbf{n}}_{K,L}$ is parallel to $\mathbf{e}_p$, for some $p = 1,2,3$.

Let $\mathcal{Y}(\mathcal{T})$ denote the space of piecewise constant functions defined on admissible mesh $\mathcal{T}$. For $w_h \in \mathcal{Y}(\mathcal{T})$, we set $w_K := w_h|_K$. Then it holds that

$$\int_{\mathbb{T}^3} w_h \,\mathrm{d}x = h^3 \sum_{K\in\mathcal{T}} w_K.$$

The value of W_h on the face $\sigma_{K,L}$ shall be denoted by $W_{\sigma_{K,L}}$, and analogously $W_{\sigma_{K,K\pm h e_p}}$ for faces $\sigma_{K,K\pm he_p}$ of cell K in $\pm\mathbf{e}_p$ direction. We also introduce a standard projection operator

$$\Pi_h : L^1(\mathbb{T}^3) \to \mathcal{Y}(\mathcal{T}), \quad (\Pi_h(\varphi))_K := \frac{1}{h^3}\int_K \varphi(x)\,\mathrm{d}x.$$

For $w_h, W_h \in \mathcal{Y}(\mathcal{T})$ we define the following discrete operators

$$\left(\widetilde{\partial_h^p} w_h\right)_K := \frac{w_L - w_J}{2h},\ \left(\partial_h^{p+} w_h\right)_K := \frac{w_L - w_K}{h},\ \left(\partial_h^{p-} w_h\right)_K := \frac{w_K - w_J}{h}, \quad L = K + h\mathbf{e}_p, J = K - h\mathbf{e}_p,$$
$$\left(\partial_h^p W_h\right)_K := \frac{W_{\sigma_{K,K+he_p}} - W_{\sigma_{K,K-he_p}}}{h}, \quad p = 1,2,3.$$

The discrete Laplace and divergence operators are defined as follows

$$(\Delta_h w_h)_K := \frac{1}{h^2}\sum_{L\in\mathcal{N}(K)}(w_L - w_K) = \sum_{p=1}^{3}(\Delta_h^s w_h)_K,$$
$$\left(\widetilde{\operatorname{div}}_h \mathbf{w}_h\right)_K := \sum_{p=1}^{3}\left(\widetilde{\partial_h^p} w_h^p\right)_K, \quad (\operatorname{div}_h \mathbf{W}_h)_K := \sum_{p=1}^{3}(\partial_h^p W_h^p)_K.$$

Furthermore, on the face $\sigma = \sigma_{K,L}$, we define the jump and mean value operators

$$[\![w_h]\!]_\sigma := w_L \overrightarrow{\mathbf{n}}_{K,L} + w_K \overrightarrow{\mathbf{n}}_{L,K}, \quad (\overline{w_h})_\sigma := \frac{w_K + w_L}{2}, \quad L = K + h\mathbf{e}_p,\ p = 1, 2, 3,$$

respectively. Here $\overrightarrow{\mathbf{n}}_{K,L}$, $\overrightarrow{\mathbf{n}}_{L,K}$ denote the unit outer normal oriented from K and L, respectively. Finally, we introduce the mean value of $w_h \in \mathcal{Y}(\mathcal{T})$ in cell K in the direction of $\mathbf{e}_p$ by

$$(\widetilde{w_h})_K^p := \frac{w_L + w_J}{2}, \quad L = K + h\mathbf{e}_p,\ J = K - h\mathbf{e}_p.$$

2.5.2. Entropy stable flux and the scheme

Note that constructing and analyzing numerical schemes for the deterministic counterpart of the underlying system of equations (1.1) and (1.2) has a long tradition. Usually the schemes are developed to satisfy certain additional properties like entropy condition and kinetic energy stability which can be important for turbulent flows. To that context, Tadmor [42] proposed the idea of entropy conservative numerical fluxes which can then be combined with some dissipation terms using entropy variables to obtain a scheme that respects the entropy condition, *i.e.*, the scheme must produce entropy in accordance with the second law of thermodynamics. Such a flux is called *entropy stable flux*.

In order to introduce the finite volume numerical scheme for the underlying system of equations, let us first recast the system of equations (2.3) and (2.4) in the following form:

$$\mathrm{d}\mathbf{U}(t) + \operatorname{div}\mathbf{f}(\mathbf{U})\,\mathrm{d}t = \mathbb{H}(\varrho, \mathbf{m})\,\mathrm{d}W(t),$$
$$\mathbf{U}(t, 0) = \mathbf{U}_0,$$

where we introduced the variables $\mathbf{U} = [\varrho, \mathbf{m}]$, $\mathbf{f}(\mathbf{U}) = [\mathbf{m}, \frac{\mathbf{m}\otimes\mathbf{m}}{\varrho} + p(\varrho)\mathbb{I}]$, $\mathbb{H}(\varrho, \mathbf{m}) = [0, \Psi(\varrho, \mathbf{m})]$, and $\mathbf{U}_0 = [\varrho_0, \mathbf{m}_0]$ with $\varrho_0 > 0$.

We propose the following semi-discrete (in space) finite volume scheme approximating the underlying system of equations (2.3) and (2.4)

$$\begin{aligned} \mathrm{d}\mathbf{U}_K(t) + (\operatorname{div}_h \mathbf{F_h}(t))_K\,\mathrm{d}t &= \mathbb{H}(\varrho_K(t), \mathbf{m}_K(t))\,\mathrm{d}W(t),\ t > 0,\ K \in \mathcal{T}, \\ \mathbf{U}_K(0) &= (\Pi_h(\mathbf{U}_0))_K,\ K \in \mathcal{T}. \end{aligned} \tag{2.15}$$

Note that (2.15) is a stochastic differential equation in $V := L^\gamma(\mathbb{T}^3)\times L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^3)$. Let us now specify the numerical flux $\mathbf{F_h} := \mathbf{F_h}(\mathbf{U}_K, \mathbf{U}_L)$ associated to the flux function f. Indeed, we want $\mathbf{F_h}$ to satisfy the following properties:

(a) (Consistency) The function $\mathbf{F_h}$ satisfies $\mathbf{F_h}(a, a) = f(a)$, for all $a \in \mathbb{R}^d$.
(b) (Local Lipschitz continuity) The function $\mathbf{F_h}$ is a local Lipschitz continuous function.
(c) (Entropy stability) The flux $\mathbf{F_h}$ is entropy stable.

Note that there are plethora of numerical fluxes available in literature satisfying the above three conditions. However, to illustrate the main ideas, we will consider a scheme with a Lax–Friedrichs-type numerical flux $\mathbf{F}_h$ (which is entropy stable) whose value on a face $\sigma = \sigma_{K,L}$ is given by

$$\mathbf{F}_\sigma := \left(\overline{\mathbf{f}(\mathbf{U}_h)}\right)_\sigma - \lambda_\sigma [\![\mathbf{U}_h]\!]_\sigma. \tag{2.16}$$

Here the global diffusion coefficient is $\lambda_\sigma \equiv \lambda := \max_{K\in\mathcal{T}} \max_{s=1,\dots,N} |\lambda^s(\mathbf{U}_K)|$, while the local diffusion coefficient is $\lambda_\sigma := \max_{s=1,\dots,N} \max(|\lambda^s(\mathbf{U}_K)|, |\lambda^s(\mathbf{U}_L)|)$. Note that λ^s is the s-th eigenvalue of the corresponding Jacobian matrix $\mathbf{f}'(\mathbf{U}_h)$. We mention that we restrict ourselves to the case of constant numerical viscosities. However, one can easily extend the results to local diffusion case, as presented in [24]. Using the above notations, we can rewrite the scheme (2.15) in the following explicit form

$$\mathrm{d}\varrho_K(t) + \left(\widetilde{\mathrm{div}_h}\, \mathbf{m}_h(t)\right)_K - \lambda h(\Delta_h\, \varrho_h(t))_K = 0, \tag{2.17a}$$

$$\begin{aligned}\mathrm{d}\mathbf{m}_K(t) &+ \left(\widetilde{\mathrm{div}_h}\left(\frac{\mathbf{m}_h(t)\otimes\mathbf{m}_h(t)}{\varrho_h(t)} + p_h(t)\mathbb{I}\right)\right)_K - \lambda h(\Delta_h\, \mathbf{m}_h(t))_K \\ &= \Psi(\varrho_K(t), \mathbf{m}_K(t))\,\mathrm{d}W(t),\ \ t>0\ K\in\mathcal{T}.\end{aligned} \tag{2.17b}$$

Existence of numerical solutions. Note that the set of equations (2.17) represent a system of stochastic differential equations. The discrete problem (2.17) admits a unique (probabilistically) strong solution $(\varrho_h, \mathbf{m}_h)$ which is continuous in time. This follows from a classical argument of stochastic differential equations with local Lipschitz non-linearities (see *e.g.*, Baldi [2], Sect. 9.5, p.270), thanks to the positivity of the density ϱ_K. For more details, we refer to Section 4 of [24].

2.6. Statements of main results

We now state main results of this paper. To begin with, regarding the convergence of solutions of the numerical scheme, we have the following theorem.

Theorem 2.15 (Convergence of numerical solutions)**.** *Assume that* $\mathbb{E}[\int_{\mathbb{T}^3}(\frac{1}{2}\frac{|\mathbf{m}_0|^2}{\varrho_0} + \frac{\varrho_0^\gamma}{\gamma-1})\,\mathrm{d}x] \le C$. *Suppose that the approximate solutions* $\{\mathbf{U}_h = (\varrho_h(t), \mathbf{m}_h(t))\}_{h>0}$ *be generated by the scheme* (2.17) *for the stochastic Euler system. Moreover, assume that*

$$0 < \underline{\varrho} \le \varrho_h \le \varrho_1, \quad |\mathbf{m}_h| \le \boldsymbol{m}_1,\ \mathbb{P}\text{-a.s. uniformly for } h\to 0, \tag{2.18}$$

for some $\varrho_1 \in L^r(\Omega)$, $\boldsymbol{m}_1 \in L^q(\Omega)$ *with* $q>1$, $r > \max\{1, (\gamma-1)/2\}$ *and positive constant* $\underline{\varrho}$. *Then* $\{\mathbf{U}_h\}_{h>0}$ *generates a dissipative measure-valued martingale solution to the barotropic Euler system in the sense of Definition 2.10.*

Next, we make use of the $\mathcal{K}$-convergence in the context of Young measures to conclude the following pointwise convergence of averages of numerical solutions to a dissipative martingale solution to (2.3) and (2.4) (see Def. 2.13).

Theorem 2.16 (Convergence to the dissipative martingale solution)**.** *Suppose that the approximate solutions* $\{\mathbf{U}_h = (\varrho_h(t), \mathbf{m}_h(t))\}_{h>0}$ *to* (2.17) *for the stochastic Euler system generate a dissipative measure-valued martingale solution* $[(\widetilde{\Omega}, \widetilde{\mathbb{F}}, (\widetilde{\mathbb{F}}_t)_{t\ge0}, \widetilde{\mathbb{P}}); \widetilde{\mathcal{V}}^\omega_{t,x}, \widetilde{W}]$ *in the sense of Definition 2.10. Then there exists a sequence of approximate solutions* $\{\widetilde{\mathbf{U}}_h = (\widetilde{\varrho}_h(t), \widetilde{\mathbf{m}}_h(t))\}_{h>0}$ *to* (2.17) *on the probability space* $(\widetilde{\Omega}, \widetilde{\mathbb{F}}, (\widetilde{\mathbb{F}}_t)_{t\ge0}, \widetilde{\mathbb{P}})$ *for which following holds true,*

(1) $\widetilde{\mathbb{P}}$-*a.s.*

$$\widetilde{\varrho}_{h_k} \to \left\langle \widetilde{\mathcal{V}}^\omega_{t,x}; \widetilde{\varrho}\right\rangle \text{ in } C_w\big([0,T], L^\gamma(\mathbb{T}^3)\big),$$
$$\widetilde{\mathbf{m}}_{h_k} \to \left\langle \widetilde{\mathcal{V}}^\omega_{t,x}; \widetilde{\boldsymbol{m}}\right\rangle \text{ in } C_w\Big([0,T], L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^3)\Big).$$

(2) $\widetilde{\mathbb{P}}$-*a.s., there exists a subsequence* $\{\widetilde{\mathbf{U}}_{h_k} = (\widetilde{\varrho}_{h_k}(t), \widetilde{\mathbf{m}}_{h_k}(t))\}_{h_k>0}$ *such that*

$$\frac{1}{N}\sum_{k=1}^{N}\widetilde{\varrho}_h \to \left\langle \widetilde{\mathcal{V}}^{\omega}_{t,x}; \widetilde{\varrho} \right\rangle, \text{ as } N \to \infty \text{ a.e. in } (0,T)\times\mathbb{T}^3,$$
$$\frac{1}{N}\sum_{k=1}^{N}\widetilde{\mathbf{m}}_h \to \left\langle \widetilde{\mathcal{V}}^{\omega}_{t,x}; \widetilde{\boldsymbol{m}} \right\rangle, \text{ as } N \to \infty \text{ a.e. in } (0,T)\times\mathbb{T}^3.$$

Finally, making use of the weak (measure-valued)–strong uniqueness principle (cf. Thm. 6.2), we prove the following result justifying the strong convergence to the regular solution.

Theorem 2.17 (Convergence of numerical solutions to a strong solution)**.** *Suppose that the approximate solutions $\{\mathbf{U}_h\}_{h>0}$ to (2.17) for the stochastic Euler system generate a dissipative measure-valued martingale solution $[(\widetilde{\Omega},\widetilde{\mathbb{F}},(\widetilde{\mathbb{F}}_t)_{t\geq 0},\widetilde{\mathbb{P}});\widetilde{\mathcal{V}}^{\omega}_{t,x},\widetilde{W}]$ in the sense of Definition 2.10. In addition, let the Euler equations (2.3) and (2.4) possess the unique strong (continuously differentiable) solution $(\bar{\mathbf{U}},(\mathfrak{t}_R)_{R\in\mathbb{N}},\mathfrak{t}) = ([\bar{\varrho},\bar{\mathbf{m}}],(\mathfrak{t}_R)_{R\in\mathbb{N}},\mathfrak{t})$ defined on the same stochastic basis $(\widetilde{\Omega},\widetilde{\mathbb{F}},(\widetilde{\mathbb{F}}_t)_{t\geq 0},\widetilde{\mathbb{P}},\widetilde{W})$ emanating from the same initial data (1.2). Then there exists a sequence of approximate solutions $\widetilde{\mathbf{U}}_h = (\widetilde{\varrho}_h(t),\widetilde{\mathbf{m}}_h(t))\}_{h>0}$ to (2.17) on the probability space $(\widetilde{\Omega},\widetilde{\mathbb{F}},(\widetilde{\mathbb{F}}_t)_{t\geq 0},\widetilde{\mathbb{P}})$ such that $\widetilde{\mathbb{P}}$-a.s.*

$$\widetilde{\varrho}_h(\cdot\wedge\mathfrak{t}_R) \to \bar{\varrho}(\cdot\wedge\mathfrak{t}_R) \text{ weakly-(*) in } L^\infty\big(0,T;L^\gamma(\mathbb{T}^3)\big) \text{ and strongly in } L^1\big((0,T)\times\mathbb{T}^3\big),$$
$$\widetilde{\mathbf{m}}_h(\cdot\wedge\mathfrak{t}_R) \to \bar{\mathbf{m}}(\cdot\wedge\mathfrak{t}_R) \text{ weakly-(*) in } L^\infty\Big(0,T;L^{2\gamma/(\gamma+1)}(\mathbb{T}^3)\Big) \text{ and strongly in } L^1\big((0,T)\times\mathbb{T}^3;\mathbb{R}^3\big).$$

Remark 2.18. Note that the results stated in Theorem 2.17 are unconditional provided that:

(1) The limit system admits a smooth solution.
(2) Numerical approximations generate a dissipative measure-valued martingale solution.

3. Stability of the numerical scheme

We show the stability of the numerical scheme defined in the previous Section by deriving *a priori* estimates. To do so, we closely follow the arguments given in [24].

3.1. *A priori* estimates for the stochastic Euler system

The approximate solutions resulting from scheme (2.17) enjoy the following properties:

(1) **Conservation of mass:** multiplying the equation of continuity in (2.17) by h^3 for all $K \in \mathcal{T}$, and integrating in time yields the total mass conservation, *i.e.*, $\mathbb{P}$-a.s.

$$\int_{\mathbb{T}^3}\varrho_h(t,\cdot)\,\mathrm{d}x = \int_{\mathbb{T}^3}\varrho_h^0\,\mathrm{d}x,\ t\geq 0.$$

(2) **Conditional positivity of numerical density:** we show positivity of the density under an additional hypothesis on the approximate velocity. We assume that $\mathbb{P}$-a.s.

$$\mathbf{u}_h \equiv \frac{\mathbf{m}_h(t)}{\varrho_h(t)} \in L^2\big(0,T;L^\infty(\mathbb{T}^3)\big). \tag{3.1}$$

Thus the first two equations of the numerical scheme for the Euler system read,

$$\mathrm{d}\varrho_K(t) + \Big(\widetilde{\mathrm{div}}_h\,(\varrho_h(t)\mathbf{u}_h(t))\Big)_K\mathrm{d}t - \lambda h(\Delta_h\,\varrho_h(t))_K\,\mathrm{d}t = 0, \tag{3.2a}$$
$$\mathrm{d}(\varrho_K(t)\mathbf{u}_K(t)) + \Big(\widetilde{\mathrm{div}}_h\,(\varrho_h(t)(\mathbf{u}_h(t)\otimes\mathbf{u}_h(t)) + p_h(t)\mathbb{I})\Big)_K\mathrm{d}t - \lambda h(\Delta_h\,(\varrho_h(t)\mathbf{u}_h(t)))_K\,\mathrm{d}t$$
$$= \Psi(\varrho_K(t),\varrho_K(t)\mathbf{u}_K(t))\,\mathrm{d}W(t), \tag{3.2b}$$

equipped with the relevant initial conditions.

