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CONVERGENCE OF A SECOND-ORDER SCHEME FOR NON-LOCAL CONSERVATION LAWS

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Abstract. In this article, we present the convergence analysis of a second-order numerical scheme for traffic flow models that incorporate non-local conservation laws. We combine a MUSCL-type spatial reconstruction with strong stability preserving Runge-Kutta time-stepping to devise a fully discrete second-order scheme. The resulting scheme is shown to converge to a weak solution by establishing the maximum principle, bounded variation estimates and L^1 Lipschitz continuity in time. Further, using a space-step dependent slope limiter, we prove its convergence to the entropy solution. We also propose a MUSCL-Hancock type second-order scheme which requires only one intermediate stage unlike the Runge-Kutta schemes and is easier to implement. The performance of the proposed second-order schemes in comparison to a first-order scheme is demonstrated through several numerical experiments.

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1. Introduction

In recent years, conservation laws with non-local flux have emerged as a vital area of research due to its wide-ranging applications in various physical models. These models include sedimentation [13], crowd dynamics [17,31–33], supply chains [8], conveyor belts [49,64], weakly coupled oscillators [5], gradient constrained equations [6], traffic flow [9,14,20,23,28,67], structured population dynamics in biology [62], etc. Various aspects of these non-local conservation laws are extensively studied in the literature, such as their well-posedness addressed in [3,7,14,18,20,30,38,47,64] and non-local to local limits studied in [15,16,29,34–37,58]. Extended studies include non-local balance laws in [1,57,59] and the non-local pair interaction model in [39,40], and references therein. In this article, we specifically focus on the traffic flow models governed by non-local conservation laws, originally proposed in [14] and further studied in [21, 24, 26, 42, 46]. These models account for the interaction between drivers and the surrounding density of vehicles and have been widely studied in the literature. In particular, stability estimates for non-local traffic flow equations have been derived in [24, 54] and regularity results for solutions of these equations have been investigated in [10]. Also, there is a recent trend to derive general numerical schemes for non-local conservation laws, see [4, 45, 55]. Furthermore, the traveling wave solutions for

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non-local traffic flow models were discussed in [63, 65]. Different versions of non-local traffic flow equations also have been proposed and analyzed in the literature. That include models for one-to-one junctions [26], models with on-off ramps [27], multilane models [43], network models [44] and non-local conservation law models with discontinuous flux [22].

Our main aim here is to devise a second-order numerical scheme and show its convergence analysis for the one-dimensional non-local traffic-flow modeling equations studied in [20, 41, 42]. In the field of computational fluid dynamics, first-order methods are generally considered robust and reliable, and they aid in establishing well-posedness of problems. However, second- or high-order methods offer the advantage of considerably more accurate solutions with the same computing cost, particularly for two- or three-dimensional problems. As a result, there has been a surge of research activities aimed at improving these high-order methods. For instance, in the context of non-local conservation laws, [25, 46] treat second-order schemes, while [19, 41] propose and compare high-order discontinuous Galerkin methods and central WENO schemes. These studies suggest that high-order schemes offer better solutions than the low-order schemes. In an attempt to obtain better resolution for solutions of non-local conservation laws, the authors in [25] studied the Lagrangian anti-diffusive scheme.

In this work, to derive a second-order scheme, we employ a MUSCL-type spatial reconstruction [68] along with a strong stability preserving Runge-Kutta time stepping method [50, 51]. Such schemes are commonly used to discretize local conservation laws, for a detailed description we refer to [50, 66]. In [25], a second-order scheme of this type was presented for non-local multi-class traffic flow problems, showing its positivity preserving property analytically and through numerical examples. In this work, our focus is on the convergence analysis of this second-order scheme which we denote by RK-2. To obtain the convergence results, we also utilize a suitable numerical integration rule for approximating the convolution term. The convergence analysis involves two main stages. In the first stage, we aim to show that the proposed scheme converges to a weak solution. This is achieved by deriving a sequence of results that establish L^∞ estimates, Lipschitz continuity property in time and total variation (TV) bounds on the family of approximate solutions. We then use a version of the Kolmogorov's theorem to extract a subsequence that converges to a weak solution. Through the classical Lax-Wendroff type argument [60] we show that the limit of the convergent sequence is a weak solution of the given problem. Importantly, in a specific case (where $g(\rho) = \rho$, as we see later), weak solutions are already unique and an entropy condition is not necessary, see [57]. In the second stage, for the more general case, building on the ideas presented in [73], we employ a space-step dependent slope limiter (see [2, 71]) to establish the convergence to the entropy solution. Furthermore, we also consider a different type of non-local traffic-flow model, referred to as the downstream velocity model, proposed in [42]. We note that the convergence analysis presented in this work is applicable to this model as well.

In addition to this, we propose a MUSCL-Hancock type second-order scheme for the non-local problems of [20, 41, 42]. We denote this scheme by MH. The MUSCL-Hancock scheme, initially introduced in [69] and subsequently explored in [11, 70], is widely recognized for its simplicity and accurate shock capturing capabilities. In our work, we have tailored this approach to create a second-order scheme through appropriate numerical integration of the non-local flux term. Our analysis shows that the MH scheme provides a solution that is comparable to that of RK-2, while requiring only one stage of spatial reconstruction. This characteristic would save computational time, particularly when dealing with two-dimensional problems. However, the convergence analysis of this scheme will be reserved for a future work.

The rest of the paper is organized as follows: Section 2 describes the mean-downstream density traffic flow problem and discusses the notion of weak and entropy solutions of the underlying problem. Next, in Section 3, we present a second-order scheme that combines a MUSCL-type spatial reconstruction and Runge-Kutta time stepping to solve the underlying problem numerically. Further, we demonstrate that the approximate solutions obtained using this scheme possess desirable properties such as the maximum principle, a BV estimate and L^1-Lipschitz continuity in time. In Section 4, we prove the existence of a subsequence which converges to a weak solution of the given problem. In Section 5, we establish the convergence of the scheme to the entropy solution. In Section 6, we propose a MUSCL-Hancock type second-order accurate scheme for the approximation of non-local traffic flow problems. Finally, in Section 7, we provide numerical experiments to illustrate the performance of the

second-order schemes in comparison to a first-order scheme. We conclude with our final remarks in Section 8. An Appendix includes the proof of Kolmogorov's theorem adapted to our context, a description of the mean downstream velocity model and results that aid in proving the convergence to the entropy solution.

2. Mean downstream density traffic flow model

We consider the following initial value problem for the non-local conservation law, originally proposed in [14, 20]:

$$\begin{aligned} \partial_t \rho + \partial_x \big(g(\rho) v(\rho * w_\eta)\big) = 0, \quad & x \in \mathbb{R},\, t \in (0,T], \\ \rho(0,x) = \rho_0(x), \qquad & x \in \mathbb{R}, \end{aligned} \tag{2.1}$$

which describes the evolution of the vehicle density $\rho(t,x)$. Here, g and the velocity v are given functions of the density such that the map $\rho \mapsto g(\rho)v(\rho)$ is the corresponding local conservation law flux. The function w_η is a convolution kernel with compact support in $[0,\eta]$ for some $\eta > 0$. The convolution term $\rho * w_\eta$ is defined as

$$\rho * w_\eta(t,x) := \int_x^{x+\eta} \rho(t,y) w_\eta(y-x)\,\mathrm{d}y. \tag{2.2}$$

We denote $\mathbb{R}^+ := [0,+\infty)$ and make the hypotheses on the functions v, g and w_η as follows: $v \in \mathrm{C}^2(I;\mathbb{R}^+)$ with $v' \le 0, g \in \mathrm{C}^1(I;\mathbb{R}^+)$ with $g' \ge 0$, $w_\eta \in \mathrm{C}^1([0,\eta];\mathbb{R}^+)$ with $w_\eta' \le 0, \int_0^\eta w_\eta(x)\,\mathrm{d}x = 1$, where $I = [0,\rho_{\max}] \subseteq \mathbb{R}^+, \rho_{\max} > 0$. Further, we assume $\rho_0 \in \mathrm{BV}(\mathbb{R};[0,\rho_{\max}])$. Through the convolution of the density profile ρ with the kernel w_η, the non-local flux function denoted by $f(t,x,\rho) := g(\rho)v(\rho * w_\eta)$ takes into account the reaction of drivers to the neighbouring density of vehicles. In the case of traffic flow, the assessment of surrounding density generally happens only in the downstream direction by looking ahead, giving greater importance to closer vehicles. In this context, at a given time t, the velocity of cars at the point x has to be thought of as a function of not just the density $\rho(t,x)$ but of a weighted mean of the density in a right neighbourhood of x. This leads to the mean downstream density term $\rho * w_\eta$ as a convolution with a non-increasing kernel function w_η in the domain $[x, x+\eta]$ (see [14]). Through this mechanism, the non-local conservation law model turns to be suitable for describing traffic flow in a congested or heterogeneous road network. In general, the solutions of (2.1) need not be smooth, necessitating the definition of a weak solution given in the following lines.

Definition 2.1. (Weak solution) A function $\rho \in (\mathrm{L}^\infty \cap \mathrm{L}^1)([0,T) \times \mathbb{R};\mathbb{R})$, $T > 0$, is a weak solution of (2.1) if

$$\int_0^T \int_{-\infty}^{+\infty} \big(\rho \partial_t \varphi + g(\rho) v(\rho * w_\eta) \partial_x \varphi\big)(t,x)\,\mathrm{d}x\,\mathrm{d}t + \int_{-\infty}^{+\infty} \rho_0(x)\varphi(0,x)\,\mathrm{d}x = 0 \tag{2.3}$$

for all $\varphi \in \mathrm{C}_c^1([0,T) \times \mathbb{R};\mathbb{R})$.

Further, we consider the following definition of entropy solution of (2.1) given in [20, 42].

Definition 2.2. (Entropy solution) A function $\rho \in (\mathrm{L}^\infty \cap \mathrm{L}^1)([0,T) \times \mathbb{R};\mathbb{R})$, $T > 0$, is an entropy weak solution of (2.1) if

$$\begin{aligned} &\int_0^T \int_{-\infty}^{+\infty} \Big(|\rho - \kappa| \partial_t \varphi + \operatorname{sgn}(\rho - \kappa)\big(g(\rho) - g(\kappa)\big) v(\rho * w_\eta) \partial_x \varphi \\ &- \operatorname{sgn}(\rho - \kappa) g(\kappa) v'(\rho * w_\eta) \partial_x (\rho * w_\eta) \varphi \Big)(t,x)\,\mathrm{d}x\,\mathrm{d}t + \int_{-\infty}^{+\infty} |\rho_0(x) - \kappa| \varphi(0,x)\,\mathrm{d}x \ge 0 \end{aligned} \tag{2.4}$$

for all $\varphi \in \mathrm{C}_c^1([0,T) \times \mathbb{R};\mathbb{R}^+)$ and $\kappa \in I = [0,\rho_{\max}]$, where sgn is the sign function.

Note that, with this definition, uniqueness of entropy solutions of problem (2.1) follows from Theorem 2.1 of [20].

Remark 2.3. In a recent paper by Friedrich *et al.* [42], a new non-local conservation law model for describing traffic flow, known as the mean downstream velocity model, was introduced and studied. This model posits that drivers adjust their velocity based on the average velocity of vehicles in their vicinity. In our study, we will primarily examine the mean downstream density model, but it is worth noting that all of the results we present can be extended to the downstream velocity model as well. Additional information on this point can be found in Appendix B.

3. Second-order numerical scheme

To begin with, we discretize the spatial domain using a uniform mesh of size Δx, ensuring that the length of the convolution kernel's support $\eta = N\Delta x$ for some $N \in \mathbb{N}$. The spatial domain can then be represented as a union of cells, given by $[x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$, where $x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}} = \Delta x$ for all $j \in \mathbb{Z}$. The time domain is discretized with grid points $t^n = n\Delta t$, where Δt is a time step which is chosen according to a CFL condition that will be specified later. Also, the ratio $\lambda = \frac{\Delta t}{\Delta x}$ is kept as a constant. Finally, we denote $w_\eta^k := w_\eta(k\Delta x)$ for $k = 0, \cdots, N$ and note the following properties

$$w_\eta^0 \geq w_\eta^k \text{ for all } k = 1, \cdots, N \tag{3.1}$$

and

$$\Delta x \sum_{k=0}^{N-1} w_\eta^k \leq w_\eta^0 N \Delta x = w_\eta^0 \eta. \tag{3.2}$$

Moving on to the numerical scheme, given the cell-average solution $\rho_j(t)$ at time t, we proceed with reconstructing a piecewise polynomial denoted as $\tilde{\rho}_j(t,x)$ which is given by

$$\tilde{\rho}_j(t,x) = \rho_j(t) + \frac{(x - x_j)}{\Delta x}\sigma_j(t) \text{ for } x \in (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}), \quad j \in \mathbb{Z}. \tag{3.3}$$

Here,

$$\sigma_j(t) = \text{minmod}\left((\rho_j(t) - \rho_{j-1}(t)),\ \frac{1}{2}(\rho_{j+1}(t) - \rho_{j-1}(t)), (\rho_{j+1}(t) - \rho_j(t))\right), \quad j \in \mathbb{Z} \tag{3.4}$$

represent the slopes obtained using the minmod limiter, where the minmod function is defined as

$$\text{minmod}(a_1, \cdots, a_m) := \begin{cases} \text{sgn}(a_1) \min\limits_{1 \leq k \leq m} \{|a_k|\}, & \text{if } \text{sgn}(a_1) = \cdots = \text{sgn}(a_m), \\ 0, & \text{otherwise.} \end{cases} \tag{3.5}$$

At each interface $x_{j+\frac{1}{2}}$, the terms $\rho_{j+\frac{1}{2},-}(t) := \rho_j(t) + \dfrac{\sigma_j(t)}{2}$ and $\rho_{j+\frac{1}{2},+}(t) := \rho_{j+1}(t) - \dfrac{\sigma_{j+1}(t)}{2}$ denote the left and right values of the reconstructed linear polynomial $\tilde{\rho}(t,x)$. With a finite volume integration, a spatially second-order semi-discrete scheme is obtained as

$$\frac{d\rho_j(t)}{dt} = -\frac{f_{j+\frac{1}{2}}(t) - f_{j-\frac{1}{2}}(t)}{\Delta x}, \quad \rho_j(0) = \frac{1}{\Delta x}\int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \rho_0(x)\,\mathrm{d}x \quad \text{for } j \in \mathbb{Z}, \tag{3.6}$$

where $f_{j+\frac{1}{2}}$ is the numerical flux. An immediate choice of the numerical flux is the Lax-Friedrich flux [14, 20]. However, we deal with a more accurate Godunov-type flux proposed in [42], given as

$$f_{j+\frac{1}{2}}(t) = g\left(\rho_{j+\frac{1}{2},-}(t)\right) V_{j+\frac{1}{2}}(t), \quad j \in \mathbb{Z},$$

where $V_{j+\frac{1}{2}}(t) := v\big(R_{j+\frac{1}{2}}(t)\big)$ and $R_{j+\frac{1}{2}}(t)$ denotes the approximation of the convolution term $R(t,x) := \rho * w_\eta(t,x)$ at the interface $x_{j+\frac{1}{2}}$. The terms $R_{j+\frac{1}{2}}(t)$ are computed using the trapezoidal rule as

$$R_{j+\frac{1}{2}}(t) = \frac{\Delta x}{2}\sum_{k=0}^{N-1}\left(\rho_{j+k+\frac{1}{2},+}(t)w_\eta^k + \rho_{j+k+\frac{3}{2},-}(t)w_\eta^{k+1}\right), \quad j\in\mathbb{Z}. \tag{3.7}$$

Finally, to obtain a second-order fully-discrete scheme, we evolve the semi-discrete formulation (3.6) in time using the strong stability preserving (SSP) Runge-Kutta method (as in [50,66]). The resulting scheme is written as

$$\begin{aligned}
\rho_j^{(1)} &= \rho_j^n - \lambda\left(g(\rho^n_{j+\frac{1}{2},-})V^n_{j+\frac{1}{2}} - g(\rho^n_{j-\frac{1}{2},-})V^n_{j-\frac{1}{2}}\right),\\
\rho_j^{(2)} &= \rho_j^{(1)} - \lambda\left(g(\rho^{(1)}_{j+\frac{1}{2},-})V^{(1)}_{j+\frac{1}{2}} - g(\rho^{(1)}_{j-\frac{1}{2},-})V^{(1)}_{j-\frac{1}{2}}\right),\\
\rho_j^{n+1} &= \frac{1}{2}\left(\rho_j^n + \rho_j^{(2)}\right), \quad j\in\mathbb{Z}.
\end{aligned} \tag{3.8}$$

The second-order scheme (3.8) can also be written in the conservative form as

$$\rho_j^{n+1} = \rho_j^n - \lambda(F^n_{j+\frac{1}{2}} - F^n_{j-\frac{1}{2}}), \text{ where } F^n_{j+\frac{1}{2}} = \frac{1}{2}\left(g(\rho^n_{j+\frac{1}{2},-})V^n_{j+\frac{1}{2}} + g(\rho^{(1)}_{j+\frac{1}{2},-})V^{(1)}_{j+\frac{1}{2}}\right), \quad j\in\mathbb{Z}. \tag{3.9}$$

We denote the corresponding approximate solution as $\rho_{\Delta x}(t,x) := \rho_j^n$ for $(t,x)\in[t^n,t^{n+1})\times(x_{j-\frac{1}{2}},x_{j+\frac{1}{2}}]$ and $\rho^n_{\Delta x}(x) := \rho_{\Delta x}(t^n,x)$. Also, define $\rho^{(l)}_{\Delta x}(t,x) := \rho_j^{(l)}$ for $(t,x)\in[t^n,t^{n+1})\times(x_{j-\frac{1}{2}},x_{j+\frac{1}{2}}]$ and $l=1,2$. Here, $\rho_j^{(1)}$ and $\rho_j^{(2)}$ are taken to be the values computed from ρ_j^n for all $j\in\mathbb{Z}$, when $(t,x)\in[t^n,t^{n+1})\times(x_{j-\frac{1}{2}},x_{j+\frac{1}{2}}]$.

Remark 3.1. The linear reconstruction procedure (3.3) possesses the following properties which play an important role in the convergence analysis:

(i) The interface values have the property

$$\rho_{j+\frac{1}{2},-}(t), \rho_{j+\frac{1}{2},+}(t) \in \left[\min\{\rho_j(t),\rho_{j+1}(t)\}, \max\{\rho_j(t),\rho_{j+1}(t)\}\right], \quad j\in\mathbb{Z}. \tag{3.10}$$

(ii) The functions ρ and $\tilde\rho$ defined as $\rho(t,x) := \rho_j(t)$, $\tilde\rho(t,x) := \tilde\rho_j(t,x)$, $x\in(x_{j-\frac{1}{2}},x_{j+\frac{1}{2}}), j\in\mathbb{Z}$ satisfy the following equality (see Lem. 3.1, Chap. 4, [48]) on the total variation (TV):

$$\mathrm{TV}\big(\tilde\rho(t,\cdot)\big) = \mathrm{TV}\big(\rho(t,\cdot)\big) = \sum_{j\in\mathbb{Z}}|\rho_{j+1}(t)-\rho_j(t)|. \tag{3.11}$$

(iii) The slopes $\{\sigma_j(t)\}_{j\in\mathbb{Z}}$ satisfy

$$|\sigma_{j+1}(t)-\sigma_j(t)| \le |\rho_{j+1}(t)-\rho_j(t)|, \quad \text{for all } j\in\mathbb{Z}. \tag{3.12}$$

Remark 3.2. To ensure non-negative velocity terms, we require that the convolution terms $R_{j+\frac{1}{2}}(t)$ in (3.7) fall within the range $[0,\rho_{\max}]$. In cases where the trapezoidal quadrature rule used to compute $R_{j+\frac{1}{2}}(t)$ in (3.7) is not exact for the given kernel function, *i.e.*, $\frac{\Delta x}{2}\sum_{k=0}^{N-1}\left(w_\eta^k + w_\eta^{k+1}\right) \neq 1$, we adopt the same approach as in [41]. In this context, we define $Q_{\Delta x} := \frac{\Delta x}{2}\sum_{k=0}^{N-1}\left(w_\eta^k + w_\eta^{k+1}\right)$ and replace w_η^k by

$$\tilde w_\eta^k = \frac{w_\eta^k}{Q_{\Delta x}}$$

so that

$$\frac{\Delta x}{2}\sum_{k=0}^{N-1}\Big(\tilde{w}_\eta^k+\tilde{w}_\eta^{k+1}\Big)=1.$$

This allows us to write the term $R_{j+\frac{1}{2}}(t)$ in (3.7) as a convex combination of the density values $\rho_{j+k+\frac{1}{2},+}(t)$ and $\rho_{j+k+\frac{3}{2},-}(t)$, where $k=0,1,\ldots,N-1$. As a result, the convolution terms fall in the desired range provided the density values can be made to lie within the range $[0,\rho_{\max}]$, which we will see later. Consequently, the velocity terms remain non-negative. Moreover, replacing the terms w_η^k by $\tilde{w}_\eta^k$ does not affect the order of accuracy of the approximation as we have $Q_{\Delta x}\approx 1$. In addition, let us observe that, the modified terms $\tilde{w}_\eta^k$ also preserve the non-increasing property of w_η^k. With these observations, henceforth, in our convergence analysis of the RK-2 scheme (3.8), we replace w_η^k by $\tilde{w}_\eta^k$, while still denoting it as w_η^k.

3.1. Maximum principle

We establish that the approximate solution constructed using the scheme (3.8) satisfies the maximum principle. Initially, we examine the first-order forward Euler time stepping involved in the RK-2 scheme (3.8). Subsequently, we demonstrate the maximum principle for the RK-2 scheme (3.8). Note that throughout this paper $\|\cdot\|$ denotes the supremum norm $\|\cdot\|_{\mathrm{L}^\infty}$. Also, for any $a,b\in\mathbb{R}$, we denote the interval $\mathcal{I}(a,b):=(\min\{a,b\},\max\{a,b\})$ and BV denotes the space of bounded variation functions. Now, we have the following lemma.

Lemma 3.3. *Let $\rho_j^n\in[\rho_m,\rho_M]\subseteq[0,\rho_{\max}]$ for all $j\in\mathbb{Z}$. Assume that the CFL condition*

$$\frac{\Delta t}{\Delta x}\le\frac{1}{2(\|g\|\|v'\|\frac{\Delta x}{2}w_\eta^0+\|v\|\|g'\|)}\tag{3.13}$$

holds. Then the approximate solution obtained using the first-order Euler forward time step

$$\rho_j^{n+1}=\rho_j^n-\lambda\Big(g(\rho_{j+\frac{1}{2},-}^n)V_{j+\frac{1}{2}}^n-g(\rho_{j-\frac{1}{2},-}^n)V_{j-\frac{1}{2}}^n\Big)\tag{3.14}$$

satisfies $\rho_m\le\rho_j^{n+1}\le\rho_M$ for all $j\in\mathbb{Z}$.

