ESAIM: M2AN 57 (2023) 3637–3668
https://doi.org/10.1051/m2an/2023091

ESAIM: Mathematical Modelling and Numerical Analysis
www.esaim-m2an.org

ERROR ANALYSIS OF FOURIER–LEGENDRE AND FOURIER–HERMITE SPECTRAL-GALERKIN METHODS FOR THE VLASOV–POISSON SYSTEM

Xiaolong Zhang[1], Li-Lian Wang[2,*] and Hongli Jia[3]

Abstract. We conduct a rigorous error analysis of the spectral-Galerkin methods for the 1D-1V Vlasov–Poisson system with the velocity variable in both finite and infinite domains. The estimates significantly improve the very limited existing results. We also provide numerical results to demonstrate the effectiveness of the analysed methods.

Mathematics Subject Classification. 76M22, 65N35, 65N12, 35Q83.

Received October 23, 2022. Accepted November 6, 2023.

1. Introduction

The Vlasov–Poisson (VP) system is a model that describes the evolution of charged particles under a self-consistent electric field. It involves two unknown fields of a spatial variable $x \in \mathbb{R}^d$ and/or a velocity variable $v \in \mathbb{R}^d$ $(d = 1, 2, 3)$, together with the temporal variable, which amounts to a nonlinear system of partial differential equations (PDEs) in $1 + 2d$ dimensions. The development of effective numerical methods for this high dimensional model is still an active research topic that attracts much recent attention.

In general, there are three classes of numerical methods for solving the Vlasov-type equations. The first category is the Particle-in-Cell (PIC) methods. The PIC technique is usually built upon the discretisation in the phase space *via* macro-particles that evolve by Newton's equations in the self-consistent electromagnetic field (see *e.g.*, [5, 17]). The essential idea is to approximate the plasma by a finite number of pseudo-particles and computes their trajectories. Secondly, instead of following the particle trajectories, the Eulerian–Vlasov method directly solving the Vlasov equation on a grid of phase space is an efficient alternative to the PIC method in some cases [8, 14]. As with the Eulerian approaches, the semi-Lagrangian methods based on the phase-space approximation are developed in *e.g.*, [11, 19, 27]. A third class of methods is the transformation (spectral) methods, which is based on the expansions of distribution function and electric field function in global spectral basis functions. Remarkably, spectral methods enjoy the spectral accuracy with relative smaller number of degree of freedoms when the underlying solutions are smooth. In particular, the well-developed

Keywords and phrases. The Vlasov–Poisson system, Fourier–Legendre spectral method, Fourier–Hermite spectral method, error analysis.

[1] MOE-LCSM, CHP-LCOCS, School of Mathematics and Statistics, Hunan Normal University, Changsha 410081, P.R. China.

[2] Division of Mathematical Sciences, School of Physical and Mathematical Sciences, Nanyang Technological University, Singapore 637371, Singapore.

[3] Department of Statistics, School of Science, Donghua University, Shanghai 200063, P.R. China.

*Corresponding author: `lilian@ntu.edu.sg`

spectral methods in unbounded domains (*cf.* [28], Chap. 7) turn out attractive for dealing with the velocity variable. Although the Hermite and Fourier spectral methods were developed for solving such models in [2] about half a century ago, they become popular in solutions of plasma physics problems only in the last decade, see *e.g.*, [4, 6, 10, 12, 13, 15, 18, 21–23, 25, 26] and the references therein.

While there have been much development and applications of Fourier and Hermite spectral methods, the works on the error analysis are very limited. Pulvirenti *et al.* [26] was perhaps the first to analyze the Fourier–Hermite method for the VP system where some convergence results were obtained through the analysis of the expansion coefficients. Manzini *et al.* [24] analysed the Fourier–Legendre spectral method (under the assumption that the velocity variable v is in a finite domain) and Fourier–Hermite spectral methods for the 1D-1V Vlasov–Poisson system (*i.e.*, $d = 1$). It is noteworthy that the L^2-orthogonal projections or direct expansions of the L^2-orthogonal basis were used in the error analysis resulted in the error estimates in L^2-norm of the order $(M + N^2)(M^{1-r+\epsilon} + N^{3/2-s+2\epsilon})$ (resp. $(M + \sqrt{N})(M^{1-r+\epsilon} + N^{(1-s+\epsilon)/2})$) under some typical assumptions of the Sobolev regularity of the underlying solution, where M and N denote the highest degree of Fourier modes and Legendre polynomials (resp., Hermite functions), respectively. On the other hand, the constant in the error bound of the Fourier–Hermite Galerkin method depends on exponential of the interval length of the velocity variable v, where the infinite interval should be understood as a "finite" interval $(v_{\min}, v_{\max})$ when the solution and its partial derivatives decay very fast in v.

In this paper, we conduct a rigorous error analysis of the Fourier–Legendre and Fourier–Hermite Galerkin methods for the VP system. In the analysis, we introduce suitable orthogonal projections, and derive some new approximation results and inverse inequalities. Through different arguments and more delicate analysis, we are able to obtain the L^2-convergence orders: $M^{1-r} + N^{1-s}$ and $M^{1-r} + N^{1-s/2}$ for Fourier–Legendre and Fourier–Hermite approximations, respectively, under weaker regularity conditions than those in [24]. Unfortunately, in the Hermite case, the constant in the estimate still depends on the exponential of $\sqrt[4]{N}$. It seems that it cannot be removed with the argument using the Grönwall-type inequality.

The paper is organized as follows. In Section 2, we introduce the problem of interest, and present some important properties of the model. We then introduce the semi-discrete Fourier–Legendre and Fourier–Hermite Galerkin schemes in a uniform formulation. In Section 3, we conduct the error analysis of the Fourier–Legendre spectral method in detail. We then analyse the Fourier–Hermite spectral method with a focus on the estimates that the imbedding properties do not hold on unbounded domains in Section 4. We present in Section 5 the full discrete scheme using the Crank–Nicolson finite difference in time, and provide some numerical results to show the effectiveness of the proposed method in preserving some intrinsic properties of this model. We conclude the paper with some final remarks.

2. Spectral-Galerkin method for the Vlasov–Poisson system

Let us introduce the following notation to be used throughout the paper.

- For any nonnegative integer r, $H^r(\Omega)$ and $H_0^r(\Omega)$ denote the usual Sobolev spaces on a bounded domain with a Lipschitz boundary and with the norm $\|\cdot\|_{H^r(\Omega)}$. In particular, the norm and inner product of $L^2(\Omega)$ are denoted by $\|\cdot\|_\Omega$ and $(\cdot,\cdot)_\Omega$.
- To account for periodic functions on $\Omega_x = (0, 2\pi)$, we define

$$H_{\mathfrak{p}}^r(\Omega_x) = \big\{v \in H^r(\Omega_x) : v^{(k)}(x) = v^{(k)}(x + 2\pi),\ 0 \le k \le r - 1\big\}.$$

2.1. Statement of the problem

To fix the idea, we consider 1D-1V Vlasov–Poisson system on $\Omega = \Omega_x \times \Omega_v$, where the spatial variable $x \in \Omega_x = (0, 2\pi)$ and the velocity variable $v \in \Omega_v = (a, b)$ or $v \in \Omega_v = \mathbb{R}$. The model for the electron distribution function $f(x,v,t)$ and the electric field $E(x,t)$ reads

$$\begin{cases} \partial_t f + v\partial_x f - E\partial_v f = 0, & (x,v)\in\Omega, \quad t\in(0,T], \\ \partial_x E = 1-\rho, \quad \text{where} \quad \rho = \int_{\Omega_v} f\,\mathrm{d}v, & x\in\Omega_x, \quad t\in(0,T]. \end{cases} \tag{2.1}$$

We can reformulate (2.1) as

$$\begin{cases} \partial_t f + \boldsymbol{F}\cdot\nabla f = 0, & (x,v)\in\Omega, \quad t\in(0,T], \\ \partial_x E = 1-\rho, & x\in\Omega_x, \quad t\in(0,T], \end{cases} \tag{2.2}$$

where

$$\boldsymbol{F} = \begin{pmatrix} v \\ -E \end{pmatrix} \quad \text{s.t.} \quad \nabla\cdot\boldsymbol{F} = \partial_x v - \partial_v E = 0. \tag{2.3}$$

The VP system (2.1) (or (2.2)) is supplemented with the initial condition: $f(x,v,0) = f_0(x,v)$ in Ω, periodic boundary condition in x:

$$f(0,v,t) = f(2\pi,v,t), \quad E(0,t) = E(2\pi,t), \quad v\in\Omega_v, \quad t\in[0,T], \tag{2.4}$$

and the homogeneous Dirichlet boundary conditions in v:

$$f(x,a,t) = f(x,b,t) = 0, \quad x\in\Omega_x, \quad t\in[0,T]. \tag{2.5}$$

It is noteworthy that if $\Omega_v = \mathbb{R}$, we understand (2.5) as $a\to-\infty$ and $b\to\infty$. We remark that the VP system is usually set in $\mathbb{R}$ for the velocity variable v, while the hard truncation of the unbounded domain into $\Omega_v = (a,b)$ with (2.5) is feasible when f decays fast and the interesting physics is roughly confined in a finite interval.

According to the hypothesis of neutrality of the plasma, we assume that the total charge in the system is zero (*cf.* [6,23]), *i.e.*,

$$\int_{\Omega_x} E(x,t)\,\mathrm{d}x = 0, \quad t\in(0,T], \tag{2.6}$$

which is equivalent to the Fourier coefficient of the zeroth mode $\widehat{E}_0(t) = 0$ in the Fourier expansion of E:

$$E(x,t) = \sum_{|m|=0}^{\infty} \widehat{E}_m(t)\,\mathrm{e}^{\mathrm{i}mx}, \quad \widehat{E}_m(t) = \frac{1}{2\pi}\int_0^{2\pi} E(x,t)\,\mathrm{e}^{-\mathrm{i}mx}\,\mathrm{d}x. \tag{2.7}$$

2.2. Some conservation laws and *a priori* bounds

The weak formulation of (2.1) is to seek $f\in H^1_{\mathfrak{p}}\big(\Omega_x; H^1_0(\Omega_v)\big)$ and $E\in H^1_{\mathfrak{p}}(\Omega_x)$ for $t>0$ such that

$$\begin{cases} (\partial_t f, u)_\Omega + (\boldsymbol{F}\cdot\nabla f, u)_\Omega = 0, & \forall u\in H^1_{\mathfrak{p}}\big(\Omega_x; H^1_0(\Omega_v)\big), \\ (\partial_x E, w)_{\Omega_x} = (1-\rho, w)_{\Omega_x}, & \forall w\in H^1_{\mathfrak{p}}(\Omega_x), \end{cases} \tag{2.8}$$

together with the initial condition: $f(x,v,0) = f_0(x,v)$ in Ω, and the condition (2.6).

Proposition 2.1. *Let f be the exact solution of (2.1). Then, there holds*

$$\int_\Omega f(x,v,t)\,\mathrm{d}\Omega = \int_{\Omega_x} \rho(x,t)\,\mathrm{d}x = 2\pi, \quad \int_\Omega v\,\partial_v f(x,v,t)\,\mathrm{d}\Omega = -2\pi, \tag{2.9}$$

and

$$\|f(\cdot,\cdot,t)\|_\Omega = \|f_0\|_\Omega, \quad t\geq 0. \tag{2.10}$$

Moreover, if f^p with $p\geq 2$ is well-defined, then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega f^p(x,v,t)\,\mathrm{d}\Omega = 0. \tag{2.11}$$

Proof. It is evident that by the periodicity of E in (2.4), we have

$$\int_{\Omega_x} \partial_x E(x,t)\,\mathrm{d}x = 0.$$

Taking $w = 1$ in (2.8), we obtain the first identity in (2.9) immediately. Then we can further derive from (2.5) and integration by parts the second identity.

Using the property $\nabla \cdot \boldsymbol{F} = 0$, we find readily that

$$\nabla \cdot (\boldsymbol{F} f^2) = 2f(\nabla f)\cdot \boldsymbol{F} + f^2\,\nabla\cdot\boldsymbol{F} = 2f\boldsymbol{F}\cdot\nabla f. \tag{2.12}$$

Taking $u = 2f$ in (2.8), we obtain from the divergence theorem, equation (2.12) and direct calculation that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega f^2\,\mathrm{d}\Omega &= -2\int_\Omega f\,\boldsymbol{F}\cdot\nabla f\,\mathrm{d}\Omega = -\int_\Omega \nabla\cdot\left(\boldsymbol{F}f^2\right)\mathrm{d}\Omega = -\int_{\partial\Omega}\boldsymbol{n}\cdot\left(\boldsymbol{F}f^2\right)\mathrm{d}s\\ &= -\int_{\partial\Omega} f^2\,\boldsymbol{n}\cdot\boldsymbol{F}\,\mathrm{d}s = 0,\end{aligned} \tag{2.13}$$

which implies (2.10).

Similarly, taking $u = pf^{p-1}$ in (2.8) and using the divergence theorem and boundary conditions (2.4) and (2.5), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega f^p\,\mathrm{d}\Omega &= -p\int_\Omega f^{p-1}\,\boldsymbol{F}\cdot(\nabla f)\,\mathrm{d}\Omega = -\int_\Omega \nabla\cdot(\boldsymbol{F}f^p)\,\mathrm{d}\Omega = -\int_{\partial\Omega}\boldsymbol{n}\cdot(\boldsymbol{F}f^p)\,\mathrm{d}s\\ &= -\int_{\partial\Omega} f^p\,\boldsymbol{n}\cdot\boldsymbol{F}\,\mathrm{d}s = 0.\end{aligned} \tag{2.14}$$

This ends the proof. □

We have the following priori estimates of the bound.

Proposition 2.2. *Let f, E be the solution of the Vlasov–Poisson system (2.1). Then there holds*

$$\|E\|_{\Omega_x} \le \|\partial_x E\|_{\Omega_x} \le \sqrt{|\Omega_v|}\;\|f\|_\Omega, \tag{2.15}$$

where the second inequality is valid for a finite interval Ω_v.

Proof. Taking $w = \partial_x E$ in the second equation of (2.8), we can derive

$$\begin{aligned}(\partial_x E, \partial_x E)_{\Omega_x} &= (1-\rho, \partial_x E)_{\Omega_x} = -(\rho, \partial_x E)_{\Omega_x} \le \|\partial_x E\|_{\Omega_x}\cdot\|\rho\|_{\Omega_x}\\ &\le \|\partial_x E\|_{\Omega_x}\left(\int_{\Omega_x}\left(\int_{\Omega_v} f^2\mathrm{d}v\int_{\Omega_v} 1^2\mathrm{d}v\right)\mathrm{d}x\right)^{1/2}\\ &= \sqrt{|\Omega_v|}\,\|\partial_x E\|_{\Omega_x}\cdot\|f\|_\Omega.\end{aligned} \tag{2.16}$$

This leads to the bound of $\|\partial_x E\|_{\Omega_x}$ in (2.15). Thanks to (2.6) and (2.7), it follows from the Parseval's identity that

$$\|E\|_{\Omega_x} \le \|\partial_x E\|_{\Omega_x}. \tag{2.17}$$

This ends the proof. □

2.3. Spectral-Galerkin scheme in space

We consider the semi-discretized spectral-Galerkin scheme for (2.8): find $f_* := f_M^N \in V_M^N := \mathcal{F}_M \times \mathcal{P}_N^0$ and $E_* := E_M \in \mathcal{F}_M^0$ for all $t \in (0,T]$, such that

$$\begin{cases} (\partial_t f_*, \psi)_\Omega + (\boldsymbol{F}_* \cdot \nabla f_*, \psi)_\Omega = 0, \quad \boldsymbol{F}_* := (v, -E_*), \quad \forall \psi \in V_M^N, \\ (\partial_x E_*, \varphi)_{\Omega_x} = (1-\rho_*, \varphi)_{\Omega_x}, \quad \rho_* := \int_{\Omega_v} f_* \,\mathrm{d}v, \qquad \forall \varphi \in \mathcal{F}_M^0, \end{cases} \tag{2.18}$$

together with a suitable initial approximation: $f_*(x,v,0) = f_{0,*}(x,v)$ to be specified later. Here, $E_* \in \mathcal{F}_M^0$ is to exactly impose the condition (2.6), so it automatically satisfies

$$\int_{\Omega_x} E_*(x,t)\,\mathrm{d}x = 0, \quad t \in (0,T], \tag{2.19}$$

i.e., the coefficient of the zeroth mode vanishes.

Note that the properties in Propositions 2.1 and 2.2 are also valid for the discrete problem.

Proposition 2.3. *Let f_*, E_* be the numerical solution of the Vlasov–Poisson system (2.18). Then there hold*

$$\int_\Omega f_*(x,v,t)\,\mathrm{d}\Omega = \int_{\Omega_x} \rho_*(x,t)\,\mathrm{d}x = 2\pi, \quad \int_\Omega v\,\partial_v f_*(x,v,t)\,\mathrm{d}\Omega = -2\pi, \tag{2.20}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega f_*^2(x,v,t)\,\mathrm{d}\Omega = 0. \tag{2.21}$$

Moreover, we have

$$\|E_*\|_{\Omega_x} \le \|\partial_x E_*\|_{\Omega_x} \le \sqrt{|\Omega_v|}\,\|f_*\|_\Omega, \tag{2.22}$$

where the second inequality is valid on finite interval Ω_v.

3. Error estimates of the Fourier–Legendre spectral-Galerkin method

In this section, we assume Ω_v to be a finite domain, and analyse the Fourier–Legendre spectral-Galerkin method for the VP system. We first present some necessary approximation results and then carry out the error analysis.