Lemma 3.1. *Let $\varrho_h(0) > 0$, and let the numerical solution $(\varrho_h(t), \mathbf{u}_h(t))$, $t > 0$ satisfy the discrete continuity equation (3.2a), where we assume $\mathbf{u}_h$ satisfies (3.1). Then $\mathbb{P}$-a.s.*

$$\varrho_K(t) > \underline{\varrho} > 0, \quad t \in [0,T],\ K \in \mathcal{T}.$$

Proof. To establish the proof, one can follow [24] modulo cosmetic changes. The details are left to the interested reader. $\square$

Note that, under the hypothesis (3.1), setting $\mathbf{m}_h \equiv \varrho_h \mathbf{u}_h$ and comparing (3.2a) with (2.17a), we conclude that both formulations are equivalent.

(3) **Energy estimates**: first observe that the positivity of $\varrho_h(t)$ implies that $\mathbb{P}$-almost surely $\varrho_h \in L^\infty(0,T;L^1(\Omega))$. Next, we show that the underlying entropy stable finite volume scheme (2.17) produces the discrete entropy inequality. In fact, we can use similar arguments presented in [42] to prove the entropy inequality. To see this, let us denote by

$$\eta(\mathbf{U}_K) = \frac{1}{2}\frac{|\mathbf{m}_K|^2}{\varrho_K} + P(\varrho_K),$$

where $\mathbf{U}_K(t)$ solves the equation (2.15), and $P(\varrho) = \frac{\varrho^\gamma}{\gamma-1}$. Now by applying Itô formula to the function $\eta(\mathbf{U}_K(t))$, we get $\mathbb{P}$-almost surely, for all $t \in [0,T]$,

$$\begin{aligned} \mathrm{d}\eta(\mathbf{U}_K(t)) &= -\nabla\eta(\mathbf{U}_K)\cdot(\mathrm{div}_h\,\mathbf{F_h}(t))_K\,\mathrm{d}t + \nabla\eta(\mathbf{U}_K)\cdot\mathbb{H}(\varrho_K(t),\mathbf{m}_K(t))\,\mathrm{d}W(t) \\ &\quad + \frac{1}{2}\mathbb{H}(\varrho_K(t),\mathbf{m}_K(t))\cdot\nabla^2\eta(\mathbf{U}_K)\cdot\mathbb{H}(\varrho_K(t),\mathbf{m}_K(t))\,\mathrm{d}t. \end{aligned}$$

By using entropy stability properties of numerical flux functions ([42], Example 5.2), we obtain the discrete energy inequality, *i.e.*, $\mathbb{P}$-almost surely, for $t \in [0,T]$,

$$\begin{aligned} &\mathrm{d}\eta(\mathbf{U}_K(t)) + (\mathrm{div}_h\,\mathbf{Q}_h(t))_K\,\mathrm{d}t \\ &\quad \le \sum_{k=1}^\infty \Psi_k(\varrho_K(t),\mathbf{m}_K(t))\cdot\mathbf{u}_K(t)\,\mathrm{d}W_k(t) + \frac{1}{2}\sum_{k=1}^\infty \varrho_K(t)^{-1}|\Psi_k(\varrho_K(t),\mathbf{m}_K(t))|^2\,\mathrm{d}t, \end{aligned} \tag{3.3}$$

where $\mathbf{Q}_h$ is a entropy stable flux. Observe that the right hand side of (3.3) corresponds to usual Itô integral term and Itô correction term respectively.
Since the numerical entropy flux is conservative, *i.e.*, $\sum_{K\in\mathcal{T}}(\mathrm{div}_h\,\mathbf{Q}_h)_K = 0$, the integral of (3.3) yields $\mathbb{P}$-a.s.

$$\begin{aligned} \int_{\mathbb{T}^3}\eta(\mathbf{U}_h(t))\,\mathrm{d}x &\le \int_{\mathbb{T}^3}\eta(\mathbf{U}_h(0))\,\mathrm{d}x + \int_0^t\int_{\mathbb{T}^3}\sum_{k=1}^\infty \Psi_k(\varrho_h(s),\mathbf{m}_h(s))\cdot\mathbf{u}_h(s)\ \mathrm{d}x\,\mathrm{d}W_k(s) \\ &\quad + \frac{1}{2}\int_0^t\int_{\mathbb{T}^3}\sum_{k=1}^\infty \varrho_h(s)^{-1}|\Psi_k(\varrho_h(s),\mathbf{m}_h(s))|^2\,\mathrm{d}x\,\mathrm{d}s. \end{aligned} \tag{3.4}$$

We can apply the p-th power on both sides of (3.4), and then take expectation to obtain usual energy bounds. In particular, we have following uniform bounds, for $p \ge 1$

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left\|\frac{\mathbf{m}_h(t)}{\sqrt{\varrho_h(t)}}\right\|^p_{L^2(\mathbb{T}^3)}\right] \le C, \tag{3.5}$$

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|\varrho_h(t)\|^p_{L^\gamma(\mathbb{T}^3)}\right] \le C, \tag{3.6}$$

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|\mathbf{m}_h(t)\|^p_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^3)}\right] \le C. \tag{3.7}$$

Remark 3.2. Note that above estimates are natural in the context of stochastic compressible Euler equations.

Let $\psi \in C_c^\infty([0,T))$, $\psi \ge 0$. By applying Itô product formula to the function $\eta(\mathbf{U}_K(t))\psi(t)$, we have $\mathbb{P}$-almost surely,

$$\mathrm{d}\psi(t)\eta(\mathbf{U}_K(t)) = \partial_t\psi(t)\eta(\mathbf{U}_K(t))\,\mathrm{d}t + \psi(t)\,\mathrm{d}\eta(\mathbf{U}_K(t)).$$

It implies that

$$\begin{aligned}\mathrm{d}\psi(t)\eta(\mathbf{U}_K(t)) &= \partial_t\psi(t)\eta(\mathbf{U}_K(t))\,\mathrm{d}t - \psi(t)\nabla\eta(\mathbf{U}_K(t))\cdot(\mathrm{div}_h\mathbf{F}_h(t))_K\,\mathrm{d}t + \psi(t)\nabla\eta(\mathbf{U}_K(t))\\ &\quad\cdot\mathbb{H}(\varrho_K(t),\mathbf{m}_K(t))\,\mathrm{d}W(t) + \psi(t)\frac{1}{2}\mathbb{H}(\varrho_K(t),m_K(t))\cdot\nabla^2\eta(\mathbf{U}_K(t))\cdot\mathbb{H}(\varrho_K(t),\mathbf{m}_K(t))\,\mathrm{d}t\end{aligned}$$

holds $\mathbb{P}$-a.s., for all $\psi \in C_c^\infty([0,T))$, $\psi \ge 0$. Using entropy stability properties of numerical flux functions ([42], Example 5.2), we have energy inequality

$$\begin{aligned}-\int_0^T \partial_t\psi\int_{\mathbb{T}^3}\left[\frac{1}{2}\frac{|\mathbf{m}_h|^2}{\varrho_h} + \frac{\varrho_h^\gamma}{\gamma-1}\right]\mathrm{d}x\,\mathrm{d}s &\le \psi(0)\int_{\mathbb{T}^3}\left[\frac{1}{2}\frac{|\mathbf{m}_h(0)|^2}{\varrho_h(0)} + \frac{\varrho_h^\gamma(0)}{\gamma-1}\right]\mathrm{d}x\\ &\quad+\sum_{k=1}^\infty\int_0^T\psi\left(\int_{\mathbb{T}^3}\Psi_k(\varrho_h,\mathbf{m}_h)\cdot\mathbf{u}_h\,\mathrm{d}x\right)\mathrm{d}W_k + \frac{1}{2}\sum_{k=1}^\infty\int_0^T\psi\int_{\mathbb{T}^3}\varrho_h^{-1}|\Psi_k(\varrho_h,\mathbf{m}_h)|^2\,\mathrm{d}x\,\mathrm{d}s\end{aligned}$$

holds $\mathbb{P}$-a.s., for all $\psi \in C_c^\infty([0,T))$, $\psi \ge 0$.

(4) **Additional estimates**: regarding the regularity estimates for the discrete numerical solution, we have the following lemma.

Lemma 3.3. *Let $\Gamma = \frac{2\gamma}{\gamma+1}$. For all $p \in [1,\infty)$, there exists constant $C \equiv C(p,\Gamma) > 0$ such that*

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left\|\widetilde{\mathrm{div}_h}\mathbf{m}_h(t)\right\|^p_{W^{-1,\Gamma}(\mathbb{T}^3)}\right] \le C, \tag{3.8}$$

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|\Delta_h\mathbf{m}_h(t)\|^p_{W^{-2,\gamma}(\mathbb{T}^3)}\right] \le C, \tag{3.9}$$

and

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|\Delta_h\varrho_h(t)\|^p_{W^{-2,\gamma}(\mathbb{T}^3)}\right] \le C. \tag{3.10}$$

Proof. We note that for any test function $\varphi \in W^{1,\Gamma'}(\mathbb{T}^3)$

$$\begin{aligned}\left\langle\widetilde{\mathrm{div}_h}\,\mathbf{m}_h(t),\varphi\right\rangle &= h^3\sum_{K\in\mathcal{T}}\left(\widetilde{\mathrm{div}_h}\,\mathbf{m}_h(t)\right)_K(\Pi_h\varphi)_K\\ &= -\sum_{K\in\mathcal{T}}\sum_{s=1}^3 m_K^s(t)\left(\int_K\frac{\varphi(x+h\mathbf{e}_s)-\varphi(x-h\mathbf{e}_s)}{2h}\,\mathrm{d}x\right)\\ &= -\sum_{K\in\mathcal{T}}\sum_{s=1}^3 m_K^s(t)\left(\frac{1}{2h}\int_K\int_{x+h\mathbf{e}_s}^{x-h\mathbf{e}_s}\varphi_x(\theta)\,\mathrm{d}\theta\,\mathrm{d}x\right) \le \frac{1}{2h}\sum_{K\in\mathcal{T}}\sum_{s=1}^3 m_K^s(t)\left(\int_K h\varphi_x\,\mathrm{d}x\right)\end{aligned}$$

$$\leq \frac{h^{1+1/\Gamma}}{2h} \sum_{K\in\mathcal{T}} \sum_{s=1}^{3} m_K^s(t) \left(\int_K \varphi_x^{\Gamma'} \,\mathrm{d}x\,\mathrm{d}x \right)^{\frac{1}{\Gamma'}} \leq C(\Gamma)\, \|\mathbf{m}_h(t)\|_{L^\Gamma(\mathbb{T}^3)} \|\varphi\|_{W^{1,\Gamma'}(\mathbb{T}^3)}.$$

It implies that $\mathbb{P}$-almost surely,

$$\left\| \widetilde{\mathrm{div}}_h \mathbf{m}_h(t) \right\|_{W^{-1,\Gamma}(\mathbb{T}^3)} \leq C(\Gamma) \sup_{t\in[0,T]} \|\mathbf{m}_h(t)\|_{L^\Gamma(\mathbb{T}^3)}.$$

By making using of uniform estimate (3.7), we conclude that for all $p \in [1,\infty)$, there exists $C \equiv C(p,\Gamma) > 0$ such that

$$\mathbb{E}\left[\sup_{t\in[0,T]} \left\| \widetilde{\mathrm{div}}_h \mathbf{m}_h(t) \right\|_{W^{-1,\Gamma}(\mathbb{T}^3)}^p \right] \leq C.$$

This confirms the first estimate. A similar argument yields the results for the discrete Laplacian. □

4. Consistency of the numerical scheme

In this section we show consistency of the entropy stable finite volume scheme. In addition, we also exhibit consistency of the energy inequality.

4.1. Consistency formulation of continuity and momentum equations

We begin by multiplying the continuity equation (2.17a) by $h^3 (\Pi_h \varphi)_K$, with $\varphi \in C^3(\mathbb{T}^3)$, and the momentum equation or (2.17b) by $h^3(\Pi_h \boldsymbol{\varphi})_K$, with $\boldsymbol{\varphi} \in C^3(\mathbb{T}^3;\mathbb{R}^3)$. Then we sum the resulting equations over $K \in \mathcal{T}$ and integrate in time. For time derivatives in the continuity and momentum equations, it is straightforward to observe that

$$h^3 \int_0^t \sum_{K\in\mathcal{T}} \mathrm{d}\varrho_K(t)\,(\Pi_h\varphi)_K = \int_0^t \mathrm{d}\left(\int_{\mathbb{T}^3} \varrho_h(t)\varphi(x)\,\mathrm{d}x \right) = \langle \varrho_h(t), \varphi\rangle - \langle \varrho_h(0), \varphi\rangle,$$

$$h^3 \int_0^t \sum_{K\in\mathcal{T}} \mathrm{d}\mathbf{m}_K(t)\cdot(\Pi_h\boldsymbol{\varphi})_K = \int_0^t \mathrm{d}\left(\int_{\mathbb{T}^3} \mathbf{m}_h(t)\cdot\boldsymbol{\varphi}(x)\,\mathrm{d}x \right) = \langle \mathbf{m}_h(t), \boldsymbol{\varphi}\rangle - \langle \mathbf{m}_h(0), \boldsymbol{\varphi}\rangle.$$

To handle the convective terms, we shall make use of the discrete integration by parts and the Taylor expansion. For the continuity equation, we have

$$\begin{aligned} h^3 \int_0^t \sum_{K\in\mathcal{T}} \left(\widetilde{\mathrm{div}}_h\, \mathbf{m}_h(t) \right)_K (\Pi_h\varphi)_K &= -\int_0^t \sum_{K\in\mathcal{T}} \sum_{s=1}^{3} m_K^s(t) \left(\int_K \frac{\varphi(x+h\mathbf{e}_s) - \varphi(x-h\mathbf{e}_s)}{2h}\,\mathrm{d}x \right) \\ &= -\int_0^t \int_{\mathbb{T}^3} \mathbf{m}_h(t)\cdot\nabla\,\varphi(x)\,\mathrm{d}x + \mathcal{R}_1(h,\varphi), \end{aligned}$$

where term $\mathcal{R}_1(h,\varphi)$ is estimated as follows

$$\mathcal{R}_1(h,\varphi) \leq C(\varphi) h \|\mathbf{m}_h\|_{L_t^\infty L_x^1}, \ \mathbb{P} \text{ a.s.}$$

Similarly, for the convective term in the momentum equations, we have

$$\begin{aligned} & h^3 \int_0^t \sum_{K\in\mathcal{T}} \left(\widetilde{\mathrm{div}}_h \left(\frac{\mathbf{m}_h(t)\otimes\mathbf{m}_h(t)}{\varrho_h(t)} + p_h(t)\mathbb{I} \right) \right)_K (\Pi_h\boldsymbol{\varphi})_K \\ &= -\int_0^t \sum_{K\in\mathcal{T}} \sum_{s=1}^{3} \sum_{z=1}^{3} \left(\frac{m_h^s(t) m_h^z(t)}{\varrho_h(t)} + p_h(t) \right) \left(\int_K \frac{\varphi^z(x+h\mathbf{e}_s) - \varphi^z(x-h\mathbf{e}_s)}{2h}\,\mathrm{d}x \right) \end{aligned}$$

$$= -\int_0^t \int_{\mathbb{T}^3} \left(\frac{\mathbf{m}_h(t) \otimes \mathbf{m}_h(t)}{\varrho_h(t)} + p_h(t)\mathbb{I} \right) \cdot \nabla \boldsymbol{\varphi}(x)\,\mathrm{d}x + \mathcal{R}_2(h, \boldsymbol{\varphi}),$$

where the term $\mathcal{R}_2(h, \boldsymbol{\varphi})$ is bounded by

$$\mathcal{R}_2(h, \boldsymbol{\varphi}) \le C(\boldsymbol{\varphi}) h \Big\{ \|\sqrt{\varrho_h}\mathbf{u}_h\|_{L_t^\infty L_x^2} + \|p_h\|_{L_t^\infty L_x^1} \Big\},\ \mathbb{P} \text{ a.s.}$$

Next, regarding the numerical diffusion term with global numerical diffusion coefficients λ, we have

$$\begin{aligned} h^4 \int_0^t \lambda \sum_{K \in \mathcal{T}} (\Delta_h \mathbf{U}_h(t))_K (\Pi_h \bar{\boldsymbol{\varphi}})_K &= h \int_0^t \lambda \sum_{K \in \mathcal{T}} \mathbf{U}_K(t) \left(\int_K \sum_{s=1}^N \frac{\bar{\boldsymbol{\varphi}}(x + h\mathbf{e}_s) - 2\bar{\boldsymbol{\varphi}}(x) + \bar{\boldsymbol{\varphi}}(x - h\mathbf{e}_s)}{h^2}\,\mathrm{d}x \right) \\ &= h \int_0^t \lambda \int_{\mathbb{T}^3} \mathbf{U}_h(t) \Delta \bar{\boldsymbol{\varphi}}(x)\,\mathrm{d}x + \mathcal{N}(h, \bar{\boldsymbol{\varphi}}), \end{aligned}$$

where $\bar{\boldsymbol{\varphi}} = (\varphi, \boldsymbol{\varphi})$ and the term $\mathcal{N}(h, \bar{\boldsymbol{\varphi}})$ is bounded by

$$|\mathcal{N}(h, \bar{\boldsymbol{\varphi}})| \le C(\bar{\boldsymbol{\varphi}}) h \|\mathbf{U}_h\|_{L_t^\infty L_x^1} \int_0^t \lambda\,\mathrm{d}t,\ \mathbb{P} \text{ a.s.}$$

In order to control the integral $\int_0^t \lambda\,\mathrm{d}t$, we need bounds on both discrete density and momentum. It can be deduced from the total energy bound if we make extra hypothesis (2.18), namely

$$0 < \underline{\varrho} \le \varrho_h \le \varrho_1, \quad |\mathbf{m}_h| \le \mathbf{m}_1,\ \mathbb{P}\text{-a.s. uniformly for } h \to 0,$$

for some $\varrho_1 \in L^r(\Omega)$, $\mathbf{m}_1 \in L^q(\Omega)$ with $q > 1$, $r > \max\{1, (\gamma - 1)/2\}$ and positive constant $\underline{\varrho}$. Indeed, for the barotropic Euler equations it holds that $\lambda(\mathbf{U}_K) \le \max_{K \in \mathcal{T}} (|\mathbf{u}_K| + \sqrt{\gamma \varrho_K^{\gamma-1}}, |\mathbf{u}_L| + \sqrt{\gamma \varrho_L^{\gamma-1}})$, $\mathbb{P}$-a.s. It implies $\mathbb{P}$-a.s.

$$\lambda(\mathbf{U}_h) \le \frac{\mathbf{m}_1}{\underline{\varrho}} + \sqrt{\gamma \varrho_1^{\gamma-1}}.$$

Under these circumstances and making use of Hölder inequality, we easily deduce from (3.5) to (3.7) the following bound

$$\mathbb{E}|\mathcal{N}(h, \bar{\boldsymbol{\varphi}})| \le h\, C\big(\underline{\varrho}, \gamma, \bar{\boldsymbol{\varphi}}\big) \left(\left(\mathbb{E}\|\mathbf{U}_h\|_{L_t^\infty L_x^1}^{q'} \right)^{1/q'} (\mathbb{E}(\mathbf{m}_1^q))^{1/q} + \left(\mathbb{E}\|\mathbf{U}_h\|_{L_t^\infty L_x^1}^{r'} \right)^{1/r'} (\mathbb{E}(\varrho_1^r))^{1/r} \right) \le C(\bar{\boldsymbol{\varphi}})\, h \tag{4.1}$$

where q' is conjugate of q and $r' = \frac{2r}{2r - \gamma + 1}$.