Proof. Using the mean value theorem we write

$$V_{j-\frac{1}{2}}^n-V_{j+\frac{1}{2}}^n=v(R_{j-\frac{1}{2}}^n)-v(R_{j+\frac{1}{2}}^n)=-v'(\zeta_j)(R_{j+\frac{1}{2}}^n-R_{j-\frac{1}{2}}^n),\tag{3.15}$$

for some $\zeta_j\in\mathcal{I}(R_{j-\frac{1}{2}}^n,R_{j+\frac{1}{2}}^n)$. The difference of convolution terms reads as

$$R_{j+\frac{1}{2}}^n-R_{j-\frac{1}{2}}^n=\frac{\Delta x}{2}\sum_{k=0}^{N-1}\Big(w_\eta^k(\rho_{j+k+\frac{1}{2},+}^n-\rho_{j+k-\frac{1}{2},+}^n)+w_\eta^{k+1}(\rho_{j+k+\frac{3}{2},-}^n-\rho_{j+k+\frac{1}{2},-}^n)\Big).\tag{3.16}$$

Using summation by parts yields

$$\begin{aligned}R_{j+\frac{1}{2}}^n-R_{j-\frac{1}{2}}^n=&\frac{\Delta x}{2}\left(w_\eta^{N-1}\rho_{j+N-\frac{1}{2},+}^n-w_\eta^0\rho_{j-\frac{1}{2},+}^n+\sum_{k=1}^{N-1}\rho_{j+k-\frac{1}{2},+}^n(w_\eta^{k-1}-w_\eta^k)\right)\\&+\frac{\Delta x}{2}\left(w_\eta^N\rho_{j+N+\frac{1}{2},-}^n-w_\eta^1\rho_{j+\frac{1}{2},-}^n+\sum_{k=1}^{N-1}\rho_{j+k+\frac{1}{2},-}^n(w_\eta^k-w_\eta^{k+1})\right).\end{aligned}\tag{3.17}$$

Upon substituting the expression (3.17) into (3.15) and considering the assumptions that w_η is non-increasing, $w_\eta \geq 0$ and $v' \leq 0$ as well as the property (3.10) of the reconstructed values, it follows that

$$\begin{aligned} V^n_{j-\frac{1}{2}} - V^n_{j+\frac{1}{2}} &\leq -v'(\zeta_j)\frac{\Delta x}{2}\left(w_\eta^{N-1}\rho_M - w_\eta^0\rho^n_{j-\frac{1}{2},+} + \rho_M\sum_{k=1}^{N-1}(w_\eta^{k-1} - w_\eta^k)\right) \\ &\quad - v'(\zeta_j)\frac{\Delta x}{2}\left(w_\eta^{N}\rho_M - w_\eta^1\rho^n_{j+\frac{1}{2},-} + \rho_M\sum_{k=1}^{N-1}(w_\eta^{k} - w_\eta^{k+1})\right) \\ &\leq -v'(\zeta_j)\frac{\Delta x}{2}\left(w_\eta^{N-1}\rho_M - w_\eta^0\rho^n_{j-\frac{1}{2},+} + \rho_M(w_\eta^0 - w_\eta^{N-1})\right) \\ &\quad - v'(\zeta_j)\frac{\Delta x}{2}\left(w_\eta^{N}\rho_M - w_\eta^1\rho^n_{j+\frac{1}{2},-} + \rho_M(w_\eta^1 - w_\eta^{N})\right) \\ &= -v'(\zeta_j)\frac{\Delta x}{2}\left(w_\eta^0(\rho_M - \rho^n_{j-\frac{1}{2},+}) + w_\eta^1(\rho_M - \rho^n_{j+\frac{1}{2},-})\right). \end{aligned} \tag{3.18}$$

Subsequently, we get that $V^n_{j-\frac{1}{2}} - V^n_{j+\frac{1}{2}} \leq \|v'\|\Delta x w_\eta^0\rho_{\max}$. With a similar argument, we can show that

$$V^n_{j-\frac{1}{2}} - V^n_{j+\frac{1}{2}} \geq -v'(\zeta_j)\frac{\Delta x}{2}\left(w_\eta^0(\rho_m - \rho^n_{j-\frac{1}{2},+}) + w_\eta^1(\rho_m - \rho^n_{j+\frac{1}{2},-})\right), \tag{3.19}$$

which yields $V^n_{j-\frac{1}{2}} - V^n_{j+\frac{1}{2}} \geq -\|v'\|\Delta x w_\eta^0\rho_{\max}$. Consequently, we have

$$|V^n_{j-\frac{1}{2}} - V^n_{j+\frac{1}{2}}| \leq \|v'\|\Delta x w_\eta^0\rho_{\max}. \tag{3.20}$$

Multiplying the inequality in (3.18) with $g(\rho_M)$ and subtracting $g(\rho^n_{j+\frac{1}{2},-})V^n_{j+\frac{1}{2}}$, we get

$$\begin{aligned} V^n_{j-\frac{1}{2}}g(\rho_M) - V^n_{j+\frac{1}{2}}g(\rho^n_{j+\frac{1}{2},-}) &\leq \|g\|\|v'\|\frac{\Delta x}{2}\left(w_\eta^0(\rho_M - \rho^n_{j-\frac{1}{2},+}) + w_\eta^1(\rho_M - \rho^n_{j+\frac{1}{2},-})\right) \\ &\quad + V^n_{j+\frac{1}{2}}\left(g(\rho_M) - g(\rho^n_{j+\frac{1}{2},-})\right) \\ &\leq \|g\|\|v'\|\frac{\Delta x}{2}\left(w_\eta^0(\rho_M - \rho^n_{j-\frac{1}{2},+}) + w_\eta^1(\rho_M - \rho^n_{j+\frac{1}{2},-})\right) \\ &\quad + \|v\|\|g'\|(\rho_M - \rho^n_{j+\frac{1}{2},-}). \end{aligned} \tag{3.21}$$

Given that the CFL condition (3.13) holds, the observation $\rho^n_j = \frac{1}{2}(\rho^n_{j-\frac{1}{2},+} + \rho^n_{j+\frac{1}{2},-})$ together with the monotonicity of g and the estimate (3.21) lead to the following estimate:

$$\begin{aligned} \rho^{n+1}_j &= \rho^n_j + \lambda\left(V^n_{j-\frac{1}{2}}g(\rho^n_{j-\frac{1}{2},-}) - V^n_{j+\frac{1}{2}}g(\rho^n_{j+\frac{1}{2},-})\right) \\ &\leq \rho^n_j + \lambda\left(V^n_{j-\frac{1}{2}}g(\rho_M) - V^n_{j+\frac{1}{2}}g(\rho^n_{j+\frac{1}{2},-})\right) \\ &\leq \frac{(\rho^n_{j-\frac{1}{2},+} + \rho^n_{j+\frac{1}{2},-})}{2} + \lambda\|g\|\|v'\|\frac{\Delta x}{2}\left(w_\eta^0(\rho_M - \rho^n_{j-\frac{1}{2},+}) + w_\eta^1(\rho_M - \rho^n_{j+\frac{1}{2},-})\right) \\ &\quad + \lambda\|v\|\|g'\|(\rho_M - \rho^n_{j+\frac{1}{2},-}) \\ &\leq \left(\frac{\rho^n_{j-\frac{1}{2},+}}{2} + \lambda\|g\|\|v'\|\frac{\Delta x}{2}w_\eta^0(\rho_M - \rho^n_{j-\frac{1}{2},+})\right) \\ &\quad + \left(\frac{\rho^n_{j+\frac{1}{2},-}}{2} + \lambda\left(\|g\|\|v'\|\frac{\Delta x}{2}w_\eta^1 + \|v\|\|g'\|\right)(\rho_M - \rho^n_{j+\frac{1}{2},-})\right) \\ &\leq \left(\frac{\rho^n_{j-\frac{1}{2},+}}{2} + \frac{\rho_M - \rho^n_{j-\frac{1}{2},+}}{2}\right) + \left(\frac{\rho^n_{j+\frac{1}{2},-}}{2} + \frac{\rho_M - \rho^n_{j+\frac{1}{2},-}}{2}\right) = \rho_M. \end{aligned} \tag{3.22}$$

Similarly, using (3.19) we can see that the following inequality holds

$$V^n_{j-\frac{1}{2}} g(\rho_m) - V^n_{j+\frac{1}{2}} g(\rho^n_{j+\frac{1}{2},-}) \geq \|g\|\|v'\| \frac{\Delta x}{2} \left(w^0_\eta (\rho_m - \rho^n_{j-\frac{1}{2},+}) + w^1_\eta (\rho_m - \rho^n_{j+\frac{1}{2},-}) \right) + \|v\|\|g'\|(\rho_m - \rho^n_{j+\frac{1}{2},-}).$$

Using this inequality, in the same line of argument as in (3.22) we get the lower bound $\rho^{n+1}_j \geq \rho_m$, provided the CFL condition (3.13) holds. This concludes the proof of the lemma. □

Theorem 3.4. *Let $\rho^0_j \in [\rho_m, \rho_M] \subset [0, \rho_{\max}]$ for all $j \in \mathbb{Z}$. Assume that the CFL condition (3.13) holds. Then for all $n \in \mathbb{N}$ the approximate solution obtained using the second-order scheme (3.8) satisfies*

$$\rho_m \leq \rho^n_j \leq \rho_M \quad \text{for all } \ j \in \mathbb{Z}. \tag{3.23}$$

Proof. Proof of this theorem uses the principle of mathematical induction. The base case $n = 0$ is trivially satisfied. For the inductive step, we assume that the inequality (3.23) holds for $n \in \mathbb{N}$, and show that it also holds for $n + 1$. We use Lemma 3.3 to show that the first and second stages of the second-order scheme (3.8) also satisfy the maximum principle. First, we apply the Euler forward step to ρ^n_j to obtain $\rho^{(1)}_j$. By Lemma 3.3, we have $\rho_m \leq \rho^{(1)}_j \leq \rho_M$ for all $j \in \mathbb{Z}$. Next, we apply the Euler forward step to $\rho^{(1)}_j$ to obtain $\rho^{(2)}_j$. Again, by Lemma 3.3, we have $\rho^{(1)}_m \leq \rho^{(2)}_j \leq \rho^{(1)}_M$, where $\rho^{(1)}_M := \sup_{j\in\mathbb{Z}} \rho^{(1)}_j$ and $\rho^{(1)}_m := \inf_{j\in\mathbb{Z}} \rho^{(1)}_j$. Finally, as $\rho^{n+1}_j = \frac{1}{2}(\rho^n_j + \rho^{(2)}_j)$, the result holds true for the case $n + 1$. This completes the proof of the theorem. □

Remark 3.5. As a consequence of Theorem 3.4, it follows that the second-order scheme (3.8) is positivity preserving, in the sense that when the initial datum is positive then the approximate solution remains positive with evolution in time. Further, provided that the initial datum ρ_0 is positive, by using the conservative form (3.9) and the positivity property, it is immediate to see that the approximate solution $\rho_{\Delta x}$ obtained using the scheme (3.8) satisfies the following

$$\|\rho_{\Delta x}(t, \cdot)\|_{\mathrm{L}^1(\mathbb{R})} = \|\rho_{\Delta x}(0, \cdot)\|_{\mathrm{L}^1(\mathbb{R})} = \|\rho_0\|_{\mathrm{L}^1(\mathbb{R})}, \quad \text{for all } t > 0.$$

3.2. Total variation estimate

We derive an estimate on the total variation of the approximate solutions obtained through the second-order scheme (3.8), when applied to the problem (2.1). This is done by initially examining the Euler forward step (3.14) and subsequently extending to the second-order scheme.

Lemma 3.6. *If $\rho^n_{\Delta x} \in \mathrm{BV}(\mathbb{R}; [0, \rho_{\max}])$ and the CFL condition (3.13) holds, then $\rho^{n+1}_{\Delta x}$ computed using the Euler forward step (3.14) has the space total variation estimate*

$$\sum_{j\in\mathbb{Z}} |\rho^{n+1}_{j+1} - \rho^{n+1}_j| \leq (1 + C\Delta t) \sum_{j\in\mathbb{Z}} |\rho^n_{j+1} - \rho^n_j|, \tag{3.24}$$

where $C := w^0_\eta \rho_{\max} \|v'\|\|g'\| + 10\rho_{\max}(w^0_\eta)^2 \|v''\|\|g\|\eta + 3w^0_\eta \|g\|\|v'\|$.

Proof. Let ρ^{n+1}_j be computed through (3.14). We proceed by subtracting and adding the term $-\lambda\big(V^n_{j+\frac{3}{2}} g(\rho^n_{j+\frac{1}{2},-}) + V^n_{j-\frac{1}{2}} g(\rho^n_{j+\frac{1}{2},-})\big)$ to the difference $\rho^{n+1}_{j+1} - \rho^{n+1}_j$. By using the observation $\rho^n_j = \frac{1}{2}(\rho^n_{j-\frac{1}{2},+} +$

$\rho^n_{j+\frac{1}{2},-}$) and rearranging the terms, we obtain:

$$\begin{aligned}
\rho^{n+1}_{j+1} - \rho^{n+1}_{j} &= (\rho^n_{j+1} - \rho^n_j) - \lambda \Big[V^n_{j+\frac{3}{2}} \big(g(\rho^n_{j+\frac{3}{2},-}) - g(\rho^n_{j+\frac{1}{2},-}) \big) - V^n_{j-\frac{1}{2}} \big(g(\rho^n_{j+\frac{1}{2},-}) - g(\rho^n_{j-\frac{1}{2},-}) \big) \\
&\quad + g(\rho^n_{j+\frac{1}{2},-}) (V^n_{j+\frac{3}{2}} - 2V^n_{j+\frac{1}{2}} + V^n_{j-\frac{1}{2}}) \Big] \\
&= (\rho^n_{j+1} - \rho^n_j) - \lambda \Big[V^n_{j+\frac{3}{2}} g'(\theta_{j+1,-}) \big(\rho^n_{j+\frac{3}{2},-} - \rho^n_{j+\frac{1}{2},-} \big) - V^n_{j-\frac{1}{2}} g'(\theta_{j,-}) \big(\rho^n_{j+\frac{1}{2},-} - \rho^n_{j-\frac{1}{2},-} \big) \\
&\quad + g(\rho^n_{j+\frac{1}{2},-}) \big(V^n_{j+\frac{3}{2}} - 2V^n_{j+\frac{1}{2}} + V^n_{j-\frac{1}{2}} \big) \Big] \\
&= \frac{1}{2} (\rho^n_{j+\frac{1}{2},+} - \rho^n_{j-\frac{1}{2},+}) + (\rho^n_{j+\frac{3}{2},-} - \rho^n_{j+\frac{1}{2},-}) \left(\frac{1}{2} - \lambda V^n_{j+\frac{3}{2}} g'(\theta_{j+1,-}) \right) \\
&\quad + \lambda V^n_{j-\frac{1}{2}} g'(\theta_{j,-}) \big(\rho^n_{j+\frac{1}{2},-} - \rho^n_{j-\frac{1}{2},-} \big) - \lambda g(\rho^n_{j+\frac{1}{2},-}) \big(V^n_{j+\frac{3}{2}} - 2V^n_{j+\frac{1}{2}} + V^n_{j-\frac{1}{2}} \big),
\end{aligned} \tag{3.25}$$

where $\theta_{j,-} \in \mathcal{I}(\rho^n_{j-\frac{1}{2},-}, \rho^n_{j+\frac{1}{2},-})$. We can write

$$\begin{aligned}
V^n_{j+\frac{3}{2}} - 2V^n_{j+\frac{1}{2}} + V^n_{j-\frac{1}{2}} &= (V^n_{j+\frac{3}{2}} - V^n_{j+\frac{1}{2}}) - (V^n_{j+\frac{1}{2}} - V^n_{j-\frac{1}{2}}) \\
&= v'(\zeta_{j+1})(R^n_{j+\frac{3}{2}} - R^n_{j+\frac{1}{2}}) - v'(\zeta_j)(R^n_{j+\frac{1}{2}} - R^n_{j-\frac{1}{2}}) \\
&= \big(v'(\zeta_{j+1}) - v'(\zeta_j) \big) (R^n_{j+\frac{3}{2}} - R^n_{j+\frac{1}{2}}) + v'(\zeta_j) \big(R^n_{j+\frac{3}{2}} - 2R^n_{j+\frac{1}{2}} + R^n_{j-\frac{1}{2}} \big) \\
&= v''(\zeta_{j+\frac{1}{2}})(\zeta_{j+1} - \zeta_j) \big(R^n_{j+\frac{3}{2}} - R^n_{j+\frac{1}{2}} \big) + v'(\zeta_j) \big(R^n_{j+\frac{3}{2}} - 2R^n_{j+\frac{1}{2}} + R^n_{j-\frac{1}{2}} \big),
\end{aligned} \tag{3.26}$$

where $\zeta_i \in \mathcal{I}(R^n_{i-\frac{1}{2}}, R^n_{i+\frac{1}{2}})$ and $\zeta_{i+\frac{1}{2}} \in \mathcal{I}(\zeta_i, \zeta_{i+1})$. By inserting the identity (3.26) into the expression (3.25), applying the CFL condition (3.13) and taking the modulus, it follows that

$$\begin{aligned}
|\rho^{n+1}_{j+1} - \rho^{n+1}_{j}| &\le \frac{1}{2} |\tilde{\Delta}^n_{j,+}| + |\tilde{\Delta}^n_{j+1,-}| \left(\frac{1}{2} - \lambda V^n_{j+\frac{3}{2}} g'(\zeta_{j+1,-}) \right) + \lambda V^n_{j-\frac{1}{2}} g'(\zeta_{j,-}) |\tilde{\Delta}^n_{j,-}| \\
&\quad + \lambda g(\rho^n_{j+\frac{1}{2},-}) |v''(\zeta_{j+\frac{1}{2}})| |\zeta_{j+1} - \zeta_j| |R^n_{j+\frac{3}{2}} - R^n_{j+\frac{1}{2}}| \\
&\quad + \lambda g(\rho^n_{j+\frac{1}{2},-}) |v'(\zeta_j)| \big| R^n_{j+\frac{3}{2}} - 2R^n_{j+\frac{1}{2}} + R^n_{j-\frac{1}{2}} \big|, \quad \text{where } \tilde{\Delta}^n_{j,\pm} := \rho^n_{j+\frac{1}{2},\pm} - \rho^n_{j-\frac{1}{2},\pm}.
\end{aligned} \tag{3.27}$$

Rearranging the terms of the expression (3.17) obtained through summation by parts, we can write

$$\begin{aligned}
R^n_{j+\frac{3}{2}} - 2R^n_{j+\frac{1}{2}} + R^n_{j-\frac{1}{2}} &= (R^n_{j+\frac{3}{2}} - R^n_{j+\frac{1}{2}}) - (R^n_{j+\frac{1}{2}} - R^n_{j-\frac{1}{2}}) \\
&= \frac{\Delta x}{2} \Big(w^{N-1}_\eta (\rho^n_{j+N+\frac{1}{2},+} - \rho^n_{j+N-\frac{1}{2},+}) - w^0_\eta (\rho^n_{j+\frac{1}{2},+} - \rho^n_{j-\frac{1}{2},+}) \\
&\quad + \sum_{k=1}^{N-1} (\rho^n_{j+k+\frac{1}{2},+} - \rho^n_{j+k-\frac{1}{2},+}) (w^{k-1}_\eta - w^k_\eta) \Big) \\
&\quad + \frac{\Delta x}{2} \Big(w^N_\eta (\rho^n_{j+N+\frac{3}{2},-} - \rho^n_{j+N+\frac{1}{2},-}) - w^1_\eta (\rho^n_{j+\frac{3}{2},-} - \rho^n_{j+\frac{1}{2},-}) \\
&\quad + \sum_{k=1}^{N-1} (\rho^n_{j+k+\frac{3}{2},-} - \rho^n_{j+k+\frac{1}{2},-}) (w^k_\eta - w^{k+1}_\eta) \Big).
\end{aligned} \tag{3.28}$$

Since $\zeta_j \in \mathcal{I}(R^n_{j-\frac{1}{2}}, R^n_{j+\frac{1}{2}})$, for some $\alpha, \beta \in (0,1)$ we can write

$$\begin{aligned}
\zeta_{j+1} - \zeta_j &= \alpha R^n_{j+\frac{3}{2}} + (1-\alpha) R^n_{j+\frac{1}{2}} - \beta R^n_{j+\frac{1}{2}} - (1-\beta) R^n_{j-\frac{1}{2}} \\
&= \alpha \frac{\Delta x}{2} \sum_{k=0}^{N-1} \left(\rho^n_{j+k+\frac{3}{2},+} w^k_\eta + \rho^n_{j+k+\frac{5}{2},-} w^{k+1}_\eta\right) + (1-\alpha)\frac{\Delta x}{2} \sum_{k=0}^{N-1} \left(\rho^n_{j+k+\frac{1}{2},+} w^k_\eta + \rho^n_{j+k+\frac{3}{2},-} w^{k+1}_\eta\right) \\
&\quad - \beta \frac{\Delta x}{2} \sum_{k=0}^{N-1} \left(\rho^n_{j+k+\frac{1}{2},+} w^k_\eta + \rho^n_{j+k+\frac{3}{2},-} w^{k+1}_\eta\right) - (1-\beta)\frac{\Delta x}{2} \sum_{k=0}^{N-1} \left(\rho^n_{j+k-\frac{1}{2},+} w^k_\eta + \rho^n_{j+k+\frac{1}{2},-} w^{k+1}_\eta\right) \\
&= \alpha \frac{\Delta x}{2} \sum_{k=1}^{N} \left(\rho^n_{j+k+\frac{1}{2},+} w^{k-1}_\eta + \rho^n_{j+k+\frac{3}{2},-} w^{k}_\eta\right) + (1-\alpha)\frac{\Delta x}{2} \sum_{k=0}^{N-1} \left(\rho^n_{j+k+\frac{1}{2},+} w^k_\eta + \rho^n_{j+k+\frac{3}{2},-} w^{k+1}_\eta\right) \\
&\quad - \beta \frac{\Delta x}{2} \sum_{k=0}^{N-1} \left(\rho^n_{j+k+\frac{1}{2},+} w^k_\eta + \rho^n_{j+k+\frac{3}{2},-} w^{k+1}_\eta\right) - (1-\beta)\frac{\Delta x}{2} \sum_{k=-1}^{N-2} \left(\rho^n_{j+k+\frac{1}{2},+} w^{k+1}_\eta + \rho^n_{j+k+\frac{3}{2},-} w^{k+2}_\eta\right) \\
&= \frac{\Delta x}{2} (\mathcal{Q}_1 + \mathcal{Q}_2),
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{Q}_1 &:= \sum_{k=1}^{N-2} \rho^n_{j+k+\frac{1}{2},+} \Big(\alpha w^{k-1}_\eta + (1-\alpha) w^k_\eta - \beta w^k_\eta - (1-\beta) w^{k+1}_\eta\Big) \\
&\quad + \sum_{k=1}^{N-2} \rho^n_{j+k+\frac{3}{2},-} \Big(\alpha w^{k}_\eta + (1-\alpha) w^{k+1}_\eta - \beta w^{k+1}_\eta - (1-\beta) w^{k+2}_\eta\Big), \\
\mathcal{Q}_2 &:= \alpha \left(\rho^n_{j+N-\frac{1}{2},+} w^{N-2}_\eta + \rho^n_{j+N+\frac{1}{2},-} w^{N-1}_\eta + \rho^n_{j+N+\frac{1}{2},+} w^{N-1}_\eta + \rho^n_{j+N+\frac{3}{2},-} w^{N}_\eta\right) \\
&\quad + (1-\alpha) \left(\rho^n_{j+\frac{1}{2},+} w^{0}_\eta + \rho^n_{j+\frac{3}{2},-} w^{1}_\eta + \rho^n_{j+N-\frac{1}{2},+} w^{N-1}_\eta + \rho^n_{j+N+\frac{1}{2},-} w^{N}_\eta\right) \\
&\quad - \beta \left(\rho^n_{j+\frac{1}{2},+} w^{0}_\eta + \rho^n_{j+\frac{3}{2},-} w^{1}_\eta + \rho^n_{j+N-\frac{1}{2},+} w^{N-1}_\eta + \rho^n_{j+N+\frac{1}{2},-} w^{N}_\eta\right) \\
&\quad - (1-\beta) \left(\rho^n_{j-\frac{1}{2},+} w^{0}_\eta + \rho^n_{j+\frac{1}{2},-} w^{1}_\eta + \rho^n_{j+\frac{1}{2},+} w^{1}_\eta + \rho^n_{j+\frac{3}{2},-} w^{2}_\eta\right).
\end{aligned}$$

As the function w_η is non-increasing, $w^k_\eta \geq w^{k+1}_\eta$ for each $k = 0, \ldots, N-1$ and it follows that

$$\alpha w^{k-1}_\eta + (1-\alpha) w^k_\eta - \beta w^k_\eta - (1-\beta) w^{k+1}_\eta \geq 0. \tag{3.29}$$

Now using (3.29), property (3.1) and property (3.10) of the linear reconstruction, we have the following bound

$$\begin{aligned}
|\mathcal{Q}_1| &\leq \rho_{\max} \Bigg[\sum_{k=1}^{N-2} \Big(\alpha w^{k-1}_\eta + (1-\alpha) w^k_\eta - \beta w^k_\eta - (1-\beta) w^{k+1}_\eta\Big) \\
&\qquad + \sum_{k=1}^{N-2} \Big(\alpha w^{k}_\eta + (1-\alpha) w^{k+1}_\eta - \beta w^{k+1}_\eta - (1-\beta) w^{k+2}_\eta\Big)\Bigg] \\
&\leq \rho_{\max} \Big[\Big(\alpha w^0_\eta + (1-\beta) w^1_\eta - \alpha w^{N-2}_\eta - (1-\beta) w^{N-1}_\eta\Big) \\
&\qquad + \Big(\alpha w^1_\eta + (1-\beta) w^2_\eta - \alpha w^{N-1}_\eta - (1-\beta) w^{N}_\eta\Big)\Big] \\
&\leq 4 w^0_\eta \rho_{\max}.
\end{aligned}$$

In a similar way, using property (3.1) and since $0 < \alpha, \beta < 1$, we obtain the following bound for Q_2

$$|\mathcal{Q}_2| \leq 16 w_\eta^0 \rho_{\max}.$$

Thus, we have

$$|\zeta_{j+1} - \zeta_j| \leq \frac{\Delta x}{2}\big(|\mathcal{Q}_1| + |\mathcal{Q}_2|\big) \leq 10 \Delta x w_\eta^0 \rho_{\max}. \tag{3.30}$$