3.1. Fourier approximation results on $\boldsymbol{\Omega_x}$

Let us begin with the approximate spaces

$$\mathcal{F}_M = \mathrm{span}\{\mathrm{e}^{\mathrm{i}kx} : 0 \le |k| \le M\}, \quad \mathcal{F}_M^0 = \mathrm{span}\{\mathrm{e}^{\mathrm{i}kx} : 1 \le |k| \le M\}. \tag{3.1}$$

Then, the L^2-orthogonal projection $\Pi_M : L^2(\Omega_x) \to \mathcal{F}_M$ is defined by

$$(\Pi_M u - u, \phi)_{\Omega_x} = 0, \quad \forall\,\phi \in \mathcal{F}_M, \tag{3.2}$$

which is also the H^m-orthogonal projection for any integer $m \ge 1$. Noted that $\Pi_M u$ is the truncated Fourier series:

$$\Pi_M u(x) = \sum_{|k|=0}^{M} \hat{u}_k\,\mathrm{e}^{\mathrm{i}kx} \quad \text{with} \quad \hat{u}_k = \big(u, \mathrm{e}^{\mathrm{i}kx}\big)_{\Omega_x} = \frac{1}{2\pi}\int_0^{2\pi} u(x)\,\mathrm{e}^{-\mathrm{i}kx}\mathrm{d}x.$$

For any $u \in H_{\mathfrak{p}}^m(\Omega_x)$, we have the commutability (*cf.* [28], p. 34):

$$\partial_x^l(\Pi_M u) = \Pi_M\big(\partial_x^l u\big), \quad 0 \le l \le m. \tag{3.3}$$

To approximate the electric field E, we need a special projection that can incorporate the condition (2.6). For this purpose, we define

$$L_0^2(\Omega_x) = \{u \in L^2(\Omega_x) : \bar{u} = 0\}, \quad \text{where} \quad \bar{u} := \frac{1}{2\pi}\int_{\Omega_x} u(x)\mathrm{d}x. \tag{3.4}$$

One derives readily from the Parseval's identity the following Poincare's inequality

$$\|u\|_{\Omega_x} \le \|u'\|_{\Omega_x}, \quad \forall\, u \in H^1_{\mathfrak{p},0}(\Omega_x) := H^1_{\mathfrak{p}}(\Omega_x) \cap L_0^2(\Omega_x). \tag{3.5}$$

Define the L_0^2-orthogonal projection $\Pi_M^0 : L_0^2(\Omega_x) \to \mathcal{F}_M^0$ such that

$$\big(\Pi_M^0 u - u, \phi\big)_{\Omega_x} = 0, \quad \forall\, \phi \in \mathcal{F}_M^0. \tag{3.6}$$

Introduce the Sobolev space $H^r_{\mathfrak{p}}(\Omega_x)$ with real $r \ge 0$, equipped with the norm and semi-norm (*cf.* [28]):

$$\|u\|_{H^r_{\mathfrak{p}}(\Omega_x)} = \left(\sum_{|k|=0}^{\infty} \big(1+k^2\big)^r |\hat{u}_k|^2\right)^{\frac{1}{2}}, \quad |u|_{H^r_{\mathfrak{p}}(\Omega_x)} = \left(\sum_{|k|=1}^{\infty} k^{2r} |\hat{u}_k|^2\right)^{\frac{1}{2}}, \tag{3.7}$$

where $\{\hat{u}_k\}$ are the Fourier coefficients of u. In particular, if r is a positive integer, they are equivalent to

$$\|u\|_{H^r_{\mathfrak{p}}(\Omega_x)} \sim \left(\sum_{k=0}^{r} \big\|\partial_x^k u\big\|^2_{\Omega_x}\right)^{\frac{1}{2}}, \quad |u|_{H^r_{\mathfrak{p}}(\Omega_x)} \sim \|\partial_x^r u\|_{\Omega_x}. \tag{3.8}$$

We denote by c a generic positive constant which is independent of the underlying function and discretisation parameters.

Recall the Nikolski's and Bernstein inequalities stated in the following two lemmas, respectively.

Lemma 3.1 (see [28], Lem. 2.4)**.** *For any $u \in \mathcal{F}_M$ and $1 \le p \le q \le \infty$,*

$$\|u\|_{L^q(\Omega_x)} \le \left(\frac{Mp_0 + 1}{2\pi}\right)^{\frac{1}{p}-\frac{1}{q}} \|u\|_{L^p(\Omega_x)}, \tag{3.9}$$

where p_0 is the least even integer $\ge p$.

Lemma 3.2 (see [28], Lem. 2.5)**.** *For any $u \in \mathcal{F}_M$ and $1 \le p \le \infty$,*

$$\|\partial_x^r u\|_{L^p(\Omega_x)} \le M^r \|u\|_{L^p(\Omega_x)}, \quad r \ge 1. \tag{3.10}$$

The main approximation result on the projection errors for Π_M and Π_M^0 are stated as follows.

Lemma 3.3. *For any $u \in H^r_{\mathfrak{p}}(\Omega_x)$ with $0 \le \mu \le r$,*

$$\|\Pi_M u - u\|_{H^\mu_{\mathfrak{p}}(\Omega_x)} \le cM^{\mu - r} |u|_{H^r_{\mathfrak{p}}(\Omega_x)}, \tag{3.11}$$

and for any $r > \frac{1}{2}$,

$$\|\Pi_M u - u\|_{L^\infty(\Omega_x)} \le \sqrt{\frac{2}{2r-1}}\, M^{\frac{1}{2}-r} |u|_{H^r_{\mathfrak{p}}(\Omega_x)}. \tag{3.12}$$

Moreover, for $u \in H^r_{\mathfrak{p}}(\Omega_x) \cap L_0^2(\Omega_x)$, the estimates (3.11) and (3.12) are also valid for the orthogonal projection Π_M^0 defined in (3.6).

Proof. The estimates (3.11) and (3.12) can be found in Section 2.2 of [28]. The same results for the orthogonal projection Π_M^0 can be derived by following the same lines as in the proofs of (3.11) and (3.12). □

3.2. Legendre approximation results on Ω_v

In what follows, we shall recap on some relevant estimates on L^2- and H_0^1-orthogonal projections on the standard interval $\Lambda := (-1, 1)$, and show a very important result on L^∞-estimate.

For $\alpha, \beta > -1$, let $P_n^{(\alpha,\beta)}(\xi)$ be the Jacobi polynomials of degree n on Λ, which are orthogonal with respect to the weight function $\omega^{(\alpha,\beta)}(\xi) = (1-\xi)^\alpha(1+\xi)^\beta$ and normalised as in Szegö [29]. For simplicity of notation, we denote $\omega^{(\alpha)}(\xi) = \omega^{(\alpha,\alpha)}(\xi)$. In particular, for $\alpha = \beta = 0$, we define the Legendre polynomial by $P_n(\xi)$. Moreover, we denote $\mathcal{P}_N$ be the set of polynomials of degree at most N.

Consider the H_0^1-orthogonal projection $\pi_N^{1,0} : H_0^1(\Lambda) \to \mathcal{P}_N^0 := H_0^1 \cap \mathcal{P}_N$ defined by

$$(\partial_\xi(\pi_N^{1,0}u - u), \partial_\xi\varphi)_\Lambda = 0, \quad \forall\, \varphi \in \mathcal{P}_N^0. \tag{3.13}$$

Lemma 3.4. *For any $u \in H_0^1(\Lambda) \cap H^s(\Lambda)$ with integer $1 \le s \le N+1$, we have*

$$\|\partial_\xi(\pi_N^{1,0}u - u)\|_{L^2(\Lambda)} + N\|\pi_N^{1,0}u - u\|_{L^2(\Lambda)} \le cN^{1-s}|u|_{H^s(\Lambda)}, \tag{3.14}$$

and

$$\|\pi_N^{1,0}u - u\|_{L^\infty(\Lambda)} \le cN^{\frac{1}{2}-s}|u|_{H^s(\Lambda)}, \tag{3.15}$$

where c is a positive constant independent of u and N.

The first statement can be found in [7], p. 288, the L^∞-estimate is derived directly by the Sobolev inequality ([28], Appendix B.8).

In the error analysis, we also need to use the $W^{1,\infty}$-estimate that appears unavailable in literature, so we provide its proof in Appendix A.

Lemma 3.5. *For any $u \in H_0^1(\Lambda) \cap H^s(\Lambda)$ with integer $2 \le s \le N+1$, we have*

$$\|\partial_\xi(\pi_N^{1,0}u - u)\|_{L^\infty(\Lambda)} \le cN^{\frac{7}{4}-s}\,|u|_{H^s(\Lambda)}. \tag{3.16}$$

With the above estimates, we consider the H_0^1-orthogonal projection $\Pi_N^{1,0} : H_0^1(\Omega_v) \to \mathcal{P}_N^0 := H_0^1(\Omega_v) \cap \mathcal{P}_N$ on the velocity (finite) subdomain $\Omega_v = (a, b)$ such that

$$(\partial_v(\Pi_N^{1,0}u - u), \partial_v\varphi)_{\Omega_v} = 0, \quad \forall\, \varphi \in \mathcal{P}_N^0, \tag{3.17}$$

and summarize the results.

Theorem 3.1. *For any $u \in H_0^1(\Omega_v) \cap H^s(\Omega_v)$ with integer $1 \le s \le N+1$, we have*

$$\|\partial_v(\Pi_N^{1,0}u - u)\|_{L^2(\Omega_v)} + N\|\Pi_N^{1,0}u - u\|_{L^2(\Omega_v)} \le cN^{1-s}|u|_{H^s(\Omega_v)}. \tag{3.18}$$

Moreover, for any $u \in H_0^1(\Omega_v) \cap H^s(\Omega_v)$ with integer $2 \le s \le N+1$, we have

$$\|\partial_v(\Pi_N^{1,0}u - u)\|_{L^\infty(\Omega_v)} \le cN^{\frac{7}{4}-s}\,|u|_{H^s(\Omega_v)}, \tag{3.19}$$

where c is a positive constant that is independent on s, u and N.

Proof. With the change of variable

$$v(\xi) = \frac{b-a}{2}\xi + \frac{b+a}{2} \quad \text{with} \quad \xi \in (-1, 1), \;\; v \in (a, b), \tag{3.20}$$

we can derive these estimates from Lemmas 3.4 and 3.5 directly. □

3.3. The Fourier–Legendre approximation results on $\Omega = \Omega_x \times \Omega_v$

We denote the tensorial subspace $V_M^N := \mathcal{F}_M \times \mathcal{P}_N^0$, and define the orthogonal projection

$$\Pi_M^N := \Pi_M \circ \Pi_N^{1,0} : \quad H_{\mathfrak{p}}^1\big(\Omega_x; H_0^1(\Omega_v)\big) \to V_M^N.$$

Then, we arrive at the following approximation result.

Theorem 3.2. *For real $r \geq 1$ and integer $1 \leq s \leq N+1$, we have the following estimates.*

(i) *If $f \in H_{\mathfrak{p}}^{r-1}(\Omega_x; L^2(\Omega_v)) \cap H^{s-1}(\Omega_v; L_{\mathfrak{p}}^2(\Omega_x))$, then*

$$\big\|f - \Pi_M^N f\big\|_\Omega \leq cM^{1-r}\|f\|_{H_{\mathfrak{p}}^{r-1}(\Omega_x;L^2(\Omega_v))} + cN^{1-s}\|f\|_{H^{s-1}\left(\Omega_v;L_{\mathfrak{p}}^2(\Omega_x)\right)}. \tag{3.21}$$

(ii) *If $f \in H_{\mathfrak{p}}^{r}(\Omega_x; L^2(\Omega_v)) \cap H^{s-1}(\Omega_v; H_{\mathfrak{p}}^1(\Omega_x))$, then*

$$\big\|\partial_x\big(f - \Pi_M^N f\big)\big\|_\Omega \leq cM^{1-r}\|f\|_{H_{\mathfrak{p}}^{r}(\Omega_x;L^2(\Omega_v))} + cN^{1-s}\|f\|_{H^{s-1}\left(\Omega_v;H_{\mathfrak{p}}^1(\Omega_x)\right)}. \tag{3.22}$$

(iii) *If $f \in H_{\mathfrak{p}}^{r-1}(\Omega_x; H^1(\Omega_v)) \cap H^{s}(\Omega_v; L^2(\Omega_x))$, then*

$$\big\|\partial_v\big(f - \Pi_M^N f\big)\big\|_\Omega \leq cM^{1-r}\|f\|_{H_{\mathfrak{p}}^{r-1}(\Omega_x;H^1(\Omega_v))} + cN^{1-s}\|f\|_{H^{s}(\Omega_v;L^2(\Omega_x))}. \tag{3.23}$$

Proof. We can derive from Lemma 3.3, Theorem 3.1 and the triangle inequality that

$$\begin{aligned}
\|f - \Pi_M^N f\|_\Omega &= \|f - \Pi_M \circ \Pi_N^{1,0} f\|_\Omega \\
&\leq \|f - \Pi_M f\|_\Omega + \|\Pi_M(f - \Pi_N^{1,0} f)\|_\Omega \\
&\leq cM^{1-r}\|f\|_{H_{\mathfrak{p}}^{r-1}(\Omega_x;L^2(\Omega_v))} + cN^{1-s}\|f\|_{H^{s-1}(\Omega_v;L_{\mathfrak{p}}^2(\Omega_x))}.
\end{aligned}$$

Similarly, by (3.3),

$$\begin{aligned}
\|\partial_x(f - \Pi_M^N f)\|_\Omega &= \|\partial_x(f - \Pi_M \circ \Pi_N^{1,0} f)\|_\Omega \\
&\leq \|\partial_x(f - \Pi_M f)\|_\Omega + \|\partial_x \Pi_M(f - \Pi_N^{1,0} f)\|_\Omega \\
&\leq \|\partial_x(f - \Pi_M f)\|_\Omega + \|\Pi_M \partial_x(f - \Pi_N^{1,0} f)\|_\Omega \\
&\leq cM^{1-r}\|f\|_{H_{\mathfrak{p}}^{r}(\Omega_x;L^2(\Omega_v))} + cN^{1-s}\|f\|_{H^{s-1}(\Omega_v;H_{\mathfrak{p}}^1(\Omega_x))}
\end{aligned}$$

and

$$\begin{aligned}
\|\partial_v(f - \Pi_M^N f)\|_\Omega &= \|\partial_v(f - \Pi_M \circ \Pi_N^{1,0} f)\|_\Omega \\
&\leq \|\partial_v(f - \Pi_M f)\|_\Omega + \|\partial_v \Pi_M(f - \Pi_N^{1,0} f)\|_\Omega \\
&= \|\partial_v f - \Pi_M \partial_v f\|_\Omega + \|\Pi_M \partial_v(f - \Pi_N^{1,0} f)\|_\Omega \\
&\leq \|\partial_v f - \Pi_M \partial_v f\|_\Omega + \|\partial_v(f - \Pi_N^{1,0} f)\|_\Omega \\
&\leq cM^{1-r}\|f\|_{H_{\mathfrak{p}}^{r-1}(\Omega_x;H^1(\Omega_v))} + cN^{1-s}\|f\|_{H^{s}(\Omega_v;L^2(\Omega_x))}.
\end{aligned}$$

This ends the proof. □

Then the error bounds in the L^∞-norm are given as follows.

Theorem 3.3. *We have the following estimates.*

(i) *If $f \in H^r_{\mathfrak{p}}(\Omega_x; L^\infty(\Omega_v)) \cap H^s(\Omega_v; H^{\frac{1}{2}+\epsilon}_{\mathfrak{p}}(\Omega_x))$ with $r > \frac{1}{2}$, integer $s \geq 1$, and $\epsilon > 0$, then*

$$\|f - \Pi^N_M f\|_{L^\infty(\Omega)} \leq c\, M^{\frac{1}{2}-r} \|f\|_{H^r_{\mathfrak{p}}(\Omega_x; L^\infty(\Omega_v))} + cN^{\frac{1}{2}-s}(\|f\|_{H^s(\Omega_v; L^\infty(\Omega_x))} + M^{-\epsilon}\|f\|_{H^s(\Omega_v; H^{\frac{1}{2}+\epsilon}_{\mathfrak{p}}(\Omega_x))}).$$

(ii) *If $f \in H^r_{\mathfrak{p}}(\Omega_x; L^\infty(\Omega_v)) \cap H^{s-1}(\Omega_v; H^{\frac{3}{2}+\epsilon}_{\mathfrak{p}}(\Omega_x))$ with $r > \frac{3}{2}$, integer $s \geq 2$, and $\epsilon > 0$, then*

$$\|\partial_x(f - \Pi^N_M f)\|_{L^\infty(\Omega)} \leq cM^{\frac{3}{2}-r} \|f\|_{H^r_{\mathfrak{p}}(\Omega_x; L^\infty(\Omega_v))} + cN^{\frac{3}{2}-s}(\|f\|_{H^{s-1}(\Omega_v; H^{\frac{3}{2}+\epsilon}_{\mathfrak{p}}(\Omega_x))} + M^{-\epsilon}\|f\|_{H^{s-1}(\Omega_v; H^{\frac{3}{2}+\epsilon}_{\mathfrak{p}}(\Omega_x))}).$$

(iii) *If $f \in H^{r-1}_{\mathfrak{p}}(\Omega_x; H^1(\Omega_v)) \cap H^s(\Omega_v; H^{\frac{1}{2}+\epsilon}_{\mathfrak{p}}(\Omega_x))$ with $r > \frac{3}{2}$, integer $s \geq 2$, and $\epsilon > 0$, then*

$$\|\partial_v(f - \Pi^N_M f)\|_{L^\infty(\Omega)} \leq cM^{\frac{3}{2}-r} \|f\|_{H^{r-1}_{\mathfrak{p}}(\Omega_x; H^1(\Omega_v))} + cN^{\frac{7}{4}-s}(\|f\|_{H^s(\Omega_v; L^\infty(\Omega_x))} + M^{-\epsilon}\|f\|_{H^s(\Omega_v; H^{\frac{1}{2}+\epsilon}_{\mathfrak{p}}(\Omega_x))}).$$

Proof. From Lemmas 3.3 and 3.4, the triangle inequality, and the fact $H^{\frac{1}{2}+\epsilon}(\Omega_x) \subset L^\infty(\Omega_x)$, it follows that

$$\begin{aligned}
\|f - \Pi^N_M f\|_{L^\infty(\Omega)} &\leq \|f - \Pi_M f\|_{L^\infty(\Omega)} + \|\Pi_M(f - \Pi^{1,0}_N f)\|_{L^\infty(\Omega)} \\
&\leq cM^{\frac{1}{2}-r}\|f\|_{H^r_{\mathfrak{p}}(\Omega_x; L^\infty(\Omega_v))} + \|f - \Pi^{1,0}_N f\|_{L^\infty(\Omega)} \\
&\quad + c\, M^{-\epsilon}\|f - \Pi^{1,0}_N f\|_{L^\infty(\Omega_v; H^{\frac{1}{2}+\epsilon}_{\mathfrak{p}}(\Omega_x))} \\
&\leq c\, M^{\frac{1}{2}-r}\|f\|_{H^r_{\mathfrak{p}}(\Omega_x; L^\infty(\Omega_v))} + cN^{\frac{1}{2}-s}\|f\|_{H^s(\Omega_v; L^\infty(\Omega_x))} \\
&\quad + cN^{\frac{1}{2}-s}M^{-\epsilon}\|f\|_{H^s(\Omega_v; H^{\frac{1}{2}+\epsilon}_{\mathfrak{p}}(\Omega_x))}.
\end{aligned}$$

Using Lemmas 3.3 and 3.4 and the triangle inequality again, yields

$$\begin{aligned}
\|\partial_x(f - \Pi^N_M f)\|_{L^\infty(\Omega)} &\leq \|\partial_x(f - \Pi_M f)\|_{L^\infty(\Omega)} + \|\partial_x(\Pi_M f - \Pi_M \circ \Pi^{1,0}_N f)\|_{L^\infty(\Omega)} \\
&\leq cM^{\frac{3}{2}-r}\|f\|_{H^r_{\mathfrak{p}}(\Omega_v; L^\infty(\Omega_v))} + \|\partial_x(f - \Pi^{1,0}_N f)\|_{L^\infty(\Omega)} \\
&\quad + M^{-\epsilon}\|\partial_x(f - \Pi^{1,0}_N f)\|_{L^\infty(\Omega_v; H^{\frac{1}{2}+\epsilon}_{\mathfrak{p}}(\Omega_x))} \\
&\leq cM^{\frac{3}{2}-r}\|f\|_{H^r_{\mathfrak{p}}(\Omega_v; L^\infty(\Omega_v))} + cN^{\frac{3}{2}-s}\|f\|_{H^{s-1}(\Omega_v; H^{\frac{3}{2}+\epsilon}_{\mathfrak{p}}(\Omega_x))} \\
&\quad + cM^{-\epsilon}N^{\frac{3}{2}-s}\|f\|_{H^{s-1}(\Omega_v; H^{\frac{3}{2}+\epsilon}_{\mathfrak{p}}(\Omega_x))}.
\end{aligned}$$

We follow the procedure similar to the above to obtain

$$\begin{aligned}
\|\partial_v(f - \Pi^N_M f)\|_{L^\infty(\Omega)} &\leq \|\partial_v f - \Pi_M \partial_v f\|_{L^\infty(\Omega)} + \|\partial_v(f - \Pi^{1,0}_N f)\|_{L^\infty(\Omega)} \\
&\quad + M^{-\epsilon}\|\partial_v(f - \Pi^{1,0}_N f)\|_{L^\infty(\Omega_v; H^{\frac{1}{2}+\epsilon}_{\mathfrak{p}}(\Omega_x))} \\
&\leq cM^{\frac{3}{2}-r}\|f\|_{H^{r-1}_{\mathfrak{p}}(\Omega_x; H^1(\Omega_v))} + cN^{\frac{7}{4}-s}\|f\|_{H^s(\Omega_v; L^\infty(\Omega_x))} \\
&\quad + cM^{-\epsilon}N^{\frac{7}{4}-s}\|f\|_{H^s(\Omega_v; H^{\frac{1}{2}+\epsilon}_{\mathfrak{p}}(\Omega_x))},
\end{aligned}$$

where we used (3.19). $\square$

3.4. Error estimates

In what follows, we carry out the error analysis for the Fourier–Legendre spectral-Galerkin method (2.18). For notational convenience, we introduce

$$e_f = f - \Pi_M^N f, \quad e_E = E - \Pi_M^0 E, \quad e_f^* = f_* - \Pi_M^N f, \quad e_E^* = E_* - \Pi_M^0 E, \tag{3.24}$$

where Π_M^0 is defined in (3.6) and $\Pi_M^N = \Pi_M \circ \Pi_N^{1,0}$. The main result on the error estimates is presented as follows.