Finally, regarding the stochastic term, we have the following

$$h^3 \int_0^t \sum_{K \in \mathcal{T}} \Psi(\varrho_K(t), \mathbf{m}_K(t))\, (\Pi_h \boldsymbol{\varphi})_K\,\mathrm{d}W(t) = \int_0^t \langle \Psi(\varrho_h, \mathbf{m}_h), \boldsymbol{\varphi} \rangle\,\mathrm{d}W(t).$$

Remark 4.1. Here one can notice that the additional hypothesis of pathwise boundedness on both approximate discrete density ϱ_h and momentum $\mathbf{m}_h$ helps to prove consistency of semi discrete scheme. In this study, we use this hypothesis to prove only consistency of semi-discrete scheme. Other mathematical task is based on natural stability bounds and approximate equations. Suppose consistency of scheme holds, than our whole machinery works smoothly without use of pathwise bounds on both discrete density ϱ_h and momentum $\mathbf{m}_h$.

Let us summarize the consistency results derived in this section.

Consistency formulation for the stochastic Euler system

The consistency formulation of the numerical schemes (2.17) for the barotropic Euler equations reads:

(1) For all $\varphi \in C^\infty(\mathbb{T}^3)$ and $\boldsymbol{\varphi} \in C^\infty(\mathbb{T}^3)$ we have $\mathbb{P}$-a.s. for all $t \in [0,T]$

$$\langle \varrho_h(t), \varphi \rangle = \langle \varrho_h(0), \varphi \rangle + \int_0^t \langle \mathbf{m}_h, \nabla\varphi \rangle \, \mathrm{d}s + h \int_0^t \lambda \langle \varrho_h(t), \Delta\varphi \rangle \, \mathrm{d}s + \mathcal{R}_1(h,\varphi) + \mathcal{N}_1(h,\varphi) \tag{4.2}$$

$$\langle \mathbf{m}_h(t), \boldsymbol{\varphi} \rangle = \langle \mathbf{m}_h(0), \boldsymbol{\varphi} \rangle + \int_0^t \left\langle \left(\frac{\mathbf{m}_h \otimes \mathbf{m}_h}{\varrho_h} + p(\varrho_h)\mathbb{I} \right), \nabla\boldsymbol{\varphi} \right\rangle \mathrm{d}s + h \int_0^t \lambda \langle \mathbf{m}_h(t), \Delta\boldsymbol{\varphi} \rangle \, \mathrm{d}s$$
$$+ \int_0^t \langle \Psi(\varrho_h, \mathbf{m}_h), \boldsymbol{\varphi} \rangle \, \mathrm{d}W(s) + \mathcal{R}_2(h,\boldsymbol{\varphi}) + \mathcal{N}_2(h,\boldsymbol{\varphi}). \tag{4.3}$$

(2) The energy inequality

$$- \int_0^T \partial_t \psi \int_{\mathbb{T}^3} \left[\frac{1}{2} \frac{|\mathbf{m}_h|^2}{\varrho_h} + \frac{\varrho_h^\gamma}{\gamma - 1} \right] \mathrm{d}x \, \mathrm{d}s \le \psi(0) \int_{\mathbb{T}^3} \left[\frac{1}{2} \frac{|\mathbf{m}_h(0)|^2}{\varrho_h(0)} + \frac{\varrho_h^\gamma(0)}{\gamma - 1} \right] \mathrm{d}x$$
$$+ \sum_{k=1}^\infty \int_0^T \psi \left(\int_{\mathbb{T}^3} \Psi_k(\varrho_h, \mathbf{m}_h) \cdot \mathbf{u}_h \, \mathrm{d}x \right) \mathrm{d}W_k(s) + \frac{1}{2} \sum_{k=1}^\infty \int_0^T \psi \int_{\mathbb{T}^3} \varrho_h^{-1} |\Psi_k(\varrho_h, \mathbf{m}_h)|^2 \, \mathrm{d}x \, \mathrm{d}s \tag{4.4}$$

holds $\mathbb{P}$-a.s., for all $\psi \in C_c^\infty([0,T))$, $\psi \ge 0$.

Here

$$\mathcal{R}_1(\varrho_h, \mathbf{m}_h, h, \varphi) := \int_0^t \sum_{K \in \mathcal{T}} \sum_{i=1}^3 m_K^i(s) \int_K \left(\partial_{x_i} \varphi - \frac{\varphi(x + h\mathbf{e}_i) - \varphi(x - h\mathbf{e}_i)}{2h} \right) \mathrm{d}x \, \mathrm{d}s,$$

$$\mathcal{R}_2(\varrho_h, \mathbf{m}_h, h, \boldsymbol{\varphi}) := \int_0^t \sum_{K \in \mathcal{T}} \sum_{i=1}^3 \sum_{j=1}^3 \left(\frac{m_h^i(s) m_h^j(s)}{\varrho_h(s)} + p_h(s) \right) \int_K \left(\partial_{x_i} \varphi^j - \frac{\varphi^j(x + h\mathbf{e}_i) - \varphi^j(x - h\mathbf{e}_i)}{2h} \right) \mathrm{d}x \, \mathrm{d}s,$$

$$\mathcal{N}_1(\varrho_h, \mathbf{m}_h, h, \varphi) := h \int_0^t \lambda \sum_{K \in \mathcal{T}} \varrho_K(s) \int_K \sum_{i=1}^3 \left(\partial_{x_i x_i} \varphi - \frac{\varphi(x + h\mathbf{e}_i) - 2\varphi(x) + \varphi(x - h\mathbf{e}_i)}{h^2} \right) \mathrm{d}x \, \mathrm{d}s,$$

$$\mathcal{N}_2(\varrho_h, \mathbf{m}_h, h, \boldsymbol{\varphi}) := h \int_0^t \lambda \sum_{K \in \mathcal{T}} \sum_{i=1}^3 \sum_{j=1}^3 m_K^j(s)$$
$$\times \int_K \left(\partial_{x_i x_i} \varphi^j - \frac{\varphi^j(x + h\mathbf{e}_i) - 2\varphi^j(x) + \varphi^j(x - h\mathbf{e}_i)}{h^2} \right) \mathrm{d}x \, \mathrm{d}s. \tag{4.5}$$

5. Proof of Theorem 2.15: Existence of measure-valued solution

We shall make use of the given *a priori* estimates (3.5)–(3.7) to pass to the limit in the parameter h. In what follows, we begin by the following compactness argument.

5.1. Compactness and almost sure representations

Note that, in general, securing a result of compactness in the probability variable (ω-variable) is a non-trivial task. To that context, to obtain strong (a.s.) convergence in the ω-variable, we make use of Skorokhod–Jakubowski's representation theorem (cf. [31]). We remark that the classical Skorokhod representation theorem does not work in our setup since our path spaces are not Polish spaces. To overcome this, we use Jakubowski version of Skorokhod representation Theorem 2.5 which works for quasi-Polish spaces. Let k_1 and k_2 two positive integer. Let $H : \mathbb{R}\times\mathbb{R} \to \mathbb{R}^{k_1}$ and $K : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{R}^{k_2}$ be two Borel measurable functions, satisfying the following growth condition

$$|H(\varrho,\mathbf{m})| \le C(1+|\varrho|^{r_1}+|\mathbf{m}|^{r_2}), \tag{5.1}$$

$$|K(\varrho,\mathbf{m})| = \mathcal{O}\Big(1+|\varrho|^{\gamma}+|\mathbf{m}|^{\frac{2\gamma}{\gamma+1}}\Big), \text{as } \varrho,\mathbf{m}\to\infty. \tag{5.2}$$

for some fixed $r_1, r_2 > 0$. As usual, to establish the tightness of the laws generated by the approximations, we first denote the path space $\mathcal{Y}$ to be the product of the following spaces:

$$\begin{aligned}
&\mathcal{Y}_\varrho = C_w\big([0,T];L^\gamma(\mathbb{T}^3)\big), && \mathcal{Y}_N = C([0,T];\mathbb{R}),\\
&\mathcal{Y}_{\mathbf{m}} = C_w([0,T];L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^3)), && \mathcal{Y}_W = C([0,T];\mathfrak{W}_0),\\
&\mathcal{Y}_K = \big(L^\infty((0,T);M_b(\mathbb{T}^3)),w^*\big), && \mathcal{Y}_H = \big(L^r\big((0,T)\times\mathbb{T}^3\big),w\big)\\
&\mathcal{Y}_{\mathcal{V}} = \big(L^\infty\big((0,T)\times\mathbb{T}^3;\mathcal{P}\big(\mathbb{R}^4\big)\big),w^*\big), && \mathcal{Y}_{\varrho_1,\mathbf{m}_1} = \mathbb{R}^2,
\end{aligned}$$

where $r\in\mathbb{R}$ with $1<r\le \frac{\gamma}{r_1}\wedge\frac{2\gamma}{r_2(\gamma+1)}$. Let us denote by μ_{ϱ_h}, $\mu_{\mathbf{m}_h}$, μ_W and $\mu_{\varrho_1,\hat{\mathbf{m}}_1}$ respectively, the law of ϱ_h, $\mathbf{m}_h$, W and $(\varrho_1,\mathbf{m}_1)$ on the corresponding path space. Moreover, let $\mu_{\mathcal{V}_h}$, μ_{N_h}, μ_{H_h}, and μ_{K_h} denote the law of $\mathcal{V}_h := \delta_{[\varrho_h,\mathbf{m}_h]}$, $N_h := \sum_{k\ge1}\int_0^t\int_{\mathbb{T}^3}\mathbf{u}_h\cdot\Psi_k(\varrho_h,\mathbf{m}_h)\,\mathrm{d}x\,\mathrm{d}W$, $H_h = H(\varrho_h,\mathbf{m}_h)$, and $K_h = K(\varrho_h,\mathbf{m}_h)$, respectively on the corresponding path spaces. Finally, let μ^h denotes joint law of all the variables on $\mathcal{Y}$. To proceed further, it is necessary to establish tightness of $\{\mu^h;\, h\in(0,1)\}$. To this end, we observe that tightness of μ_W and $(\varrho_1,\mathbf{m}_1)$ are immediate. So we show tightness of other variables.

Proposition 5.1. *The sets $\{\mu_{\varrho_h};\, h\in(0,1)\}$, and $\{\mu_{\boldsymbol{m}_h};\, h\in(0,1)\}$ are tight on path spaces $\mathcal{Y}_\varrho$, and $\mathcal{Y}_{\boldsymbol{m}}$ respectively.*

Proof. Proof of this proposition is straightforward, by making use of the *a priori* bounds (3.5)–(3.10) and compactness results ([9], Thm. 1.8.5).

From the equations (2.17a) and (2.17b) and with help of *a priori* bounds (3.5)–(3.10), we conclude that for all $p\in[1,\infty)$,

$$\mathbb{E}\Bigg[\sup_{t\in[0,T]}\|\varrho_h(t)\|^p_{W^{-2,\gamma}(\mathbb{T}^3)}\Bigg]\le C, \text{ and } \mathbb{E}\Bigg[\sup_{t\in[0,T]}\|\mathbf{m}_h(t)\|^p_{W^{-2,\frac{2\gamma}{\gamma+1}}(\mathbb{T}^3)}\Bigg]\le C. \tag{5.3}$$

By compact embedding ([9], Thm. 1.8.5), we know that

$$L^\infty\big(0,T;L^\gamma\big(\mathbb{T}^3\big)\big)\cap C\big([0,T];W^{-2,\gamma}\big(\mathbb{T}^3\big)\big) \subset\subset C_w\big([0,T];L^\gamma\big(\mathbb{T}^3\big)\big)$$
$$L^\infty\Big(0,T;L^{\frac{2\gamma}{\gamma+1}}\Big)\cap C\Big([0,T];W^{-2,\frac{2\gamma}{\gamma+1}}\big(\mathbb{T}^3\big)\Big) \subset\subset C_w\Big([0,T];L^{\frac{2\gamma}{\gamma+1}}\big(\mathbb{T}^3\big)\Big).$$

With the help of above compact embedding, Markov inequality and *a priori* bounds (3.5)–(3.10), one can easily prove the tightness of $\{\mu_{\varrho_h};\, h\in(0,1)\}$, and $\{\mu_{\mathbf{m}_h}; h\in(0,1)\}$. ☐

Proposition 5.2. *The set $\{\mu_{\mathcal{V}_h};\, h\in(0,1)\}$ is tight on the path space $\mathcal{Y}_{\mathcal{V}}$.*

Proof. Our aim is to apply the compactness criterion in $(L^\infty((0,T)\times\mathbb{T}^3;\mathcal{P}(\mathbb{R}^4)),w^*)$. Define the set

$$B_R := \Bigg\{\mathcal{V}\in\big(L^\infty\big((0,T)\times\mathbb{T}^3;\mathcal{P}\big(\mathbb{R}^4\big)\big),w^*\big);\int_0^T\int_{\mathbb{T}^3}\int_{\mathbb{R}^4}\Big(|\xi_1|^\gamma+|\xi_2|^{\frac{2\gamma}{\gamma+1}}\Big)\,\mathrm{d}\mathcal{V}_{t,x}(\xi)\,\mathrm{d}x\,\mathrm{d}t\le R\Bigg\},$$

which is relatively compact in $\big(L^\infty((0,T)\times\mathbb{T}^3;\mathcal{P}(\mathbb{R}^4)),w^*\big)$. Note that

$$\mathcal{L}[\mathcal{V}_h](B_R^c) = \mathbb{P}\Bigg(\int_0^T\int_{\mathbb{T}^3}\int_{\mathbb{R}^4}\Big(|\xi_1|^\gamma+|\xi_2|^{\frac{2\gamma}{\gamma+1}}\Big)\,\mathrm{d}\mathcal{V}_{t,x}(\xi)\,\mathrm{d}x\,\mathrm{d}t>R\Bigg)$$

$$= \mathbb{P}\left(\int_0^T \int_{\mathbb{T}^3} \left(|\varrho_h|^\gamma + |\mathbf{m}_h|^{\frac{2\gamma}{\gamma+1}} \right) \mathrm{d}x \, \mathrm{d}t > R \right) \leq \frac{1}{R} \mathbb{E}\left[\|\varrho_h\|^\gamma_{L^\gamma} + \|\mathbf{m}_h\|^{\frac{2\gamma}{\gamma+1}}_{L^{\frac{2\gamma}{\gamma+1}}} \right] \leq \frac{C}{R}.$$

The proof is complete. ☐

Proposition 5.3. *The set $\{\mu_{N_h}; h \in (0,1)\}$ is tight on the path space $\mathcal{Y}_N$.*