Now, from (3.27) we write

$$\begin{aligned}\sum_{j\in\mathbb{Z}} |\rho_{j+1}^{n+1} - \rho_j^{n+1}| \leq \frac{1}{2}\sum_{j\in\mathbb{Z}} |\tilde{\Delta}_{j,+}^n| + \sum_{j\in\mathbb{Z}} |\tilde{\Delta}_{j+1,-}^n| \left(\frac{1}{2} - \lambda V_{j+\frac{3}{2}}^n g'(\zeta_{j+1,-})\right) + \sum_{j\in\mathbb{Z}} \lambda V_{j-\frac{1}{2}}^n g'(\zeta_{j,-}) |\tilde{\Delta}_{j,-}^n| \\ + \sum_{j\in\mathbb{Z}} \lambda g(\rho_{j+\frac{1}{2},-}^n) |v''(\zeta_{j+\frac{1}{2}})| |\zeta_{j+1} - \zeta_j| |R_{j+\frac{3}{2}}^n - R_{j+\frac{1}{2}}^n| \\ + \sum_{j\in\mathbb{Z}} \lambda g(\rho_{j+\frac{1}{2},-}^n) |v'(\zeta_j)| |R_{j+\frac{3}{2}}^n - 2R_{j+\frac{1}{2}}^n + R_{j-\frac{1}{2}}^n| \leq \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4,\end{aligned}$$

where

$$\begin{aligned}\mathcal{A}_1 &:= \frac{1}{2}\sum_{j\in\mathbb{Z}} |\tilde{\Delta}_{j,+}^n|, \qquad \mathcal{A}_2 := \sum_{j\in\mathbb{Z}} |\tilde{\Delta}_{j+1,-}^n| \left(\frac{1}{2} + \lambda (V_{j+\frac{1}{2}}^n - V_{j+\frac{3}{2}}^n) g'(\zeta_{j+1,-})\right), \\ \mathcal{A}_3 &:= \sum_{j\in\mathbb{Z}} \lambda g(\rho_{j+\frac{1}{2},-}^n) |v''(\zeta_{j+\frac{1}{2}})| |\zeta_{j+1} - \zeta_j| \left(\sum_{k=0}^{N-1} \frac{\Delta x}{2} w_\eta^k |\tilde{\Delta}_{j+k+1,+}^n| + \sum_{k=0}^{N-1} \frac{\Delta x}{2} w_\eta^{k+1} |\tilde{\Delta}_{j+k+2,-}^n|\right), \\ \mathcal{A}_4 &:= \sum_{j\in\mathbb{Z}} \lambda g(\rho_{j+\frac{1}{2},-}^n) |v'(\zeta_j)| \left[\frac{\Delta x}{2}\left(w_\eta^{N-1} |\tilde{\Delta}_{j+N,+}| + w_\eta^0 |\tilde{\Delta}_{j,+}| + \sum_{k=1}^{N-1} |\tilde{\Delta}_{j+k,+}^n| (w_\eta^{k-1} - w_\eta^k)\right)\right. \\ &\quad \left. + \frac{\Delta x}{2}\left(w_\eta^N |\tilde{\Delta}_{j+N+1,-}^n| + w_\eta^1 |\tilde{\Delta}_{j+1,-}^n| + \sum_{k=1}^{N-1} |\tilde{\Delta}_{j+k+1,-}^n| (w_\eta^k - w_\eta^{k+1})\right)\right].\end{aligned}$$

Note that $\mathcal{A}_2$ is obtained by shifting the index and grouping the second and third terms in (3.27), $\mathcal{A}_3$ is obtained using (3.16) and $\mathcal{A}_4$ using (3.28). The property (3.11) of the reconstruction reads as $\sum_{j\in\mathbb{Z}} |\tilde{\Delta}_{j,\pm}^n| \leq \mathrm{TV}(\rho_{\Delta x}^n)$. Consequently, using the estimate (3.20) the following bound holds

$$|\mathcal{A}_2| \leq \left(\frac{1}{2} + \Delta t \|v'\| w_\eta^0 \|g'\| \rho_{\max}\right) \mathrm{TV}(\rho_{\Delta x}^n).$$

Further, using the estimate (3.30), property (3.11) and using the fact that $0 \leq \sum_{k=0}^{N-1} \Delta x w_\eta^k \leq w_\eta^0 N \Delta x = w_\eta^0 \eta$, we obtain

$$|\mathcal{A}_3| \leq 10 \rho_{\max} \Delta t \|g\| \|v''\| (w_\eta^0)^2 \eta \mathrm{TV}(\rho_{\Delta x}^n).$$

By using property (3.11) and observing that $0 \leq \sum_{k=1}^{N-1} (w_\eta^{k-1} - w_\eta^k) = w_\eta^0 - w_\eta^{N-1} \leq w_\eta^0$ and $0 \leq \sum_{k=1}^{N-1} (w_\eta^k - w_\eta^{k+1}) = w_\eta^1 - w_\eta^N \leq w_\eta^1 \leq w_\eta^0$, we have the bound $|\mathcal{A}_4| \leq 3\|g\| \|v'\| w_\eta^0 \mathrm{TV}(\rho_{\Delta x}^n)$. Finally, we can write

$$\sum_{j\in\mathbb{Z}} |\rho_{j+1}^{n+1} - \rho_j^{n+1}| \leq \mathrm{TV}(\rho_{\Delta x}^n) \left(1 + \Delta t \Big(w_\eta^0 \rho_{\max} \|v'\| \|g'\| + 10 \rho_{\max} (w_\eta^0)^2 \|v''\| \|g\| \eta + 3 w_\eta^0 \|g\| \|v'\|\Big)\right).$$

Hence we obtain the desired bound on the total variation as

$$\mathrm{TV}(\rho_{\Delta x}^{n+1}) \leq (1 + C\Delta t) \mathrm{TV}(\rho_{\Delta x}^n),$$

where $C = w_\eta^0 \rho_{\max} \|v'\| \|g'\| + 10 \rho_{\max} (w_\eta^0)^2 \|v''\| \|g\| \eta + 3 w_\eta^0 \|g\| \|v'\|$. □

Theorem 3.7. *(BV estimate in space) Let the initial data $\rho_0 \in \mathrm{BV}(\mathbb{R};[0,\rho_{\max}])$ and the CFL condition (3.13) holds. Then for every $T>0$ the approximate solution $\rho_{\Delta x}$ obtained using the second-order scheme (3.8) satisfies the space total variation estimate*

$$\mathrm{TV}\big(\rho_{\Delta x}(T,\cdot)\big) \le \exp(2TC)\mathrm{TV}(\rho_0), \tag{3.31}$$

where $C := w_\eta^0 \rho_{\max}\|v'\|\|g'\| + 10\rho_{\max}(w_\eta^0)^2\|v''\|\|g\|\eta + 3w_\eta^0\|g\|\|v'\|$.

Proof. Let $\{\rho_j^{n+1}\}_{j\in\mathbb{Z}}$ be calculated using the second-order scheme (3.8). We can write

$$\mathrm{TV}(\rho_{\Delta x}^{n+1}) = \sum_{j\in\mathbb{Z}}|\rho_{j+1}^{n+1}-\rho_j^{n+1}| \le \frac{1}{2}\Big(\sum_{j\in\mathbb{Z}}|\rho_{j+1}^{n}-\rho_j^{n}| + \sum_{j\in\mathbb{Z}}|\rho_{j+1}^{(2)}-\rho_j^{(2)}|\Big).$$

Applying Lemma 3.6 on the two Euler forward steps in (3.8) we get the following bound,

$$\sum_{j\in\mathbb{Z}}|\rho_{j+1}^{(2)}-\rho_j^{(2)}| \le (1+C\Delta t)\sum_{j\in\mathbb{Z}}|\rho_{j+1}^{(1)}-\rho_j^{(1)}| \le (1+C\Delta t)^2\sum_{j\in\mathbb{Z}}|\rho_{j+1}^{n}-\rho_j^{n}|.$$

Therefore,

$$\begin{aligned}\mathrm{TV}(\rho_{\Delta x}^{n+1}) &\le \frac{1}{2}\Big(\sum_{j\in\mathbb{Z}}|\rho_{j+1}^{n}-\rho_j^{n}| + (1+C\Delta t)^2\sum_{j\in\mathbb{Z}}|\rho_{j+1}^{n}-\rho_j^{n}|\Big) \le (1+C\Delta t)^2\sum_{j\in\mathbb{Z}}|\rho_{j+1}^{n}-\rho_j^{n}| \\ &\le (1+C\Delta t)^{2(n+1)}\sum_{j\in\mathbb{Z}}|\rho_{j+1}^{0}-\rho_j^{0}| \le \exp\big(2\Delta t(n+1)C\big)\mathrm{TV}(\rho_0) \le \exp(2TC)\mathrm{TV}(\rho_0),\end{aligned}$$

whenever $(n+1)\Delta t \le T$. Thus we have $\mathrm{TV}\big(\rho_{\Delta x}(T,\cdot)\big) \le \exp(2TC)\mathrm{TV}(\rho_0)$. □

3.3. L^1-Lipschitz continuity in time

Lemma 3.8. *Let $\rho_{\Delta x}^n \in \mathrm{BV}(\mathbb{R};[0,\rho_{\max}])$ be the piecewise constant function given by $\rho_{\Delta x}^n(x) = \rho_j^n$ for $x \in (x_{j-\frac12}, x_{j+\frac12}]$. If $\rho_{\Delta x}^{n+1}$ is computed using the Euler forward step (3.14) with the CFL condition (3.13), then the following estimate holds*

$$\|\rho_{\Delta x}^{n+1}-\rho_{\Delta x}^{n}\|_{\mathrm{L}^1(\mathbb{R})} \le \Delta t\Big(\|g\|\|v'\|w_\eta^0\eta + \|g'\|\|v\|\Big)\mathrm{TV}(\rho_{\Delta x}^n). \tag{3.32}$$

Proof. From (3.14) we write

$$\|\rho_{\Delta x}^{n+1}-\rho_{\Delta x}^{n}\|_{\mathrm{L}^1(\mathbb{R})} = \Delta x\sum_{j\in\mathbb{Z}}|\rho_j^{n+1}-\rho_j^n| = \Delta t\sum_{j\in\mathbb{Z}}|\bar F_{j-\frac12}^n - \bar F_{j+\frac12}^n|,$$

where $\bar F_{j+\frac12} := g(\rho_{j+\frac12,-}^n)V_{j+\frac12}^n$. Subtracting and adding $g(\rho_{j-\frac12,-}^n)V_{j+\frac12}^n$ to the term $\bar F_{j-\frac12}^n - \bar F_{j+\frac12}^n$ and using the mean value theorem, we deduce that

$$\begin{aligned}\Big|\bar F_{j-\frac12}^n - \bar F_{j+\frac12}^n\Big| &= \Big|g(\rho_{j-\frac12,-}^n)(V_{j-\frac12}^n - V_{j+\frac12}^n) + \big(g(\rho_{j-\frac12,-}^n) - g(\rho_{j+\frac12,-}^n)\big)V_{j+\frac12}^n\Big| \\ &= \Big|-v'(\zeta_j)g(\rho_{j+\frac12,-}^n)(R_{j+\frac12}^n - R_{j-\frac12}^n) + g'(\theta_{j,-})\big(\rho_{j-\frac12,-}^n - \rho_{j+\frac12,-}^n\big)V_{j+\frac12}^n\Big| \\ &\le \|g\|\|v'\|\frac{\Delta x}{2}\sum_{k=0}^{N-1}\Big(w_\eta^k|\rho_{j+k+\frac12,+}^n - \rho_{j+k-\frac12,+}^n| + w_\eta^{k+1}|\rho_{j+k+\frac32,-}^n - \rho_{j+k+\frac12,-}^n|\Big) \\ &\quad + \|g'\|\|v\||\rho_{j-\frac12,-}^n - \rho_{j+\frac12,-}^n| \quad \text{for } \zeta_j \in \mathcal{I}(R_{j-\frac12}^n, R_{j+\frac12}^n) \text{ and } \theta_{j,-} \in \mathcal{I}(\rho_{j-\frac12,-}^n, \rho_{j+\frac12,-}^n).\end{aligned}$$

Further, invoking the properties (3.11) and (3.2) yields

$$\begin{aligned}
\|\rho^{n+1}_{\Delta x} - \rho^n_{\Delta x}\|_{\mathrm{L}^1(\mathbb{R})} &= \Delta t \sum_{j\in\mathbb{Z}} |\bar{F}^n_{j-\frac{1}{2}} - \bar{F}^n_{j+\frac{1}{2}}| \\
&\le \Delta t \left[\|g\|\|v'\| \frac{\Delta x}{2} \sum_{k=0}^{N-1} \left(w^k_\eta \sum_{j\in\mathbb{Z}} |\rho^n_{j+k+\frac{1}{2},+} - \rho^n_{j+k-\frac{1}{2},+}| + w^{k+1}_\eta \sum_{j\in\mathbb{Z}} |\rho^n_{j+k+\frac{3}{2},-} - \rho^n_{j+k+\frac{1}{2},-}| \right) \right. \\
&\qquad \left. + \|g'\|\|v\| \sum_{j\in\mathbb{Z}} |\rho^n_{j-\frac{1}{2},-} - \rho^n_{j+\frac{1}{2},-}| \right] \\
&\le \Delta t \left(\|g\|\|v'\| \mathrm{TV}(\rho^n_{\Delta x}) w^0_\eta N \Delta x + \|g'\|\|v\| \mathrm{TV}(\rho^n_{\Delta x}) \right) \\
&= \Delta t \left(\|g\|\|v'\| w^0_\eta \eta + \|g'\|\|v\| \right) \mathrm{TV}(\rho^n_{\Delta x}).
\end{aligned}$$

□

Theorem 3.9. *(L^1- Lipschitz continuity in time) Let $\rho_0 \in \mathrm{BV}(\mathbb{R}; [0, \rho_{\max}])$ and the CFL condition (3.13) holds. Then the approximate solution constructed using the second-order scheme (3.8) is an L^1- Lipschitz continuous function of time, i.e, for any $T > 0$, there exists a constant κ_T such that*

$$\|\rho_{\Delta x}(t,\cdot) - \rho_{\Delta x}(s,\cdot)\|_{\mathrm{L}^1(\mathbb{R})} \le \kappa_T(|t-s| + \Delta t) \quad \textit{for } t, s \in [0, T]. \tag{3.33}$$

Proof. For the second-order scheme (3.8), we see that

$$\|\rho^{n+1}_{\Delta x} - \rho^n_{\Delta x}\|_{\mathrm{L}^1(\mathbb{R})} = \Delta x \sum_{j\in\mathbb{Z}} \left| \frac{\rho^n_j + \rho^{(2)}_j}{2} - \rho^n_j \right| = \frac{\Delta x}{2} \sum_{j\in\mathbb{Z}} |\rho^{(2)}_j - \rho^n_j|. \tag{3.34}$$

Upon subtracting and adding $\rho^{(1)}_j$ to the term $\rho^{(2)}_j - \rho^n_j$, employing Lemma 3.8 together with Lemma 3.6 and subsequently using Theorem 3.7, it follows from (3.34) that

$$\begin{aligned}
\|\rho^{n+1}_{\Delta x} - \rho^n_{\Delta x}\|_{\mathrm{L}^1(\mathbb{R})} &\le \frac{\Delta x}{2} \left(\sum_{j\in\mathbb{Z}} |\rho^{(2)}_j - \rho^{(1)}_j| + \sum_{j\in\mathbb{Z}} |\rho^{(1)}_j - \rho^n_j| \right) \\
&\le \frac{\Delta t}{2} \left(\mathrm{TV}\big(\rho^{(1)}_{\Delta x}(t^n,\cdot)\big)(\|g\|\|v'\| w^0_\eta \eta + \|g'\|\|v\|) + \mathrm{TV}\big(\rho^n_{\Delta x}\big)(\|g\|\|v'\| w^0_\eta \eta + \|g'\|\|v\|) \right) \\
&\le \frac{\Delta t}{2} (\|g\|\|v'\| w^0_\eta \eta + \|g'\|\|v\|) \left((1 + C\Delta t) \mathrm{TV}\big(\rho^n_{\Delta x}\big) + \mathrm{TV}\big(\rho^n_{\Delta x}\big) \right) \\
&\le \frac{\Delta t}{2} (\|g\|\|v'\| w^0_\eta \eta + \|g'\|\|v\|) \mathrm{TV}\big(\rho^n_{\Delta x}\big) \Big(2 + C\Delta t\Big) \\
&\le \Delta t (\|g\|\|v'\| w^0_\eta \eta + \|g'\|\|v\|) \exp(2TC) \mathrm{TV}(\rho_0) \left(1 + C\frac{\Delta t}{2}\right),
\end{aligned}$$

provided that the CFL condition (3.13) holds. Furthermore, we can write

$$\|\rho^m_{\Delta x} - \rho^n_{\Delta x}\|_{\mathrm{L}^1(\mathbb{R})} \le \|\rho^m_{\Delta x} - \rho^{m-1}_{\Delta x}\|_{\mathrm{L}^1(\mathbb{R})} + \cdots + \|\rho^{n+1}_{\Delta x} - \rho^n_{\Delta x}\|_{\mathrm{L}^1(\mathbb{R})} \le \Delta t \tilde{\kappa}_T + \cdots + \Delta t \tilde{\kappa}_T = \tilde{\kappa}_T |n-m| \Delta t,$$

with $\tilde{\kappa}_T = \big(\|g\|\|v'\| w^0_\eta \eta + \|g'\|\|v\|\big) \exp(2TC) \mathrm{TV}(\rho_0) \Big(1 + \frac{\Delta t}{2} C\Big)$ and for $m, n \in \mathbb{N}, m > n$ with $m\Delta t < T$ and $n\Delta t < T$. Thus we can conclude that for sufficiently small Δt,

$$\|\rho^n_{\Delta x} - \rho^m_{\Delta x}\|_{\mathrm{L}^1(\mathbb{R})} \le \kappa_T |n-m| \Delta t \tag{3.35}$$

for $n, m \in \mathbb{N}$ with $n\Delta t < T$ and $m\Delta t < T$, where

$$\kappa_T = \big(\|g\|\|v'\|w_\eta^0\eta + \|g'\|\|v\|\big)\exp(2TC)\mathrm{TV}(\rho_0)(1+C).$$

Now, for $t, s \in [0,T]$, let n, m be such that $t \in [t^n, t^{n+1})$ and $s \in [t^m, t^{m+1})$. Since $|n-m|\Delta t \le |t-s| + \Delta t$, from (3.35) it follows that

$$\|\rho_{\Delta x}(t,\cdot) - \rho_{\Delta x}(s,\cdot)\|_{\mathrm{L}^1(\mathbb{R})} \le \kappa_T(|t-s| + \Delta t) \quad \text{for } t, s \in [0,T]. \tag{3.36}$$

□

4. Convergence to a weak solution

The results in Theorems 3.4, 3.7 and 3.9 allow us to use the Kolmogorov's theorem (as described in Theorem A.8 of [53]) to extract a convergence subsequence of approximate solutions obtained using the second-order scheme (3.8). We have adapted the Kolmogorov's theorem to fit our specific context. For the sake of completeness, we state the modified theorem here and the proof is given in the Appendix A. Further, we use a Lax-Wendroff type argument to show that the limit is a weak solution.

Theorem 4.1. *Let $u_\xi : [0,+\infty) \times \mathbb{R} \to \mathbb{R}$ be a family of functions such that for each positive T,*

$$|u_\xi(t,x)| \le \mu_T \tag{4.1}$$

for $(t,x) \in [0,T) \times \mathbb{R}$ and a constant μ_T independent of ξ. Assume in addition for all compact set $B \subset \mathbb{R}$ and for $t \in [0,T]$ that

$$\sup_{|\zeta| \le |\tau|} \int_B |u_\xi(t, x+\zeta) - u_\xi(t,x)|\,\mathrm{d}x \le \nu_T^B|\tau|, \tag{4.2}$$

for a modulus of continuity ν_T^B . Furthermore, assume for s and t in $[0,T]$ that

$$\int_B |u_\xi(t,x) - u_\xi(s,x)|\,\mathrm{d}x \le \omega_T^B(|t-s|) + \mathcal{O}(\xi), \tag{4.3}$$

as $\xi \to 0$ for some modulus of continuity ω_T^B. Then there exists a sequence $\xi_j \to 0$ such that for each $t \in [0,T]$ the sequence $\{u_{\xi_j}(t,\cdot)\}$ converges to a function $u(t,\cdot)$ in $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})$. Furthermore, the convergence is in $\mathrm{C}([0,T]; \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}))$.

Now, in the following theorem we establish the convergence of a subsequence of the approximate solutions to a weak solution of the problem (2.1).

Theorem 4.2. *(Convergence to a weak solution) Let $\rho_0 \in \mathrm{BV}(\mathbb{R}; [0, \rho_{\max}])$ and let $\rho_{\Delta x}$ be the approximate solution obtained using the second-order scheme (3.8) under the CFL condition (3.13). Then corresponding to any sequence $\Delta x_k \to 0$, there exists a subsequence, still denoted by Δx_k, such that $\rho_{\Delta x_k}$ converges in $\mathrm{C}([0,T]; \mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}))$ to a weak solution of (2.1).*

Proof. Firstly, the existence of a convergent subsequence is proven by using the Kolmogorov's Theorem 4.1 invoking the estimates derived in Theorems 3.4, 3.7 and 3.9 given by

$$\|\rho_{\Delta x}\| \le \|\rho_0\|, \tag{4.4}$$

$$\mathrm{TV}(\rho_{\Delta x}(t,\cdot)) \le \exp(2TC)\mathrm{TV}(\rho_0) \quad \text{for } t \in [0,T] \tag{4.5}$$

and

$$\|\rho_{\Delta x}(t,\cdot)-\rho_{\Delta x}(s,\cdot)\|_{\mathrm{L}^1(\mathbb{R})}\le \kappa_T(|t-s|+\Delta t)\quad \text{for } t,s\in[0,T], \tag{4.6}$$

respectively. Under the CFL conditon (3.13), the family $\{\rho_{\Delta x}\}$ obtained from the second-order scheme (3.8) satisfies (4.1) with $\mu_T=\|\rho_0\|$. By Lemma A.1 of [53], the total variation bound (4.5) ensures that the family satisfies (4.2) with $\nu_T^B=\exp(2TC)\mathrm{TV}(\rho_0)$. Additionally, using (4.6), we observe that the family $\{\rho_{\Delta x}\}$ satisfies (4.3) with $\omega_T^B=\kappa_T$. Now by Theorem 4.1, corresponding to any sequence $\Delta x_k\to 0$, there exists a subsequence, still denoted by Δx_k, such that $\rho_{\Delta x_k}$ converges to a function ρ in $\mathrm{C}([0,T];\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R}))$ and consequently in $\mathrm{L}^1_{\mathrm{loc}}([0,T)\times\mathbb{R})$.

Our next step is to show that the limit ρ is a weak solution of (2.1). Typically, we will use a Lax-Wendroff type argument [60], with certain modifications to deal with the numerical flux which also depends on Δx. Denote the convergent subsequence obtained above by $\rho_{\Delta x}$. Let $\varphi\in\mathrm{C}^1_c([0,T)\times\mathbb{R})$. Let T_φ be such that $0\le T_\varphi<T$ and $\varphi(t,x)=0$ for $t\ge T_\varphi$ and let n_T be such that $T_\varphi\in(n_T\Delta t,(n_T+1)\Delta t]$. Multiplying the conservative form (3.9) by $\varphi(t^n,x_j)$ and summing over n and j yields

$$\sum_{n=0}^{n_T}\sum_{j\in\mathbb{Z}}\varphi(t^n,x_j)(\rho_j^{n+1}-\rho_j^n)=-\lambda\sum_{n=0}^{n_T}\sum_{j\in\mathbb{Z}}\varphi(t^n,x_j)(F^n_{j+\frac12}-F^n_{j-\frac12}).$$

Further, summing by parts we get

$$\sum_{j\in\mathbb{Z}}\varphi(0,x_j)\rho_j^0+\sum_{n=0}^{n_T}\sum_{j\in\mathbb{Z}}\Big(\varphi(t^{n+1},x_j)-\varphi(t^n,x_j)\Big)\rho_j^{n+1}+\lambda\sum_{n=0}^{n_T}\sum_{j\in\mathbb{Z}}F^n_{j+\frac12}\Big(\varphi(t^n,x_{j+1})-\varphi(t^n,x_j)\Big)=0.$$

Now, multiplying the above expression by Δx we see that

$$\mathcal{P}_1+\mathcal{P}_2+\mathcal{P}_3=0, \tag{4.7}$$

where we define the terms

$$\mathcal{P}_1\coloneqq\Delta x\sum_{j\in\mathbb{Z}}\varphi(0,x_j)\rho_j^0,\quad \mathcal{P}_2\coloneqq\Delta t\Delta x\sum_{n=0}^{n_T}\sum_{j\in\mathbb{Z}}\frac{(\varphi(t^{n+1},x_j)-\varphi(t^n,x_j))}{\Delta t}\rho_j^{n+1},$$
$$\mathcal{P}_3\coloneqq\Delta t\Delta x\sum_{n=0}^{n_T}\sum_{j\in\mathbb{Z}}F^n_{j+\frac12}\frac{(\varphi(t^n,x_{j+1})-\varphi(t^n,x_j))}{\Delta x}.$$

We can also write

$$\mathcal{P}_1+\mathcal{P}_2=\int_{-\infty}^{+\infty}\rho_{\Delta x}(0,x)\varphi_{\Delta x}(0,x)\,\mathrm{d}x+\int_{t=0}^{T}\int_{-\infty}^{+\infty}\rho_{\Delta x}(t+\Delta t,x)\partial_t\varphi_{\Delta x}(t,x)\,\mathrm{d}x\,\mathrm{d}t, \tag{4.8}$$

where

$$\varphi_{\Delta x}(0,x)\coloneqq\varphi(0,x_j)\quad\text{for } x\in(x_{j-\frac12},x_{j+\frac12}],$$
$$\partial_t\varphi_{\Delta x}(t,x)\coloneqq\varphi_t(\bar t,x_j)\quad\text{for } t\in[t^n,t^{n+1}),x\in(x_{j-\frac12},x_{j+\frac12}]\text{ for some }\bar t\in(t^n,t^{n+1}).$$

By the dominated convergence theorem it follows that

$$\lim_{\Delta x\to 0}(\mathcal{P}_1+\mathcal{P}_2)=\int_{-\infty}^{+\infty}\rho(0,x)\varphi(0,x)\,\mathrm{d}x+\int_{t=0}^{T}\int_{-\infty}^{+\infty}\rho(t,x)\varphi_t(t,x)\,\mathrm{d}x\,\mathrm{d}t.$$