Theorem 3.4 (Error for Fourier–Legendre spectral-Galerkin method)**.** *Let f and f_* be the solution of the Galerkin formulation (2.8) and (2.18), respectively, where the domain $\Omega = \Omega_x \times \Omega_v$ with finite $\Omega_v = (a,b)$. If the distribution function $f(x,v,t)$ satisfies*

$$\begin{aligned} f \in L^\infty(0,T;H_{\mathfrak{p}}^r(\Omega_x;L^2(\Omega_v))) \cap L^2(0,T;H_{\mathfrak{p}}^{r-1}(\Omega_x;H^1(\Omega_v))) \\ \cap L^\infty(0,T;H^s(\Omega_v;L_{\mathfrak{p}}^2(\Omega_x))) \cap L^2(0,T;H^{s-1}(\Omega_v;H_{\mathfrak{p}}^1(\Omega_x))) \end{aligned}$$

and $\partial_t f \in L^2(0,T;H_{\mathfrak{p}}^{r-1}(\Omega_x;L^2(\Omega_v))) \cap L^2(0,T;H^{s-1}(\Omega_v;L_{\mathfrak{p}}^2(\Omega_x)))$, then for any $r \geq \frac{3}{2}$ and integer $s \geq 2$, we have the following error estimates:

$$\|f(\cdot,\cdot,t) - f_*(\cdot,\cdot,t)\|_\Omega \leq c(M^{-r}B_f^1 + N^{-s}B_f^2 + M^{1-r}B_f^3 + N^{1-s}B_f^4), \tag{3.25}$$

where

$$\begin{aligned} B_f^1 &:= \|f\|_{L^\infty(0,T;H_{\mathfrak{p}}^r(\Omega_x;L^2(\Omega_v)))}, \quad B_f^2 := \|f\|_{L^\infty(0,T;H^s(\Omega_v;L_{\mathfrak{p}}^2(\Omega_x)))}, \\ B_f^3 &:= \int_0^t (\|\partial_\tau f\|_{H_{\mathfrak{p}}^{r-1}(\Omega_x;L^2(\Omega_v))} + \max\{|a|,|b|\}\|f\|_{H_{\mathfrak{p}}^r(\Omega_x;L^2(\Omega_v))}\cdot \\ &\quad + \cdot\sqrt{b-a}\,\|f\|_\Omega \cdot \|f\|_{H_{\mathfrak{p}}^{r-1}(\Omega_x;H^1(\Omega_v))})\, \mathrm{e}^{\int_\tau^t \alpha(\eta)\mathrm{d}\eta}\, \mathrm{d}\tau, \\ B_f^4 &:= \int_0^t (\|\partial_\tau f\|_{H^{s-1}(\Omega_v;L_{\mathfrak{p}}^2(\Omega_x))} + \max\{|a|,|b|\}\|f\|_{H^{s-1}(\Omega_v;H_{\mathfrak{p}}^1(\Omega_x))}\cdot \\ &\quad + \cdot\sqrt{b-a}\|f\|_{H^s(\Omega_v;L_{\mathfrak{p}}^2(\Omega_x))})\, \mathrm{e}^{\int_\tau^t \alpha(\eta)\mathrm{d}\eta}\, \mathrm{d}\tau, \end{aligned}$$

and

$$\alpha(t) := \sqrt{b-a}\,(cM^{3/2-r}\|f\|_{H_{\mathfrak{p}}^{r-1}(\Omega_x;H^1(\Omega_v))} + c\sqrt{M}N^{1-s}\|f\|_{L_{\mathfrak{p}}^2(\Omega_x;H^s(\Omega_v))} + 2\|\partial_v f\|_\infty) = \mathcal{O}(1),$$

where c is a positive constant that is independent of M, N and f.

Proof. From (2.8) and (2.18), we obtain the error equations:

$$\begin{cases} (\partial_t(f_* - f), \psi)_\Omega + (\boldsymbol{F}_* \cdot \nabla f_* - \boldsymbol{F} \cdot \nabla f, \psi)_\Omega = 0, & \forall \psi \in V_M^N, \\ (\partial_x(E_* - E), \varphi)_{\Omega_x} = (\rho - \rho_*, \varphi)_{\Omega_x}, & \forall \varphi \in \mathcal{F}_M^0. \end{cases} \tag{3.26}$$

Since $f_* - f = e_f^* - e_f$ and $E_* - E = e_E^* - e_E$, we can rewrite the above system as

$$\begin{cases} (\partial_t e_f^*, \psi)_\Omega + (\boldsymbol{F}_* \cdot \nabla e_f^*, \psi)_\Omega = (\partial_t e_f, \psi)_\Omega + (\boldsymbol{F}_* \cdot \nabla e_f, \psi)_\Omega + ((\boldsymbol{F} - \boldsymbol{F}_*) \cdot \nabla f, \psi)_\Omega, \\ (\partial_x e_E^*, \varphi)_{\Omega_x} + (\int_{\Omega_v} e_f^*\, \mathrm{d}v, \varphi)_{\Omega_x} = (\partial_x e_E, \varphi)_{\Omega_x} + (\int_{\Omega_v} e_f\, \mathrm{d}v, \varphi)_{\Omega_x}, \end{cases} \tag{3.27}$$

for all $\psi \in V_M^N$ and all $\varphi \in \mathcal{F}_M^0$. We take two steps to carry out the proof.

Step 1. *We first deal with the second equation of* (3.27). Taking $\varphi = \partial_x e_E^*$ in the second equation of (3.27) leads to

$$\|\partial_x e_E^*\|_{\Omega_x}^2 \leq |(\partial_x e_E, \partial_x e_E^*)_{\Omega_x}| + |(\int_{\Omega_v} e_f^* \,\mathrm{d}v, \partial_x e_E^*)_{\Omega_x}| + |(\int_{\Omega_v} e_f \,\mathrm{d}v, \partial_x e_E^*)_{\Omega_x}|. \tag{3.28}$$

From (3.3), we find that Π_M^0 in (3.6) also holds for the $H_{\mathfrak{p}}^1(\Omega_x)$-orthogonal projection upon $\mathcal{F}_M^0$, so we have

$$(\partial_x e_E, \partial_x e_E^*)_{\Omega_x} = 0. \tag{3.29}$$

The orthogonality of the Legendre polynomials implies

$$\int_{\Omega_v} e_f \,\mathrm{d}v = 0, \tag{3.30}$$

which, together with (3.28), (3.29) and the Cauchy–Schwarz inequality, leads to

$$\|\partial_x e_E^*\|_{\Omega_x}^2 |(\int_{\Omega_v} e_f^* \,\mathrm{d}v, \partial_x e_E^*)_{\Omega_x}| \leq \sqrt{|\Omega_v|}\ \|\partial_x e_E^*\|_{\Omega_x} \big\|e_f^*\big\|_{\Omega}, \tag{3.31}$$

so we have

$$\|\partial_x e_E^*\|_{\Omega_x} \leq \sqrt{|\Omega_v|}\ \big\|e_f^*\big\|_{\Omega}. \tag{3.32}$$

As $e_E^* \in \mathcal{F}_M^0$, we have $\|e_E^*\|_{\Omega_x} \leq \|\partial_x e_E^*\|_{\Omega_x}$. Thus, by (2.15) and (3.32),

$$\|e_E^*\|_{\Omega_x} \leq \sqrt{|\Omega_v|}\ \big\|e_f^*\big\|_{\Omega}, \tag{3.33}$$

which illustrates that the error of electric field is bounded by the error of distribution function. That means we only need to estimate the distribution function error.

Step 2. *We next deal with the first equation in* (3.27). Taking $\psi = 2e_f^*$ in the first equation of (3.27), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\big\|e_f^*\big\|_{\Omega}^2 + 2\big(\boldsymbol{F}_* \cdot \nabla e_f^*, e_f^*\big)_{\Omega} = 2\big(\partial_t e_f, e_f^*\big)_{\Omega} + 2\big(\boldsymbol{F}_* \cdot \nabla e_f, e_f^*\big)_{\Omega} + 2\big((\boldsymbol{F} - \boldsymbol{F}_*) \cdot \nabla f, e_f^*\big)_{\Omega}. \tag{3.34}$$

Since $\nabla \cdot \boldsymbol{F}_* = 0$, the identity (2.12) holds with $\boldsymbol{F}_*$ and e_f^* in place of $\boldsymbol{F}$ and f, respectively. By the same argument as in (2.13), we find $(\boldsymbol{F}_* \cdot \nabla e_f^*, 2e_f^*)_{\Omega} = 0$. Consequently, we need to estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\big\|e_f^*\big\|_{\Omega}^2 \leq \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3, \tag{3.35}$$

where

$$\mathcal{T}_1 = 2\,|(\partial_t e_f, e_f^*)_{\Omega}|, \quad \mathcal{T}_2 = 2\,|(\boldsymbol{F}_* \cdot \nabla e_f, e_f^*)_{\Omega}|, \quad \mathcal{T}_3 = 2\,|((\boldsymbol{F} - \boldsymbol{F}_*) \cdot \nabla f, e_f^*)_{\Omega}|. \tag{3.36}$$

From (3.21) and the Cauchy–Schwarz inequality, we derive

$$\begin{aligned} \mathcal{T}_1 &\leq 2\big|\big(\partial_t e_f, e_f^*\big)_{\Omega}\big| \leq 2\|\partial_t e_f\|_{\Omega} \cdot \big\|e_f^*\big\|_{\Omega} \\ &\leq \Big(c\,M^{1-r}\|\partial_t f\|_{H_{\mathfrak{p}}^{r-1}(\Omega_x; L^2(\Omega_v))} + c\,N^{1-s}\|\partial_t f\|_{H^{s-1}\left(\Omega_v; L_{\mathfrak{p}}^2(\Omega_x)\right)}\Big)\big\|e_f^*\big\|_{\Omega}. \end{aligned} \tag{3.37}$$

We continue to estimate $\mathcal{T}_2$ and deduce from (2.18), (3.24), (3.33) and the Cauchy–Schwarz inequality that

$$\begin{aligned} \mathcal{T}_2 &= 2|(\boldsymbol{F}_* \cdot \nabla e_f, e_f^*)_{\Omega}| \leq 2|(v\partial_x e_f, e_f^*)_{\Omega}| + 2|(E_* \partial_v e_f, e_f^*)_{\Omega}| \\ &\leq 2|(v\partial_x e_f, e_f^*)_{\Omega}| + 2|(e_E^*\,\partial_v e_f, e_f^*)_{\Omega}| + 2|(\Pi_M^0 E\,\partial_v e_f, e_f^*)_{\Omega}| \\ &\leq 2\|v\partial_x e_f\|_{\Omega} \cdot \|e_f^*\|_{\Omega} + 2\|e_E^*\|_{\infty} \cdot \|\partial_v e_f\|_{\Omega} \cdot \|e_f^*\|_{\Omega} + 2\|\Pi_M^0 E\|_{\infty} \cdot \|\partial_v e_f\|_{\Omega} \cdot \|e_f^*\|_{\Omega}. \end{aligned} \tag{3.38}$$

We now estimate the terms in (3.38). From (2.15) and (3.12), we obtain

$$\left\|\Pi_M^0 E\right\|_\infty \le \|E\|_\infty + \sqrt{2}M^{-\frac{1}{2}}|E|_{H^1_{\mathfrak{p}}(\Omega_x)} \le c\,\|\partial_x E\|_{\Omega_x} \le c\,\sqrt{|\Omega_v|}\,\|f\|_\Omega, \tag{3.39}$$

where we used the fact that: $H^1_{\mathfrak{p}}(\Omega_x) \hookrightarrow L^\infty(\Omega_x)$. Using (3.33) and (3.9) leads to

$$\|e_E^*\|_\infty \|\partial_v e_f\|_\Omega \left\|e_f^*\right\|_\Omega \le \sqrt{|\Omega_v|}\sqrt{(2M+1)/2\pi}\,\|\partial_v e_f\|_\Omega \cdot \left\|e_f^*\right\|_\Omega^2. \tag{3.40}$$

Similarly, in view of (3.39), we can show that

$$2\left\|\Pi_M^0 E\right\|_\infty \|\partial_v e_f\|_\Omega \cdot \left\|e_f^*\right\|_\Omega \le 2c\,\sqrt{|\Omega_v|}\cdot\|f\|_\Omega \cdot \|\partial_v e_f\|_\Omega \cdot \left\|e_f^*\right\|_\Omega. \tag{3.41}$$

Then, we derive from (3.38) to (3.41) that

$$\begin{aligned} \mathcal{T}_2 \le{}& 2\sqrt{|\Omega_v|}\sqrt{(2M+1)/2\pi}\,\|\partial_v e_f\|_\Omega \cdot \left\|e_f^*\right\|_\Omega^2 + 2\Big(\max\{|a|,|b|\}\|\partial_x e_f\|_\Omega \\ &+ c\sqrt{|\Omega_v|}\cdot\|f\|_\Omega\cdot\|\partial_v e_f\|_\Omega\Big)\left\|e_f^*\right\|_\Omega. \end{aligned} \tag{3.42}$$

It remains to estimate $\mathcal{T}_3$. Using (2.18), (3.24), (3.33), and the triangle inequality leads to

$$\begin{aligned} \mathcal{T}_3 &= 2\left|\left((E_* - E)\partial_v f, e_f^*\right)_\Omega\right| = 2\left|\left((e_E^* - e_E)\partial_v f, e_f^*\right)_\Omega\right| \\ &\le 2\|\partial_v f\|_\infty\left(\|e_E^*\|_{\Omega_x} + \|e_E\|_{\Omega_x}\right)\left\|e_f^*\right\|_\Omega \\ &\le 2\sqrt{|\Omega_v|}\|\partial_v f\|_\infty\left\|e_f^*\right\|_\Omega^2 + 2\|\partial_v f\|_\infty\|e_E\|_{\Omega_x}\left\|e_f^*\right\|_\Omega. \end{aligned} \tag{3.43}$$

By the second equation in (2.1), we can see that if $f \in H^r_{\mathfrak{p}}(\Omega_x; L^2(\Omega_v))$, then $E \in H^{r+1}_{\mathfrak{p}}(\Omega_x)$. A combination of (3.3), (3.35), (3.37), (3.42) and (3.43) yields

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left\|e_f^*\right\|_\Omega^2 \le{}& 2\Big(\|\partial_t e_f\|_\Omega + \max\{|a|,|b|\}\|\partial_x e_f\|_\Omega + c\sqrt{|\Omega_v|}\cdot\|f\|_\Omega\cdot\|\partial_v e_f\|_\Omega \\ &+ \|\partial_v f\|_\infty \cdot \|e_E\|_{\Omega_x}\Big)\left\|e_f^*\right\|_\Omega + 2\,\sqrt{|\Omega_v|}\Big(\sqrt{M}\|\partial_v e_f\|_\Omega + \|\partial_v f\|_\infty\Big)\left\|e_f^*\right\|_\Omega^2, \end{aligned} \tag{3.44}$$

which implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\|e_f^*\right\|_\Omega \le \alpha(t)\left\|e_f^*\right\|_\Omega + \beta(t,M,N), \tag{3.45}$$

where

$$\begin{aligned} \alpha(t) :={}& \sqrt{\Omega_v}\left(cM^{3/2-r}\|f\|_{H^{r-1}_{\mathfrak{p}}(\Omega_x;H^1(\Omega_v))} + c\sqrt{M}N^{1-s}\|f\|_{L^2_{\mathfrak{p}}(\Omega_x;H^s(\Omega_v))} + 2\|\partial_v f\|_\infty\right) = \mathcal{O}(1), \\ \beta(t,M,N) :={}& \|\partial_t e_f\|_\Omega + \max\{|a|,|b|\}\|\partial_x e_f\|_\Omega + c\sqrt{|\Omega_v|}\cdot\|f\|_\Omega\cdot\|\partial_v e_f\|_\Omega + \|\partial_v f\|_\infty \cdot \|e_E\|_{\Omega_x} \\ \le{}& cM^{1-r}\|\partial_t f\|_{H^{r-1}_{\mathfrak{p}}(\Omega_x;L^2(\Omega_v))} + cN^{1-s}\|\partial_t f\|_{H^{s-1}\left(\Omega_v;L^2_{\mathfrak{p}}(\Omega_x)\right)} \\ &+ c\max\{|a|,|b|\}\left(M^{1-r}\|f\|_{H^r_{\mathfrak{p}}(\Omega_x;L^2(\Omega_v))} + N^{1-s}\|f\|_{H^{s-1}\left(\Omega_v;H^1_{\mathfrak{p}}(\Omega_x)\right)}\right) \\ &+ c\sqrt{b-a}\|f\|_\Omega\left(M^{1-r}\|f\|_{H^{r-1}_{\mathfrak{p}}(\Omega_x;H^1(\Omega_v))} + N^{1-s}\|f\|_{H^s\left(\Omega_v;L^2_{\mathfrak{p}}(\Omega_x)\right)}\right) \\ &+ c\|\partial_v f\|_\infty M^{-(r+1)}\|E\|_{H^{r+1}_{\mathfrak{p}}(\Omega_x)} \\ \approx{}& \mathcal{O}\left(M^{1-r} + N^{1-s}\right). \end{aligned}$$

Integrating (3.45) with respect to t and using $e_f^*(0) = f_*(x,v,0) - \Pi_M^N f_0 = 0$, we obtain

$$\left\|e_f^*(t)\right\|_\Omega \le \int_0^t \alpha(\tau)\left\|e_f^*(\tau)\right\|_\Omega \mathrm{d}\tau + \int_0^t \beta(s,M,N)\,\mathrm{d}s. \tag{3.46}$$

Finally, the assertion of the theorem follows from the Grönwall inequality. □

Remark 3.1. In [24], the authors proved that for any f which belongs to the Sobolev space $L^\infty\big((0,T];H^r(\Omega_x),H^s(\Omega_v)\big)$ and $\partial_t f \in L^2(0,T;H^{r-1}(\Omega_x;L^2(\Omega_v)))\cap L^2(0,T;H^{s-1}(\Omega_v;L^2(\Omega_x)))$ with $r \ge 2+\epsilon$, $s \ge \frac{7}{2}+\epsilon$, $\epsilon > 0$, the error bound is

$$\|f(\cdot,\cdot,t) - f_*(\cdot,\cdot,t)\|_\Omega \le c\big(M+N^2\big)\Big(M^{1-r+\epsilon} + N^{\frac{3}{2}-s+\epsilon}\Big).$$

However, with a weaker regularity assumption in Theorem 3.4, we derived

$$\|f(\cdot,\cdot,t) - f_*(\cdot,\cdot,t)\|_\Omega \le c\big(M^{1-r} + N^{1-s}\big),$$

for $r \ge \frac{3}{2}$ and integer $s \ge 2$, which significantly improved the estimate in [24]. Such an improvement is largely due to the use of a more suitable orthogonal projection in the v-direction together with a more delicate analysis.