Proof. First observe that, for each h, $N_h(t) = \sum_{k \geq 1} \int_0^t \int_{\mathbb{T}^3} \mathbf{u}_h \cdot \Psi_k(\varrho_h, \mathbf{m}_h) \, \mathrm{d}x \, \mathrm{d}W$ is a square integrable martingale. Note that for $r > 2$

$$\mathbb{E}\left[\left| \sum_{k \geq 1} \int_s^t \int_{\mathbb{T}^3} \mathbf{u}_h \cdot \Psi_k(\varrho_h, \mathbf{m}_h) \right|^r \right] \leq \mathbb{E}\left[\int_s^t \sum_{k=1}^\infty \left| \int_{\mathbb{T}^3} \mathbf{u}_h \cdot \Psi_k(\varrho_h, \mathbf{m}_h) \right|^2 \right]^{r/2}$$
$$\leq |t-s|^{r/2} \left(1 + \mathbb{E}\left[\sup_{0 \leq t \leq T} \|\sqrt{\varrho_h} u_h\|^r_{L^2} \right] \right) \leq C|t-s|^{r/2},$$

and the Kolmogorov continuity criterion *i.e.*, Lemma 2.6 applies. This in particular implies that, for all $r > 2$, there exist constants, $\alpha \in (0, \frac{1}{2})$ and $C > 0$ (independent of h) such that

$$\mathbb{E}\|N_h\|^r_{C^\alpha([0,T];\mathbb{R})} \leq C.$$

Therefore, tightness of law follows from the compact embedding of $C^\alpha([0,T];\mathbb{R})$ into $C([0,T];\mathbb{R})$. ☐

Proposition 5.4. *The set $\{\mu_{H_h};\, h \in (0,1)\}$ and $\{\mu_{K_h};\, h \in (0,1)\}$ are tight on the path spaces $\mathcal{Y}_H$ and $\mathcal{Y}_K$.*

Proof. By growth condition (5.1) and (5.2), we can conclude that for all $p \geq 1$,

$$\mathbb{E}\Big[\|H_h\|^r_{L^r((0,T)\times\mathbb{T}^3)} \Big] \leq C \mathbb{E}\left[\sup_{t \in [0,T]} \|\varrho_h\|^\gamma_{L^\gamma(\mathbb{T}^3)} \right] + \mathbb{E}\left[\sup_{t \in [0,T]} \|\mathbf{m}_h\|^{\frac{2\gamma}{\gamma+1}}_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^3)} \right]$$

and

$$\mathbb{E}\big[\|K_h\|_{L^\infty((0,T);M_b(\mathbb{T}^3))} \big] \leq C \mathbb{E}\left[\sup_{t \in [0,T]} \|\varrho_h\|^\gamma_{L^\gamma(\mathbb{T}^3)} \right] + \mathbb{E}\left[\sup_{t \in [0,T]} \|\mathbf{m}_h\|^{\frac{2\gamma}{\gamma+1}}_{L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^3)} \right].$$

With the help of uniform bounds (3.6) and (3.7), we conclude that there exists positive constant C (independent of h) such that

$$\mathbb{E}\Big[\|H_h\|^r_{L^r((0,T)\times\mathbb{T}^3)} \Big] \leq C \text{ and } \mathbb{E}\big[\|K_h\|_{L^\infty((0,T);M_b(\mathbb{T}^3))} \big] \leq C.$$

Therefore, tightness of law can be proved with the help of Banach–Alaoglu theorem and Markov's inequality. ☐

Combining all the informations obtained from Propositions 5.1–5.3, we conclude that

Corollary 5.5. *The set $\{\mu^h; h \in (0,1)\}$ is tight on $\mathcal{Y}$.*

At this point, we are ready to apply Jakubowski–Skorokhod representation Theorem 2.5 to extract a.s convergence on a new probability space. In what follows, passing to a weakly convergent subsequence μ^h (and denoting by μ the limit law) we infer the following result:

Proposition 5.6. *There exists a subsequence μ^h (not relabelled), a probability space $(\widetilde{\Omega}, \widetilde{\mathbb{F}}, \widetilde{\mathbb{P}})$ with $\mathcal{Y}$-valued Borel measurable random variables $(\widetilde{\varrho}_h, \widetilde{\mathbf{m}}_h, \widetilde{W}_h, \widetilde{N}_h, \widetilde{\mathcal{V}}_h, \widetilde{\varrho}_{1,h}, \widetilde{\boldsymbol{m}}_{1,h}, \widetilde{H}_h, \widetilde{K}_h)$, $h \in (0,1)$, and $(\widetilde{\varrho}, \widetilde{\mathbf{m}}, \widetilde{W}, \widetilde{N}, \widetilde{\mathcal{V}}, \widetilde{\varrho}_1, \widetilde{\mathbf{m}}_1, \widetilde{H_1}, \widetilde{K_1})$ such that*

(1) *the law of* $(\widetilde{\varrho}_h, \widetilde{\mathbf{m}}_h, \widetilde{W}_h, \widetilde{N}_h, \widetilde{\mathcal{V}}_h, \widetilde{\varrho}_{1,h}, \widetilde{\boldsymbol{m}}_{1,h}, \widetilde{H}_h, \widetilde{K}_h)$ *is given by* μ^h, $h \in (0,1)$,

(2) *the law of* $(\widetilde{\varrho}_b, \widetilde{\mathbf{m}}_b, \widetilde{W}, \widetilde{N}, \widetilde{\mathcal{V}}, \widetilde{\varrho}_1, \widetilde{\mathbf{m}}_1, \widetilde{H}_b, \widetilde{K}_b)$*, denoted by* μ*, is a Radon measure,*

(3) $(\widetilde{\varrho}_h, \widetilde{\mathbf{m}}_h, \widetilde{W}_h, \widetilde{N}_h, \widetilde{\mathcal{V}}_h, \widetilde{\varrho}_{1,h}, \widetilde{\boldsymbol{m}}_{1,h}, \widetilde{H}_h, \widetilde{K}_h)$ *converges* $\widetilde{\mathbb{P}}$*-almost surely to* $(\widetilde{\varrho}_b, \widetilde{\mathbf{m}}_b, \widetilde{\boldsymbol{u}}, \widetilde{W}, \widetilde{N}, \widetilde{\mathcal{V}}, \widetilde{\varrho}_1, \widetilde{\mathbf{m}}_1, \widetilde{H}_b, \widetilde{K}_b)$ *in the topology of* $\mathcal{Y}$*, i.e.,*

$$\widetilde{\varrho}_h \to \widetilde{\varrho}_b \text{ in } C_w\big([0,T]; L^\gamma(\mathbb{T}^3)\big), \qquad \widetilde{\boldsymbol{m}}_h \to \widetilde{\boldsymbol{m}}_b \text{ in } C_w\Big([0,T]; L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^3)\Big),$$
$$\widetilde{N}_h \to \widetilde{N} \text{ in } C([0,T];\mathbb{R}), \qquad \widetilde{W}_h \to \widetilde{W} \text{ in } C([0,T];\mathfrak{W}_0),$$
$$\widetilde{\mathcal{V}}_h \to \widetilde{\mathcal{V}} \text{ weak-}* \text{ in } L^\infty\big((0,T)\times\mathbb{T}^3; \mathcal{P}(\mathbb{R}^4)\big), \qquad (\widetilde{\varrho}_{1,h}, \widetilde{\mathbf{m}}_{1,h}) \to (\widetilde{\varrho}_1, \widetilde{\mathbf{m}}_1) \text{ in } \mathbb{R}^2,$$
$$\widetilde{H}_h \to H_b \text{ weakly in } L^r\big((0,T)\times\mathbb{T}^3\big), \qquad \widetilde{K}_h \to \widetilde{K}_b \text{ weak-}* \text{ in } L^\infty\big(0,T; M_b(\mathbb{T}^3)\big).$$

(4) *For every* h*, we have* $\widetilde{W}_h = \widetilde{W}$*, and* $(\widetilde{\varrho}_{1,h}, \widetilde{\mathbf{m}}_{1,h}) = (\widetilde{\varrho}_1, \widetilde{\mathbf{m}}_1)$*,* $\widetilde{\mathbb{P}}$*-a.s.*

(5) $\widetilde{\mathbb{P}}$*-a.s.,* $\widetilde{\varrho}_b = \langle \widetilde{\mathcal{V}}^\omega_{t,x}; \widetilde{\varrho}\rangle$*,* $\widetilde{\mathbf{m}}_b = \langle \widetilde{\mathcal{V}}^\omega_{t,x}; \widetilde{\boldsymbol{m}}\rangle$*,* $H_b = \langle \widetilde{\mathcal{V}}^\omega_{(\cdot,\cdot)}; H(\widetilde{\varrho}, \widetilde{\boldsymbol{m}})\rangle$*, and* $K_b = \langle \widetilde{\mathcal{V}}^\omega_{(\cdot,\cdot)}; K(\widetilde{\varrho}, \widetilde{\boldsymbol{m}})\rangle \,\mathrm{d}x\,\mathrm{d}t + \widetilde{\mu}_K$*,, where* μ_K *is the concentration defect measure associated to the function* K*.*

Proof. Proof of the items (1)–(4) directly follow from Theorem 2.5. For item (5), by making use of the fundamental theorem of Young measure ([38], Thm. 4.2.1, Cor. 4.2.10) (Young measure captures the weak limits), we conclude that $\widetilde{\mathbb{P}}$-a.s.

$$\widetilde{\varrho}_b = \Big\langle \widetilde{\mathcal{V}}^\omega_{t,x}; \widetilde{\varrho}\Big\rangle, \quad \widetilde{\mathbf{m}}_b = \Big\langle \widetilde{\mathcal{V}}^\omega_{t,x}; \widetilde{\mathbf{m}}\Big\rangle, \quad \text{and} \quad H_b = \Big\langle \widetilde{\mathcal{V}}^\omega_{(\cdot,\cdot)}; H(\widetilde{\varrho}, \widetilde{\mathbf{m}})\Big\rangle. \tag{5.4}$$

With the help of concentration defect measures, thanks to the discussion in Section 2.1.1, we conclude that $\widetilde{\mathbb{P}}$-a.s.

$$K_b = \Big\langle \widetilde{\mathcal{V}}^\omega_{(\cdot,\cdot)}; K(\widetilde{\varrho}, \widetilde{\mathbf{m}})\Big\rangle \,\mathrm{d}x\,\mathrm{d}t + \widetilde{\mu}_K.$$

□

5.1.1. Passage to the limit

We shall now make use of the above convergences to pass to the limit in approximate equations (4.2), (4.3), and the energy inequality (4.4). To that context, let us first show that the approximations $\widetilde{\varrho}_h, \widetilde{u}_h$ solve equations (4.2) and (4.3) on the new probability space $(\widetilde{\Omega}, \widetilde{\mathbb{F}}, \widetilde{\mathbb{P}})$. Note that, since $(\varrho_h, \mathbf{m}_h, N_h)$ are random variables with values in $C([0,T]; L^\gamma(\mathbb{T}^3)) \times C([0,T]; L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^3)) \times C([0,T];\mathbb{R})$. By Lemma A.3 of [43] and Corollary A.2 of [41], $(\widetilde{\varrho}_h, \widetilde{\mathbf{m}}_h, \widetilde{N}_h)$ are also random variables with values in $C([0,T], L^\gamma(\mathbb{T}^3)) \times C([0,T], L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^3)) \times C([0,T];\mathbb{R})$. Let $(\widetilde{\mathbb{F}}^h_t)$ be the $\widetilde{\mathbb{P}}$-augmented canonical filtration of the process $(\widetilde{\varrho}_h, \widetilde{\mathbf{m}}_h, \widetilde{W}, \widetilde{N}_h)$, that is

$$\widetilde{\mathbb{F}}^h_t = \sigma\Big(\sigma\Big(\mathbf{r}_t\widetilde{\varrho}_h,\, \mathbf{r}_t\widetilde{\mathbf{m}}_h,\, \mathbf{r}_t\widetilde{W}, \mathbf{r}_t\widetilde{N}_h\Big) \cup \Big\{N \in \widetilde{\mathbb{F}};\ \widetilde{\mathbb{P}}(N) = 0\Big\}\Big), \quad t \in [0,T],$$

where we denote by $\mathbf{r}_t$ the operator of restriction to the interval $[0,t]$ acting on various path spaces. Let us remark that by assuming that the initial filtration $(\mathbb{F}_t)$ is the one generated by W, by Lemma A.6 of [43], one can consider $(\widetilde{\mathbb{F}}^h_t) = (\widetilde{\mathbb{F}}_t)$ is the filtration generated by $\widetilde{W}$.

Proposition 5.7. *For every* $h \in (0,1)$*,* $((\widetilde{\Omega}, \widetilde{\mathbb{F}}, (\widetilde{\mathbb{F}}_t), \widetilde{\mathbb{P}}), \widetilde{\varrho}_h, \widetilde{\mathbf{u}}_h, \widetilde{W})$ *is a finite energy weak martingale*[3] *solution to* (4.2) *and* (4.3) *with the initial law* $\Lambda_h = \mathcal{L}(\widetilde{\varrho}_{0,h}, \widetilde{\mathbf{m}}_{0,h})$*, that is, the following equations and the energy inequality hold;*

[3] For definition of martingale solution, see Definition 2.9 in [30].

– *for all* $\varphi \in C^\infty(\mathbb{T}^3)$ *and* $\boldsymbol{\varphi} \in C^\infty(\mathbb{T}^3)$ *we have* $\widetilde{\mathbb{P}}$*-a.s. for all* $t \in [0,T]$

$$\langle \widetilde{\varrho}_h(t), \varphi\rangle = \langle \widetilde{\varrho}_h(0), \varphi\rangle + \int_0^t \langle \widetilde{\mathbf{m}}_h, \nabla\varphi\rangle \,\mathrm{d}s + h\int_0^t \lambda\,\langle \widetilde{\varrho}_h(t), \Delta\varphi\rangle\,\mathrm{d}s + \mathcal{R}_1(\widetilde{\varrho}_h, \widetilde{\mathbf{m}}_h, h, \varphi) + \mathcal{N}_1(\widetilde{\varrho}_h, \widetilde{\mathbf{m}}_h, h, \varphi) \tag{5.5}$$

$$\langle \widetilde{\mathbf{m}}_h(t), \boldsymbol{\varphi}\rangle = \langle \widetilde{\mathbf{m}}_h(0), \boldsymbol{\varphi}\rangle + \int_0^t \left\langle \left(\frac{\widetilde{\mathbf{m}}_h \otimes \widetilde{\mathbf{m}}_h}{\widetilde{\varrho}_h} + \widetilde{p}_h\mathbb{I}\right), \nabla\boldsymbol{\varphi}\right\rangle \mathrm{d}s + h\int_0^t \lambda\langle \widetilde{\mathbf{m}}_h(t), \Delta\boldsymbol{\varphi}\rangle\,\mathrm{d}s + \int_0^t \langle \Psi(\widetilde{\varrho}_h, \widetilde{\mathbf{m}}_h), \boldsymbol{\varphi}\rangle\,\mathrm{d}\widetilde{W} + \mathcal{R}_2(\widetilde{\varrho}_h, \widetilde{\mathbf{m}}_h, h, \boldsymbol{\varphi}) + \mathcal{N}_2(\widetilde{\varrho}_h, \widetilde{\mathbf{m}}_h, h, \boldsymbol{\varphi}), \tag{5.6}$$

where $\mathcal{R}_1, \mathcal{R}_2, \mathcal{N}_1$*, and* $\mathcal{N}_2$ *are defined as in* (4.5)*, in the new probability space.*

– *The energy inequality holds*

$$-\int_0^T \partial_t\psi \int_{\mathbb{T}^3}\left[\frac{1}{2}\frac{|\widetilde{\mathbf{m}}_h|^2}{\widetilde{\varrho}_h} + \frac{\widetilde{\varrho}_h^\gamma}{\gamma-1}\right]\mathrm{d}x\,\mathrm{d}s \le \psi(0)\int_{\mathbb{T}^3}\left[\frac{1}{2}\frac{|\widetilde{\mathbf{m}}_h(0)|^2}{\widetilde{\varrho}_h(0)} + \frac{\widetilde{\varrho}_h^\gamma(0)}{\gamma-1}\right]\mathrm{d}x + \sum_{k=1}^\infty \int_0^T \psi\left(\int_{\mathbb{T}^3} \Psi_k(\widetilde{\varrho}_h, \widetilde{\mathbf{m}}_h)\cdot \widetilde{\mathbf{u}}_h\,\mathrm{d}x\right)\mathrm{d}\widetilde{W}_k + \frac{1}{2}\sum_{k=1}^\infty\int_0^T \psi\int_{\mathbb{T}^3} \widetilde{\varrho}_h^{-1}|\Psi_k(\widetilde{\varrho}_h, \widetilde{\mathbf{m}}_h)|^2\,\mathrm{d}x\,\mathrm{d}s \tag{5.7}$$

holds $\widetilde{\mathbb{P}}$*-a.s., for all* $\psi \in C_c^\infty([0,T)), \psi \ge 0$.