We define $R_j^n := \Delta x \sum_{k=0}^{N-1} \hat{w}_\eta^k \rho_{j+k}^n$, $R_j^{(1)} := \Delta x \sum_{k=0}^{N-1} \hat{w}_\eta^k \rho_{j+k}^{(1)}$, $V_j^n := v(R_j^n)$ and $V_j^{(1)} := v(R_j^{(1)})$, where $\hat{w}_\eta^k := \dfrac{w_\eta^k}{\hat{Q}_{\Delta x}}$ and $\hat{Q}_{\Delta x} := \Delta x \displaystyle\sum_{k=0}^{N-1} w_\eta^k$. Note that $\hat{Q}_{\Delta x} \approx \int_0^\eta w_\eta(y)\,\mathrm{d}y = 1$. Further, there exists a constant $L > 0$ such that $\left|\dfrac{\hat{Q}_{\Delta x} - 1}{\hat{Q}_{\Delta x}}\right| \le L\Delta x$ for sufficiently small Δx. Here we use the modified weights $\hat{w}_\eta^k$ to ensure that R_j^n and $R_j^{(1)}$ fall in the range $[0, \rho_{\max}]$. By adding and subtracting $\frac{1}{2}\Delta t\big(g(\rho_j^n)V_j^n + g(\rho_j^{(1)})V_j^{(1)}\big)\big(\varphi(t^n, x_{j+1}) - \varphi(t^n, x_j)\big)$ into the term $\mathcal{P}_3$ in (4.7), it reads as

$$\mathcal{P}_3 = \mathcal{S}_1 + \mathcal{S}_2, \tag{4.9}$$

where we define

$$\begin{aligned}
\mathcal{S}_1 &:= \Delta t \Delta x \sum_{n=0}^{n_T} \sum_{j\in\mathbb{Z}} \left(F_{j+\frac{1}{2}}^n - \frac{g(\rho_j^n)V_j^n + g(\rho_j^{(1)})V_j^{(1)}}{2} \right) \frac{\big(\varphi(t^n, x_{j+1}) - \varphi(t^n, x_j)\big)}{\Delta x}, \\
\mathcal{S}_2 &:= \Delta t \Delta x \sum_{n=0}^{n_T} \sum_{j\in\mathbb{Z}} \left(\frac{g(\rho_j^n)V_j^n + g(\rho_j^{(1)})V_j^{(1)}}{2} \right) \frac{\big(\varphi(t^n, x_{j+1}) - \varphi(t^n, x_j)\big)}{\Delta x}.
\end{aligned}$$

Further, by the mean value theorem we observe that

$$\begin{aligned}
g(\rho_j^{(1)}) &= g\left(\rho_j^n - \lambda\big(g(\rho_{j+\frac{1}{2},-}^n)V_{j+\frac{1}{2}}^n - g(\rho_{j-\frac{1}{2},-}^n)V_{j-\frac{1}{2}}^n\big)\right) \\
&= g(\rho_j^n) - \lambda g'(\zeta_j)\left(g(\rho_{j+\frac{1}{2},-}^n)V_{j+\frac{1}{2}}^n - g(\rho_{j-\frac{1}{2},-}^n)V_{j-\frac{1}{2}}^n\right) \quad \text{for some } \zeta_j \in \mathcal{I}(\rho_j^n, \rho_j^{(1)}).
\end{aligned} \tag{4.10}$$

Similarly,

$$\begin{aligned}
V_j^{(1)} &= v\left(\Delta x \sum_{k=0}^{N-1} \hat{w}_\eta^k \rho_{j+k}^{(1)}\right) \\
&= v\left(\Delta x \sum_{k=0}^{N-1} \hat{w}_\eta^k \rho_{j+k}^n - \Delta t \sum_{k=0}^{N-1} \hat{w}_\eta^k \Big(g(\rho_{j+k+\frac{1}{2},-}^n)V_{j+k+\frac{1}{2}}^n - g(\rho_{j+k-\frac{1}{2},-}^n)V_{j+k-\frac{1}{2}}^n\Big)\right) \\
&= v\left(\Delta x \sum_{k=0}^{N-1} \hat{w}_\eta^k \rho_{j+k}^n\right) - \Delta t v'(\theta_j) \sum_{k=0}^{N-1} \hat{w}_\eta^k \left(g(\rho_{j+k+\frac{1}{2},-}^n)V_{j+k+\frac{1}{2}}^n - g(\rho_{j+k-\frac{1}{2},-}^n)V_{j+k-\frac{1}{2}}^n\right) \\
&= V_j^n - \Delta t v'(\theta_j) \sum_{k=0}^{N-1} \hat{w}_\eta^k \left(g(\rho_{j+k+\frac{1}{2},-}^n)V_{j+k+\frac{1}{2}}^n - g(\rho_{j+k-\frac{1}{2},-}^n)V_{j+k-\frac{1}{2}}^n\right) \quad \text{for some } \theta_j \in \mathcal{I}(R_j^n, R_j^{(1)}).
\end{aligned}$$

As a result, the term $\mathcal{S}_2$ in (4.9) can be written as

$$\mathcal{S}_2 = \mathcal{T}_1 + \mathcal{T}_2 + \mathcal{T}_3 + \mathcal{T}_4, \tag{4.11}$$

where

$$\mathcal{T}_1 \coloneqq \Delta t \Delta x \sum_{n=0}^{n_T} \sum_{j\in\mathbb{Z}} g(\rho_j^n) V_j^n \frac{\big(\varphi(t^n, x_{j+1}) - \varphi(t^n, x_j)\big)}{\Delta x},$$

$$\mathcal{T}_2 \coloneqq -\frac{1}{2}\Delta t^2 \Delta x \sum_{n=0}^{n_T} \sum_{j\in\mathbb{Z}} \sum_{k=0}^{N-1} g(\rho_j^n) v'(\theta_j) \ell(n,j,k) \frac{\big(\varphi(t^n, x_{j+1}) - \varphi(t^n, x_j)\big)}{\Delta x},$$

$$\mathcal{T}_3 \coloneqq -\frac{1}{2}\Delta t \Delta x \sum_{n=0}^{n_T} \sum_{j\in\mathbb{Z}} \lambda V_j^n g'(\zeta_j) \left(g(\rho^n_{j+\frac{1}{2},-}) V^n_{j+\frac{1}{2}} - g(\rho^n_{j-\frac{1}{2},-}) V^n_{j-\frac{1}{2}} \right) \frac{\big(\varphi(t^n, x_{j+1}) - \varphi(t^n, x_j)\big)}{\Delta x},$$

$$\mathcal{T}_4 \coloneqq \frac{1}{2}\Delta t^2 \Delta x \sum_{n=0}^{n_T} \sum_{j\in\mathbb{Z}} \sum_{k=0}^{N-1} \lambda g'(\zeta_j) \left(g(\rho^n_{j+\frac{1}{2},-}) V^n_{j+\frac{1}{2}} - g(\rho^n_{j-\frac{1}{2},-}) V^n_{j-\frac{1}{2}} \right) v'(\theta_j) \ell(n,j,k) \frac{\big(\varphi(t^n, x_{j+1}) - \varphi(t^n, x_j)\big)}{\Delta x},$$

with the definition $\ell(n,j,k) \coloneqq \hat{w}_\eta^k \left(g(\rho^n_{j+k+\frac{1}{2},-}) V^n_{j+k+\frac{1}{2}} - g(\rho^n_{j+k-\frac{1}{2},-}) V^n_{j+k-\frac{1}{2}} \right)$.
Note that, the term $\mathcal{T}_1$ can also be written as

$$\mathcal{T}_1 = \int_0^T \int_{-\infty}^{+\infty} g\left(\rho_{\Delta x}(t,x)\right) v\left(R_{\Delta x}(t,x)\right) \partial_x \varphi_{\Delta x}(t,x) \,\mathrm{d}x \,\mathrm{d}t,$$

where

$$R_{\Delta x}(t,x) \coloneqq R_j^n \quad \text{for } x \in (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}], t \in [t^n, t^{n+1}),$$
$$\partial_x \varphi_{\Delta x}(t,x) \coloneqq \varphi_x(t^n, \bar{x}) \quad \text{for } x \in (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}],\ t \in [t^n, t^{n+1}) \text{ for some } \bar{x} \in (x_j, x_{j+1}).$$

Note that

$$R_{\Delta x}(t,x) = \int_{x_{j-\frac{1}{2}}}^{x_{j-\frac{1}{2}}+\eta} \rho_{\Delta x}(t,y) w_{\eta,\Delta x}(y - x_{j-\frac{1}{2}}) \,\mathrm{d}y \quad \text{for } x \in (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}], t \in [t^n, t^{n+1}),$$

where $w_{\eta,\Delta x}(x) \coloneqq \hat{w}_\eta^k$ for $x \in (k\Delta x, (k+1)\Delta x]$. By the dominated convergence theorem, it is clear that $R_{\Delta x}(t,x)$ converges to $\int_x^{x+\eta} \rho(t,y) w_\eta(y-x) \,\mathrm{d}y$ as $\Delta x \to 0$. Now, applying the dominated convergence theorem again, we have

$$\lim_{\Delta x \to 0} \mathcal{T}_1 = \int_0^T \int_{-\infty}^{+\infty} g\big(\rho(t,x)\big) v\big(\rho * w_\eta(t,x)\big) \varphi_x(t,x) \,\mathrm{d}x \,\mathrm{d}t.$$

We will now show that the terms $\mathcal{T}_2, \mathcal{T}_3$ and $\mathcal{T}_4$ in (4.11) tend to 0 as $\Delta x \to 0$. To proceed further, we consider the following fact

$$\begin{aligned} \left| g(\rho^n_{j+\frac{1}{2},-}) V^n_{j+\frac{1}{2}} - g(\rho^n_{j-\frac{1}{2},-}) V^n_{j-\frac{1}{2}} \right| &\le \|g\| \|v'\| \Delta x w_\eta^0 \rho_{\max} + \|v\| \|g'\| |\rho^n_{j+\frac{1}{2},-} - \rho^n_{j-\frac{1}{2},-}| \\ &\le \|g\| \|v'\| \Delta x w_\eta^0 \rho_{\max} + 2\|v\| \|g'\| |\rho_j^n - \rho_{j-1}^n|, \end{aligned} \tag{4.12}$$

which is obtained by writing

$$\begin{aligned} g(\rho^n_{j+\frac{1}{2},-}) V^n_{j+\frac{1}{2}} - g(\rho^n_{j-\frac{1}{2},-}) V^n_{j-\frac{1}{2}} &= g(\rho^n_{j+\frac{1}{2},-})(V^n_{j+\frac{1}{2}} - V^n_{j-\frac{1}{2}}) + V^n_{j-\frac{1}{2}} \big(g(\rho^n_{j+\frac{1}{2},-}) - g(\rho^n_{j-\frac{1}{2},-}) \big) \\ &= g(\rho^n_{j+\frac{1}{2},-})(V^n_{j+\frac{1}{2}} - V^n_{j-\frac{1}{2}}) + V^n_{j-\frac{1}{2}} g'(\zeta_{j,-})(\rho^n_{j+\frac{1}{2},-} - \rho^n_{j-\frac{1}{2},-}), \end{aligned}$$

for some $\zeta_{j,-} \in \mathcal{I}(\rho^n_{j-\frac{1}{2},-}, \rho^n_{j+\frac{1}{2},-})$ and using (3.20) and property (3.12). Now, using summation by parts, the term in $\mathcal{T}_2$ can be reformulated as

$$\sum_{k=0}^{N-1} \ell(n,j,k) = \hat{w}_\eta^{N-1} g(\rho^n_{j+N-\frac{1}{2},-}) V^n_{j+N-\frac{1}{2}} - \hat{w}_\eta^0 g(\rho^n_{j-\frac{1}{2},-}) V^n_{j-\frac{1}{2}} + \sum_{k=1}^{N-1} g(\rho^n_{j+k-\frac{1}{2},-}) V^n_{j+k-\frac{1}{2}} (\hat{w}_\eta^{k-1} - \hat{w}_\eta^k).$$

Taking absolute values and using property (3.1), we deduce that

$$\begin{aligned} \left| \sum_{k=0}^{N-1} \ell(n,j,k) \right| &\le 2\hat{w}_\eta^0 \|g\|\|v\| + \|g\|\|v\| \sum_{k=1}^{N-1} (\hat{w}_\eta^{k-1} - \hat{w}_\eta^k) \\ &\le 2\hat{w}_\eta^0 \|g\|\|v\| + \|g\|\|v\| \hat{w}_\eta^0 \le 3\hat{w}_\eta^0 \|g\|\|v\|. \end{aligned} \tag{4.13}$$

Let $R > 0$ be such that $\varphi(t,x) = 0$ for $|x| > R$. Let $j_0, j_1 \in \mathbb{Z}$ such that $-R \in (x_{j_0-\frac{1}{2}}, x_{j_0+\frac{1}{2}}]$ and $R \in (x_{j_1-\frac{1}{2}}, x_{j_1+\frac{1}{2}}]$. By using the estimate (4.13) and the mean value theorem, we obtain a bound on the term $\mathcal{T}_2$ as

$$\begin{aligned} |\mathcal{T}_2| &\le \frac{1}{2} \Delta t^2 \Delta x \|g\|\|v'\|\|\varphi_x\| \sum_{n=0}^{n_T} \sum_{j=j_0}^{j_1} \left| \sum_{k=0}^{N-1} \ell(n,j,k) \right| \\ &\le \frac{3}{2} \hat{w}_\eta^0 \Delta t \|g\|^2 \|v'\|\|v\|\|\varphi_x\| \sum_{n=0}^{n_T} \sum_{j=j_0}^{j_1} \Delta t \Delta x \le 3\hat{w}_\eta^0 \Delta t \|g\|^2 \|v'\|\|v\|\|\varphi_x\| RT. \end{aligned}$$

Additionally, using (4.12) and Theorem 3.7, the term $\mathcal{T}_3$ in (4.11) can be bounded as

$$\begin{aligned} |\mathcal{T}_3| &\le \frac{1}{2} \Delta t \Delta x \|\varphi_x\| \sum_{n=0}^{n_T} \sum_{j=j_0}^{j_1} \lambda V_j^n g'(\zeta_j) \left| g(\rho^n_{j+\frac{1}{2},-}) V^n_{j+\frac{1}{2}} - g(\rho^n_{j-\frac{1}{2},-}) V^n_{j-\frac{1}{2}} \right| \\ &\le \frac{1}{2} \Delta t \Delta x \|\varphi_x\| \sum_{n=0}^{n_T} \sum_{j=j_0}^{j_1} \lambda V_j^n g'(\zeta_j) \left(\|g\|\|v'\| \Delta x w_\eta^0 \rho_{\max} + 2\|v\|\|g'\| |\rho_j^n - \rho_{j-1}^n| \right) \\ &\le \frac{1}{2} \lambda \|\varphi_x\|\|v\|\|g'\|\|g\|\|v'\| \Delta x w_\eta^0 \rho_{\max} \sum_{n=0}^{n_T} \sum_{j=j_0}^{j_1} \Delta t \Delta x + \Delta t \Delta x \lambda \|\varphi_x\|\|v\|^2 \|g'\|^2 \sum_{n=0}^{n_T} \sum_{j=j_0}^{j_1} |\rho_j^n - \rho_{j-1}^n| \\ &\le \frac{1}{2} \Delta t \|\varphi_x\|\|v\|\|g'\|\|g\|\|v'\| w_\eta^0 \rho_{\max} 2RT + \lambda \|\varphi_x\|\|v\|^2 \|g'\|^2 \Delta x \int_0^T \int_{-R}^R \frac{|\rho_{\Delta x}(t,x) - \rho_{\Delta x}(t, x - \Delta x)|}{\Delta x} \, \mathrm{d}x \, \mathrm{d}t \\ &\le \Delta t \|\varphi_x\|\|v\|\|g'\|\|g\|\|v'\| w_\eta^0 \rho_{\max} RT + \Delta t \|\varphi_x\|\|v\|^2 \|g'\|^2 \exp(2TC) \mathrm{TV}(\rho_0) T, \end{aligned}$$

where C is as given in Theorem 3.7. Furthermore, using the Theorem 3.7 and the estimates (4.12) and (4.13), we obtain a bound for the term $\mathcal{T}_4$ in (4.11) as

$$\begin{aligned} |\mathcal{T}_4| &\le \frac{1}{2} \Delta t^2 \Delta x \sum_{n=0}^{n_T} \sum_{j=j_0}^{j_1} \lambda \|g'\|\|v'\| \left(\|g\|\|v'\| \Delta x w_\eta^0 \rho_{\max} + 2\|v\|\|g'\| |\rho_j^n - \rho_{j-1}^n| \right) 3\hat{w}_\eta^0 \|g\|\|v\|\|\varphi_x\| \\ &\le \frac{3}{2} \Delta t \lambda \|g'\|\|v'\| \left(\|g\|\|v'\| \Delta x w_\eta^0 \rho_{\max} \right) \hat{w}_\eta^0 \|g\|\|v\|\|\varphi_x\| 2RT \\ &\quad + 3\Delta t \lambda \|g'\|\|v'\|\|v\|\|g'\| \hat{w}_\eta^0 \|g\|\|v\|\|\varphi_x\| \Delta x \int_{-R}^R \int_0^T \frac{|\rho_{\Delta x}(t,x) - \rho_{\Delta x}(t, x - \Delta x)|}{\Delta x} \, \mathrm{d}x \, \mathrm{d}t \\ &\le \frac{3}{2} \Delta t^2 \|g'\|\|v'\|^2 \|g\|^2 \|v\| w_\eta^0 \hat{w}_\eta^0 \|\varphi_x\| 2RT \rho_{\max} \\ &\quad + 3\Delta t^2 \|g'\|^2 \|v'\|\|g\|\|v\|^2 \hat{w}_\eta^0 \|\varphi_x\| \exp(2TC) \mathrm{TV}(\rho_0) T. \end{aligned}$$

Thus, from the estimates obtained for the terms $\mathcal{T}_2, \mathcal{T}_3$ and $\mathcal{T}_4$, we can conclude that

$$\lim_{\Delta x\to 0}\mathcal{T}_2 = \lim_{\Delta x\to 0}\mathcal{T}_3 = \lim_{\Delta x\to 0}\mathcal{T}_4 = 0.$$

Therefore,

$$\lim_{\Delta x\to 0}\mathcal{S}_2 = \lim_{\Delta x\to 0}\Big(\mathcal{T}_1+\mathcal{T}_2+\mathcal{T}_3+\mathcal{T}_4\Big) = \int_0^T\int_{-\infty}^{+\infty} g\big(\rho(t,x)\big)v\big(\rho * w_\eta(t,x)\big)\varphi_x(t,x)\,\mathrm{d}x\,\mathrm{d}t. \tag{4.14}$$

Finally, we show that the term $\mathcal{S}_1$ in (4.9) converges to 0 as $\Delta x\to 0$. Now, using the form of $F^n_{j+\frac{1}{2}}$ in (3.9), the term $\mathcal{S}_1$ can be expressed as

$$\mathcal{S}_1 = \Delta t\Delta x\sum_{n=0}^{n_T}\sum_{j=j_0}^{j_1}(\mathcal{D}_1+\mathcal{D}_2)\frac{(\varphi(t^n,x_{j+1})-\varphi(t^n,x_j))}{\Delta x}, \tag{4.15}$$

where the terms $\mathcal{D}_1$ and $\mathcal{D}_2$ are defined as

$$\mathcal{D}_1 := \frac{\Big(g(\rho^n_{j+\frac{1}{2},-})V^n_{j+\frac{1}{2}} - g(\rho^n_j)V^n_j\Big)}{2},\quad \mathcal{D}_2 := \frac{\Big(g(\rho^{(1)}_{j+\frac{1}{2},-})V^{(1)}_{j+\frac{1}{2}} - g(\rho^{(1)}_j)V^{(1)}_j\Big)}{2}.$$

Next, with the observation $\rho^{(1)}_j = \frac{1}{2}(\rho^{(1)}_{j-\frac{1}{2},+}+\rho^{(1)}_{j+\frac{1}{2},-})$ and using the property (3.12) in conjunction with the Lemmas 3.3 and 3.6, we obtain a bound on the distance between $R^{(1)}_{j+\frac{1}{2}}$ and $R^{(1)}_j$as

$$\begin{aligned}
|R^{(1)}_{j+\frac{1}{2}} - R^{(1)}_j| &= \frac{\Delta x}{2}\Bigg|\sum_{k=0}^{N-1} w^k_\eta\left(\rho^{(1)}_{j+k+\frac{1}{2},+} - \frac{\rho^{(1)}_{j+k-\frac{1}{2},+}}{\hat{Q}_{\Delta x}}\right) + \sum_{k=0}^{N-1} w^{k+1}_\eta\left(\rho^{(1)}_{j+k+\frac{3}{2},-} - \frac{\rho^{(1)}_{j+k+\frac{1}{2},-}}{\hat{Q}_{\Delta x}}\right)\\
&\quad + \sum_{k=0}^{N-1}\frac{\rho^{(1)}_{j+k+\frac{1}{2},-}}{\hat{Q}_{\Delta x}}(w^{k+1}_\eta - w^k_\eta)\Bigg|\\
&= \frac{\Delta x}{2}\Bigg|\sum_{k=0}^{N-1} w^k_\eta\left(\rho^{(1)}_{j+k+\frac{1}{2},+} - \rho^{(1)}_{j+k-\frac{1}{2},+} + \rho^{(1)}_{j+k-\frac{1}{2},+}(\frac{\hat{Q}_{\Delta x}-1}{\hat{Q}_{\Delta x}})\right)\\
&\quad + \sum_{k=0}^{N-1} w^{k+1}_\eta\left(\rho^{(1)}_{j+k+\frac{3}{2},-} - \rho^{(1)}_{j+k+\frac{1}{2},-} + \rho^{(1)}_{j+k+\frac{1}{2},-}(\frac{\hat{Q}_{\Delta x}-1}{\hat{Q}_{\Delta x}})\right)\\
&\quad + \sum_{k=0}^{N-1}\frac{\rho^{(1)}_{j+k+\frac{1}{2},-}}{\hat{Q}_{\Delta x}}(w^{k+1}_\eta - w^k_\eta)\Bigg|\\
&\le \frac{\Delta x}{2}\Bigg(w^0_\eta\sum_{k\in\mathbb{Z}}|\rho^{(1)}_{j+k+\frac{1}{2},+} - \rho^{(1)}_{j+k-\frac{1}{2},+}| + w^0_\eta\|\rho^{(1)}\|L\sum_{k=0}^{N-1}\Delta x\\
&\quad + w^0_\eta\sum_{k\in\mathbb{Z}}|\rho^{(1)}_{j+k+\frac{3}{2},-} - \rho^{(1)}_{j+k+\frac{1}{2},-}| + w^0_\eta\|\rho^{(1)}\|L\sum_{k=0}^{N-1}\Delta x + \|\rho^{(1)}\|\frac{w^0_\eta}{\hat{Q}_{\Delta x}}\Bigg)\\
&\le \frac{\Delta x}{2}\left(2w^0_\eta\sum_{k\in\mathbb{Z}}|\rho^{(1)}_{j+k+1} - \rho^{(1)}_{j+k}| + 2w^0_\eta\|\rho^{(1)}\|\eta L + 2w^0_\eta\sum_{k\in\mathbb{Z}}|\rho^{(1)}_{j+k+1} - \rho^{(1)}_{j+k}| + \|\rho^{(1)}\|\frac{w^0_\eta}{\hat{Q}_{\Delta x}}\right)\\
&\le \frac{\Delta x}{2}\left(4w^0_\eta(1+C\Delta t)\mathrm{TV}(\rho^n_{\Delta x}) + 2w^0_\eta\rho_{\max}\eta L + \rho_{\max}\hat{w}^0_\eta\right).
\end{aligned} \tag{4.16}$$

Subsequently, by subtracting and adding $g(\rho^{(1)}_{j+\frac{1}{2},-})V^{(1)}_j$ to the term $\mathcal{D}_2$ and applying the estimate (4.16) as well as the property (3.12), we obtain

$$\begin{aligned}
2|\mathcal{D}_2| &\leq \left|g\big(\rho^{(1)}_{j+\frac{1}{2},-}\big)\big(V^{(1)}_{j+\frac{1}{2}} - V^{(1)}_j\big)\right| + \left|\big(g(\rho^{(1)}_{j+\frac{1}{2},-}) - g(\rho^{(1)}_j)\big)V^{(1)}_j\right| \\
&\leq \|g\|\|v'\||R^{(1)}_{j+\frac{1}{2}} - R^{(1)}_j| + \|g'\|\|v\||\rho^{(1)}_{j+\frac{1}{2},-} - \rho^{(1)}_j| \\
&\leq \|g\|\|v'\|\frac{\Delta x}{2}\left(4w^0_\eta(1+C\Delta t)\mathrm{TV}(\rho^n_{\Delta x}) + 2w^0_\eta\rho_{\max}\eta L + \rho_{\max}\hat{w}^0_\eta\right) + \|g'\|\|v\|\frac{|\sigma^{(1)}_j|}{2} \\
&\leq \|g\|\|v'\|\frac{\Delta x}{2}\left(4w^0_\eta(1+C\Delta t)\mathrm{TV}(\rho^n_{\Delta x}) + 2w^0_\eta\rho_{\max}\eta L + \rho_{\max}\hat{w}^0_\eta\right) + \|g'\|\|v\|\frac{|\rho^{(1)}_{j+1} - \rho^{(1)}_j|}{2}.
\end{aligned}$$