4. Error estimates of the Fourier–Hermite spectral-Galerkin method

In this section, we consider the velocity variable on the whole line $\Omega_v = \mathbb{R}$ and conduct the error analysis of the Fourier–Hermite spectral-Galerkin method for the VP system. It is seen from Theorem 3.4 that the size of Ω_v appears in the error bounds. Such a dependence also occurred in the unbounded case, see [24], where it was assumed that the solution and its derivatives decay very fast to alleviate this factor. To overcome this, we shall adopt a different argument and derive some new inverse inequalities and approximation results on the unbounded domains. The estimates to be obtained improve the existing ones.

4.1. Hermite approximation results on $\mathbb{R}$

We now consider approximations by Hermite functions on $\Omega_v = \mathbb{R}$. To this end, the generalized Hermite function of degree n is defined by

$$\widehat{H}_n(v) = \frac{(-1)^n}{\pi^{\frac{1}{4}}\sqrt{2^n n!}}\mathrm{e}^{\frac{v^2}{2}}\,\frac{\mathrm{d}^n}{\mathrm{d}v^n}\mathrm{e}^{-v^2} = \frac{1}{\pi^{\frac{1}{4}}\sqrt{2^n n!}}\mathrm{e}^{-\frac{v^2}{2}}H_n(v), \quad v\in\mathbb{R}, \tag{4.1}$$

which is normalized so that

$$\int_{\mathbb{R}} \widehat{H}_n(v)\widehat{H}_m(v)\,\mathrm{d}v = \delta_{mn}. \tag{4.2}$$

Define the approximation space

$$V_N = \mathrm{span}\big\{\phi : \phi = \mathrm{e}^{-\frac{v^2}{2}}\psi,\ \forall\psi\in\mathcal{P}_N\big\}. \tag{4.3}$$

Consider the L^2-orthogonal projection $\hat{\Pi}_N : L^2(\mathbb{R}) \to V_N$ such that

$$\big(u - \hat{\Pi}_N u, \varphi\big)_{\mathbb{R}} = 0, \quad \forall\varphi\in V_N, \tag{4.4}$$

or equivalently,

$$\hat{\Pi}_N u(v) = \sum_{n=0}^N \hat{u}_n \widehat{H}_n(v) \quad \text{where} \quad \hat{u}_n = \int_{\mathbb{R}} u(v)\widehat{H}_n(v)\,\mathrm{d}v. \tag{4.5}$$

To characterize the regularity of the functions, we define the derivative operator $\hat{\partial}_v = \partial_v + v$ and introduce

$$H_A^s(\mathbb{R}) := \{u : u \in L^2(\mathbb{R}),\ \hat{\partial}_v^k u \in L^2(\mathbb{R}),\ 0 \le k \le s,\ k, s \in \mathbb{N}\},$$

equipped with the norm and semi-norm

$$\|u\|_{H_A^s(\mathbb{R})} = \left(\sum_{k=0}^{s} \|\hat{\partial}_v^k u\|_{L^2(\mathbb{R})}^2\right)^{\frac{1}{2}}, \quad |u|_{H_A^s(\mathbb{R})} = \|\hat{\partial}_v^s u\|_{L^2(\mathbb{R})}. \tag{4.6}$$

The Hermite function approximations under H^l-norm is stated in Theorem 7.14 from [28]: *if* $u \in H_A^s(\mathbb{R})$ *with* $0 \le l \le s \le N+1$, *then*

$$\|\hat{\Pi}_N u - u\|_{H^l(\mathbb{R})} \le c N^{\frac{l-s}{2}} |u|_{H_A^s(\mathbb{R})}, \quad l = 0, 1, 2, \tag{4.7}$$

where c *is a positive constant independent of* s, N *and* u.

The following estimate is useful in the error analysis whose proof is postponed to Appendix B.

Theorem 4.1. *Let* $\hat{\partial}_v := \partial_v + v$. *For any* $\hat{\partial}_v^s u \in L^2(\mathbb{R})$ *with* $1 \le s \le N+1$, *we have*

$$\|v\hat{\partial}_v^l(\hat{\Pi}_N u - u)\|_{L^2(\mathbb{R})} \le c\sqrt{\frac{(N-s+1)!}{(N-l)!}}\,|u|_{H_A^s(\mathbb{R})}, \quad 0 \le l \le s-1, \tag{4.8}$$

and

$$\|v\partial_v^l(\hat{\Pi}_N u - u)\|_{L^2(\mathbb{R})} \le c\sqrt{\frac{(N-s+1)!}{(N-l)!}}|u|_{H_A^s(\mathbb{R})}, \quad l = 0, 1. \tag{4.9}$$

In the error analysis, we also need the following L^∞-estimates.

Theorem 4.2. *For any* $\hat{\partial}_v^s u \in L^2(\mathbb{R})$ *with* $1 \le s \le N+1$,

$$\|\hat{\Pi}_N u - u\|_{L^\infty(\mathbb{R})} \le c N^{\frac{1}{4}-\frac{s}{2}} |u|_{H_A^s(\mathbb{R})}. \tag{4.10}$$

For any $u \in H^1(\mathbb{R}) \cap H_A^s(\mathbb{R})$ *with* $2 \le s \le N+1$, *we have*

$$\|\partial_v(\hat{\Pi}_N u - u)\|_{L^\infty(\mathbb{R})} \le c N^{\frac{3}{4}-\frac{s}{2}} |u|_{H_A^s(\mathbb{R})}. \tag{4.11}$$

Proof. The first estimate can be derived from (4.7) and the Sobolev inequality (C.1) straightforwardly. The proof of the second result is given in Appendix C. $\square$

To estimate the integration of the projection error, we introduce another Sobolev-type space. For any real $s \ge 0$, we define

$$H_B^s(\mathbb{R}) := \big\{u : \|u\|_{H_B^s(\mathbb{R})} < \infty\big\}, \tag{4.12}$$

equipped with the norm and semi-norm

$$\|u\|_{H_B^s(\mathbb{R})} = \left(\sum_{n=0}^{\infty} (1+n)^s \left|\hat{u}_n\right|^2\right)^{1/2}, |u|_{H_B^s(\mathbb{R})} = \left(\sum_{n=1}^{\infty} n^s \left|\hat{u}_n\right|^2\right)^{1/2}, \tag{4.13}$$

where $\{\hat{u}_n\}$ are the coefficients of Hermite function expansion for $u(v)$. There holds the following equivalence of the norms when s is an integer. We provide its proof in Appendix D.

Lemma 4.1. *For integer* $s \ge 0$, *we have the equivalence*

$$\|u\|_{H_A^s(\mathbb{R})} \sim \|u\|_{H_B^s(\mathbb{R})}. \tag{4.14}$$

On a finite interval, the imbedding $L^2 \hookrightarrow L^1$ is obvious, but it does not hold on $\mathbb{R}$. However, we can derive the inverse inequality (4.15), which, together with the approximation result (4.16), is critical in dealing with the integral terms of the VP system. It is noteworthy that the factor $O(N^{1/4})$ in (4.15) seems optimal (see Rem. 4.1 below).

Theorem 4.3. *For any $\phi \in V_N$ given in (4.3), we have*

$$\left|\int_{\mathbb{R}} \phi(v)\,\mathrm{d}v\right| \le \sqrt{3}\,(2N)^{\frac14}\|\phi\|_{L^2(\mathbb{R})}. \tag{4.15}$$

For any $u \in H^s_A(\mathbb{R})$ with integer $s \ge 1$, we have the following estimate for the orthogonal projection $\hat{\Pi}_N$ defined in (4.4):

$$\left|\int_{\mathbb{R}} \Big(\hat{\Pi}_N u - u\Big)(v)\,\mathrm{d}v\right| \le cN^{-\frac14}\|\hat{\partial}_v(\hat{\Pi}_N u - u)\|_{L^2(\mathbb{R})} \le cN^{\frac14-\frac{s}{2}}\|u\|_{H^s_A(\mathbb{R})}. \tag{4.16}$$

Proof. We first prove (4.15). For any $\phi \in V_N$, we have

$$\int_{\mathbb{R}} \phi(v)\,\mathrm{d}v = \sum_{n=0}^{N} \hat{\phi}_n \int_{\mathbb{R}} \widehat{H}_n(v)\,\mathrm{d}v = \sum_{n=0}^{[N/2]} \hat{\phi}_{2n} \int_{\mathbb{R}} \widehat{H}_{2n}(v)\,\mathrm{d}v, \tag{4.17}$$

where we noted that $\widehat{H}_{2n+1}(v)$ is an odd function. Recall that the identity (*cf.* [16], 7.373):

$$\int_{\mathbb{R}} \widehat{H}_{2n}(v)\,\mathrm{d}v = \sqrt[4]{4\pi}\cdot\frac{\sqrt{(2n)!}}{n!2^n}. \tag{4.18}$$

One verifies readily that the sequence $\left\{\frac{(2n)!\sqrt{\pi n}}{(n!)^2 2^{2n}}\right\}$ is monotonically increasing and $\frac{(2n)!\sqrt{\pi n}}{(n!)^2 2^{2n}} \to 1$, as $n \to \infty$. Thus we obtain

$$\frac{(2n)!}{(n!)^2 2^{2n}} \le \frac{1}{\sqrt{\pi}}\cdot\frac{1}{\sqrt{n}}, \quad n \ge 1. \tag{4.19}$$

Using the Cauchy–Schwarz inequality, we find from the above that

$$\begin{aligned}\left|\int_{\mathbb{R}} \phi(v)\,\mathrm{d}v\right| &\le \sqrt[4]{4\pi}\sum_{n=0}^{[N/2]} |\hat{\phi}_{2n}|\frac{\sqrt{(2n)!}}{n!2^n} \le \sqrt[4]{4\pi}\left(\sum_{n=0}^{[N/2]}\left|\hat{\phi}_{2n}\right|^2\right)^{1/2}\left(\sum_{n=0}^{[N/2]}\frac{(2n)!}{(n!)^2 2^{2n}}\right)^{1/2}\\ &\le \sqrt[4]{4\pi}\|\phi\|_{L^2(\mathbb{R})}\times\left(\frac{3}{\sqrt{\pi}}\sqrt{\frac{N}{2}}\right)^{1/2} = \sqrt{3}\,(2N)^{\frac14}\|\phi\|_{L^2(\mathbb{R})},\end{aligned}$$

where we used the fact

$$\sum_{n=0}^{N}\frac{(2n)!}{(n!)^2 2^{2n}} = (2N+1)\frac{(2N)!}{(N!)^2\,2^{2N}} \le \frac{3}{\sqrt{\pi}}\sqrt{N}.$$

This ends the proof of (4.15).

We now turn to the proof of (4.16). For notational simplicity, we denote $e_N = u - \hat{\Pi}_N u$, and obtain

$$\int_{\mathbb{R}} e_N(v)\,\mathrm{d}v = \sum_{n=N+1}^{\infty} \hat{u}_n \int_{\mathbb{R}} \widehat{H}_n(v)\,\mathrm{d}v = \sum_{n=[N/2]+1}^{\infty} \hat{u}_{2n} \int_{\mathbb{R}} \widehat{H}_{2n}(v)\,\mathrm{d}v.$$

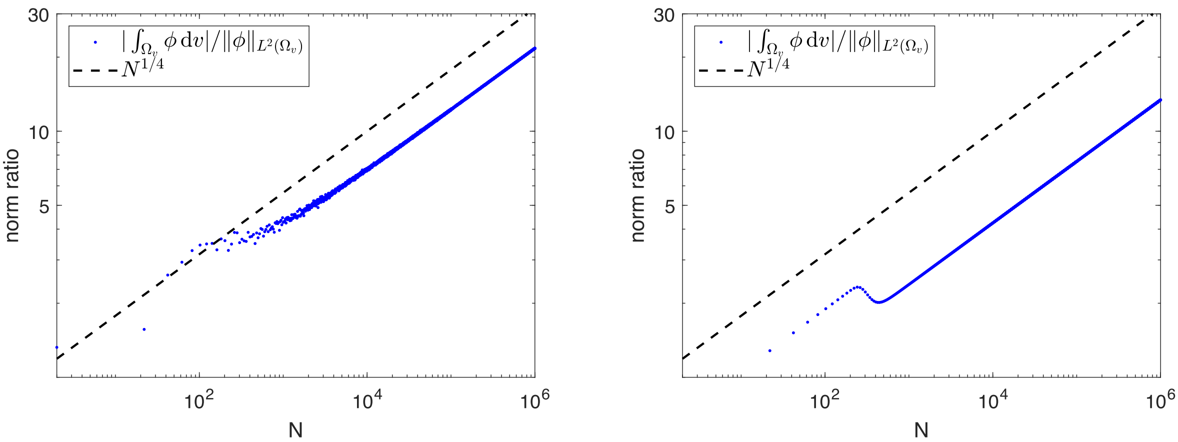

FIGURE 1. The ratio $|\int_{\Omega_v} \phi \,\mathrm{d}v|/\|\phi\|_{L^2(\Omega_v)}$ with the expansion coefficients of $\phi \in V_N$ in $[0,1]$ randomly generated by "`rand()`" (*left*) or "`normpdf()`" (*right*).

Using (4.18), (4.19) and the Cauchy–Schwarz inequality, we obtain from (4.13) and Lemma 4.1 that

$$
\begin{aligned}
\left|\int_{\mathbb{R}} e_N(v)\,\mathrm{d}v\right| &\le \sqrt[4]{4\pi} \sum_{n=[N/2]+1}^{\infty} |\hat{u}_{2n}| \frac{\sqrt{(2n)!}}{n!2^n} \\
&\le \sqrt{2}\Bigg(\sum_{n=[N/2]+1}^{\infty} n|\hat{u}_{2n}|^2\Bigg)^{1/2}\Bigg(\sum_{n=[N/2]+1}^{\infty} \frac{1}{n^{3/2}}\Bigg)^{1/2} \\
&\le c\,N^{-\frac{1}{4}} |e_N|_{H^1_B(\mathbb{R})} \le c\,N^{-\frac{1}{4}} \|e_N\|_{H^1_A(\mathbb{R})} \le cN^{-\frac{1}{4}} \|\hat{\partial}_v e_N\|_{L^2(\mathbb{R})},
\end{aligned}
\tag{4.20}
$$

where we used the bound

$$
\sum_{n=[N/2]+1}^{\infty} \frac{1}{n^{3/2}} \le \int_{[N/2]}^{\infty} \frac{1}{x^{3/2}}\,\mathrm{d}x \le cN^{-\frac{1}{2}}.
$$

Then the desired estimate (4.16) follows from the known bound of $\|\hat{\partial}_v e_N\|_{L^2(\mathbb{R})}$ given in Corollary 7.3 from [28]. $\square$

Remark 4.1. The order $O(N^{1/4})$ in (4.15) appears optimal. Indeed, there exist many $\phi \in V_N$ that can nearly attain this bound as we illustrated below. We examine the ratio $\mathcal{R}_N[\phi] := |\int_{\mathbb{R}} \phi \,\mathrm{d}v|/|\phi\|_{L^2(\mathbb{R})}$ by taking the expansion coefficients of $\phi \in V_N$ to be $\{\hat{\phi}_n = Aa_n\}_{n=0}^N$ with the vector of $a_n \in [0,1]$ randomly generated by `rand(N+1,1)` or `normpdf(`$\boldsymbol{x}$`)` (with $x_j = -5 + 10j/N, j = 0, 1, \cdots, N$) in Matlab. Note that the nonzero constant A is scaled by the ratio. In Figure 1, we depict the ratio $\mathcal{R}_N[\phi]$ and the reference line $N^{1/4}$ in log–log scale from which we observe $\mathcal{R}_N[\phi] \sim N^{1/4}$.

4.2. Fourier–Hermite approximation

Define the tensorial subspace $V_M^N := \mathcal{F}_M \times V_N$, and the orthogonal projection:

$$
\Pi_M^N := \Pi_M \circ \hat{\Pi}_N : H^1_{\mathtt{p}}\big(\Omega_x; H^1(\mathbb{R})\big) \to V_M^N.
$$

The Hermite function approximation results stated in the following two theorems will be directly used in the error analysis. As their derivations are very similar to those for the Legendre cases in Theorems 3.2 and 3.3, so we omit the details to avoid repetition.