Proof. For the proof, we follow the similar arguments as in the proof of Theorem 2.9.1 from [9]. Let $\psi \in C_c^\infty([0,T))$ and $\varphi \in C^\infty(\mathbb{T}^3)$ be test functions. By using the integration by parts formula with (4.2), we obtain $\mathbb{P}$-almost surely,

$$-\int_0^T \partial_t\psi(t)\langle \varrho_h(t), \varphi\rangle\,\mathrm{d}t = \psi(0)\langle \varrho_h(0), \varphi\rangle + \int_0^T \langle \mathbf{m}_h, \nabla\varphi\rangle\psi(t)\,\mathrm{d}t + h\int_0^T \lambda\langle \varrho_h(t), \Delta\varphi\rangle\psi(t)\,\mathrm{d}t + \int_0^T \partial_t\mathcal{R}_1(\varrho_h, \mathbf{m}_h, h, \varphi)\psi(t)\mathrm{d}t + \int_0^T \partial_t\mathcal{N}_1(\varrho_h, \mathbf{m}_h, h, \varphi)\psi(t)\,\mathrm{d}t. \tag{5.8}$$

For fixed $h > 0$, we define Borel measurable function on $C_w([0,T]; L^\gamma(\mathbb{T}^3)) \times C_w([0,T]; L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^3))$ as follows:

$$L(\varrho, \mathbf{m}) = \int_0^T \partial_t\psi(t)\langle \varrho(t), \varphi\rangle\,\mathrm{d}t + \psi(0)\langle \varrho(0), \varphi\rangle + \int_0^T \langle \mathbf{m}, \nabla\varphi\rangle\psi(t)\,\mathrm{d}t + h\int_0^T \lambda\,\langle \varrho(t), \Delta\varphi\rangle\psi(t)\,\mathrm{d}t + \int_0^T \partial_t\mathcal{R}_1(\varrho, \mathbf{m}, h, \varphi)\psi(t)\,\mathrm{d}t + \int_0^T \partial_t\mathcal{N}_1(\varrho, \mathbf{m}, h, \varphi)\psi(t)\,\mathrm{d}t$$

where

$$\mathcal{R}_1(\varrho, \mathbf{m}, h, \varphi) := \int_0^t\int_{\mathbb{T}^3}\sum_{i=1}^3 m^i(s)\left(\partial_{x_i}\varphi - \frac{\varphi(x+h\mathbf{e}_i) - \varphi(x-h\mathbf{e}_i)}{2h}\right)\mathrm{d}x\,\mathrm{d}s,$$

$$\mathcal{N}_1(\varrho, \mathbf{m}, h, \varphi) := h\int_0^t\int_{\mathbb{T}^3}\lambda\,\varrho(s)\sum_{i=1}^3\left(\partial_{x_ix_i}\varphi - \frac{\varphi(x+h\mathbf{e}_i) - 2\varphi(x) + \varphi(x-h\mathbf{e}_i)}{h^2}\right)\mathrm{d}x\,\mathrm{d}s. \tag{5.9}$$

From (5.8), we have $\mathbb{P}$-almost surely

$$L(\varrho_h, \mathbf{m}_h) = 0.$$

It implies that

$$\mathcal{L}\{L(\varrho_h, \mathbf{m}_h)\}(\{0\}) = \mathbb{P}\{L(\varrho_h, \mathbf{m}_h) = 0\} = 1.$$

By the equality of law as given in Proposition 5.6, we can conclude that

$$\mathcal{L}\{L(\varrho_h, \mathbf{m}_h)\} = \mathcal{L}\{L(\widetilde{\varrho}_h, \widetilde{\mathbf{m}}_h)\},$$

and

$$\mathbb{P}\{L(\varrho_h, \mathbf{m}_h) = 0\} = \widetilde{\mathbb{P}}\{L(\widetilde{\varrho}_h, \widetilde{\mathbf{m}}_h) = 0\} = 1.$$

It implies that $\widetilde{\mathbb{P}}$-almost surely,

$$\begin{aligned}
-\int_0^T \partial_t \psi(t) \langle \widetilde{\varrho}_h(t), \varphi \rangle \, \mathrm{d}t = \psi(0) \langle \widetilde{\varrho}_h(0), \varphi \rangle + \int_0^T \langle \widetilde{\mathbf{m}}_h, \nabla \varphi \rangle \psi(t) \, \mathrm{d}t + h \int_0^T \lambda \, \langle \widetilde{\varrho}_h(t), \Delta \varphi \rangle \psi(t) \, \mathrm{d}t \\
+ \int_0^T \partial_t \mathcal{R}_1(\widetilde{\varrho}_h, \widetilde{\mathbf{m}}_h, h, \varphi) \psi(t) \, \mathrm{d}t + \int_0^T \partial_t \mathcal{N}_1(\widetilde{\varrho}_h, \widetilde{\mathbf{m}}_h, h, \varphi) \psi(t) \, \mathrm{d}t.
\end{aligned}$$

By using the standard argument and continuity in time variable, we can conclude that for all $\varphi \in C^\infty(\mathbb{T}^3)$ and $\boldsymbol{\varphi} \in C^\infty(\mathbb{T}^3)$, $\widetilde{\mathbb{P}}$-a.s. for all $t \in [0, T]$,

$$\langle \widetilde{\varrho}_h(t), \varphi \rangle = \langle \widetilde{\varrho}_h(0), \varphi \rangle + \int_0^t \langle \widetilde{\mathbf{m}}_h, \nabla \varphi \rangle \, \mathrm{d}s + h \int_0^t \lambda \langle \widetilde{\varrho}_h(t), \Delta \varphi \rangle \, \mathrm{d}s + \mathcal{R}_1(\widetilde{\varrho}_h, \widetilde{\mathbf{m}}_h, h, \varphi) + \mathcal{N}_1(\widetilde{\varrho}_h, \widetilde{\mathbf{m}}_h, h, \varphi).$$

For momentum equation (5.6), let $\psi \in C_c^\infty\big([0, T)\big)$ and $\boldsymbol{\varphi} \in C^\infty(\mathbb{T}^3; \mathbb{R}^3)$ be test functions. We apply Itô formula with (4.3) to conclude that $\mathbb{P}$-almost surely,

$$\begin{aligned}
-\int_0^T \partial_t \psi(t) \langle \mathbf{m}_h(t), \boldsymbol{\varphi} \rangle \, \mathrm{d}t = \psi(0) \langle \mathbf{m}_h(0), \boldsymbol{\varphi} \rangle + \int_0^T \psi(t) \Big\langle \Big(\frac{\mathbf{m}_h \otimes \mathbf{m}_h}{\varrho_h} + p_h \mathbb{I} \Big), \nabla \boldsymbol{\varphi} \Big\rangle \, \mathrm{d}t \\
+ h \int_0^T \psi(t) \lambda \langle \mathbf{m}_h(t), \Delta \boldsymbol{\varphi} \rangle \, \mathrm{d}t + \int_0^T \psi(t) \langle \Psi(\varrho_h, \mathbf{m}_h), \boldsymbol{\varphi} \rangle \, \mathrm{d}W \\
+ \int_0^T \psi(t) \partial_t \mathcal{R}_2(\varrho_h, \mathbf{m}_h, h, \boldsymbol{\varphi}) \, \mathrm{d}t + \int_0^T \psi(t) \partial_t \mathcal{N}_2(\varrho_h, \mathbf{m}_h, h, \boldsymbol{\varphi}) \, \mathrm{d}t.
\end{aligned} \tag{5.10}$$

Let us define function $L_1 : L^2_{\mathcal{F}}(\Omega \times [0, T]; L^\gamma(\mathbb{T}^3)) \times L^2_{\mathcal{F}}(\Omega \times [0, T]; L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^3)) \to L^2(\Omega)$ as follows[4];

$$\begin{aligned}
L_1(\varrho, \mathbf{m}) = \int_0^T \partial_t \psi(t) \langle \mathbf{m}(t), \boldsymbol{\varphi} \rangle \mathrm{d}t + \psi(0) \langle \mathbf{m}(0), \boldsymbol{\varphi} \rangle + \int_0^T \psi(t) \Big\langle \Big(\frac{\mathbf{m} \otimes \mathbf{m}}{\varrho} + p_h \mathbb{I} \Big), \nabla \boldsymbol{\varphi} \Big\rangle \, \mathrm{d}t \\
+ \int_0^T \psi(t) \lambda \langle \mathbf{m}(t), \Delta \boldsymbol{\varphi} \rangle \, \mathrm{d}t + \int_0^T \psi(t) \langle \Psi(\varrho, \mathbf{m}), \boldsymbol{\varphi} \rangle \, \mathrm{d}W(t) + \int_0^T \psi(t) \partial_t \mathcal{R}_2(\varrho, \mathbf{m}, h, \boldsymbol{\varphi}) \, \mathrm{d}t \\
+ \int_0^T \psi(t) \partial_t \mathcal{N}_2(\varrho, \mathbf{m}, h, \boldsymbol{\varphi}) \, \mathrm{d}t
\end{aligned} \tag{5.11}$$

where $\mathcal{R}_2, \mathcal{N}_2$ are defined as follow

$$\mathcal{R}_2(\varrho, \mathbf{m}, h, \boldsymbol{\varphi}) := \int_0^t \int_{\mathbb{T}^3} \sum_{i=1}^3 \sum_{j=1}^3 \Big(\frac{m^i(s) m^j(s)}{\varrho(s)} + p(s) \Big) \Big(\partial_{x_i} \varphi^j - \frac{\varphi^j(x + h\mathbf{e}_i) - \varphi^j(x - h\mathbf{e}_i)}{2h} \Big) \, \mathrm{d}x \, \mathrm{d}s,$$

$$\mathcal{N}_2(\varrho, \mathbf{m}, h, \boldsymbol{\varphi}) := h \int_0^t \int_{T^3} \lambda \sum_{i=1}^3 \sum_{j=1}^3 m^j(s) \Big(\partial_{x_i x_i} \varphi^j - \frac{\varphi^j(x + h\mathbf{e}_i) - 2\varphi^j(x) + \varphi^j(x - h\mathbf{e}_i)}{h^2} \Big) \, \mathrm{d}x \, \mathrm{d}s.$$

[4]Here, $L^2_{\mathcal{F}}$ refer to denote the space of predictable process

From (5.10), we have $\mathbb{P}$-almost surely

$$L_1(\varrho_h, \mathbf{m}_h) = 0.$$

It implies that

$$\mathcal{L}\{L_1(\varrho_h, \mathbf{m}_h)\}\big(\{0\}\big) = \mathbb{P}\{L_1(\varrho_h, \mathbf{m}_h) = 0\} = 1.$$

On the new probability space, similarly we defined $\widetilde{L}_1 : L^2_{\mathcal{F}}(\widetilde{\Omega} \times [0,T]; L^\gamma(\mathbb{T}^3)) \times L^2_{\mathcal{F}}(\widetilde{\Omega} \times [0,T]; L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^3)) \to L^2(\widetilde{\Omega})$ as defined L_1. Now we can follow the similar lines (the method of cut-off function and regularization) as done in the proof of Theorem 2.9.1 from [9], and using equality of laws, we can conclude that

$$\mathcal{L}\{L_1(\varrho_h, \mathbf{m}_h)\} = \mathcal{L}\Big\{\widetilde{L}_1(\widetilde{\varrho}_h, \widetilde{\mathbf{m}}_h)\Big\},$$

and

$$\widetilde{\mathbb{P}}\Big\{\widetilde{L}_1(\widetilde{\varrho}_h, \widetilde{\mathbf{m}}_h) = 0\Big\} = \mathbb{P}\{L_1(\varrho_h, \mathbf{m}_h) = 0\} = 1.$$

We conclude that $\widetilde{\mathbb{P}}$-almost surely,

$$\begin{aligned}
-\int_0^T \partial_t\psi(t)\langle \widetilde{\mathbf{m}}_h(t), \boldsymbol{\varphi}\rangle \,\mathrm{d}t = \psi(0)\langle \widetilde{\mathbf{m}}_h(0), \boldsymbol{\varphi}\rangle + \int_0^T \psi(t)\Big\langle \Big(\frac{\widetilde{\mathbf{m}}_h \otimes \widetilde{\mathbf{m}}_h}{\widetilde{\varrho}_h} + \widetilde{p}_h\mathbb{I}\Big), \nabla\boldsymbol{\varphi}\Big\rangle \,\mathrm{d}t \\
+ h\int_0^T \psi(t)\lambda\,\langle \widetilde{\mathbf{m}}_h(t), \Delta\boldsymbol{\varphi}\rangle \,\mathrm{d}t + \int_0^T \psi(t)\langle \Psi(\widetilde{\varrho}_h, \widetilde{\mathbf{m}}_h), \boldsymbol{\varphi}\rangle \,\mathrm{d}\widetilde{W} \\
+ \int_0^T \psi(t)\partial_t\mathcal{R}_2(\widetilde{\varrho}_h, \widetilde{\mathbf{m}}_h, h, \boldsymbol{\varphi}) \,\mathrm{d}t + \int_0^T \psi(t)\partial_t\mathcal{N}_2(\widetilde{\varrho}_h, \widetilde{\mathbf{m}}_h, h, \boldsymbol{\varphi}).
\end{aligned}$$

By using continuity in time variable and standard argument with appropriate test function in time variable, we can conclude that for all $\varphi \in C^\infty(\mathbb{T}^3)$ and $\boldsymbol{\varphi} \in C^\infty(\mathbb{T}^3)$, $\widetilde{\mathbb{P}}$-a.s. for all $t \in [0,T]$

$$\begin{aligned}
\langle \widetilde{\mathbf{m}}_h(t), \boldsymbol{\varphi}\rangle = \langle \widetilde{\mathbf{m}}_h(0), \boldsymbol{\varphi}\rangle + \int_0^t \Big\langle \Big(\frac{\widetilde{\mathbf{m}}_h \otimes \widetilde{\mathbf{m}}_h}{\widetilde{\varrho}_h} + \widetilde{p}_h\mathbb{I}\Big), \nabla\boldsymbol{\varphi}\Big\rangle \,\mathrm{d}s + h\int_0^t \lambda\,\langle \widetilde{\mathbf{m}}_h(t), \Delta\boldsymbol{\varphi}\rangle \,\mathrm{d}s \\
+ \int_0^t \langle \Psi(\widetilde{\varrho}_h, \widetilde{\mathbf{m}}_h), \boldsymbol{\varphi}\rangle \,\mathrm{d}\widetilde{W} + \mathcal{R}_2(\widetilde{\varrho}_h, \widetilde{\mathbf{m}}_h, h, \boldsymbol{\varphi}) + \mathcal{N}_2(\widetilde{\varrho}_h, \widetilde{\mathbf{m}}_h, h, \boldsymbol{\varphi}).
\end{aligned}$$

With the help of similar arguments as previous used, one can easily prove the energy inequality (5.7). ☐

Next we would like to pass to the limit in h in (5.5)–(5.7). To do this, we first recall that *a priori* estimates (3.5)–(3.7) continue to hold for the new random variables. Thus, making use of the item (5) of Proposition 5.6, we conclude that $\widetilde{\mathbb{P}}$-a.s.,

$$\widetilde{\varrho}_h \rightharpoonup \Big\langle \widetilde{\mathcal{V}}^\omega_{t,x}; \widetilde{\varrho}\Big\rangle, \text{ weakly in } L^\gamma\big((0,T)\times\mathbb{T}^3\big), \tag{5.12}$$

$$\widetilde{\mathbf{m}}_h \rightharpoonup \Big\langle \widetilde{\mathcal{V}}^\omega_{t,x}; \widetilde{\mathbf{m}}\Big\rangle, \text{ weakly in } L^{\frac{2\gamma}{\gamma+1}}\big((0,T)\times\mathbb{T}^3\big). \tag{5.13}$$

In order to pass to the limit in the nonlinear terms present in the equations, we first introduce the corresponding concentration defect measures

$$\widetilde{\mu}_C = \widetilde{C} - \left\langle \widetilde{\mathcal{V}}^{\omega}_{(\cdot,\cdot)}; \frac{\widetilde{\mathbf{m}} \otimes \widetilde{\mathbf{m}}}{\widetilde{\varrho}} \right\rangle \mathrm{d}x\,\mathrm{d}t, \qquad \widetilde{\mu}_P = \widetilde{P} - \langle \widetilde{\mathcal{V}}^{\omega}_{(\cdot,\cdot)}; p(\widetilde{\varrho}) \rangle \,\mathrm{d}x\,\mathrm{d}t,$$

$$\widetilde{\mu}_E = \widetilde{E} - \left\langle \widetilde{\mathcal{V}}^{\omega}_{(\cdot,\cdot)}; \frac{1}{2}\frac{|\widetilde{\mathbf{m}}|^2}{\widetilde{\varrho}} + P(\widetilde{\varrho}) \right\rangle \mathrm{d}x, \qquad \widetilde{\mu}_D = \widetilde{D} - \left\langle \widetilde{\mathcal{V}}^{\omega}_{(\cdot,\cdot)}; \sum_{k\ge 1} \frac{|\Psi_k(\widetilde{\varrho},\widetilde{\mathbf{m}})|^2}{\widetilde{\varrho}} \right\rangle \mathrm{d}x\,\mathrm{d}t.$$

With the help of these concentration defect measures, thanks to item (5) of Proposition 5.6, we can conclude that $\widetilde{\mathbb{P}}$-a.s.

$$\begin{aligned}
\widetilde{C}_h &\rightharpoonup \left\langle \widetilde{\mathcal{V}}^{\omega}_{(\cdot,\cdot)}; \frac{\widetilde{\mathbf{m}} \otimes \widetilde{\mathbf{m}}}{\widetilde{\varrho}} \right\rangle \mathrm{d}x\,\mathrm{d}t + \widetilde{\mu}_C, \text{ weak-}* \text{ in } L^\infty\big(0,T;\mathcal{M}_b(\mathbb{T}^3)\big),\\
\widetilde{D}_h &\rightharpoonup \left\langle \widetilde{\mathcal{V}}^{\omega}_{(\cdot,\cdot)}; \sum_{k\ge 1} \frac{|\Psi_k(\widetilde{\varrho},\widetilde{\mathbf{m}})|^2}{\widetilde{\varrho}} \right\rangle \mathrm{d}x\,\mathrm{d}t + \widetilde{\mu}_D, \text{ weak-}* \text{ in } L^\infty\big(0,T;\mathcal{M}_b(\mathbb{T}^3)\big),\\
\widetilde{E}_h &\rightharpoonup \left\langle \widetilde{\mathcal{V}}^{\omega}_{(\cdot,\cdot)}; \frac{1}{2}\frac{|\widetilde{\mathbf{m}}|^2}{\widetilde{\varrho}} + P(\widetilde{\varrho}) \right\rangle \mathrm{d}x + \widetilde{\mu}_E, \text{ weak-}* \text{ in } L^\infty\big(0,T;\mathcal{M}_b^+(\mathbb{T}^3)\big),\\
\widetilde{P}_h &\rightharpoonup \left\langle \widetilde{\mathcal{V}}^{\omega}_{(\cdot,\cdot)}; P(\widetilde{\varrho}) \right\rangle \mathrm{d}x\,\mathrm{d}t + \widetilde{\mu}_P, \text{ weak-}* \text{ in } L^\infty\big(0,T;\mathcal{M}_b(\mathbb{T}^3)\big).
\end{aligned}$$

Remark 5.8. Note that under additional hypothesis (2.18), thanks to the equality of joint laws, we have $\underline{\varrho} \le \widetilde{\varrho}_h \le \widetilde{\varrho}_1$ and $|\widetilde{\mathbf{m}}_h| \le \widetilde{\mathbf{m}}_1$ for all $h \in (0,1)$. As a result, all the nonlinearities appearing in the consistency formulation (5.5) and (5.6) are $\widetilde{\mathbb{P}}$-almost surely, weakly precompact in $L^1([0,T]\times\mathbb{T}^3)$. This implies that all the concentration defect measures will disappear in the limit. However, to provide a general convergence analysis framework, based on only energy bounds, we assume the presence of defect measures in the upcoming analysis though they vanish in our present settings.