To proceed further, we observe the following inequality

$$\begin{aligned}
\Delta t\Delta x\sum_{n=0}^{n_T}\sum_{j=j_0}^{j_1}|\rho^{(1)}_{j+1} - \rho^{(1)}_j| &\leq \Delta t\Delta x(1+C\Delta t)\sum_{n=0}^{n_T}\sum_{j_0}^{j_1}|\rho^n_{j+1} - \rho^n_j| \\
&= (1+C\Delta t)\int_0^T\int_{-R}^{R}|\rho_{\Delta x}(t,x+\Delta x) - \rho_{\Delta x}(t,x)|\,\mathrm{d}x\,\mathrm{d}t \\
&\leq (1+C\Delta t)\int_0^T \Delta x\mathrm{TV}(\rho_{\Delta x}(t,\cdot))\,\mathrm{d}t \\
&\leq \Delta x(1+C\Delta t)\exp(2TC)\mathrm{TV}(\rho_0)T. \qquad (4.17)
\end{aligned}$$

Now, using (4.17) we obtain the bound

$$\begin{aligned}
&|\Delta t\Delta x\sum_{n=0}^{n_T}\sum_{j=j_0}^{j_1}\mathcal{D}_2\frac{\big(\varphi(t^n,x_{j+1}) - \varphi(t^n,x_j)\big)}{\Delta x}| \\
&\qquad\leq \Delta t\Delta x\|\varphi_x\|\sum_{n=0}^{n_T}\sum_{j_0}^{j_1}\|g\|\|v'\|\frac{\Delta x}{4}\left(4w^0_\eta(1+C\Delta t)\mathrm{TV}(\rho^n_{\Delta x}) + 2w^0_\eta\rho_{\max}\eta L + \rho_{\max}\hat{w}^0_\eta\right) \\
&\qquad\quad + \Delta t\Delta x\|\varphi_x\|\sum_{n=0}^{n_T}\sum_{j_0}^{j_1}\|g'\|\|v\|\frac{|\rho^{(1)}_{j+1} - \rho^{(1)}_j|}{4} \\
&\qquad\leq \frac{\Delta x}{4}\|\varphi_x\|\|g\|\|v'\|\left(4w^0_\eta(1+C\Delta t)\exp(2TC)\mathrm{TV}(\rho_0) + 2w^0_\eta\rho_{\max}\eta L + \rho_{\max}\hat{w}^0_\eta\right)2RT \\
&\qquad\quad + \frac{\Delta x}{4}\|\varphi_x\|\|g'\|\|v\|(1+C\Delta t)\exp(2TC)\mathrm{TV}(\rho_0)T. \qquad (4.18)
\end{aligned}$$

In a similar way, a bound can be obtained on the term involving $\mathcal{D}_1$ in (4.15) as follows

$$\begin{aligned}
&\left|\Delta t\Delta x\sum_{n=0}^{n_T}\sum_{j=j_0}^{j_1}\mathcal{D}_1\frac{\big(\varphi(t^n,x_{j+1}) - \varphi(t^n,x_j)\big)}{\Delta x}\right| \\
&\qquad\leq \frac{\Delta x}{4}\|\varphi_x\|\|g\|\|v'\|\left(4w^0_\eta\exp(2TC)\mathrm{TV}(\rho_0) + 2w^0_\eta\rho_{\max}\eta L + \rho_{\max}\hat{w}^0_\eta\right)2RT \\
&\qquad\quad + \frac{\Delta x}{4}\|\varphi_x\|\|g'\|\|v\|\exp(2TC)\mathrm{TV}(\rho_0)T. \qquad (4.19)
\end{aligned}$$

Combining the estimates (4.18) and (4.19), we see that

$$\lim_{\Delta x\to 0}\mathcal{S}_1 = 0. \qquad (4.20)$$

Finally, collecting the results (4.8), (4.14) and (4.20), we can conclude that

$$\begin{aligned} 0 = \lim_{\Delta x \to 0} (\mathcal{P}_1 + \mathcal{P}_2 + \mathcal{P}_3) &= \lim_{\Delta x \to 0} (\mathcal{P}_1 + \mathcal{P}_2) + \lim_{\Delta x \to 0} (\mathcal{S}_1 + \mathcal{S}_2) \\ &= \int_{-\infty}^{+\infty} \rho(0,x)\varphi(0,x)dx + \int_0^T \int_{-\infty}^{+\infty} \rho(t,x)\varphi_t(t,x)\,\mathrm{d}x\,\mathrm{d}t \\ &\quad + \int_0^T \int_{-\infty}^{+\infty} g\big(\rho(t,x)\big) v\big(\rho * w_\eta(t,x)\big)\varphi_x(t,x)\,\mathrm{d}x\,\mathrm{d}t. \end{aligned}$$

This reveals that the limit is a weak solution of the problem (2.1). □

Remark 4.3. It is important to note that in the case when $g(\rho) = \rho$, the weak solutions are unique and no entropy condition is required, as discussed in [57]. Hence, in this specific case, we can conclude that the second-order scheme (3.8) converges to the unique weak solution without any entropy condition. However, for the general case, it is required to prove the convergence to the entropy solution. This will be discussed in the next section.

5. Convergence to the entropy solution

To prove convergence to the entropy solution, we shall use the same approach as outlined in [73], also see [72]. These ideas can be combined to form the following theorem which is analogous to the Theorem 3.1 of [72].

Theorem 5.1. *Suppose that a scheme can be written in the form:*

$$\rho_j^{n+1} = \tilde{\rho}_j^{n+1} - a_{j+\frac{1}{2}}^{n+1} + a_{j-\frac{1}{2}}^{n+1}, \tag{5.1}$$

where

(i) $\tilde{\rho}_j^{n+1}$ *is computed from* ρ_j^n*, using a scheme which yields a sequence of approximate solutions converging in* $\mathrm{L}^1_{\mathrm{loc}}$ *to the entropy solution of* (2.1).

(ii) $|a_{j+\frac{1}{2}}^{n+1}| \le K\Delta x^\delta$ *for some constant* K *which is independent of* Δx *and for some* $\delta \in (0,1)$.

(iii) The approximate solutions $\rho_{\Delta x}$ *obtained using* (5.1) *are in* BV, L^∞ *and admits* L^1*- Lipschitz continuity in time.*
Then the approximate solutions generated by the scheme (5.1) converges in $\mathrm{L}^1_{\mathrm{loc}}$ *to the entropy solution of* (2.1).

Remark 5.2. Proof of Theorem 5.1 follows along the same lines as that of Theorem 3.1 of [72]. Specifically, the hypothesis (iii) of Theorem 5.1 ensures that the approximate solutions generated by the scheme (5.1) converges in $\mathrm{L}^1_{\mathrm{loc}}$. To prove that the limit solution satisfies the entropy condition (2.4), we mainly use two facts. Firstly, we utilize the discrete entropy inequality of the scheme $\tilde{\rho}_j^{n+1}$ in hypothesis (i) of Theorem 5.1, which is provided later in equation (5.5) of Theorem 5.4. Secondly, we make use of the boundedness of the terms $|a_{j+\frac{1}{2}}^{n+1}| \le K\Delta x^\delta$ as mentioned in hypothesis (ii) of Theorem 5.1, and the BV and L^∞ estimates of ρ_j^{n+1}. By adding an appropriate term to both sides of the discrete entropy inequality (5.5) of Theorem (5.4), we get a similar expression as in the inequality (3.18) of [72]. Further, by employing a similar argument as presented in [72], one can show that the right-hand side of the obtained expression has a limit supremum that is non-positive as the mesh size approaches zero. Consequently, the required result follows.

In this scenario, first we consider the first-order in space and second-order in time scheme (FSST) obtained by setting the slopes $\sigma_j(t) = 0$ for all $j \in \mathbb{Z}$ in (3.8):

$$\begin{aligned}
\rho_j^{(1)} &= \rho_j^n - \lambda \left(g(\rho_j^n) V_{j+\frac{1}{2}}^n - g(\rho_{j-1}^n) V_{j-\frac{1}{2}}^n \right), \\
\rho_j^{(2)} &= \rho_j^{(1)} - \lambda \left(g(\rho_j^{(1)}) V_{j+\frac{1}{2}}^{(1)} - g(\rho_{j-1}^{(1)}) V_{j-\frac{1}{2}}^{(1)} \right), \\
\rho_j^{n+1} &= \frac{1}{2} \left(\rho_j^n + \rho_j^{(2)} \right),
\end{aligned} \tag{5.2}$$

where

$$\begin{aligned}
&V_{j+\frac{1}{2}}^n = v(R_{j+\frac{1}{2}}^n), \quad V_{j+\frac{1}{2}}^{(1)} = v(R_{j+\frac{1}{2}}^{(1)}), \\
&R_{j+\frac{1}{2}}^n = \Delta x \sum_{k=0}^{N-1} \rho_{j+k+1}^n \frac{(w_\eta^k + w_\eta^{k+1})}{2} \text{ and } R_{j+\frac{1}{2}}^{(1)} = \Delta x \sum_{k=0}^{N-1} \rho_{j+k+1}^{(1)} \frac{(w_\eta^k + w_\eta^{k+1})}{2}.
\end{aligned}$$

Further, for some $K > 0$ and $\delta \in (0,1)$, we modify the slopes $\sigma_j(t)$ defined in (3.4) by adding the term $K\Delta x^\delta$ in its definition, *i.e.*,

$$\sigma_j(t) = \text{minmod}((\rho_j(t) - \rho_{j-1}(t)),\ \frac{1}{2}(\rho_{j+1}(t) - \rho_{j-1}(t)), (\rho_{j+1}(t) - \rho_j(t)),\ \text{sgn}(\rho_j(t) - \rho_{j-1}(t)) K\Delta x^\delta). \tag{5.3}$$

Now, the second-order scheme (3.8) with the modified slope (5.3) can be written as a predictor-corrector scheme in the form

$$\rho_j^{n+1} = \tilde{\rho}_j^{n+1} - a_{j+\frac{1}{2}}^{n+1} + a_{j-\frac{1}{2}}^{n+1}, \tag{5.4}$$

where $a_{j+\frac{1}{2}}^{n+1} = \lambda(F_{j+\frac{1}{2}}^n - \tilde{F}_{j+\frac{1}{2}}^n)$ with $F_{j+\frac{1}{2}}^n$ as in (3.9) and $\tilde{\rho}_j^{n+1}$ is a predictor step obtained from ρ_j^n using the FSST scheme (5.2), written as

$$\tilde{\rho}_j^{n+1} = \rho_j^n - \lambda(\tilde{F}_{j+\frac{1}{2}}^n - \tilde{F}_{j-\frac{1}{2}}^n), \text{ where } \tilde{F}_{j+\frac{1}{2}}^n = \frac{1}{2} \left(g(\rho_j^n) \tilde{V}_{j+\frac{1}{2}}^n + g(\tilde{\rho}_j^{(1)}) \tilde{V}_{j+\frac{1}{2}}^{(1)} \right)$$

and

$$\begin{aligned}
&\tilde{\rho}_j^{(1)} = \rho_j^n - \lambda \left(g(\rho_j^n) \tilde{V}_{j+\frac{1}{2}}^n - g(\rho_{j-1}^n) \tilde{V}_{j-\frac{1}{2}}^n \right), \ \tilde{V}_{j+\frac{1}{2}}^n = v(\tilde{R}_{j+\frac{1}{2}}^n), \ \ \tilde{V}_{j+\frac{1}{2}}^{(1)} = v(\tilde{R}_{j+\frac{1}{2}}^{(1)}) \\
&\tilde{R}_{j+\frac{1}{2}}^n = \Delta x \sum_{k=0}^{N-1} \rho_{j+k+1}^n \frac{(w_\eta^k + w_\eta^{k+1})}{2}, \ \tilde{R}_{j+\frac{1}{2}}^{(1)} = \Delta x \sum_{k=0}^{N-1} \tilde{\rho}_{j+k+1}^{(1)} \frac{(w_\eta^k + w_\eta^{k+1})}{2} \quad \text{for all } j \in \mathbb{Z}.
\end{aligned}$$

We now state our final result in the following theorem which ensures convergence to the entropy solution and it will be proved using Theorem 5.1.

Theorem 5.3. *(Convergence to the entropy solution) Let $\rho_0 \in \text{BV}(\mathbb{R}; [0, \rho_{\max}])$ and let $\rho_{\Delta x}$ be the approximate solution obtained using the second-order scheme (3.8) under the CFL condition (3.13), with a space-step dependent slope limiter (5.3). Then, the corresponding sequence of approximate solutions $\rho_{\Delta x}$ converges in $\text{L}^1_{\text{loc}}([0,T) \times \mathbb{R})$ to the unique entropy solution of (2.1) as $\Delta x \to 0$.*

As the first step in proving Theorem 5.3, we show that the numerical solutions obtained by the scheme (5.2) converges to the entropy solution of (2.1).

Theorem 5.4. *Let $\rho_0 \in \mathrm{BV}(\mathbb{R};[0,\rho_{\max}])$ and let $\rho_{\Delta x}$ be the approximate solution obtained using the FSST scheme (5.2) under the CFL condition (3.13). Then $\rho_{\Delta x}$ converges in $\mathrm{L}^1_{\mathrm{loc}}([0,T)\times\mathbb{R})$ to the unique entropy solution of (2.1) as $\Delta x \to 0$.*

Proof. It is clear that the convergence analysis (Thm. 4.2) presented in Section 4 holds for the FSST scheme by setting $\sigma_j(t) = 0$ for all j in the scheme (3.8). Therefore, it is only left to prove that the limit function ρ obtained from the FSST scheme (5.2) satisfies the entropy condition (2.4). To prove this, first we observe that the first-order time steps in the scheme (5.2) satisfy the following discrete entropy inequalities (see [42]):

$$|\rho_j^{(1)} - \kappa| - |\rho_j^n - \kappa| + \lambda\left(F^{\kappa}_{j+\frac12}(\rho_j^n) - F^{\kappa}_{j-\frac12}(\rho_{j-1}^n)\right) + \lambda g(\kappa)\mathrm{sgn}(\rho_j^{(1)} - \kappa)(V^n_{j+\frac12} - V^n_{j-\frac12}) \le 0,$$
$$|\rho_j^{(2)} - \kappa| - |\rho_j^{(1)} - \kappa| + \lambda\left(F^{\kappa,(1)}_{j+\frac12}(\rho_j^{(1)}) - F^{\kappa,(1)}_{j-\frac12}(\rho_{j-1}^{(1)})\right) + \lambda g(\kappa)\mathrm{sgn}(\rho_j^{(2)} - \kappa)(V^{(1)}_{j+\frac12} - V^{(1)}_{j-\frac12}) \le 0,$$

where $\kappa \in I = [0,\rho_{\max}]$, $F^{\kappa}_{j+\frac12}(\rho) := \Big(g(\rho\wedge\kappa) - g(\rho\vee\kappa)\Big)V^n_{j+\frac12}$, $F^{\kappa,(1)}_{j+\frac12}(\rho) := \Big(g(\rho\wedge\kappa) - g(\rho\vee\kappa)\Big)V^{(1)}_{j+\frac12}$, $a\wedge b := \max\{a,b\}$ and $a\vee b := \min\{a,b\}$. Combining these, we obtain a discrete entropy inequality for the FSST scheme (5.2) as follows

$$\begin{aligned}|\rho_j^{n+1} - \kappa| - |\rho_j^n - \kappa| &+ \frac{\lambda}{2}\left[F^{\kappa,(1)}_{j+\frac12}(\rho_j^{(1)}) + F^{\kappa}_{j+\frac12}(\rho_j^n) - F^{\kappa,(1)}_{j-\frac12}(\rho_{j-1}^{(1)}) - F^{\kappa}_{j-\frac12}(\rho_{j-1}^n)\right]\\ &+ \frac{\lambda}{2}g(\kappa)\mathrm{sgn}(\rho_j^{(2)} - \kappa)(V^{(1)}_{j+\frac12} - V^{(1)}_{j-\frac12}) + \frac{\lambda}{2}g(\kappa)\mathrm{sgn}(\rho_j^{(1)} - \kappa)(V^n_{j+\frac12} - V^n_{j-\frac12}) \le 0. \qquad (5.5)\end{aligned}$$

From (5.5), we prove that the approximate solutions converge to the entropy solution as in (2.4) of Definition 2.2.

Now, consider a non-negative test function $\varphi \in \mathrm{C}^1_c([0,T)\times\mathbb{R};\mathbb{R}^+)$. Let T_φ be such that $0 \le T_\varphi < T$ and $\varphi(t,x) = 0$ for $t \ge T_\varphi$ and let n_T be such that $T_\varphi \in (n_T\Delta t, (n_T+1)\Delta t]$. Multiplying (5.5) by $\Delta x\varphi(t^n,x_j)$, summing over n, j and using summation by parts, we obtain

$$\mathcal{E}_0 + \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 + \mathcal{E}_4 \ge 0, \qquad (5.6)$$

where

$$\mathcal{E}_0 := \Delta x\sum_{j\in\mathbb{Z}}\varphi(0,x_j)|\rho_j^0 - \kappa| + \Delta t\Delta x\sum_{n=0}^{n_T}\sum_{j\in\mathbb{Z}}|\rho_j^{n+1} - \kappa|\frac{\big(\varphi(t^{n+1},x_j) - \varphi(t^n,x_j)\big)}{\Delta t},$$
$$\mathcal{E}_1 := \frac{\Delta t\Delta x}{2}\sum_{n=0}^{n_T}\sum_{j\in\mathbb{Z}}F^{\kappa,(1)}_{j+\frac12}(\rho_j^{(1)})\frac{\big(\varphi(t^n,x_{j+1}) - \varphi(t^n,x_j)\big)}{\Delta x},$$
$$\mathcal{E}_2 := \frac{\Delta t\Delta x}{2}\sum_{n=0}^{n_T}\sum_{j\in\mathbb{Z}}F^{\kappa,n}_{j+\frac12}(\rho_j^n)\frac{\big(\varphi(t^n,x_{j+1}) - \varphi(t^n,x_j)\big)}{\Delta x},$$
$$\mathcal{E}_3 := -\frac{\Delta t\Delta x}{2}g(\kappa)\sum_{n=0}^{n_T}\sum_{j\in\mathbb{Z}}\mathrm{sgn}(\rho_j^{(1)} - \kappa)\frac{(V^n_{j+\frac12} - V^n_{j-\frac12})}{\Delta x}\varphi(t^n,x_j),$$
$$\mathcal{E}_4 := -\frac{\Delta t\Delta x}{2}g(\kappa)\sum_{n=0}^{n_T}\sum_{j\in\mathbb{Z}}\mathrm{sgn}(\rho_j^{(2)} - \kappa)\frac{(V^{(1)}_{j+\frac12} - V^{(1)}_{j-\frac12})}{\Delta x}\varphi(t^n,x_j).$$

First, consider the term $\mathcal{E}_0$ in (5.6) which can be written as

$$\mathcal{E}_0 = \int_{-\infty}^{+\infty}\varphi_{\Delta x}(0,x)|\rho_{\Delta x}(0,x) - \kappa|\,\mathrm{d}x + \int_0^T\int_{-\infty}^{+\infty}|\rho_{\Delta x}(t+\Delta t,x) - \kappa|\partial_t\varphi_{\Delta x}(t,x)\,\mathrm{d}x\,\mathrm{d}t,$$

where $\partial_t \varphi_{\Delta x}(t,x) := \varphi_t(\bar{t}, x_j)$ for $x \in (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}], t \in [t^n, t^{n+1})$, for some $\bar{t} \in (t^n, t^{n+1})$. Through the dominated convergence theorem it follows that

$$\lim_{\Delta x \to 0} \mathcal{E}_0 = \int_{-\infty}^{\infty} |\rho_0(x) - \kappa| \varphi(0,x)\, \mathrm{d}x + \int_0^T \int_{-\infty}^{\infty} |\rho(t,x) - \kappa| \partial_t \varphi(t,x)\, \mathrm{d}x\, \mathrm{d}t. \tag{5.7}$$

Let $R > 0$ be such that $\varphi(t,x) = 0$ for $|x| > R$. Let $j_0, j_1 \in \mathbb{Z}$ such that $-R \in (x_{j_0-\frac{1}{2}}, x_{j_0+\frac{1}{2}}]$ and $R \in (x_{j_1-\frac{1}{2}}, x_{j_1+\frac{1}{2}}]$. Next, we consider the term $\mathcal{E}_1$ in (5.6) which writes as

$$\mathcal{E}_1 = \frac{\Delta t \Delta x}{2} \sum_{n=0}^{n_T} \sum_{j \in \mathbb{Z}} \left(g(\rho_j^{(1)} \wedge \kappa) - g(\rho_j^{(1)} \vee \kappa) \right) V_{j+\frac{1}{2}}^{(1)} \frac{\left(\varphi(t^n, x_{j+1}) - \varphi(t^n, x_j)\right)}{\Delta x}. \tag{5.8}$$

Using the definition of $\rho^{(1)}$ and applying the mean value theorem, it follows that

$$V_{j+\frac{1}{2}}^{(1)} = V_{j+\frac{1}{2}}^n - \Delta t v'(\theta_{j+\frac{1}{2}}) \sum_{k=0}^{N-1} \frac{(w_\eta^k + w_\eta^{k+1})}{2} \left(g(\rho_{j+k+1}^n) V_{j+k+\frac{3}{2}}^n - g(\rho_{j+k}^n) V_{j+k+\frac{1}{2}}^n \right),$$

for some $\theta_{j+\frac{1}{2}} \in \mathcal{I}(R_{j+\frac{1}{2}}^n, R_{j+\frac{1}{2}}^{(1)})$. Thus the term $\mathcal{E}_1$ can be written as $\mathcal{E}_1 = \mathcal{E}_1^a + \mathcal{E}_1^b$ where

$$\mathcal{E}_1^a := \frac{\Delta t \Delta x}{2} \sum_{n=0}^{n_T} \sum_{j=j_0}^{j_1} \left(g(\rho_j^{(1)} \wedge \kappa) - g(\rho_j^{(1)} \vee \kappa) \right) V_{j+\frac{1}{2}}^n \frac{\left(\varphi(t^n, x_{j+1}) - \varphi(t^n, x_j)\right)}{\Delta x},$$

$$\mathcal{E}_1^b := -\frac{\Delta t^2 \Delta x}{2} \sum_{n=0}^{n_T} \sum_{j=j_0}^{j_1} \left(g(\rho_j^{(1)} \wedge \kappa) - g(\rho_j^{(1)} \vee \kappa) \right) v'(\theta_{j+\frac{1}{2}}) \sum_{k=0}^{N-1} \hat{\ell}(n,j,k) \frac{\left(\varphi(t^n, x_{j+1}) - \varphi(t^n, x_j)\right)}{\Delta x},$$

with the definition, $\hat{\ell}(n,j,k) := \dfrac{(w_\eta^k + w_\eta^{k+1})}{2} \left(g(\rho_{j+k+1}^n) V_{j+k+\frac{3}{2}}^n - g(\rho_{j+k}^n) V_{j+k+\frac{1}{2}}^n \right)$.
Now, summation by parts yields

$$\sum_{k=0}^{N-1} \hat{\ell}(n,j,k) = \frac{w_\eta^{N-1} + w_\eta^N}{2} g(\rho_{j+N}^n) V_{j+N+\frac{1}{2}}^n - \frac{w_\eta^0 + w_\eta^1}{2} g(\rho_j^n) V_{j+\frac{1}{2}}^n + \sum_{k=1}^{N-1} g(\rho_{j+k}^n) V_{j+k+\frac{1}{2}}^n \frac{(w_\eta^{k-1} - w_\eta^{k+1})}{2},$$

which implies that

$$\left| \sum_{k=0}^{N-1} \hat{\ell}(n,j,k) \right| \le 2 w_\eta^0 \|g\| \|v\| + \|g\| \|v\| \sum_{k=1}^{N-1} \frac{(w_\eta^{k-1} - w_\eta^{k+1})}{2} \le 2 w_\eta^0 \|g\| \|v\| + \|g\| \|v\| w_\eta^0 \le 3 w_\eta^0 \|g\| \|v\|.$$

Therefore we have the following bound on $|\mathcal{E}_1^b|$:

$$|\mathcal{E}_1^b| \le 3 \Delta t \|\varphi_x\| \|g\|^2 \|v\| \|v'\| w_\eta^0 \sum_{n=0}^{n_T} \sum_{j=j_0}^{j_1} \Delta t \Delta x \le 6 \Delta t \|\varphi_x\| \|g\|^2 \|v\| \|v'\| w_\eta^0 RT. \tag{5.9}$$

Since g is an increasing function, $g(\rho_j^{(1)} \wedge \kappa) - g(\rho_j^{(1)} \vee \kappa) = |g(\rho_j^{(1)}) - g(\kappa)|$. Now, note that $\mathcal{E}_1^a$ can be written as follows

$$\mathcal{E}_1^a = \frac{1}{2} \int_0^T \int_{-\infty}^{+\infty} \left(\left| g\left(\rho_{\Delta x}^{(1)}(t,x)\right) - g(\kappa) \right| v\left(R_{\Delta x}(t, x + \Delta x)\right) \right) \partial_x \varphi_{\Delta x}(t,x)\, \mathrm{d}x\, \mathrm{d}t,$$

where

$$R_{\Delta x}(t,x) := R^n_{j-\frac{1}{2}} \quad \text{for } x \in (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}], t \in [t^n, t^{n+1})$$
$$\partial_x \varphi_{\Delta x}(t,x) := \varphi_x(t^n, \bar{x}) \quad \text{for } x \in (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}],\ t \in [t^n, t^{n+1}) \text{ for some } \bar{x} \in (x_j, x_{j+1}).$$