Theorem 4.4. *Let $1 \le r \le M+1$ and $1 \le s \le N+1$. If $f \in H_{\mathfrak{p}}^{r-1}(\Omega_x; L^2(\mathbb{R})) \cap H_A^{s-1}(\mathbb{R}; L^2(\Omega_x))$,*

$$\|f - \Pi_M^N f\|_\Omega \le cM^{1-r}\|f\|_{H_{\mathfrak{p}}^{r-1}(\Omega_x;L^2(\mathbb{R}))} + cN^{\frac{1-s}{2}}\|f\|_{H_A^{s-1}(\mathbb{R};L^2(\Omega_x))}. \tag{4.21}$$

If $f \in H_{\mathfrak{p}}^{r}(\Omega_x; L^2(\mathbb{R})) \cap H_A^{s-1}(\mathbb{R}; H_{\mathfrak{p}}^1(\Omega_x))$, then

$$\|\partial_x(f - \Pi_M^N f)\|_\Omega \le cM^{1-r}\|f\|_{H_{\mathfrak{p}}^{r}(\Omega_x;L^2(\mathbb{R}))} + cN^{\frac{1-s}{2}}\|f\|_{H_A^{s-1}\left(\mathbb{R};H_{\mathfrak{p}}^1(\Omega_x)\right)}. \tag{4.22}$$

If $f \in H_{\mathfrak{p}}^{r-1}(\Omega_x; H^1(\mathbb{R})) \cap H_A^{s}(\mathbb{R}; L^2(\Omega_x))$, then

$$\|\partial_v(f - \Pi_M^N f)\|_\Omega \le cM^{1-r}\|f\|_{H_{\mathfrak{p}}^{r-1}(\Omega_x;H^1(\mathbb{R}))} + cN^{\frac{1-s}{2}}\|f\|_{H_A^{s}(\mathbb{R};L^2(\Omega_x))}, \tag{4.23}$$

where c is a positive constant independent of real r, integer s, M, N and f.

Theorem 4.5. *We have the following estimates. If $f \in H_{\mathfrak{p}}^{r}(\Omega_x; L^\infty(\mathbb{R})) \cap H_A^{s}(\mathbb{R}; H_{\mathfrak{p}}^{\frac{1}{2}+\epsilon}(\Omega_x))$ with $r > \frac{1}{2}$, integer $s \ge 1$, and $\epsilon > 0$, we have*

$$\begin{aligned}\|f - \Pi_M^N f\|_{L^\infty(\Omega)} \le\ & cM^{\frac{1}{2}-r}\|f\|_{H_{\mathfrak{p}}^{r}(\Omega_x;L^\infty(\mathbb{R}))} + cN^{\frac{1}{4}-\frac{s}{2}}\|f\|_{H_A^{s}(\mathbb{R};L^\infty(\Omega_x))} \\ & + cM^{-\epsilon}N^{\frac{1}{4}-\frac{s}{2}}\|f\|_{H_A^{s}\left(\mathbb{R};H_{\mathfrak{p}}^{\frac{1}{2}+\epsilon}(\Omega_x)\right)}.\end{aligned}$$

If $f \in H_{\mathfrak{p}}^{r}(\Omega_x; L^\infty(\mathbb{R})) \cap H_A^{s-1}(\mathbb{R}; H_{\mathfrak{p}}^{\frac{3}{2}+\epsilon}(\Omega_x))$ with $r > \frac{3}{2}$, integer $s \ge 2$, and $\epsilon > 0$, we have

$$\begin{aligned}\|\partial_x(f - \Pi_M^N f)\|_{L^\infty(\Omega)} \le\ & cM^{\frac{3}{2}-r}\|f\|_{H_{\mathfrak{p}}^{r}\left(\Omega_x;L^\infty(\mathbb{R})\right)} + cN^{\frac{3}{4}-\frac{s}{2}}\|f\|_{H_A^{s-1}\left(\mathbb{R};H_{\mathfrak{p}}^{\frac{3}{2}+\epsilon}(\Omega_x)\right)} \\ & + cM^{-\epsilon}N^{\frac{3}{4}-\frac{s}{2}}\|f\|_{H_A^{s-1}\left(\mathbb{R};H_{\mathfrak{p}}^{\frac{3}{2}+\epsilon}(\Omega_x)\right)}.\end{aligned}$$

If $f \in H_{\mathfrak{p}}^{r-1}(\Omega_x; H^1(\mathbb{R})) \cap H_A^{s}(\mathbb{R}; H_{\mathfrak{p}}^{\frac{1}{2}+\epsilon}(\Omega_x))$ with $r > \frac{3}{2}$, integer $s \ge 2$ and $\epsilon > 0$, we have

$$\begin{aligned}\|\partial_v(f - \Pi_M^N f)\|_{L^\infty(\Omega)} \le\ & cM^{\frac{3}{2}-r}\|f\|_{H_{\mathfrak{p}}^{r-1}(\Omega_x;H^1(\mathbb{R}))} + cN^{\frac{3}{4}-\frac{s}{2}}\|f\|_{H_A^{s}(\Omega_v;L^\infty(\Omega_x))} \\ & + cM^{-\epsilon}N^{\frac{3}{4}-\frac{s}{2}}\|f\|_{H_A^{s}\left(\mathbb{R};H_{\mathfrak{p}}^{\frac{1}{2}+\epsilon}(\Omega_x)\right)}.\end{aligned}$$

4.3. Error estimates

Like (3.24), we introduce similar notation for the Hermite case

$$e_f = f - \Pi_M^N f, \quad e_E = E - \Pi_M^0 E, \quad e_f^* = f_* - \Pi_M^N f, \quad e_E^* = E_* - \Pi_M^0 E, \tag{4.24}$$

where Π_M^0 is defined in (3.6) and $\Pi_M^N = \Pi_M \circ \hat{\Pi}_N$.

We are now ready to conduct the convergence analysis and present the main results as follows.

Theorem 4.6 (Error for Fourier–Hermite)**.** *Let f and f_* be the solutions of (2.8) and (2.18), respectively, on the domain $\Omega = \Omega_x \times \Omega_v$ with $\Omega_v = \mathbb{R}$. If*

$$\begin{aligned} f \in\ & L^\infty\big(0,T;H_{\mathfrak{p}}^r\big(\Omega_x;L^2(\mathbb{R})\big)\big) \cap L^2\big(0,T;H_{\mathfrak{p}}^{r-1}\big(\Omega_x;H_A^1(\mathbb{R})\big)\big) \cap L^2\big(0,T;H_{\mathfrak{p}}^{r-1}\big(\Omega_x;H^1(\mathbb{R})\big)\big) \\ & \cap L^\infty\big(0,T;H_A^s\big(\mathbb{R};L_{\mathfrak{p}}^2(\Omega_x)\big)\big) \cap L^2\big(0,T;H_A^{s-1}\big(\mathbb{R};H_{\mathfrak{p}}^1(\Omega_x)\big)\big),\end{aligned}$$

$\partial_t f \in L^2(0,T;H^{r-1}_{\mathfrak{p}}(\Omega_x;L^2(\mathbb{R}))) \cap L^2(0,T;H^{s-1}_A(\mathbb{R};L^2_{\mathfrak{p}}(\Omega_x))$ *and* $vf \in L^2(0,T;H^r_{\mathfrak{p}}(\Omega_x;L^2(\mathbb{R})))$, *then for any real* $r>1$ *and integer* $s \geq 2$, *we have*

$$\begin{aligned}\|f-f_*\|_\Omega \leq c\Big(M^{-r}\|f\|_{H^r_{\mathfrak{p}}(\Omega_x;L^2(\mathbb{R}))}+N^{-\frac{s}{2}}\|f\|_{H^s_A\left(\mathbb{R};L^2_{\mathfrak{p}}(\Omega_x)\right)}\Big)+c\Big(M^{1-r}\hat{B}^1_f+N^{\frac{1-s}{2}}\hat{B}^2_f\\ +M^{\frac{5}{2}-2r}N^{-\frac{1}{4}}\hat{B}^3_f+M^{\frac{1}{2}}N^{\frac{3}{4}-s}\hat{B}^4_f+N^{\frac{2-s}{2}}\hat{B}^5_f+M^{-(r+1)}\hat{B}^6_f\Big),\end{aligned} \tag{4.25}$$

where

$$\begin{aligned}\hat{B}^1_f &= \int_0^t \Big(\|\partial_t f\|_{H^{r-1}_{\mathfrak{p}}(\Omega_x;L^2(\mathbb{R}))}+\|vf\|_{H^r_{\mathfrak{p}}(\Omega_x;L^2(\mathbb{R}))}+\big\|\Pi^0_M E\big\|_{L^\infty(\Omega_x)}\|f\|_{H^{r-1}_{\mathfrak{p}}(\Omega_x;H^1(\mathbb{R}))}\\ &\quad + N^{-\frac{1}{4}}\|\partial_v f\|_{L^\infty(\Omega_x;L^2(\mathbb{R}))}\|f\|_{H^{r-1}_{\mathfrak{p}}\left(\Omega_x;H^1_A(\mathbb{R})\right)}\Big)\,\mathrm{e}^{\int_\tau^t \hat{\alpha}(\eta)\,\mathrm{d}\eta}\,\mathrm{d}\tau,\\ \hat{B}^2_f &= \int_0^t \Big(\|\partial_t f\|_{H^{s-1}_A\left(\mathbb{R};L^2_{\mathfrak{p}}(\Omega_x)\right)}+\Big(\big\|\Pi^0_M E\big\|_{L^\infty(\Omega_x)}+N^{-\frac{1}{4}}\|\partial_v f\|_{L^\infty(\Omega_x;L^2(\mathbb{R}))}\Big)\|f\|_{H^s_A\left(\mathbb{R};L^2_{\mathfrak{p}}(\Omega_x)\right)}\Big)\\ &\quad\times \mathrm{e}^{\int_\tau^t \hat{\alpha}(\eta)\,\mathrm{d}\eta}\,\mathrm{d}\tau,\\ \hat{B}^3_f &= \int_0^t \Big(\|f\|^2_{H^{r-1}_{\mathfrak{p}}\left(\Omega_x;H^1_A(\mathbb{R})\right)}+\|f\|^2_{H^{r-1}_{\mathfrak{p}}(\Omega_x;H^1(\mathbb{R}))}\Big)\,\mathrm{e}^{\int_\tau^t \hat{\alpha}(\eta)\,\mathrm{d}\eta}\,\mathrm{d}\tau,\\ \hat{B}^4_f &= \int_0^t \|f\|^2_{H^s_A\left(\mathbb{R};L^2_{\mathfrak{p}}(\Omega_x)\right)}\,\mathrm{e}^{\int_\tau^t \hat{\alpha}(\eta)\,\mathrm{d}\eta}\,\mathrm{d}\tau,\\ \hat{B}^5_f &= \int_0^t \|f\|_{H^{s-1}_A\left(\mathbb{R};H^1_{\mathfrak{p}}(\Omega_x)\right)}\,\mathrm{e}^{\int_\tau^t \hat{\alpha}(\eta)\,\mathrm{d}\eta}\,\mathrm{d}\tau,\\ \hat{B}^6_f &= \int_0^t \|\partial_v f\|_\infty \|E\|_{H^{r+1}_{\mathfrak{p}}(\Omega_x)},\end{aligned}$$

with

$$\begin{aligned}\hat{\alpha}(t) = \sqrt{3}\Big(M^{\frac{3}{2}-r}(2N)^{\frac{1}{4}}\|f\|_{H^{r-1}_{\mathfrak{p}}(\Omega_x;H^1(\mathbb{R}))}+\sqrt[4]{2}M^{\frac{1}{2}}N^{\frac{3}{4}-\frac{s}{2}}\|f\|_{H^s_A(\mathbb{R};L^2(\Omega_x))}\Big)\\ +\|\partial_v f\|_{L^\infty(\Omega_x;L^2(\mathbb{R}))}N^{\frac{1}{4}} = \mathcal{O}(\sqrt[4]{N}).\end{aligned}$$

Proof. Following the argument in Section 3.4, we also split the error into three terms

$$\frac{\mathrm{d}}{\mathrm{d}t}\big\|e^*_f\big\|^2_\Omega \leq \mathcal{T}_1+\mathcal{T}_2+\mathcal{T}_3, \tag{4.26}$$

where

$$\mathcal{T}_1 = 2\,|(\partial_t e_f, e^*_f)_\Omega|,\quad \mathcal{T}_2 = 2|(\boldsymbol{F}_*\cdot\nabla e_f, e^*_f)_\Omega|,\quad \mathcal{T}_3 = 2|((\boldsymbol{F}-\boldsymbol{F}_*)\cdot\nabla f, e^*_f)_\Omega|.$$

The upper bound of $\mathcal{T}_1$ can be obtained by using the Cauchy–Schwarz inequality

$$\mathcal{T}_1 \leq 2\|\partial_t e_f\|_\Omega\cdot\big\|e^*_f\big\|_\Omega, \tag{4.27}$$

similar to (3.37). However, the estimate of $\mathcal{T}_2$ is more involved for two reasons. Firstly, the interval Ω_v becomes infinite, so the estimates (3.33) involving the length $|\Omega_v|$ cannot be used. Moreover, the identity (3.30) is not valid. To obtain a meaningful upper bound of $\mathcal{T}_2$, the estimate of $\partial_x e^*_E$ is essential. Using Theorem 4.3, we derive

$$\begin{aligned}\left|\left(\int_{\mathbb{R}} e_f\,\mathrm{d}v, \partial_x e^*_E\right)_{\Omega_x}\right| = \left|\int_{\Omega_x}\int_{\mathbb{R}} e_f\,\mathrm{d}v\,\partial_x e^*_E\,\mathrm{d}x\right| \leq \left(\int_{\Omega_x}(\partial_x e^*_E)^2\,\mathrm{d}x\right)^{1/2}\left(\int_{\Omega_x}\left(\int_{\mathbb{R}} e_f\,\mathrm{d}v\right)^2\mathrm{d}x\right)^{1/2}\\ \leq cN^{-\frac{1}{4}}\big\|\hat{\partial}_v e_f\big\|_\Omega\cdot\|\partial_x e^*_E\|_{\Omega_x}.\end{aligned}$$

In a similar way, we also can derive that

$$\left|\left(\int_{\mathbb{R}} e_f^*\,\mathrm{d}v, \partial_x e_E^*\right)_{\Omega_x}\right| \le \sqrt{3}\,(2N)^{\frac{1}{4}}\left\|e_f^*\right\|_\Omega \cdot \|\partial_x e_E^*\|_{\Omega_x}.$$

Taking $\varphi = \partial_x e_E^*$ in (3.27) leads to

$$\begin{aligned}\|\partial_x e_E^*\|_{\Omega_x}^2 &\le \left|\left(\int_{\Omega_v} e_f^*\,\mathrm{d}v, \partial_x e_E^*\right)_{\Omega_x}\right| + \left|\left(\int_{\Omega_v} e_f\,\mathrm{d}v, \partial_x e_E^*\right)_{\Omega_x}\right| \\ &\le cN^{-\frac{1}{4}}\left\|\hat{\partial}_v e_f\right\|_\Omega \cdot \|\partial_x e_E^*\|_{\Omega_x} + \sqrt{3}(2N)^{\frac{1}{4}}\left\|e_f^*\right\|_\Omega \cdot \|\partial_x e_E^*\|_{\Omega_x},\end{aligned}$$

which yields

$$\|\partial_x e_E^*\|_{\Omega_x} \le cN^{-\frac{1}{4}}\left\|\hat{\partial}_v e_f\right\|_\Omega + \sqrt{3}(2N)^{\frac{1}{4}}\left\|e_f^*\right\|_\Omega. \tag{4.28}$$

Using the identity $E_* = e_E^* + \Pi_M^0 E$, we estimate $\mathcal{T}_2$ as follows

$$\begin{aligned}\mathcal{T}_2 &= 2|(F_* \cdot \nabla e_f, e_f^*)_\Omega| \le 2|(v\partial_x e_f, e_f^*)_\Omega| + 2|(E_*\partial_v e_f, e_f^*)_\Omega| \\ &\le 2\|v\partial_x e_f\|_\Omega \cdot \|e_f^*\|_\Omega + 2\|e_E^*\|_\infty \cdot \|\partial_v e_f\|_\Omega \cdot \|e_f^*\|_\Omega + 2\|\Pi_M^0 E\|_{L^\infty(\Omega_x)} \cdot \|\partial_v e_f\|_\Omega \cdot \|e_f^*\|_\Omega.\end{aligned} \tag{4.29}$$

Noting that $\|e_E^*\|_{\Omega_x} \le \|\partial_x e_E^*\|_{\Omega_x}$ and substituting the inequalities (3.9) and (4.28) into (4.29), leads to

$$\begin{aligned}\mathcal{T}_2 &= 2\Big(\|v\partial_x e_f\|_\Omega + M^{\frac{1}{2}}N^{-\frac{1}{4}}\|\partial_v e_f\|_\Omega \cdot \left\|\hat{\partial}_v e_f\right\|_\Omega + \left\|\Pi_M^0 E\right\|_{L^\infty(\Omega_x)} \cdot \|\partial_v e_f\|_\Omega\Big)\left\|e_f^*\right\|_\Omega \\ &\quad + 2\sqrt{3}\,M^{\frac{1}{2}}\,(2N)^{\frac{1}{4}}\|\partial_v e_f\|_\Omega \cdot \left\|e_f^*\right\|_\Omega^2.\end{aligned} \tag{4.30}$$

Different from the previous Legendre case, the estimate of $\|v\partial_x e_f\|$ whose factor v cannot be bounded by a constant any more in this case. Here, we used a tight bound that leads to a tradeoff by $\sqrt[4]{N}$. We next estimate $\mathcal{T}_3$. Using the triangle inequality, equations (2.3), (3.24) and (4.28), we obtain

$$\begin{aligned}\mathcal{T}_3 &= 2|((\boldsymbol{F} - \boldsymbol{F}_*) \cdot \nabla f, e_f^*)_\Omega| = 2|((E_* - E)\partial_v f, e_f^*)_\Omega| \\ &= 2|((e_E^* - e_E)\partial_v f, e_f^*)| \\ &\le 2\|\partial_v f\|_\infty(\|e_E^*\|_{\Omega_x} + \|e_E\|_{\Omega_x})\|e_f^*\|_\Omega \\ &\le 2\|\partial_v f\|_\infty(cN^{-\frac{1}{4}}\|\hat{\partial}_v e_f\|_\Omega + \sqrt{3}(2N)^{\frac{1}{4}}\|e_f^*\|_\Omega + \|e_E\|_{\Omega_x})\|e_f^*\|_\Omega,\end{aligned} \tag{4.31}$$

which brings about the factor $\sqrt[4]{N}$ in $\alpha(M,N,t)$. A combination of (4.27), (4.30) and (4.31) yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|e_f^*\|_\Omega &\le (\sqrt{3}M^{\frac{1}{2}}(2N)^{\frac{1}{4}}\|\partial_v e_f\|_\Omega + c\|\partial_v f\|_{L^\infty(\Omega_x;L^2(\mathbb{R}))}N^{\frac{1}{4}})\|e_f^*\|_\Omega + \|\partial_t e_f\|_\Omega + \|v\partial_x e_f\|_\Omega \\ &\quad + cM^{\frac{1}{2}}N^{-\frac{1}{4}}\|\partial_v e_f\|_\Omega \cdot \|\hat{\partial}_v e_f\|_\Omega + \|\Pi_M^0 E\|_{L^\infty(\Omega_x)} \cdot \|\partial_v e_f\|_\Omega \\ &\quad + cN^{-\frac{1}{4}}\|\partial_v f\|_{L^\infty(\Omega_x;L^2(\mathbb{R}))}\|\hat{\partial}_v e_f\|_\Omega + \|\partial_v f\|_\infty\|e_E\|_{\Omega_x}.\end{aligned}$$

By Theorems 4.1, 4.3, 4.4, and applying the Grönwall's inequality, one can obtain the desired bounds. □

Remark 4.2. Manzini *et al.* [24] analysed the Fourier–Hermite spectral Galerkin method under the regularity condition $f \in L^\infty(0,T;H^r(\Omega_x) \times H^s(\mathbb{R}))$ and $\partial_t f \in L^2(0,T;H^{r-1}(\Omega_x) \times L^2(\Omega_v)) \cap L^2(0,T;H^{s-1}(\Omega_v) \times L^2(\Omega_x))$with $r, s \ge 2 + \epsilon$, $\epsilon > 0$, with the error bound

$$\|f(t) - f_*(t)\|_\Omega \le C\big(|\Omega_v|, f\big)\big(M + \sqrt{N}\big)\big(M^{1-r+\epsilon} + N^{(1-s+2\epsilon)/2}\big),$$

where the constant $C(|\Omega_v|, f)$ depends on the exponential of the length of Ω_v and the regularity of f as in the Legendre case. The essential argument therein is that $\Omega_v = \mathbb{R}$ can be understood as a "finite" interval $(v_{\min}, v_{\max})$ if the solution (and its partial derivatives) decays sufficiently fast.