Note that, collecting all the previous informations, we can pass to the limit in equation (5.5) to get $\widetilde{\mathbb{P}}$-a.s.

$$\int_{\mathbb{T}^3} \left\langle \widetilde{\mathcal{V}}^{\omega}_{\tau,x}; \widetilde{\varrho} \right\rangle \varphi \,\mathrm{d}x - \int_{\mathbb{T}^3} \left\langle \widetilde{\mathcal{V}}^{\omega}_{0,x}; \widetilde{\varrho} \right\rangle \varphi \,\mathrm{d}x = \int_0^\tau \int_{\mathbb{T}^3} \left\langle \widetilde{\mathcal{V}}^{\omega}_{s,x}; \widetilde{\mathbf{m}} \right\rangle \cdot \nabla_x \varphi \,\mathrm{d}x\,\mathrm{d}s$$

holds for all $\tau \in [0,T)$, and for all $\varphi \in C^\infty(\mathbb{T}^3)$.

Next, we move onto the martingale term $\widetilde{M}_h := \int_0^t \langle \Psi(\widetilde{\varrho}_h, \widetilde{\mathbf{m}}_h), \boldsymbol{\varphi}\rangle \,\mathrm{d}\widetilde{W}$ coming from the momentum equation. Note that thanks to compact embedding given in Lemma 2.1, we conclude that for each t, $\tilde{M}_h(t) \to \tilde{M}(t)$, $\widetilde{\mathbb{P}}$-a.s. in the topology of $W^{-m,2}(\mathbb{T}^N)$. However, we are interested in identifying $\tilde{M}(t)$. Indeed, we may apply item (5) of Proposition 5.6, to the composition $\Psi_k(\widetilde{\rho}_h, \widetilde{\mathbf{m}}_h)$, $k \in \mathbb{N}$. This gives

$$\Psi_k(\widetilde{\rho}_h, \widetilde{\mathbf{m}}_h) \rightharpoonup \left\langle \widetilde{\mathcal{V}}^{\omega}_{t,x}; \Psi_k(\widetilde{\rho}, \widetilde{\mathbf{m}}) \right\rangle \text{ weakly in } L^q\big((0,T)\times\mathbb{T}^3\big),$$

$\widetilde{\mathbb{P}}$-a.s., for some $q > 1$. Moreover, for $m > 3/2$, we have by Sobolev embedding

$$\widetilde{\mathbb{E}}\left[\int_0^T \|\Psi(\widetilde{\rho}_h, \widetilde{\mathbf{m}}_h)\|^2_{L_2(\mathcal{U};W^{-m,2})} \,\mathrm{d}t\right] \le \widetilde{\mathbb{E}}\left[\int_0^T (\widetilde{\rho}_h)_{\mathbb{T}^3} \int_{\mathbb{T}^3} \Big(\widetilde{\rho}_h + \widetilde{\rho}_h |\widetilde{\mathbf{u}}_h|^2\Big) \,\mathrm{d}x\,\mathrm{d}t\right] \le c(r).$$

This implies that

$$\Psi_k(\widetilde{\rho}, \widetilde{\mathbf{m}}) \rightharpoonup \left\langle \widetilde{\mathcal{V}}^{\omega}_{t,x}; \Psi_k(\widetilde{\rho}, \widetilde{\mathbf{m}}) \right\rangle \text{ weakly in } L^2\big(\Omega\times[0,T]; W^{-m,2}(\mathbb{T}^3)\big).$$

Note that for any $t \in [0,T]$, the Itô integral

$$I_t : \varphi \to \int_0^t \varphi(s)\,\mathrm{d}\widetilde{W}(s)$$

is a linear and continuous (hence weakly continuous) map from $L^2(\Omega\times[0,T];W^{-m,2}(\mathbb{T}^3))$ to $L^2(\Omega;W^{-m,2}(\mathbb{T}^3))$. Therefore, we can make use of weak continuity of Itô integral, and item (4) of Proposition 5.6, to conclude $I_t(\Psi_k(\widetilde{\rho}_h,\widetilde{\mathbf{m}}_h))$ converges weakly to $I_t(\langle\widetilde{\mathcal{V}}^\omega_{(\cdot,\cdot)};\Psi_k(\widetilde{\rho},\widetilde{\mathbf{m}})\rangle)$ in $L^2(\Omega;W^{-m,2}(\mathbb{T}^3))$. Collecting all above informations, we can conclude that

$$\begin{aligned}\int_{\widetilde{\Omega}}\bigg(\int_{\mathbb{T}^3}\Big\langle\widetilde{\mathcal{V}}^\omega_{\tau,x};\widetilde{\mathbf{m}}\Big\rangle\cdot\boldsymbol{\varphi}\,\mathrm{d}x-\int_{\mathbb{T}^3}\Big\langle\widetilde{\mathcal{V}}^\omega_{0,x};\widetilde{\mathbf{m}}\Big\rangle\cdot\boldsymbol{\varphi}\,\mathrm{d}x\bigg)\alpha(\omega)\,\mathrm{d}\widetilde{\mathbb{P}}(\omega)\\ =\int_{\widetilde{\Omega}}\bigg(\int_0^\tau\int_{\mathbb{T}^3}\bigg[\bigg\langle\widetilde{\mathcal{V}}^\omega_{t,x};\frac{\widetilde{\mathbf{m}}\otimes\widetilde{\mathbf{m}}}{\widetilde{\varrho}}\bigg\rangle:\nabla\boldsymbol{\varphi}+\Big\langle\widetilde{\mathcal{V}}^\omega_{t,x};p(\widetilde{\varrho})\Big\rangle\operatorname{div}\boldsymbol{\varphi}\bigg]\,\mathrm{d}x\,\mathrm{d}t\\ +\int_0^\tau\Big\langle\Big\langle\widetilde{\mathcal{V}}^\omega_{t,x};\Psi(\widetilde{\varrho},\widetilde{\mathbf{m}})\Big\rangle,\boldsymbol{\varphi}\Big\rangle\,\mathrm{d}\widetilde{W}+\int_0^\tau\int_{\mathbb{T}^3}\nabla\boldsymbol{\varphi}:\mathrm{d}(\widetilde{\mu}_C+\widetilde{\mu}_P\mathbb{I})\bigg)\alpha(\omega)\,\mathrm{d}\widetilde{\mathbb{P}}(\omega),\end{aligned}\tag{5.14}$$

holds for all $\tau\in[0,T)$, for all $\alpha\in L^2(\widetilde{\Omega})$ and for all $\boldsymbol{\varphi}\in C^\infty(\mathbb{T}^3;\mathbb{R}^3)$. Since $C^\infty(\mathbb{T}^3)$ is separable space with sup norm, above equality (5.14) implies that $\widetilde{\mathbb{P}}$-a.s.

$$\begin{aligned}\int_{\mathbb{T}^3}\Big\langle\widetilde{\mathcal{V}}^\omega_{\tau,x};\widetilde{\mathbf{m}}\Big\rangle\cdot\boldsymbol{\varphi}\,\mathrm{d}x-\int_{\mathbb{T}^3}\Big\langle\widetilde{\mathcal{V}}^\omega_{0,x};\widetilde{\mathbf{m}}\Big\rangle\cdot\boldsymbol{\varphi}\,\mathrm{d}x=\int_0^\tau\int_{\mathbb{T}^3}\bigg[\bigg\langle\widetilde{\mathcal{V}}^\omega_{t,x};\frac{\widetilde{\mathbf{m}}\otimes\widetilde{\mathbf{m}}}{\widetilde{\varrho}}\bigg\rangle:\nabla\boldsymbol{\varphi}+\Big\langle\widetilde{\mathcal{V}}^\omega_{t,x};p(\widetilde{\varrho})\Big\rangle\operatorname{div}\boldsymbol{\varphi}\bigg]\,\mathrm{d}x\,\mathrm{d}t\\ +\int_0^\tau\Big\langle\Big\langle\widetilde{\mathcal{V}}^\omega_{t,x};\Psi(\widetilde{\varrho},\widetilde{\mathbf{m}})\Big\rangle,\boldsymbol{\varphi}\Big\rangle\,\mathrm{d}\widetilde{W}+\int_0^\tau\int_{\mathbb{T}^3}\nabla\boldsymbol{\varphi}:\mathrm{d}(\widetilde{\mu}_C+\widetilde{\mu}_P\mathbb{I})\end{aligned}$$

holds for all $\tau\in[0,T)$, and for all $\boldsymbol{\varphi}\in C^\infty(\mathbb{T}^3;\mathbb{R}^3)$. where $(\widetilde{\mu}_C+\widetilde{\mu}_P\mathbb{I})\in L^\infty_{w^*}\big([0,T];\mathcal{M}_b(\mathbb{T}^3),\widetilde{\mathbb{P}}$-a.s., is tensor-valued measure. Therefore we conclude that (2.9) and (2.10) holds.

Regarding the convergence of martingale term $\widetilde{N}_h$, appearing in the energy inequality, we have following proposition.

Proposition 5.9. *For each t, $\widetilde{N}_h(t)\to\widetilde{N}(t)$ in $\mathbb{R}$, $\mathbb{P}$-a.s., and $\widetilde{N}(t)$ is a real valued square-integrable martingale.*

Proof. Note that, thanks to Proposition 5.6, we have the information $\widetilde{N}_h\to\widetilde{N}$, $\widetilde{\mathbb{P}}$-a.s. in $C([0,T];\mathbb{R})$. To conclude that $\widetilde{N}(t)$ is a martingale, We have to show that, $\widetilde{\mathbb{P}}$-a.s.

$$\widetilde{\mathbb{E}}\Big[\widetilde{N}(t)|\widetilde{\mathcal{F}}_s\Big]=\widetilde{N}(s),$$

for all $t,s\in[0,T]$ with $s\le t$. To prove this, it is sufficient to show that, for all $A\in\widetilde{\mathcal{F}}_s$

$$\widetilde{\mathbb{E}}\Big[\mathcal{I}_A\Big(\widetilde{N}(t)-\widetilde{N}(s)\Big)\Big]=0.$$

Now using the fact that $\widetilde{N}_h(t)$ is a martingale, we know that

$$\widetilde{\mathbb{E}}\Big[\mathcal{I}_A\Big(\widetilde{N}_h(t)-\widetilde{N}_h(s)\Big)\Big]=0,$$

for all $A\in\widetilde{\mathcal{F}}_s$. Note that for all $t\in[0,T]$, $\widetilde{N}_h(t)$ is uniformly bounded in $L^2(\widetilde{\Omega})$. Therefore, thanks to Vitali's convergence theorem, we can pass to the limit in h to conclude that $\widetilde{N}(t)$ is a martingale. □

Lemma 5.10. *The concentration defect $0\le\widetilde{\mathcal{D}}(\tau):=\widetilde{\mu}_E(\tau)(\mathbb{T}^3)$ dominates defect measures $\widetilde{\mu}_D$ in the sense of Lemma 2.3. More precisely, there exists a constant $C>0$ such that*

$$\int_0^\tau\int_{\mathbb{T}^3}\mathrm{d}|\widetilde{\mu}_C|+\int_0^\tau\int_{\mathbb{T}^3}\mathrm{d}|\widetilde{\mu}_D|+\int_0^\tau\int_{\mathbb{T}^3}\mathrm{d}|\widetilde{\mu}_P|\le C\int_0^\tau\widetilde{\mathcal{D}}(\tau)\,\mathrm{d}t,$$

for a.e. $\tau\in(0,T)$, $\widetilde{\mathbb{P}}$-a.s.

Proof. Following deterministic argument we can conclude that $\widetilde{\mu}_E$ dominates defect measures $\widetilde{\mu}_C, \widetilde{\mu}_P$. To show the dominance of $\widetilde{\mu}_E$ over $\widetilde{\mu}_D$, observe that by virtue of hypotheses (2.1), (2.2), the function

$$[\varrho, \mathbf{m}] \mapsto \sum_{k\geq 1} \frac{|\Psi_k(\varrho, \mathbf{m})|^2}{\varrho} \text{ is continuous,}$$

and as such dominated by the total energy

$$\sum_{k\geq 1} \frac{|\Psi_k(\varrho, \mathbf{m})|^2}{\varrho} \leq c\Big(\varrho + \frac{|\mathbf{m}|^2}{\varrho}\Big) \leq c\Big(\frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho)\Big) + 1.$$

Hence, a simple application of the Lemma 2.3 finishes the proof of the lemma. $\square$

To conclude (2.11), we proceed as follows. First note that we can pass to the limit in $h \to 0$ in (5.7) to obtain the following energy inequality in the new probability space.

$$\begin{aligned} -\int_0^T \partial_t\psi \left[\int_{\mathbb{T}^3} \left\langle \widetilde{\mathcal{V}}^\omega_{t,x}; \frac{|\widetilde{\mathbf{m}}|^2}{\widetilde{\varrho}} + P(\widetilde{\varrho}) \right\rangle \mathrm{d}x + \widetilde{\mathcal{D}}(s)\right] \mathrm{d}s \leq \psi(0) \int_{\mathbb{T}^3} \left[\left\langle \widetilde{\mathcal{V}}^\omega_{0,x}; \frac{1}{2}\frac{|\widetilde{\mathbf{m}}|^2}{\widetilde{\varrho}} + P(\widetilde{\varrho}) \right\rangle\right] \mathrm{d}x \\ + \frac{1}{2}\int_0^T \psi \int_{\mathbb{T}^3} \mathrm{d}\tilde{\mu}_D + \int_0^T \psi\, \mathrm{d}\widetilde{N} + \frac{1}{2}\sum_{k=1}^\infty \int_0^T \psi \int_{\mathbb{T}^3} \left\langle \widetilde{\mathcal{V}}_{t,x}; \widetilde{\varrho}^{-1} |\Psi_k(\widetilde{\varrho}, \widetilde{\mathbf{m}})|^2 \right\rangle \mathrm{d}x\, \mathrm{d}s \end{aligned} \tag{5.15}$$

holds $\widetilde{\mathbb{P}}$-a.s., for all $\psi \in C_c^\infty([0,T)), \psi \geq 0$. Fix any s and t such that $0<s<t<T$. For any $r>0$ with $0<s-r<t+r<T$, let ψ_r be a Lipschitz function that is linear on $[s-r,s]$ or $[t,t+r]$ and satisfies

$$\psi_r(\tau) = \begin{cases} 0, & \text{if } \tau \in [0, s-r] \text{ or } \tau \in [t+r, T] \\ 1, & \text{if } \tau \in [s,t]. \end{cases}$$

Then, ψ_r is an admissible test function in (5.15), *via* a standard regularization argument. From (5.15) with ψ_r as test function, we have $\widetilde{\mathbb{P}}$-a.s, for all $t \in [0,T]$

$$\begin{aligned} &\frac{1}{r}\int_t^{t+r} \left(\int_{\mathbb{T}^3} \left\langle \widetilde{\mathcal{V}}^\omega_{\tau,x}; \frac{1}{2}\frac{|\widetilde{\mathbf{m}}|^2}{\widetilde{\varrho}} + P(\widetilde{\varrho}) \right\rangle \mathrm{d}x + \widetilde{\mathcal{D}}(\tau)\right) \mathrm{d}\tau \\ &\quad\leq \frac{1}{r}\int_{s-r}^{s} \left(\int_{\mathbb{T}^3} \left\langle \widetilde{\mathcal{V}}^\omega_{\tau,x}; \frac{1}{2}\frac{|\widetilde{\mathbf{m}}|^2}{\widetilde{\varrho}} + P(\widetilde{\varrho}) \right\rangle \mathrm{d}x + \widetilde{\mathcal{D}}(\tau)\right) \mathrm{d}\tau + \frac{1}{2}\int_{s-r}^{t+r} \psi_r(\tau) \int_{\mathbb{T}^3} \left\langle \widetilde{\mathcal{V}}^\omega_{\tau,x}; \widetilde{\varrho}^{-1} |\Psi_k(\widetilde{\varrho}, \widetilde{\mathbf{m}})|^2 \right\rangle \mathrm{d}x \\ &\quad\quad + \frac{1}{2}\int_{s-r}^{t+r}\int_{\mathbb{T}^3} \psi_r(\tau)\, \mathrm{d}\widetilde{\mu}_D(x,\tau) + \int_{s-r}^{t+r} \psi_r(\tau)\, \mathrm{d}\widetilde{N}(\tau). \end{aligned} \tag{5.16}$$