Observe that

$$R_{\Delta x}(t, x+\Delta x) = \frac{1}{2}\int_{x_{j-\frac{1}{2}}}^{x_{j-\frac{1}{2}}+\eta} \rho_{\Delta x}(t, y+\Delta x)\left(w_{\eta,\Delta x}(y - x_{j-\frac{1}{2}} + \Delta x) + w_{\eta,\Delta x}(y - x_{j-\frac{1}{2}} + 2\Delta x)\right) \mathrm{d}y$$

for $x \in (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}], t \in [t^n, t^{n+1})$, where

$$w_{\eta,\Delta x}(x) := w^k_\eta \quad \text{for } x \in (k\Delta x, (k+1)\Delta x], \quad w_{\eta,\Delta x}(0) := w_\eta(0).$$

By using Lemma C.1 (see Appendix C) and applying the dominated convergence theorem, it follows that $R_{\Delta x}(t, x+\Delta x)$ converges to $\int_x^{x+\eta} \rho(t,y) w_\eta(y-x)\,\mathrm{d}y$ as $\Delta x \to 0$. Using Lemma C.2 (see Appendix C) and the dominated convergence theorem, we deduce that

$$\lim_{\Delta x \to 0} \mathcal{E}^a_1 = \frac{1}{2}\int_0^T \int_{-\infty}^{+\infty} \left(\left|g\big(\rho(t,x)\big) - g(\kappa)\right| v\big(\rho * w_\eta(t,x)\big)\right) \varphi_x(t,x)\,\mathrm{d}x\,\mathrm{d}t. \tag{5.10}$$

The term $\mathcal{E}_2$ in (5.6) can be expressed as follows

$$\begin{aligned}\mathcal{E}_2 &= \frac{\Delta t \Delta x}{2}\sum_{n=0}^{n_T}\sum_{j\in\mathbb{Z}} \left(g(\rho^n_j \wedge \kappa) - g(\rho^n_j \vee \kappa)\right) V^n_{j+\frac{1}{2}} \frac{\big(\varphi(t^n, x_{j+1}) - \varphi(t^n, x_j)\big)}{\Delta x} \\ &= \frac{1}{2}\int_0^T \int_{-\infty}^{+\infty} \Big(\left|g\big(\rho_{\Delta x}(t,x)\big) - g(\kappa)\right| v\big(R_{\Delta x}(t, x+\Delta x)\big)\Big) \partial_x \varphi_{\Delta x}(t,x)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Using the convergence of $\rho_{\Delta x}$ to ρ and similar arguments as in the case of $\mathcal{E}^a_1$, we observe that

$$\lim_{\Delta x \to 0} \mathcal{E}_2 = \frac{1}{2}\int_0^T \int_{-\infty}^{+\infty} \left(\left|g\big(\rho(t,x)\big) - g(\kappa)\right| v\big(\rho * w_\eta(t,x)\big)\right) \varphi_x(t,x)\,\mathrm{d}x\,\mathrm{d}t. \tag{5.11}$$

Further, the term $\mathcal{E}_3$ in (5.6) can be written as

$$\begin{aligned}\mathcal{E}_3 &= -\frac{\Delta t \Delta x}{2} g(\kappa) \sum_{n=0}^{n_T}\sum_{j\in\mathbb{Z}} \operatorname{sgn}\big(\rho^{(1)}_j - \kappa\big) v'(\bar{R}^n_j) \frac{(R^n_{j+\frac{1}{2}} - R^n_{j-\frac{1}{2}})}{\Delta x} \varphi(t^n, x_j) \\ &= -\frac{1}{2} g(\kappa) \int_0^T \int_{-\infty}^{+\infty} \operatorname{sgn}\left(\rho^{(1)}_{\Delta x}(t,x) - \kappa\right) v'\big(\bar{R}_{\Delta x}(t,x)\big) \frac{\big(R_{\Delta x}(t, x+\Delta x) - R_{\Delta x}(t,x)\big)}{\Delta x} \varphi_{\Delta x}(t,x)\,\mathrm{d}x\,\mathrm{d}t,\end{aligned}$$

where $\bar{R}^n_j \in \mathcal{I}(R^n_{j-\frac{1}{2}}, R^n_{j+\frac{1}{2}})$, $\bar{R}_{\Delta x}(t,x) := \bar{R}^n_j$ for $x \in (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}], t \in [t^n, t^{n+1})$, and $\varphi_{\Delta x} := \varphi(t^n, x_j)$ for $x \in (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}], t \in [t^n, t^{n+1})$.
Using summation by parts, we obtain

$$\frac{R^n_{j+\frac{1}{2}} - R^n_{j-\frac{1}{2}}}{\Delta x} = \frac{w^{N-1}_\eta + w^N_\eta}{2}\rho^n_{j+N} - \frac{w^0_\eta + w^1_\eta}{2}\rho^n_j - \Delta x \sum_{k=1}^{N-1} \rho^n_{j+k} \frac{w^{k+1}_\eta - w^{k-1}_\eta}{2\Delta x}, \tag{5.12}$$

which enables us to write

$$\begin{aligned}\frac{R_{\Delta x}(t,x+\Delta x)-R_{\Delta x}(t,x)}{\Delta x} &= \rho_{\Delta x}(t^n,x_{j+N})\left(\frac{w_{\eta,\Delta x}(\eta-\Delta x)+w_{\eta,\Delta x}(\eta)}{2}\right)\\ &\quad-\rho_{\Delta x}(t^n,x_j)\left(\frac{w_{\eta,\Delta x}(0)+w_{\eta,\Delta x}(\Delta x)}{2}\right)\\ &\quad-\int_{x_{j-\frac{1}{2}}+\Delta x}^{x_{j-\frac{1}{2}}+\eta}\rho_{\Delta x}(t,y)w'_{\eta,\Delta x}\big(y-(x_{j-\frac{1}{2}}+\Delta x)\big)\,\mathrm{d}y,\end{aligned}\tag{5.13}$$

for $x\in(x_{j-\frac{1}{2}},x_{j+\frac{1}{2}}], t\in[t^n,t^{n+1})$, where $w'_{\eta,\Delta x}(x):=w'_\eta(\bar{x}),\ x\in(k\Delta x,(k+1)\Delta x]$ for some $\bar{x}\in(k\Delta x,(k+2)\Delta x)$. Defining $R(t,x):=\int_x^{x+\eta}\rho(t,y)w_\eta(y-x)\,\mathrm{d}y$ and differentiating yields

$$\frac{\partial R(t,x)}{\partial x}=\rho(t,x+\eta)w_\eta(\eta)-\rho(t,x)w_\eta(0)-\int_x^{x+\eta}\rho(t,y)w'_\eta(y-x)\,\mathrm{d}y.\tag{5.14}$$

Using the dominated convergence theorem in (5.13), we have the following for a.e. $(t,x)\in[0,T)\times\mathbb{R}$,

$$\lim_{\Delta x\to 0}\frac{\big(R_{\Delta x}(t,x+\Delta x)-R_{\Delta x}(t,x)\big)}{\Delta x}=\frac{\partial R(t,x)}{\partial x}.$$

Now, using Lemma C.2 (see Appendix C) together with the arguments in Lemmas 4.3 and 4.4 of [56], the following holds for $\kappa\in I=[0,\rho_{\max}]$

$$\lim_{\Delta x\to 0}\mathcal{E}_3=-\frac{1}{2}\int_0^T\int_{-\infty}^{+\infty}\operatorname{sgn}\big(\rho(t,x)-\kappa\big)g(\kappa)v'\left(\rho*w_\eta(t,x)\right)\partial_x\big(\rho*w_\eta(t,x)\big)\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t.\tag{5.15}$$

Further, we consider the term $\mathcal{E}_4$ in (5.6) which can be expressed as follows

$$\mathcal{E}_4=-\frac{1}{2}g(\kappa)\int_0^T\int_{-\infty}^{+\infty}\operatorname{sgn}\big(\rho^{(2)}_{\Delta x}(t,x)-\kappa\big)v'\big(\bar{R}^{(1)}_{\Delta x}(t,x)\big)\frac{\big(R^{(1)}_{\Delta x}(t,x+\Delta x)-R^{(1)}_{\Delta x}(t,x)\big)}{\Delta x}\varphi_{\Delta x}(t,x)\,\mathrm{d}x\,\mathrm{d}t,$$

where $\bar{R}^{(1)}_{\Delta x}(t,x):=\bar{R}^{(1)}_j,\ x\in(x_{j-\frac{1}{2}},x_{j+\frac{1}{2}}], t\in[t^n,t^{n+1})$ for some $\bar{R}^{(1)}_j\in\mathcal{I}(R^{(1)}_{j-\frac{1}{2}},R^{(1)}_{j+\frac{1}{2}})$ and $R^{(1)}_{\Delta x}(t,x):=R^{(1)}_{j-\frac{1}{2}}$ for $x\in(x_{j-\frac{1}{2}},x_{j+\frac{1}{2}}], t\in[t^n,t^{n+1})$. Proceeding in a way similar to the derivation of (5.15), we get

$$\lim_{\Delta x\to 0}\mathcal{E}_4=-\frac{1}{2}\int_0^T\int_{-\infty}^{+\infty}\operatorname{sgn}\big(\rho(t,x)-\kappa\big)g(\kappa)v'\big(\rho*w_\eta(t,x)\big)\partial_x\big(\rho*w_\eta(t,x)\big)\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t.\tag{5.16}$$

Finally, collecting the expressions (5.7), (5.9), (5.10), (5.11), (5.15) and (5.16), we obtain the desired entropy inequality

$$\begin{aligned}&\int_0^T\int_{-\infty}^{+\infty}\Big(|\rho-\kappa|\partial_t\varphi+\operatorname{sgn}(\rho-\kappa)\big(g(\rho)-g(\kappa)\big)v(\rho*w_\eta)\partial_x\varphi\\ &-\operatorname{sgn}(\rho-\kappa)g(\kappa)v'(\rho*w_\eta)\partial_x(\rho*w_\eta)\varphi\Big)(t,x)\,\mathrm{d}x\,\mathrm{d}t+\int_{-\infty}^{+\infty}|\rho_0(x)-\kappa|\varphi(0,x)\,\mathrm{d}x\geq 0.\end{aligned}$$

□

In the following lemma we show that the bound in condition (ii) of Theorem 5.1 holds for the terms $a^{n+1}_{j+\frac{1}{2}}$ in (5.4).

Lemma 5.5. *Consider the second-order scheme with the modified slope (5.3) written in the form (5.4). Then* $|a^{n+1}_{j+\frac{1}{2}}| \le \tilde{K}\Delta x^{\delta}$ *for some constant* $\tilde{K}$ *which is independent of* Δx *and* $\delta \in (0,1)$ *as in (5.3).*

Proof. We can write

$$\begin{aligned}
a^{n+1}_{j+\frac{1}{2}} &= \frac{\lambda}{2}\Big(g(\rho^n_{j+\frac{1}{2},-})V^n_{j+\frac{1}{2}} + g(\rho^{(1)}_{j+\frac{1}{2},-})V^{(1)}_{j+\frac{1}{2}} - g(\rho^n_j)\tilde{V}^n_{j+\frac{1}{2}} - g(\tilde{\rho}^{(1)}_j)\tilde{V}^{(1)}_{j+\frac{1}{2}}\Big)\\
&= \frac{\lambda}{2}\Big(\big(g(\rho^n_{j+\frac{1}{2},-}) - g(\rho^n_j)\big)V^n_{j+\frac{1}{2}} + g(\rho^n_j)(V^n_{j+\frac{1}{2}} - \tilde{V}^n_{j+\frac{1}{2}})\\
&\quad + \big(g(\rho^{(1)}_{j+\frac{1}{2},-}) - g(\rho^{(1)}_j)\big)V^{(1)}_{j+\frac{1}{2}} + g(\rho^{(1)}_j)(V^{(1)}_{j+\frac{1}{2}} - \tilde{V}^{(1)}_{j+\frac{1}{2}}) + \big(g(\rho^{(1)}_j) - g(\tilde{\rho}^{(1)}_j)\big)\tilde{V}^{(1)}_{j+\frac{1}{2}}\Big)\\
&= \frac{\lambda}{2}\Big(g'(\xi_j)(\rho^n_{j+\frac{1}{2},-} - \rho^n_j)V^n_{j+\frac{1}{2}} + g(\rho^n_j)v'(\eta_{j+\frac{1}{2}})(R^n_{j+\frac{1}{2}} - \tilde{R}^n_{j+\frac{1}{2}})\\
&\quad + g'(\xi^{(1)}_j)(\rho^{(1)}_{j+\frac{1}{2},-} - \rho^{(1)}_j)V^{(1)}_{j+\frac{1}{2}} + g(\rho^n_j)v'(\eta^{(1)}_{j+\frac{1}{2}})(R^{(1)}_{j+\frac{1}{2}} - \tilde{R}^{(1)}_{j+\frac{1}{2}}) + g'(\bar{\xi}^{(1)}_j)(\rho^{(1)}_j - \tilde{\rho}^{(1)}_j)\tilde{V}^{(1)}_{j+\frac{1}{2}}\Big),
\end{aligned}$$

for some suitable $\xi_j \in \mathcal{I}(\rho^n_{j+\frac{1}{2},-}, \rho^n_j), \eta_{j+\frac{1}{2}} \in \mathcal{I}(R^n_{j+\frac{1}{2}}, \tilde{R}^n_{j+\frac{1}{2}}), \xi^{(1)}_j \in \mathcal{I}(\rho^{(1)}_{j+\frac{1}{2},-}, \rho^{(1)}_j), \bar{\xi}^{(1)}_j \in \mathcal{I}(\rho^{(1)}_j, \tilde{\rho}^{(1)}_j)$ and $\eta^{(1)}_{j+\frac{1}{2}} \in \mathcal{I}(R^{(1)}_{j+\frac{1}{2}}, \tilde{R}^{(1)}_{j+\frac{1}{2}})$ by the mean value theorem. Also, note that the term $\rho^{(1)}_j - \tilde{\rho}^{(1)}_j$ can be written as

$$\begin{aligned}
\rho^{(1)}_j - \tilde{\rho}^{(1)}_j = -\lambda\Big(&g'(\zeta_j)(\rho^n_{j+\frac{1}{2},-} - \rho^n_j)V^n_{j+\frac{1}{2}} + g(\rho^n_j)v'(\theta_{j+\frac{1}{2}})(R^n_{j+\frac{1}{2}} - \tilde{R}^n_{j+\frac{1}{2}})\\
&+ g'(\zeta_{j-1})(\rho^n_{j-1} - \rho^n_{j-\frac{1}{2},-})V^n_{j+\frac{1}{2}} + g(\rho^n_j)v'(\theta_{j-\frac{1}{2}})(\tilde{R}^n_{j-\frac{1}{2}} - R^n_{j-\frac{1}{2}})\Big),
\end{aligned}$$

for some $\zeta_j \in \mathcal{I}(\rho^n_{j+\frac{1}{2},-}, \rho^n_j), \theta_{j+\frac{1}{2}} \in \mathcal{I}(R^n_{j+\frac{1}{2}}, \tilde{R}^n_{j+\frac{1}{2}})$. By the definition of slopes (5.3) and using property (3.2), we can easily see that

$$\begin{aligned}
&|\rho^n_{j+\frac{1}{2}-} - \rho^n_j|, \ \ |\rho^{(1)}_{j+\frac{1}{2}-} - \rho^{(1)}_j| \le \frac{1}{2}K\Delta x^{\delta},\\
&|R^n_{j+\frac{1}{2}} - \tilde{R}^n_{j+\frac{1}{2}}| \le K_1\Delta x^{\delta}, \ \ |R^{(1)}_{j+\frac{1}{2}} - \tilde{R}^{(1)}_{j+\frac{1}{2}}| \le K_2\Delta x^{\delta}, \ \ |\rho^{(1)}_j - \tilde{\rho}^{(1)}_j| \le K_3\Delta x^{\delta},
\end{aligned}$$

where $K_1 := \frac{1}{2}Kw^0_\eta\eta$, $K_2 := w^0_\eta\eta\Big(\lambda K\big(\|g'\|\|v\| + \|g\|\|v'\|w^0_\eta\eta\big) + \frac{K}{2}\Big)$ and $K_3 := \lambda K\big(\|g'\|\|v\| + \|g\|\|v'\|w^0_\eta\eta\big)$. Now, defining $K_4 := \max\{\frac{K}{2}, K_1, K_2, K_3\}$ and $\tilde{K} := \frac{\lambda}{2}K_4\big(3\|g'\|\|v\| + 2\|g\|\|v'\|\big)$, we can conclude that

$$|a^{n+1}_{j+\frac{1}{2}}| \le \tilde{K}\Delta x^{\delta}, \ \ \delta \in (0,1).$$

This completes the proof. □

Proof of Theorem 5.3: The second-order scheme (3.8) with the modified slope (5.3) can be written in the form (5.1). Further, the hypotheses (i) and (ii) of the Theorem 5.1 are satisfied through Theorem 5.4 and Lemma 5.5, respectively. Theorems 3.4, 3.7, and 3.9 hold for the scheme (3.8) even with the modified slopes (5.3), thereby proving hypothesis (iii) of Theorem 5.1. Thus, using Theorem 5.1 we can conclude that with the modified slope (5.3), the second-order scheme (3.8) converges to the unique entropy solution of (2.1).

Remark 5.6. In fact, the modification in the slope is needed only for the analysis, in implementation it is not needed (see [52,61,71–73]). Specifically, it is mentioned just below equation (26), page number 158 of [61] and just below Figure 3, page 68 of [72]. Also see the Remark in page 577 of [71].

6. A MUSCL-Hancock type scheme

In this section, we propose a MUSCL- Hancock type second-order accurate method for approximating the problem given in (2.1). We discretize the domain with parameters Δx and Δt as in Section 3. Given the cell average values ρ_j^n at time $t = t^n$, we reconstruct a piecewise linear function denoted by $\tilde{\rho}^n$ as

$$\tilde{\rho}^n(x) = \rho_j^n + \frac{(x - x_j)}{\Delta x}\sigma_j^n \text{ for } x \in (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}),$$

where the slope σ_j^n is chosen as in (3.4). To compute the solution at the next time level t^{n+1}, we follow two steps.

Step 1. The left and right face values at each interface are evolved in time by a unit of $\frac{\Delta t}{2}$ using the Taylor expansion:

$$\begin{aligned} \rho_{j+\frac{1}{2},-}^{n+\frac{1}{2}} &= \rho_{j+\frac{1}{2},-}^n - \frac{\lambda}{2}\left(g(\rho_{j+\frac{1}{2},-}^n)v(R_{j+\frac{1}{2},-}^n) - g(\rho_{j-\frac{1}{2},+}^n)v(R_{j-\frac{1}{2},+}^n)\right), \\ \rho_{j+\frac{1}{2},+}^{n+\frac{1}{2}} &= \rho_{j+\frac{1}{2},+}^n - \frac{\lambda}{2}\left(g(\rho_{j+\frac{3}{2},-}^n)v(R_{j+\frac{3}{2},-}^n) - g(\rho_{j+\frac{1}{2},+}^n)v(R_{j+\frac{1}{2},+}^n)\right), \end{aligned} \tag{6.1}$$

where the convolution terms $R_{j+\frac{1}{2},\pm}^n$ are computed as

$$R_{j+\frac{1}{2},-}^n = \Delta x \sum_{k=0}^{N-1} w_\eta^k \rho_{j+k+\frac{1}{2},-}^n \quad \text{and} \quad R_{j+\frac{1}{2},+}^n = \Delta x \sum_{k=0}^{N-1} w_\eta^k \rho_{j+k+\frac{1}{2},+}^n.$$

Step 2. The updated approximate solution at time t^{n+1} is given by

$$\rho_j^{n+1} = \rho_j^n - \lambda(f_{j+\frac{1}{2}}^n - f_{j-\frac{1}{2}}^n),$$

where $f_{j+\frac{1}{2}}^n$ is the numerical flux. Here, we use a Godunov-type numerical flux (as given in [42]) defined by

$$f_{j+\frac{1}{2}}^n = g(\rho_{j+\frac{1}{2},-}^{n+\frac{1}{2}})v(R_{j+\frac{1}{2}}^{n+\frac{1}{2}}), \tag{6.2}$$

where the the convolution term approximation $R_{j+\frac{1}{2}}^{n+\frac{1}{2}}$ needs to be evaluated carefully. For this, we consider the following piecewise linear function

$$\hat{\rho}^{n+\frac{1}{2}}(x) := \rho_{j-\frac{1}{2},+}^{n+\frac{1}{2}} + \frac{(x - x_{j-\frac{1}{2}})}{\Delta x}\left(\rho_{j+\frac{1}{2},-}^{n+\frac{1}{2}} - \rho_{j-\frac{1}{2},+}^{n+\frac{1}{2}}\right) \text{ for } x \in (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}).$$

Subsequently, the term $R_{j+\frac{1}{2}}^{n+\frac{1}{2}}$ can be defined as

$$R_{j+\frac{1}{2}}^{n+\frac{1}{2}} := \sum_{k=0}^{N-1} \gamma_k \rho_{j+k+\frac{1}{2},+}^{n+\frac{1}{2}} + \frac{1}{\Delta x}\sum_{k=0}^{N-1} \chi_k \left(\rho_{j+k+\frac{3}{2},-}^{n+\frac{1}{2}} - \rho_{j+k+\frac{1}{2},+}^{n+\frac{1}{2}}\right),$$

where $\gamma_k := \int_{k\Delta x}^{(k+1)\Delta x} w_\eta(y)\,\mathrm{d}y$ and $\chi_k := \int_0^{\Delta x} y w_\eta(y+k\Delta x)\,\mathrm{d}y$ for $k=0,\dots,N-1$. This approximation of the convolution term is motivated by writing

$$\begin{aligned}
R(x_{j+\frac{1}{2}}, t^{n+\frac{1}{2}}) &= \sum_{k=0}^{N-1} \int_{x_{j+k+\frac{1}{2}}}^{x_{j+k+\frac{3}{2}}} \rho(y, t^{n+\frac{1}{2}}) w_\eta(y - x_{j+\frac{1}{2}})\,\mathrm{d}y \\
&\approx \sum_{k=0}^{N-1} \int_{x_{j+k+\frac{1}{2}}}^{x_{j+k+\frac{3}{2}}} \hat{\rho}^{n+\frac{1}{2}}(y) w_\eta(y - x_{j+\frac{1}{2}})\,\mathrm{d}y \\
&= \sum_{k=0}^{N-1} \rho^{n+\frac{1}{2}}_{j+k+\frac{1}{2},+} \int_{x_{j+k+\frac{1}{2}}}^{x_{j+k+\frac{3}{2}}} w_\eta(y - x_{j+\frac{1}{2}})\,\mathrm{d}y \\
&\quad + \frac{1}{\Delta x} \sum_{k=0}^{N-1} (\rho^{n+\frac{1}{2}}_{j+k+\frac{3}{2},-} - \rho^{n+\frac{1}{2}}_{j+k+\frac{1}{2},+}) \int_{x_{j+k+\frac{1}{2}}}^{x_{j+k+\frac{3}{2}}} (y - x_{j+k+\frac{1}{2}}) w_\eta(y - x_{j+\frac{1}{2}})\,\mathrm{d}y \\
&= \sum_{k=0}^{N-1} \gamma_k \rho^{n+\frac{1}{2}}_{j+k+\frac{1}{2},+} + \frac{1}{\Delta x} \sum_{k=0}^{N-1} \chi_k \left(\rho^{n+\frac{1}{2}}_{j+k+\frac{3}{2},-} - \rho^{n+\frac{1}{2}}_{j+k+\frac{1}{2},+} \right).
\end{aligned}$$

Remark 6.1. Analogous to Remark 3.2, if the quadrature rule used to compute $R^n_{j+\frac{1}{2},\pm}$ in Step 1 is not exact for the given kernel function (*i.e.*, if $\Delta x \sum_{k=0}^{N-1} w_\eta^k \neq 1$), then we replace w_η^k by $\tilde{w}_\eta^k = \dfrac{w_\eta^k}{Q_{\Delta x}}$, where we choose $Q_{\Delta x} := \Delta x \sum_{k=0}^{N-1} w_\eta^k$.