Note that in our error bound, the dependence becomes exponential of $\sqrt[4]{N}$ (which appears sharp), and we improve the convergence order under weaker regularity conditions:

$$\|f(t) - f_*(t)\|_\Omega \leq C\big(\sqrt[4]{N}, f\big)\big(M^{1-r} + N^{1-\frac{s}{2}}\big),$$

where $r \geq 1, s \geq 2$.

5. Full discretisation schemes and numerical results

In this section, we provide some numerical results to demonstrate the accuracy of the proposed spectral algorithms and simulate some interesting dynamics of the VP system.

5.1. Full discretisation scheme

Here, we employ the Crank–Nicolson scheme in time and proposed spectral-Galerkin method in space and show that full discretisation scheme enjoys the mass conservation.

Let τ be the time stepping size and $\tau_k = k\tau, k = 0, 1, \cdots, K := [T/\tau]$. Introduce the finite difference operator and grid average

$$\delta_\tau^+ f^k = \frac{f^{k+1} - f^k}{\tau}, \quad f^{k+\frac{1}{2}} = \frac{f^{k+1} + f^k}{2}\cdot$$

The Crank–Nicolson Fourier–Legendre (CNFL) scheme (resp., Crank–Nicolson Fourier–Hermite (CNFH) scheme) for (2.18) is to find $f_*^k \in \mathcal{F}_M \times \mathcal{P}_N^0$ (resp., $f_*^k \in \mathcal{F}_M \times V_N$) and $E_*^k \in \mathcal{F}_M$ such that

$$\begin{cases} \big(\delta_\tau^+ f_*^k, \psi\big)_\Omega + \Big(v\partial_x f_*^{k+\frac{1}{2}}, \psi\Big)_\Omega - \Big(E_*^{k+\frac{1}{2}}\partial_v f_*^{k+\frac{1}{2}}, \psi\Big)_\Omega = 0, & \forall\, \psi \in V_M^N, \\ \Big(\partial_x E_*^{k+\frac{1}{2}}, \varphi\Big)_{\Omega_x} = \Big(1 - \int_{\Omega_v} f_*^{k+\frac{1}{2}}\,\mathrm{d}v, \varphi\Big)_{\Omega_x}, & \forall\, \varphi \in \mathcal{F}_M, \\ f_*^0 = \Pi_M^N f_0, & (x, v) \in \Omega := \Omega_x \times \Omega_v, \end{cases} \tag{5.1}$$

which preserves the mass in the following sense.

Theorem 5.1. *Let f_*^{k+1} be the solution of (5.1). Then for all $k = 0, \cdots, K-1$, we have*

$$\big\|f_*^{k+1}\big\|_\Omega = \big\|f_*^k\big\|_\Omega = \cdots = \big\|f_*^0\big\|_\Omega. \tag{5.2}$$

Proof. In view of the periodic condition in x, we have

$$2\Big(v\partial_x f_*^{k+\frac{1}{2}}, f_*^{k+\frac{1}{2}}\Big)_\Omega = \int_{\Omega_x \times \Omega_v} v\partial_x\Big(f_*^{k+\frac{1}{2}}\Big)^2 \mathrm{d}x\,\mathrm{d}v = -\int_{\Omega_v} v\Big(f_*^{k+\frac{1}{2}}\Big)^2\Big|_{\partial\Omega_x} \mathrm{d}v = 0,$$

and similarly, in view of the homogeneous boundary in v, we obtain

$$2\Big(E_*^{k+\frac{1}{2}}\partial_v f_*^{k+\frac{1}{2}}, f_*^{k+\frac{1}{2}}\Big)_\Omega = \int_{\Omega_x} E_*^{k+\frac{1}{2}}\Big(f_*^{k+\frac{1}{2}}\Big)^2\Big|_{\partial\Omega_v} \mathrm{d}x = 0.$$

Thus, taking $\psi = 2f_*^{k+\frac{1}{2}}$ in the first equation of (5.1) leads to

$$2\Big(\delta_\tau^+ f_*^k, f_*^{k+\frac{1}{2}}\Big)_\Omega = \frac{1}{\tau}\Big(\big\|f_*^{k+1}(\cdot,\cdot)\big\|_\Omega^2 - \big\|f_*^k(\cdot,\cdot)\big\|_\Omega^2\Big) = 0,$$

which implies the desired property (5.2). □

It is seen that the scheme (5.1) is implicit, so we use the Newton iteration method to solve the resulting nonlinear systems in the following numerical tests.

5.2. Numerical results

For the accuracy test, we consider the VP system

$$\begin{cases} \partial_t f + v\partial_x f - E\partial_v f = g & \text{in } \Omega_x \times \mathbb{R}, \\ \partial_x E = 1 - \rho \ \text{ with } \ \rho = \int_{\Omega_v} f \, \mathrm{d}v & \text{in } \Omega_x, \end{cases} \tag{5.3}$$

with the exact solution

$$f(x, v, t) = \frac{1}{\alpha}(1 + \epsilon \cos(\omega x + t))|v|^s \mathrm{e}^{-\frac{v^2}{2}}, \quad \alpha := 2^{\frac{s+1}{2}}\Gamma\Big(\frac{s+1}{2}\Big), \quad s \geq 0, \tag{5.4}$$

where ϵ, ω are given constants. From direct calculation, we find the corresponding source term is

$$\begin{aligned} g(x, v, t) = & -\frac{\epsilon}{\alpha}(1 + \omega v)\sin(\omega x + t)\,|v|^s\,\mathrm{e}^{-\frac{v^2}{2}} + \frac{\epsilon}{\alpha}\omega\sin(\omega x + t)(1 + \epsilon\cos(\omega x + t))\,\mathrm{e}^{-\frac{v^2}{2}} \\ & \times \big(s|v|^{s-1}\mathrm{sign}(v) - |v|^s\big). \end{aligned} \tag{5.5}$$

Accordingly, the initial distribution function is

$$f_0(x, v) = \frac{1}{\alpha}(1 + \epsilon\cos(\omega x))\,|v|^s\,\mathrm{e}^{-\frac{v^2}{2}}. \tag{5.6}$$

As a matter of fact, it can serve as a multipurpose example to test the accuracy of the scheme and simulate the dynamics. More precisely, a road-map to conduct the numerical tests is as follows:

(i) **Case-I:** accuracy test**.** We apply the CNFL/CNFH scheme (5.1) to (5.3) with smooth or non-smooth exact solutions given in (5.4). Here, we choose $s = 0$ and $s = 1, 2.1, 3$, respectively. For the CNFL method, we extract from the exact solution the boundary conditions at $v = a, b$.

(ii) **Case-II:** Landau damping**.** Landau damping is a classical kinetic effect in plasmas, due to the resonance of particles with an initial wave perturbation. To simulate such dynamics, we set $g \equiv 0$ in (5.3), and take the initial distribution function to be

$$f_0(x, v) = \frac{1}{\sqrt{2\pi}}(1 + \epsilon\cos(\omega x))\,\mathrm{e}^{-\frac{v^2}{2}}, \tag{5.7}$$

as in [9, 13, 23]. In this case, there is no exact solution, and for the CNFL method, we choose suitable a, b, and impose homogeneous Dirichlet boundary conditions at $v = a, b$.

(iii) **Case-III:** two-stream instability**.** Two-stream instability is a common phenomenon in plasma physics. It is therefore of interest to numerically study this phenomenon *via* the proposed numerical scheme. We use a setting similar to that in Case-II, but take the following initial distribution function:

$$f_0(x, v) = \frac{1}{\sqrt{2\pi}}(1 + \epsilon\cos(\omega x))\,v^2\,\mathrm{e}^{-\frac{v^2}{2}}. \tag{5.8}$$

For clarity, we report the numerical results for the above three cases obtained by the CNFL and CNFH schemes separately in the following two subsections.

5.2.1. *Numerical results for the CNFL scheme*

As the first example for the accuracy test, we consider the VP problem (5.3)–(5.6) with $s = 0$, and take $\epsilon = 1, \omega = 1$, $\Omega_x = (0, 2\pi)$, $\Omega_v = (-8, 8)$. To verify the second-order convergence in time, we choose $N = 50$ and $M = 6$ (note: $2M + 1$ Fourier modes in total) to ensure the spectral accuracy in space. In Table 1, we tabulate the discrete L^∞- and L^2-errors and order of convergence various τ at $t = 1$, which clearly indicates a second-order convergence.

Table 1. L^∞- and L^2-errors and convergence in time (CNFL).

τ	L^∞-error	Order	L^2-error	Order
1/2	1.397e-2	–	3.240e-2	–
1/4	3.429e-3	2.0263	8.218e-3	1.9793
1/8	8.649e-4	1.9871	2.064e-3	1.9936
1/16	2.167e-4	1.9970	5.165e-4	1.9983
1/32	5.419e-5	1.9993	1.292e-4	1.9996
1/64	1.355e-5	1.9999	3.229e-5	1.9999

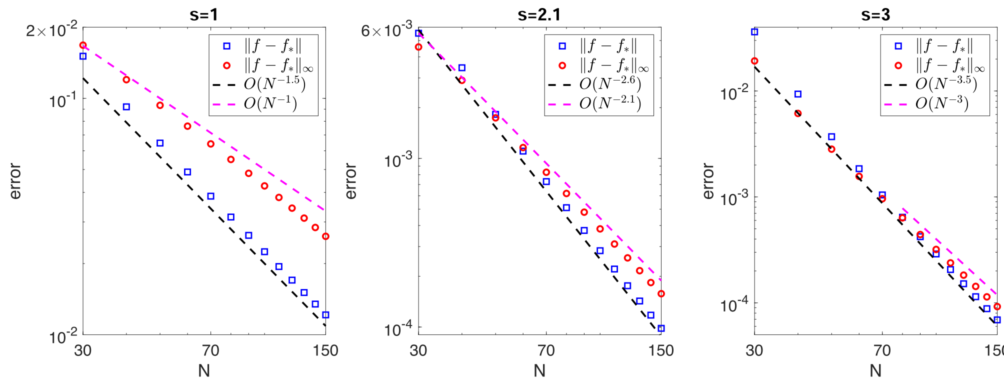

Figure 2. The convergence rate of the CNFL solution on the direction of v. We take (5.4) as the exact solution with $s = 1, 2.1, 3$, $\epsilon = \omega = 1$ and $\tau = 0.005$.

In the second test, we consider the VP system (5.3) with $s = 1, 2.1, 3$ in the exact solution (which has limited regularity). To test the convergence rate on direction of v, we take $\epsilon = \omega = 1, \tau = 0.005, \Omega_x = (0, 2\pi)$, $\Omega_v = (-8, 8)$ and fix $2M+1 = 13$ Fourier modes, but run up $T = 0.1$. In Figure 2, it shows that the convergence orders in L^2 norm, L^∞ norm are $s + \frac{1}{2}$ and s, respectively, and the convergence behaviours are consistent with the Legendre approximation results for such functions with limited regularity (see *e.g.*, [20]).

With above accuracy tests, we are now ready to simulate the Landau damping in Case-II, where we set $g \equiv 0$ in (5.3) and take the initial condition (5.7). Though we do not know the exact solution in this case, we can verify the conservation in Theorem 5.1. Here, we take $\epsilon = 0.01, \omega = 0.5, \Omega_x = (0, 4\pi), \Omega_v = (-8, 8)$, and choose the discretisation parameters in the scheme (5.1) to be $N = 50, M = 25, \tau = 0.005$. In Figure 3 (left), we plot the difference $\|f_*^k\|_\Omega - \|f^0\|_\Omega$ (but in the discrete L^2-norms) for $t \le T = 20$. We see that the property (5.2) is well-preserved in the sense that there is no error in time and it is spectrally accurate in space. In Figure 3 (right), we record the evolution of the numerical electric field E_*^k in discrete L^1-, L^2- and L^∞-norms, where the dashed line has a slope: $\nu = -0.1533$, so the electric filed in such norms has an exponential decay $O(e^{\nu t})$ as observed numerically in [3, 13, 14].

In the two-stream instability test in Case-III, we take $g = 0$ and choose the initial function to be (5.8). It is known that the VP system with the such initial input will develop some instability vortex structures as time evolves [3, 9]. To demonstrate this, we plot the distribution function at time $t = 0, 10, 15, 20$ in Figure 4. Indeed, we observe the same instability pattern for slightly large t as shown in these literature.

5.2.2. *Numerical results for the CNFH scheme*

We next conduct similar numerical tests for the CNFH scheme as for the CHFL scheme. In this case, $\Omega_v = \mathbb{R}$. As before, we first test the accuracy and consider the system (5.3) having the exact solution (5.4) with $s = 0$

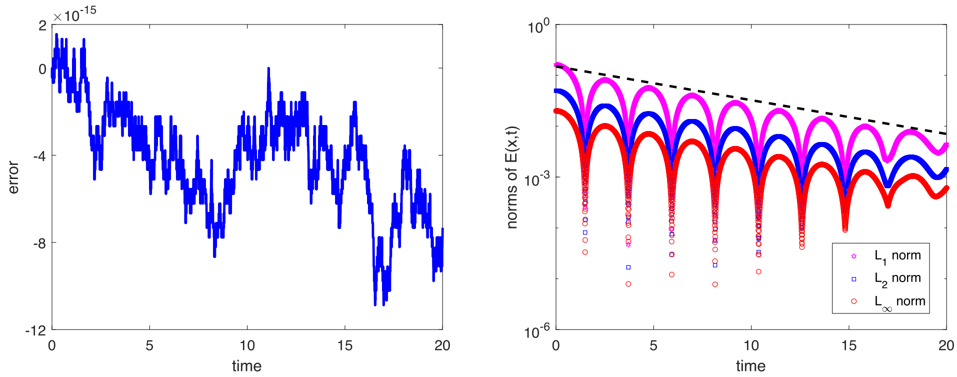

FIGURE 3. Weak Landau damping of the initial distribution (5.7) with $\epsilon = 0.01$, $\omega = 0.5$ and $\tau = 0.005$. *Left*: time evolution of the efforts between the discrete L^2-norm of the numerical distribution f_*^k and initial distribution f^0. *Right*: time evolution of the numerical electric field E in discrete L^1-, L^2- and L^∞-norms.

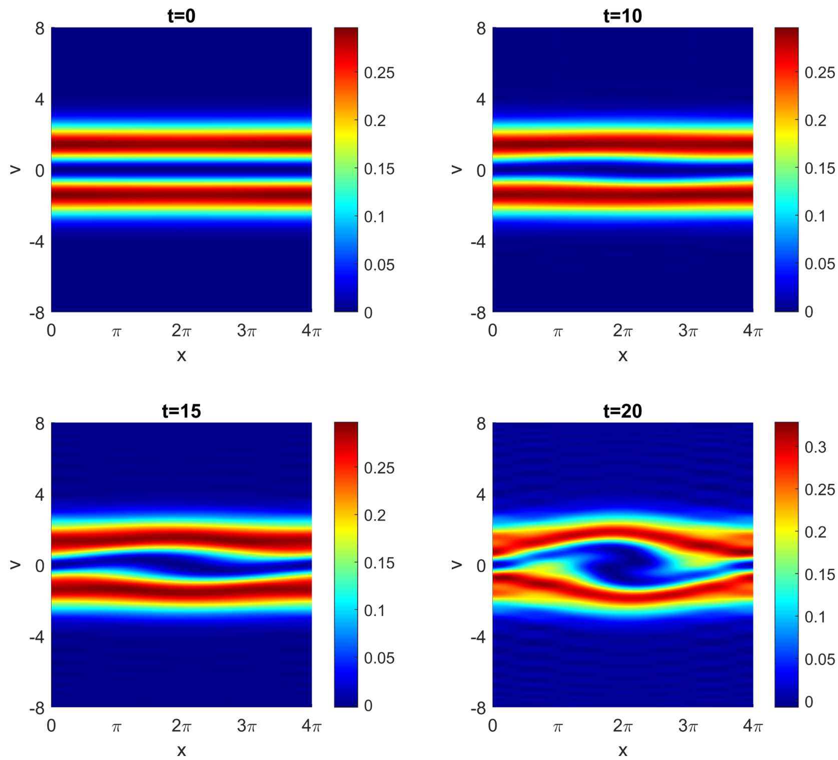

FIGURE 4. Evolution of the distribution function f for time $t = 0, 10, 15, 20$.

TABLE 2. L^∞- and L^2-errors and convergence in time (CNFH).