Now letting limit as $r \to 0^+$ in (5.17), then we have $\widetilde{\mathbb{P}}$-a.s, for all $t \in [0,T]$

$$\begin{aligned} &\liminf_{r\to 0^+} \frac{1}{r}\int_t^{t+r} \left(\int_{\mathbb{T}^3} \left\langle \widetilde{\mathcal{V}}^\omega_{\tau,x}; \frac{1}{2}\frac{|\widetilde{\mathbf{m}}|^2}{\varrho} + P(\widetilde{\varrho}) \right\rangle \mathrm{d}x + \widetilde{\mathcal{D}}(\tau)\right) \mathrm{d}\tau \\ &\quad\leq \liminf_{r\to 0^+} \frac{1}{r}\int_{s-r}^{s} \left(\int_{\mathbb{T}^3} \left\langle \widetilde{\mathcal{V}}^\omega_{\tau,x}; \frac{1}{2}\frac{|\widetilde{\mathbf{m}}|^2}{\widetilde{\varrho}} + P(\widetilde{\varrho}) \right\rangle \mathrm{d}x + \widetilde{\mathcal{D}}(\tau)\right) \mathrm{d}\tau + \frac{1}{2}\int_s^t \int_{\mathbb{T}^3} \left\langle \mathcal{V}^\omega_{\tau,x}; \widetilde{\varrho}^{-1} |\Psi_k(\widetilde{\varrho}, \widetilde{\mathbf{m}})|^2 \right\rangle \mathrm{d}x\, \mathrm{d}\tau \\ &\quad\quad + \frac{1}{2}\int_s^t \int_{\mathbb{T}^3} \mathrm{d}\widetilde{\mu}_D(x,\tau) + \int_s^t \mathrm{d}\widetilde{N}(\tau). \end{aligned}$$

Thus we conclude that (2.11) holds. If $s = 0$, then we need a different test function to conclude the result. In this case we take

$$\psi_r(\tau) = \begin{cases} 1, & \text{if } \tau \in [0, t] \\ \text{linear}, & \text{if } \tau \in [t, t+r] \\ 0, & \text{otherwise.} \end{cases}$$

Then, ψ_r is an admissible test function in (5.15), *via* a standard regularization argument. From (5.15) with ψ_r as test function, we have $\widetilde{\mathbb{P}}$-a.s, for all $t \in [0, T]$

$$\begin{aligned} &\frac{1}{r}\int_t^{t+r}\left(\int_{\mathbb{T}^3}\left\langle \tilde{\mathcal{V}}^\omega_{\tau,x}; \frac{1}{2}\frac{|\widetilde{\mathbf{m}}|^2}{\widetilde{\varrho}} + P(\widetilde{\varrho})\right\rangle \mathrm{d}x + \widetilde{\mathcal{D}}(\tau)\right)\mathrm{d}\tau \\ &\quad \le \int_{\mathbb{T}^3}\left\langle \widetilde{\mathcal{V}}^\omega_{0,x}; \frac{1}{2}\frac{|\widetilde{\mathbf{m}}|^2}{\widetilde{\varrho}} + P(\widetilde{\varrho})\right\rangle \mathrm{d}x + \frac{1}{2}\int_0^{t+r}\psi_r(\tau)\int_{\mathbb{T}^3}\left\langle \widetilde{\mathcal{V}}^\omega_{\tau,x}; \widetilde{\varrho}^{-1}|\Psi_k(\widetilde{\varrho},\widetilde{\mathbf{m}})|^2\right\rangle \mathrm{d}x \\ &\qquad + \frac{1}{2}\int_0^{t+r}\int_{\mathbb{T}^3}\psi_r(\tau)\,\mathrm{d}\widetilde{\mu}_D(x,\tau) + \int_0^{t+r}\psi_r(\tau)\,\mathrm{d}\widetilde{N}(\tau). \end{aligned} \tag{5.17}$$

Now letting limit as $r \to 0^+$ in (5.17), then we have $\widetilde{\mathbb{P}}$-a.s, for all $t \in [0, T]$

$$\begin{aligned} &\liminf_{r\to 0^+}\frac{1}{r}\int_t^{t+r}\left(\int_{\mathbb{T}^3}\left\langle \widetilde{\mathcal{V}}^\omega_{\tau,x}; \frac{1}{2}\frac{|\widetilde{\mathbf{m}}|^2}{\varrho} + P(\widetilde{\varrho})\right\rangle \mathrm{d}x + \widetilde{\mathcal{D}}(\tau)\right)\mathrm{d}\tau \le \int_{\mathbb{T}^3}\left\langle \widetilde{\mathcal{V}}^\omega_{0,x}; \frac{1}{2}\frac{|\widetilde{\mathbf{m}}|^2}{\widetilde{\varrho}} + P(\widetilde{\varrho})\right\rangle \mathrm{d}x \\ &\quad + \frac{1}{2}\int_0^t\int_{\mathbb{T}^3}\left\langle \mathcal{V}^\omega_{\tau,x}; \widetilde{\varrho}^{-1}|\Psi_k(\widetilde{\varrho},\widetilde{\mathbf{m}})|^2\right\rangle \mathrm{d}x\,\mathrm{d}\tau + \frac{1}{2}\int_0^t\int_{\mathbb{T}^3}\mathrm{d}\widetilde{\mu}_D(x,\tau) + \int_0^t \mathrm{d}\widetilde{N}(\tau). \end{aligned}$$

Thus we conclude that (2.11) holds.

6. Weak–strong uniqueness principle

In this section, we establish pathwise weak (measure-valued)–strong uniqueness principle for dissipative measure-valued martingale solutions. In what follows, we first introduce the relative energy functional which plays a pivotal role in the proof of weak (measure-valued)–strong uniqueness principle. In the context of compressible Euler equations, the relative energy functional reads

$$\begin{aligned} \mathfrak{E}'_{\mathrm{mv}}\big(\varrho,\mathbf{m}\,\big|\,\kappa,\mathbf{Q}\big)(t) := &\int_{\mathbb{T}^3}\left\langle \mathcal{V}^\omega_{t,x}; \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho)\right\rangle \mathrm{d}x - \int_{\mathbb{T}^3}\left\langle \mathcal{V}^\omega_{t,x}; \mathbf{m}\right\rangle \cdot \mathbf{Q}(\mathbf{x})\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{T}^3}\left\langle \mathcal{V}^\omega_{t,x}; \varrho\right\rangle |\mathbf{Q}(\mathbf{x})|^2\,\mathrm{d}x \\ &- \int_{\mathbb{T}^3}\left\langle \mathcal{V}^\omega_{t,x}; \varrho\right\rangle P'(\kappa(x))\,\mathrm{d}x - \int_{\mathbb{T}^3}[P'(\kappa(x))\kappa(x) - P(\kappa(x))]\,\mathrm{d}x + \mathcal{D}(t), \end{aligned} \tag{6.1}$$

where $\kappa \in C([0,T], W^{1,q}(\mathbb{T}^3))$, $\mathbf{Q} \in C([0,T]; W^{1,q}(\mathbb{T}^3))$ $\mathbb{P} -$ a.s. In view of the energy inequality (2.13), it is clear that the above energy functional (6.1) is defined for all $t \in [0,T] \setminus \mathcal{A}$, where the set $\mathcal{A}$, may depends on ω, has Lebesgue measure zero. We also define relative energy function for all $t \in \mathcal{A}$ as follows

$$\begin{aligned} \mathfrak{E}''_{\mathrm{mv}}\big(\varrho,\mathbf{m}\,\big|\,\kappa,\mathbf{Q}\big)(t) := &\liminf_{r\to 0}\frac{1}{r}\int_t^{t+r}\left[\int_{\mathbb{T}^3}\left\langle \mathcal{V}^\omega_{s,x}; \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + P(\varrho)\right\rangle \mathrm{d}x + \mathcal{D}(s)\right]\mathrm{d}s - \int_{\mathbb{T}^3}\left\langle \mathcal{V}^\omega_{t,x}; \mathbf{m}\right\rangle \cdot \mathbf{Q}(\mathbf{x})\,\mathrm{d}x \\ &+ \frac{1}{2}\int_{\mathbb{T}^3}\left\langle \mathcal{V}^\omega_{t,x}; \varrho\right\rangle |\mathbf{Q}(\mathbf{x})|^2\mathrm{d}x - \int_{\mathbb{T}^3}\left\langle \mathcal{V}^\omega_{t,x}; \varrho\right\rangle P'(\kappa(x))\,\mathrm{d}x \\ &- \int_{\mathbb{T}^3}[P'(\kappa(x))\kappa(x) - P(\kappa)]\,\mathrm{d}x. \end{aligned} \tag{6.2}$$

Existence of lim inf in the above identity can be justified by the help of energy inequality (2.11). Using relative energy functionals (6.1) and (6.2), we define relative energy functional for all time $t \in [0,T]$ as follows

$$\mathfrak{E}_{\mathrm{mv}}\big(\varrho, \mathbf{m} \,\big|\, \kappa, \mathbf{Q}\big)(t) := \begin{cases} \mathfrak{E}'_{\mathrm{mv}}\big(\varrho, \mathbf{m} \,\big|\, \kappa, \mathbf{Q}\big)(t), & \text{if } t \in [0,T] \setminus \mathcal{A} \\ \mathfrak{E}''_{\mathrm{mv}}\big(\varrho, \mathbf{m} \,\big|\, \kappa, \mathbf{Q}\big)(t), & \text{if } t \in \mathcal{A}. \end{cases} \tag{6.3}$$

With the help of the above definition of relative energy functional, we are now in a position to derive the following relative energy inequality.

Proposition 6.1 (Relative energy inequality)**.** *Let $[(\Omega, \mathbb{F}, (\mathbb{F}_t)_{t\geq 0}, \mathbb{P}); \mathcal{V}^{\omega}_{t,x}, W]$ be a dissipative measure-valued martingale solution to the system (1.1) and (1.2). Suppose $(\kappa, \mathbf{Q})$ be a pair of stochastic processes which are adapted to the filtration $(\mathbb{F}_t)_{t\geq 0}$ and which satisfies*

$$\begin{aligned} \mathrm{d}\kappa &= \kappa_1 \mathrm{d}t + \kappa_2 \,\mathrm{d}W, \\ \mathrm{d}\mathbf{Q} &= \mathbf{Q}_1 \mathrm{d}t + \mathbf{Q}_2 \,\mathrm{d}W \end{aligned}$$

with

$$\begin{aligned} &\kappa \in C\big([0,T], W^{1,q}(\mathbb{T}^3)\big), \;\; \mathbf{Q} \in C\big([0,T]; W^{1,q}(\mathbb{T}^3)\big) \;\; \mathbb{P}\text{-a.s.} \\ &\mathbb{E}\left[\sup_{t\in[0,T]} \|\kappa\|^2_{W^{1,q}(\mathbb{T}^3)}\right]^q + \mathbb{E}\left[\sup_{t\in[0,T]} \|\mathbf{Q}\|^2_{W^{1,q}}\right]^q \leq C \text{ for all } 2 \leq q < \infty, \\ &0 < r_1 \leq \kappa(t,x) \leq r_2 \;\; \mathbb{P}\text{-a.s.} \end{aligned}$$

Moreover, $\kappa_i, \mathbf{Q}_i$ for $i = 1, 2$, satisfy

$$\begin{aligned} &\kappa_1, \mathbf{Q}_1 \in L^q\big(\Omega; L^q\big(0,T; W^{1,q}(\mathbb{T}^3)\big)\big) \;\; \kappa_2, \mathbf{Q}_2 \in L^2\big(\Omega; L^2\big((0,T); L_2\big(\mathfrak{U}; L^2(\mathbb{T}^3)\big)\big)\big), \\ &\left(\sum_{k\geq 1} |\kappa_2(e_k)|^q\right)^{1/q}, \left(\sum_{k\geq 1} |\mathbf{Q}_2(e_k)|^q\right)^{1/q} \in L^q\big(\Omega; L^q\big(0,T; L^q(\mathbb{T}^3)\big)\big). \end{aligned}$$

Then the following relative energy inequality holds $\mathbb{P}$-a.s., for all $t \in [0,T]$

$$\mathfrak{E}_{\mathrm{mv}}\big(\varrho, \mathbf{m} \,\big|\, \kappa, \mathbf{Q}\big)(t) \leq \mathfrak{E}_{\mathrm{mv}}\big(\varrho, \mathbf{m} \,\big|\, \kappa, \mathbf{Q}\big)(0) + \mathcal{M}_{\mathrm{RE}}(t) + \int_0^t \mathfrak{R}_{\mathrm{mv}}\big(\varrho, \mathbf{m} \,\big|\, \kappa, \mathbf{Q}\big)(\tau)\,\mathrm{d}\tau \tag{6.4}$$

where

$$\begin{aligned} \mathfrak{R}_{\mathrm{mv}}\big(\varrho, \mathbf{m} \,\big|\, \kappa, \mathbf{Q}\big)(t) = &\int_{\mathbb{T}^3} \big\langle \mathcal{V}^{\omega}_{t,x}; \varrho\mathbf{Q} - \mathbf{m}\big\rangle \cdot [\mathbf{Q}_1 + \nabla\mathbf{Q}\cdot\mathbf{Q}]\,\mathrm{d}x + \int_{\mathbb{T}^3} \left\langle \mathcal{V}^{\omega}_{t,x}; \frac{(\mathbf{m} - \varrho\mathbf{Q}) \otimes (\varrho\mathbf{Q} - \mathbf{m})}{\varrho} \right\rangle : \nabla\mathbf{Q}\,\mathrm{d}x \\ &+ \int_{\mathbb{T}^3} \big[\big(\kappa - \langle \mathcal{V}^{\omega}_{t,x}; \varrho\rangle P''(\kappa)\kappa_1 + \nabla_x P'(\kappa) \cdot \big(s\mathbf{Q} - \langle \mathcal{V}^{\omega}_{t,x}); \mathbf{m}\rangle\big)\big]\,\mathrm{d}x \\ &+ \int_{\mathbb{T}^3} \big[p(\kappa) - \langle \mathcal{V}^{\omega}_{t,x}; p(\kappa)\rangle\big] \mathrm{div}(\mathbf{Q})\,\mathrm{d}x + \frac{1}{2}\sum_{k\geq 1} \int_{\mathbb{T}^3} \left\langle \mathcal{V}^{\omega}_{t,x}; \varrho \left| \frac{\Psi_k(\varrho, \mathbf{m})}{\varrho} - \mathbf{Q}_2(e_k) \right|^2 \right\rangle \mathrm{d}x \\ &+ \frac{1}{2}\sum_{k\geq 1} \int_{\mathbb{T}^3} \langle \mathcal{V}^{\omega}_{t,x}; \varrho\rangle P''(\kappa)|s_2(e_k)|^2\,\mathrm{d}x + \frac{1}{2}\sum_{k\geq 1} \int_{\mathbb{T}^3} p''(\kappa)|\kappa_2(e_k)|^2\,\mathrm{d}x \\ &- \int_{\mathbb{T}^3} \nabla\mathbf{Q} : \mathrm{d}\mu_m + \frac{1}{2}\int_{\mathbb{T}^3} \mathrm{d}\mu_e. \end{aligned} \tag{6.5}$$

Here $\mathcal{M}_{\mathrm{RE}}$ is a real valued square integrable martingale. Moreover, P and p satisfy the relation (2.6).