7. Numerical results

In this section, we consider several test cases to demonstrate the performance of the proposed RK-2 and MH schemes described in Sections 3 and 6, respectively, by comparing it with the first-order Godunov-type scheme of [25, 42], which we denote by FO-Godunov. For all the test cases, we use the same CFL as that of RK-2 scheme, given in (3.13). Also, we choose $g(\rho) = \rho$ unless otherwise specified. Consider a uniform partition $\{I_j\}_{j=1}^M$ of the spatial domain $[a,b]$ with $\Delta x = \frac{b-a}{M}$. We will consider two types of boundary conditions: periodic and absorbing. In order to implement these boundary conditions, we will introduce ghost cells on either side of the domain. The ghost cell values, ρ_0^n and ρ_{M+j}^n for $j=1,\dots,N$, where $N = \eta/\Delta x$, are taken as follows. For periodic boundary conditions,

$$\rho_0^n = \rho_M^n \quad \text{and} \quad \rho_{M+j}^n = \rho_j^n \text{ for } j = 1,\dots,N,$$

and for absorbing boundary conditions,

$$\rho_0^n = \rho_1^n \quad \text{and} \quad \rho_{M+j}^n = \rho_M^n \text{ for } j = 1,\dots,N,$$

where $\{\rho_j^n\}_{j=1}^M$ denote the solution in real cells. In all the test cases, as the analytical solutions of (2.1) are not available, we use the RK-2 scheme (3.8) with fine mesh to generate reference solutions. These are used to determine the numerical errors and the experimental order of accuracy. The L^1-error for the cell average solution at time $t = t^n$ is given by

$$e(\Delta x) := \Delta x \sum_{j=1}^{M} |\rho_j^n - \rho_j^{n,ref}|,$$

where ρ_j^n and $\rho_j^{n,ref}$ are the cell averages of the numerical and the reference solutions, respectively. The experimental order of accuracy (E.O.A) is determined as

$$\Theta(\Delta x) := \log_2\Big(e(\Delta x)/e(\Delta x/2)\Big).$$

7.1. Mean downstream density model

In this part, we consider test cases with various initial data to solve the downstream density model equation given in (2.1):

$$\begin{aligned} \partial_t \rho + \partial_x\big(g(\rho)v(\rho * w_\eta)\big) &= 0, \quad x \in \mathbb{R},\, t \in (0,T], \\ \rho(0,x) &= \rho_0(x), \qquad x \in \mathbb{R}. \end{aligned}$$

Example 1 (smooth test case): To verify the order of accuracy of the proposed RK-2 and MH schemes, we consider the problem (2.1) with a smooth initial datum (see [25])

$$\rho_0(x) = 0.5 + 0.4 \sin \pi x. \tag{7.1}$$

The numerical solutions are computed in the domain $[-1,1]$ with periodic boundary conditions. We choose $v(\rho) = 1-\rho$ and the convolution parameter $\eta = 0.1$. Here, the reference solution is computed using a mesh size of $\Delta x = \frac{1}{1280}$. The solutions are computed up to time $T = 0.15$ for three different kernel functions $w_\eta(x) = \frac{1}{\eta}$, $w_\eta(x) = \frac{2(\eta - x)}{\eta^2}$ and $w_\eta(x) = \frac{3(\eta^2 - x^2)}{2\eta^3}$ with time steps $\Delta t = \frac{\Delta x}{2+10\Delta x}$, $\Delta t = \frac{\Delta x}{2+20\Delta x}$ and $\Delta t = \frac{\Delta x}{2+15\Delta x}$ respectively. From Table 1, we observe that both the RK-2 and MH schemes exhibit the desired experimental order of accuracy. In Figure 1, we provide the L^1 error *versus* CPU time plots for the RK-2 and MH schemes, corresponding to the initial data (7.1) and considering the three kernel functions mentioned above. Here, we use the mesh-sizes $\Delta x = 0.1, 0.05, 0.025, 0.0125, 0.00625$ and 0.003125. The results indicate that the MH scheme is computationally more efficient when compared to the RK-2 scheme.

Example 2: We consider the problem (2.1) with a discontinuous initial datum as given in [46],

$$\rho_0(x) = \begin{cases} 0.8, & \text{if} -0.5 < x < -0.1, \\ 0, & \text{otherwise}, \end{cases} \tag{7.2}$$

described in the computational domain $[-1,1]$ and using absorbing boundary conditions. The velocity and convolution parameters are set as $v(\rho) = 1-\rho$ and $\eta = 0.1$, respectively. The numerical solutions are computed at time T = 0.5 with a mesh size of $\Delta x = 0.01$ for two different kernel functions: $w_\eta(x) = \frac{1}{\eta}$ and $w_\eta(x) = \frac{2(\eta - x)}{\eta^2}$, with the respective time steps $\Delta t = \frac{\Delta x}{2+10\Delta x}$ and $\Delta t = \frac{\Delta x}{2+20\Delta x}$. The reference solutions are computed using a mesh-size of $\Delta x = \frac{1}{2560}$. The results are depicted in Figures 2a and 2c, respectively. Also, zoomed images of the region $[0.3, 0.55] \times [-0.01, 0.19]$ for both the cases are given in Figures 2b and 2d, respectively. It is observed that both the RK-2 and MH schemes provide better resolution than the first-order Godunov type scheme. Moreover, the RK-2 and MH solutions are comparable, with the MH solution giving slightly better resolution near the right discontinuity as seen in Figures 2b and 2d. The L^1 error *versus* CPU time plots for the RK-2 and MH schemes corresponding to the initial datum (7.2), for the kernel functions $w_\eta(x) = \frac{1}{\eta}$ and $w_\eta(x) = \frac{2(\eta - x)}{\eta^2}$ are given in Figures 5a and 5b, respectively. The solutions are computed with mesh-sizes $\Delta x = 0.1, 0.05, 0.025, 0.0125, 0.00625$ and 0.003125. The results show that the MH scheme is computationally more efficient when compared to the RK-2 scheme.

Example 3: In this example we consider the same initial datum (7.2), particularly to see the behaviour of the solutions at two different times. Here, we use this initial condition to simulate the scalar problem (2.1) in the computational domain $[-1,2]$. The velocity v is given by $v(\rho) = v_{\max}(1-\rho)$ with $v_{\max} = 0.8$, and the convolution kernel $w_\eta(x) = \frac{2(\eta - x)}{\eta^2}$ with $\eta = 0.3$. Numerical solutions are computed at two different time levels,

TABLE 1. Example 1. L^1-errors and E.O.A. obtained using the FO-Godunov, RK-2 and MH schemes to solve the problem (2.1) with smooth initial condition (7.1) and three different kernel functions: $w_\eta(x) = \frac{1}{\eta}$, $w_\eta(x) = \frac{2(\eta - x)}{\eta^2}$ and $w_\eta(x) = \frac{3(\eta^2 - x^2)}{2\eta^3}$ where $\eta = 0.1$. Numerical solutions are computed up to time $T = 0.15$.

$w_\eta(x)$		FO-Godunov		RK-2		MH	
	Δx	L^1-error	E.O.A.	L^1-error	E.O.A.	L^1-error	E.O.A.
$\frac{1}{\eta}$	0.05	0.013011	–	0.001686	–	0.001474	–
	0.025	0.006478	1.006117	0.000463	1.862867	0.000374	1.978590
	0.0125	0.003199	1.017847	0.000122	1.924356	9.635912e-05	1.956891
	0.00625	0.001591	1.007311	3.240261e-05	1.914917	2.486745e-05	1.954162
	0.003125	0.000794	1.001897	8.062984e-06	2.006724	5.996242e-06	2.052127
$\frac{2(\eta-x)}{\eta^2}$	0.05	0.014857	–	0.004348	–	0.001489	–
	0.025	0.007085	1.068269	0.001151	1.917093	0.000369	2.009029
	0.0125	0.003436	1.044055	0.000299	1.943794	9.558809e-05	1.952366
	0.00625	0.001687	1.026122	7.636725e-05	1.970390	2.448845e-05	1.964729
	0.003125	0.000835	1.013861	1.880892e-05	2.021536	5.907713e-06	2.051429
$\frac{3(\eta^2-x^2)}{2\eta^3}$	0.05	0.014294	–	0.003977	–	0.001452	–
	0.025	0.006894	1.051945	0.001024	1.956857	0.000366	1.987019
	0.0125	0.003358	1.037732	0.000265	1.949414	9.510829e-05	1.945937
	0.00625	0.001654	1.021031	6.804842e-05	1.962939	2.450048e-05	1.956760
	0.003125	0.000820	1.011442	1.679244e-05	2.018749	5.900419e-06	2.053920

$T = 1.0, 2.0$ with a mesh size of $\Delta x = 0.0125$ and time step $\Delta t = \frac{\Delta x}{2 + \frac{20}{3}\Delta x}$ using absorbing boundary conditions. The results are shown in Figure 3. For the reference solution, we use the RK-2 scheme with a fine mesh of size $\Delta x = 0.0025$. It is observed that at both times the RK-2 and MH schemes give better resolution than the first-order scheme.

Example 4: In this example, we evolve (2.1) for a quadratic kernel function with the initial datum (7.2) in the computational domain $x \in [-1, 1]$. Further, we choose: $v(\rho) = 1 - \rho$, $w_\eta(x) = \frac{3(\eta^2 - x^2)}{2\eta^3}$, $\eta = 0.1$, $\Delta x = 0.0025$ and $\Delta t = \frac{\Delta x}{2 + 15\Delta x}$. Numerical solutions are computed at time $T = 0.1$ using absorbing boundary conditions and are given in Figure 4. Reference solution is obtained using the RK-2 scheme with a mesh size of $\Delta x = 0.000625$. Figure 4b is the enlarged view of the region $[-0.04, 0.04] \times [-0.025, 0.400]$ in Figure 4a. It is observed that the RK-2 and MH solutions give a better resolution compared to the Godunov-type scheme. The L^1 error *versus* CPU time plot for the RK-2 and MH schemes corresponding to the initial datum (7.2) with the kernel function $w_\eta(x) = \frac{3(\eta^2 - x^2)}{2\eta^3}$ is given in Figure 5c. The solutions are computed using the mesh-sizes $\Delta x = 0.1, 0.05, 0.025, 0.0125, 0.00625$ and 0.003125. Here also, we observe that the MH scheme is more efficient compared to the RK-2 scheme.

7.2. Mean downstream velocity model

We consider the initial value problem for the downstream velocity model (B.1) outlined in Appendix B

$$\partial_t \rho + \partial_x \Big(g(\rho) \big(v(\rho) * w_\eta \big) \Big) = 0, \quad x \in \mathbb{R}, t \in (0, T],$$
$$\rho(x, 0) = \rho_0(x), \qquad x \in \mathbb{R},$$

with three test cases. The first test case involves a smooth function and will be used to confirm the order of accuracy of the schemes. The remaining test cases involve initial data with discontinuities and will be used to compare the performance of first-order and second-order schemes.

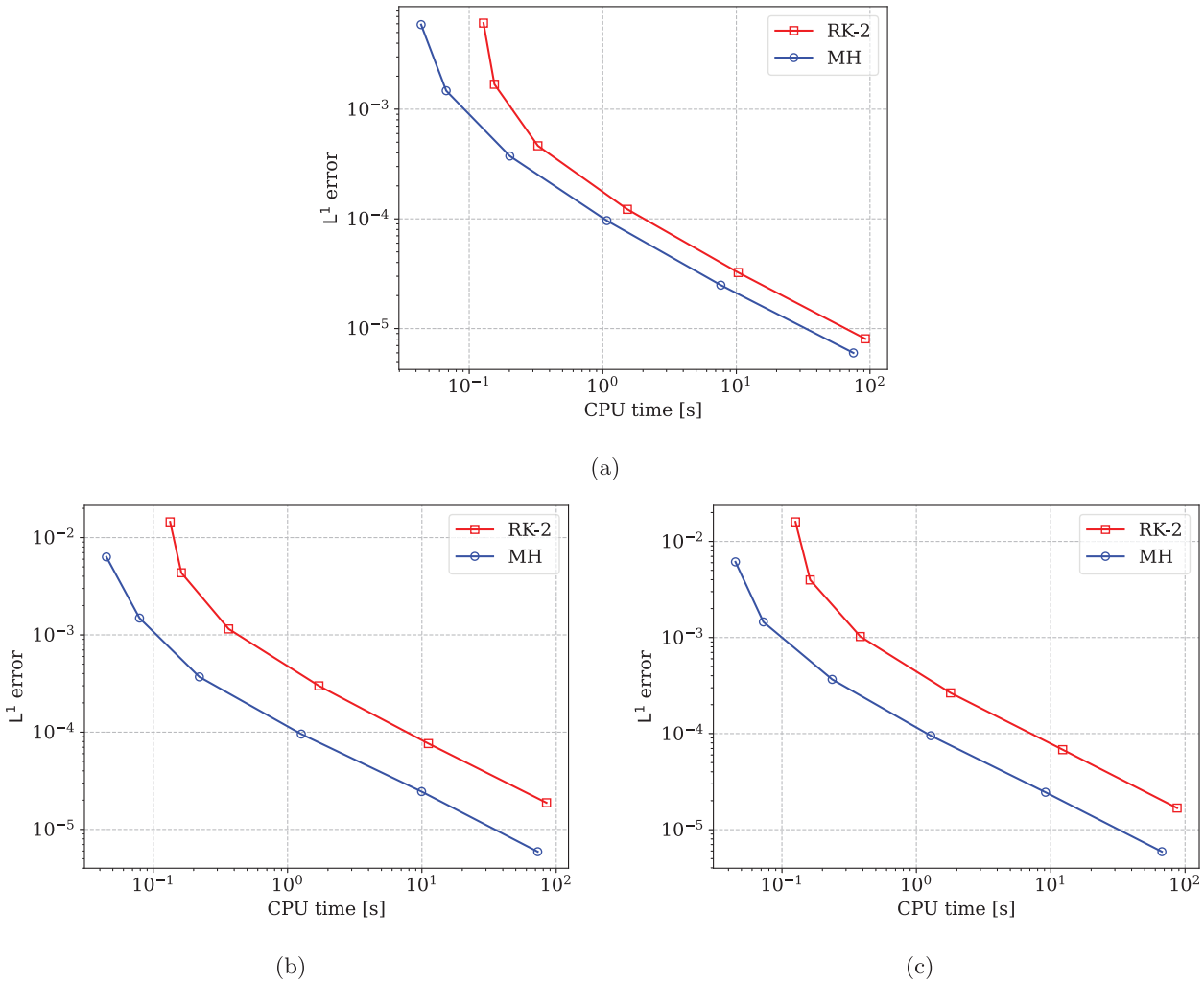


FIGURE 1. Example 1. L^1 error *versus* CPU time plots in loglog scale for the RK-2 and MH schemes to solve the problem (2.1) with initial datum (7.1) and three different kernel functions (a) $w_\eta(x) = \frac{1}{\eta}$, (b) $w_\eta(x) = \frac{2(\eta - x)}{\eta^2}$ and (c) $w_\eta(x) = \frac{3(\eta^2 - x^2)}{2\eta^3}$. The solutions are computed at time $T = 0.15$.

Example 5 (smooth test case): To verify the desired order of accuracy of the proposed RK-2 and MH schemes, we consider a test case with the same smooth initial datum as in (7.1) to evolve (B.1), in the computational domain $x \in [-1, 1]$ together with periodic boundary conditions. The velocity function v is taken as $v(\rho) = 1 - \rho^2$ and $g(\rho) = \rho^2$. Numerical solutions are computed upto time $T = 0.15$ for the kernel functions $w_\eta(x) = \frac{1}{\eta}$, $w_\eta(x) = \frac{2(\eta - x)}{\eta^2}$ and $w_\eta(x) = \frac{3(\eta^2 - x^2)}{2\eta^3}$ where $\eta = 0.1$ and the respective time steps are $\Delta t = \frac{\Delta x}{2 + 10\Delta x}$, $\Delta t = \frac{\Delta x}{2 + 20\Delta x}$ and $\Delta t = \frac{\Delta x}{2 + 15\Delta x}$. Here, we use a reference solution generated with a mesh of size $\Delta x = \frac{1}{1280}$. The results are tabulated in Table 2.

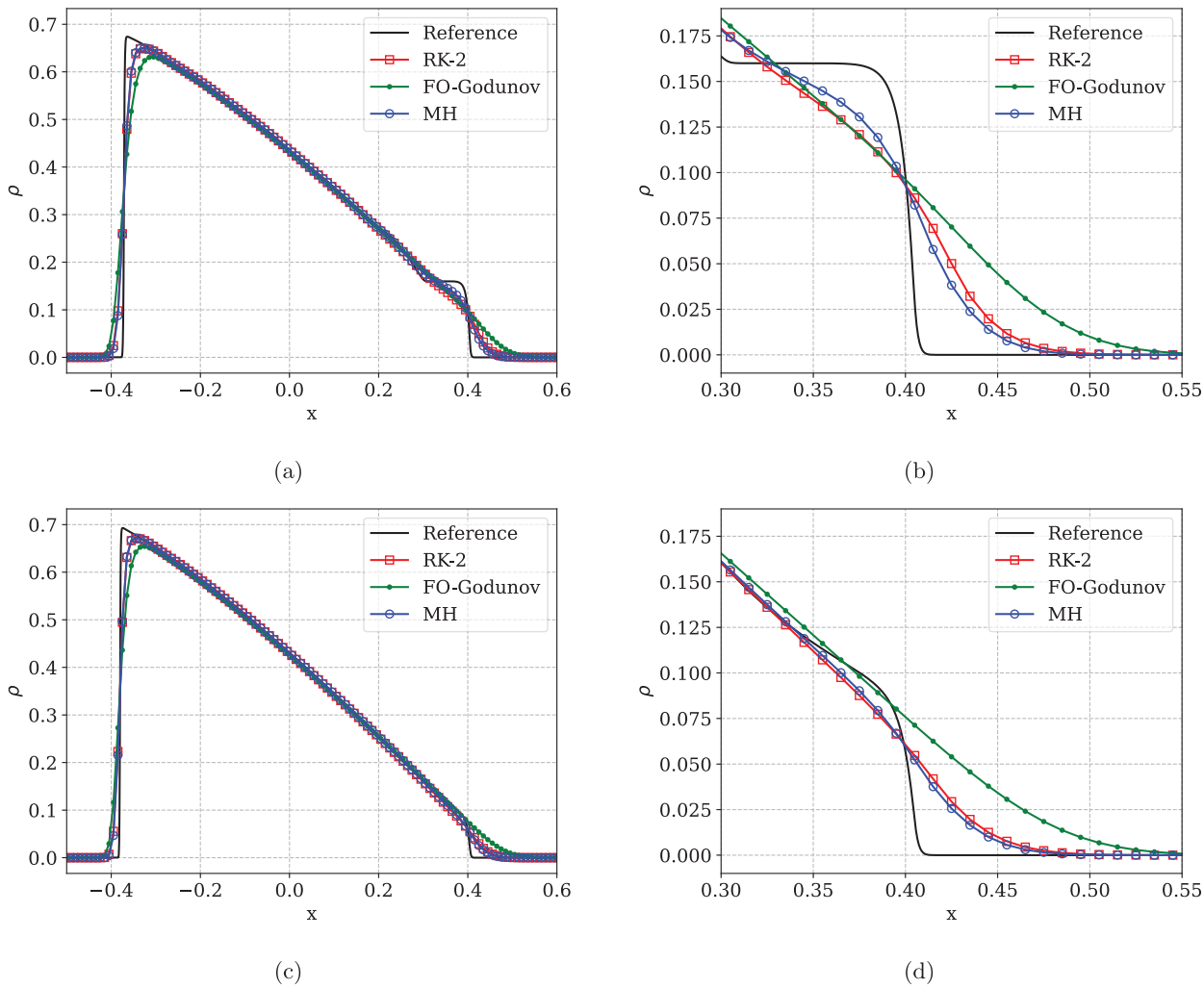
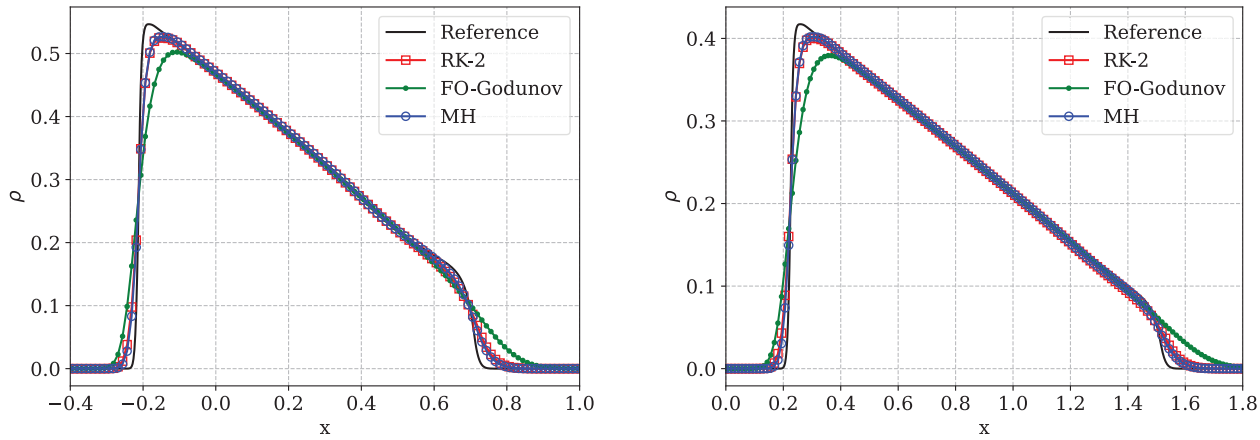


FIGURE 2. Example 2. Numerical solutions of (2.1) at time T = 0.5 with the initial condition (7.2) for two different kernel functions (a) $w_\eta(x) = \frac{1}{\eta}$ and (c) $w_\eta(x) = \frac{2(\eta - x)}{\eta^2}$, where $\eta = 0.1$. (b) Zoomed image of the region $[0.3, 0.55] \times [-0.01, 0.19]$ in (a), (d) Zoomed image of the region $[0.3, 0.55] \times [-0.01, 0.19]$ in (c). The velocity function $v(\rho) = 1 - \rho$. A mesh size of $\Delta x = 0.01$ is chosen with the times steps $\Delta t = \frac{\Delta x}{2+10\Delta x}$ and $\Delta t = \frac{\Delta x}{2+20\Delta x}$ for the two kernels (a) and (c), respectively.

Example 6 (Non-linear velocity): In this test case, we choose a discontinuous initial datum (see [42]), for the problem (B.1)

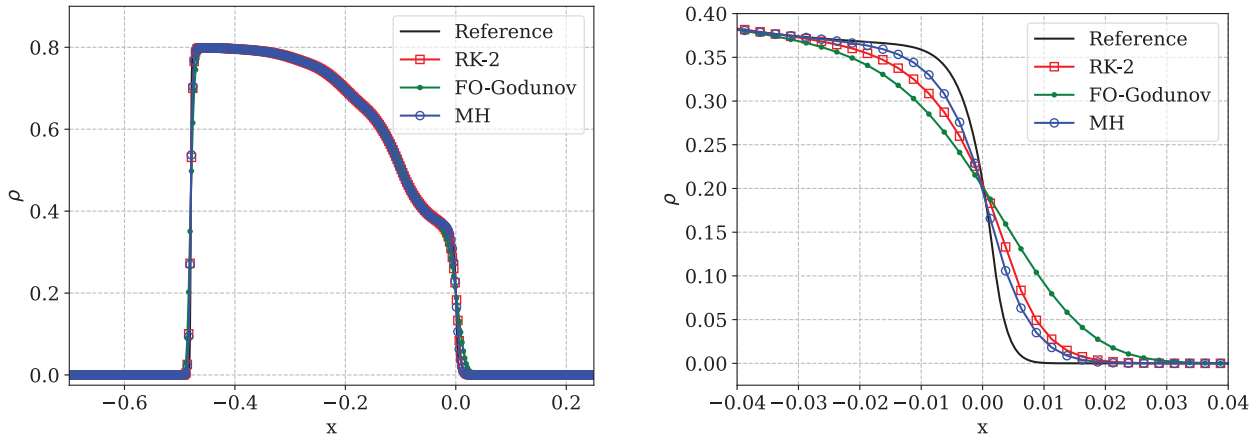
$$\rho_0(x) = \begin{cases} 1, & \text{if } 1/3 < x < 2/3, \\ \frac{1}{3}, & \text{otherwise}, \end{cases} \tag{7.3}$$

in the domain $[0, 1]$. Further, we choose the convolution kernel: $w_\eta(x) = \frac{3(\eta^2 - x^2)}{2\eta^3}$ with $\eta = 0.1$, velocity function $v(\rho) = 1 - \rho^2$, $g(\rho) = \rho^2$ and time step $\Delta t = \frac{\Delta x}{2+15\Delta x}$ with a mesh size of $\Delta x = 0.01$. By imposing periodic



(a) (b)

FIGURE 3. Example 3. Numerical solutions of (2.1) with the initial condition (7.2) at two different times: (a) $T = 1.0$ and (b) $T = 2.0$. Mesh size $\Delta x = 0.0125$, $\Delta t = \frac{\Delta x}{2+\frac{20}{3}\Delta x}$, $v(\rho) = 0.8(1-\rho)$, $w_\eta(x) = \frac{2(\eta - x)}{\eta^2}$, $\eta = 0.3$.



(a) (b)

FIGURE 4. Example 4. (a) Numerical solutions of (2.1) with initial datum (7.2) at time $T = 0.1$, where $v(\rho) = 1 - \rho$, $w_\eta(x) = \frac{3(\eta^2 - x^2)}{2\eta^3}$, $\eta = 0.1$, $\Delta x = \frac{1}{400}$ and $\Delta t = \frac{\Delta x}{2+15\Delta x}$. (b) Enlarged view of the region $[-0.04, 0.04] \times [-0.025, 0.400]$ in plot (a).

boundary conditions, numerical solutions are computed at time $T = 0.1$ and time-step $\Delta t = \frac{\Delta x}{2+15\Delta x}$. The results are plotted in Figure 6, where the reference solution is computed with a mesh size of $\Delta x = \frac{1}{1000}$. As illustrated in Figure 6b, an enlarged view of the region $[0.71, 0.82] \times [0.3, 0.67]$ reveals that the RK-2 and MH schemes provide higher resolution than the first-order Godunov-type scheme.