τ	L^∞-error	Order	L^2-error	Order
1/2	1.422e-2	–	1.288e-2	–
1/4	3.496e-3	2.0242	3.268e-3	1.9791
1/8	8.677e-4	2.0102	8.205e-4	1.9936
1/16	2.166e-4	2.0024	2.054e-4	1.9983
1/32	5.412e-5	2.0006	5.135e-5	1.9996
1/64	1.353e-5	2.0001	1.284e-5	1.9999

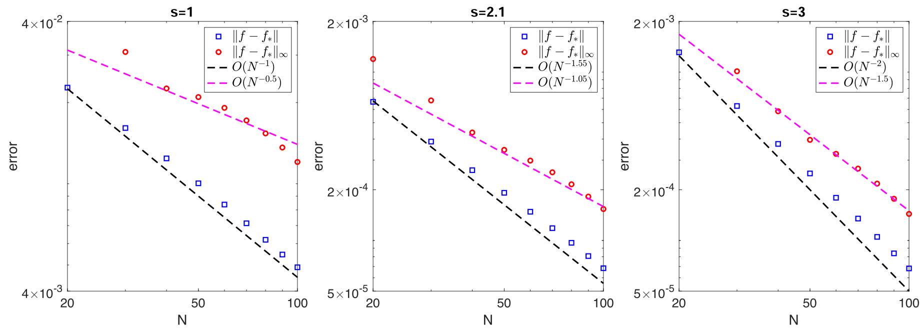

FIGURE 5. The convergence rate of the CNFH solution on the direction of v. We choose (5.6) as the initial value with $\epsilon = 1, \omega = 1, \tau = 0.005$. The cases that $s = 1$ (*left*), $s = 2.1$ (*middle*) and $s = 3$ (*right*) are considered.

and $\epsilon = \omega = 1$. As the solution is smooth, we take $N = 10$ and $M = 6$. In Table 2, we tabulate the discrete L^∞- and L^2-errors and order of convergence various τ at $t = 1$, which clearly indicates a second-order convergence.

Similar to the second test in the CNFL method, we choose $s = 1, 2.1, 3$ in the system (5.3) with (5.6) and (5.5) to test the convergence order on the direction v by the Hermite function approximation. Here, we take $\epsilon = \omega = 1, \tau = 0.005, \Omega_x = (0, 2\pi)$. From Figure 5, we can see that the order of convergence in L^2 norm and L^∞ norm at $t = 0.1$ are $\frac{s+1}{2}$ and $\frac{s}{2}$, respectively. As expected, they are half of orders for the Legendre approximation as shown in Figure 2.

In the third test, we consider the Landau damping in Case-II, and follow the same setting as for the CNFL scheme. In Figure 6 (left), we plot the difference $\|f_*^k\|_\Omega - \|f^0\|_\Omega$ for $t \le T = 20$. We see that the property (5.2) is well-preserved in the sense that there is no error in time and it is spectrally accurate in space. In Figure 6 (right), we record the evolution of the numerical electric field E_*^k in discrete L^1-, L^2- and L^∞-norms, where the dashed line has a slope: $O(\mathrm{e}^{\nu t}), \nu = -0.1533$, but the norm of E_*^k grows after $t = 13$. Although the CNFH approximation preserves the L^2 norm of f_*^k (see Fig. 6 (left)), it cannot ensure the same exponential decay rate of the electric field for long-time simulation.

Finally, we demonstrate the two-stream instability phenomenon of the distribution function f in Case-III. As with the last test for the CNFL scheme, we choose (5.8) with $\epsilon = 0.01$ and $\omega = 0.5$ as the initial distribution function. The computation domain is $\Omega = (0, 4\pi) \times \mathbb{R}$. We take $N = 51, M = 25$ and $\tau = 0.01$ in the scheme (5.1). Similar to the results obtained by the CNFL method, the CNFH method can also capture the two-stream instability vortex structure when t is large as shown in Figure 7.

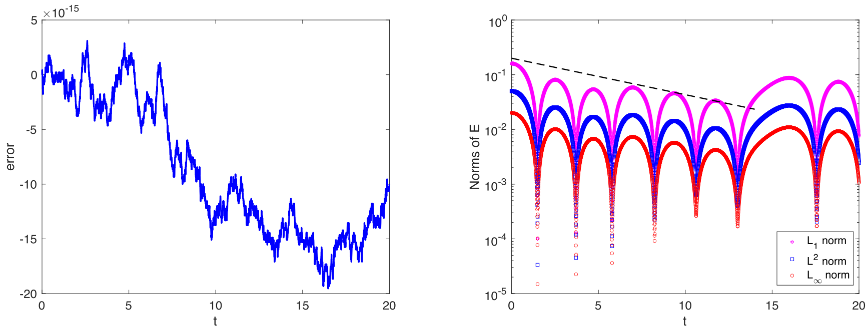

FIGURE 6. Weak Landau damping of the initial distribution (5.6) solved by the CNFH method with $s = 0$, $\epsilon = 0.01$, $\omega = 0.5$ and $\tau = 0.005$. *Left*: time evolution of the efforts between the discrete L^2-norm of the numerical distribution f_*^k and initial distribution f^0. *Right*: time evolution of the numerical electric field E in discrete L^1-, L^2- and L^∞-norms.

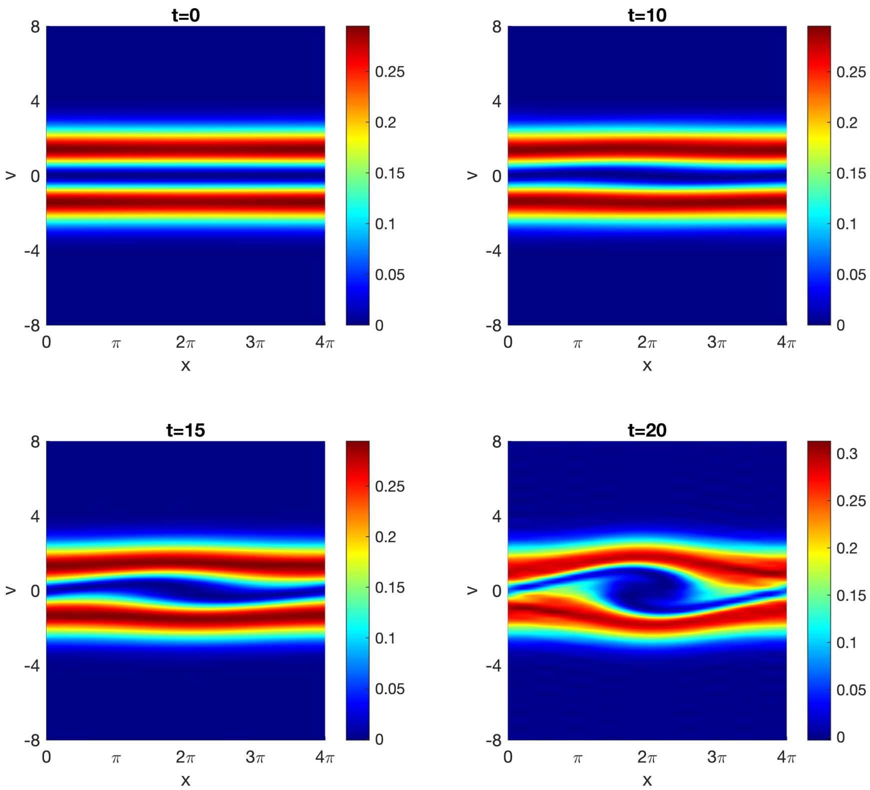

FIGURE 7. Evolution of the distribution function f for time $t = 0, 10, 15, 20$, which is obtained by the CNFH method.

5.3. Concluding remarks

In summary, we provided some improved error estimates for the spectral approximations of the Vlasov–Poisson system with the velocity variable both in a finite interval or on the whole line. The arguments and main tools can be extended to the analysis of spectral methods for the general dD-dV Vlasov–Poisson system.

Although we did not analyse the full discrete scheme, it is not difficult to mimic the analysis of the semi-discrete scheme to obtain the estimates for the Crank–Nicolson discretisation in time. It is noteworthy that we analyzed herein the symmetric Hermite function approximation in the velocity variable which conserves the mass and has the non-weighted L^2-error estimates. In fact, it is advantageous to use the asymmetric Hermite functions (see *e.g.*, [4,13,18]) that lead to simultaneous conservation of mass, momentum and total energy. However, the error estimates are only available in certain weighted L^2_ω-spaces. It appears open to obtain the usual non-weighted L^2-estimates.

Appendix A. The proof of Lemma 3.5

In order to prove Lemma 3.5, we first introduce the L^2-orthogonal projection $\pi_N : L^2(\Lambda) \to \mathcal{P}_N$ defined by

$$(\pi_N u - u, \varphi)_\Lambda = 0, \quad \forall\, \varphi \in \mathcal{P}_N. \tag{A.1}$$

Then, the corresponding approximation result ([28], Sect. 3.5) is: for any $u \in H^s(\Lambda)$ with integer $0 \le s \le N+1$,

$$\|\pi_N u - u\|_{L^2(\Lambda)} \le cN^{-s}|u|_{H^s(\Lambda)}, \tag{A.2}$$

and for $1 \le s \le N+1$,

$$\left\|(\pi_N u - u)'\right\|_{L^2(\Lambda)} \le cN^{\frac{3}{2}-s}|u|_{H^s(\Lambda)}, \tag{A.3}$$

where c is a generic positive constant independent of u and N. In Section 3.5 of [28], the authors also gave the error estimates when $s > N+1$, but here $s \le N+1$ is considered.

Recall the Sobolev inequality (see *e.g.*, [1]): for any $u \in H^1(\Lambda)$, we have

$$\|u\|_{L^\infty(\Lambda)} \le 2\|u\|^{\frac{1}{2}}_{H^1(\Lambda)}\|u\|^{\frac{1}{2}}_{L^2(\Lambda)}. \tag{A.4}$$

From (A.4), we obtain

$$\begin{aligned}\left\|\partial_\xi\left(\pi_N^{1,0}u - u\right)\right\|_{L^\infty(\Lambda)} &\le 2\left\|\partial_\xi\left(\pi_N^{1,0}u - u\right)\right\|^{\frac{1}{2}}_{H^1(\Lambda)}\left\|\partial_\xi\left(\pi_N^{1,0}u - u\right)\right\|^{\frac{1}{2}}_{L^2(\Lambda)}\\ &\le 2\left(\left\|\partial_\xi^2\left(\pi_N^{1,0}u - u\right)\right\|^2_{L^2(\Lambda)} + \left\|\partial_\xi\left(\pi_N^{1,0}u - u\right)\right\|^2_{L^2(\Lambda)}\right)^{\frac{1}{4}}\left\|\partial_\xi\left(\pi_N^{1,0}u - u\right)\right\|^{\frac{1}{2}}_{L^2(\Lambda)}.\end{aligned} \tag{A.5}$$

In view of Lemma 3.4, it suffices to estimate $\|\partial_\xi^2(\pi_N^{1,0}u - u)\|_{L^2(\Lambda)}$. For this purpose, it is essential to show that

$$\partial_\xi \pi_N^{1,0}u(\xi) = \pi_{N-1}\partial_\xi u(\xi), \quad \forall\, u \in H_0^1(\Lambda). \tag{A.6}$$

To derive this property, we define

$$\Phi_n(\xi) = P_{n+2}(\xi) - P_n(\xi), \quad n \in \mathbb{N}. \tag{A.7}$$

Using the derivative recurrence relations of Legendre polynomials ([28], (3.176a)):

$$(2n+1)P_n(\xi) = P'_{n+1}(\xi) - P'_{n-1}(\xi), \quad n \ge 1,$$

and

$$(1-\xi^2)P'_n(\xi) = \frac{n(n+1)}{(2n+1)}(P_{n-1}(\xi) - P_{n+1}(\xi)) = \frac{1}{2}(n+1)(1-\xi^2)P^{(1,1)}_{n-1}(\xi),$$

we find

$$\Phi_n(\xi) = (2n+3)\int_{-1}^{\xi} P_{n+1}(\eta)\,\mathrm{d}\eta = a_n(1-\xi^2)P_n^{(1,1)}(\xi), \quad a_n = -\frac{(2n+3)}{2(n+1)}. \tag{A.8}$$

By the orthogonality of Jacobi polynomials $\{P_n^{(1,1)}\}$, we have the orthogonality:

$$\int_{-1}^{1} \Phi_n(\xi)\Phi_m(\xi)\big(1-\xi^2\big)^{-1}\,\mathrm{d}\xi = \gamma_n\delta_{nm}, \quad \gamma_n = \frac{2(2n+3)}{(n+2)(n+1)}.$$

Note that $H_0^1(\Lambda) \subseteq L^2_{\omega^{-1}}(\Lambda)$ (*cf.* [28], (B.40)). For any $u \in H_0^1(\Lambda)$, we write

$$u(\xi) = \sum_{n=0}^{\infty} \hat{u}_n \Phi_n(\xi), \quad \hat{u}_n = \frac{1}{\gamma_n}\int_{-1}^{1} u(\xi)\Phi_n(\xi)\big(1-\xi^2\big)^{-1}\,\mathrm{d}\xi. \tag{A.9}$$

We can expand $\partial_\xi u$ in series of Legendre polynomials:

$$\partial_\xi u(\xi) = \sum_{n=0}^{\infty} \hat{u}_n^{(1)} P_n(\xi), \quad \hat{u}_n^{(1)} = \frac{2n+1}{2}\int_{-1}^{1} \partial_\xi u(\xi) P_n(\xi)\,\mathrm{d}\xi.$$

Integrating it on both sides and using (A.8), leads to

$$\begin{aligned} u(\xi) &= \sum_{n=0}^{\infty} \hat{u}_n^{(1)} \int_{-1}^{\xi} P_n(\eta)\,\mathrm{d}\eta = (1+\xi)\hat{u}_0^{(1)} + \sum_{n=0}^{\infty} \hat{u}_{n+1}^{(1)} \int_{-1}^{\xi} P_{n+1}(\eta)\,\mathrm{d}\eta \\ &= (1+\xi)\hat{u}_0^{(1)} + \sum_{n=0}^{\infty} \frac{\hat{u}_{n+1}^{(1)}}{2n+3}\Phi_n(\xi), \end{aligned} \tag{A.10}$$

which, together with the fact: $u(\pm 1) = 0$ and (A.10), implies

$$\hat{u}_0^{(1)} = 0; \quad \hat{u}_n^{(1)} = (2n+1)\hat{u}_{n-1}, \ \ n \geq 1.$$

As a result, we can write

$$\partial_\xi u(\xi) = \sum_{n=1}^{\infty} \hat{u}_n^{(1)} P_n(\xi) = \sum_{n=1}^{\infty} (2n+1)\hat{u}_{n-1} P_n(\xi) = \sum_{n=0}^{\infty} (2n+3)\hat{u}_n P_{n+1}(\xi). \tag{A.11}$$

Since $\mathcal{P}_N^0 = \operatorname{span}\{\Phi_n : 0 \leq n \leq N-2\}$, we infer from (A.9) that

$$\pi_N^{1,0} u(\xi) = \sum_{n=0}^{N-2} \hat{u}_n \Phi_n(\xi). \tag{A.12}$$

By (A.8) and (A.11), (A.12), we have that for any $u \in H_0^1(\Lambda)$,

$$\partial_\xi \pi_N^{1,0} u(\xi) = \sum_{n=0}^{N-2} \hat{u}_n \Phi_n'(\xi) = \sum_{n=0}^{N-2} (2n+3)\hat{u}_n P_{n+1}(\xi) = \pi_{N-1}\partial_\xi u(\xi). \tag{A.13}$$

Thus, from (A.3), we conclude that for integer $s \geq 2$,

$$\left\| \partial_\xi^2 \Big(\pi_N^{1,0} u - u \Big) \right\|_{L^2(\Lambda)} = \|\partial_\xi(\pi_{N-1}\partial_\xi u - \partial_\xi u)\|_{L^2(\Lambda)} \leq c N^{\frac{5}{2}-s} |u|_{H^s(\Lambda)}. \tag{A.14}$$

Hence, the estimate (3.16) follows directly from Lemma 3.4, (A.5) and (A.14).

Appendix B. The proof of Theorem 4.1

We first introduce the Hermite-weighted Sobolev space. Denote $\omega := \mathrm{e}^{-v^2}$ and define

$$H^s_\omega(\mathbb{R}) := \big\{u : \partial_v^k u \in L^2_\omega(\mathbb{R}),\ \ k = 0, 1, \cdots, s\big\},$$

with integer $s \geq 0$, which is equipped with the norm and semi-norm

$$\|u\|_{H^s_\omega(\mathbb{R})} := \left(\sum_{k=0}^{s} \big\|\partial_v^k u\big\|^2_{L^2_\omega(\mathbb{R})}\right)^{\frac{1}{2}}, \quad |u|_{H^s_\omega(\mathbb{R})} := \|\partial_v^s u\|_{L^2_\omega(\mathbb{R})}.$$

We then give a lemma that will be used in this proof.