Proof. The proof of this proposition is a consequence of generalized Itô formula, which is similar to the Lemma 4.1 in [30]. However, strictly speaking, the proof given in [30] is based on a slightly different notion of dissipative measure-valued martingale solutions. Therefore, for the sake of completeness, we briefly mention the proof. Note that the given conditions on the stochastic processes allows us to apply Itô formula to compute $\int_{\mathbb{T}^3} \langle \mathcal{V}^\omega_{t,x}; \mathbf{m}\rangle \cdot \mathbf{Q}\,\mathrm{d}x$. The result is

$$\begin{aligned}\mathrm{d}\left(\int_{\mathbb{T}^3} \langle \mathcal{V}^\omega_{t,x}; \mathbf{m}\rangle \cdot \mathbf{Q}\mathrm{d}x\right) &= \int_{\mathbb{T}^3}\left(\langle \mathcal{V}^\omega_{t,x}; \mathbf{m}\rangle \cdot \mathbf{Q}_1 + \left\langle \mathcal{V}^\omega_{t,x}; \frac{\mathbf{m}\otimes\mathbf{m}}{\varrho}\right\rangle : \nabla\mathbf{Q} + \langle \mathcal{V}^\omega_{t,x}; p(\varrho)\rangle \mathrm{div}\mathbf{Q}\right)\mathrm{d}x\,\mathrm{d}t\\ &\quad + \sum_{k\geq 1}\int_{\mathbb{T}^3} \mathbf{Q}_2(e_k)\cdot\langle \mathcal{V}^\omega_{t,x}; \Psi_k(\varrho,\mathbf{m})\rangle\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{T}^3}\nabla\mathbf{Q} : \mathrm{d}\mu_m\,\mathrm{d}t + \mathrm{d}M_1(t)\end{aligned} \tag{6.6}$$

where

$$M_1(t) = \int_0^t\int_{\mathbb{T}^3} \mathbf{Q}\cdot\langle \mathcal{V}^\omega_{t,x}; \Psi(\varrho,\mathbf{m})\rangle\,\mathrm{d}W + \int_0^t\int_{\mathbb{T}^3}\langle \mathcal{V}^\omega_{t,x}; \mathbf{m}\rangle\cdot\mathbf{Q}_2\,\mathrm{d}x\,\mathrm{d}W.$$

Similarly, we get

$$\begin{aligned}\mathrm{d}\left(\int_{\mathbb{T}^3} \frac{1}{2}\langle \mathcal{V}^\omega_{t,x}; \varrho\rangle|\mathbf{Q}|^2\mathrm{d}x\right) &= \int_{\mathbb{T}^3}\langle \mathcal{V}^\omega_{t,x}; \mathbf{m}\rangle\cdot\nabla\mathbf{Q}\cdot\mathbf{Q}\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{T}^3}\langle \mathcal{V}^\omega_{t,x}; \varrho\rangle\mathbf{Q}\cdot\mathbf{Q_1}\,\mathrm{d}x\,\mathrm{d}t\\ &\quad + \frac{1}{2}\sum_{k\geq 1}\int_{\mathbb{T}^3}\langle \mathcal{V}^\omega_{t,x}; \varrho\rangle|\mathbf{Q}_2(e_k)|^2\,\mathrm{d}x\,\mathrm{d}t + \mathrm{d}M_2(t),\end{aligned} \tag{6.7}$$

where

$$M_2(t) = \int_0^t\int_{\mathbb{T}^3}\langle \mathcal{V}^\omega_{t,x}; \varrho\rangle\mathbf{Q}\cdot\mathbf{Q}_2\,\mathrm{d}x\,\mathrm{d}W,$$

and

$$\begin{aligned}\mathrm{d}\left(\int_{\mathbb{T}^3}(P'(\kappa)s - P(\kappa))\,\mathrm{d}x\right) = \mathrm{d}\left(\int_{\mathbb{T}^3} p(\kappa)\,\mathrm{d}x\right) &= \int_{\mathbb{T}^3} p'(\kappa)\kappa_1\,\mathrm{d}x\,\mathrm{d}t\\ &\quad + \frac{1}{2}\sum_{k\geq 1}\int_{\mathbb{T}^3} p''(\kappa)|\kappa_2(e_k)|^2\,\mathrm{d}x\,\mathrm{d}t + \mathrm{d}M_3(t),\end{aligned} \tag{6.8}$$

where

$$M_3(t) = \int_0^t\int_{\mathbb{T}^3} p'(\kappa)\kappa_2\,\mathrm{d}x\,\mathrm{d}W,$$

and

$$\begin{aligned}\mathrm{d}\left(\int_{\mathbb{T}^3}\langle \mathcal{V}^\omega_{t,x}; \varrho\rangle P'(\kappa)\mathrm{d}x\right) &= \int_{\mathbb{T}^3}\langle \mathcal{V}^\omega_{t,x}; \mathbf{m}\rangle\cdot\nabla_x P'(\kappa)\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{T}^3}\langle \mathcal{V}^\omega_{t,x}; \varrho\rangle P''(\kappa)\kappa_1\,\mathrm{d}x\,\mathrm{d}t\\ &\quad + \frac{1}{2}\sum_{k\geq 1}\int_{\mathbb{T}^3}\langle \mathcal{V}^\omega_{t,x}; \varrho\rangle P''(\kappa)|\kappa_2(e_k)|^2\,\mathrm{d}x\,\mathrm{d}t + \mathrm{d}M_4(t)\end{aligned} \tag{6.9}$$

where

$$M_4(t) = \int_0^t\int_{\mathbb{T}^3}\langle \mathcal{V}^\omega_{t,x}; \varrho\rangle P''(\kappa)\kappa_2\,\mathrm{d}x\,\mathrm{d}W.$$

Now we can combine (6.6)–(6.9) with (2.11), define the square integrable real-valued martingale $\mathcal{M}_{\mathrm{RE}}(t) := M_1(t) + M_2(t) + M_3(t) + M_4(t) + M_E(t)$, and summing up the resulting expressions and adding the sum with (2.11) to obtain (6.4). ☐

With the help of the Proposition 6.1, we now briefly describe the proof of the weak (measure-valued)–strong uniqueness principle. For detials of the proof, we refer to Chapter 6 of [9] and [30].

Theorem 6.2 (Weak–strong uniqueness). *Let* $\big[\big(\Omega, \mathbb{F}, (\mathbb{F}_t)_{t\geq 0}, \mathbb{P}\big); \mathcal{V}^{\omega}_{t,x}, W\big]$ *be a dissipative measure-valued martingale solution to the system (1.1) and (1.2). On the same stochastic basis* $\big(\Omega, \mathbb{F}, (\mathbb{F}_t)_{t\geq 0}, \mathbb{P}\big)$*, let us consider the unique maximal strong pathwise solution to the Euler system (1.1) and (1.2) given by* $(\bar\varrho, \bar{\mathbf{u}}, (\mathfrak{t}_R)_{R\in\mathbb{N}}, \mathfrak{t})$ *driven by the same cylindrical Wiener process* W *with the initial data* $\bar\varrho(0), \bar\varrho\bar{\mathbf{u}}(0)$ *satisfies*

$$\mathcal{V}^{\omega}_{0,x} = \delta_{\bar\varrho(0,x,\omega),(\bar\varrho\bar{\mathbf{u}})(0,x,\omega)}, \ \mathbb{P}\text{-a.s.}, \quad \text{for a.e. } x \in \mathbb{T}^3.$$

Then for a.e. $t \in [0,T]$, $\mathcal{D}(t\wedge \mathfrak{t}_R) = 0$, $\mathbb{P}$*-a.s., and* $\mathbb{P}$*-a.s.,*

$$\mathcal{V}^{\omega}_{t\wedge\mathfrak{t}_R,x} = \delta_{\bar\varrho(t\wedge\mathfrak{t}_R,x,\omega),(\bar\varrho\bar{\mathbf{u}})(t\wedge\mathfrak{t}_R,x,\omega)}, \quad \text{for a.e. } (t,x) \in (0,T)\times\mathbb{T}^3. \tag{6.10}$$

Proof. Since $(\bar\varrho(\cdot\wedge\mathfrak{t}_R), \bar{\mathbf{u}})$ is the strong pathwise solution to stystem (1.1) and (1.2), so we can replace $(\kappa, \mathbf{Q})$ by $(\bar\varrho(\cdot\wedge\mathfrak{t}_R), \bar{\mathbf{u}})$ in the relative energy inequality (6.4). Then we have $\mathbb{P}$-a.s., for all $t\in[0,T]$,

$$\mathfrak{E}_{\mathrm{mv}}\big(\varrho, \mathbf{m} \,\big|\, \bar\varrho, \bar{\mathbf{u}}\big)(t\wedge\mathfrak{t}_R) \leq \mathfrak{E}_{\mathrm{mv}}\big(\varrho, \mathbf{m} \,\big|\, \bar\varrho, \bar{\mathbf{u}}\big)(0) + \mathcal{M}_{\mathrm{RE}}(t\wedge\mathfrak{t}_R) + \int_0^{t\wedge\mathfrak{t}_R} \mathfrak{R}_{\mathrm{mv}}\big(\varrho, \mathbf{m} \,\big|\, \bar\varrho, \bar{\mathbf{u}}\big)(s)\,\mathrm{d}s, \tag{6.11}$$

where $\mathfrak{R}_{\mathrm{mv}}(\varrho, \mathbf{m} \,|\, \bar\varrho, \bar{\mathbf{u}})$ is given by (6.5) after replacing $(\kappa, \mathbf{Q})$ by $(\bar\varrho(\cdot\wedge\mathfrak{t}_R), \bar{\mathbf{u}})$. Following [30] and Chapter 6 of [9], one can verify that

$$\int_0^{t\wedge\mathfrak{t}_R} \mathfrak{R}_{\mathrm{mv}}\big(\varrho, \mathbf{m} \,\big|\, \bar\varrho, \bar{\mathbf{u}}\big)(s)\,\mathrm{d}s \leq c(R) \int_0^{t\wedge\mathfrak{t}_R} \mathfrak{E}_{\mathrm{mv}}\big(\varrho, \mathbf{m} \,\big|\, \bar\varrho, \bar{\mathbf{u}}\big)(s)\,\mathrm{d}s. \tag{6.12}$$

In light of (6.11) and (6.12), a straightforward consequence of Gronwall's lemma yields, for all $t\in[0,T]$

$$\mathbb{E}\big[\mathfrak{E}_{\mathrm{mv}}\big(\varrho, \mathbf{m} \,\big|\, \bar\varrho, \bar{\mathbf{u}}\big)(t\wedge\mathfrak{t}_R)\big] \leq c(R)\,\mathbb{E}\big[\mathfrak{E}_{\mathrm{mv}}\big(\varrho, \mathbf{m} \,\big|\, \bar\varrho, \bar{\mathbf{u}}\big)(0)\big].$$

Since initial data are same for both solutions, right hand side of above inequality equals to zero. Therefore it implies that for all $t\in[0,T]$

$$\mathbb{E}\big[\mathfrak{E}_{\mathrm{mv}}\big(\varrho, \mathbf{m} \,\big|\, \bar\varrho, \bar{\mathbf{u}}\big)(t\wedge\mathfrak{t}_R)\big] = 0.$$

This also implies that

$$\lim_{r\to 0^+} \frac{1}{r}\int_t^{t+r} \mathbb{E}\big[\mathfrak{E}_{\mathrm{mv}}\big(\varrho, \mathbf{m} \,\big|\, \bar\varrho, \bar{\mathbf{u}}\big)(s\wedge\mathfrak{t}_R)\big]\mathrm{d}s = 0.$$

In view of *a priori* estimates (3.5)–(3.7) (which are preserved in the limit), energy inequality (2.13), a usual Lebesgue point argument, and application of Fubini's theorem reveals that for a.e. $t\in[0,T]$,

$$\mathbb{E}\big[\mathfrak{E}'_{\mathrm{mv}}\big(\varrho, \mathbf{m} \,\big|\, \bar\varrho, \bar{\mathbf{u}}\big)(t\wedge\mathfrak{t}_R)\big] = 0.$$

Since the defect measure $\mathcal{D}\geq 0$, we have for a.e. $t\in[0,T]$, $\mathcal{D}(t\wedge\mathfrak{t}_R) = 0$, $\mathbb{P}$-a.s. Moreover, $\mathbb{P}$-a.s.

$$\mathcal{V}^{\omega}_{t\wedge\mathfrak{t}_R,x} = \delta_{\bar\varrho(t\wedge\mathfrak{t}_R,x),(\bar\varrho\bar{\mathbf{u}})(t\wedge\mathfrak{t}_R,x)}, \quad \text{for a.e. } (t,x)\in(0,T)\times\mathbb{T}^3.$$

This finishes the proof of the theorem. $\square$

7. Proof of Theorem 2.16: Convergence to a dissipative martingale solution

In view of the Proposition 5.6 and convergence results given by (5.12) and (5.13), we conclude that there is subsequence $\{(\widetilde{\varrho}_{h_k}(t), \widetilde{\mathbf{m}}_{h_k}(t))\}_{h_k>0}$ such that $\widetilde{\mathbb{P}}$-a.s,

$$\widetilde{\varrho}_{h_k} \to \left\langle \widetilde{\mathcal{V}}^{\omega}_{t,x}; \widetilde{\varrho} \right\rangle \text{ in } C_w\big([0,T], L^{\gamma}(\mathbb{T}^3)\big),$$
$$\widetilde{\mathbf{m}}_{h_k} \to \left\langle \widetilde{\mathcal{V}}^{\omega}_{t,x}; \widetilde{\mathbf{m}} \right\rangle \text{ in } C_w\Big([0,T], L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^3)\Big).$$

We can also conclude that there exists a full probability subset $\overline{\Omega} \subset \widetilde{\Omega}$ such that for all $w \in \overline{\Omega}$,

$$\widetilde{\varrho}_{h_k}(\omega) \rightharpoonup \left\langle \widetilde{\mathcal{V}}^{\omega}_{t,x}; \widetilde{\varrho} \right\rangle \text{ in } L^{\gamma}\big([0,T] \times \mathbb{T}^3\big),$$
$$\widetilde{\mathbf{m}}_{h_k}(\omega) \rightharpoonup \left\langle \widetilde{\mathcal{V}}^{\omega}_{t,x}; \widetilde{\mathbf{m}} \right\rangle \text{ in } L^{\frac{2\gamma}{\gamma+1}}\big([0,T] \times \mathbb{T}^3\big).$$

For the pointwise convergence of numerical approximations, we can make use of Proposition 2.4. Indeed, we obtain that for any fixed $\omega \in \overline{\Omega}$, there exists a subsequence $\{(\widetilde{\varrho}_{h_k}(\omega), \widetilde{\mathbf{m}}_{h_k}(\omega))\}_{h_k>0}$ (here indices h_k depends on ω) such that

$$\frac{1}{N}\sum_{k=1}^{N} \widetilde{\varrho}_{h_k}(\omega) \to \langle \mathcal{V}^{\omega}_{t,x}; \widetilde{\varrho} \rangle, \text{ as } N \to \infty \text{ a.e. in } (0,T) \times \mathbb{T}^3,$$
$$\frac{1}{N}\sum_{k=1}^{N} \widetilde{\mathbf{m}}_{h_k}(\omega) \to \left\langle \widetilde{\mathcal{V}}^{\omega}_{t,x}; \widetilde{\mathbf{m}} \right\rangle, \text{ as } N \to \infty \text{ a.e. in } (0,T) \times \mathbb{T}^3.$$

8. Proof of Theorem 2.17: Convergence to a regular solution

We have proven that the numerical solutions $\{\widetilde{\mathbf{U}}_h\}_{h>0}$ to (2.17) for the stochastic Euler system converges to the dissipative measure-valued martingale solution, in the sense of Definition 2.10. Employing the corresponding weak (measure-valued)–strong uniqueness results (cf. Thm. 6.2), we can show the strong convergence of numerical approximations to a strong solution of the system on its lifespan.

First note that, Proposition 5.6 and Theorem 6.2 gives the required weak-$*$ convergence. Indeed, from Proposition 5.6, we have $\widetilde{\mathbb{P}}$-a.s.,

$$\widetilde{\varrho}_h(\cdot \wedge \mathfrak{t}_R) \to \left\langle \widetilde{\mathcal{V}}^{\omega}_{t,x}; \widetilde{\varrho} \right\rangle(\cdot \wedge \mathfrak{t}_R) \text{ in } C_w\big([0,T], L^{\gamma}(\mathbb{T}^3)\big),$$
$$\widetilde{\mathbf{m}}_h(\cdot \wedge \mathfrak{t}_R) \to \left\langle \widetilde{\mathcal{V}}^{\omega}_{t,x}; \widetilde{\mathbf{m}} \right\rangle(\cdot \wedge \mathfrak{t}_R) \text{ in } C_w\Big([0,T], L^{\frac{2\gamma}{\gamma+1}}(\mathbb{T}^3)\Big).$$

Combination of above convergence and Theorem 6.2 gives the required weak-* convergence. For the proof of strong convergence of density and momentum in $L^1(\mathbb{T}^3)$, we make use of Proposition 5.6, Theorem 6.2, energy bounds (3.5)–(3.7), and the fact that limit Young measure of any subsequence $(\delta_{\widetilde{\varrho}_{h_k}(\cdot\wedge\mathfrak{t}_R), \widetilde{\mathbf{m}}_{h_k}(\cdot\wedge\mathfrak{t}_R)})_{k\ge1}$ is $\delta_{\bar{\varrho}(\cdot\wedge\mathfrak{t}_R), \bar{\mathbf{m}}(\cdot\wedge\mathfrak{t}_R)}$. Therefore, we have $\widetilde{\mathbb{P}}$-a.s., sequence of young measure converges to dirac Young measure, *i.e.*, $\widetilde{\mathbb{P}}$-a.s.

$$\delta_{\widetilde{\varrho}_h(\cdot\wedge\mathfrak{t}_R), \widetilde{\mathbf{m}}_h(\cdot\wedge\mathfrak{t}_R)} \to \delta_{\bar{\varrho}(\cdot\wedge\mathfrak{t}_R), \bar{\mathbf{m}}(\cdot\wedge\mathfrak{t}_R)}, \text{ in weak-}* \text{ in } L^{\infty}\big((0,T) \times \mathbb{T}^3; \mathcal{P}(\mathbb{R}^4)\big).$$

By theory of Young measure ([1], Prop. 4.16), it implies that, $\widetilde{\mathbb{P}}$-a.s. $\widetilde{\varrho}_h(\cdot \wedge \mathfrak{t}_R)$, $\widetilde{\mathbf{m}}_h(\cdot \wedge \mathfrak{t}_R)$ converges to $\bar{\varrho}(\cdot \wedge \mathfrak{t}_R), \bar{\mathbf{m}}(\cdot \wedge \mathfrak{t}_R)$ in measure respectively. Note that, $\widetilde{\mathbb{P}}$-a.s. sequence $(\widetilde{\varrho}_h(\cdot \wedge \mathfrak{t}_R), \widetilde{\mathbf{m}}_h(\cdot \wedge \mathfrak{t}_R))$ is uniformly integrable and converges in measure, therefore Vitali's convergence theorem implies that $\widetilde{\mathbb{P}}$-a.s,

$$\widetilde{\varrho}_h(\cdot \wedge \mathfrak{t}_R) \to \bar{\varrho}(\cdot \wedge \mathfrak{t}_R), \quad \text{strongly in } L^1\big((0,T) \times \mathbb{T}^3\big),$$
$$\widetilde{\mathbf{m}}_h(\cdot \wedge \mathfrak{t}_R) \to \bar{\mathbf{m}}(\cdot \wedge \mathfrak{t}_R), \quad \text{strongly in } L^1\big((0,T) \times \mathbb{T}^3; \mathbb{R}^3\big).$$

This finishes the proof of the theorem.

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