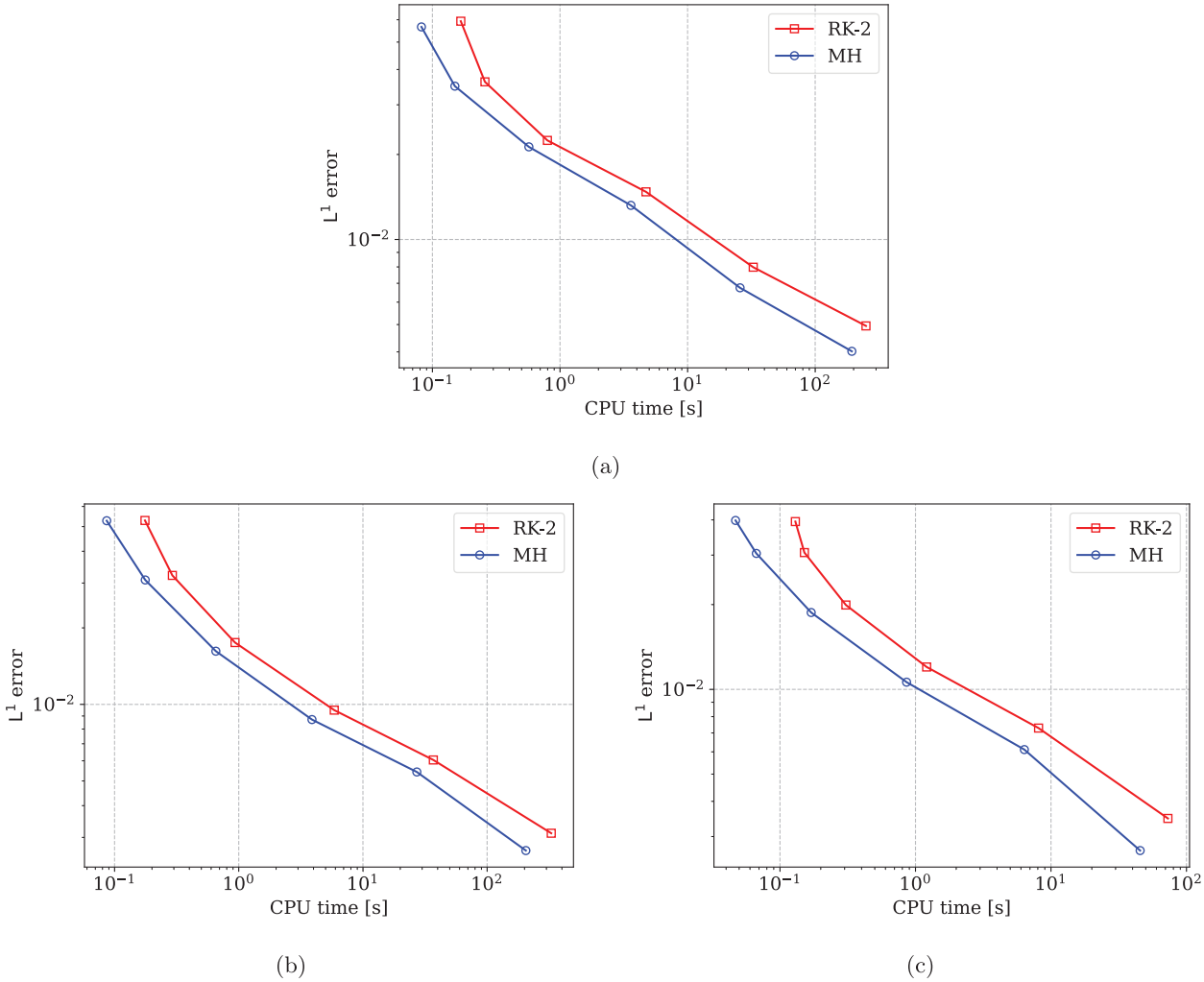
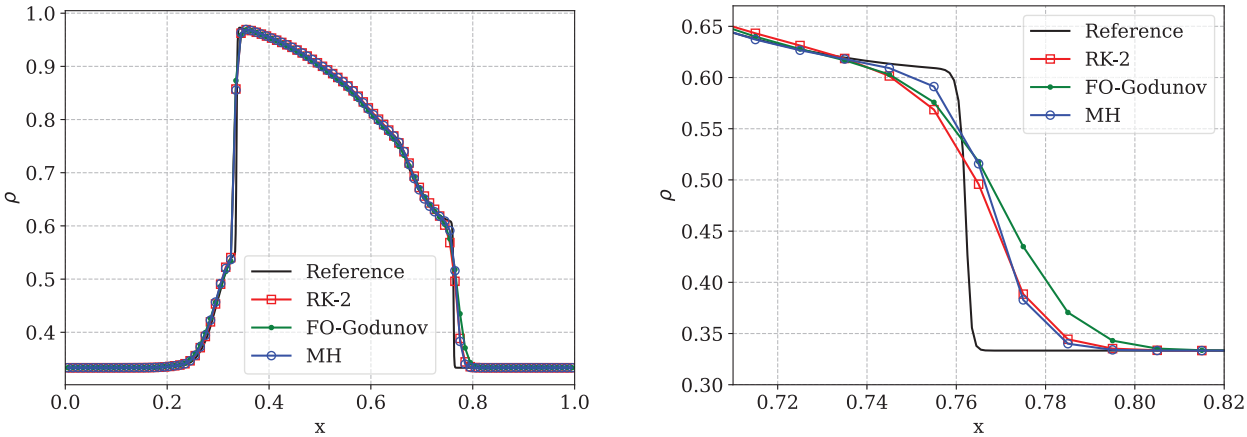


FIGURE 5. Examples 2 and 4. L^1 error *versus* CPU time plots in loglog scale for the RK-2 and MH schemes to solve the problem (2.1) with the discontinuous initial data (7.2) as described in: (a) Example 2 with $w_\eta(x) = \frac{1}{\eta}$ at time $T = 0.5$, (b) Example 2 with $w_\eta(x) = \frac{2(\eta - x)}{\eta^2}$ at time $T = 0.5$ and (c) Example 4 with $w_\eta(x) = \frac{3(\eta^2 - x^2)}{2\eta^3}$ at time $T = 0.1$.

Example 7 (Non-linear velocity): We now consider an example with a non-linear velocity function $v(\rho) = 1 - \rho^5$, described in [42]. The problem (B.1) is simulated in the computational domain $[0, 1]$ with the initial datum (7.3) and the kernel function $w_\eta(x) = \frac{1}{\eta}$, where $\eta = 0.1$. Numerical solutions are computed at time $T = 0.05$ with periodic boundary conditions and a mesh of size $\Delta x = \frac{1}{100}$. The time-step is chosen as $\Delta t = \frac{\Delta x}{2 + 10\Delta x}$. The results can be seen in Figure 7, where the reference solution is computed with a fine mesh of size $\Delta x = \frac{1}{1000}$. The plots indicate that the second-order RK-2 and MH schemes produce better resolution than the Godunov-type scheme. For a better visualization, we have provided an enlarged view of the region $[0.66, 0.77] \times [0.3, 0.8]$ in Figure 7b.

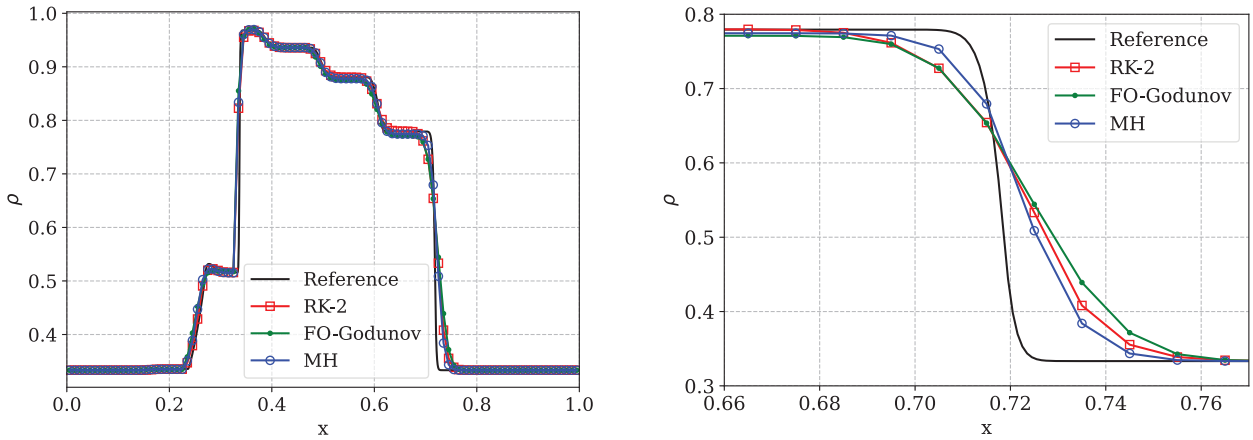
TABLE 2. Example 5. L^1-errors and E.O.A. obtained for the RK-2 and MH schemes to solve the problem (B.1) with smooth initial condition (7.1) at $T = 0.15$, $\eta = 0.1$ and for the kernel functions: $w_\eta(x) = \frac{1}{\eta}$, $w_\eta(x) = \frac{2(\eta - x)}{\eta^2}$ and $w_\eta(x) = \frac{3(\eta^2 - x^2)}{2\eta^3}$.

$w_\eta(x)$		FO-Godunov		RK-2		MH	
	Δx	L^1-error	E.O.A.	L^1-error	E.O.A.	L^1-error	E.O.A.
$\frac{1}{\eta}$	0.05	0.014125	–	0.001002	–	0.001230	–
	0.025	0.007099	0.992521	0.000280	1.839409	0.000330	1.895510
	0.0125	0.003542	1.002772	7.413608e-05	1.917842	8.579633e-05	1.947201
	0.00625	0.001767	1.003282	1.893899e-05	1.968816	2.169048e-05	1.983853
	0.003125	0.000882	1.002633	4.501158e-06	2.072991	5.206407e-06	2.058702
$\frac{2(\eta-x)}{\eta^2}$	0.05	0.016040	–	0.002595	–	0.001413	–
	0.025	0.007745	1.050226	0.000744	1.801440	0.000355	1.991035
	0.0125	0.003780	1.034750	0.000197	1.912197	8.939415e-05	1.992234
	0.00625	0.001864	1.020026	5.068842e-05	1.964803	2.266346e-05	1.979811
	0.003125	0.000925	1.010639	1.250372e-05	2.019297	5.805551e-06	1.964863
$\frac{3(\eta^2-x^2)}{2\eta^3}$	0.05	0.015555	–	0.002308	–	0.001331	–
	0.025	0.007569	1.039206	0.000621	1.893337	0.000340	1.965531
	0.0125	0.003716	1.026320	0.000161	1.939921	8.587823e-05	1.988833
	0.00625	0.001837	1.015781	4.141757e-05	1.967297	2.176546e-05	1.980251
	0.003125	0.000913	1.008295	1.024745e-05	2.014977	5.485051e-06	1.988463



(a) (b)

FIGURE 6. Example 6. (a) Simulation of (B.1) at time $T = 0.1$ with $v(\rho) = 1 - \rho^2$, $g(\rho) = \rho^2$, $w_\eta(x) = \frac{3(\eta^2 - x^2)}{2\eta^3}$, $\eta = 0.1$, $\Delta x = 0.01$ and $\Delta t = \frac{\Delta x}{2 + 15\Delta x}$. (b) Enlarged view of the region $[0.71, 0.82] \times [0.30, 0.67]$ in plot (a).



(a) (b)

FIGURE 7. Example 7. Behaviour of first-order and second-order schemes with non-linear velocity $v(\rho) = 1 - \rho^5$. (a) Numerical solution of (B.1) at time $T = 0.1$ with $w_\eta(x) = \frac{1}{\eta}$, $\eta = 0.1$, $\Delta x = 0.01$ and $\Delta t = \frac{\Delta x}{2+15\Delta x}$. (b) Enlarged view of the region $[0.66, 0.77] \times [0.3, 0.8]$ in (a).

8. Conclusion

We have conducted a study on the numerical approximation of a class of non-local conservation laws modelling traffic-flow problems, with more emphasis on the convergence analysis. We use a MUSCL-type spatial reconstruction and strong stability preserving Runge-Kutta time stepping to derive a second-order scheme, denoted by RK-2. The resulting scheme is shown to converge to a weak solution of the given problem. In addition, using a space-step dependent slope limiter we show that the scheme converges to the unique entropy solution. Further, we have proposed a MUSCL-Hancock type (MH) second-order scheme which requires only one intermediate stage in the time evolution, unlike the RK schemes. We observe that both the second-order schemes produce stable and accurate solutions. Additionally, we notice that the MH scheme gives slightly better results compared to the RK-2 scheme, for example see Figures 2b, 2d, 6b and 7b of Examples 2, 6 and 7. Further, the L^1 error *versus* CPU time plots in Figures 1 and 5 indicate that the MH scheme is computationally more efficient in comparison to the RK-2 scheme. We wish to study the convergence analysis of this MUSCL-Hancock type scheme in a future work.

Appendix A. Proof of Theorem 4.1

Proof. By Theorem A.8 of [53]), for each fixed $t \in [0, T]$ and for any sequence $\xi_j \to 0$ there exists a subsequence, again denoted by ξ_j, such that $\{u_{\xi_j}(t)\}$ converges to a function $u(t)$ in $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})$.

Now, consider a countable dense subset E of the interval $[0, T]$. By a diagonalization argument, we can extract again a subsequence (still denoted by ξ_j) such that

$$\int_B |u_{\xi_j}(t, x) - u(t, x)| \, \mathrm{d}x \to 0 \text{ as } \xi_j \to 0, \text{ for } t \in E. \tag{A.1}$$

Let $\epsilon > 0$ be given. Then there exists a positive δ such that $\omega_T^B \tilde{\delta} \leq \epsilon$ for all $\tilde{\delta} \leq \delta$. Fix $t \in [0, T]$. By the denseness of E, there exists a $t_k \in E$ with $|t_k - t| \leq \delta$. Therefore, by (4.3)

$$\int_B |u_{\tilde{\xi}}(t,x) - u(t_k,x)|\,\mathrm{d}x \le \omega_T^B(|t-t_k|) + \mathcal{O}(\tilde{\xi})$$
$$\le \epsilon + \mathcal{O}(\tilde{\xi}) \text{ for } \tilde{\xi} \le \xi$$

and by (A.1)

$$\int_B |u_{\xi_{j_1}}(t_k,x) - u_{\xi_{j_2}}(t_k,x)|\,\mathrm{d}x \le \epsilon \text{ for } \xi_{j_1}, \xi_{j_2} \le \xi \text{ and } t_k \in E.$$

Further, applying the triangle inequality, it yields

$$\begin{aligned}\int_B |u_{\xi_{j_1}}(t,x) - u_{\xi_{j_2}}(t,x)|\,\mathrm{d}x &\le \int_B |u_{\xi_{j_1}}(t,x) - u_{\xi_{j_1}}(t_k,x)|\,\mathrm{d}x + \int_B |u_{\xi_{j_1}}(t_k,x) - u_{\xi_{j_2}}(t_k,x)|\,\mathrm{d}x \\ &\quad + \int_B |u_{\xi_{j_2}}(t_k,x) - u_{\xi_{j_2}}(t,x)|\,\mathrm{d}x \\ &\le \omega_T^B(|t-t_k|) + \mathcal{O}(\xi_{j_1}) + \epsilon + \omega_T^B(|t-t_k|) + \mathcal{O}(\xi_{j_2}) \\ &\le 3\epsilon + \mathcal{O}(\xi).\end{aligned}$$

This shows that $u_\xi(t) \to u(t)$ in $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})$ for each $t \in [0,T]$. Finally, by the dominated convergence theorem it follows that

$$\sup_{t\in[0,T]} \int_B |u_\xi(t,x) - u(t,x)|\,\mathrm{d}x \text{ as } \xi \to 0.$$

This completes the proof. □

Appendix B. Mean downstream velocity model

Now, we consider the mean downstream velocity model of non-local traffic proposed in [42]. The corresponding model is given by

$$\begin{aligned}\partial_t \rho + \partial_x\Big(g(\rho)\big(v(\rho) * w_\eta\big)\Big) &= 0, \quad x \in \mathbb{R}, t \in (0,T], \\ \rho(x,0) &= \rho_0(x), \qquad x \in \mathbb{R},\end{aligned} \tag{B.1}$$

where $v(\rho) * w_\eta(t,x) = \int_x^{x+\eta} v(\rho(t,y)) w_\eta(y-x)\,\mathrm{d}y$ and the terms g, v are as in Section 2. This model assumes that the drivers adapt their velocity by evaluating the average velocity of vehicles in a neighbourhood in front of them, giving greater importance to closer vehicles.

B.1. Second-order scheme

Here we extend the second-order scheme considered in Section 3 to approximate (B.1). We proceed as in Section 3, where the main difference is in evaluating the convolution term. Now, the second-order scheme is written as

$$\begin{aligned}\rho_j^{(1)} &= \rho_j^n - \lambda\left(g(\rho^n_{j+\frac{1}{2},-})V^n_{j+\frac{1}{2}} - g(\rho^n_{j-\frac{1}{2},-})V^n_{j-\frac{1}{2}}\right), \\ \rho_j^{(2)} &= \rho_j^{(1)} - \lambda\left(g(\rho^{(1)}_{j+\frac{1}{2},-})V^{(1)}_{j+\frac{1}{2}} - g(\rho^{(1)}_{j-\frac{1}{2},-})V^{(1)}_{j-\frac{1}{2}}\right), \\ \rho_j^{n+1} &= \frac{1}{2}\left(\rho_j^n + \rho_j^{(2)}\right),\end{aligned}$$

where $V^n_{j+\frac{1}{2}}$ and $V^{(1)}_{j+\frac{1}{2}}$ are the approximations of the convolution term $v(\rho) * w_\eta$ at the respective Runge-Kutta time steps, which are given by

$$V^n_{j+\frac{1}{2}} := \frac{\Delta x}{2} \sum_{k=0}^{N-1} \left(v(\rho^n_{j+k+\frac{1}{2},+}) w^k_\eta + v(\rho^n_{j+k+\frac{3}{2},-}) w^{k+1}_\eta \right), \tag{B.2}$$

$$V^{(1)}_{j+\frac{1}{2}} := \frac{\Delta x}{2} \sum_{k=0}^{N-1} \left(v(\rho^{(1)}_{j+k+\frac{1}{2},+}) w^k_\eta + v(\rho^{(1)}_{j+k+\frac{3}{2},-}) w^{k+1}_\eta \right). \tag{B.3}$$

Remark B.1. If the quadrature rule used in (B.2) and (B.3) is not exact for the given kernel function, (*i.e.*, if $\frac{\Delta x}{2} \sum_{k=0}^{N-1} \left(w^k_\eta + w^{k+1}_\eta \right) \neq 1$), then we replace w^k_η by $\tilde{w}^k_\eta = \frac{w^k_\eta}{Q_{\Delta x}}$ in (B.2) and (B.3), where $Q_{\Delta x} := \frac{\Delta x}{2} \sum_{k=0}^{N-1} \left(w^k_\eta + w^{k+1}_\eta \right)$.

B.2. MUSCL-Hancock type scheme

To propose a MUSCL-Hancock scheme for the model (B.1), we proceed exactly as in Section 6, where $v(R^n_{j+\frac{1}{2},\pm})$ and $v(R^{n+\frac{1}{2}}_{j+\frac{1}{2}})$ in (6.1) and (6.2) are replaced by $V^n_{j+\frac{1}{2},\pm}$ and $V^{n+\frac{1}{2}}_{j+\frac{1}{2}}$, respectively. These terms are defined as follows

$$V^n_{j+\frac{1}{2},-} := \Delta x \sum_{k=0}^{N-1} v(\rho^n_{j+k+\frac{1}{2},-}) w^k_\eta, \tag{B.4}$$

$$V^n_{j+\frac{1}{2},+} := \Delta x \sum_{k=0}^{N-1} v(\rho^n_{j+k+\frac{1}{2},+}) w^k_\eta, \tag{B.5}$$

$$V^{n+\frac{1}{2}}_{j+\frac{1}{2}} := \sum_{k=0}^{N-1} \gamma_k v(\rho^{n+\frac{1}{2}}_{j+k+\frac{1}{2},+}) + \frac{1}{\Delta x} \sum_{k=0}^{N-1} \chi_k \left(v(\rho^{n+\frac{1}{2}}_{j+k+\frac{3}{2},-}) - v(\rho^{n+\frac{1}{2}}_{j+k+\frac{1}{2},+}) \right).$$

Remark B.2. If the quadrature rule used in (B.4) and (B.5) is not exact for the given kernel function (*i.e.*, if $\Delta x \sum_{k=0}^{N-1} w^k_\eta \neq 1$), then we replace w^k_η by $\tilde{w}^k_\eta := \frac{w^k_\eta}{Q_{\Delta x}}$ in (B.4) and (B.5), where $Q_{\Delta x} := \Delta x \sum_{k=0}^{N-1} w^k_\eta$.

Appendix C. Results for the FSST scheme

We state a technical lemma (Lem. A.1, page 32, [12]) without proof, which will be used in the next lemma.

Lemma C.1. *Consider a sequence of functions $\psi_{\Delta x} : \mathbb{R} \to \mathbb{R}$ such that there exists a uniform bound $\|\psi_{\Delta x}\| \leq C$. Also assume that $\psi_{\Delta x}(x)$ converges to a function ψ in $\mathrm{L}^1_{\mathrm{loc}}(\mathbb{R})$. Then for all $\zeta \in \mathbb{R}$, $\psi_{\Delta x}(x + \zeta \Delta x)$ converges to $\psi(x)$ as $\Delta x \to 0$ for a.e. $x \in \mathbb{R}$.*

Let us recall the notations $\rho_{\Delta x}(t, x) := \rho^n_j$, and $\rho^{(l)}_{\Delta x}(t, x) := \rho^{(l)}_j$ for $(t, x) \in [t^n, t^{n+1}) \times (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$, $l = 1, 2$ with $\rho^{(l)}_j$ computed from ρ^n_j for all $j \in \mathbb{Z}$. Now, we present a lemma which is used in the proof of Theorem 5.4.

Lemma C.2. *Assume that $\rho_{\Delta x}$ and $\rho^{(l)}_{\Delta x}$, $l = 1, 2$ obtained from (5.2) are uniformly bounded and that $\rho_{\Delta x}$ converges to a function ρ in $\mathrm{L}^1_{\mathrm{loc}}([0, T) \times \mathbb{R})$. Then $\rho^{(l)}_{\Delta x}$ converges to ρ in $\mathrm{L}^1_{\mathrm{loc}}([0, T) \times \mathbb{R})$ for $l = 1, 2$.*

Proof. From the first time-step in (5.2), we can write

$$\rho^{(1)}_{\Delta x}(t,x) = \rho_{\Delta x}(t,x) - \lambda\left(f_{\Delta x}\big(t, x+\frac{\Delta x}{2}\big) - f_{\Delta x}\big(t, x-\frac{\Delta x}{2}\big)\right), \tag{C.1}$$

where $f_{\Delta x}(t,x) := g(\rho^n_j)V^n_{j+\frac{1}{2}}$ for $(t,x) \in [t^n, t^{n+1}) \times (x_j, x_{j+1}]$. Clearly

$$f_{\Delta x}(t, x+\frac{\Delta x}{2}) = g\left(\rho_{\Delta x}(t,x)\right) v\left(R_{\Delta x}(t, x+\Delta x)\right),$$

where

$$R_{\Delta x}(t,x) := R^n_{j-\frac{1}{2}} \quad \text{for } x \in (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}], t \in [t^n, t^{n+1}).$$

Note that

$$R_{\Delta x}(t, x+\Delta x) = \frac{1}{2}\int_{x_{j-\frac{1}{2}}}^{x_{j-\frac{1}{2}}+\eta} \rho_{\Delta x}(t, y+\Delta x)\left(w_{\eta,\Delta x}(y - x_{j-\frac{1}{2}} + \Delta x) + w_{\eta,\Delta x}(y - x_{j-\frac{1}{2}} + 2\Delta x)\right) \mathrm{d}y,$$

for $x \in (x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}], t \in [t^n, t^{n+1})$, where $w_{\eta,\Delta x}(x) := w^k_\eta$ for $x \in (k\Delta x, (k+1)\Delta x]$ and $w_{\eta,\Delta x}(0) := w_\eta(0)$. Using Lemma C.1 and employing the dominated convergence theorem, we can conclude that the term $R_{\Delta x}(t, x+\Delta x)$ converges to $\int_x^{x+\eta} \rho(t,y) w_\eta(y-x)\,\mathrm{d}y$ as $\Delta x \to 0$. Now, using the continuity of g and v, we can conclude that $f_{\Delta x}\big(t, x+\frac{\Delta x}{2}\big)$ converges to $g\big(\rho(t,x)\big)v\big(\int_x^{x+\eta} \rho(t,y)w_\eta(y-x)\,\mathrm{d}y\big)$ *a.e.* Similarly, it can be shown that $f_{\Delta x}\big(t, x-\frac{\Delta x}{2}\big)$ also converges to $g(\rho(t,x))v\big(\int_x^{x+\eta} \rho(t,y)w_\eta(y-x)\,\mathrm{d}y\big)$ *a.e.* Thus, taking the limit $\Delta x \to 0$ in (C.1), we conclude that $\rho^{(1)}_{\Delta x}$ converges to ρ *a.e.* Additionally, as $\rho^{(1)}_{\Delta x}$ is uniformly bounded by hypothesis, the dominated convergence theorem implies that $\rho^{(1)}_{\Delta x}$ converges to ρ in $\mathrm{L}^1_{\mathrm{loc}}([0,T)\times\mathbb{R})$. Following similar arguments, we can show that $\rho^{(2)}_{\Delta x}$ converges to ρ in $\mathrm{L}^1_{\mathrm{loc}}([0,T)\times\mathbb{R})$. □

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