Lemma B.1. *Let $\omega(v) = \mathrm{e}^{-v^2}$. Then for any $u \in H^2_\omega(\mathbb{R})$, we have*

$$\big\|v^2 u\big\|_{L^2_\omega(\mathbb{R})} \leq c\|u\|_{H^2_\omega(\mathbb{R})}. \tag{B.1}$$

Proof. By Lemma B.6 of [28], there has

$$\|vu\|_{L^2_\omega(\mathbb{R})} \leq \|u\|_{H^1_\omega(\mathbb{R})}, \quad \|v\partial_v u\|_{L^2_\omega(\mathbb{R})} \leq c\|u\|_{H^2_\omega(\mathbb{R})}. \tag{B.2}$$

Applying integration by parts and the Schwarz inequality yields,

$$\begin{aligned} \int_{-\infty}^{\infty} v^3 u^2(v)\omega(v)\,\mathrm{d}v &= \frac{1}{2}\int_{-\infty}^{\infty} \partial_v\big(v^2 u^2(v)\big)\omega(v)\,\mathrm{d}v \\ &\leq \int_{-\infty}^{\infty} v u^2(v)\omega(v)\,\mathrm{d}v + \|vu(v)\|_{L^2_\omega(\mathbb{R})} \cdot \|v\partial_v u(v)\|_{L^2_\omega(\mathbb{R})}. \end{aligned} \tag{B.3}$$

To demonstrate the integration above on the left is bounded, one only needs to estimate the first term on the right side of (B.3). Using the integration by parts and the Schwarz inequality again, it is not hard to see that

$$\int_{-\infty}^{\infty} v u^2(v)\omega(v)\,\mathrm{d}v \leq \|u\|_{L^2_\omega(\mathbb{R})}|u|_{H^1_\omega(\mathbb{R})}.$$

Thus,

$$\int_{-\infty}^{\infty} v^3 u^2(v)\omega(v)\,\mathrm{d}v \leq \|u\|_{L^2_\omega(\mathbb{R})}|u|_{H^1_\omega(\mathbb{R})} + \|vu(v)\|_{L^2_\omega(\mathbb{R})} \cdot \|v\partial_v u\|_{L^2_\omega(\mathbb{R})},$$

which indicates that $v^3u^2(v)\omega(v) \to 0$ as $v \to +\infty$. Therefore, one has

$$\begin{aligned} \int_{-\infty}^{\infty} \big(v^2 u(v)\big)^2\omega(v)\,\mathrm{d}v &= \frac{1}{2}\int_{-\infty}^{\infty} \big(3v^2u^2(v) + 2v^3 u(v)\partial_v u(v)\big)\omega(v)\,\mathrm{d}v \\ &\leq \frac{3}{2}\|vu\|^2_{L^2_\omega(\mathbb{R})} + \frac{1}{2}\big\|v^2 u\big\|^2_{L^2_\omega(\mathbb{R})} + \frac{1}{2}\|v\partial_v u\|^2_{L^2_\omega(\mathbb{R})}, \end{aligned} \tag{B.4}$$

which finally gives (B.1). ☐

Proof of Theorem 4.1. We find from (4.5) that

$$\hat{\Pi}_N u := \mathrm{e}^{-\frac{v^2}{2}}\Pi_N\Big(u\,\mathrm{e}^{\frac{v^2}{2}}\Big) \in V_N, \tag{B.5}$$

where $\hat{\Pi}_N$ and V_N are defined in (4.4) and (4.3), respectively. Here the operator $\Pi_N : L^2_\omega(\mathbb{R}) \to \mathcal{P}_N$ is the L^2-orthogonal projection associated with the Hermite polynomial expansion. According to Corollary 7.3 of [28], we have the approximation error for the projection $\hat{\Pi}_N$: for any $\hat{\partial}_v^s u \in L^2(\mathbb{R})$, with $0 \le l \le s \le N+1$,

$$\left\|\hat{\partial}_v^l\left(\hat{\Pi}_N u - u\right)\right\|_{L^2(\mathbb{R})} \le c\sqrt{\frac{(N-s+1)!}{(N-l+1)!}}\left\|\hat{\partial}_v^s u\right\|_{L^2(\mathbb{R})}. \tag{B.6}$$

With the aid of (B.6), we will derive an upper bound of L^2_ω-orthogonal projection error that multiplied by v. By the definition of $\hat{\Pi}_N$, it is clear that

$$\left\|v\partial_v^l\left(\Pi_N\left(\mathrm{e}^{\frac{v^2}{2}}u\right) - \mathrm{e}^{\frac{v^2}{2}}u\right)\right\|_{L^2_\omega(\mathbb{R})} = \left\|v\partial_v^l\left(\mathrm{e}^{\frac{v^2}{2}}\left(\hat{\Pi}_N u - u\right)\right)\right\|_{L^2_\omega(\mathbb{R})}.$$

Notice that $\partial_v^l(\mathrm{e}^{\frac{v^2}{2}}(\hat{\Pi}_N u - u)) \in L^2_\omega(\mathbb{R})$. Using the inequality ([28], Lem. B.6), there has

$$\begin{aligned}\left\|v\partial_v^l\left(\mathrm{e}^{\frac{v^2}{2}}\left(\hat{\Pi}_N u - u\right)\right)\right\|_{L^2_\omega(\mathbb{R})} &\le c\left\|\partial_v^l\left(\mathrm{e}^{\frac{v^2}{2}}\left(\hat{\Pi}_N u - u\right)\right)\right\|_{H^1_\omega(\mathbb{R})}\\ &= c\left(\left\|\hat{\partial}_v^l\left(\hat{\Pi}_N u - u\right)\right\|^2_{L^2(\mathbb{R})} + \left\|\hat{\partial}_v^{l+1}\left(\hat{\Pi}_N u - u\right)\right\|^2_{L^2(\mathbb{R})}\right)^{\frac{1}{2}}.\end{aligned}$$

Then thanks to (B.6), one finally derives that

$$\begin{aligned}\left\|v\partial_v^l\left(\Pi_N\left(\mathrm{e}^{\frac{v^2}{2}}u\right) - \mathrm{e}^{\frac{v^2}{2}}u\right)\right\|_{L^2_\omega(\mathbb{R})} &\le c\left\|\hat{\partial}_v^{l+1}\left(\hat{\Pi}_N u - u\right)\right\|_{L^2(\mathbb{R})}\\ &\le c\sqrt{\frac{(N-s+1)!}{(N-l)!}}|u|_{H^s_A(\mathbb{R})}.\end{aligned}$$

It notes that

$$\left\|v\hat{\partial}_v^l\left(\hat{\Pi}_N u - u\right)\right\|_{L^2(\mathbb{R})} = \left\|v\partial_v^l\left(\Pi_N\left(\mathrm{e}^{\frac{v^2}{2}}u\right) - \mathrm{e}^{\frac{v^2}{2}}u\right)\right\|_{L^2_\omega(\mathbb{R})}. \tag{B.7}$$

Then the first statement in (4.8) follows.

Next we will prove the second statement. The estimate (4.9) with $l=0$ follows from the (4.8) with $l=0$. For $l=1$, we find

$$v\partial_v\left(\hat{\Pi}_N u - u\right) = v\,\mathrm{e}^{-\frac{v^2}{2}}\partial_v\left(\Pi_N\left(\mathrm{e}^{\frac{v^2}{2}}u\right) - \mathrm{e}^{\frac{v^2}{2}}u\right) - v^2\,\mathrm{e}^{-\frac{v^2}{2}}\left(\Pi_N\left(\mathrm{e}^{\frac{v^2}{2}}u\right) - \mathrm{e}^{\frac{v^2}{2}}u\right).$$

Hence, by Lemma B.1 and Theorem 7.13 of [28],

$$\begin{aligned}\left\|v\partial_v\left(\hat{\Pi}_N u - u\right)\right\|_{L^2(\mathbb{R})} &\le \left\|v\,\mathrm{e}^{-\frac{v^2}{2}}\partial_v\left(\Pi_N\left(\mathrm{e}^{\frac{v^2}{2}}u\right) - \mathrm{e}^{\frac{v^2}{2}}u\right)\right\|_{L^2(\mathbb{R})} + \left\|v^2\,\mathrm{e}^{-\frac{v^2}{2}}\left(\Pi_N\left(\mathrm{e}^{\frac{v^2}{2}}u\right) - \mathrm{e}^{\frac{v^2}{2}}u\right)\right\|\\ &\le c\left\|\Pi_N\left(\mathrm{e}^{\frac{v^2}{2}}u\right) - \mathrm{e}^{\frac{v^2}{2}}u\right\|_{H^1_\omega(\mathbb{R})} + \left\|\Pi_N\left(\mathrm{e}^{\frac{v^2}{2}}u\right) - \mathrm{e}^{\frac{v^2}{2}}u\right\|_{H^2_\omega(\mathbb{R})}\\ &\le c\sqrt{\frac{(N-s+1)!}{(N-l)!}}|u|_{H^s_A(\mathbb{R})},\end{aligned}$$

which completes the proof. $\square$

Appendix C. The proof of Theorem 4.2

Recall the Sobolev inequality ([28], Lem. B.6): for any $u, \partial_v u \in L^2_\omega(\mathbb{R})$,

$$\left\| \mathrm{e}^{-\frac{v^2}{2}} u \right\|_{L^\infty(\mathbb{R})} \leq 2\|u\|^{\frac{1}{2}}_{L^2_\omega(\mathbb{R})} |u|^{\frac{1}{2}}_{H^1_\omega(\mathbb{R})}. \tag{C.1}$$

In view of (B.5), by triangle inequality,

$$\begin{aligned}
\left\|\partial_v\left(\hat{\Pi}_N u - u\right)\right\|_{L^\infty(\mathbb{R})} &= \left\|\partial_v\left(\mathrm{e}^{-\frac{v^2}{2}}\left(\Pi_N\left(u\,\mathrm{e}^{\frac{v^2}{2}}\right) - u\,\mathrm{e}^{\frac{v^2}{2}}\right)\right)\right\|_{L^\infty(\mathbb{R})} \\
&\leq \left\|\mathrm{e}^{-\frac{v^2}{2}} v\left(\Pi_N\left(u\,\mathrm{e}^{\frac{v^2}{2}}\right) - u\,\mathrm{e}^{\frac{v^2}{2}}\right)\right\|_{L^\infty(\mathbb{R})} + \left\|\mathrm{e}^{-\frac{v^2}{2}}\partial_v\left(\Pi_N\left(u\,\mathrm{e}^{\frac{v^2}{2}}\right) - u\,\mathrm{e}^{\frac{v^2}{2}}\right)\right\|_{L^\infty(\mathbb{R})} \\
&=: I_1 + I_2.
\end{aligned}$$

By (4.7), (4.8) and the Sobolev inequality (C.1), we obtain that

$$\begin{aligned}
I_1 &\leq 2\left\|v\left(\Pi_N\left(u\,\mathrm{e}^{\frac{v^2}{2}}\right) - u\,\mathrm{e}^{\frac{v^2}{2}}\right)\right\|^{\frac{1}{2}}_{L^2_\omega(\mathbb{R})}\left\|\partial_v\left(v\Pi_N\left(u\,\mathrm{e}^{\frac{v^2}{2}}\right) - v\left(u\,\mathrm{e}^{\frac{v^2}{2}}\right)\right)\right\|^{\frac{1}{2}}_{L^2_\omega(\mathbb{R})} \\
&\leq 2\left\|v\left(\hat{\Pi}_N u - u\right)\right\|^{\frac{1}{2}}_{L^2(\mathbb{R})}\left(\left\|\hat{\Pi}_N u - u\right\|_{L^2(\mathbb{R})} + \left\|v\hat{\partial}_v\left(\hat{\Pi}_N u - u\right)\right\|_{L^2(\mathbb{R})}\right)^{\frac{1}{2}} \\
&\leq cN^{\frac{3}{4}-\frac{s}{2}}|u|_{H^s_A(\mathbb{R})}.
\end{aligned}$$

Similarly, we have from (4.7) and (C.1) that

$$\begin{aligned}
I_2 &\leq 2\left\|\partial_v\left(\Pi_N\left(u\,\mathrm{e}^{\frac{v^2}{2}}\right) - u\,\mathrm{e}^{\frac{v^2}{2}}\right)\right\|^{\frac{1}{2}}_{L^2_\omega(\mathbb{R})}\left\|\partial_v^2\left(\Pi_N\left(u\,\mathrm{e}^{\frac{v^2}{2}}\right) - u\,\mathrm{e}^{\frac{v^2}{2}}\right)\right\|^{\frac{1}{2}}_{L^2_\omega(\mathbb{R})} \\
&\leq N^{\frac{3}{4}-\frac{s}{2}}\left\|\hat{\partial}_v^s u\right\|_{L^2(\mathbb{R})} = N^{\frac{3}{4}-\frac{s}{2}}|u|_{H^s_A(\mathbb{R})}.
\end{aligned}$$

A combination of the above facts leads to the desired result (4.11).

Appendix D. Proof of Lemma 4.1

Using the derivative formula of the Hermite polynomial (*cf.* [28], (7.126)):

$$\partial_v^k H_n(v) = \frac{2^k n!}{(n-k)!} H_{n-k}(v),$$

we obtain from (4.1) that

$$\begin{aligned}
\hat{\partial}_v^k \widehat{H}_n(v) = \mathrm{e}^{-\frac{v^2}{2}}\partial_v^k\left(\mathrm{e}^{\frac{v^2}{2}}\widehat{H}_n(v)\right) &= \frac{2^k n!}{(n-k)!}\frac{1}{\pi^{\frac{1}{4}}\sqrt{n!2^n}}\,\mathrm{e}^{-\frac{v^2}{2}} H_{n-k}(v) \\
&= 2^{\frac{k}{2}}\sqrt{\frac{n!}{(n-k)!}}\widehat{H}_{n-k}(v), \quad n \geq k.
\end{aligned}$$

In view of the orthogonality (4.2), we have

$$\int_{\mathbb{R}} \hat{\partial}_v^k \widehat{H}_n(v)\hat{\partial}_v^k \widehat{H}_m(v)\,\mathrm{d}v = \frac{2^k n!}{(n-k)!}\delta_{mn}. \tag{D.1}$$

Thus, by the Parseval's identity,

$$\begin{aligned}\|u\|^2_{H^s_A(\mathbb{R})} &= \sum_{k=0}^{s} \|\hat{\partial}^k_v u\|^2_{L^2(\mathbb{R})} = \sum_{k=0}^{s} \|\sum_{n=0}^{\infty} \hat{u}_n\, \hat{\partial}^k_v \widehat{H}_n\|^2_{L^2(\mathbb{R})} = \sum_{k=0}^{s} 2^k \sum_{n=k}^{\infty} \frac{n!}{(n-k)!} |\hat{u}_n|^2 \\ &= \sum_{n=0}^{s}\sum_{k=0}^{n} \frac{2^k n!}{(n-k)!}|\hat{u}_n|^2 + \sum_{n=s+1}^{\infty}\sum_{k=0}^{s} \frac{2^k n!}{(n-k)!}|\hat{u}_n|^2 \sim \|u\|^2_{H^s_B(\mathbb{R})}.\end{aligned}$$

This ends the proof.

Acknowledgements. The research of the first author is supported in part by the National Natural Science Foundation of China (No. 12101229) and the Hunan Provincial Natural Science Foundation of China (No. 2021JJ40331). The research of the second author is partially supported by Singapore MOE AcRF Tier 1 Grant: MOE2021-T1-RG15/21. The first author would like to acknowledge the support of China Scholarship Council (CSC, No. 202106720024) for visiting NTU. This work was initiated while the last author was visiting NTU with the support of CSC.

References

[1] R.A. Adams and J.J.F. Fournier, Sobolev Spaces. Vol. 140. Elsevier (2003).

[2] T.P. Armstrong, R.C. Harding, G. Knorr and D. Montgomery, Solution of Vlasov's equation by transform methods. *Methods Comput. Phys.* **9** (1970) 29–86.

[3] N. Besse and E. Sonnendrücker, Semi-Lagrangian schemes for the Vlasov equation on an unstructured mesh of phase space. *J. Comput. Phys.* **191** (2003) 341–376.

[4] M. Bessemoulin-Chatard and F. Filbet, On the stability of conservative discontinuous Galerkin/Hermite spectral methods for the Vlasov–Poisson system. *J. Comput. Phys.* **451** (2022) 110881.

[5] C.K. Birdsall and A.B. Langdon, Plasma Physics via Computer Simulation. CRC Press, Boca Raton (1998).

[6] E. Camporeale, G.L. Delzanno, B.K. Bergen and J.D. Moulton, On the velocity space discretization for the Vlasov–Poisson system: comparison between implicit Hermite spectral and Particle-in-Cell methods. *Comput. Phys. Commun.* **198** (2016) 47–58.

[7] C. Canuto, M.Y. Hussaini, A. Quarteroni and T.A. Zang, Spectral Methods: Fundamentals in Single Domains. Springer (2006).

[8] C.Z. Cheng and G. Knorr, The integration of the Vlasov equation in configuration space. *J. Comput. Phys.* **22** (1976) 330–351.

[9] N. Crouseilles, G. Latu and E. Sonnendrücker, A parallel Vlasov solver based on local cubic spline interpolation on patches. *J. Comput. Phys.* **228** (2009) 1429–1446.

[10] G.L. Delzanno, Multi-dimensional, fully-implicit, spectral method for the Vlasov-Maxwell equations with exact conservation laws in discrete form. *J. Comput. Phys.* **301** (2015) 338–356.

[11] L. Einkemmer and C. Lubich, A quasi-conservative dynamical low-rank algorithm for the Vlasov equation. *SIAM J. Sci. Comput.* **41** (2019) B1061–B1081.

[12] L. Fatone, D. Funaro and G. Manzini, A Semi-Lagrangian spectral method for the Vlasov–Poisson system based on Fourier, Legendre and Hermite polynomials. *Commun. Appl. Math. Comput.* **1** (2019) 333–360.

[13] F. Filbet and T. Xiong, Conservative discontinuous Galerkin/Hermite spectral method for the Vlasov–Poisson system. *Commun. Appl. Math. Comput.* **4** (2022) 34–59.

[14] F. Filbet, E. Sonnendrücker and P. Bertrand, Conservative numerical schemes for the Vlasov equation. *J. Comput. Phys.* **172** (2001) 166–187.

[15] D. Funaro and G. Manzini, Stability and conservation properties of Hermite-based approximations of the Vlasov–Poisson system. *J. Sci. Comput.* **88** (2021) 29.

[16] I.S. Gradshteyn, I.M. Ryzhik, D. Zwillinger and V. Moll, Table of Integrals, Series, and Products, 8th edition. Academic Press, Amsterdam (2015).

[17] R.W. Hockney and J.W. Eastwood, Computer Simulation Using Particles. CRC Press, Boca Raton (1988).

[18] J.P. Holloway, Spectral velocity discretizations for the Vlasov-Maxwell equations. *Transp. Theory Stat. Phys.* **25** (1996) 1–32.

[19] K. Kormann, A semi-Lagrangian Vlasov solver in tensor train format. *SIAM J. Sci. Comput.* **37** (2015) B613–B632.

[20] W. Liu, L.-L. Wang and B. Wu, Optimal error estimates for Legendre expansions of singular functions with fractional derivatives of bounded variation. *Adv. Comput. Math.* **47** (2021) 1–32.

[21] N.F. Loureiro, A.A. Schekochihin and A. Zocco, Fast collisionless reconnection and electron heating in strongly magnetized plasmas. *Phys. Rev. Lett.* **111** (2013) 025002.

[22] N.F. Loureiro, W. Dorland, L. Fazendeiro, A. Kanekar, A. Mallet, M.S. Vilelas and A. Zocco, Viriato: a Fourier–Hermite spectral code for strongly magnetized fluid-kinetic plasma dynamics. *Comput. Phys. Commun.* **206** (2016) 45–63.

[23] G. Manzini, G.L. Delzanno, J. Vencels and S. Markidis, A Legendre-Fourier spectral method with exact conservation laws for the Vlasov–Poisson system. *J. Comput. Phys.* **317** (2016) 82–107.

[24] G. Manzini, D. Funaro and G.L. Delzanno, Convergence of spectral discretizations of the Vlasov–Poisson system. *SIAM J. Numer. Anal.* **55** (2017) 2312–2335.

[25] J.T. Parker and P.J. Dellar, Fourier–Hermite spectral representation for the Vlasov–Poisson system in the weakly collisional limit. *J. Plasma Phys.* **81** (2015) 305810203.

[26] M. Pulvirenti and J. Wick, Convergence of Galerkin approximation for two-dimensional Vlasov–Poisson equation. *Z. Angew. Math. Phys.* **35** (1984) 790–801.

[27] J.M. Qiu and C.W. Shu, Positivity preserving semi-Lagrangian discontinuous Galerkin formulation: theoretical analysis and application to the Vlasov–Poisson system. *J. Comput. Phys.* **230** (2011) 8386–8409.

[28] J. Shen, T. Tang and L.L. Wang, Spectral Methods: Algorithms, Analysis and Applications. Vol. 41. Springer Science & Business Media (2011).

[29] G. Szegö, Orthogonal Polynomials, 4th edition. Amer. Math. Soc., Providence, RI (1975).